Multiscale Operator

Steven Chiu

The complete idea is elaborated on Teams-ML2.

Ideas for Multiscale Operator

Important papers:

- Basis Operator Network (Hua and Lu 2023)
- Convolutional Neural Operator (Raonić et al. 2023)

and

- Multi-scale DNN (Cai and Xu 2019)

Function Encoder

Proposed by Hua and Lu (2023), function encoder approximate a function with neural basis. It is invariant to the resolution of sensor grids.

- **Neural Basis Decomposition**: Suppose u is the function we want to approximate, and given a set of basis function $\{\phi_i(x) : \mathbb{R}^d \to \mathbb{R}^1 | i = 1, \dots, N\}$. Suppose N is large enough, we can approximate u as

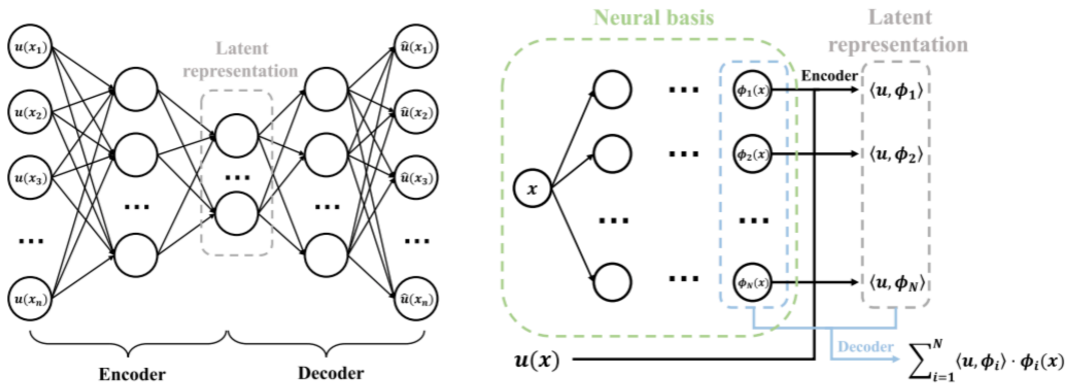
$$u \approx \sum_{i=1}^{N} \langle u, \phi_i \rangle \phi_i$$

  where $\langle u, \phi_i \rangle := \int_D u(x)\phi_i(x)dx$. (proof of convergence on N is in Hua and Lu (2023).).

  - Suppose we have a function u with uniform discretization $x_1, \dots, x_{n_1}$ and $u(x_1), \dots, u(x_{n_1})$
  - **Encoding**: $x \in \mathbb{R}^d \to \phi_i(x) \in \mathbb{R}^1$ for $i = 1, \dots, N$, and get $\langle u, \phi_i \rangle = \int_D u(x)\phi_i(x)dx$ with trapezoidal rule.

