INEQUALITIES AND EQUALITIES FOR $\ell = 2$ (SYLVESTER), $\ell = 3$ (FROBENIUS), AND $\ell > 3$ MATRICES

Néstor Thome
Instituto Universitario de Matemática Multidisciplinar
Universitat Politècnica de València, València, Spain
njthome@mat.upv.es

Abstract

This paper gives simple proofs of Sylvester ($\ell = 2$) and Frobenius ($\ell = 3$) inequalities. Moreover, a new sufficient condition for the equality of the Frobenius inequality is provided. In addition, an extension for $\ell > 3$ matrices and an application to generalized inverses are provided.

AMS Classification: 15A45; 15A09
Keywords: Sylvester inequality, Frobenius inequality, Moore-Penrose inverse, rank

1 Introduction

Two famous inequalities in Matrix Analysis are Sylvester inequality:

$$\operatorname{rank}(\mathbf{AB}) \geq \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}) - n,$$

and Frobenius inequality:

$$\operatorname{rank}(\mathbf{ABC}) \geq \operatorname{rank}(\mathbf{AB}) + \operatorname{rank}(\mathbf{BC}) - \operatorname{rank}(\mathbf{B}),$$

where the complex matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ have adequate sizes to be accordingly multiplied and n is the number of columns of $\mathbf{A}$ and rows of $\mathbf{B}$.

This paper is devoted to revisited proofs of Sylvester and Frobenius inequalities and conditions for their equalities. It is shown that the basic setting needed to obtain our main results is the rank normal form. In addition, an extension of these results to a finite number of matrices and an application of them to generalized inverses are given.

2 Sylvester's inequality and equality

We first prove the Sylvester inequality [4].

Theorem 1. *Let* $\mathbf{A} \in \mathbb{C}^{m\times n}$ *and* $\mathbf{B} \in \mathbb{C}^{n\times p}$. *Then*

$$\operatorname{rank}(\mathbf{AB}) \geq \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}) - n. \tag{1}$$

Proof. Let consider a rank normal form of $\mathbf{B}$, that is

$$\mathbf{B} = \mathbf{P}\begin{bmatrix} \mathbf{I}_r & \mathbf{O}_{r\times(p-r)} \\ \mathbf{O}_{(n-r)\times r} & \mathbf{O}_{(n-r)\times(p-r)} \end{bmatrix}\mathbf{Q},$$

where $\mathbf{P} \in \mathbb{C}^{n\times n}$ and $\mathbf{Q} \in \mathbb{C}^{p\times p}$ are nonsingular. Partitioning $\mathbf{AP}$ according to the sizes of blocks in $\mathbf{B}$ we can write

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}\mathbf{P}^{-1},$$

where $\mathbf{A}_1 \in \mathbb{C}^{r\times r}$, $\mathbf{A}_2 \in \mathbb{C}^{r\times(n-r)}$, $\mathbf{A}_3 \in \mathbb{C}^{(m-r)\times r}$, and $\mathbf{A}_4 \in \mathbb{C}^{(m-r)\times(n-r)}$. Some simple computations give

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{O} \\ \mathbf{A}_3 & \mathbf{O} \end{bmatrix}\mathbf{Q},$$

and so

$$\operatorname{rank}(\mathbf{AB}) = \operatorname{rank}\left(\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_3 \end{bmatrix}\right).$$

Since

$$\begin{aligned}\operatorname{rank}\left(\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}\right) &\leq \operatorname{rank}\left(\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_3 \end{bmatrix}\right) + \operatorname{rank}\left(\begin{bmatrix} \mathbf{A}_2 \\ \mathbf{A}_4 \end{bmatrix}\right) \\ &\leq \operatorname{rank}\left(\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_3 \end{bmatrix}\right) + (n-r),\end{aligned}$$

it then follows

$$\begin{aligned}\operatorname{rank}(\mathbf{A})+\operatorname{rank}(\mathbf{B}) &\leq \operatorname{rank}\left(\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_3\end{bmatrix}\right)+n\\ &= \operatorname{rank}(\mathbf{AB})+n.\end{aligned}$$

The inequality is then proved. □

Now, we proceed with the equality. We denote by $\mathcal{N}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A})$ the null space and the range space of a complex matrix $\mathbf{A}$, respectively.

Theorem 2. *Let $\mathbf{A}\in\mathbb{C}^{m\times n}$ and $\mathbf{B}\in\mathbb{C}^{n\times p}$. Then*

$$\operatorname{rank}(\mathbf{AB})=\operatorname{rank}(\mathbf{A})+\operatorname{rank}(\mathbf{B})-n \tag{2}$$

holds if and only if $\mathcal{N}(\mathbf{A})\subseteq\mathcal{R}(\mathbf{B})$.

