obtained by pure deduction, employing only hypotheses already accepted as fundamental in wave mechanics. Starting from the principle that the wave equation expresses the conservation of probability in a steady state, it is pointed out that, owing to interchangeability, the conservation of probability for two charges is not a simple extension to eight dimensions of the result for one charge in four dimensions. It is necessary to add a new direction of flow of probability between the two configurations corresponding to interchange. The Fermi-Dirac law of antisymmetry is deduced, and it is shown that this quantises the flow in the new direction. It is found that the same state (*i.e.*, distribution of probability) repeats itself 137 times as frequently along the cyclic co-ordinate corresponding to interchange compared with the other cyclic co-ordinates corresponding to excitation. The matrix associated with the interchange energy differs from that given in the earlier papers, and in special cases it reduces to the form $\frac{1}{2}(1 + (\sigma, \sigma'))$ commonly employed in non-relativity theory.

The Viscosity of a Fluid Containing Small Drops of Another Fluid.

By G. I. Taylor, F.R.S.

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The viscosity of a fluid in which small solid spheres are suspended has been studied by Einstein as a problem in theoretical hydrodynamics.* Einstein's paper gave rise to many experimental researches on the viscosity of fluids containing solid particles, and it soon became clear that though complete agreement with the theory might be expected when the particles are true spheres, some modification is necessary when the particles are flattened or elongated. The theory of such systems was developed by G. B. Jeffery,† who calculated the motion of ellipsoidal particles in a viscous fluid and their effect on the mean viscosity. Some of his conclusions have been verified by observation.‡

* 'Ann. Physik.,' vol. 19, p. 289 (1906), and a correction to that paper, 'Ann. Physik.,' vol. 34, p. 591 (1911).

† 'Proc. Roy. Soc.,' A, vol. 102, p. 161 (1922).

‡ G. I. Taylor, 'Proc. Roy. Soc.,' A, vol. 103, p. 58 (1923).

So far no one seems to have extended Einstein's work to liquids containing small drops of another liquid in suspension. The difficulties in the way of a complete theory when solid particles are replaced by fluid drops are almost insuperable, partly because the correct boundary conditions are not known, and partly because a fluid drop would deform under the combined action of viscous forces and surface tension. Even if the boundary conditions were known to be those commonly used in hydrodynamical theory, the calculation of the shape of the deformed drop would be exceedingly difficult. When the radius of the suspended drops or the velocity of distortion of the fluid are small, surface tension may be expected to keep them nearly spherical, and in that case Einstein's analysis may be extended so as to include the case of liquid drops. For this purpose the following assumptions will be made :—

(1) The drops are so small that they remain nearly spherical.
(2) There is no slipping at the surface of the drop.
(3) The tangential stress parallel to the surface is continuous at the surface of the drop, so that any film which may exist between the two liquids merely transmits tangential stress from one fluid to the other.

A general method for analysing the slow motion of viscous fluids has been given by Lamb,* according to which the components of velocity may be regarded as containing three types of terms. The complete expression for one component of velocity is

$$u = \left[\frac{1}{\mu}\Sigma\frac{r^2}{2(2n+1)}\frac{\partial p_u}{\partial x} + \frac{nr^{2n+3}}{(n+1)(2n+1)(2n+3)}\frac{\partial}{\partial u}\left(\frac{p_n}{r^{2n+1}}\right)\right] + \left[\Sigma\frac{\partial \phi_n}{\partial x}\right] + \left[\Sigma\left(z\frac{\partial \chi_n}{\partial y} - y\frac{\partial \chi_n}{\partial z}\right)\right] \quad (1)$$

The terms in the first square bracket are connected with the pressure distribution, which is represented by $p = \Sigma p_n$, p_n being a solid harmonic function of degree n. The terms in the second bracket represent an irrotational motion which can exist in a field of uniform pressure. The terms in the third bracket represent vortex motion which can exist in a field of uniform pressure. χ_n is an arbitrary function of degree n.

