MTH4022

Partial Differential Equations

Lecture Notes

Jérôme Droniou[1]

22 July 2014

[1]School of Mathematical Sciences, Monash University, Victoria 3800, Australia. email: jerome.droniou@monash.edu

Contents

1 Introduction

1.1 Motivation: miscible fluid flow in porous media

References:

- J. Bear, Dynamics of fluids in porous media, New York: American Elsevier, 1972.
- D.W. Peaceman. Improved treatment of dispersion in numerical calculation of multidimen- sional miscible displacement. Soc. Pet. Eng. J., 6(3):213216, 1966.

A particular method of oil recovery in underground reservoir, when natural pressure and movement of oil stopped, is to inject a solvent in order to increase the oil mobility and recover more of it. The solvent mixes with the oil and the mathematical model for this single-phase miscible displacement of one fluid by another in a porous medium (the reservoir), in the case where the fluids are considered incompressible, is an elliptic-parabolic coupled system.

Let Ω be a bounded domain of $\mathbb{R}^d$ ($d = 2$ or 3) representing the reservoir and $(0, T)$ be the time interval. The unknowns of the problem are p the pressure in the mixture, $\mathbf{U}$ its Darcy velocity and c the concentration of the invading fluid.

We denote by $\Phi(x)$ and $\mathbf{K}(x)$ the porosity and the absolute permeability tensor of the porous medium, $\mu(c)$ the viscosity of the fluid mixture, $\widehat{c}$ the injected concentration, q^+ and q^- the injection and the production source terms (wells). If we neglect gravity, the model reads

$$\begin{cases} \operatorname{div}(\mathbf{U}) = q^+ - q^- & \text{in } (0,T)\times\Omega, \\ \mathbf{U} = -\dfrac{\mathbf{K}(x)}{\mu(c)}\nabla p & \text{in } (0,T)\times\Omega, \end{cases} \tag{1.1}$$

$$\Phi(x)\partial_t c - \operatorname{div}(D(x,\mathbf{U})\nabla c - c\mathbf{U}) + q^- c = q^+\widehat{c} \quad \text{in } (0,T)\times\Omega \tag{1.2}$$

where D is the diffusion-dispersion tensor including molecular diffusion and mechanical dispersion

$$D(x,\mathbf{U}) = \Phi(x)\Big(d_m\mathbf{I} + |\mathbf{U}|\Big(d_l E(\mathbf{U}) + d_t(\mathbf{I} - E(\mathbf{U}))\Big)\Big) \tag{1.3}$$

with $\mathbf{I}$ the identity matrix, $d_m > 0$ the molecular diffusion, $d_l > 0$ and $d_t > 0$ the longitudinal and transverse dispersion coefficients and $E(\mathbf{U}) = (\frac{\mathbf{U}_i\mathbf{U}_j}{|\mathbf{U}|^2})_{1\le i,j\le d}$.

In reservoir simulation, the boundary $\partial\Omega$ is typically impermeable. Therefore, if $\mathbf{n}$ denotes the exterior normal to $\partial\Omega$, the system (1.1)—(1.2) is supplemented with no flow boundary conditions

$$\begin{cases} \mathbf{U}\cdot\mathbf{n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ D(x,\mathbf{U})\nabla c\cdot\mathbf{n} = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{1.4}$$

An initial condition is also prescribed

$$c(x,0) = c_0(x) \text{ in } \Omega. \tag{1.5}$$

Because of the homogeneous Neumann boundary conditions on $\mathbf{U}$, the injection and production source terms have to satisfy the compatibility condition $\int_\Omega q^+(.,x)\ dx =$

$\int_\Omega q^-(.,x)\ dx$ in $(0,T)$, and since the pressure is defined only up to an arbitrary constant, we normalize p by the following condition

$$\int_\Omega p(.,x)\ dx = 0 \text{ in } (0,T). \tag{1.6}$$

The viscosity μ is usually determined by the following mixing rule

$$\mu(c) = \mu(0)\Big(1 + \Big(M^{1/4} - 1\Big)c\Big)^{-4} \text{ in } [0,1], \tag{1.7}$$

where $M = \frac{\mu(0)}{\mu(1)}$ is the mobility ratio (μ can be extended to $\mathbb{R}$ by letting $\mu = \mu(0)$ on $(-\infty, 0)$ and $\mu = \mu(1)$ on $(1,\infty)$). The porosity Φ and the permeability $\mathbf{K}$ are in general assumed to be bounded from above and from below by positive constants (or positive multiples of $\mathbf{I}$ for the tensor $\mathbf{K}$).

Two important characteristics of the data deserved to be mentioned:

- The permeability $\mathbf{K}$ and porosity ϕ depend on the position x and may vary abruptly (be discontinuous) from one geological layer to the other – the nature and property of rocks in each layer are not the same.
- The wellbore diameter is a few dozens of centimeters. The reservoir is a few hundreds of meters or a few kilometers wide. At the reservoir scale, q^+ and q^- seem to only act, as source terms, on lines (not on a whole subdomain of Ω).

These lead to particular issues in finding a proper mathematical framework for (1.1)—(1.2): how can we differentiate quantities which seem discontinuous? How can we handle source terms which seem to be 0 nearly everywhere?

1.2 Energy estimates

Let us take a look at a very simplified model, where we basically only look at the pressure equation. This model however retains all of the aforementioned issues...
We therefore consider the elliptic equation

$$\begin{cases} -\mathrm{div}(A\nabla p) = f \text{ in } \Omega, \\ A\nabla p \cdot \mathbf{n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.8}$$

where Ω is a bounded open set in $\mathbb{R}^N$ and $A : \Omega \to M_N(\mathbb{R})$ is a matrix-valued function (in practice, $A(x)$ is symmetric for all x).
Multiply the equation by p and integrate "by parts", i.e. using Stokes formula. Taking into account the boundary conditions, this leads to

$$\int_\Omega A\nabla p \cdot \nabla p = \int_\Omega fp.$$

Let us concentrate on the left-hand side, and assume the simpler case where $A \equiv \mathrm{Id}$. Then this left-hand side is just

$$\int |\nabla p|^2$$

(where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$). This integral of the square of the gradient of p appears as a natural nonnegative quantity – an energy – that seems maybe estimable if p is a solution to the PDE.

Remark 1.2.1 *This energy is also a natural estimate if A is* coercive*, i.e. there exists $\alpha > 0$ such that, for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$, $A(x)\xi \cdot \xi \geq \alpha|\xi|^2$ (where $X \cdot Y$ is the Euclidean product).*

A proper mathematical framework for (1.8) will therefore need to:

1. make use of this energy $\int |\nabla p|^2$ as a main estimate on the unknown p,
2. give the tools to establish the existence of a solution to (1.8) that may be non-smooth (to take into account the possible discontinuities of A),
3. be expandable to the case where f is no longer a classical function but one with a "very thin" support (a line in 3D).

There are basically only two ways, in mathematics, to establish the existence of a solution without explicitly constructing it (which is not doable, under real-world constraints, for (1.8) or (1.1)—(1.2)): compactness or completeness.
Completeness is the one we will be using (to start with, as compactness is also paramount when looking at non-linear equations). We are therefore lead to finding/using a complete space, whose norm is strongly related with the natural energy of the problem (i.e. $\int |\nabla p|^2$) and which allows us to handle derivatives of function seemingly not differentiable...

2 Some results about Lebesgue spaces that everyone must know

References:

- W. Rudin, *Real and complex analysis* (chapters 1, 2, 3).
- H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations* (chapter 4).
- M. Capinski & E. Kopp, *Measure, Integral and Probability* (all chapters!)

2.1 Vanilla version of the Lebesgue integration theory

2.1.1 Definition of the integral

A N-cell is a set $C \subset \mathbb{R}^N$ of the form $\prod_{i=1}^N]a_i, b_i[$. The measure of W is $\mathrm{meas}(W) = \prod_{i=1}^N (b_i - a_i)$ (this can be infinite).
If $N = 2$, W is a rectangle and $\mathrm{meas}(W)$ is its area. If $N = 3$, W is a (generalised) hypercube and its measure is its volume.

Definition 2.1.1 ***(Null set / set of measure 0)*** *A null set $A \subset \mathbb{R}^N$ is a set that can be covered a sequence of N-cells of arbitrary small total measure: for all $\varepsilon > 0$, there exists N-cells $(C_n)_{n\geq 1}$ such that*

$$A \subset \bigcup_{n=1}^{\infty} C_n \qquad \textit{and} \qquad \sum_{n=1}^{\infty} \mathrm{meas}(C_n) < \varepsilon.$$

Example 2.1.2

1. Any countable set is a null set.
2. The triadic Cantor set (in $\mathbb{R}$) is a null set.

Definition 2.1.3 ***(a.e. property)*** *Let $A \subset \mathbb{R}^N$. A property $P(x)$ depending on a point $x \in A$ is said to hold almost everywhere (a.e.) if it holds for all $x \in A$ except in a null set, i.e. there exists a null set $B \subset A$ such that, for all $x \in A \backslash B$, $P(x)$ is true.*

Example 2.1.4

1. The function f defined by $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \notin \mathbb{Q}$ is equal to 1 a.e.
2. The sequence of functions $f_n(x) = x^n$ converges a.e. to 0 on $[0, 1]$.

A very useful property of null sets:

Proposition 2.1.5 *If $(A_n)_{n\geq 1}$ is a countable family of null sets, then $\cup_{n\geq 1} A_n$ is a null set.*

There exists a theory of integration such that the following results hold.

FACT 1 We can integrate *nearly* all nonnegative functions on *nearly* all subset of $\mathbb{R}^N$.

$$\int_A f(x)\, dx \qquad \left(\int_A f \quad \text{in shorthand} \right) \tag{2.1}$$

makes sense provided that:

- $A \subset \mathbb{R}^N$ is *measurable*, which happens if A is open, closed, or formed from open and closed sets by sequences of possibly infinite intersections and/or unions, and
- $f : A \to [0, \infty]$ is *measurable*, which happens if it is continuous, piecewise continuous, or the a.e. limit of measurable functions.