Proof. Continuing with the same notation as in the proof of Theorem 1, it is clear that the equality in (1) is equivalent to

$$\mathcal{R}\left(\begin{bmatrix}\mathbf{A}_1 & \mathbf{O}\\ \mathbf{A}_3 & \mathbf{O}\end{bmatrix}\right)\cap\mathcal{R}\left(\begin{bmatrix}\mathbf{O} & \mathbf{A}_2\\ \mathbf{O} & \mathbf{A}_4\end{bmatrix}\right)=\{\mathbf{0}\} \quad \text{and} \quad \operatorname{rank}\left(\begin{bmatrix}\mathbf{A}_2\\ \mathbf{A}_4\end{bmatrix}\right)=n-r. \tag{3}$$

On the other hand, it is easy to see that $\mathcal{N}(\mathbf{A})\subseteq\mathcal{R}(\mathbf{B})$ is satisfied if and only if

$$\mathcal{N}\left(\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_3 & \mathbf{A}_4\end{bmatrix}\right)\subseteq\mathcal{R}\left(\begin{bmatrix}\mathbf{I}_r & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{bmatrix}\right)=\left\{\begin{bmatrix}\mathbf{z}\\ \mathbf{0}_{(n-r)\times 1}\end{bmatrix}:\mathbf{z}\in\mathbb{C}^{r\times 1}\right\}. \tag{4}$$

Now, in order to prove this theorem we must show that (3) is equivalent to (4). In fact, assume that (3) holds and let

$$\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}\in\mathcal{N}\left(\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_3 & \mathbf{A}_4\end{bmatrix}\right).$$

Then

$$\begin{bmatrix}\mathbf{A}_2\\ \mathbf{A}_4\end{bmatrix}\mathbf{x}_2=-\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_3\end{bmatrix}\mathbf{x}_1.$$

So,

$$\mathcal{R}\left(\begin{bmatrix}\mathbf{A}_2\\ \mathbf{A}_4\end{bmatrix}\mathbf{x}_2\right)=\mathcal{R}\left(\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_3\end{bmatrix}\mathbf{x}_1\right)\subseteq\mathcal{R}\left(\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_3\end{bmatrix}\right)\cap\mathcal{R}\left(\begin{bmatrix}\mathbf{A}_2\\ \mathbf{A}_4\end{bmatrix}\right)=\{\mathbf{0}\},$$

from where

$$\begin{bmatrix} \mathbf{A}_2 \\ \mathbf{A}_4 \end{bmatrix} \mathbf{x}_2 = \mathbf{0}.$$

The linearly independence of the columns of $\begin{bmatrix} \mathbf{A}_2 \\ \mathbf{A}_4 \end{bmatrix}$ leads to $\mathbf{x}_2 = \mathbf{0}$. It then follows that

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{bmatrix} \in \mathcal{R}\left(\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \right).$$

Assume now that (4) holds. This condition can be rewritten as follows:

$$\text{If } \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \text{ satisfies } \begin{bmatrix} \mathbf{A}_2 \\ \mathbf{A}_4 \end{bmatrix} \mathbf{x}_2 = - \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_3 \end{bmatrix} \mathbf{x}_1 \qquad \Longrightarrow \qquad \mathbf{x}_2 = \mathbf{0}. \tag{5}$$

Now, let

$$\mathbf{y} \in \mathcal{R}\left(\begin{bmatrix} \mathbf{A}_1 & \mathbf{O} \\ \mathbf{A}_3 & \mathbf{O} \end{bmatrix} \right) \cap \mathcal{R}\left(\begin{bmatrix} \mathbf{O} & \mathbf{A}_2 \\ \mathbf{O} & \mathbf{A}_4 \end{bmatrix} \right).$$

Thus,

$$\mathbf{y} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{O} \\ \mathbf{A}_3 & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{v}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_3 \end{bmatrix} \mathbf{u}_1$$

and

$$\mathbf{y} = \begin{bmatrix} \mathbf{O} & \mathbf{A}_2 \\ \mathbf{O} & \mathbf{A}_4 \end{bmatrix} \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_2 \\ \mathbf{A}_4 \end{bmatrix} \mathbf{v}_2.$$

Hence, the vector

$$\begin{bmatrix} -\mathbf{u}_1 \\ \mathbf{v}_2 \end{bmatrix}$$

satisfies

$$\begin{bmatrix} \mathbf{A}_2 \\ \mathbf{A}_4 \end{bmatrix} \mathbf{v}_2 = - \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_3 \end{bmatrix} (-\mathbf{u}_1).$$

By (5), it then follows $\mathbf{v}_2 = \mathbf{0}$. Consequently, $\mathbf{y} = \mathbf{0}$.