In Einstein's equations the co-ordinate axes were chosen parallel to the principal axes of distortion, and with this choice of axes terms containing χ_n disappear. Einstein used the most general type of flow near a solid sphere,

* "Hydrodynamica," chap. 11.

but little loss of generality is suffered by taking the special case when the mean motion of the whole system is two-dimensional. In this case, if the origin of co-ordinates is taken at the centre of a drop, the flow at great distances from it may be represented by the irrotational flow $\phi_2 = \frac{1}{4}\alpha(x^2 - y^2)$, the constant $\frac{1}{4}\alpha$ being chosen so that the flow is identical, except for a rotation of the whole field, with the familiar case of uniformly shearing laminar flow which is commonly represented by taking the axis of x in the direction of flow when the system is $(u = \alpha y, v = 0)$.

Choice of Functions ϕ_n and p_n.

Outside the drop the appropriate functions are

$$\phi_2 = \tfrac{1}{4}\alpha(x^2 - y^2), \quad \phi_{-3} = \mathrm{B}_{-3}\, a^5 \frac{x^2 - y^2}{r^5}, \quad p_{-3} = \mu \mathrm{A}_{-3}\, a^3 \frac{x^2 - y^2}{r^5} \tag{2}$$

while inside the drop

$$\phi'_2 = \mathrm{B}_2 (x^2 - y^2), \qquad p_2 = \mu' \mathrm{A}_2\, a^{-2} (x^2 - y^2) \tag{3}$$

where B_{-3}, A_{-3}, B_2, A_2, are constants to be determined by the boundary conditions, μ and μ' are the viscosities of the main body of fluid and the drop, $r^2 = x^2 + y^2 + z^2$, and a is the radius of the drop.

Substituting these expressions in (1), the components of velocity outside the drop are

$$\left.\begin{aligned}
u &= \tfrac{1}{2}\mathrm{A}_{-3}\, a^3 x \frac{x^2 - y^2}{r^5} + \mathrm{B}_{-3}\, a^5 \left[-\frac{5x(x^2 - y^2)}{r^7} + \frac{2x}{r^5} \right] + \tfrac{1}{2}\alpha x \\
v &= \tfrac{1}{2}\mathrm{A}_{-3}\, a^3 y \frac{x^2 - y^2}{r^5} + \mathrm{B}_{-3}\, a^5 \left[-\frac{5y(x^2 - y^2)}{r^7} - \frac{2y}{r^5} \right] - \tfrac{1}{2}\alpha y \\
w &= \tfrac{1}{2}\mathrm{A}_{-3}\, a^3 z \frac{x^2 - y^2}{r^5} + \mathrm{B}_{-3}\, a^5 \left[-\frac{5z(x^2 - y^2)}{r^7} \right]
\end{aligned}\right\} \tag{4}$$

Similarly inside the drop

$$\left.\begin{aligned}
u' &= \mathrm{A}_2\, a^{-2} \left[-\frac{5}{21} x r^2 - \frac{2}{21} x (x^2 - y^2) \right] + 2\mathrm{B}_2\, x \\
v' &= \mathrm{A}_2\, a^{-2} \left[-\frac{5}{21} y r^2 - \frac{2}{21} y (x^2 - y^2) \right] - 2\mathrm{B}_2\, y \\
w' &= \mathrm{A}_2\, a^{-2} \left[-\frac{2}{21} z (x^2 - y^2) \right]
\end{aligned}\right\} \tag{5}$$

Boundary Conditions at $r = a$.

