DSC5211C QUANTITATIVE RISK MANAGEMENT

SESSION 6

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Stochastic Programming

Objectives

- Introduction to stochastic programming
- The General Recourse Model
- Probabilistic Constraints
- Data requirements.

Introduction to Stochastic Programming

- We analyze how to use linear programming to model problems with a stochastic component.
- We consider models with a simple resource structure in which decisions in a second period are taken after the observation of the events that were uncertain in the first period.
- In some problems there is a sequence of decisions and stochastic events that are interdependent. We present a model for such problems.
- Stochastic programming also considers problems in which the constraints are only binding with some known probability.

Importance of Uncertainty in Model Building

- There are some naïve ways of dealing with uncertainty:
  - We can look at the *expected values* only and maximize (minimize) expected profit (cost).
  - We can do a what-if analysis and compare the optimal decisions under different scenarios for the stochastic variables.
- The naïve approaches do not fully address the possible impact of uncertainty in the problem we are modeling.
- Instead we need to use a modeling framework that explicitly considers uncertainty when optimizing the decisions.

Minimizing Cost

- Let us look at the problem of minimizing cost in an inventory management problem.

x- order quantity of a certain product
d – demand for the product
c – ordering cost per unit
b – re-ordering cost per unit if demand is larger than x
b > c > 0
F – total cost

$^{+}$ - states that the variable (or expression) is non-negative

$$F(x,d) = cx + b(d-x)^+ + h(x-d)^+$$

$b(d\text{-}x)^+$ represents the re-ordering cost when demand is *larger* than the orders

$h(x\text{-}d)^+$ represents the holding cost when demand is *less* than the orders

- The optimization problem is:

$$\underset{x \geq 0}{Min}\ F(x,d)$$

- If demand is known the minimum is attended at $d = x^*$.

Minimizing Expected Values

- The optimization problem is:

$$\underset{x \geq 0}{Min}\ F(x, E(D))$$

D: represents a known probability distribution of demand and

E(D): stands for the expected value of this distribution.

Minimizing Expected Values - Example

$$\begin{cases} \Pr(D=125)=1/2 \\ \Pr(D=75)=1/2 \end{cases}$$

$c = 10$
$b = 20$
$h = 1$

$$F(x,D)=10x+20(D-x)^{+}+(x-D)^{+}$$

$$\underset{x\geq 0}{Min}\ F(x, E(D)) =$$

$$\underset{x\geq 0}{Min}\left[10x + 20(E(D) - x)^{+} + (x - E(D))^{+}\right] =$$

$$\underset{x\geq 0}{Min}\left[10x + 20(100 - x)^{+} + (x - 100)^{+}\right]$$

Solution: $x^{*} = 100$

F(x, E(D)) = 1000

Min. Expected Values ISSUES - Example

Scenario 1: $d = 125$

$$F(x=100, d=125) = 10\times 100 + 20\times(125-100) + 0 = 1500$$

Scenario 2: $d = 75$

$$F(x=100, d=75) = 10\times 100 + 0 + (100-75) = 1025$$

$$\mathrm{E}\left[F(x=100, D)\right] = 0.5\times 1500 + 0.5\times 1025 = 1262.2$$

$$\boxed{F(x, E(D)) \neq E\left[F(x, D)\right]}$$

Minimizing Expected Cost

- The optimization problem is:

$$\underset{x\geq 0}{Min}\ \mathrm{E}\left[F(x,D)\right]$$

D – represents a known probability distribution of demand.

x – the order is made in stage 1

d – demand for the product is revealed in stage 2
b – re-ordering cost per unit. Re-ordering is made after demand is revealed if larger than *x*.
h – if demand is smaller than *x* we keep the remaining items in storage.

Two-Stage Recourse Models

- This is a two-stage recourse model:
  - In stage 1 we decide the order
  - In stage 2 we decide the recourse actions, level of storage or re-ordering quantities in order to cope with actual demand.

Two-Stage Recourse Models - Solution

- In some cases we can derive a closed form solution for this problem. However in most practical applications such a solution is not possible.

- In this case, we can build scenarios and solve the problem numerically.

- S – number of scenarios

$$\mathrm{E}\left[F(x,D)\right]=\sum_{s=1}^{S} p_s F(x,d_s)$$

- p_s – probability of scenario s

- v_s – free variable

Two-Stage Recourse Models – Solution (Cont.)

$$\underset{x, v_1, \ldots, v_S}{Min} \sum_{s=1}^{S} p_s v_s$$

$$s.t.$$

$$v_s \geq (c-b)x + bd_s, \qquad \text{for } s = 1,\ldots,S$$

$$v_s \geq (c+h)x - hd_s, \qquad \text{for } s = 1,\ldots,S$$

$$x \geq 0$$

- This is equivalent to minimize the expected cost subject to the constraint that, at each scenario, the cost (and binding constraint) is the larger of the two possibilities (to re-order or to hold inventory).