Lastly, in order to prove that

$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{A}_2 \\ \mathbf{A}_4 \end{bmatrix}\right) = n - r,$$

suppose that $\mathbf{x} \in \mathbb{C}^{(n-r)\times 1}$ satisfies

$$\begin{bmatrix} \mathbf{A}_2 \\ \mathbf{A}_4 \end{bmatrix} \mathbf{x} = \mathbf{0}.$$

It is clear that

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{x} \end{bmatrix} \in \mathcal{N}\left(\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}\right).$$

By (4) we get $\mathbf{x} = \mathbf{0}$. □

On the other hand, it is well known that

$$\mathbf{AB} = \mathbf{O} \qquad \Longleftrightarrow \qquad \mathcal{R}(\mathbf{B}) \subseteq \mathcal{N}(\mathbf{A}),$$

for any $\mathbf{A} \in \mathbb{C}^{m\times n}$ and $\mathbf{B} \in \mathbb{C}^{n\times p}$. What can we conclude when the opposite inclusion holds?

Corollary 3. *Let $\mathbf{A} \in \mathbb{C}^{m\times n}$ and $\mathbf{B} \in \mathbb{C}^{n\times p}$ such that $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{B})$ holds. Then*

$$\mathbf{AB} = \mathbf{O} \qquad \Longleftrightarrow \qquad \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}) = n.$$

Comparing Corollary 3 to Proposition 17.5 in [6] (for square matrices), we can see that Corollary 3 gives us an extension of that version of the Cochran's Theorem without assuming $\mathbf{A} + \mathbf{B} = \mathbf{I}$. Indeed, for squares matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, it is easy to see that $\mathbf{A} + \mathbf{B} = \mathbf{I}$ implies $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{B})$. However, not always $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{B})$ implies $\mathbf{A} + \mathbf{B} = \mathbf{I}$; it sufficient to consider $\mathbf{A} = \mathbf{B} = \mathbf{I}$. Our Corollary 3 can be applied to $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but $\mathbf{A} + \mathbf{B} \neq \mathbf{I}$.

A simple proof can be also given for the known upper bound of the rank of a matrix product.

Lemma 4. *Let* $\mathbf{A} \in \mathbb{C}^{m\times n}$ *and* $\mathbf{B} \in \mathbb{C}^{n\times p}$*. Then*

$$\operatorname{rank}(\mathbf{AB}) \leq \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}. \tag{6}$$

Proof. Simple computations give

$$\begin{bmatrix} \mathbf{I}_m & -\mathbf{A} \\ \mathbf{O} & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{O} & \mathbf{A} \\ \mathbf{B} & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{I}_p & \mathbf{O} \\ -\mathbf{B} & \mathbf{I}_n \end{bmatrix} = \begin{bmatrix} -\mathbf{AB} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_n \end{bmatrix}.$$

Since the first and the third matrices on the left side are nonsingular,

$$\begin{aligned} \operatorname{rank}\left(\begin{bmatrix} -\mathbf{AB} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_n \end{bmatrix}\right) &= \operatorname{rank}\left(\begin{bmatrix} \mathbf{O} & \mathbf{A} \\ \mathbf{B} & \mathbf{I}_n \end{bmatrix}\right) \\ &= \operatorname{rank}\left(\begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{B} & \mathbf{O} \end{bmatrix} + \begin{bmatrix} \mathbf{O} & \mathbf{A} \\ \mathbf{O} & \mathbf{I}_n \end{bmatrix}\right) \\ &\leq \operatorname{rank}(\mathbf{B}) + n. \end{aligned}$$

Then, $\operatorname{rank}(\mathbf{AB}) + n \leq \operatorname{rank}(\mathbf{B}) + n$, that is, $\operatorname{rank}(\mathbf{AB}) \leq \operatorname{rank}(\mathbf{B})$ holds.

On the other hand, denoting by $\mathbf{A}^T$ the transpose of $\mathbf{A}$ we get

$$\operatorname{rank}(\mathbf{AB}) = \operatorname{rank}(\mathbf{AB})^T = \operatorname{rank}(\mathbf{B}^T\mathbf{A}^T) \leq \operatorname{rank}(\mathbf{A}^T) = \operatorname{rank}(\mathbf{A}).$$

Hence, the result follows directly. $\square$

3 Frobenius inequality and equality

In the following result we provide a simple proof also for the Frobenius inequality [2].

