DSC5211C QUANTITATIVE RISK MANAGEMENT

SESSION 10

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Catastrophic Risk Management

Objectives

- Introduction to:
  - Catastrophic Risk Exposure
  - Measures of Risk Aversion
  - Utility functions and Risk Aversion
  - Mean-Risk Models
  - Coherent Risk Measures
- Robust optimization

Catastrophic Risks and Insurance

- A Catastrophic event has a very low probability but a very high impact risks: accidents.
- As we said before this type of risk should be transferred to a third party.
    - **Transfer**: these are low-probability, high-impact risks. The risk can be transferred with an insurance or contract with a third party.

- Example of insurance contracts for these types of risk:
    - Term life insurance. The insurer pays a certain sum in case of death of the ensured person. The probability of death is very low and therefore the insurance premium is also low.
    - Accident insurance. The insurer provides cover for financial loss associated with accidents.

Catastrophic Risks

- People often fail to purchase insurance against low-probability high-loss events even when it is offered at favorable premiums (Kunreuther and Pauly, 2004).

- One obvious characterization of such behavior is that it represents non-optimal decision-making.

- OR: individuals maximize their utility (they are rational) but there are transaction costs and information opacity that prevent an optimal behavior.

Catastrophic Risks and Insurance

- In order for people to insurer against catastrophic risks:
  - More information on the probability of loss needs to be freely available.
  - The profit margins need to be explicit and transparent.
  - Aggregation of catastrophic risk insurance into “bundles” that includes several risks increases the perceived probability of loss and therefore the willingness to buy insurance.

Catastrophic Risks and Insurance

- Catastrophes may produce losses that are rare and highly correlated in space and time: earthquakes, floods, fires.
- Catastrophic risks may ruin many insurers if their portfolio is not geographically diversified.
- Catastrophe modeling needs to take into account the possible correlations between the different assets in the portfolio in order to estimate correctly the risk exposure of the insurance companies.

Catastrophic Risks and Portfolio Theory

- Portfolio theory proves that risk can be reduced by diversification, e.g., Bodie, Kane and Marcus (2002).

- ***Hedging*** is the holding positions in assets (projects) with payoffs that offset exposure to a particular source of risk.

- ***Diversification*** limits portfolio risk as by holding a variety of assets (projects) the portfolio limits the exposure to a specific source of risk.

- This conclusion is important for the management of catastrophic risks. Insurance companies may reduce their exposure to this risks by having a large portfolio of clients with different risk profiles.

Catastrophic Risks and Portfolio Theory

- Assume that an insurance company has a portfolio of *n* clients; each one of which has an expected value of claims r_i: in this case this represents the expected payout per contract/year.

- The expected payout for all the *n* insurance contracts will be (w_i is the proportion of contracts with expected payout r_i):

$$E(r_p) = \sum_{i=1}^{n} w_i . r_i$$

- The portfolio payout variance is given by

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)$$

Catastrophic Risks – The Insurance Problem

- If all the risks are very highly correlated (correlation equal to 1) the variance is given by:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)$$

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij}\sigma_i\sigma_j$$

If all $\rho_{ij} = 1$

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_i\sigma_j$$

- *In this case the portfolio fails to reduce risk. In the extreme case the variance of the portfolio is the same as the one of an individual contract.*

Measures of Extreme Risk

- V@R
- CV@R
- Worst-case Analysis

Minimizing Expected Cost

- In the stochastic program that minimizes expected cost we are optimizing a random outcome on *average*.

- This approach can justified when:
  - The law of large numbers can be applied
  - We are interested in the long-term performance, irrespectively of short-term fluctuations.

- The optimization problem is:

$$\underset{x \geq 0}{Min}\ \mathrm{E}\left[F(x)\right]$$

Mean-Risk Models

- We characterize the uncertain outcome by two scalar characteristics:
  - Mean E(F)
  - Risk - a measure of dispersion $\Phi[F]$

  which measures the uncertainty of the outcome.