The integral in (2.1) is an element of $[0, \infty]$. A measurable (not necessarily nonnegative) function $f : A \to \mathbb{R}$ is said *integrable* on the measurable set A if $\int_A |f| < \infty$.

Fact 2 The vocabulary is thankfully (for once!) coherent: we can integrate integrable functions. If A is measurable and $f : A \to \mathbb{R}$ is integrable we define

$$\int_A f = \int_A f^+ - \int_A f^- \tag{2.2}$$

where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are the positive and negative parts of f (note that each integral on the right-hand side of (2.2) is finite, so we do not write $\infty - \infty$).

Fact 3 If $\int_A |f| = 0$ then $f = 0$ a.e. on A. In particular, if $\int_A |f - g| = 0$ then $f = g$ a.e. on A.

Fact 4 All the classical algebra of integration holds, provided all integrals are defined (i.e. involve measurable sets and measurable nonnegative or integrable functions):

- The integral is the area/volume/measure under the function. In particular, $\int_A 1 =$"measure" of A (known if A is a N-cell...).
- Linearity: $\int_A(\lambda f + \mu g) = \lambda \int_A f + \mu \int_A g$,
- Chasles' relation: if A and B are disjoint then $\int_{A\cup B} f = \int_A f + \int_B f$. In fact, if $(A_n)_{n\geq 1}$ is a sequence of pairwise disjoint measurable sets, then $\int_{\cup_n A_n} f = \sum_{n=1}^{\infty} \int_{A_n} f$.
- Behaviour with respect to the absolute value: $|\int_A f| \leq \int_A |f|$.
- Comparison: if $f \leq g$ a.e. on A then $\int_A f \leq \int_A g$ *(note that the inequality $f \leq g$ only needs to hold a.e.: the integral does not "see" what is happening on null sets).*
- Relation with finite values: if $\int_A |f| < \infty$ then $|f| < \infty$ a.e. on A.
- In 1D, the fundamental theorem of calculus and integration: if f is continuous, $\frac{d}{dx}\int_{[0,x]} f = \frac{d}{dx}\int_0^x f = f(x)$.

From now on, all sets and functions are assumed to be measurable.

2.1.2 Convergence theorems

Note: we denote sequences by plain brackets. Hence, (f_n) denotes the sequence of functions f_n, not just one function f_n.

Theorem 2.1.6 ***(Beppo-Levi's/Lebesgue's Monotone convergence theorem)*** *If (f_n) is a sequence of nonnegative functions $A \to [0, \infty]$ such that $f_n \leq f_{n+1}$, then (f_n) converges a.e. to some function $f : A \to [0, \infty]$ and*

$$\int_A f_n \to \int_A f \text{ as } n \to \infty.$$

Theorem 2.1.7 ***(Fatou's lemma)*** *Let (f_n) be a sequence of nonnegative functions $A \to [0, \infty]$. Then*

$$\int_A \liminf f_n \leq \liminf \int_A f_n.$$

Theorem 2.1.8 ***(Lebesgue's Dominated convergence theorem)*** *Let $f : A \to \mathbb{R}$ and (f_n) be a sequence of functions $A \to \mathbb{R}$ satisfying:*

- *$f_n \to f$ a.e. on A as $n \to \infty$,*
- *there exists an integrable function $g : A \to [0, \infty]$ such that, for all n, $|f_n| \leq g$ a.e. on A.*

Then

$$\int_A |f_n - f| \to 0 \text{ as } n \to \infty.$$

Remark 2.1.9 *Compare this theorem with the convergence theorem you know for the Riemann integral. Consider e.g. $f_n(x) = x^n$ on $[0,1]$...*

2.2 Lebesgue spaces

2.2.1 Definitions

Definition 2.2.1 *For $p \in [1, \infty)$ we define*

$$||f||_{L^p(A)} = \left(\int_A |f|^p \right)^{1/p}.$$

If $p = \infty$, we define

$$||f||_{L^\infty(A)} = \inf\{M \geq 0 \text{ such that } |f| \leq M \text{ a.e. on } A\}.$$

We denote $||f||_p$ for short when there is no ambiguity on A.

These are defined (possibly infinite) for any measurable function f.

Exercise 2.2.2 We have $|f| \leq ||f||_\infty$ a.e. *(Hint: use the fact that a countable union of null sets is a null set).*

Theorem 2.2.3 ***(Hölder's inequality)*** *If $p, q \in [1, \infty]$ are* conjugate exponents *in the sense that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$||fg||_1 \leq ||f||_p ||g||_q$$

Proof

We can take $f, g \geq 0$. If we assume $p, q < \infty$, we have, by convexity of exp,

$$fg = e^{\ln(fg)} = e^{\ln(f)+\ln(g)} = e^{\frac{1}{p}\ln(f^p)+\frac{1}{q}\ln(g^q)} \leq \frac{1}{p}e^{\ln(f^p)} + \frac{1}{q}e^{\ln(g^q)} = \frac{1}{p}f^p + \frac{1}{q}g^q.$$

Integrating that gives, if $||f||_p = ||g||_q = 1$, $||fg||_1 \leq \frac{1}{p} + \frac{1}{q} = 1 = ||f||_p ||g||_q$. Otherwise we apply this to $f/||f||_p$ and $g/||g||_q$.

If $p = \infty$ then $q = 1$ and $fg \leq ||f||_\infty g$ a.e. Integrate that to get the result. ■

Exercise 2.2.4 (Brezis, chap 4, ex 4.2) Prove that if A has a finite measure (i.e. $\int_A 1 = \text{meas}(A)$ is finite), $f \in L^p(A)$ and $q \leq p$ then $f \in L^q(A)$.
(hint: use Hölder's inequality).

Corollary 2.2.5 ***(Minkowski inequality)*** *For all $p \in [1, \infty]$, $|| \cdot ||_p$ satisfies the triangular inequality:*

$$||f + g||_p \leq ||f||_p + ||g||_p.$$

PROOF

Since $|f + g| \leq |f| + |g|$, we can consider $f, g \geq 0$. We write, for $p < \infty$,

$$(f + g)^p = (f + g)(f + g)^{p-1} = f(f + g)^{p-1} + g(f + g)^{p-1}$$

then integrate and use Hölder inequality

$$||f + g||_p^p \leq ||f||_p||(f + g)^{p-1}||_q + ||g||_p||(f + g)^{p-1}||_q.$$

Recalling that $q = p/(p-1)$, we see that $||(f + g)^{p-1}||_q = ||f + g||_p^{p-1}$. We then divide throughout by $||f + g||_p^{p-1}$, taking care of the case where this quantity is infinite...

If $||f + g||_p = \infty$ we need to prove that $||f||_p + ||g||_p = \infty$ (i.e. at least one is infinite). We still consider $f, g \geq 0$ and start with the convexity of $s \to s^p$ to get $(\frac{f+g}{2})^p \leq \frac{f^p}{2} + \frac{g^p}{2}$, that is $(f+g)^p \leq 2^{p-1}f^p + 2^{p-1}g^p$. We then integrate and find $\infty \leq 2^{p-1}\int f^p + 2^{p-1}\int g^p$, which shows that either $\int f^p$ or $\int g^p$ (or both) is infinite.

If $p = \infty$, we just write $|f + g| \leq |f| + |g| \leq ||f||_\infty + ||g||_\infty$ a.e. and thus $||f + g||_\infty \leq ||f||_\infty + ||g||_\infty$. ■

We want a space where $|| \cdot ||_p$ is a norm. We start with

$$\mathcal{L}^p(A) = \{f \mid ||f||_p < \infty\}.$$

This is a vector space, but $|| \cdot ||_p$ do not separate points: if $||f||_p = 0$ then we only have $f = 0$ a.e. We therefore change a bit $\mathcal{L}^p$ by quotienting it by the equivalence relation "$f \sim g$ if $f = g$ a.e.". In other words, our "$= 0$" (or any "$=$") will always be "a.e.":

$$L^p(A) = (\mathcal{L}^p(A)/\sim) = \{f \text{ considered defined only a.e. and s.t. } ||f||_p < \infty\}.$$

We therefore do not make any difference between f and g in $L^p(A)$ if $f = g$ a.e. In this space, $|| \cdot ||_p$ is a norm since "$f = 0$ a.e." precisely mean that $f = 0$ in L^p.

Remark 2.2.6 *We also define, if Ω is an open set of $\mathbb{R}^N$,*

$$L^1_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{R} \mid \int_K |f| < \infty \textit{ for all } K \textit{ compact subset of } \Omega\}.$$

It is not a normed *space (it is a metric space though).*
Example: $e^x \in L^1_{\text{loc}}(\mathbb{R})$, but e^x belongs to no $L^p(\mathbb{R})$.

2.2.2 Completeness

Theorem 2.2.7 ***(Dominated convergence in** L^p**)*** *Let $p < \infty$. If (f_n) is a sequence of functions in L^p such that $f_n \to f$ a.e. and such that there exists $g \in L^p$ satisfying $|f_n| \le g$ a.e. for all n, then $f_n \to f$ in L^p.*

PROOF
Based on $|f_n - f|^p \le (|f_n| + |f|)^p \le (g^p + g^p)$. ■

Theorem 2.2.8 ***(Completeness of** L^p**)*** *For all $p \in [1, \infty]$, L^p is a Banach (complete) space.*

PROOF
Assume first $p < \infty$. Take a Cauchy sequence (f_n) and a subsequence such that $||f_{n_{k+1}} - f_{n_k}||_p < 1/2^k$. Let

$$g = \sum_{k \ge 1} |f_{n_{k+1}} - f_{n_k}|.$$

We first show that $g < \infty$ a.e. We have $g = \lim_{K \to \infty} g_K$ with

$$g_K = \sum_{k=1}^{K} |f_{n_{k+1}} - f_{n_k}|.$$

By the triangular inequality, $||g_K||_p \le \sum_{k=1}^{K} ||f_{n_{k+1}} - f_{n_k}||_p \le \sum 1/2^k = 1$. Using Fatou we thus get $\int_A g^p \le \liminf \int_A g_K^p \le 1$. Hence $g < \infty$ a.e.