Theorem 5. *Let* $\mathbf{A} \in \mathbb{C}^{m\times n}$*,* $\mathbf{B} \in \mathbb{C}^{n\times p}$*, and* $\mathbf{C} \in \mathbb{C}^{p\times q}$*. Then*

$$\operatorname{rank}(\mathbf{ABC}) \geq \operatorname{rank}(\mathbf{AB}) + \operatorname{rank}(\mathbf{BC}) - \operatorname{rank}(\mathbf{B}). \tag{7}$$

Proof. Let consider a rank normal form of $\mathbf{B}$, that is

$$\mathbf{B} = \mathbf{P} \begin{bmatrix} \mathbf{I}_r & \mathbf{O}_{r\times(p-r)} \\ \mathbf{O}_{(n-r)\times r} & \mathbf{O}_{(n-r)\times(p-r)} \end{bmatrix} \mathbf{Q},$$

where $\mathbf{P} \in \mathbb{C}^{n\times n}$ and $\mathbf{Q} \in \mathbb{C}^{p\times p}$ are nonsingular. Partitioning $\mathbf{AP}$ and $\mathbf{QC}$ according to the sizes of blocks in $\mathbf{B}$ we can write

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix} \mathbf{P}^{-1} \qquad \text{and} \qquad \mathbf{C} = \mathbf{Q}^{-1} \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_3 \end{bmatrix},$$

where $\mathbf{A}_1 \in \mathbb{C}^{m\times r}$, $\mathbf{A}_2 \in \mathbb{C}^{m\times(n-r)}$, $\mathbf{C}_1 \in \mathbb{C}^{r\times q}$, and $\mathbf{C}_3 \in \mathbb{C}^{(p-r)\times q}$. Some simple computations give

$$\mathbf{AB} = \left[\begin{array}{cc} \mathbf{A}_1 & \mathbf{O}_{m\times(p-r)} \end{array}\right] \mathbf{Q},$$

$$\mathbf{BC} = \mathbf{P} \left[\begin{array}{c} \mathbf{C}_1 \\ \mathbf{O}_{(n-r)\times q} \end{array}\right],$$

and

$$\mathbf{ABC} = \mathbf{A}_1\mathbf{C}_1.$$

By the Sylvester inequality and using the invariance of the rank of a product by a nonsingular matrix we get

$$\begin{array}{rcl} \operatorname{rank}(\mathbf{ABC}) & = & \operatorname{rank}(\mathbf{A}_1\mathbf{C}_1) \\ & \geq & \operatorname{rank}(\mathbf{A}_1) + \operatorname{rank}(\mathbf{C}_1) - r \\ & = & \operatorname{rank}\left(\left[\begin{array}{cc} \mathbf{A}_1 & \mathbf{O}_{m\times(p-r)} \end{array}\right] \mathbf{Q}\right) + \operatorname{rank}\left(\mathbf{P}\left[\begin{array}{c} \mathbf{C}_1 \\ \mathbf{O}_{(n-r)\times q} \end{array}\right]\right) - r \\ & = & \operatorname{rank}(\mathbf{AB}) + \operatorname{rank}(\mathbf{BC}) - \operatorname{rank}(\mathbf{B}). \end{array}$$

$\square$

Next result provides a new (as far as we know) sufficient condition to obtain the equality in the Frobenius inequality. Notice that condition (8) below is a natural extension of that for the equality in the Sylvester inequality (see Theorem 2).

Theorem 6. *Let* $\mathbf{A} \in \mathbb{C}^{m\times n}$*,* $\mathbf{B} \in \mathbb{C}^{n\times n}$ *be an idempotent matrix, and* $\mathbf{C} \in \mathbb{C}^{n\times q}$*. If*

$$\mathcal{N}(\mathbf{AB}) \subseteq \mathcal{R}(\mathbf{BC}) \tag{8}$$

then

$$\operatorname{rank}(\mathbf{ABC}) = \operatorname{rank}(\mathbf{AB}) + \operatorname{rank}(\mathbf{BC}) - \operatorname{rank}(\mathbf{B}) \tag{9}$$

holds.

Proof. Assume that $\mathcal{N}(\mathbf{AB}) \subseteq \mathcal{R}(\mathbf{BC})$. Following the same notation as in the proof of Theorem 5, it can be taken $\mathbf{Q} = \mathbf{P}^{-1}$ since $\mathbf{B} \in \mathbb{C}^{n\times n}$ is idempotent. Thus, it is easy to see that condition (8) is equivalent to

$$\mathcal{N}(\begin{bmatrix} \mathbf{A}_1 & \mathbf{O}_{m\times(n-r)} \end{bmatrix}) \subseteq \mathcal{R}\left(\begin{bmatrix} \mathbf{C}_1 \\ \mathbf{O}_{(n-r)\times q} \end{bmatrix}\right).$$