Continuity of velocity requires

$$u = u', \quad v = v', \quad w = w' \tag{6}$$

and the drop remains spherical if

$$ux + vy + wz = 0 \tag{7}$$

At $r = a$,

$$u = \tfrac{1}{2} \mathrm{A}_{-3}\, a^{-2} x\, (x^2 - y^2) + \mathrm{B}_{-3}\, a^{-2} [-5x\,(x^2 - y^2) + 2a^2 x] + \tfrac{1}{2}\, \alpha x$$

and

$$u' = \frac{5}{21} \mathrm{A}_2 x - \frac{2}{21} \mathrm{A}_2 a^{-2}\, x\, (x^2 - y^2) + 2\mathrm{B}_2 x$$

so that $u = u'$ if

$$\tfrac{1}{2} \mathrm{A}_{-3} - 5\mathrm{B}_{-3} = -\frac{2}{21} \mathrm{A}_2 \tag{8}$$

and

$$2\mathrm{B}_{-3} + \tfrac{1}{2}\, \alpha = \frac{5}{21} \mathrm{A}_2 + 2\mathrm{B}_2 ; \tag{9}$$

when (8) and (9) are satisfied it will be found that $v = v'$ and $w = w'$, so that all three conditions (6) are satisfied.

To satisfy (7) a formula given by Lamb may be used, namely,

$$xu + yv + zw = \frac{1}{\mu} \Sigma \frac{nr^2}{2\,(2n+3)}\, p_n + \Sigma u\, \phi_n$$

This provides a third equation between the four undetermined constants, namely,

$$\tfrac{1}{2} \mathrm{A}_{-3} - 3\mathrm{B}_{-3} + \tfrac{1}{2}\, \alpha = 0 \tag{10}$$

and the problem will be soluble in this form if only one further equation is necessary in order to satisfy both conditions for continuity of tangential stress.

The components of stress acting across unit area of a spherical surface are p_{rx}, p_{ry}, p_{rz}. The general expression* for p_{rx} is

$$r p_{rx} = \Sigma \left\{ \frac{n-1}{2n+1}\, r^2 \frac{\partial p_n}{\partial x} + \frac{2n^2 + 4n + 3}{(n+1)\,(2n+1)\,(2n+3)}\, r^{2n+3} \frac{\partial}{\partial x} \left(\frac{p_n}{r^{2n+1}} \right) \right\}$$
$$+ 2\mu\, \Sigma\, (n-1) \frac{\partial \phi_n}{\partial x} + \mu\, \Sigma\, (n-1) \left(y \frac{\partial \chi_n}{\partial z} - z \frac{\partial \chi_n}{\partial y} \right) \tag{11}$$

In the present case this reduces outside the drop to

$$\frac{r p_{rx}}{\mu} = \mathrm{A}_{-3}\, a^3 \left[\frac{x}{r^3} - \frac{4x}{r^5} (x^2 - y^2) \right] - 8\mathrm{B}_{-3}\, a^5 \left[\frac{2x}{r^5} - \frac{5x}{r^7} (x^2 - y^2) \right]$$
$$+ \alpha x \tag{12}$$

* See Lamb, *l.c.*

and inside the drop to

$$\frac{rp_{rx}}{\mu} = A_2\, a^{-2}\left[\frac{16}{21}\, r^2\, x - \frac{19}{21}\, x\, (x^2 - y^2)\right] + 4B_2\, x, \tag{13}$$

with somewhat similar expressions for p_{ry} and p_{rz}.

Writing

$$\left.\begin{array}{ll} A_{-3} - 16B_{-3} + \alpha = \gamma, & \frac{16}{21} A_2 + 4B_2 = \gamma' \\ 4A_{-3} - 40B_{-3} \quad = \beta, & \frac{19}{21} A_2 \qquad\quad = \beta' \end{array}\right\}, \tag{14}$$

the stress components on the outside of the surface $r = a$ are

$$\left.\begin{array}{l} \frac{ap_{rx}}{\mu} = \gamma x - \beta a^{-2} x\, (x^2 - y^2) \\ \frac{ap_{ry}}{\mu} = -\gamma y - \beta a^{-2} y\, (x^2 - y^2) \\ \frac{ap_{rz}}{\mu} = -\beta a^{-2} z\, (x^2 - y^2) \end{array}\right\}, \tag{15}$$

while identical expressions serve to express the stress components inside, provided μ, β, γ are replaced by μ', β', γ'. It will be noticed that it is not possible to ensure continuity of all three components of stress. To do so would require both $\mu\beta = \mu'\beta'$ and $\mu\gamma = \mu'\gamma'$, and these equations cannot both be satisfied as well as (8), (9) and (10).

