ADMM-based Optimal Quantum State Discrimination

Neha (Bsz258211), Shadab Anjum (2024phs7155)

***Abstract*—Quantum State Discrimination (QSD) is a fundamental task in quantum communication systems, where the objective is to identify the quantum state transmitted through a noisy channel with minimum probability of error. The optimal measurement strategy for QSD is obtained by solving a semidefinite program (SDP), which becomes computationally prohibitive for large-dimensional Hilbert spaces or a large number of hypotheses. This makes traditional SDP-based approaches unsuitable for real-time quantum receivers. In this work, we study an efficient iterative algorithm—the Alternating Direction Method of Multipliers (ADMM)—for computing optimal POVMs for minimum-error state discrimination. By splitting the optimization into an affine projection step and a positive semidefinite (PSD) projection step, ADMM significantly reduces computational complexity while retaining near-optimal performance. Numerical simulations for both pure and mixed quantum states, including practical 4-PSK coherent-state ensembles under thermal noise, demonstrate that ADMM achieves accuracy comparable to SDP but with orders-of-magnitude faster execution time. These results highlight the potential of ADMM methods for enabling real-time quantum detection in future quantum communication networks.**

Keywords: Quantum State Discrimination (QSD), Alternating Direction Method of Multipliers (ADMM), Semidefinite Programming (SDP), Quantum Communication.

I. Quantum State Discrimination

A. *Introduction*

Quantum State Discrimination (QSD) is a fundamental task in quantum communication systems. The core objective of QSD is for the receiver to identify the specific quantum state transmitted through a noisy channel with the minimum probability of error.

Why is QSD challenging? Classical information is encoded into quantum states (like photon polarizations or coherent states) by the transmitter. The receiver then performs a measurement to determine which state was sent.

Key challenges:

- **Non-Orthogonality:** If two states are not orthogonal, no measurement can perfectly distinguish them, leading to unavoidable errors.
- **Optimal Measurement:** Minimum error is achieved using an optimal POVM $\{\Pi_i\}$.
- **Computational Bottleneck:** Finding the optimal POVM requires solving an SDP. Its complexity scales as $\mathcal{O}(d^9)$, making it unsuitable for real-time applications.

Solution: ADMM The Alternating Direction Method of Multipliers (ADMM) is an iterative algorithm that efficiently computes optimal POVMs. It splits the optimization into simpler steps: an affine projection step and a PSD projection step. Complexity per iteration: $O(md^3)$.

Numerical simulations show ADMM achieves near-optimal accuracy with much faster execution time.

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B. *Core Task in Quantum Communication*

The communication process:

1) A classical message a is selected from alphabet $\mathcal{A} = \{a_1, \dots, a_m\}$.
2) Transmitter encodes a into quantum state $|\gamma_a\rangle$.
3) Noisy quantum channel maps $|\gamma_a\rangle \to \rho_a$.
4) Receiver discriminates among m possible states $\{\rho_i\}$ using POVM $\{\Pi_i\}$.

Average probability of correct detection:

$$P_{\text{correct}} = \sum_{i=1}^{m} p_i \operatorname{Tr}(\rho_i \Pi_i)$$

Probability of error:

$$P_e = 1 - \sum_{i=1}^{m} p_i \operatorname{Tr}(\rho_i \Pi_i)$$

C. *Importance of QSD in Quantum Technologies*

1) Quantum Key Distribution (QKD): Non-orthogonal states prevent eavesdroppers from perfect discrimination. Legitimate receivers must minimize error for high key rates.

2) Quantum Sensing and Metrology: Discriminating output states detects the presence or strength of a signal.

3) Quantum Receivers: In optical communication, optimal discrimination improves data rate and energy efficiency. Practical receivers approximate POVM with photodetectors, homodyne detection, or displacement operations.