Two-Stage Recourse Models - Example

$$\begin{cases} \Pr(D = 125) = 1/2 \\ \Pr(D = 75) = 1/2 \end{cases}$$

c = 10 $\quad b = 20 \quad h = 1$

$$\underset{x, v_1, v_2}{Min}\ 0.5 v_1 + 0.5 v_2$$

$$s.t.$$

$$v_1 \geq (10 - 20)x + 20 \times 125$$

$$v_1 \geq (10 + 1)x - 1 \times 125$$

$$v_2 \geq (10 - 20)x + 20 \times 75$$

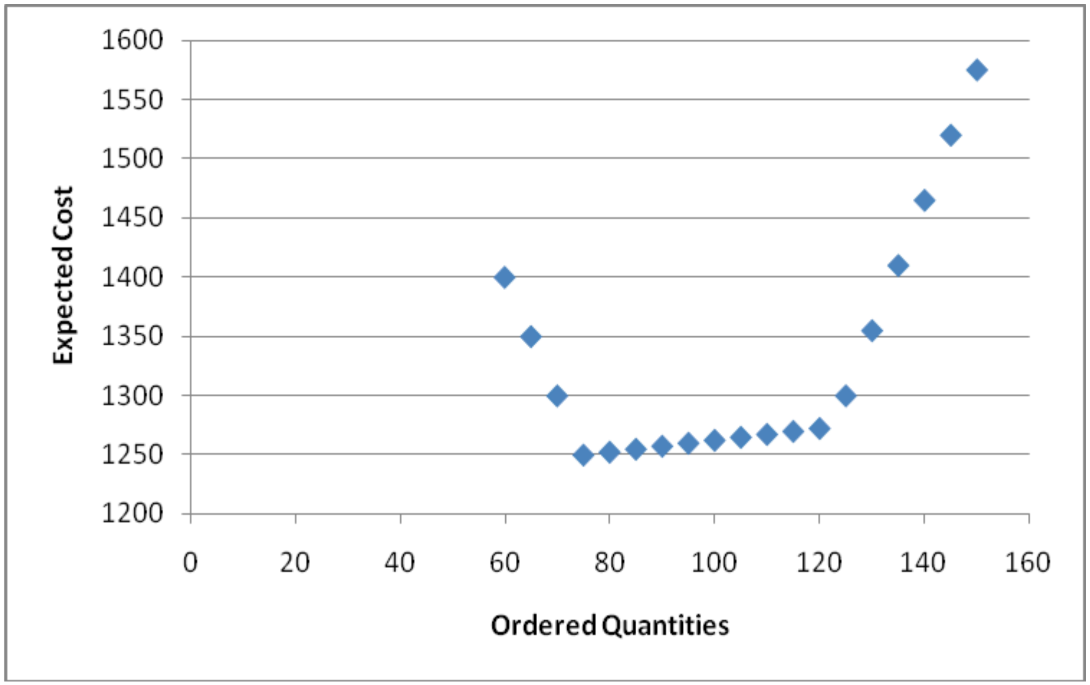
$$v_2 \geq (10 + 1)x - 1 \times 75$$

$$x \geq 0$$

Example Solution

$v_1^* = 1750 \qquad v_2^* = 750 \qquad x^* = 75$

$$\mathrm{E}\left[F(x^*, D) \right] = 1250$$



Probabilistic Constraints

- One of the consequences of uncertainty is the possibility of infeasibility in the future.

- In some circumstances it may be appropriate to accept the possibility of infeasibility with some probability. Examples:

  - A customer, in a given day, is not visited by the delivery van with a probability of 1/1000.
  - A priority mail letter arrives delayed with a probability of 1%.
  - An airline customer is refused boarding with 2% probability.

- We can try to incorporate directly in the problem constraints that are binding with some probability.

Probabilistic Constraints – A model

$$
\begin{aligned}
&Min\ c(x) \\
&s.t. \\
&\Pr\left[g_j(x,Y) \le 0,\ j = 1\ldots,J \right] \ge p
\end{aligned}
$$

$c(x)$ – cost function

x – vector of decision variables

Y – vector of random variables

Probabilistic Constraints – Example (Portfolio Selection)

- Suppose we want to invest a capital W_0 in n assets by investing an amount x_i in asset i, for $i = 1, \ldots, n$.