This shows that, for a.e. x, the series $g(x)$ converges in $\mathbb{R}$ and that the resulting function g belongs to L^p. The series $\sum_k (f_{n_{k+1}}(x) - f_{n_k}(x))$ is therefore convergent for a.e. x, which shows that f_{n_k} converges a.e. to some f. Now, $|f_{n_k}| \le g(x) + |f_{n_0}| \in L^p$ and the Dominated Convergence theorem shows that $f_{n_k} \to f$ in L^p.

(f_n) is a Cauchy sequence in L^p which has a convergence subsequence: it converges, and the proof is complete. ■

Corollary 2.2.9 ***(Partial converse of the dominated convergence theorem)*** *Let $p \in [1, \infty]$. If $f_n \to f$ in $L^p(A)$ then there exists a subsequence $(f_{n_k})_k$ and a function $g \in L^p$ such that $f_{n_k} \to f$ a.e. and $|f_{n_k}| \le g$ a.e.*

Example 2.2.10 The whole sequence does not necessarily converge a.e., see the sliding step.

Exercise 2.2.11 Prove that L^∞ is a Banach space and that if $f_n \to f$ in L^∞ then $f_n \to f$ a.e.

Exercise 2.2.12 Let $H : \mathbb{R} \to \mathbb{R}$ be continuous such that $|H(s)| \le C(1 + |s|)$ for all $s \in \mathbb{R}$, where C does not depend on s. Let $(f_n) \subset L^p(A)$ such that $f_n \to f$ in $L^p(A)$, where A has finite measure.

1. Prove that $H(f_n)$ and $H(f)$ belong to $L^p(A)$ and that $H(f_n) \to H(f)$ in $L^p(A)$. *(hint: separate the cases $p = \infty$ and $p \in [1, \infty)$. In the second case, reason by contradiction and use the partial reciprocal of the dominated convergence theorem in L^p.).*

2. What about this convergence if we remove the assumption "$|H(s)| \le C(1 + |s|)$"? And what if we do not assume that A has finite measure?

2.2.3 Regularisation

$A = \Omega$ open subset of $\mathbb{R}^N$.

Exercise 2.2.13 Let Ω be an open bounded set and (γ_n) a sequence of functions $\Omega \to [0, 1]$ such that

$$\gamma_n(x) = 1 \text{ whenever } \operatorname{dist}(x, \partial\Omega) \ge \frac{1}{n}.$$

Let $f \in L^p(\Omega)$.

1. If $p < \infty$, prove that $f\gamma_n \to f$ in $L^p(\Omega)$.

2. Find a counter-example to this convergence when $p = \infty$.

A corollary to this exercise: if $p < \infty$, the set $S = \{f \in L^p(\Omega) \mid f \text{ has a compact support}\}$ is dense in $L^p(\Omega)$. We can do better:

Theorem 2.2.14 ***(Density of smooth functions)*** *If $p < \infty$, $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

Exercise 2.2.15 Why is it false if $p = \infty$?

Exercise 2.2.16

1. **(Continuity under the integral sign)** Assume that $f : U \times \Omega \to \mathbb{R}$ (U open set of $\mathbb{R}^N$) satisfies: for all $x \in \Omega$, $f(\cdot, x)$ is continuous on U and there exists $g \in L^1(\Omega)$ such that for all $(z, x) \in U \times \Omega$, $|f(z, x)| \le g(x)$. Then $F(z) = \int_\Omega f(z, x)\,dx$ is continuous on U.

2. **(Differentiating under the integral sign)** Assume that $f : I \times \Omega \to \mathbb{R}$ (I open interval of $\mathbb{R}$) satisfies: for all $x \in \Omega$, $f(\cdot, x)$ is differentiable and there exists $g \in L^1(\Omega)$ such that $|\frac{\partial f}{\partial t}(t, x)| \le g(x)$ for all (t, x). Then $F(t) = \int_\Omega f(t, x)\,dx$ is differentiable and $F'(t) = \int_\Omega \frac{\partial f}{\partial t}(t, x)\,dx$. *(Hint: use the definition of $F'(t)$ as the limit of "increment of F / increment of t" along a sequence $t_n \to t$, the mean value theorem and the dominated convergence theorem).*

Proof

We just need to approximate a function $f \in S$ by some smooth compactly-supported functions. We use a regularisation kernel $(\rho_n) \in C_c^\infty(\mathbb{R}^N)$ such that $\rho_n \geq 0$, $\int_{\mathbb{R}^N} \rho_n = 1$ and $\mathrm{supp}(\rho_n) \subset B(0, 1/n)$. We define

$$f_n(x) = \int_\Omega f(y)\rho_n(x-y)\,dy.$$

f_n is well defined ($f \in L^p$ with compact support implies that $f \in L^1$ – see Exercise 2.2.4 – and ρ_n is a bounded function) and smooth (use Exercise 2.2.16 to show that all partial derivatives of f_n exist and are continuous).

We can prove, using the "continuity of translations", that $f_n \to f$ in $L^p(\Omega)$. Since the support of f_n is contained in the closure of $\mathrm{supp}(f) + B(0, 1/n)$, it is compact in Ω if n is large enough and that concludes the proof. ■

3 Sobolev spaces and Elliptic variational PDEs

References:

- H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations* (chapter 9),
- R.A. Adams & J. J.F. Fournier, *Sobolev Spaces 2nd ed*, Academic Press, 2003 (chapters 1 to 6),
- L.C. Evans, *Partial Differential Equations*, American Mathematical Society, 2010 (chapter 5)

We mix together results on Sobolev spaces and on elliptic equations, proving just what is necessary on Sobolev spaces to apply them to elliptic PDE, one result at a time.
The tags [S] and [PDE] makes clear which section presents generic results on Sobolev spaces and which sections talk about elliptic PDEs.

3.1 [S] Weak partial derivatives and distributions

How to differentiate non-differentiable functions, based on integration-by-parts. If f, φ are regular and φ has a compact support in Ω,

$$\int_\Omega \partial_i f \varphi = -\int_\Omega f \partial_i \varphi.$$

If $f \in L^1_{\mathrm{loc}}$ then the left-hand side does not make sense, but the right-hand side does. We therefore *define*, when it exists, $\partial_i f$ as the function $g_i \in L^1_{\mathrm{loc}}$ such that, for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega g_i \varphi = -\int_\Omega f \partial_i \varphi.$$

Questions: does g_i exist? Is it unique?

Exercise 3.1.1 Let the Heavyside function $H : \mathbb{R} \to \mathbb{R}$ be defined by $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 1$. Prove that there exists no $g \in L^1_{\text{loc}}$ such that $\int g\varphi = -\int H\varphi'$ for all $\varphi \in C^\infty_c(\mathbb{R})$.

We see that the derivative is defined by duality. Put it another way, it is strongly based linear forms $\varphi \to \int h\varphi$ rather than on functions h.

Definition 3.1.2 ***(Distribution)*** *We call distribution on Ω any linear form $T : C^\infty_c(\Omega) \to \mathbb{R}$ and denote by $\mathcal{D}'(\Omega)$ the space of all distributions on Ω.*

Remark 3.1.3 *Beware, this definition is not standard, I removed the topology...*

Proposition 3.1.4 ***(L^1_{loc} functions are distributions)*** *The linear transformation $f \in L^1_{\text{loc}} \to T_f \in \mathcal{D}'$ defined by $\langle T_f, \varphi\rangle = \int f\varphi$ is one-to-one and continuous (for the convergence in the sense of distributions).*
In other words, any function in L^1_{loc} can be considered as a distribution: $L^1_{\text{loc}} \subset \mathcal{D}'$.

Proof

first need to prove that if $T_f = 0$ then $f = 0$. Let $\gamma \in C^\infty_c(\Omega)$ and ρ_n a regularising kernel. We know that $f_n = (\gamma f) * \rho_n \to \gamma f$ in $L^1(\Omega)$. But

$$f_n(x) = \int_\Omega \gamma(y)f(y)\rho_n(x-y)\,dy = \int_\Omega f(y)\varphi_x(y)\,dy$$

with $\varphi_x(y) = \gamma(y)\rho_n(x-y)$ regular and equal to 0 as soon as $y \notin \text{supp}(\gamma)$. Hence $\varphi_x \in C^\infty_c(\Omega)$ and thus, since $T_f = 0$, $\int_\Omega f\varphi_x = 0$. This proves that $f_n = 0$ and, since it converges to γf, that $\gamma f = 0$. Since this is true for any $\gamma \in C^\infty_c(\Omega)$, this gives $f = 0$. ■

Definition 3.1.5 ***(Distribution derivative)*** *If $T \in \mathcal{D}'$, we define $\partial_i T$ the distribution such that, for all φ, $\langle \partial_i T, \varphi\rangle = -\langle T, \partial_i\varphi\rangle$ (well defined and unique!).*

We can therefore differentiate any L^1_{loc} function f, but $\partial_i f$ is a distribution, not a function in general.

Definition 3.1.6 ***(Convergence in the sense of distributions)*** *We say that a sequence of distributions (T_n) converges to a distribution T in the sense of distributions if, for all $\varphi \in C^\infty_c$, $\langle T_n, \varphi\rangle \to \langle T, \varphi\rangle$.*
This is therefore the pointwise convergence of linear forms (functions).

It is then straightforward that

Proposition 3.1.7 *If $T_n \to T$ in the sense of distributions, then $\partial_i T_n \to \partial_i T$ in the sense of distributions.*

Proposition 3.1.8 *If (f_n) converges to f in L^1_{loc} (i.e. converges in $L^1(K)$ for all compact $K \subset \Omega$) then $f_n \to f$ in the sense of distribution.*

Proof

Just write, for $\varphi \in C_c^\infty(\Omega)$ and $K = \mathrm{supp}(\varphi)$, $|\int f_n\varphi - \int f\varphi| = |\int (f_n - f)\varphi| \leq ||\varphi||_\infty \int_K |f_n - f|$. ■

Exercise 3.1.9 If $f_n \to f$ in L^p then the convergence also holds in L^1_{loc}.