To apply the condition of continuity of tangential stress it is necessary to transform the stress components p_{rx}, p_{ry}, p_{rz} into components p_{rr} acting normal to the surface of the sphere, $p_{r\theta}$ parallel to the surface of the sphere and in a plane passing through the axis of x, $p_{r\phi}$ parallel to the surface of the sphere and perpendicular to the axis of x. The scheme of direction cosines for this transformation is

	p_{rx}.	p_{ry}.	$p_{r\theta}$.
p_{rr}	x/r	y/r	z/r
$p_{r\theta}$	$-(1 - x^2/a^2)^{\frac{1}{2}}$	$xy\, a^{-2}\, (1 - x^2/a^2)^{\frac{1}{2}}$	$xz\, a^{-2}\, (1 - x^2/a^2)^{\frac{1}{2}}$
$p_{r\phi}$	0	$-z\, a^{-1}\, (1 - x^2/a^2)^{\frac{1}{2}}$	$y\, a^{-1}\, (1 - x^2/a^2)^{\frac{1}{2}}$

Applying this transformation, the stresses at $r = a$ in the outer fluid are

$$\left.\begin{aligned} p_{rr} &= \mu a^{-2}(x^2 - y^2)(\gamma - \beta) \\ p_{r\theta} &= \frac{\mu\gamma x(x^2 - y^2 - a^2)}{a^2\sqrt{a^2 - x^2}} \\ p_{r\phi} &= \frac{\mu\gamma yz}{a\sqrt{a^2 - x^2}} \end{aligned}\right\}. \tag{16}$$

Identical expressions serve to represent the stress at $r = a$ inside the drop if β, γ, μ are replaced by β', γ', μ'. It will be seen from (16) that continuity of both $p_{r\theta}$ and $p_{r\phi}$ is ensured if

$$\mu\gamma = \mu'\gamma' \text{ or } A_{-3} - 16\, B_{-3} + \alpha = \frac{\mu'}{\mu}\left(\frac{16}{21} A_2 + 4B_2\right), \tag{17}$$

but when (17) is satisfied p_{rr} is discontinuous at $r = a$.

Determination of A_{-3}, B_{-3}, A_2, B_2.

The four equations (8), (9), (10), (17) can now be used to determine the constants. The solution is

$$A_{-3} = -\frac{5\alpha}{2}\left(\frac{\mu' + \frac{2}{5}\mu}{\mu' + \mu}\right), \quad B_{-3} = -\frac{\alpha}{4}\,\frac{\mu'}{\mu' + \mu}, \quad A_2 = \frac{21\alpha\mu}{4(\mu' + \mu)},$$

$$B_2 = -\frac{3\alpha\mu}{8(\mu' + \mu)} \tag{18}$$

Viscosity of the Suspension.

The effect of the presence of solid spheres in suspension on the viscosity of a fluid was shown by Einstein to depend only on p_{-3}, and the same reasoning is still true when the spheres are liquid. In the case of a solid sphere $A_{-3} = -\frac{5\alpha}{2}$; thus it will be seen from (18) that Einstein's expression*

$$\mu^{\times} = \mu(1 + 2\cdot5\ \Phi) \tag{19}$$

for the viscosity of a fluid containing solid spheres must be replaced by

$$\mu^{\times} = \mu\left\{1 + 2\cdot5\,\Phi\left(\frac{\mu' + \frac{2}{5}\mu}{\mu' + \mu}\right)\right\} \tag{20}$$

when the spheres are fluid. In these formulæ $\mu^{\times}$ is the mean viscosity and Φ

* 'Ann. Physik,' vol. 34, p. 592 (1911).

is the small proportion of the whole volume occupied by the spheres. The two formulæ are identical, as would be expected when μ' becomes infinite.

