DSC5211C QUANTITATIVE RISK MANAGEMENT

SESSION 9

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Optimisation of the Conditional Value at Risk

Objectives

- Introduction of the Concept of VaR
- Introduction of the Concept of CVaR
- Comparing VaR and CVaR
- Calculation of VaR and CVaR:
    - Historical Data
    - Simulation
    - Optimization
- Comparison of the Advantages and Disadvantages of VaR and CVaR

Value at Risk - Introduction

Origin of VaR stems from financial derivatives losses:

• Question: how much the company could possibly lose by using derivatives – In theory, the company could lose everything.

• How can you communicate the extent of risk exposure?
Using the Value at Risk.

Value at Risk - Definition

- VaR summarizes the worst loss over a target horizon with a given level of confidence

- VaR defines the *quantile* of the projected distribution of gains and losses over the target horizon

- The VaR corresponds to the α tail of the probability distribution (the quantile).

$$\text{VaR} = Q(\alpha)$$

Value at Risk – Alternative Definition

- VaR is the maximum loss that will occur within a given time period T and with a given probability α:

$$prob\left(S_{t+T} - S_t \leq -VaR\right) = \alpha$$

- This is the definition used in most financial models.

Value at Risk – in Finance

Market risk in financial institutions is primarily measured by value at risk (VaR):

- Quantification of losses due to movements in financial market variables (interest rates, foreign exchange rates, equities, and commodities) in given time horizon with certain probability.

- Statistical measure of potential downside risk.

- Simple to explain – one number aggregates the risks across the whole company.

- Time horizons usually from 1 day to 6 months. Confidence level of 95 or 99 percent.

Value at Risk – Historical Data

- It refers to the process of calculating the hypothetical distribution of profit and losses of the current portfolio based on applying historical asset returns.
- The advantage of this approach is that it does not use estimated variances and covariances, and it does not assume anything about the distribution of risk factors.
- The main disadvantage is the assumption that the future risk is much like the past risk. That is less frequent in today's fast changing business environment.

VaR – Historical Data – Advantages/Disadvantages

- **Advantages:**
  - intuitive, simple, easy to report
  - assumes no theoretical distribution
  - can accommodate for fat tails

- **Disadvantages:**
  - data must be representative
  - outliers unlikely to recur
  - we are always looking at the past
  - the sample may be too small.

Value at Risk – Monte Carlo Simulation

- It is the process of calculating the distribution of profit and losses of the current portfolio based on randomly generated movements in risk factors from a given distribution.

- This method is the most flexible, but also carries an enormous computational burden.

- It requires users to make assumptions about the stochastic process and to understand the sensitivity of the results to these assumptions.

Estimating the Value at Risk using Simulation

To estimate the VaR quantile for a risky business use these steps:

1. Develop a stochastic simulation model of the risky business decision

2. Validate stochastic variables and validate the model

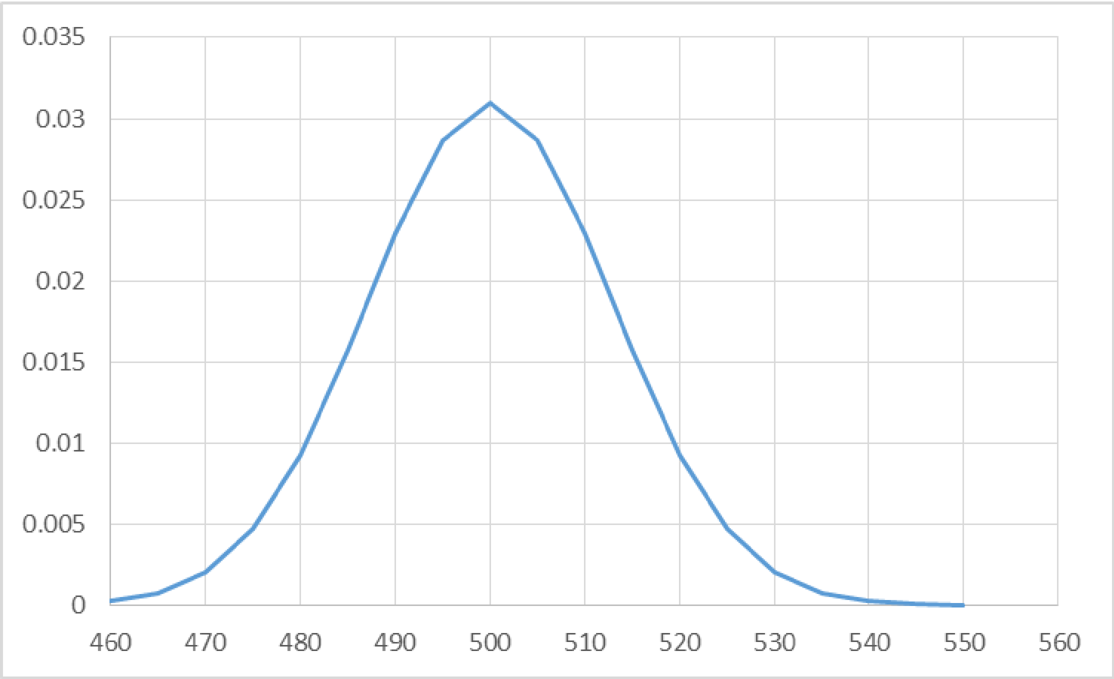
3. Simulate the model

4. Calculate the VaR, i.e., the quantile for the α value

Example: Cost Function (Mean = 500, SD = 12.896)

