

Proofs and Problem Solving

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1 Upper bounds and Least upper bound

1.1 Upper and Lower Bounds

Definition 1.1 (Upper and Lower Bounds). M is an upper bound of a set A if $M \geq a$ for all $a \in A$. N is a lower bound of a set A if $N \leq a$ for all $a \in A$.

A set is bounded if it has both an upper bound and a lower bound.

1.2 Least Upper Bounds and Greatest Lower Bounds

Definition 1.2 (Least Upper Bound). Given a subset $A \subseteq \mathbb{R}$, a number L is a **least upper bound (LUB)** or supremum for A if and only if:

1. L is an upper bound for A , that is, $x \leq L$ for all $x \in A$;
2. $L \leq M$ for every upper bound M of A **or**
for all $t < L$, there exist $x \in A$ such that $x > t$.

This is definition 5.2 in the textbook.

Definition 1.3 (Greatest Lower Bound).

1.3 Completeness Axiom for the real numbers

Every nonempty subset of \mathbb{R} that is bounded above has a *least upper bound*.

Theorem 1.1 (The Archimedean Property). *For any $x, y \in \mathbb{R}$ with $x, y > 0$, there is some $n \in \mathbb{N}$ such that $ny > x$. (Theorem 5.1 in the textbook).*

Observatio 1.1. Could be prove by contradiction

2 Limits

2.1 The $\epsilon - \mathbb{N}$ definition of a limit.

Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers x_1, x_2, \dots and let $L \in \mathbb{R}$. We say that x_n **converge** to L , or L is the **limit** of the sequence (x_n) , if for all $\epsilon > 0$, there exists a natural number N such that for all $n > N$,

$$|x_n - L| < \epsilon,$$

When x_n converges to L , we write $x_n \rightarrow L$ as $n \rightarrow \infty$, or $\lim_{n \rightarrow \infty} x_n = L$.

2.2 Bounded Sequence

Definition 2.1. We say that a sequence (a_n) is **bounded** if the set of values a_1, a_2, \dots is a bounded set. i.e. there are m, M such that $m \leq a_n \leq M$ for all n .

Proposition 2.1.1. Suppose the sequence (a_n) converges. Then it is bounded.

Proposition 2.1.2. Suppose that $x_n \rightarrow L$ as $n \rightarrow \infty$ and that $k \in \mathbb{N}$. Then $x_n^k \rightarrow L^k$ as $n \rightarrow \infty$.

2.3 Application: the existence of roots

Theorem 6.6

Let $x > 0$ and $k \in \mathbb{N}$. Then there is a unique $y > 0$ such that $y^k = x$.

2.4 Infinite limits

Definition 2.2. Let x_n be a sequence of real numbers.

1. We say (x_n) tends to ∞ or **diverges** to ∞ , and write $x_n \rightarrow \infty$ as $n \rightarrow \infty$ if for all $M > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $x_n \geq M$.
2. Similar for $x_n \rightarrow -\infty$

3 The monotone Convergence Theorem

3.1 Monotonic

A sequence $(a_n)_{n=1}^{\infty}$ is **increasing** if $a_{n+1} \geq a_n$ for all n . It is **decreasing** if $a_{n+1} \leq a_n$ for all n . We say a_n is **monotonic** if it is either increasing or decreasing.

Monotone Convergence Theorem (MCT)

Let (a_n) be an **increasing** sequence of real numbers that is bounded above (i.e. there is M so that $a_n \leq M$ for all n). Then (a_n) converges to some limit. Similar to **decreasing** sequence.

3.2 iteratively defined sequence

The value of x_{n+1} is specified in terms of previous values of the sequence x_1, \dots, x_n , (usually x_n).

4 Decimals and Series

Every real number has a decimal expansion, and conversely, every decimal expansion gives rise to a real number.

4.1 Decimals

Theorem 8.1

Given sequence (a_n) with $a_n \in 0, 1, \dots, 9$ for all n , the decimal expansion $x = 0.a_1a_2a_3a_4\dots a_n\dots$ given by the limit of the sequence

$$x_n = \frac{a_1}{10} + \frac{a_2}{100} + \dots + \frac{a_n}{10^n} \quad (1)$$

defines a real number x satisfying $0 \leq x \leq 1$.

Theorem 8.3

Let $0 \leq x \leq 1$ and suppose that $x = 0.a_1a_2\dots = 0.b_1b_2\dots$. Let l be the smallest integer for which $a_l \neq b_l$ and suppose that $a_l < b_l$. Then $b_l = a_l + 1$, and for all $k > l$ we have $b_k = 0$ and $a_k = 9$.

Corollary 8.5 - of the proof.

If p and q are positive integers, the rational number p/q has a decimal expansion with period at most q .

Theorem 8.7

A number $x \geq 0$ is rational \Leftrightarrow it has periodic decimal expansion.

Corollary 8.8

A number $x \geq 0$ is irrational \Leftrightarrow it has aperiodic decimal expansion

Definition 8.2

Given a sequence a_j , we say that the infinite series $\sum_{j=1}^{\infty} a_j$ *converges* if the sequence of partial sums

$$s_n = \sum_{j=1}^n a_j \quad (2)$$

converges as $n \rightarrow \infty$. When (s_n) converges we denote its limit by $\sum_{j=1}^{\infty} a_j$.

