Term Paper

ADMM based optimal Quantum state discrimination

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Why Quantum State Discrimination?

- Core task in quantum communication: identifying which quantum state was sent.
- Fundamental to:
  - Quantum key distribution (QKD)
  - Quantum sensing
  - Quantum receivers and detection theory
- Non-orthogonality of quantum states makes perfect detection impossible.
- Goal: design measurements that minimize detection error.

QSD Problem

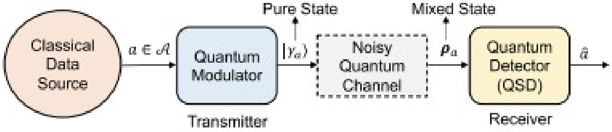

Figure: Schematic of QSD application in quantum communications

Quantum State Discrimination Setup

- A classical message a is selected from an alphabet $\mathcal{A}$
- Transmitter encodes a into a pure quantum state $|\gamma_a\rangle$
- Noisy quantum channel maps $|\gamma_a\rangle \to \rho_a$ (a mixed state)
- Receiver must discriminate among m possible states $\{\rho_i\}$
- Each density operator ρ_i is PSD and satisfies $\mathrm{Tr}(\rho_i) = 1$
- Prior probabilities $p_i > 0$ with $\sum_{i=1}^{m} p_i = 1$
- All states lie in a d-dimensional Hilbert space $\mathcal{H}$

Problem formulation

The probability of error for QSD is:

$$P_e = 1 - \sum_{i=1}^{m} p_i \operatorname{Tr}(\rho_i \Pi_i)$$

The optimization problem:

$$\min_{\{\Pi_i\}} \quad 1 - \sum_{i=1}^{m} p_i \operatorname{Tr}(\rho_i \Pi_i)$$

such that:

$$\Pi_i \succeq 0 \qquad \forall\, i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} \Pi_i = I$$

Optimal POVM via Semidefinite Programming

- This problem is a **semidefinite program (SDP)** and can be solved using standard solvers (CVX).
- However, SDP becomes computationally expensive as the number of POVM elements m or the Hilbert-space dimension d increases.
- Example: For $m = 100$, $d = 10$, solving the SDP can take over 10 seconds on a typical desktop computer.
- This high complexity makes SDP impractical for real-time quantum communication systems.

Alternating Direction Method of Multipliers (ADMM)

- ADMM is an algorithm that solves constrained convex optimization problems by splitting them into smaller, more manageable parts.
- It combines the strengths of two other methods
- Augmented Lagrangian method
- Dual decomposition.

ADMM Working

- ADMM is an effective optimization method for problems that can be split into simpler subproblems.

$$\min_{x,y} \; f(x) + g(y)$$

- At each iteration, ADMM performs three steps:
  - x-update: minimize the augmented Lagrangian w.r.t. x
  - y-update: minimize w.r.t. y
  - Dual update: update the Lagrange multiplier

Key Features

Key features

- Splits problems: ADMM breaks down complex problems into simpler subproblems that can be solved more easily.
- Combines methods: It merges the robustness of the augmented Lagrangian method with the parallelizability of dual decomposition.
- Iterative updates: The algorithm iteratively updates the primal variables (x and y) and the dual variables (Lagrange multipliers) to converge to a solution
- Robustness: ADMM is known for its robustness and can converge under mild assumptions.
- Wide applicability: While originating in the 1970s, ADMM has seen a recent surge in popularity and is widely used in modern fields like image processing, machine learning, and statistical learning.

ADMM: Optimization Formulation

ADMM is applied to problems of the form:

$$\min_{x,z} \; f(x) + g(z) \quad \text{subject to} \quad Ax + Bz = c$$

- $f(x)$ and $g(z)$ are convex functions.
- x and z are optimization variables.
- A, B, and c define the linear constraint.

Augmented Lagrangian:
For penalty parameter $\rho > 0$, the augmented Lagrangian is:

$$L_\rho(x, z, u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|_2^2$$

where u is the dual variable (Lagrange multiplier).

ADMM Iterations

ADMM proceeds for iterations $k = 1, 2, 3, \ldots$ using three steps:

1. x-update

$$x^{(k)} = \operatorname*{argmin}_x L_\rho\Big(x,\ z^{(k-1)},\ u^{(k-1)}\Big)$$

2. z-update

$$z^{(k)} = \operatorname*{argmin}_z L_\rho\Big(x^{(k)},\ z,\ u^{(k-1)}\Big)$$

3. Dual variable update

$$u^{(k)} = u^{(k-1)} + \rho\big(Ax^{(k)} + Bz^{(k)} - c\big)$$

These updates alternate between the primal variables (x, z) and the dual variable u.