Exercise 3.1.10 Let $f \in C(\mathbb{R}^2)$ such that f has continuous bounded partial derivatives $\partial_i f$ on $\mathbb{R} \times (0, \infty)$ and on $\mathbb{R} \times (-\infty, 0)$. Prove that the *distribution* derivatives of f on $\mathbb{R}^2$ are the $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ functions equal to $\partial_i f$ on $\mathbb{R} \times (0, \infty)$ and on $\mathbb{R} \times (-\infty, 0)$.
What if you remove the assumption $f \in C(\mathbb{R}^2)$, only assuming that the restrictions of f to $\mathbb{R} \times (0, \infty)$ and to $\mathbb{R} \times (-\infty, 0)$ are continuous up to $y = 0$ (but with possibly different values as $y \to 0^+$ and $y \to 0^-$)?

3.2 [S] Sobolev spaces

If $f \in L^1_{\mathrm{loc}}$, $\partial_i f$ is a distribution in general. When we restrict ourselves to the case where the derivatives are functions, we obtain Sobolev spaces.

Definition 3.2.1 ***(Sobolev spaces)*** *Let $p \in [1, \infty]$. We define*

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) \mid \nabla f = (\partial_1 f, \ldots, \partial_N f) \in (L^p(\Omega))^N\}$$

and we endow this space with the norm

$$||f||_{W^{1,p}(\Omega)} = ||f||_{1,p} = ||f||_p + ||\,|\nabla f|\,||_p.$$

Remark 3.2.2 *That is one possible choice of norm, there are plenty other which give equivalent norms. For example, if $p < \infty$,*

$$||f||_{1,p} = \left(||f||_p^p + ||\,|\nabla f|\,||_p^p\right)^{1/p}.$$

Theorem 3.2.3 ***(Completeness)*** *$W^{1,p}(\Omega)$ is complete.*

Proof

If (f_n) is a Cauchy sequence in $W^{1,p}$, then it is a Cauchy sequence in L^p and ∇f_n is a Cauchy sequence in $(L^p)^N$. We therefore have $f_n \to f$ in L^p and $\nabla f_n \to G$ in $(L^p)^N$. We therefore have $f_n \to f$ in $\mathcal{D}'$ and thus $\nabla f_n \to \nabla f$ in $(\mathcal{D}')^N$. But we also have $\nabla f_n \to G$ in $(\mathcal{D}')^N$, and thus $G = \nabla f$. This proves that $f \in W^{1,p}$ and that $f_n \to f$ in $W^{1,p}$. ■

Exercise 3.2.4 We define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x) = |\ln |x||^\gamma$, where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^2$ and $\gamma > 0$. Let B be the unit ball in $\mathbb{R}^2$.

1. Is f continuous or bounded on B?

2. Find the γ such that $f \in L^2(B)$.
   (hint: use polar coordinates and the fact that $r \to r^\alpha \ln(r)^\gamma$ is integrable on $(0, 1)$ if and only if $\alpha > -1$ or $\alpha = -1$ and $\gamma < -1$.)

3. We have, on $B\backslash\{0\}$, $\nabla f = -\gamma\frac{x}{|x|}|\ln|x||^{\gamma-1}$, in the classical sense (f is smooth on $B\backslash\{0\}$). Find the γ such that $\nabla f \in L^2(B)$ and prove that, for those γ, $f \in H^1(B)$. *(hint: you will have to prove that the distribution gradient of f on B coincides with ∇f (defined a.e. on B). Take a test function $\varphi \in C_c^\infty(B)$, define B_ε as the ball centred at 0 with radius ε, write $\int_B \partial_i\varphi f = \lim_{\varepsilon\to 0}\int_{B\backslash B_\varepsilon}\partial_i\varphi f$ (why does that hold?) and use the classical Stoke's formula on the ring $B\backslash B_\varepsilon$.)*

Exercise 3.2.5 Let Ω be a bounded connected open subset and $u \in L^p(\Omega)$ such that $\nabla u = 0$. Show that u is constant.
(hint: use the regularisation technique to approximate u by functions u_n which are constant on $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq 2/n\}$.)

A weak manner to take into account a condition "0 on $\partial\Omega$":

Definition 3.2.6 *We define $W_0^{1,p}(\Omega)$ as the closure in $W^{1,p}$ of functions in $C_c^\infty(\Omega)$.*

This means that a function u belongs to $W_0^{1,p}(\Omega)$ if it belongs to $W^{1,p}(\Omega)$ and if there exists a sequence $(\varphi_n)_{n\geq 1}$ of functions in $C_c^\infty(\Omega)$ such that $\varphi_n \to u$ in $W^{1,p}(\Omega)$ (i.e. $\varphi_n \to u$ in $L^p(\Omega)$ and $\nabla\varphi_n \to \nabla u$ in $L^2(\Omega)^N$).

Exercise 3.2.7 $f = 1$ belongs to $W^{1,1}((0,1))$ but not to $W_0^{1,1}((0,1))$ *(Hint: if $\varphi \in C_c^\infty$, $\varphi(x) = \int_0^x \varphi'$ and if $\varphi_n' \to 0$ in L^1 then $\varphi_n \to 0$ in L^1.)*

Exercise 3.2.8 If $f \in W^{1,p}$ has a compact support in Ω, then $f \in W_0^{1,p}$ *(Hint: take (ρ_n) a smoothing kernel and let $f_n = f * \rho_n$. Prove that (i) $f_n \in C^\infty$, (ii) f_n has a compact support, (iii) $\nabla f_n = (\nabla f) * \rho_n$, (iv) $f_n \to f$ and $\nabla f_n \to \nabla f$ in L^p).*

3.3 Crash course on Hilbert spaces

Definition 3.3.1 *A Hilbert space H is a space endowed with a scalar product such that, for the norm $||x||_H = \sqrt{\langle x, x\rangle_H}$ stemming from this scalar product, H is complete.*

Example 3.3.2 L^2 is an Hilbert space for the inner product $\langle f, g\rangle = \int fg$.
$W^{1,2} = H^1$ is an Hilbert space for the scalar product

$$\langle f, g\rangle_{H^1} = \int fg + \int \nabla f \cdot \nabla g.$$

Proposition 3.3.3 (Projection on a closed subspace) *Let H be an Hilbert space and V be a closed subspace of H. Then for any $x \in H$ there exists a unique $y \in V$ such that $x - y \perp V$ (that is, for any $w \in V$, $\langle x - y, w\rangle_H = 0$). We call y the orthogonal projection of x on V.*

PROOF

Based on the parallelogram inequality

$$\left|\left|\frac{a+b}{2}\right|\right|^2 + \left|\left|\frac{a-b}{2}\right|\right|^2 = \frac{1}{2}(||a||^2 + ||b||^2)$$

(proved by developing these norms using the scalar product). Let $(y_n) \in V$ such that $||x - y_n|| \to \inf_{w\in V} ||x - w|| =: d$. Then since $(y_n + y_m)/2 \in V$ we have

$$\frac{1}{2}(||x-y_n||^2 + ||x-y_m||^2) = \left|\left|x - \frac{y_n+y_m}{2}\right|\right|^2 + \left|\left|\frac{y_n-y_m}{2}\right|\right|^2 \geq d^2 + \left|\left|\frac{y_n-y_m}{2}\right|\right|^2.$$

Thus,

$$\left|\left|\frac{y_n-y_m}{2}\right|\right|^2 \leq \frac{1}{2}(||x-y_n||^2 + ||x-y_m||^2) - d^2$$

and, by definition of (y_n), the right-hand side tends to $\frac{1}{2}(d^2 + d^2) - d^2 = 0$. Hence, (y_n) is a Cauchy sequence in H and therefore converges to some $y \in H$. As V is closed and each y_n belongs to V, we have $y \in V$. Finally, passing to the limit, we see that $||x - y|| = \inf_{w\in V} ||x - w||$.

The point y is therefore the closest in V to x. If $w \in V$ then $y + tw \in V$ for any $t \in \mathbb{R}$ and thus $||x-y||^2 \leq ||x - (y+tw)||^2 = ||(x-y) + tw||^2$ which gives

$$||x-y||^2 \leq ||x-y||^2 + 2t\langle x-y, w\rangle + t^2||w||^2. \tag{3.1}$$

Subtracting $||x-y||^2$, dividing by $t > 0$ and letting $t \to 0$ gives $\langle x-y, w\rangle \geq 0$. Dividing by $t < 0$ and letting $t \to 0$ gives $\langle x-y, w\rangle \leq 0$, which proves $\langle x-y, w\rangle = 0$ and concludes the existence.

Uniqueness of y is quite simple. Assume that y' is another element in V satisfying $x - y' \perp V$. Then $(x-y) - (x-y') = y' - y \perp V$, but since $y - y' \in V$, this proves $y - y' = 0$. ∎

The most important result in Hilbert spaces is probably the following.

Theorem 3.3.4 (Riesz representation theorem) *The transformation*

$$x \in H \mapsto \langle x, \cdot\rangle_H \in H'$$

is an isomorphic isomorphism between H and H'.
Put it another way, for any $l \in H'$ there exists a unique $x \in H$ such that, for any $v \in H$, $l(v) = \langle x, v\rangle$, and $||l||_{H'} = ||x||_H$.

PROOF

If $l = 0$ then $x = 0$ is the only vector that can satisfy $\langle x, v\rangle = l(v) = 0$ for any v.

If $l \neq 0$ then, since l is continuous, $V = \ker(l) = \{v \in H : l(v) = 0\}$ is a closed subspace of H and $V \neq H$. Let $x_0 \in H$ such that $l(x_0) \neq 0$ and let y the projection of x_0 on V.