– **Decoding**: $u \approx \sum_{i=1}^{N} \langle u, \phi_i \rangle \phi_i$

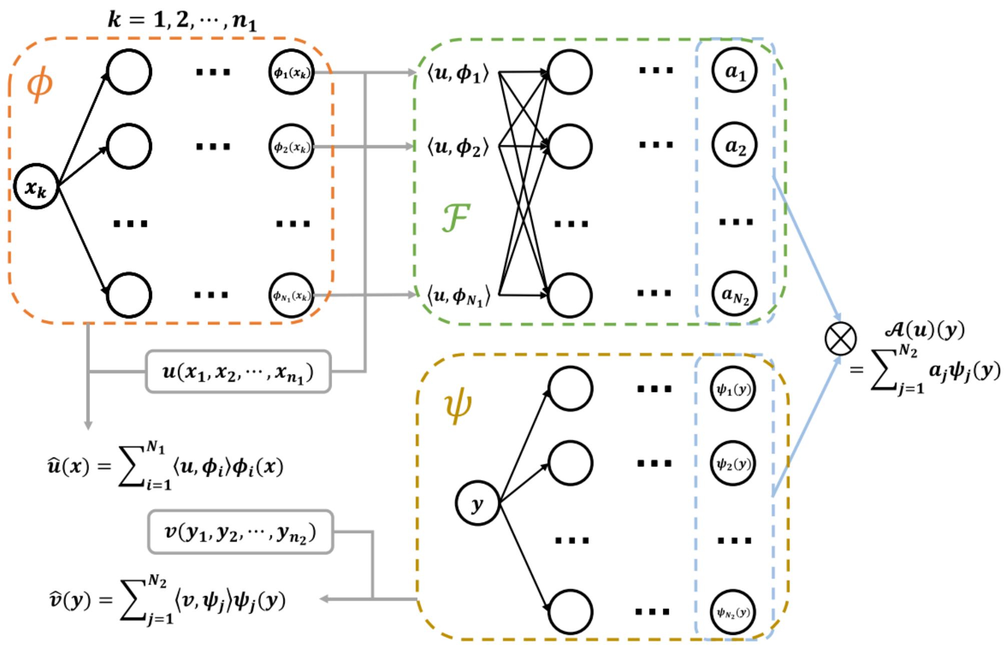
- **Orthogonal by Training**: To make $\{\phi_1, \ldots, \phi_{N_1}\}$ orthonormal, they use loss function $l(\mathcal{R}_\phi \circ \mathcal{P}_\phi, ID)$ during training to induce orthogonal bias. (The notation is missing in Hua and Lu (2023))
- **Resolution invariance**: This structure is resolution-invariance because the inner product is invariant to the number of x grids.



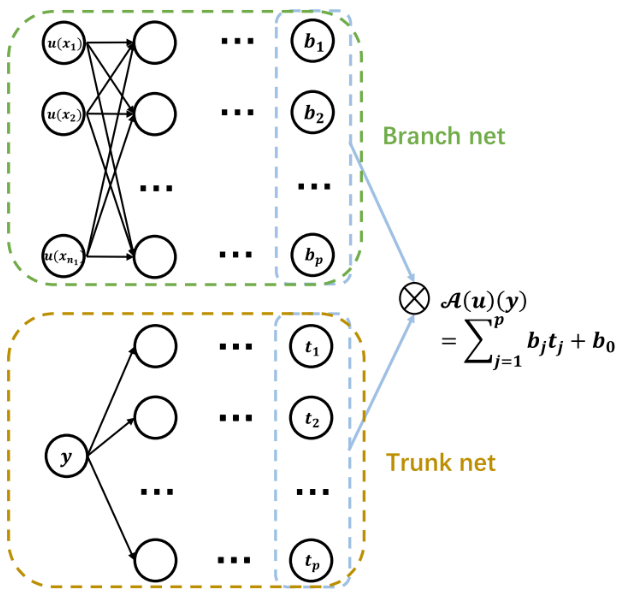
(a) Traditional autoencoder. (b) Function autoencoder.

Fig. 2. The structures of the traditional autoencoder and the function autoencoder.

Figure 1: Comparison of autoencoder and function autoencoder



(a) Architecture of BasisONet.



