Proofs and Problem Solving

Li Qianrui

February 19, 2023

Contents

1 Upper bounds and Least upper bound

1.1 Upper and Lower Bounds

Definition 5.1 Let A be a subset of $\mathbb{R}$.

- An *upper bound* for A is a real number M such that for all $x \in A$, we have $x \leqslant M$. We say that A is *bounded above* if there exists an upper bound for A, and *unbounded above* otherwise.
- A *lower bound* for A is a real number m such that for all $x \in A$, we have $m \leqslant x$. We say that A is *bounded below* if there exists a lower bound for A, and *unbounded below* otherwise.
- We say A is *bounded* if it is both bounded above and bounded below.

1.2 Least Upper Bounds and Greatest Lower Bounds

Definition 5.2
Given a subset $A \sqsubseteq \mathbb{R}$, a number L is a **least upper bound (LUB)** or supremum for A if:
(a)L is an upper bound for A, that is, $x \leq L$ for all $x \in A$;
(b)$L \leq M$ for every upper bound M of A **or**
for all $t < L$, there exist $x \in A$ such that $x > t$.

Greatest lower boudn (GLB) or infinum for A is similar to LUB

1.3 Completeness Axiom for the real numbers

Every nonempty subset of $\mathbb{R}$ that is bounded above has a *least upper bound*.

Theorem 5.1
The Archimedean property holds. That is, for any $x, y \in \mathbb{R}$ with $x, y > 0$, there is some $n \in \mathbb{N}$ such that $ny > x$.
(Could be prove by contradiction)

2 Limits

2.1 The $\epsilon - \mathbb{N}$ definition of a limit.

Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers x1,x2,... and let $L \in \mathbb{R}$. We say that x_n **converge** to L, or L us the **limit** of the sequence (x_n),if for all $\epsilon > 0$, there exists a natural numebr N such that for all $n > N$,

$$|x_n - L| < \epsilon,$$

When x_n converges to L, we write $x_n \to L$ as $n \to \infty$, or $\lim_{n\to\infty} x_n = L$.

2.2 Bounded Sequence

We say that a sequence (a_n) is **bounded** if the set of values a1,a2,... is a bounded set.
i.e. thereare m,M such that $m \leq a_n \leq M$ for all n.

Proposition 6.1
Suppose the sequence (a_n) converges. Then it is bounded.

Proposition 6.3
Suppose that $x_n \to L$ as $n \to \infty$ and that $k \in \mathbb{N}$. Then $x_n^k \to L^k$ as $n \to \infty$.

2.3 Application: the existence of roots

Theorem 6.6
Let $x > 0$ and $k \in \mathbb{N}$. Then there is a unique $y > 0$ such that $y^k = x$.

2.4 Infinite limits

Definition 6.4 Let x_n be a sequence of real numbers.

(a) We say (x_n) tends to ∞ or **diverges** to ∞, and write $x_n \to \infty$ as $n \to \infty$ if for all $M > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $x_n \geq M$.

(b) Similar for $x_n \to -\infty$

3 The monotone Convergence Theorem

3.1 Monotonic

A sequence $(a_n)_{n=1}^{\infty}$ is **increasing** if $a_{n+1} \geq a_n$ for all n. It is **decreasing** if $a_{n+1} \leq a_n$ for all n. We say a_n is **monotonic** if it is either increasing or decreasing.

Monotone Convergence Theorem (MCT)
Let (a_n) be an **increasing** sequence of real numbers that is bounded above (i.e. there is M so that $a_n \leq M$ for all n). Then (a_n) converges to some limit.
Similar to **decreasing** sequence.

3.2 literatively defined sequence

The value of x_{n+1} is specified in terms of previous values of the sequence $x_1, ..., x_n$, (usually x_n).

4 Decimals and Series

Every real number has a decinal expansion, and conversely, every decimal expansion gives rise to a real number.

4.1 Decimals

Theorem 8.1
Given sequence (a_n) with $a_n \in 0, 1, ..., 9$ for all n, the decimal expanision $x = 0.a_1a_2a_3a_4...a_n...$ given by the limit of the sequence

$$x_n = \frac{a_1}{10} + \frac{a_2}{100} + ... + \frac{a_n}{10^n} \tag{1}$$

defines a real number x satisfying $0 \leq x \leq 1$.

Theorem 8.3
Let $0 \leq x \leq 1$ and suppose that $x = 0.a_1a_2... = 0.b_1b_2....$ Let l be the smallest integer for which $a_l \neq b_l$ and suppose that $a_l < b_l$. Then $b_l = a_l + 1$, and for all $k > l$ we have $b_k = 0$ and $a_l = 9$.

Corollary 8.5 - of the proof.
If p and q are positive inegers, the rational number p/q has a decimal expansion with period at most q.

Theorem 8.7
A number $x \geq 0$ is ration $\Leftrightarrow$ it has periodic decimal expansion.

Corollary 8.8
A number $x \geq 0$ is irrational $\Leftrightarrow$ if has aperiodic decimal expansion

Definition 8.2
Given a sequence a_j, we say that the infinite series $\sum_{j=1}^{\infty} a_j$ *converges* if the sequence of partial sums

$$s_n = \sum_{j=1}^{n} a_j \tag{2}$$

converges as $n \to \infty$. When (s_n) converges we denote its limit by $\sum_{j=1}^{\infty} a_j$.