Quantum States and Prior Probabilities

Quantum States (Kets)

$$|\psi_1\rangle = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \qquad |\psi_2\rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \qquad |\psi_3\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Prior Probabilities

$$p_1 = p_2 = p, \qquad p_3 = 1 - 2p$$

Given Parameters

$$\theta = \frac{\pi}{16}, \qquad p = 0.3$$

Numerical Values:

$$p_1 = 0.3, \qquad p_2 = 0.3, \qquad p_3 = 1 - 0.6 = 0.4$$

These satisfy the normalization:

Result

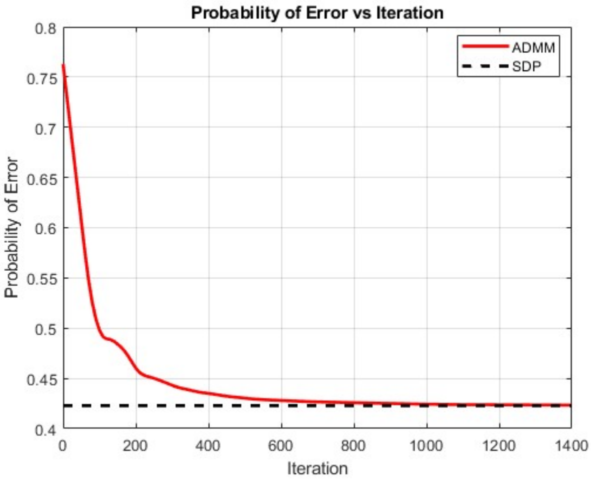

Figure: ADMM vs SDP

4-PSK Quantum State Discrimination: Setup

Modulation Scheme: Quantum m-PSK (Phase-Shift Keying)

- Mixed quantum states $\{\rho_k\}$ due to thermal noise
- Practical scenario: states are no longer pure kets but density matrices

Given Parameters for 4-PSK ($m = 4$):

- Number of phases: $m = 4$
- Average thermal photons: $N = 2$
- Signal amplitude (magnitude): $\alpha_0 = 1$
- Truncated dimension: $d = 10$ (density matrices are 10×10)
- Prior probabilities:

$$p_1 = 0.2, \quad p_2 = 0.2, \quad p_3 = 0.1, \quad p_4 = 0.5$$

Thermal noise parameter:

$$v = \frac{N}{1 + N}$$

4-PSK Density Matrices and Coherent State Parameters

Density Matrix Entries:

$$\rho_{i,j}(\alpha_k), \qquad 1 \le i \le j$$

Depend on:

- Thermal noise factor $v = \frac{N}{1+N}$
- Signal photon number $|\alpha_k|^2$
- Generalized Laguerre polynomials $L^i_{j-i}(x)$

Hermitian completion:

$$\rho_{j,i}(\alpha_k) = \rho^*_{i,j}(\alpha_k)$$

Coherent State Parameters for m-PSK:

$$\alpha_k = \alpha_0\, W_m^{k-1}, \qquad W_m = e^{i2\pi/m}$$

For 4-PSK ($m = 4$, $\alpha_0 = 1$):

$$W_4 = e^{i\pi/2} = i$$

$$\alpha_1 = 1, \quad \alpha_2 = i, \quad \alpha_3 = -1, \quad \alpha_4 = -i$$

Result

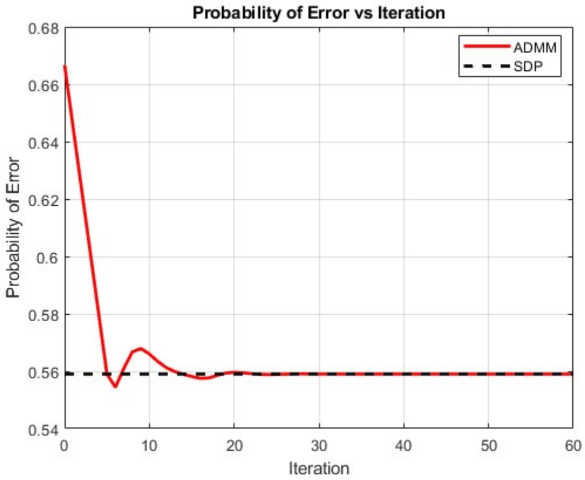

Figure: ADMM vs SDP

References

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