- Typically we use a coefficient $c \geq 0$ which plays the role of price of risk. The problem can be formulated as:

$$\underset{x \geq 0}{Min}\ \mathrm{E}[F(x)] + c.\Phi[F(x)]$$

Coherent Risk Measures - Cost

- Consider the following axioms associated with a risk measure ρ.
- Let *F* represent a random cost.
    - $F \succ F'$ is a partial order over *F* such that $F(w) > F'(w)$

      In every possible state the cost associated with *F* is larger than the one associated with *F'*.

- ***Smaller costs are better***.

Coherent Risk Measures - Cost

- **Monotonicity**

For all F, F' if $F(w) \geq F'(w)$ then $\rho(F) \geq \rho(F')$

- Higher losses mean higher risk.

- Variance is ***not*** monotonic.

If F is a constant and F' is a variable.
Then $\sigma^2(F) < \sigma^2(F')$

- V@R and CV@R are monotonic.

Coherent Risk Measures - Cost

- **Translation Equivariance**

- Increasing (or decreasing) the loss by a constant, increases (or decreases) the risk by the same amount.

For any real number α for all F

$$\rho(F+\alpha) \geq \rho(F)+\alpha$$

- V@R and CV@R are translation equivariant.

- Is Variance translation equivariant?

Coherent Risk Measures - Cost

- **Positive Homogeneity**
- The risk increases directly with the size of the exposure.
- Doubling portfolio size doubles risk.

For all F and $t > 0$ $\rho(tF) = t\rho(F)$

VARIANCE:

$$\sigma^2(tF) = t^2\sigma^2(F) \neq t\sigma^2(F)$$

- Variance is ***not*** positive homogeneous.
- V@R and CV@R are positive homogeneous.

Coherent Risk Measures - Cost

- **Convexity**

For all $F, F', t \in [0,1]$ $\qquad \rho(tF + (1-t)F') \leq t\rho(F) + (1-t)\rho(F')$

- Diversification decreases risk.
- Is variance convex?
- CV@R is convex.
- V@R is ***not*** convex.

Coherent Risk Measures - Cost

- **Convexity – Example**

Consider two possible investments A and B, with three possible scenarios w_1, w_2, w_3, with probabilities $p(w_i)$.

Losses for investments A and B:

	w_1	w_2	w_3
$p(w_i)$	0.04	0.04	0.92
A	1000	0	0
B	0	1000	0

Coherent Risk Measures - Cost

- **Convexity – Example (V@R)**

$V@R_{0.05}(A) = 0$ $\quad$ $V@R_{0.05}(B) = 0$ $\quad$ $V@R_{0.05}(A+B) = 1000$

$V@R_{0.05}(A+B) > V@R_{0.05}(A) + V@R_{0.05}(B)$

$V@R_{0.05}(0.5A+0.5B) = 500$

$V@R_{0.05}(A+B) > 0.5V@R_{0.05}(A) + 0.5V@R_{0.05}(B)$

- V@R is ***not*** convex.

Coherent Risk Measures - Cost

- **Convexity – Example (CV@R)**

$$CV@R_{0.05}(A) = CV@R_{0.05}(B) = \frac{0.04 \times 1000 + 0.01 \times 0}{0.05} = 800$$

$$CV@R_{0.05}(A+B) = \frac{0.04 \times 1000 + 0.01 \times 1000}{0.05} = 1000$$

$$CV@R_{0.05}(A+B) < CV@R_{0.05}(A) + CV@R_{0.05}(B)$$

Coherent Risk Measures - Cost

- **Convexity – Example (CV@R - Continued)**

$$CV@R_{0.05}(A) = CV@R_{0.05}(B) = \frac{0.04 \times 1000 + 0.01 \times 0}{0.05} = 800$$

$$CV@R_{0.05}(0.5A + 0.5B) = \frac{0.04 \times 1000 \times 0.5 + 0.01 \times 1000 \times 0.5}{0.05} = 500$$

$$CV@R_{0.05}(0.5A + 0.5B) < 0.5CV@R_{0.05}(A) + 0.5CV@R_{0.05}(B)$$

- ***CV@R is convex.***

Coherent Risk Measures - Cost

- A risk measure is **coherent** if it satisfies:
  - Convexity
  - Monotonicity
  - Translation equivariance
  - Positive Homogeneity

