1 Propbability

1.1 Probability that $ax^2 + bx + c = 0$ Has Real Solutions

$$
\begin{aligned}
P &= \frac{1}{8}\int_{b^2-4ac\geq 0} 1\, dadbdc \\
&= \frac{1}{72}(41 + 3\log 4) \approx 0.627207
\end{aligned}
$$

1.2 Probability that $ax^3 + bx^2 + cx + d = 0$ Has Real Solutions

$$
\begin{aligned}
P &= \frac{1}{16}\int_{b^2c^2-4ac^3-4b^3d+18abcd-27a^2d^2\geq 0} 1\, dadbdc \\
&\approx 0.217503
\end{aligned}
$$

1.3 Probability that $w^3 + aw^2 + bw + c = 0$ Has Real Solutions

1.3.1 Method 1

Such a polynomial has roots exactly when $\frac{1}{k}w^3 + \frac{a}{k}w^2 + \frac{b}{k}w + \frac{c}{k} = 0$ has real roots. Let $x_k = \frac{a}{k}, y_k = \frac{b}{k}, z_k = \frac{c}{k}$. Since the discriminant of this polynomial is $\Delta = \frac{18}{k}x_k y_k z_k + x_k^2 y_k^2 - 4x_k^3 z_k - \frac{4}{k}y_k^3 - \frac{27}{k^2}z_k^2$, hence we need to have $\Delta \geq 0$, which is equivalent to

$$
\begin{aligned}
x_k^2(y_k^2 - 4x_k z_k) &\geq \frac{27}{k^2}z_k^2 + \frac{4}{k}y_k^3 - \frac{18}{k}x_k y_k z_k \\
(y_k^2 - 4x_k z_k) &\geq \frac{1}{x_k^2}\left(\frac{27}{k^2}z_k^2 + \frac{4}{k}y_k^3 - \frac{18}{k}x_k y_k z_k\right)
\end{aligned}
$$

Let $P_k = P\left(\frac{1}{k}w^3 + x_k w^2 + y_k w + z_k = 0 \text{ has real roots}\right)$.
Since $x_k, y_k, z_k \in [-1, 1]$,

$$
\begin{aligned}
\frac{1}{x_k^2}\left(\frac{27}{k^2}z_k^2 + \frac{4}{k}y_k^3 - \frac{18}{k}x_k y_k z_k\right) &\geq 0 - \frac{4}{k} - \frac{18}{k} \\
&= \frac{-22}{k} \\
\frac{1}{x_k^2}\left(\frac{27}{k^2}z_k^2 + \frac{4}{k}y_k^3 - \frac{18}{k}x_k y_k z_k\right) &\leq \sqrt{k}\left(\frac{27}{k^2} + \frac{4}{k} + \frac{18}{k}\right) \\
&\leq \sqrt{k}\left(\frac{27}{k} + \frac{4}{k} + \frac{18}{k}\right) \\
&= \frac{49}{\sqrt{k}} \\
&\text{if } |x_k| \geq \frac{1}{\sqrt[4]{k}}
\end{aligned}
$$

hence the probability satisfies the following inequality:

$$
P\left(y_k^2 - 4x_k z_k \geq \frac{49}{k}, |x| \geq \frac{1}{\sqrt[4]{k}}\right) \leq P_k \leq P\left(y_k^2 - 4x_k z_k \geq \frac{-22}{k}\right)
$$

Given any number $\delta \geq 0$, we can find a number N_1 such that $\frac{22}{k} \leq \delta$, $\frac{-22}{k} \geq -\delta$ for $k > N_1$; we can find a number N_2 such that $\frac{49}{k} \leq \delta$ for $k > N_2$; we can find a number N_3 such that $\frac{1}{\sqrt[4]{k}} \leq \delta$ for $k > N_3$. Let $N = \max(N_1, N_2, N_3)$. Then for $k \geq N$, we have