We set $x = \alpha(x_0 - y) \in H$ for some $\alpha \in \mathbb{R}\backslash\{0\}$ to be fixed later. Then $l(x) = \alpha l(x_0) \neq 0$ (since $l(y) = 0$ as $y \in \ker(l)$) and $x \perp V$ by definition of y.

For any $v \in H$, we have $l(v - \frac{l(v)}{l(x)}x) = l(v) - l(v) = 0$ so $v - \frac{l(v)}{l(x)}x \in V$ and therefore $0 = \langle x, v - \frac{l(v)}{l(x)}x\rangle = \langle x, v\rangle - l(v)\frac{\langle x,x\rangle}{l(x)}$. This gives

$$l(v) = \frac{l(x)}{||x||^2}\langle x, v\rangle. \tag{3.2}$$

We then choose $\alpha \neq 0$ such that $l(x) = ||x||^2$, that is to say $\alpha l(x_0) = \alpha^2||x_0 - y||$, i.e. $\alpha = \frac{l(x_0)}{||x_0-y||}$ (valid choice since $x_0 \notin V$ and thus $x_0 - y \neq 0$). Equality (3.2) then shows that $l(v) = \langle x, v\rangle$ for any $v \in H$.

Uniqueness of x is pretty obvious: if x' is another vector which satisfies $l(v) = \langle x', v\rangle$, then $\langle x - x', v\rangle = 0$ for any v and thus, with $v = x - x'$, we find $x - x' = 0$. ■

And the main result, at the core of solving elliptic PDEs with irregular data...

Theorem 3.3.5 (Lax-Milgram) *Let H be an Hilbert space, $l \in H'$ and $a : H \times H \to \mathbb{R}$ satisfy:*

- *a is* bilinear*: $a(\cdot, v)$ and $a(u, \cdot)$ are linear, for any $u, v \in H$,*
- *a is* continuous*: there exists M such that, for any $u, v \in H$, $|a(u, v)| \leq M||u||\,||v||$,*
- *a is* coercive*: there exists $\alpha > 0$ such that, for any $u \in H$, $a(u, u) \geq \alpha||u||^2$.*

Then there exists one and only one solution to the problem

$$\begin{cases} \text{Find } u \in H \text{ such that:} \\ \forall v \in H\,, \quad a(u, v) = l(v). \end{cases} \tag{3.3}$$

Moreover, the solution u satisfies

$$||u||_H \leq \frac{1}{\alpha}||l||_{H'}.$$

Proof

Let us start by taking $x_l \in H$ which represents l in the sense of the Riesz representation theorem: $\langle x_l, v\rangle = l(v)$ for any $v \in H$. For any $u \in H$, since $a(u, \cdot) \in H'$ (thanks to the bilinearity and continuity of a), we have a unique element $y_u \in H$ such that $\langle y_u, v\rangle = a(u, v)$. By bilinearity of a, it is easy to check that $A : u \mapsto y_u$ is a linear transformation and that it is continuous: indeed, $||Au||^2 = \langle Au, Au\rangle = a(u, Au) \leq M||u||\,||Au||$ so $||Au|| \leq M||u||$. Moreover, by coercivity of a,

$$\langle Au, u\rangle = a(u, u) \geq \alpha||u||^2. \tag{3.4}$$

We also notice that finding u such that $a(u, \cdot) = l$ (which is (3.3)) boils down to finding $u \in H$ such that $Au = x_l$. This is equivalent, for any $\rho > 0$, to finding $u \in H$ such that $u - \rho(Au - x_l) = u$, i.e. to finding a fixed point of $f(u) = u - \rho(Au - x_l)$. We will prove that, for a suitable $\rho > 0$, f is a (strict) contraction on H. The contraction mapping theorem then establish the existence and uniqueness of a fixed point for f, and thus that (3.3) has a unique solution.

We have $f(u) - f(v) = (u - v) - \rho(Au - Av)$ and thus, by continuity of A and (3.4),

$$\begin{aligned} ||f(u) - f(v)||^2 &= \langle (u-v) - \rho A(u-v), (u-v) - \rho A(u-v) \rangle \\ &= ||u-v||^2 - 2\rho\langle u-v, A(u-v)\rangle + \rho^2 ||A(u-v)||^2 \\ &\leq ||u-v||^2 (1 - 2\rho\alpha + M^2\rho^2). \end{aligned}$$

We need to find $\rho > 0$ such that $1 - 2\rho\alpha + M^2\alpha^2 < 1$, that is $2\rho\alpha > \rho^2 M^2$, or $2\alpha > \rho M$. Any $\rho \in (0, \frac{2\alpha}{M})$ is suitable, and for those, f is a strict contraction on H. This concludes the existence and uniqueness of a solution to (3.3).

To prove the estimate on u, we just take $v = u$ in (3.3), and use the coercivity of a. ■

3.4 [PDE] Variational elliptic PDEs with Dirichlet boundary conditions

Let us start with a general definition.

Definition 3.4.1 *A matrix-valued function $A : \Omega \mapsto M_N(\mathbb{R})$ is* coercive*, or* uniformly elliptic*, if there exists $\alpha > 0$ such that*

$$\textit{for any } x \in \Omega, \textit{ for any } \xi \in \mathbb{R}^N, \quad A(x)\xi \cdot \xi \geq \alpha |\xi|^2$$

(where $\cdot$ is the Euclidean dot product in $\mathbb{R}^N$ and $|\cdot|$ the corresponding Euclidean norm). The function A is bounded *if there exists $M > 0$ such that*

$$\textit{for any } x \in \Omega, \textit{ for any } \xi \in \mathbb{R}^N, \quad |A(x)\xi| \leq M|\xi|.$$

3.4.1 Diffusion-reaction equation

We can now solve the boundary problem

$$\begin{cases} -\mathrm{div}(A\nabla u) + u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

for $f \in L^2(\Omega)$ and $A : \Omega \to \mathcal{S}_N(\mathbb{R})$ (space of symmetric matrices) is coercive and bounded. The problem can be recast (either using formal IPP or the definition of div in $\mathcal{D}'$) as

$$\forall \varphi \in C_c^\infty, \int A\nabla u \cdot \nabla\varphi + \int u\varphi = \int f\varphi.$$

All terms make sense if $u \in H^1$ and the boundary condition can be included in the space, so we arrive at

$$\begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ \forall v \in H_0^1(\Omega), \int A\nabla u \cdot \nabla v + \int uv = \int fv. \end{array} \tag{3.6}$$

(we have extended the space of test functions, by density). This is the variational formulation of (3.5).
The bilinear form

$$(u, v)_{H_0^1} = \int_\Omega A\nabla u \cdot \nabla v + \int uv$$

is a scalar product on $H_0^1(\Omega)$ which is equivalent to the standard scalar product. Hence, H_0^1 is a Hilbert space for this scalar product. Moreover, the linear form

$$v \in H_0^1 \to \int fv$$

is continuous (cf Hölder). The Riesz representation theorem then gives the existence and uniqueness of $u \in H_0^1(\Omega)$ such that, for all $v \in H_0^1$, $(u,v)_{H_0^1} = \int fv$. Thus our first existence-uniqueness result

Theorem 3.4.2 *There exists a unique solution to* (3.6)*.*

Remark 3.4.3 *We can in fact solve* (3.5) *for any $f \in (H_0^1)' = H^{-1}$, e.g. $f = \mathrm{div}(F)$ with $F \in (L^2)^N$.*

3.4.2 Removing the symmetry assumption on A

$(\cdot,\cdot)_{H_0^1}$ defined above is no longer symmetric and is therefore not an inner product and we cannot just rely on the Riesz representation theorem. However, defining $a(u,v) = \int A\nabla u \cdot \nabla v + \int uv$ and $l(v) = \int fv$, existence and uniqueness of a solution to (3.6) is a straightforward consequence of the Lax-Milgram theorem.

Theorem 3.4.2 is therefore also valid if A is not symmetric.

3.4.3 Pure diffusion equation

If we think about Peaceman's model of flow in porous media, we have to solve $-\mathrm{div}(A\nabla u) = f$ without the lower order term "$+u$". But in this case, it is not clear if $a(u,v) = \int A\nabla u \cdot \nabla v$ is coercive on H_0^1. We need for that Poincaré's inequality.

Lemma 3.4.4 ***(Poincaré's inequality)*** *If Ω is bounded we have, for all $u \in W_0^{1,p}(\Omega)$, $||u||_{L^p(\Omega)} \le \mathrm{diam}(\Omega) ||\,|\nabla u|\,||_{L^p(\Omega)}$.*

Remark 3.4.5 *The result if false in $W^{1,p}(\Omega)$.*

PROOF

We only prove the result for $p < \infty$, although it also holds for $p = \infty$.

Step 1: by density of $C_c^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$, and because both sides of the inequality are continuous with respect to the $W_0^{1,p}$ norm, we only need to prove that for $u \in C_c^\infty(\Omega)$.