- ***CV@R is a coherent measure of risk***

- V@R is not a coherent measure of risk.
  - Var is not convex.
  - Violates diversification principle.

Coherent Risk Measures - Profit

- Consider the following axioms associated with a risk measure δ.
  - Let *F* represent a random REWARD, for example, and $F \succ F'$ be a partial order over *F* such that $F(w) > F'(w)$, i.e., in every possible state the Reward associated with F is larger than the one associated with *F'*.
  - Larger rewards are preferred

$$\delta(F) = \rho(-F)$$

Coherent Risk Measures - Profit

- **Convexity**

For all $F, F', t \in [0,1]$ $\delta(tF + (1-t)F') \le t\delta(F) + (1-t)\delta(F')$

- **Monotonicity**

For all F, F' if $F(w) \ge F'(w)$ then $\delta(F) \le \delta(F')$

- **Translation Equivariance**

For any real number α for all F

$$\delta(F+\alpha) \ge \delta(F) - \alpha$$

Coherent Risk Measures - Profit

- **Positive Homogeneity**

For all F and $t > 0$ $\qquad$ $\delta(tF) = t\delta(F)$

- A risk measure is **coherent** if it satisfies:
  - Convexity
  - Monotonicity
  - Translation equivariance
  - Positive Homogeneity

Robust Optimization: Worst – Case Analysis

- The stochastic characterization of uncertainty relies on the average or expected performance of the system in the presence of uncertain effects.

- Although expected performance optimization is often adequate, it is the realization of the **worst-case** that causes the failure of the system, Rustem & Howe (2002).

- An important tool to address the inherent error for forecasting uncertainty is worst-case analysis.

- Worst-case analysis (min-max) provides robust optimal strategies that yield guaranteed performance.

Robust Optimization - II

- From the risk management point of view, minimax yields the best strategy determined simultaneously with the worst state of the underlying system and also analyses the effects of uncertain events.

- Min-max optimal strategy is determined in view of all the scenarios, rather than any single scenario.

- Min-max optimisation is more robust to the realisation of worst-case scenarios than considering a single scenario or an arbitrary pooling of scenarios.

Worst Case Approach

- The worst-case approach minimizes the objective function with respect to the worst possible outcome of the uncertain variables.

- Let $G : R^{n+m} \to R$ be a function of a decision variable x and uncertain variable w for a stochastic system.

- The general minimax optimization problem can be stated as:

$$\min_{x \in X} \quad \max_{w \in W} \quad G(x, w)$$

$$s.t \qquad X \subset R^n, W \subset R^m$$

Worst Case Approach – Discrete States

- To be on the conservative side, the decision x is required to be ***optimal with respect to each observation*** of uncertain variable w.

- x is chosen to minimize the objective function, where nature chooses w to maximize it.

- When the objective function is convex with respect to the uncertain variables the maximum will correspond to one or more vertices of the hypercube defined by the upper and lower bounds on the uncertain variables.

- If the objective function is concave with respect to the uncertainties the maximum may lie anywhere within the hypercube.

- The discrete minimax problem arises when the worst-case is to be determined over a discrete set.