Now, if $\mathbf{x}_1 \in \mathcal{N}(\mathbf{A}_1)$ then $\mathbf{A}_1\mathbf{x}_1 = \mathbf{0}$ and so

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{O}_{m\times(n-r)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{bmatrix} = \mathbf{0},$$

from where

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{bmatrix} \in \mathcal{R}\left(\begin{bmatrix} \mathbf{C}_1 \\ \mathbf{O}_{(n-r)\times q} \end{bmatrix}\right),$$

that is, $\mathbf{x}_1 = \mathbf{C}_1\mathbf{y}_1$ for some vector $\mathbf{y}_1$. Thus, $\mathcal{N}(\mathbf{A}_1) \subseteq \mathcal{R}(\mathbf{C}_1)$. This last inclusion is equivalent to the equality in the inequality $\operatorname{rank}(\mathbf{A}_1\mathbf{C}_1) \geq \operatorname{rank}(\mathbf{A}_1)+\operatorname{rank}(\mathbf{C}_1)-r$. Hence, from the proof of Theorem 5, we can deduce that equality (9) holds. $\square$

Remark 7. Condition (8) is sufficient to get (9) but, in general, the opposite is not necessarily true, in the same way that occurs with [7, Theorem 2]. This is due to the fact that $\mathcal{N}(\mathbf{A}_1) \subseteq \mathcal{R}(\mathbf{C}_1)$ in general does not imply

$$\mathcal{N}(\begin{bmatrix} \mathbf{A}_1 & \mathbf{O}_{m\times(n-r)} \end{bmatrix}) \subseteq \mathcal{R}\left(\begin{bmatrix} \mathbf{C}_1 \\ \mathbf{O}_{(n-r)\times q} \end{bmatrix}\right). \tag{10}$$

Using the condition for the equality of (2.1) in [2] we obtain the following consequence.

Corollary 8. *Let* $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times n}$ *an idempotent matrix, and* $\mathbf{C} \in \mathbb{C}^{n\times q}$ *such that* $\mathcal{N}(\mathbf{AB}) \subseteq \mathcal{R}(\mathbf{BC})$ *holds. Then*

(a) $\mathbf{ABC} = \mathbf{O}$ *if and only if* $\operatorname{rank}(\mathbf{AB}) + \operatorname{rank}(\mathbf{BC}) = \operatorname{rank}(\mathbf{B})$.

(b) there exist matrices $\mathbf{X} \in \mathbb{C}^{q\times n}$ *and* $\mathbf{Y} \in \mathbb{C}^{n\times m}$ *such that*

$$\mathbf{BCX} + \mathbf{YAB} = \mathbf{B}.$$

Remark 9. The factorizations used in Theorem 5 allow us to give the explicit general solution of $\mathbf{BCX} + \mathbf{YAB} = \mathbf{B}$ for $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times n}$, and $\mathbf{C} \in \mathbb{C}^{n\times q}$. Indeed, by using the expressions of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ given in Theorem 5 it is not hard to show that $\mathbf{BCX}+\mathbf{YAB} = \mathbf{B}$ is consistent (in the unknowns $\mathbf{X}$ and $\mathbf{Y}$) if and only if $(\mathbf{I} - \mathbf{C}_1\mathbf{C}_1^-)(\mathbf{I} - \mathbf{A}_1^-\mathbf{A}_1) = \mathbf{O}$ (see [1, 3]). Here, $\mathbf{A}^-$ denotes a $\{1\}$-generalized inverse of $\mathbf{A}$ (that is, $\mathbf{AA}^-\mathbf{A} = \mathbf{A}$). The general solution is then obtained by solving $\mathbf{C}_1\mathbf{X}_1 - \mathbf{Y}_1(-\mathbf{A}_1) = \mathbf{I}_r$, $\mathbf{C}_1\mathbf{X}_2 = \mathbf{O}$, and $\mathbf{Y}_2\mathbf{A}_1 = \mathbf{O}$ simultaneously where $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 \end{bmatrix}\mathbf{Q}$ and $\mathbf{Y} = \mathbf{P}\begin{bmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 \end{bmatrix}^T$. It then follows that

$$\mathbf{X} = \begin{bmatrix} \mathbf{C}_1^- - \mathbf{C}_1^-\mathbf{Z}\mathbf{A}_1 + (\mathbf{I} - \mathbf{C}_1^-\mathbf{C}_1)\mathbf{W} & (\mathbf{I} - \mathbf{C}_1^-\mathbf{C}_1)\mathbf{M} \end{bmatrix}\mathbf{Q}$$

and

$$\mathbf{Y} = \mathbf{P}\begin{bmatrix} (\mathbf{I} - \mathbf{C}_1^-\mathbf{C}_1)\mathbf{A}_1^- + \mathbf{Z} - (\mathbf{I} - \mathbf{C}_1\mathbf{C}_1^-)\mathbf{Z}\mathbf{A}_1\mathbf{A}_1^- \\ \mathbf{N}(\mathbf{I} - \mathbf{A}_1\mathbf{A}_1^-) \end{bmatrix}$$

for arbitrary matrices $\mathbf{Z}$, $\mathbf{W}$, $\mathbf{M}$, and $\mathbf{N}$. Notice that matrices of smaller sizes are used in our computations compared to that given in [1].