Step 2: Let $u \in C_c^\infty(\Omega)$. If $x \in \Omega$, let $\bar{x} \in \partial\Omega$ such that $x - \bar{x}$ is colinear to the first coordinate axis and $\bar{x}_1 \le x_1$. Then

$$u(x) = u(\bar{x}) + \int_{\bar{x}_1}^{x_1} \partial_1 u(s, x_2, \ldots, x_N)\, ds.$$

We have $u(\bar{x}) = 0$ (because u has compact support in Ω) and thus, by Hölder's inequality,

$$
\begin{aligned}
|u(x)| &\leq \int_{\bar{x}_1}^{x_1} |\nabla u(s, x_2, \ldots, x_N)|\, ds \\
&\leq \left(\int_{\bar{x}_1}^{x_1} |\nabla u(s, x_2, \ldots, x_N)|^p\, ds\right)^{1/p} \left(\int_{\bar{x}_1}^{x_1} 1\, dx\right)^{1/p'} \\
&\leq \left(\int_{\bar{x}_1}^{x_1} |\nabla u(s, x_2, \ldots, x_N)|^p\, ds\right)^{1/p} \times \mathrm{diam}(\Omega)^{1/p'}.
\end{aligned}
$$

Take the power p of that and integrate for $x \in S$ where $S = [a, b]^N$ is a square containing Ω (remember that u has in fact compact support in Ω) such that $b - a \leq \mathrm{diam}(\Omega)$:

$$
\begin{aligned}
||u||^p_{L^p(\Omega)} &\leq \mathrm{diam}(\Omega)^{p/p'} \int_S \int_{\bar{x}_1}^{x_1} |\nabla u(s, x_2, \ldots, x_N)|^p\, ds\, dx \\
&\leq \mathrm{diam}(\Omega)^{p/p'} \int_{[a,b]^N} \int_a^b |\nabla u(s, x_2, \ldots, x_N)|^p\, ds\, dx_1 \ldots dx_N \\
&\leq \mathrm{diam}(\Omega)^{p/p'} (b - a) \int_{[a,b]^N} |\nabla u(s, x_2, \ldots, x_N)|^p\, ds\, dx_2 \ldots dx_N \\
&\leq \mathrm{diam}(\Omega)^{p/p'+1} \int_{[a,b]^N} |\nabla u(s, x_2, \ldots, x_N)|^p\, ds\, dx_2 \ldots dx_N \\
&\leq \mathrm{diam}(\Omega)^p \int_\Omega |\nabla u|^p
\end{aligned}
$$

(recall that $\nabla u = 0$ outside Ω) and the proof is complete. ∎

Remark 3.4.6 *It also works if Ω is only bounded in one direction.*

Corollary 3.4.7 *If Ω is bounded, the quantity $||u||_{H^1_0} := ||\,|\nabla u|\,||_2$ is a norm on H^1_0, equivalent to the classical norm and coming from the scalar product $\langle u, v\rangle_{H^1_0} = \int \nabla u \cdot \nabla v$. In particular, H^1_0 is Hilbert space for this norm.*

We can then apply Lax-Milgram theorem with $a(u, v) = \int A\nabla u \cdot \nabla v$, which is coercive in H^1_0 for this norm:

$$a(u, u) = \int A\nabla u \cdot \nabla u \geq \alpha||\,|\nabla u|\,||^2_{L^2}.$$

We therefore get

Theorem 3.4.8 *If $f \in L^2(\Omega)$ and $A : \Omega \to M_N(\mathbb{R})$ is uniformly coercive there exists a unique solution to*

$$
\left\{
\begin{array}{l}
\text{Find } u \in H^1_0(\Omega) \text{ such that} \\
\forall v \in H^1_0(\Omega)\,, \displaystyle\int A\nabla u \cdot \nabla v = \int fv,
\end{array}
\right. \tag{3.7}
$$

which is the variational (or weak) form of

$$
\left\{
\begin{array}{ll}
-\mathrm{div}(A\nabla u) = f & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega.
\end{array}
\right. \tag{3.8}
$$

Moreover, the transformation $f \in L^2 \to u \in H^1_0$ solution to (3.7) is linear and continuous.

Proof

The continuity of $f \to u$ is based on the estimate obtained by taking $v = u$ in the variational formulation and using Hölder's and Poincaré's inequality:

$$\alpha|| \, |\nabla u| \, ||_2^2 \leq \int A\nabla u \cdot \nabla u = \int fu \leq ||f||_2 ||u||_2 \leq \mathrm{diam}(\Omega) ||f||_2 || \, |\nabla u| \, ||_2,$$

which gives $||u||_{H_0^1} = || \, |\nabla u| \, ||_2 \leq \mathrm{diam}(\Omega)\alpha^{-1}||f||_2$. ■

3.5 [PDE] Maximum principle

Theorem 3.5.1 ***(Chain rule)*** *Let $u \in W_0^{1,p}(\Omega)$ and $T : \mathbb{R} \to \mathbb{R}$ of class C^1 such that $T(0) = 0$ and T' is bounded. Then $T(u) \in W_0^{1,p}(\Omega)$ and $\nabla(T(u)) = T'(u)\nabla u$ a.e.*

Remark 3.5.2 *By localizing the functions inside Ω (and knowing that functions in $W^{1,p}(\Omega)$ with compact support belong to $W_0^{1,p}(\Omega)$), we see that Theorem 3.5.1 holds if we replace $W_0^{1,p}$ by $W^{1,p}$. In fact, we can even state similar results for functions in L^1_{loc} whose derivative are in L^1_{loc}.*

Proof

Let $(u_n) \in C_c^\infty$ which converges to u in $W^{1,p}$ and such that: $u_n \to u$ a.e., $\nabla u_n \to \nabla u$ a.e. and (u_n) and (∇u_n) stay dominated by a function in L^p.

We have $T(u_n) \in C^1$ and, since $T(0) = 0$ and u_n has a compact support, $T(u_n)$ has a compact support. Hence, by Exercise 3.2.8, $T(u_n) \in W_0^{1,p}$.

Since T' is bounded and $T(0) = 0$, $|T(s)| \leq C|s|$ and Exercise 2.2.12 shows that $T(u_n) \to T(u)$ in L^p. Moreover, $\nabla T(u_n) = T'(u_n)\nabla u_n \to T'(u)\nabla u$ a.e. while remaining dominated by a function in L^p (because T' is bounded). hus $\nabla T(u_n) \to T'(u)\nabla u$ in L^p, and thus in $\mathcal{D}'$. Since $T(u_n) \to T(u)$ in L^p and thus in $\mathcal{D}'$, we also have $\nabla T(u_n) \to \nabla T(u)$ in $\mathcal{D}'$. This shows that $\nabla T(u) = T'(u)\nabla u$ in $\mathcal{D}'$, and thus that $\nabla T(u) \in L^p$. We therefore get $T(u) \in W^{1,p}$ and it remains to prove that it belongs in fact to $W_0^{1,p}$.

Moreover, $T(u_n)$ and $\nabla T(u_n)$ converge respectively to $T(u)$ and $\nabla T(u)$ in L^p. Hence $T(u_n) \to T(u)$ in $W^{1,p}$ and since $T(u_n) \in W_0^{1,p}$ with $W_0^{1,p}$ closed in $W^{1,p}$, this gives $T(u) \in W_0^{1,p}$. ■

Theorem 3.5.3 ***(Maximum principle)*** *If $f \geq 0$ then the solution to* (3.7) *is also (a.e.) nonnegative.*

Proof

Let $T \in C^1$ such that $T' \geq 0$ is bounded, $T(s) < 0$ if $s < 0$ and $T(s) = 0$ if $s \geq 0$. Use $T(u)$ as a test function: since $T \leq 0$ and $T' \geq 0$,

$$\alpha \int T'(u)|\nabla u|^2 \leq \int A\nabla u \cdot \nabla(T(u)) = \int fT(u) \leq 0.$$

Let $\varphi = \int_0^s \sqrt{T'(t)}\, dt$. Then $T'(u)|\nabla u|^2 = \varphi'(u)^2|\nabla u|^2 = |\nabla(\varphi(u))|^2$ and the preceding inequality shows that $\nabla(\varphi(u)) = 0$. Since $\varphi(u) \in H_0^1$, Poincaré's inequality gives $\varphi(u) = 0$. But $\varphi(s) < 0$ if $s < 0$ and this shows that $u \geq 0$ a.e. ■

As a consequence, we can also compare solutions: if u, v are solution corresponding to f, g such that $f \geq g$ then $u \geq v$.
In particular, if the solution u_0 corresponding to $f = 1$ is bounded, then any solution corresponding to $f \in L^\infty$ is also bounded. Indeed, $||f||_\infty u_0$ is the solution with right-hand side $||f||_\infty \geq f$, so $||f||_\infty u_0 \geq u$. Likewise, $-||f||_\infty u_0$ being the solution with right-hand side $-||f||_\infty \leq f$, $-||f||_\infty u_0 \leq u$. Hence $|u| \leq ||f||_\infty |u_0| \leq M||f||_\infty$ if $u_0 \in L^\infty$.
But we can get better L^∞ estimates than that, without assuming that f is bounded.

3.6 [S] A bunch of additional property of Sobolev spaces: density of smooth functions, extension operators, Sobolev embeddings, generalised chain rule

3.6.1 Density of smooth functions if $\Omega = \mathbb{R}^N$

Theorem 3.6.1 *If $p < \infty$ then $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$. In other words, $W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$.*

The proof is a consequence of the two following exercises.

Exercise 3.6.2 Let $(\gamma_n) \in C_c^\infty(\mathbb{R}^N)$ such that $\gamma_n(x) \in [0,1]$, $\gamma_n \to 1$ on $\mathbb{R}^N$ and $\nabla(\gamma_n) \to 0$ in $L^\infty(\mathbb{R}^N)$. One possible choice is to take $\gamma_1 \in C_c^\infty(\mathbb{R}^N)$ equal to 1 on $B(0,1)$ and to define $\gamma_n(x) = \gamma_1(x/n)$.
Prove that if $p < \infty$ and $u \in W^{1,p}(\mathbb{R}^N)$ then $\gamma_n u \to u$ in $W^{1,p}(\mathbb{R}^N)$.

Exercise 3.6.3 Let (ρ_n) be a smoothing kernel and $u \in W^{1,p}(\mathbb{R}^N)$ has a compact support. Prove that $u * \rho_n \in C_c^\infty(\mathbb{R}^N)$ converges to u in $W^{1,p}(\mathbb{R}^N)$.

3.6.2 Density of smooth functions and extension operator if $\Omega =$ half space

The first idea to get the density of smooth functions in $W^{1,p}(\Omega)$ would be to take $u * \rho_n$, with (ρ_n) a smoothing kernel and $u \in W^{1,p}(\Omega)$. However, this requires to extend u by 0 outside Ω, and in doing that we lose the $W^{1,p}$ characteristic in general.