4.2 The number e

$$e := \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \quad (3)$$

5 Complex number

5.1 Definition 9.1 - Complex numbers definition

Define i to be a number such that $i^2 = -1$

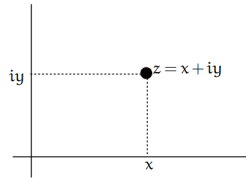
The **complex numbers** are any number of the form

$$z = x + iy \text{ where } x, y \in \mathbb{R} \quad (4)$$

The set of all complex numbers is denoted \mathbb{C}

We define $\text{Re}(z)=x$ and $\text{Im}(z)=y$ to be the **real** and **imaginary** parts of z .

5.2 Argand diagram

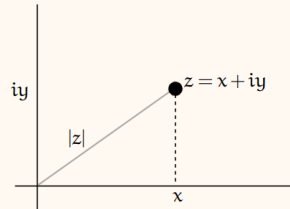


5.3 Definition 9.2 - Modulus

Definition 9.2 — Modulus. Given a complex number $z = x + iy$, the *modulus* of z is

$$|z| = |x + iy| = \sqrt{x^2 + y^2}.$$

Geometrically, the modulus of z is its distance from the origin, which is the length of the hypotenuse of a triangle of base x and height y , computed using the Pythagorean theorem.

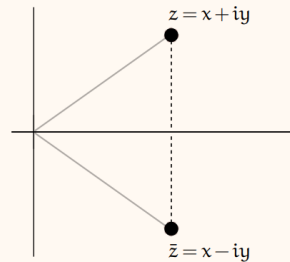


5.4 Definition 9.3 - Complex conjugate

Definition 9.3 — Complex Conjugate. Given a complex number $z = x + iy$, its *complex conjugate* is defined to be

$$\bar{z} = x - yi.$$

Geometrically, this is the complex number obtained by reflecting z across the 'x-axis'.

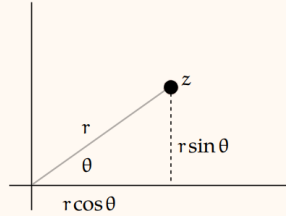


5.5 Definition 9.4 - Cartesian and Polar form

Definition 9.4 Let $z \neq 0$ be a complex number. Let $r = |z|$ and define the *argument* of z be the angle $\theta \in [0, 2\pi)$ between the line from 0 to z and the positive x-axis. We can then write

$$z = r(\cos \theta + i \sin \theta).$$

This is the *polar form* of z .



5.6 multiplication of complex

Let $z = r(\cos \theta + i \sin \theta)$ and $w = s(\cos \phi + i \sin \phi)$

$$\begin{aligned} zw &= r(\cos \theta + i \sin \theta) \times s(\cos \phi + i \sin \phi) \\ &= rs(\cos(\theta + \phi) + i \sin(\theta + \phi)) \end{aligned} \quad (5)$$

De Moivre's Theorem

If we let $z = r(\cos \theta + i \sin \theta)$, and $n \in \mathbb{N}$, then

$$\begin{aligned} z^n &= r^n(\cos n\theta + i \sin n\theta) \\ z^{-n} &= r^{-n}(\cos(-n\theta) + i \sin(-n\theta)) = \frac{1}{r^n}(\cos(n\theta) - i \sin(n\theta)) \end{aligned} \quad (6)$$

5.7 Exponential form: Euler's formula

$$e^{i\theta} = \cos(\theta) + i \sin(\theta) \quad (7)$$

Special case 1:

$$e^{i\pi} = -1$$

Special case 2:

$$z = re^{i\theta} = r(\cos \theta + i \sin \theta)$$

5.8 Theorem 9.4 - Roots of Unity

The solutions to $z^n = 1$ are $1, w, \dots, w^{n-1}$ where $w = e^{\frac{2\pi i}{n}}$. That is, they are $e^{\frac{2\pi k i}{n}}$ for $k=0, 1, \dots, n-1$.

6 Polynomials

6.1 Definition 10.1

For $n \in \mathbb{N}$, and n -degree complex polynomial p is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 \quad (8)$$

where $a_n \neq 0$ and $a_i \in \mathbb{C}$ for all i .

n is called the *degree* of the polynomial

A *root* of p is a complex number $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.

6.2 Abel-Ruffini Theorem

There is no formula for the roots of a polynomial of degree ≥ 5 .

6.3 Theorem 10.2 - Fundamental Theorem of Algebra.

Any complex polynomial has at least one root in \mathbb{C} .

6.4 Theorem 10.3 - Factorization Theorem

If p is a degree n polynomial, then there are n roots $r_1, \dots, r_n \in \mathbb{C}$ and a number $a \in \mathbb{C}$ so that

$$p(z) = a(z - r_1)(z - r_2)\dots(z - r_n). \quad (9)$$

6.5 Theorem 10.4 - Real Polynomials

Real Polynomials have *conjugated roots*.

If $p(x)$ has real coefficients and r is a root, so is \bar{r} .

6.6 Theorem 10.5 - Root-Coefficient Theorem

If $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$, has roots r_1, \dots, r_n (counting multiplicities), then

$$r_1 + \dots + r_n = -a_{n-1}, r_1 r_2 \dots r_n = (-1)^n a_0 \quad (10)$$

In general, if s_j denotes the sum of all products of j -tuples of the roots (e.g. $s_2 = r_1 r_2 + r_1 r_3 + r_2 r_3 + \dots$), then

$$s_j = (-1)^j a_{n-j}. \quad (11)$$