$$\frac{-22}{k} \geq -\delta \tag{1}$$

$$\frac{49}{k} \leq \delta \tag{2}$$

$$\frac{1}{\sqrt[4]{k}} \leq \delta \tag{3}$$

$$\begin{aligned}
P_k &\geq P\left(y_k^2 - 4x_kz_k \geq \frac{49}{k}, |x_k| \geq \frac{1}{\sqrt[4]{k}}\right) \\
&\geq P\left(y_k^2 - 4x_kz_k \geq \delta, |x_k| \geq \delta\right) \\
P_k &\leq P\left(y_k^2 - 4x_kz_k \geq \frac{-22}{k}\right) \\
&\leq P\left(y_k^2 - 4x_kz_k \geq -\delta\right)
\end{aligned}$$

Since $(x_k, y_k, z_k) \to (x, y, z)$ in distribution for $x, y, z \sim U(-1, 1)$, and δ can be arbitrarily small, hence

$$\begin{aligned}
&\limsup_{k\to\infty} P_k \leq \limsup_{k\to\infty} P\left(y_k^2 - 4x_kz_k \geq -\delta\right) = P\left(y^2 - 4xz \geq -\delta\right) \\
&\limsup_{k\to\infty} P_k = \lim_{\delta\to 0} \limsup_{k\to\infty} P_k \leq \lim_{\delta\to 0} P\left(y^2 - 4xz \geq -\delta\right) = P\left(y^2 - 4xz \geq 0\right)
\end{aligned}$$

$$\begin{aligned}
&\liminf_{k\to\infty} P_k \geq \liminf_{k\to\infty} P\left(y_k^2 - 4x_kz_k \geq \delta, |x_k| \geq \delta\right) = P\left(y^2 - 4xz \geq \delta, |x| \geq \delta\right) \\
&\liminf_{k\to\infty} P_k = \lim_{\delta\to 0} \liminf_{k\to\infty} P_k \geq \lim_{\delta\to 0} P\left(y^2 - 4xz \geq \delta, |x| \geq \delta\right) = P\left(y^2 - 4xz \geq 0\right)
\end{aligned}$$

Since $\liminf_{k\to\infty} P_k = \limsup_{k\to\infty} P_k$, thus $P_0 = \lim_{k\to\infty} P_k$ exists, and

$$\begin{aligned}
P_0 &= P\left(y_k^2 - 4x_kz_k \geq 0\right) \\
&= \frac{1}{72}(41 + 3\log 4) \approx 0.627207
\end{aligned}$$

1.3.2 Method 2

Let $f(w) = \frac{1}{k}w^3 + x_kw^2 + y_kw + z_k$, $g(w) = x_kw^2 + y_kw + z_k$. Suppose $g(w) = 0$ has 2 real roots r_1, r_2, and $r_2 < r_1$, then $y_k^2 - 4x_kz_k > 0$.

(1)$x_k = 0$

Since $y_kw + z_k = 0$ could not have 2 distinct real roots, hence this situation does not exist.

(2)$x_k > 0$

Let $R_k = \max\{|r_1|, |r_2|\}$, $\delta_k = \frac{1}{2}|\frac{y_k^2 - 4x_kz_k}{4x_k}|$. Let k be such that $k > \max\{\frac{R_k^3}{\delta_k}, \frac{(2R_k)^3}{g(-2R_k)}\}$, then $0 \leq \frac{R_k^3}{k} < \delta_k$, $\frac{1}{k}(2R_k)^3 < g(-2R_k)$.

-1) $r1 > 0, r2 > 0$.
Since $r1 > 0, r2 > 0$, thus

$$f(0) = g(0) > 0$$
$$f(r_1) = g(r_1) + \frac{1}{k}r_1^3 = 0 + \frac{1}{k}r_1^3 > 0$$
$$f(-\frac{y_k}{2x_k}) = -\frac{y_k^2 - 4x_kz_k}{4x_k} + \frac{1}{k}(-\frac{y_k}{2x_k})^3 < -\frac{y_k^2 - 4x_kz_k}{4x_k} + \frac{1}{k}R_k^3 < -\frac{y_k^2 - 4x_kz_k}{4x_k} + \delta_k < 0$$

thus there are two real solutions: one is in $(0, -\frac{y_k}{2x_k})$, and the other in $(-\frac{y_k}{2x_k}, r_1)$.