Exercise 3.6.4 Let $u \in W^{1,p}(\Omega)$ have a compact support, and define $\widetilde{u} : \mathbb{R}^N \to \mathbb{R}$ as the extension of u by 0 outside Ω. Prove that $\widetilde{u} \in W^{1,p}(\Omega)$ and that $\nabla\widetilde{u}$ is the extension of ∇u by 0 outside Ω.

There is however a trick, based on a *shifted* smoothing kernel, which allows to prove the density of smooth functions if $\Omega = \mathbb{R}^N_+ := \{(x', x_N) \in \mathbb{R}^N \mid x_N > 0\}$ is a half space. We denote by $C_c^\infty(\overline{\mathbb{R}^N_+})$ the set of restrictions to $\mathbb{R}^N_+$ of functions in $C_c^\infty(\mathbb{R}^N)$.

Proposition 3.6.5 ***(Density of smooth functions if $\Omega =$ half-space)*** *Let $p < \infty$. Then $C_c^\infty(\overline{\mathbb{R}^N_+})$ is dense in $W^{1,p}(\mathbb{R}^N_+)$.*

Proof

Let $(\rho_n)_{n\geq 1}$ a smoothing kernel such that $\mathrm{supp}(\rho_n) \subset \mathbb{R}^N_- = \{x \in \mathbb{R}^N \mid x_N < 0\}$. Let $u \in W^{1,p}(\mathbb{R}^N_+)$ and $\widetilde{u} : \mathbb{R}^N \to \mathbb{R}$ the extension of u by 0 outside $\mathbb{R}^N_+$. We know that $\widetilde{u} \in L^p(\mathbb{R}^N)$. Let $u_n = \widetilde{u} * \rho_n$. We know that $u_n \to \widetilde{u}$ in $L^p(\mathbb{R}^N)$ so, in particular, $u_n \to u$ in $L^p(\mathbb{R}^N_+)$.

Let us compute $\partial_i u_n$ in $\mathcal{D}'(\mathbb{R}^N_+)$ (**not** in $\mathcal{D}'(\mathbb{R}^N)$!). If $\varphi \in C_c^\infty(\mathbb{R}^N_+)$ we have

$$\begin{aligned}
\langle \partial_i u_n, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^N_+),\mathcal{D}(\mathbb{R}^N_+)} &= -\int_{\mathbb{R}^N_+} u_n(x)\partial_i\varphi(x)\,dx \\
&= -\int_{\mathbb{R}^N} u_n(x)\partial_i\varphi(x)\,dx,
\end{aligned}$$

because $\varphi \equiv 0$ on $\mathbb{R}^N\backslash\mathbb{R}^N_+$. Hence, with $\rho_n^\vee(x) = \rho_n(-x)$,

$$\begin{aligned}
\langle \partial_i u_n, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^N_+),\mathcal{D}(\mathbb{R}^N_+)} &= -\langle \widetilde{u} * \rho_n, \partial_i\varphi\rangle_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} \\
&= -\int_{\mathbb{R}^N} \widetilde{u}\rho_n^\vee * \partial_i\varphi \\
&= -\int_{\mathbb{R}^N} \widetilde{u}\partial_i(\rho_n^\vee * \varphi) \\
&= -\int_{\mathbb{R}^N} \widetilde{u}(x)\partial_i(\rho_n^\vee * \varphi)(x)\,dx \\
&= -\int_{\mathbb{R}^N_+} u(x)\partial_i(\rho_n^\vee * \varphi)(x)\,dx.
\end{aligned}$$

But $\mathrm{supp}(\rho_n^\vee * \varphi) \subset \mathrm{supp}(\varphi) + \mathrm{supp}(\rho_n^\vee)$ is compact in $\mathbb{R}^N_+$ ($\mathrm{supp}(\rho_n^\vee)$ is compact in $\mathbb{R}^N_+$) and thus $\rho_n^\vee * \varphi \in C_c^\infty(\mathbb{R}^N_+)$. This gives

$$\begin{aligned}
-\int_{\mathbb{R}^N_+} u(x)\partial_i(\rho_n^\vee * \varphi)(x)\,dx &= -\int_{\mathbb{R}^N_+} u\partial_i(\rho_n^\vee * \varphi) \\
&= \int_{\mathbb{R}^N_+} \partial_i u\rho_n^\vee * \varphi \\
&= \int_{\mathbb{R}^N_+} \partial_i u(x)\rho_n^\vee * \varphi(x)\,dx,
\end{aligned}$$

by definition of $\partial_i u \in L^p(\mathbb{R}^N_+)$. Denoting by $\widetilde{\partial_i u} \in L^p(\mathbb{R}^N)$ the extension of $\partial_i u$ by 0 outside $\mathbb{R}^N_+$, we get

$$\begin{aligned}
\int_{\mathbb{R}^N_+} \partial_i u(x)\rho_n^\vee * \varphi(x)\,dx &= \int_{\mathbb{R}^N} \widetilde{\partial_i u}(x)\rho_n^\vee * \varphi(x)\,dx \\
&= \int_{\mathbb{R}^N} \widetilde{\partial_i u}\rho_n^\vee * \varphi \\
&= \int_{\mathbb{R}^N} \rho_n * \widetilde{\partial_i u}\varphi.
\end{aligned}$$

Gluing together these equalities we get $\partial_i u_n = \rho_n * \widetilde{\partial_i u}$ in $\mathcal{D}'(\mathbb{R}^N_+)$. Since $\widetilde{\partial_i u} \in L^p(\mathbb{R}^N)$, that gives $\partial_i u_n \in L^p(\mathbb{R}^N)$ (so $u_n \in W^{1,p}(\mathbb{R}^N)$) and $\partial_i u_n \to \widetilde{\partial_i u}$ in $L^p(\mathbb{R}^N)$. Therefore,

$\partial_i u_n \to \partial_i u$ in $L^p(\mathbb{R}^N_+)$. We approximated u in $W^{1,p}(\mathbb{R}^N_+)$ by a sequence in $W^{1,p}(\mathbb{R}^N)$. Since we already know that $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, the proposition is proved. ■

We understood at the beginning of this section that there is somehow a link between the density of smooth functions and the extension of Sobolev functions. Here is a result, using the density theorem just proved, which states that, for the half-space, we can extend Sobolev functions.

Corollary 3.6.6 ***(Extension operator if Ω = half-space)*** *Let $E_0 : L^p(\mathbb{R}^N_+) \to L^p(\mathbb{R}^N)$ be defined by the extension by symmetry about $x_N = 0$: $E_0 u(x', x_N) = u(x', |x_N|)$ for all $u \in L^p(\mathbb{R}^N)$. Then $E_0 : W^{1,p}(\mathbb{R}^N_+) \to W^{1,p}(\mathbb{R}^N)$ is linear continuous and*

$$\begin{cases} \partial_i(E_0 u)(x', x_N) = \partial_i u(x', |x_N|) & \text{if } i \in [1, N-1], \\ \partial_N(E_0 u)(x', x_N) = \mathrm{sgn}(x_N)\partial_N u(x', |x_N|). & \end{cases} \tag{3.9}$$

Proof

If $u \in W^{1,p}(\mathbb{R}^N_+)$, $E_0 u$ is in $L^p(\mathbb{R}^N)$, with $||E_0 u||_{L^p(\mathbb{R}^N)} = 2^{1/p}||u||_{L^p(\mathbb{R}^N_+)}$. Let us compute the distribution derivatives of $E_0 u$ on $\mathbb{R}^N$, by first assuming that $u \in C_c^\infty(\overline{\mathbb{R}^N_+})$.

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$.

1) If $i < N$, using Fubini's theorem and since $x_i \to u(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{N-1}, |x_N|)$ belongs to $C^\infty(\mathbb{R})$ for all $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \mathbb{R}^{N-1}$,

$$\begin{aligned} \int_{\mathbb{R}^N} E_0 u(x)\partial_i\varphi(x)\,dx &= \int_{\mathbb{R}^{N-1}} \left(\int_{\mathbb{R}} u(x', |x_N|)\partial_i\varphi(x)\,dx_i \right) dx_1 \ldots dx_{i-1}dx_{i+1} \ldots dx_N \\ &= -\int_{\mathbb{R}^{N-1}} \left(\int_{\mathbb{R}} \partial_i u(x', |x_N|)\varphi(x)\,dx_i \right) dx_1 \ldots dx_{i-1}dx_{i+1} \ldots dx_N \\ &= -\int_{\mathbb{R}^N} \partial_i u(x', |x_N|)\varphi(x)\,dx. \end{aligned}$$

Hence, for $i \in [1, N-1]$ we have $\partial_i(E_0 u) \in L^p(\mathbb{R}^N)$ and $\partial_i(E_0 u)(x', x_N) = \partial_i u(x', |x_N|)$.

2) If $i = N$, since $x_N \to u(x', x_N)$ and $x_N \to u(x', -x_N)$ belong to $C^\infty(\mathbb{R})$ for all

$x' \in \mathbb{R}^{N-1}$, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} E_0u(x)\partial_N\varphi(x)\,dx &= \int_{\mathbb{R}^{N-1}}\left(\int_{\mathbb{R}} u(x',|x_N|)\partial_N\varphi(x)\,dx_N\right)dx' \\
&= \int_{\mathbb{R}^{N-1}}\left(\int_0^{+\infty} u(x',x_N)\partial_N\varphi(x)\,dx_N\right)dx' \\
&\quad + \int_{\mathbb{R}^{N-1}}\left(\int_{-\infty}^0 u(x',-x_N)\partial_N\varphi(x)\,dx_N\right)dx' \\
&= -\int_{\mathbb{R}^{N-1}}\left(\int_0^{+\infty} \partial_N u(x',x_N)\varphi(x)\,dx_N\right)dx' \\
&\quad -\int_{\mathbb{R}^{N-1}} u(x',0)\varphi(x',0)\,dx' + \int_{\mathbb{R}^{N-1}} u(x',0)\varphi(x',0)\,dx' \\
&\quad -\int_{\mathbb{R}^{N-1}}\left(\int_{-\infty}^0 -\partial_N u(x',-x_N)\varphi(x)\,dx_N\right)dx' \\
&= -\int_{\mathbb{R}^{N-1}}\left(\int_{\mathbb{R}} \mathrm{sgn}(x_N)\partial_N u(x',|x_N|)\varphi(x)\,dx_N\right)dx' \\
&= -\int_{\mathbb{R}^N} \mathrm{sgn}(x_N)\partial_N u(x',|x_N|)\varphi(x)\,dx.
\end{aligned}$$

We deduce that $\partial_N(E_0u) \in L^p(\mathbb{R}^N)$ with $\partial_N(E_0u)(x',x_N) = \mathrm{sgn}(x_N)\partial_N u(x',|x_N|)$.