By means of generalized inverses theory, Tian and Styan showed the following result.

Theorem 10. *[7, Theorem 2] Let $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times n}$ an idempotent matrix, and $\mathbf{C} \in \mathbb{C}^{n\times q}$. If*

$$\begin{bmatrix} \mathbf{A}^* & \mathbf{C} \end{bmatrix} \text{ has full row rank} \tag{11}$$

and

$$\mathbf{AC} = \mathbf{O} \tag{12}$$

then

$$\operatorname{rank}(\mathbf{ABC}) = \operatorname{rank}(\mathbf{AB}) + \operatorname{rank}(\mathbf{BC}) - \operatorname{rank}(\mathbf{B}) \tag{13}$$

holds.

It can be noticed that, when all matrices are square of the same size, if $\mathbf{A}$ and/or $\mathbf{C}$ are nonsingular, equality (13) holds vacuously and both sides are equal to zero. Now, condition (8) in Theorem 6 also gives a vacuous equality but both sides are rank($\mathbf{BC}$) provided that $\mathbf{A}$ is nonsingular and rank($\mathbf{AB}$) provided that $\mathbf{C}$ is nonsingular.

Proposition 11. *Let* $\mathbf{A} \in \mathbb{C}^{m\times n}$ *and* $\mathbf{C} \in \mathbb{C}^{n\times q}$*. If (11) and (12) hold then (8) is also satisfied for* $\mathbf{B} = \mathbf{I}_n$*.*

Proof. Indeed, from (11) we get $n = \operatorname{rank}\left(\begin{bmatrix} \mathbf{A}^* & \mathbf{C} \end{bmatrix}\right) \leq \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{C})$. And, from (12) we obtain $\mathcal{R}(\mathbf{C}) \subseteq \mathcal{N}(\mathbf{A})$ and then $\operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{C}) = n$. Comparing to $\operatorname{rank}(\mathbf{A}) + \dim(\mathcal{N}(\mathbf{A})) = n$ we have $\dim(\mathcal{N}(\mathbf{A})) = \operatorname{rank}(\mathbf{C})$. Thus, $\mathcal{R}(\mathbf{C}) = \mathcal{N}(\mathbf{A})$ and hence (8) is satisfied. □

4 The inequality and equality for $\ell > 3$ matrices

Next, we give the following generalization.

Theorem 12. *Let* $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_\ell$ *matrices having* $n_1, n_2, \ldots, n_\ell$ *columns, respectively, such that the product* $\mathbf{A}_1\mathbf{A}_2\ldots\mathbf{A}_\ell$ *is well-defined. Then,*

$$\begin{aligned} \operatorname{rank}(\mathbf{A}_1\mathbf{A}_2\ldots\mathbf{A}_\ell) &\geq \sum_{i=1}^{\ell-1}\operatorname{rank}(\mathbf{A}_i\mathbf{A}_{i+1}) - \sum_{i=2}^{\ell-1}\operatorname{rank}(\mathbf{A}_i) \\ &\geq \sum_{i=1}^{\ell}\operatorname{rank}(\mathbf{A}_i) - \sum_{i=1}^{\ell-1} n_i, \end{aligned} \tag{14}$$

for all $\ell \geq 3$*.*

Proof. The inequality (14) follows by induction on ℓ using Frobenius inequality. Similarly, the second one follows by induction on ℓ using (14) and Sylvester inequality. □

Corollary 13. *Let* $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_\ell$ *matrices such that the product* $\mathbf{A}_1\mathbf{A}_2\ldots\mathbf{A}_\ell$ *is well-defined. If* $\mathbf{A}_i$ *is idempotent for* $i = 2, \ldots, \ell-1$ *and* $\mathcal{N}(\prod_{j=1}^{s}\mathbf{A}_j) \subseteq \mathcal{R}(\mathbf{A}_s\mathbf{A}_{s+1})$ *for* $s = 2, 3, \ldots, \ell-1$ *then the equality in (14) holds.*

Proof. It follows by induction on ℓ and Theorem 6. □

Some applications of formulae studied in this work were given in the recent paper [2]. In what follows, we present another application.