-2) $r1 > 0, r2 = 0$.
Since $r1 > 0, r2 = 0$, thus

$$f(0) = g(0) = 0$$
$$f(r_1) = g(r_1) + \frac{1}{k}r_1^3 = 0 + \frac{1}{k}r_1^3 > 0$$
$$f(-\frac{y_k}{2x_k}) = -\frac{y_k^2 - 4x_kz_k}{4x_k} + \frac{1}{k}(-\frac{y_k}{2x_k})^3 < -\frac{y_k^2 - 4x_kz_k}{4x_k} + \frac{1}{k}R_k^3 < -\frac{y_k^2 - 4x_kz_k}{4x_k} + \delta_k < 0$$

thus there are two real solutions: one is 0, and the other in $(-\frac{y_k}{2x_k}, r_1)$.

-3) $r1 > 0, r2 < 0$.
Since $r1 > 0, r2 < 0$, thus

$$f(0) = g(0) < 0$$
$$f(r_1) = g(r_1) + \frac{1}{k}r_1^3 = 0 + \frac{1}{k}r_1^3 > 0$$
$$f(-2R_k) = g(-2R_k) + \frac{1}{k}(-2R_k)^3 = g(-2R_k) - \frac{1}{k}(2R_k)^3 > 0$$

thus there are two real solutions: one is in $(0, r_1)$, and the other in $(-2R_k, 0)$.

-4) $r1 = 0, r2 < 0$.
Since $r1 = 0, r2 < 0$, thus

$$f(0) = g(0) = 0$$
$$f(-2R_k) = g(-2R_k) + \frac{1}{k}(-2R_k)^3 = g(-2R_k) - \frac{1}{k}(2R_k)^3 > 0$$
$$f(-\frac{y_k}{2x_k}) = -\frac{y_k^2 - 4x_kz_k}{4x_k} + \frac{1}{k}(-\frac{y_k}{2x_k})^3 < -\frac{y_k^2 - 4x_kz_k}{4x_k} + 0 < 0$$

thus there are two real solutions: one is 0, and the other in $(-2R_k, -\frac{y_k}{2x_k})$.

-5) $r1 < 0, r2 < 0$.

Since $r1 < 0, r2 < 0$, thus

$$
\begin{aligned}
&f(0) = g(0) > 0 \\
&f(r_1) = g(r_1) + \frac{1}{k}(r_1)^3 = 0 + \frac{1}{k}(r_1)^3 < 0 \\
&f(r_2) = g(r_2) + \frac{1}{k}(r_2)^3 = 0 + \frac{1}{k}(r_2)^3 < 0 \\
&f(-2R_k) = g(-2R_k) + \frac{1}{k}(-2R_k)^3 = g(-2R_k) - \frac{1}{k}(2R_k)^3 > 0
\end{aligned}
$$

thus there are two real solutions: one is in $(r_1, 0)$, and the other in $(-2R_k, r_2)$.

(3)$x_k < 0$

If $x_k < 0$, we let G(w)=-g(w), F(w)=-f(w), then "$x_k > 0$" in this case. Thus similarly, we can prove that there are two roots between $(-2R_k, R_k)$ if $x_k < 0$.

Hence the cubic equation has two distinct real roots in $(-2R_k, R_k)$. From Complex Conjugate Roots Theorem, we know that $f(w)$ has 3 real roots.