- If W is a finite set, then it is called a discrete minimax problem and formulated as

$$\min_{x \in X} \quad \max_{w \in W} \quad G(x, w)$$

$$s.t$$

$$X \subset R^n$$

$$W = \{w_1, w_2, \cdots, w_m\}$$

Discrete Minimax - Solving it

- Let u denote the worst-case cost.
- The worst cost in view of all rival scenarios is the maximum.
- This is described by a set of constraints.
- Then the minimax problem can be reformulated as the following linear programming problem:

$$\min_{x \in X, u} \quad u$$

$$s.t$$

$$G(x, w) \leq u \qquad for \;\; all \quad w \in W$$

$$X \subset R^n$$

$$W = \{w_1, w_2, \cdots, w_m\}$$

Inventory Management

- The risk-neutral optimization problem is:

$$\underset{x \geq 0}{Min} \; \mathrm{E}\left[F(x, D)\right]$$

D – represents a known probability distribution of demand.
x- order quantity of a certain product
d – demand for the product
c – ordering cost per unit
b – re-ordering cost per unit if demand is larger than *x*.
h – holding cost per unit if demand is smaller than *x*.
b > c > 0
F – total cost
$^{+}$ - states that the variable (or expression) is non-negative

$$F(x,d) = cx + b(d-x)^{+} + h(x-d)^{+}$$

$b(d\text{-}x)^{+}$ represents the re-ordering cost when demand is *larger* than the orders

$h(x\text{-}d)^{+}$ represents the holding cost when demand is *less* than the orders

- The optimization problem is:

$$\underset{x \geq 0}{Min}\ F(x,d)$$

- If demand is known the minimum is attended at $d = x^{*}$:

Two-Stage Recourse Models

- This is a two-stage recourse model
  - In stage 1 we decide the order
  - In stage 2 we decide the recourse actions, level of storage or re-ordering quantities in order to cope with actual demand.

Two-Stage Recourse Models – Risk Neutral

- In some cases we can derive a closed form solution for this problem. However in most practical applications such a solution is not possible.

- In this case, we can build scenarios and solve the problem numerically

- S – number of scenarios

$$\mathrm{E}\left[F(x,D)\right]=\sum_{s=1}^{S} p_s F(x,d_s)$$

- p_s – probability of scenario s

- v_s – free variable

Two-Stage Recourse Models – Risk Neutral

$$\underset{x, v_1, \ldots, v_S}{Min} \sum_{s=1}^{S} p_s v_s$$

$$s.t.$$

$$v_s \geq (c-b)x + bd_s, \qquad \text{for } s = 1,\ldots,S$$

$$v_s \geq (c+h)x - hd_s, \qquad \text{for } s = 1,\ldots,S$$

$$x \geq 0$$

- This is equivalent to minimize the expected cost subject to the constraint that, at each scenario, the cost (and binding constraint) is the larger of the two possibilities (to re-order or to hold inventory).

Two-Stage Recourse Models – Worst-Case

$$
\begin{aligned}
&\underset{x,u}{Min}\ u \\
&s.t. \\
&u \geq (c-b)x + bd_s, && \text{for } s = 1,\ldots,S \\
&u \geq (c+h)x - hd_s, && \text{for } s = 1,\ldots,S \\
&x \geq 0
\end{aligned}
$$

- This is equivalent ***to minimize the worst-case cost subject*** to the constraint that, at each scenario, the cost (and binding constraint) is the larger of the two possibilities (to re-order or to hold inventory).