$\alpha = 0.01$ → VaR = 530

$\alpha = 0.05$ → VaR = 521.212



VaR and Capital at Risk

- VaR can be equity capital that should be set aside to cover most all potential losses with a probability of α

- Thus the VaR is the amount of capital reserves that should be available to meet shortfalls

Conditional Value at Risk (CVaR)

- Conditional Value-at-Risk (CVaR): is the expected value of the losses conditioned in being in excess of VaR.

$$CVaR = \frac{\sum \left(x_j - VaR\right)^+}{n}$$

n: number of observations in the tail.

Conditional Value at Risk – Cont.

- Represents Expected Tail Loss (ETL), Expected Shortfall (ES)

- It represents the expected loss in the α percent worst cases. Captures the frequency and magnitude of extreme losses.

CVar – Example 1

- Generate 1000 points from a distribution with mean = 500 and SD = 12.896

- Filter the lowest 50 points (5%)

- We get an estimate of the Var and Cvar for $\alpha = 0.05$

    - Estimated VaR = 521.91 (Theoretical est. 521.212)
    - The CVaR = 528.05

CVar – Example Cont.

- Generate 1000 points from a distribution with mean = 500 and SD = 12.896

- Filter the lowest 10 points (1%)

- We get an estimate of the VaR and CvaR for $\alpha = 0.01$

    - Estimated VaR = 529.85 (Theoretical est. 530)
    - The CVaR = 536.9

CVaR and VaR Minimization

- CVaR minimizes the expected loss on outcomes that exceed some value α, i.e., the VaR.
- α is chosen so that the probability of the loss being greater than α is less than $1-\beta$.
- s index the set of scenarios that represent the outcomes of some random process. There are S scenarios of equal probability.
- $c(x_s)$ be the cost associated with decision x_s in Xs.
- z_s be the above-threshold loss associated with scenario s

A Linear Program to Minimize VaR and CVaR.

$$\min_{z,x,\alpha} \quad \alpha + \frac{\sum_s z_s}{(1-\beta)S}$$

s.t.

$$z_s \geq c\left(x_s\right) - \alpha \qquad \text{for all } s$$

$$x_s \in X_s, z_s \geq 0 \qquad \text{for all } s.$$

Portfolio Theory – Expected Value and Variance

- The expected return of a project *i* is the weighted average of its return in all scenarios:

$$r_i = \sum_s P_s r_{is}$$

Portfolio Theory – Rate of Return

- The rate of return of a portfolio of projects is the weighted average of each project in the portfolio with weights equal to the project weight in the portfolio