5 An application

Let $\mathbf{S} \in \mathbb{C}^{m\times n}$ of rank r, M be a subspace of $\mathbb{C}^n$ of dimension $t \leq r$ and N be a subspace of $\mathbb{C}^m$ of dimension $m-t$. It is well known that there exists a unique $\{2\}$-generalized inverse $\mathbf{X} \in \mathbb{C}^{n\times m}$ having range space M and null space N [3, 5, 8], that is,

$$\mathbf{XSX} = \mathbf{X}, \qquad \mathcal{R}(\mathbf{X}) = M, \qquad \text{and} \qquad \mathcal{N}(\mathbf{X}) = N,$$

if and only if

$$\mathbf{S}M \oplus N = \mathbb{C}^m,$$

where $\mathbf{S}M = \{\mathbf{S}m : \ m \in M\}$. This unique matrix $\mathbf{X}$ is denoted by $\mathbf{S}^{(2)}_{M,N}$.

We also remind that, for certain special subspaces M and N, this generalized inverse $\mathbf{S}^{(2)}_{M,N}$ includes the classical inverses as particular cases: the Moore-Penrose inverse $\mathbf{S}^{\dagger}$, the weighted Moore-Penrose inverse $\mathbf{S}^{\dagger}_{\mathbf{L},\mathbf{T}}$ (for $\mathbf{L}$ and $\mathbf{T}$ being hermitian positive definite matrices of appropriate sizes), the group inverse $\mathbf{S}^{\#}$ (whenever it exists), the Drazin inverse $\mathbf{S}^{D}$, and the weighted Drazin inverse $\mathbf{S}^{D}_{\mathbf{W}}$, among others.

Proposition 14. *Let $\mathbf{A} \in \mathbb{C}^{p\times m}$, $\mathbf{S} \in \mathbb{C}^{m\times n}$ of rank r, and $\mathbf{C} \in \mathbb{C}^{m\times q}$. Assume that M is a subspace of $\mathbb{C}^n$ of dimension $t \leq r$ and N be a subspace of $\mathbb{C}^m$ of dimension $m-t$ such that $\mathbf{S}^{(2)}_{M,N}$ exists. If*

$$\mathcal{N}(\mathbf{A}\mathbf{S}\mathbf{S}^{(2)}_{M,N}) \subseteq \mathcal{R}(\mathbf{S}\mathbf{S}^{(2)}_{M,N}\mathbf{C})$$

then

$$\operatorname{rank}(\mathbf{A}\mathbf{S}\mathbf{S}^{(2)}_{M,N}\mathbf{C}) = \operatorname{rank}(\mathbf{A}\mathbf{S}\mathbf{S}^{(2)}_{M,N}) + \operatorname{rank}(\mathbf{S}^{(2)}_{M,N}\mathbf{C}) - \operatorname{rank}(\mathbf{S}^{(2)}_{M,N}). \quad (15)$$

Proof. Applying Theorem 6 with $\mathbf{B} = \mathbf{S}\mathbf{S}^{(2)}_{M,N}$ and recalling that $\mathbf{S}^{(2)}_{M,N}\mathbf{S}\mathbf{S}^{(2)}_{M,N} = \mathbf{S}^{(2)}_{M,N}$ is valid, we get $\mathbf{B}^2 = \mathbf{B}$. Again, since $\mathbf{S}^{(2)}_{M,N}$ is a $\{2\}$-generalized inverse of $\mathbf{S}$ we have $\operatorname{rank}(\mathbf{S}\mathbf{S}^{(2)}_{M,N}\mathbf{C}) = \operatorname{rank}(\mathbf{S}^{(2)}_{M,N}\mathbf{C})$. Analogously, $\operatorname{rank}(\mathbf{S}\mathbf{S}^{(2)}_{M,N}) = \operatorname{rank}(\mathbf{S}^{(2)}_{M,N})$. Replacing these terms in (9) we arrive at equality (15). $\square$

Assuming adequate sizes for all matrices and that

$$\mathcal{N}(\mathbf{A}\mathbf{S}^{(2)}_{M,N}\mathbf{S}) \subseteq \mathcal{R}(\mathbf{S}^{(2)}_{M,N}\mathbf{S}\mathbf{C})$$

holds, it then similarly follows

$$\operatorname{rank}(\mathbf{A}\mathbf{S}^{(2)}_{M,N}\mathbf{S}\mathbf{C}) = \operatorname{rank}(\mathbf{A}\mathbf{S}^{(2)}_{M,N}) + \operatorname{rank}(\mathbf{S}^{(2)}_{M,N}\mathbf{S}\mathbf{C}) - \operatorname{rank}(\mathbf{S}^{(2)}_{M,N}).$$