The factor $\dfrac{\mu' + \frac{2}{5}\mu}{\mu' + \mu}$ by which Einstein's term must be multiplied in order to take account of the currents set up inside the drop, may be compared with the factor* $\dfrac{\mu' + \frac{2}{3}\mu}{\mu' + \mu}$ by which Stokes' expression for the resistance of a solid sphere in a viscous fluid must be multiplied in order to take account of the internal currents in a liquid drop falling through another liquid.

Limits to the Size of Drops contained in a Shearing Fluid.

In order that the drops may be nearly spherical the pressure difference due to viscous forces must be small compared with that due to surface tension, namely, $2\,\mathrm{T}/a$, where T is the surface tension.

The difference in pressure between the inside and outside of the drop is

$$[p_{rr}]_{\text{inside}} - [p_{rr}]_{\text{outside}},$$

and from (16) this is

$$\mathrm{P} = [\mu'(\gamma' - \beta') - \mu(\gamma - \beta)]\frac{x^2 - y^2}{a^2},$$

hence substituting from (14) and (18)

$$\mathrm{P} = \frac{\alpha\mu}{\mu' + \mu}\left(\frac{19}{4}\mu' + 4\mu\right)\frac{x^2 - y^2}{a^2}. \tag{21}$$

The maximum and minimum values of $\dfrac{x^2 - y^2}{a^2}$ are ± 1, so that the drop will be nearly spherical so long as α is small compared with

$$\frac{\mu' + \mu}{\mu\left(\dfrac{19\mu'}{4} + 4\mu\right)}\left(\frac{2\mathrm{T}}{a}\right). \tag{22}$$

On the other hand, the drop whose radius is

$$a = \frac{2\mathrm{T}(\mu' + \mu)}{\alpha\mu\left(\dfrac{19}{4}\mu' + 4\mu\right)}, \tag{23}$$

is of such a size that the disruptive forces due to viscosity tending to burst the drop are about equal to the force due to surface tension which tends to hold it together. An approximate expression of this kind might have some

* Hadamard, 'C.R. Acad. Sci.,' Paris, vol. 152, p. 1735 (1911).

interest in connection with the mechanical formation of emulsions, but since the fluid would certainly be turbulent in any emulsifying machine, the appropriate value to be taken for α would need much consideration.

Summary.

Einstein's expression for the viscosity of a fluid containing solid spheres in suspension is extended so as to include the case when the spheres are liquid. The expression obtained is valid provided the surface tension is great enough to keep the drops nearly spherical. When the rate of distortion of the fluid or the radius of the drop is great enough, the drops tend to break up, and an approximate expression is given for determining the size of the largest drop that can exist in a fluid which is undergoing distortion at any given rate.

On the Theory of Errors and Least Squares.

By Harold Jeffreys, M.A., D.Sc., F.R.S.

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1. In my "Scientific Inference," chapter V, I found that the usual presentation of the theory of errors of observation needed some modification, even where the probability of error is distributed according to the normal law. One change made was in the distribution of the prior probability of the precision constant h. Whereas this is usually taken as uniform (or ignored), I considered it better to assume that the prior probability that the constant lies in a range dh is proportional to dh/h. This is equivalent to assuming that if $h_1/h_2 = h_3/h_4$, h is as likely to lie between h_1 and h_2 as between h_3 and h_4; this was thought to be the best way of expressing the condition that there is no previous knowledge of the magnitude of the errors. The relation must break down for very small h, comparable with the reciprocal of the whole length of the scale used, and for large h comparable with the reciprocal of the step of the scale; but for the range of practically admissible values it appeared to be the most plausible distribution.

The argument for this law can now be expressed in an alternative form. The normal law of error is supposed to hold, but the true value x and the precision constant h are unknown. Two measures are made: what is the prob-