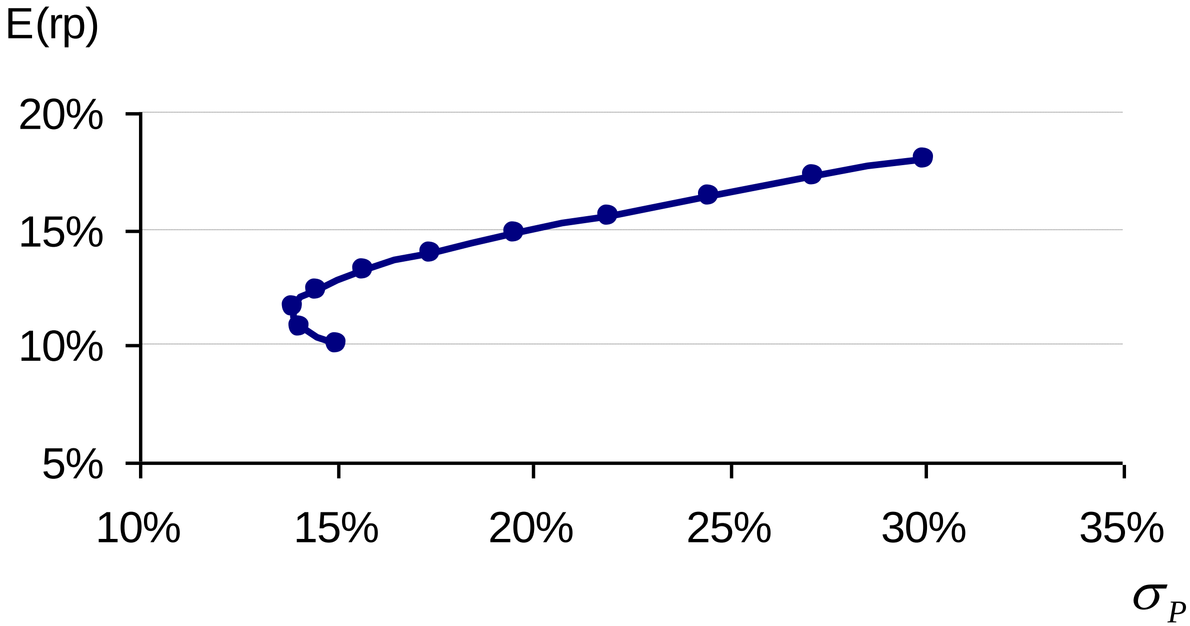
$$E(r_p) = \sum_{i=1}^{n} w_i r_i$$

- When two projects are combined into a portfolio with weights w_1 and w_2 the portfolio variance is given by

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2Cov(r_1, r_2)$$

Portfolio Theory – The Efficient Frontier

- The *efficient frontier of risky* projects: for any level of risk (standard deviation) we are only interested in projects with the highest expected return.



Portfolio Theory with *n* Projects

- The rate of return of a portfolio of *n* projects:

- The portfolio variance is given by

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j Cov\left(r_i, r_j\right) \quad \text{or} \quad \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j$$

- The risk associated to the portfolio is less than the risk of the individual projects for as long as the correlation between the projects' returns is not equal to 1.

VaR and Stochastic Constraints

$$\underset{w}{Max} \sum_{i=1}^{n} r_i w_i$$

$$s.t.$$

$$\sum_{i=1}^{n} w_i = W_0$$

$$\Pr\left[\sum_{i=1}^{n} a_i w_i \geq b\right] \geq 1-\alpha$$

$$b \text{ free, } w_i \geq 0 \quad \forall i$$

- It is a maximization problem s.t. a VaR constraint.
- If *b* is left as a free variable it will be computed to be the *VaR*.

Portfolio Selection – New Formulation

$$\underset{w}{Max} \sum_{i=1}^{n} r_i w_i$$

$$s.t.$$

$$\sum_{i=1}^{n} w_i = W_0$$

$$b - \sum_{i=1}^{n} r_i w_i + z_\alpha \sqrt{w'\Sigma w} \leq 0$$

$$b \text{ free, } w_i \geq 0 \quad \forall i$$

Comparing Var and CVAR

- Since $CVAR(r_p, \alpha) > VAR(r_p, \alpha)$.

- There is a level of significance $\alpha' > \alpha$ such that $CVAR(r_p, \alpha') < VAR(r_p, \alpha)$.

- Therefore, we can always find a VAR which imposes the same constraints as the CVAR.

Minimize VaR and CVaR for the Portfolio Problem

$$\min_{z,w,\alpha} \quad \alpha + \frac{\sum_s z_s}{(1-\beta)S}$$

s.t.

$$z_s \geq -\sum_{i=1}^{n} a_{is} w_i - \alpha$$ for profit, if for cost, should be +sum

for all s

$$\sum_i w_i = 1$$

$$w_i \geq 0$$ for all i.

$$z_s \geq 0$$ for all s.

Minimize VaR and CVaR - Parameters

$$z_s \geq -\sum_{i=1}^{n} a_{is} w_i - \alpha$$

for all *s*.

To generate the observed returns, for each scenario *s*, from a normal distribution **is not correct**, as the parameters for different assets are correlated.

$$a_{is} \sim N(r_i, \sigma_i)$$

Minimize VaR and CVaR - Parameters – Cont.

1. Compute the correlation matrix for all assets. Let *i* be the reference asset. Calculate for all asset *j*.

$$\rho_{ij}$$

2. For asset *i* let

$$y_{is} \sim N(0,1)$$

$$a_{is} = r_i + \sigma_i y_{is}$$

3. For every asset j that is not i calculate:

$$y_{js} \sim N(0,1)$$

$$q_{js} = \rho_{ij} y_{is} + \sqrt{1-\rho_{ij}^2}\, y_{js}$$

$$a_{js} = r_j + \sigma_j q_{js}$$

Result: the correlation between i and j is ρ_{ij}.

This is a simple procedure to ensure that each security maintains the historical relationship with the market index i.

Conclusions

- The Cvar is always greater than the Var, for a given α

- The Cvar is more sensitive to extreme values and captures better the risk exposure.

Reading List

Kevin Dowd, *An Introduction to Market Risk Measurement*, Wiley Finance 2002.

Philippe Jorion, *Value at Risk: The New Benchmark for Managing Financial Risk*, McGraw Hill, 2nd ed: 2001; 3rd ed: 2007.