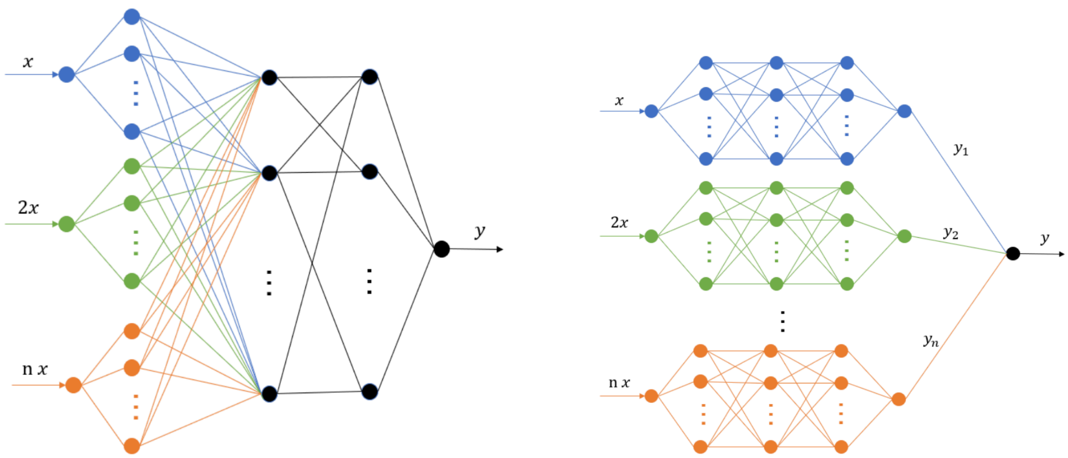
(b) Architecture of DeepONet.

Fig. 3. BasisONet versus DeepONet in architecture.

Idea

- Function autoencoder can have frequncy bias just like regular MLP

apply MscaleDNN to ϕ



(a) MscaleDNN-1 (b) MscaleDNN-2

Figure 3: Illustration of two MscaleDNN structures.

Convolutional Neural Operator

A Neural Operator composed by Convolutaional layer and UNet structure for multiscale sampling (Raonić et al. 2023).

2 Convolutional Neural Operators.

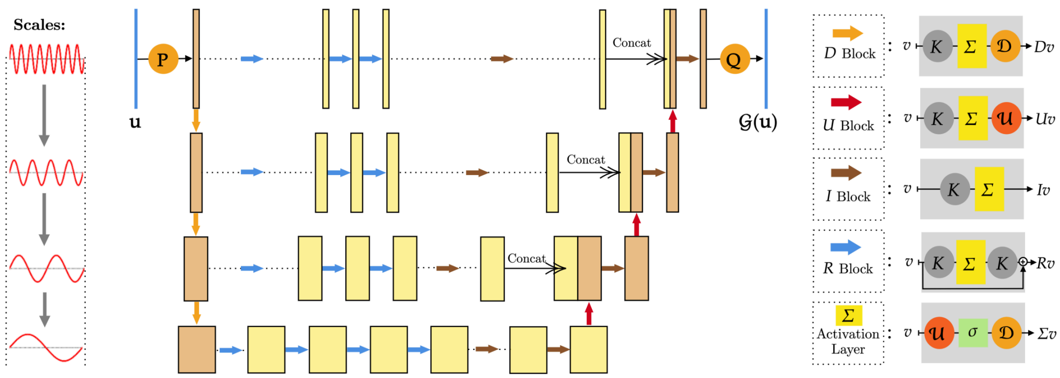


Figure 1: Schematic representation of CNO (2.3) as a modified U-Net with a sequence of layers (each identified with the relevant operators on the right, see Section 2) mapping between bandlimited functions. Rectangles represent multi-channel signals. Larger the height, larger is the resolution. Wider the rectangles, more channels are present.

- The activation function need to be properly setup. ReLU tend to create high frequency features, that create aliasing error.
- There can be a modification on multiscale NN concept to
  - Channel encoding with $x, 2x, \ldots, nx$
  - Custom design of activation function based on frequency response.

Cai, Wei, and Zhi-Qin John Xu. 2019. "Multi-Scale Deep Neural Networks for Solving High Dimensional Pdes." *arXiv Preprint arXiv:1910.11710*.

Hua, Ning, and Wenlian Lu. 2023. "Basis Operator Network: A Neural Network-Based Model for Learning Nonlinear Operators via Neural Basis." *Neural Networks* 164: 21–37.

Raonić, B, R Molinaro, TD Ryck, T Rohner, F Bartolucci, R Alaifari, S Mishra, and E de Bézenac. 2023. "Convolutional Neural Operators for Robust and Accurate Learning of PDEs. arXiv."