Let P_k be the probability that $\frac{1}{k}w^3 + x_k w^2 + y_k w + z_k = 0$ has 3 real roots. Let $m_k = \max\{\frac{R_k^3}{\delta_k}, \frac{(2R_k)^3}{g(-2R_k)}\}$. Then

$$
\begin{aligned}
P_k = &P\left(\frac{1}{k}w^3 + x_k w^2 + y_k w + z_k = 0 \text{ has 3 real roots}, k > m_k\right) \\
&+ P\left(\frac{1}{k}w^3 + x_k w^2 + y_k w + z_k = 0 \text{ has 3 real roots}, k < m_k\right) \\
&\geq P\left(y_k^2 - 4x_k z_k > 0, k > m_k\right) \\
&+ P\left(\frac{1}{k}w^3 + x_k w^2 + y_k w + z_k = 0 \text{ has 3 real roots}, k < m_k\right) \\
&\geq P\left(y_k^2 - 4x_k z_k > 0, k > m_k\right) \\
&+ P\left(\frac{1}{k}w^3 + x_k w^2 + y_k w + z_k = 0 \text{ has 3 real roots}, k < m_k\right)
\end{aligned}
$$

1.3.3 Method 3

Let $f(w) = t_1 w^3 + x_1 w^2 + y_1 w + z_1$. Let $D(f)$ be discriminant of f. The fundamental theorem for symmetric polynomials asserts that there exists a polynomial $P(w_0, w_1, w_2, w_3)$ such that $D(f) = P(t_1, x_1, y_1, z_1)$. Thus D is continuous in coefficients of polynomial.

Let $g(w) = t_2 w^3 + x_2 w^2 + y_2 w + z_2$. The solutions of $f(w)$ are in complex plane and can be covered by three open balls. Thus for all points on the boundary of the ball, $|f(w)| > 0$. Let $\delta = \min\{|f(w)| : w \text{ is on the boundary of the balls }\}$. Since there are only a finite number of balls, thus for all points z on the boundary of balls, we can find $\delta_1 > 0$ such that $|z| < \delta_1, \delta_1 > 0$.

$$\begin{aligned}|f(w)-g(w)| &= |(t_1-t_2)w^3+(x_1-x_2)w^2+(y_1-y_2)w+(z_1-z_2)| \\ &\leq |(t_1-t_2)w^3|+|(x_1-x_2)w^2|+|(y_1-y_2)w|+|(z_1-z_2)| \\ &< |t_1-t_2|\delta_1^3+|x_1-x_2|\delta_1^2+|y_1-y_|\delta_1+|z_1-z_2|\end{aligned}$$

Let $|t_1-t_2| < \frac{\delta}{4\delta_1^3}, |x_1-x_2| < \frac{\delta}{4\delta_1^2}, |y_1-y_| < \frac{\delta}{4\delta_1}, |z_1-z_2| < \frac{\delta}{4}$. Then $|f(w)-g(w)| < \delta < |f(w)|$ for all points w on the boundary of the three balls. From Rouche's Theorem, we know that f and g have same number of roots inside three balls. Thus changing coefficients of f(w) a little would also change the roots a little. Hence $f(w)$ changes from having all real roots to having complex roots or change from having complex roots to having all real roots, then there must be a solution with double roots at some point. (See proof below.)

Let $F(t,x,y,z) = 1$ if $D \geq 0$, and 0 otherwise. Then F is almost continuous except at points where $D == 0$.

Thus

$$\begin{aligned}&\lim_{k\to\infty}(P(\frac{1}{k}w^3+x_kw^2+y_kw+z_k=0 \text{ has all real roots})) \\ &= \lim_{k\to\infty}(P(F(\frac{1}{k},x_k,y_k,z_k)==1)) \\ &\to P(F(0,x,y,z)==1) \\ &= P(xw^2+yw+z=0 \text{ has all real roots}) \\ &= P(y^2-4xz>=0)\end{aligned}$$

1.4 Proof for General Case

Lemma 1.1. *$F(a_n, a_{n-1}, \ldots, a_1, a_0)$ is almost continuous on $\mathbb{R}^{n+1}$*

Proof. Let $f(w) = a_nw^n + a_{n-1}w^{n-1} + \ldots + a_1w + a_0$. Let $D(f)$ be discriminant of f. The fundamental theorem for symmetric polynomials asserts that there exists a polynomial $P(w_0, w_1, \ldots, w_n)$ such that $D(f) = P(a_n, a_{n-1}, \ldots, a_0)$. Thus D is continuous in coefficients of polynomial.

