Baker-Hausdorff 公式及Glauber 公式的简单证明

1 Baker-Hausdorff公式

如果$[A,B]\neq 0$,则有:

$$\begin{aligned}
e^{A}Be^{-A} &= B+\frac{1}{1!}[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}\Big[A,\big[A,[A,B]\big]\Big]+...\\
S_0 &= B\\
S_{n+1} &= [A,S_n]\\
\Rightarrow e^{A}Be^{-A} &= S_0+\frac{1}{1!}S_1+\frac{1}{2!}S_2+...
\end{aligned}$$

证：采用参量微分法

$$\begin{aligned}
g(t) &= e^{At}Be^{-At}\\
g(t) &= \sum_{n=0}\frac{1}{n!}\frac{d^n g(t)}{dt^n}\bigg|_{t=0} t^n\\
\frac{dg(t)}{dt} &= e^{At}ABe^{-At}-e^{At}BAe^{-At}=e^{At}[A,B]e^{-At}\\
\frac{d^2g(t)}{dt^2} &= \frac{d}{dt}\left(\frac{dg(t)}{dt}\right)=e^{At}A[A,B]e^{-At}-e^{At}[A,B]Ae^{-At}=e^{At}\big[A,[A,B]\big]e^{-At}\\
&\vdots\\
t=1 &\Rightarrow e^{A}Be^{-A}=B+\frac{1}{1!}[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}\Big[A,\big[A,[A,B]\big]\Big]+...
\end{aligned}$$

2 Glauber 公式

如果：$\big[A,[A,B]\big]=\big[B,[A,B]\big]=0$ 则：$e^{A+B}=e^{-[A,B]/2}e^{A}e^{B}$

证：采用参量微分法

$$\begin{aligned}
f(t) &= e^{At}e^{Bt}\\
\frac{df(t)}{dt} &= Ae^{At}e^{Bt}+e^{At}Be^{Bt}\\
&= Ae^{At}e^{Bt}+(e^{At}Be^{-At})e^{At}e^{Bt}\\
e^{At}Be^{-At} &= B+\frac{1}{1!}[A,B]t+\frac{1}{2!}\big[A,[A,B]\big]t^2+...\\
&= B+\frac{1}{1!}[A,B]t=B+[A,B]t
\end{aligned}$$

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\begin{aligned}
\frac{df(t)}{dt} &= \left[A+(e^{At}Be^{-At})\right]f(t) = \left[A+B+[A,B]t\right]f(t) \\
\frac{df}{f} &= \left[A+B+[A,B]t\right]dt \\
\ln f(t)-\ln f(0) &= (A+B)t+\frac{1}{2}[A,B]t \\
f(t) &= f(0)e^{(A+B)t+\frac{1}{2}[A+B]t^2} = e^{(A+B)t}e^{\frac{1}{2}[A,B]t^2} \\
t=1 &\Rightarrow e^{A+B} = e^{-[A,B]/2}e^Ae^B
\end{aligned}
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如果字符混乱，请下载阅读。