4.2 The number e

$$e := \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} (1 + \frac{1}{n})^n \tag{3}$$

5 Complex number

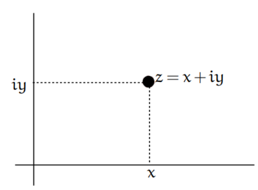
5.1 Definition 9.1 - Complex numbers definition

Define i to be a number such that $i^2 = -1$
The **comple numbers** are any number of the form

$$z = x + iywherex, y \in \mathbb{R} \tag{4}$$

The set of all complex numbers is denoted $\mathbb{C}$
We define Re(z)=x and Im(z)=y to be the **real** and **imaginary** parts of z.

5.2 Argand diagram

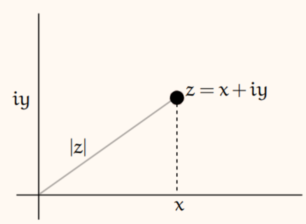


5.3 Definition 9.2 - Modulus

Definition 9.2 — Modulus. Given a complex number $z = x + iy$, the *modulus* of z is

$$|z| = |x + iy| = \sqrt{x^2 + y^2}.$$

Geometrically, the modulus of z is its distance from the origin, which is the length of the hypotenuse of a triangle of base x and height y, computed using the Pythagorean theorem.

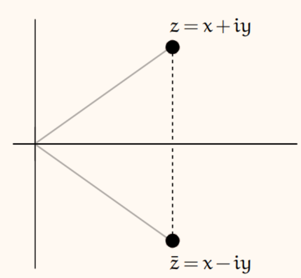


5.4 Definition 9.3 - Complex conjugate

Definition 9.3 — Complex Conjugate. Given a complex number $z = x + iy$, its *complex conjugate* is defined to be

$$\bar{z} = x - yi.$$

Geometrically, this is the complex number obtained by reflecting z across the 'x-axis'.

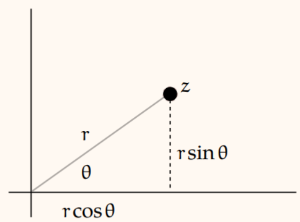


5.5 Definition 9.4 - Cartesian and Polar form

Definition 9.4 Let $z \neq 0$ be a complex number. Let $r = |z|$ and define the *argument* of z be the angle $\theta \in [0, 2\pi)$ between the line from 0 to z and the positive x-axis. We can then write

$$z = r(\cos\theta + i\sin\theta).$$

This is the *polar form* of z.



5.6 multiplication of complex

Let $z = r(cos\theta + isin\theta)$ and $w = s(cos\phi + isin\phi)$

$$\begin{aligned} zw &= r(cos\theta + isin\theta) \times s(cos\phi + isin\phi) \\ &= rs(cos(\theta + \phi) + isin(\theta + \phi)) \end{aligned} \tag{5}$$

De Movivre's Theorem
If we let $z = r(cos\theta + isin\theta)$, and $n \in \mathbb{N}$, then

$$\begin{aligned} z^n &= r^n(cosn\theta + isinn\theta) \\ z^{-n} = r^{-n}(cos(-n\theta) + isin(-n\theta)) &= \frac{1}{r^n}(cos(n\theta) - isin(n\theta)) \end{aligned} \tag{6}$$

5.7 Exponential form: Euler's formula

$$e^{i\theta} = cos(\theta) + isin(\theta) \tag{7}$$

Special case 1:

$$e^{i\pi} = -1$$

Special case 2:

$$z = re^{i\theta} = r(cos\theta + isin\theta)$$

5.8 Theorem9.4 - Roots of Unity

The solutions to $z^n = 1$ are $1, w, ...w^{n-1}$ where $w = e^{\frac{2\pi i}{n}}$, That is, they are $e^{\frac{2\pi ki}{n}}$ for k=0,1,..., n-1.

6 Polynomials

6.1 Definition 10.1

For $n \in \mathbb{N}$, and n-degree complex polynomial p is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0 \tag{8}$$

where $a_n \neq 0$ and $a_i \in \mathbb{C}$ for all i.
n is called the *degree* of the polynomial
A *root* of p is a complex number $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.

6.2 Abel-Ruffini Theorem

There is no formula for the roots of a polynomial of degree ≥ 5.

6.3 Theorem 10.2 - Fundamental Theorem of Algebra.

Any complex polynomial has at least one root in $\mathbb{C}$.

6.4 Theorem 10.3 - Factorization Theorem

If p is a degree n polynomial, then there are n roots $r_1, ..., r_n \in \mathbb{C}$ and a number $a \in \mathbb{C}$ so that

$$p(z) = a(z - r_1)(z - r_2)...(z - r_n). \tag{9}$$

6.5 Theorem 10.4 - Real Polynomials

Real Polynomials have *conjugated roots.*
If p(x) has real coefficients and r is a root, so is $\bar{(}r)$.

6.6 Theorem 10.5 - Root-Coefficient Theorem

If $p(x) = x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, has roots $r_1, ..., r_n$ (counting multiplicities), then

$$r_1 + ... + r_n = -a_{n-1} r_1 ... r_n = (-1)^a_0 \tag{10}$$

In general, if s_j denotes the sum of all products of j-tuples of the roots (e.g. $s_2 = r_1 r_2 + r_1 r_3 + r_2 r_3 + ...$), then

$$s_j = (-1)^j a_{n-j}. \tag{11}$$