Let $F(a_n, a_{n-1}, \ldots, a_1, a_0) = 1$ if $f(w)$ has all real roots, and 0 otherwise.

Suppose $F(a_n, a_{n-1}, \ldots, a_1, a_0) = 1$ at a point $(a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0)$, then $f(w) = a_nw^n + a_{n-1}w^{n-1} + \ldots + a_1w + a_0$ only has real roots. Then the solutions of $f(w)$ are on the real axis of the complex plane and can be covered by n disjoint open balls. Thus for all points on the boundary of the ball, $|f(w)| > 0$. Let $\delta = \min\{|f(w)| : w \text{ is on the boundary of the balls }\}$. Since there are only a finite number of balls, thus for all points on the boundary of balls, we can find $\delta_1 > 0$ such that $|z| < \delta_1, \delta_1 > 0$.

Let $g(w) = b_nw^n + b_{n-1}w^{n-1} + \ldots + b_1w + b_0$.

$$\begin{aligned}|f(w)-g(w)| &= |(a_n-b_n)w^n+(a_{n-1}-b_{n-1})w^{n-1}+\ldots+(a_0-b_0)| \\ &\leq |(a_n-b_n)w^n|+|(a_{n-1}-b_{n-1})w^{n-1}|+\ldots+|a_0-b_0| \\ &< |(a_n-b_n)|\delta_1^n+|(a_{n-1}-b_{n-1})|\delta_1^{n-1}+\ldots+|a_0-b_0|\end{aligned}$$

Let $b_n, b_{n-1}, \ldots, b_1, b_0$ be such that$|a_n - b_n| < \frac{\delta}{(n+1)\delta_1^n}, |a_{n-1} - b_{n-1}| < \frac{\delta}{(n+1)\delta_1^{n-1}}, \ldots, |a_1 - b_1| < \frac{\delta}{(n+1)\delta_1}, |a_0 - b_0| < \frac{\delta}{(n+1)}$. Then $|f(w) - g(w)| < \delta < |f(w)|$ for all points on the boundary of the n balls. From Rouche's Theorem, we know that f and g have same number of roots inside n balls. Since the n balls are disjoint, from Complex Conjugate Roots Theorem, we know that roots of g are also all real. Thus $F(b_n, b_{n-1}, \ldots, b_1, b_0) = 1$.

Hence F is continuous at points where $f(w)$ has distinct real roots. Similarly we can prove that F is continuous at points where $f(w)$ has complex roots and does not have double roots.

If $f(w)$ has double roots, then the double roots can be only covered by one open ball. If $|f(w) - g(w)| < \delta < |f(w)|$ for all points on the boundary of the balls, we cound know from Rouche's Theorem that f and g have same number of roots inside the balls. Since the new two roots could be either a complex conjugate pair or two distinct real roots or double roots, we know that $F(g)$ may be 0. Thus F may be discontinuous at points where D(f)==0.

Since the probability that a polynomial has double roots is 0, thus F is almost continuous.

Theorem 1.2. $\lim_{k\to\infty}(P(\frac{1}{k}w^n + a_{n-1,k}w^{n-1} + \ldots + a_{1,k}w + a_{0,k} = 0$ *has all real roots*$)) = P(a_{n-1}w^{n-1} + \ldots + a_1 w + a_0 = 0$ *has all real roots*$)$

Proof. Since F is almost continuous except at points where $f(w)$ has double roots, $D(f) = 0$, thus

$$\begin{aligned}
&\lim_{k\to\infty}(P(\frac{1}{k}w^n + a_{n-1,k}w^{n-1} + \ldots + a_{1,k}w + a_{0,k} = 0 \text{ has all real roots})) \\
&= \lim_{k\to\infty}(P(F(\frac{1}{k}, a_{n-1,k}, \ldots, a_{1,k}, a_{0,k}) == 1)) \\
&\to P(F(0, a_{n-1}, \ldots, a_1, a_0) == 1) \\
&= P(a_{n-1}w^{n-1} + \ldots + a_1 w + a_0 = 0 \text{ has all real roots})
\end{aligned}$$