- Each asset i has a rate of return R_i per period of time, which is unknown at the time of the investment.

- The total wealth after 1 period is:

$$W_1 = \sum_{i=1}^{n} a_i x_i$$

with $a_i = 1 + R_i$

Portfolio Selection - Continued

- Balance constraint $\sum_{i=1}^{n} x_i = W_0$
- $E[W_1] = \sum_{i=1}^{n} E(a_i)x_i = \sum_{i=1}^{n} u_i x_i$

$u_i = 1 + E[R_i]$

- In general we also want to control the risk associated to the investment. This, in finance models is usually represented by the Variance.

$$VAR[W_1] = \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j = x'\Sigma x$$

Σ stands for the covariance matrix of the a_i.

- Optimization problem

$$\underset{x}{Min}\ x'\Sigma x$$

$$s.t.$$

$$\sum_{i=1}^{n} x_i = W_0$$

$$\sum_{i=1}^{n} u_i x_i \geq \tau$$

$$x \geq 0$$

Portfolio Selection – Probabilistic Constraints

$$\underset{x}{Max} \ \sum_{i=1}^{n} u_i x_i$$

$$s.t.$$

$$\sum_{i=1}^{n} x_i = W_0$$

$$\Pr\left[\sum_{i=1}^{n} a_i x_i \geq b\right] \geq 1-\alpha$$

$$x \geq 0$$

- Assume that the returns follow a multivariate normal distribution. In this case W_1 also follows a normal distribution with mean $\sum_{i=1}^{n} u_i x_i$ and standard deviation $x'\Sigma x$.

Then

$$\Pr\left[\sum_{i=1}^{n} a_i x_i \geq b\right] = \Pr\left[W_1 \geq b\right] = \Pr\left[Z \geq \frac{b - \sum_{i=1}^{n} u_i x_i}{\sqrt{x'\Sigma x}}\right] = \Phi\left(\frac{\sum_{i=1}^{n} u_i x_i - b}{\sqrt{x'\Sigma x}}\right)$$

Z~*N*(0, 1) follows a standard normal distribution.

$\Phi\left(z\right) = \Pr(Z \leq z)$ is the cumulative distribution function of Z.

Then, it follows that:

$$\Pr\left[\sum_{i=1}^{n} a_i x_i \geq b\right] \geq 1-\alpha$$

$$\Phi\left(\frac{\sum_{i=1}^{n} u_i x_i - b}{\sqrt{x'\Sigma x}}\right) \geq 1-\alpha$$

$$\frac{\sum_{i=1}^{n} u_i x_i - b}{\sqrt{x'\Sigma x}} \geq \Phi^{-1}(1-\alpha)$$

$$\frac{\sum_{i=1}^{n} u_i x_i - b}{\sqrt{x'\Sigma x}} \geq z_\alpha$$

- Then we can derive a new representation of the probabilistic constraint:

$$\sum_{i=1}^{n} u_i x_i - b \geq z_\alpha \sqrt{x'\Sigma x}$$

$$\boxed{b - \sum_{i=1}^{n} u_i x_i + z_\alpha \sqrt{x'\Sigma x} \leq 0}$$

Portfolio Selection – New Formulation

$$\underset{x}{Max} \sum_{i=1}^{n} u_i x_i$$

$$s.t.$$

$$\sum_{i=1}^{n} x_i = W_0$$

$$b - \sum_{i=1}^{n} u_i x_i + z_\alpha \sqrt{x'\Sigma x} \le 0$$

$$x \ge 0$$

Data Requirements

- Cost estimates. Example:
  - Ordering cost
  - Holding cost
  - Re-ordering cost, etc.

- Demand estimates for each scenario.

- The probability of each scenario.

- The covariance matrix.

Data Requirements – Costs Estimates

- Ordering, holding, and re-ordering costs are usually known, subject to a contractual arrangement and we do not model uncertainty about them.

- Production costs may be highly uncertain:
    - For example if dependent on prices of raw materials.