D. *Non-Orthogonality Makes Perfect Detection Impossible*

Two pure states $|\psi\rangle$ and $|\phi\rangle$ are orthogonal iff

$$\langle\psi|\phi\rangle = 0$$

Otherwise, no measurement can perfectly distinguish them.

Binary Helstrom limit:

$$P_e^{\min} = \frac{1}{2}\left(1 - \|p_0\rho_0 - p_1\rho_1\|_1\right)$$

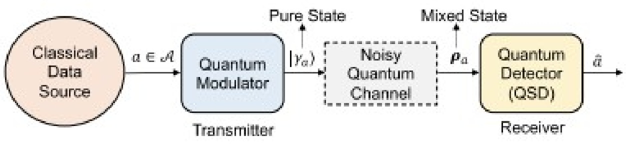

Fig. 1. Schematic of QSD application in quantum communications.

E. Designing Optimal Measurements (POVMs)

POVM constraints:

$$\Pi_i \succeq 0, \quad \sum_{i=1}^{m} \Pi_i = I$$

Optimal minimum-error problem:

$$\min_{\{\Pi_i\}} 1 - \sum_{i=1}^{m} p_i \operatorname{Tr}(\rho_i \Pi_i)$$
$$\text{s.t. } \Pi_i \succeq 0, \sum_i \Pi_i = I$$

II. Quantum State Discrimination Setup

- Classical message a from alphabet $\mathcal{A} = \{a_1, \ldots, a_m\}$
- Transmitter encodes a into pure state $|\gamma_a\rangle$
- Noisy channel: $|\gamma_a\rangle\langle\gamma_a| \to \rho_a$
- Receiver discriminates among $\{\rho_i\}$
- Density operators: $\rho_i \succeq 0$, $\operatorname{Tr}(\rho_i) = 1$
- Prior probabilities: $p_i > 0$, $\sum_i p_i = 1$
- States lie in d-dimensional Hilbert space $\mathcal{H}$

Probability of error:

$$P_e = 1 - \sum_{i=1}^{m} p_i \operatorname{Tr}(\rho_i \Pi_i)$$

Optimization problem:

$$\min_{\{\Pi_i\}} 1 - \sum_i p_i \operatorname{Tr}(\rho_i \Pi_i)$$
$$\text{s.t. } \Pi_i \succeq 0, \quad \sum_i \Pi_i = I$$

A. Why SDP is Computationally Expensive

- SDP has linear objective and PSD constraints
- Scalar decision variables: $\mathcal{O}(md^2)$
- Interior-point methods require eigen-decomposition $\to$ expensive
- Example: $m = 100$, $d = 10 \to$ 10 seconds desktop runtime
- Not suitable for real-time quantum receivers

B. Alternating Direction Method of Multipliers (ADMM)

- Splits large optimization into simpler subproblems
- Combines augmented Lagrangian method (stability) + dual decomposition (parallelizable)
- Iterative updates of primal and dual variables
- Converges under mild assumptions

1) ADMM Structure and Updates: Problem form:

$$\min_{x,z} f(x) + g(z) \quad \text{s.t. } Ax + Bz = c$$

Updates per iteration k:

1) x-update: $x^{(k)} = \arg\min_x L_\rho(x, z^{(k-1)}, u^{(k-1)})$
2) z-update: $z^{(k)} = \arg\min_z L_\rho(x^{(k)}, z, u^{(k-1)})$
3) Dual update: $u^{(k)} = u^{(k-1)} + \rho(Ax^{(k)} + Bz^{(k)} - c)$

C. ADMM for POVM Optimization

- Introduce auxiliary variables W_i; enforce $\Pi_i = W_i$
- PSD handled by W_i, affine constraint by Π_i
- Dual variables: U_i, penalty $\rho > 0$

ADMM Iteration Steps:

1) Π-update (Affine projection):

$$T_i = W_i^{(k)} - U_i^{(k)} + \frac{p_i}{\rho}\rho_i, \quad \Pi_i^{(k+1)} = T_i + \frac{1}{m}(I - \sum_i T_i)$$