Corollary 15. *Under the same assumptions as in Proposition 14, the expressions of* $\mathbf{S}^{(2)}_{M,N}$*,* $\mathbf{SS}^{(2)}_{M,N}$*, and* $\mathrm{rank}(\mathbf{S}^{(2)}_{M,N})$ *in (15) particularized to the classical generalized inverses can be simplified as shown in the following table:*

M	N	$\mathbf{S}^{(2)}_{M,N}$	$\mathbf{SS}^{(2)}_{M,N}$	$\mathrm{rank}(\mathbf{S}^{(2)}_{M,N})$
$\mathcal{R}(\mathbf{S}^*)$	$\mathcal{N}(\mathbf{S}^*)$	$\mathbf{S}^{\dagger}$	$\mathbf{SS}^{\dagger}$	$\mathrm{rank}(\mathbf{S})$
$\mathcal{R}(\mathbf{L}^{-1}\mathbf{S}^*\mathbf{T})$	$\mathcal{N}(\mathbf{L}^{-1}\mathbf{S}^*\mathbf{T})$	$\mathbf{S}^{\dagger}_{\mathbf{L},\mathbf{T}}$	$\mathbf{SS}^{\dagger}_{\mathbf{L},\mathbf{T}}$	$\mathrm{rank}(\mathbf{S})$
$\mathcal{R}(\mathbf{S})$	$\mathcal{N}(\mathbf{S})$	$\mathbf{S}^{\#}$	$\mathbf{SS}^{\#}$	$\mathrm{rank}(\mathbf{S})$
$\mathcal{R}(\mathbf{S}^k)$	$\mathcal{N}(\mathbf{S}^k)$	$\mathbf{S}^{D}$	$\mathbf{SS}^{k}$	$\mathrm{rank}(\mathbf{S}^k)$

where k *is the index of* $\mathbf{S}$*.*

Now, if $\mathbf{S} \in \mathbb{C}^{m\times n}$*,* $\mathbf{W} \in \mathbb{C}^{n\times m}$*,* k_1 *is the index of* $\mathbf{SW}$*,* k_2 *is the index of* $\mathbf{WS}$*,* $k = \max\{k_1, k_2\}$ *and* $\mathbf{U} := \mathbf{WSW}$ *then*

$$\mathbf{S}^D_{\mathbf{W}} = \mathbf{U}^{(2)}_{\mathcal{R}((\mathbf{SW})^k),\mathcal{N}((\mathbf{WS})^k)}.$$

If $\mathcal{N}(\mathbf{AUS}^D_{\mathbf{W}}) \subseteq \mathcal{R}(\mathbf{US}^D_{\mathbf{W}}\mathbf{C})$ *then*

$$\mathrm{rank}(\mathbf{AUS}^D_{\mathbf{W}}\mathbf{C}) = \mathrm{rank}(\mathbf{AUS}^D_{\mathbf{W}}) + \mathrm{rank}(\mathbf{US}^D_{\mathbf{W}}\mathbf{C}) - \mathrm{rank}(\mathbf{US}^D_{\mathbf{W}}).$$

Acknowledgements

This paper has been partially supported by Ministerio de Economía y Competitividad (Grant DGI MTM2013-43678P and Red de Excelencia MTM2015-68805-REDT). The author thanks the referees for their valuable suggestions.

References

[1] J.K. Baksalary, R. Kala, The matrix equation $AX - YB = C$, *Linear Algebra and its Applications*, 25, 41–43, 1979.

[2] O.M. Baksalary, G. Trenkler, On k-potent matrices, Electronic Journal of Linear Algebra, 26, 446–470, 2013.

[3] A. Ben-Israel, T.N.E. Greville, *Generalized Inverses: Theory and Applications*, John Wiley & Sons, Second Ed., 2003.

[4] G. Marsaglia, G.P.H. Styan, *Equalities and inequalities for ranks of matrices*, Linear and Multilinear Algebra, 2 (1974), 269–292.

[5] D. Mosić, D.S. Djordjević, *Condition number of the W-weighted Drazin inverse*, Applied Mathematics and Computation, 203, 1 (2008), 308–318.

[6] S. Puntanen, G.P.H. Styan, J. Isotalo, *Matrix Tricks for Linear Statistical Models*, Springer, 2011.

[7] Y. Tian, G.P.H. Styan, A new rank formula for idempotent matrices with applications, Comment. Math. Univ. Carolinae, 43, 2, 379–384, 2002.

[8] G. Wang, Y. Wei, S. Qiao, *Generalized Inverses: Theory and Computations*, Science Press, 2004.