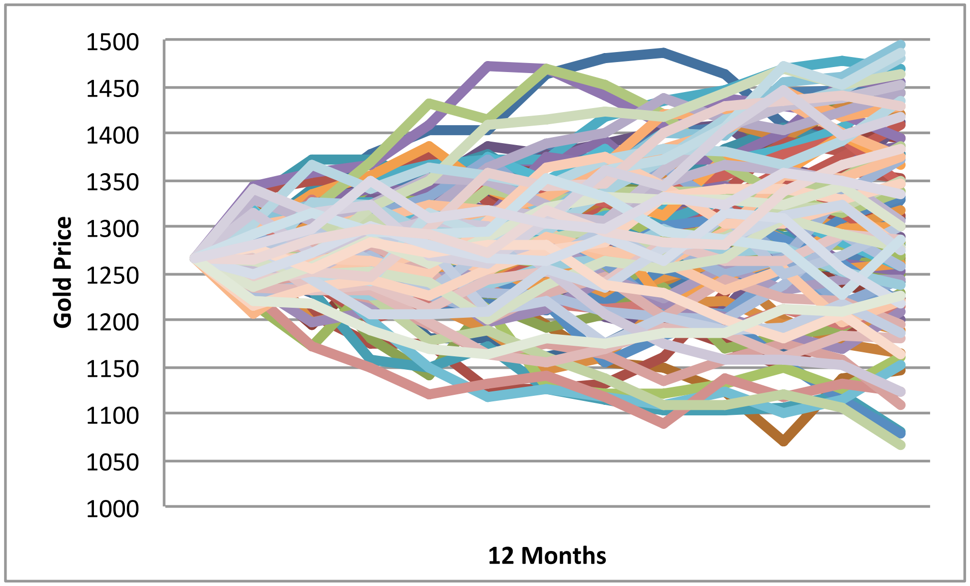
- Demand may also be very uncertain.

- Price (production cost) and Demand uncertainty can be modeled using scenarios.

Data Requirements – Scenarios

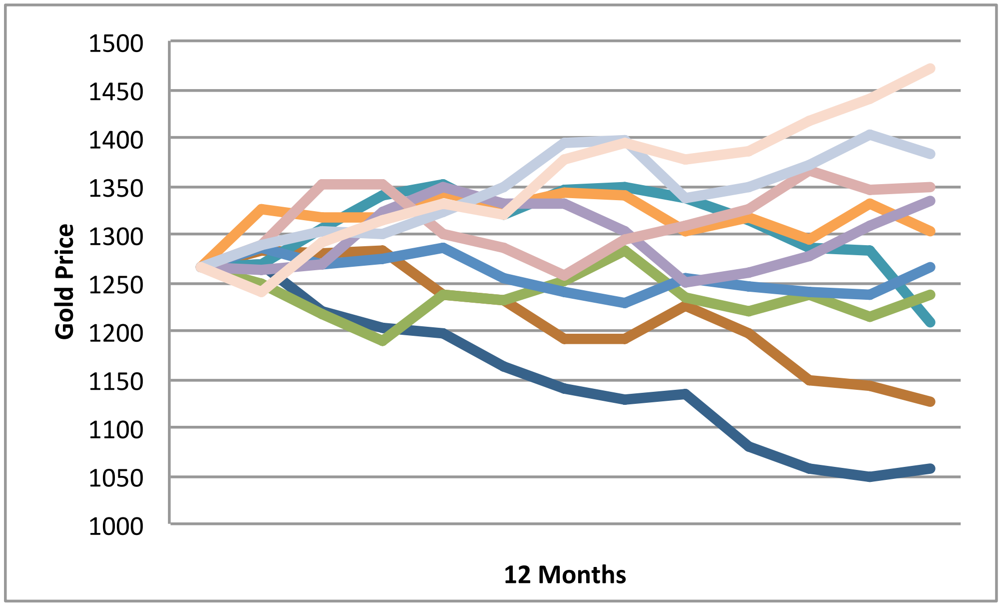
- From the estimated model, i.e., random walk, smoothing methods, ARIMA, Multivariate models, simulate *S* sample paths for the period under analysis.

  $P_t = a + P_{t-1} + NORMINV(RAND(), 0\ , SD\ Change)$



Data Requirements – Scenario Reduction Techniques

- It is also possible to aggregate the scenarios in a representative cluster assigning to each one of them a probability.



- Each one of these sample paths is then used as a scenario in the stochastic optimization model.

Data Requirements – Covariance Matrix

- When we forecast several variables simultaneously, if they are correlated, this needs to be modeled in the Monte-Carlo simulation.

- In the Portfolio Optimization model the Covariance Matrix captures this autocorrelation.

- The Covariance Matrix can be estimated from:
  - Past data. But in this case what is the best time window to use? The entire data set? The last couple of years?

  - Guessed when past data does not seem representative.

Conclusions

- We have analyzed how uncertainty affects optimization models:
  - The fallacy of taking expected values and issues with scenario analysis.
  - Problems with infeasible solutions.
- We have introduced the general two-stage recourse model.
- We have introduced probabilistic constraints:
  - We illustrate this methodology in the portfolio selection problem.
  - We show how to solve this problem.
- We have discussed the data requirements of the different methods.