2) W-update (PSD projection):

$$W_i^{(k+1)} = \text{Proj}_{\succeq 0}(\Pi_i^{(k+1)} + U_i^{(k)})$$

3) Dual update:

$$U_i^{(k+1)} = U_i^{(k)} + \Pi_i^{(k+1)} - W_i^{(k+1)}$$

4) Repeat until convergence

Primal residual:

$$r^{(k)} = \sqrt{\sum_i \|\Pi_i^{(k+1)} - W_i^{(k+1)}\|_F^2}$$

Dual residual:

$$s^{(k)} = \rho\sqrt{\sum_i \|W_i^{(k+1)} - W_i^{(k)}\|_F^2}$$

Algorithm 1 ADMM-based POVM Optimization

Input: Quantum states $\{\rho_i\}$, priors $\{p_i\}$, penalty ρ, max iterations $K_{\max}$, tolerance ϵ
Output: Optimized POVM elements $\{\Pi_i\}$
Initialize $W_i^{(0)} = 0$, $U_i^{(0)} = 0$
for $k = 0$ ***to*** $K_{\max} - 1$ **do**
    $T_i = W_i^{(k)} - U_i^{(k)} + \frac{p_i}{\rho}\rho_i$ $T_{\text{sum}} = \sum_i T_i$ $\Pi_i^{(k+1)} = T_i + \frac{1}{m}(I - T_{\text{sum}})$ $\Pi_i^{(k+1)} \leftarrow \frac{1}{2}(\Pi_i^{(k+1)} + \Pi_i^{(k+1)\dagger})$
    Eigendecompose $X_i = \Pi_i^{(k+1)} + U_i^{(k)} = Q\Lambda Q^\dagger$ Clip: $\Lambda_+ = \max(\Lambda, 0)$ $W_i^{(k+1)} = Q\Lambda_+ Q^\dagger$
    $U_i^{(k+1)} = U_i^{(k)} + \Pi_i^{(k+1)} - W_i^{(k+1)}$
    Compute residuals $r^{(k)}$, $s^{(k)}$ **if** $r^{(k)} < \epsilon$ ***and*** $s^{(k)} < \epsilon$ **then**
        break
    end
end

III. 4-PSK Quantum State Discrimination: Setup

- Mixed quantum states due to thermal noise
- $m = 4$, average thermal photons $N = 2$, amplitude $\alpha_0 = 1$
- Truncated dimension $d = 10$
- Prior probabilities: $p_1 = 0.2, p_2 = 0.2, p_3 = 0.1, p_4 = 0.5$

Thermal noise parameter: $v = \frac{N}{1+N}$
Coherent states: $\alpha_k = \alpha_0 e^{i2\pi(k-1)/m}$

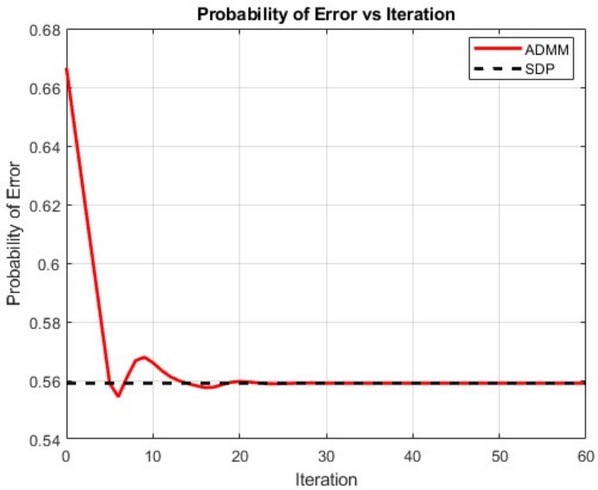

Fig. 2. ADMM vs SDP (Result 2)

IV. References

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