Two-Stage Recourse Models – Risk-Neutral

$$\begin{cases} \Pr(D = 125) = 1/2 \\ \Pr(D = 75) = 1/2 \end{cases}$$

c = 10 b = 20 h = 1

$$\min_{x, v_1, v_2} 0.5 v_1 + 0.5 v_2$$

$$s.t.$$

$$v_1 \geq (10 - 20)x + 20 \times 125$$

$$v_1 \geq (10 + 1)x - 1 \times 125$$

$$v_2 \geq (10 - 20)x + 20 \times 75$$

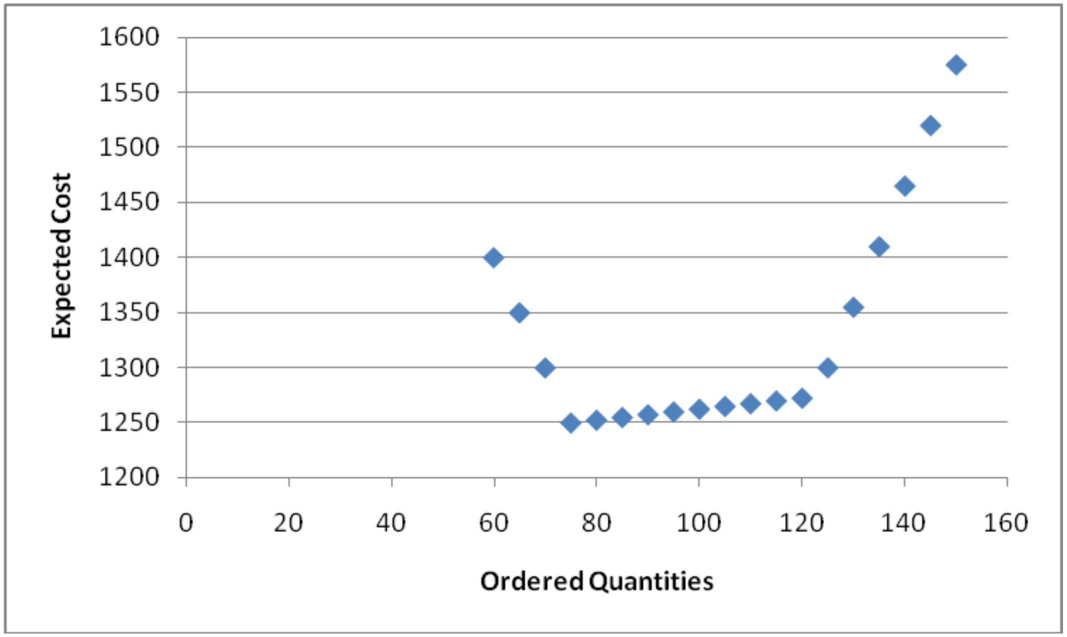
$$v_2 \geq (10 + 1)x - 1 \times 75$$

$$x \geq 0$$

Risk-Neutral Solution

$v_1^* = 1750$ $\quad$ $v_2^* = 750$ $\quad$ $x^* = 75$

$$\mathrm{E}\left[F(x^*, D) \right] = 1250$$



Two-Stage Recourse Models – Worst-Case

$$\begin{cases} \Pr(D=125)=1/2 \\ \Pr(D=75)=1/2 \end{cases}$$

c = 10 b = 20 h = 1

$$\underset{x,u}{Min}\; u$$

$$s.t.$$

$$u \geq (10-20)x + 20\times 125$$

$$u \geq (10+1)x - 1\times 125$$

$$u \geq (10-20)x + 20\times 75$$

$$u \geq (10+1)x - 1\times 75$$

$$x \geq 0$$

Worst-Case Solution

$u^* = 1273.81$ $\qquad$ $x^* = 122.619$

$$\mathrm{E}\left[F(x^*, D)\right] = 1273.81$$

- There is a trade-off between risk and expected cost.
- The firm now has a higher expected cost.
- There is no risk, as the firm knows, with certainty, the cost of ordering and storing *x* units of the product.

Conclusions

- We introduced:
    - Modeling catastrophic Risk.
    - Mean-Risk Models.

- We discuss coherent risk measures.

- We introduced the concept of robustness and worst-case analysis.

Reading List

Kevin Dowd, *An Introduction to Market Risk Measurement*, Wiley Finance 2002.

Kunreuther, H., and M. Pauly (2004) “Neglecting Disaster: why don’t people insure against large losses?” *The Journal of Risk and Uncertainty* 28 (1): 5-21.

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