Hence, if $u \in C_c^\infty(\overline{\mathbb{R}^N_+})$, $E_0u \in W^{1,p}(\mathbb{R}^N)$ and (3.9) holds.

Assume now that $u \in W^{1,p}(\mathbb{R}^N_+)$ (with $p < +\infty$) and take $(u_n)_{n\ge1} \in C_c^\infty(\overline{\mathbb{R}^N_+})$ which converges to u in $W^{1,p}(\mathbb{R}^N_+)$, a.e. on $\mathbb{R}^N_+$ and such that $\nabla u_n \to \nabla u$ a.e. on $\mathbb{R}^N_+$. We have then $E_0u_n \to E_0u$ in $L^p(\mathbb{R}^N)$ and therefore $\nabla(E_0u_n) \to \nabla(E_0u)$ in $(\mathcal{D}'(\mathbb{R}^N))^N$. Thanks to (3.9) satisfied by each u_n, we see that $\nabla(E_0u_n)$ tends in $L^p(\mathbb{R}^N)$ (and thus in $(\mathcal{D}'(\mathbb{R}^N))^N$) to the function

$$x \to (\partial_1 u(x',|x_N|),\dots,\partial_{N-1}u(x',|x_N|),\mathrm{sgn}(x_N)\partial_N u(x',x_N)).$$

Identifying the limits in $(\mathcal{D}'(\mathbb{R}^N))^N$ of $\nabla(E_0u_n)$, we infer that $E_0u \in W^{1,p}(\mathbb{R}^N)$ and that (3.9) holds. This shows in particular that $||E_0u||_{W^{1,p}(\mathbb{R}^N)} \le C||u||_{W^{1,p}(\mathbb{R}^N_+)}$. ■

The following generalisation of Exercise 3.6.4 will be useful in the following, with $U = B$ and $V = \mathbb{R}^N_+$.

Exercise 3.6.7 Let U, V be open sets and $v \in W^{1,p}(U \cap V)$ have a compact support in U (i.e. its support might touch ∂V but not ∂U). Then the extension of v to V by 0 outside U belongs to $W^{1,p}(V)$ and has a gradient which is the extension of ∇v by 0 outside U. (*Long hint/Solution: take $\gamma \in C_c^\infty(U)$ which is equal to 1 on a neighbourhood of* $\mathrm{supp}(v)$ *and, if $\widetilde{v}$ is the extension of v, write, for $\varphi \in C_c^\infty(V)$,*

$$\langle \partial_j\widetilde{v},\varphi\rangle_{\mathcal{D}'(V)} = -\int_V \widetilde{v}\partial_j\varphi = -\int_{U\cap V} v\partial_j\varphi = -\int_{U\cap V} v\partial_j(\gamma\varphi)$$

(because $\partial_j(\gamma\varphi) = (\partial_j\gamma)\varphi + \gamma\partial_j\varphi$ and $\gamma = 1$, $\partial_j\gamma = 0$ on a neighbourhood of $\mathrm{supp}(v)$*, so that $v\partial_j(\gamma\varphi) = v\partial_j\varphi$), which gives, since $\gamma\varphi \in C_c^\infty(U \cap V)$ and $v \in W^{1,p}(U \cap V)$ with*

$\text{supp}(\partial_j v) \subset \text{supp}(v)$ *on which* $\gamma = 1$,

$$\langle \partial_j \widetilde{v}, \varphi \rangle = \langle \partial_j v, \gamma\varphi \rangle_{\mathcal{D}'(U\cap V)} = \int_{U\cap V} \partial_j v \gamma \varphi = \int_{U\cap V} \partial_j v \varphi = \int_V \widetilde{\partial_j v} \varphi$$

with $\widetilde{\partial_j v}$ *the extension by* 0 *outside* U.*)*

3.6.3 Extension operator and density of smooth functions if Ω is C^1

C^1 open sets A bounded open set Ω is C^1 if for all $x \in \partial\Omega$ there exists an open set U containing x and a C^1-diffeomorphism $\varphi : U \to B = B(0,1)$ such that $\varphi(\Omega \cap U) = B_+ = B \cap \mathbb{R}^N +$ and $\varphi(\partial\Omega \cap U) = B_{N-1} = \{x \in B \,|\, x_N = 0\}$.

Example 3.6.8 A classical (equivalent) way of proving that Ω is C^1 is to find, for each $x \in \partial\Omega$, a neighbourhood U, a set of coordinates $U = V \times (-\eta, \eta)$ and a function $\gamma : V \to (-\eta, \eta)$ such that $\Omega \cap U$ is locally described as $\{y \in V \,|\, y_N > \gamma(y_1, \ldots, y_{N-1})\}$.

In this case, we can cover $\partial\Omega$ by a finite number of open sets $(U_i)_{i=1,\ldots,k}$ such that there exists diffeomorphisms $\varphi_i : U_i \to B$. Moreover, there exists a unit partition $(\zeta_i)_{i=0,\ldots,k}$ associated with this cover: $\zeta_0 \in C_c^\infty(\Omega)$, $\zeta_i \in C_c^\infty(U_i)$ for $i = 1, \ldots, k$, $0 \le \zeta_i \le 1$ and $\sum_{i=0}^k \zeta_i = 1$ on Ω. Any function $u \in W^{1,p}(\Omega)$ can then be written $u = \sum_{i=0}^k \zeta_i u$ with $\zeta_0 u$ with compact support in Ω and $\zeta_i u$ with compact support in U_i (possibly touching $\partial\Omega$).
If $\varphi : U \to V$ is a diffeomorphism then it transports Sobolev spaces. More precisely, the transformation

$$\begin{cases} W^{1,p}(V) \to W^{1,p}(U) \\ u \to u \circ \varphi \end{cases}$$

is a continuous isomorphism. Hence, any topological property we want to establish on one Sobolev space can be established in the other and transported.
These remarks allow us to prove results for functions in $W^{1,p}(\Omega)$ with compact supports in Ω or for functions in $W^{1,p}(B_+)$ and to extend these results to functions in $W^{1,p}(\Omega)$.

Example 3.6.9 A ball with a cut is not C^1. An open set with a cusp in not C^1.

Extension and density results for C^1 open sets

Theorem 3.6.10 ***(Extension operator if*** Ω ***is*** C^1***)*** *If* Ω *is* C^1 *then there exists a linear continuous operator* $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ *such that:*

1. *For all* u, $E(u) = u$ *on* Ω,
2. *For all* u, $E(u)$ *has a compact support in* $\mathbb{R}^N$.

Proof

Let $u = \sum_i \zeta_i u \in W^{1,p}$ with (ζ_i) as in Section 3.6.3. Exercise 3.6.4 shows that the extension $\widetilde{\zeta_0 u}$ of $\zeta_0 u$ by 0 outside Ω belongs to $W^{1,}(\mathbb{R}^N)$. We therefore just need to extend, for any $i \ge 1$, $\zeta_i u$ to get an extension of u.

Fix i and let $v = \zeta_i u \circ \varphi_i^{-1} \in W^{1,p}(B_+)$. v has a compact support in B (because ζ_i has a compact support in U_i) and by Exercise 3.6.7, we can see that its extension $\widetilde{v}$ to

$\mathbb{R}^N+$ by 0 outside B_+ belongs to $W^{1,p}(\mathbb{R}^N_+)$. Applying the symmetry operator E_0, we get an extension $E_0\widetilde{v} \in W^{1,p}(\mathbb{R}^N)$ with compact support in B. Hence, $w = (E_0\widetilde{v})\circ\varphi_i \in W^{1,p}(U_i)$ has a compact support in U_i and is equal to $\zeta_i u$ on $U_i \cap \Omega$. We can extend w by 0 outside U_i, getting a function in $W^{1,p}(\mathbb{R}^N)$ which is equal to $\zeta_i u$ on Ω (on $U_i \cap \Omega$ by construction, and on $\Omega \backslash U_i$ because $\widetilde{w} = \zeta_i u = 0$ outside U_i).

Since each $\zeta_i u$ has been extended, this gives an extension of u, and it is straightforward to check that this extension is linear and continuous (all transport and extension operations used above are linear and continuous). ■

For Ω bounded, we define $C^\infty(\overline{\Omega})$ the set of restrictions to Ω of functions in $C^\infty(\mathbb{R}^N)$.

Theorem 3.6.11 ***(Density of smooth functions if Ω is C^1)*** *Let $p < \infty$. If Ω is C^1 then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$.*

Remark 3.6.12 *We do not really need the C^1 property on Ω, see the book by Adams & Fournier.*

Proof

Let $u \in W^{1,p}(\Omega)$ and $Pu \in W^{1,p}(\mathbb{R}^N)$ an extension of u. By Theorem 3.6.1, we can find $(u_n) \subset C_c^\infty(\mathbb{R}^N)$ which converge to Pu in $W^{1,p}(\mathbb{R}^N)$. Then the restrictions of u_n to Ω belong to $C^\infty(\overline{\Omega})$ and converge to $Pu_{|\Omega} = u$ in $W^{1,p}(\Omega)$. ■

3.6.4 Sobolev embeddings

These will be our main tool to get regularity results on solutions to elliptic equations.

Theorem 3.6.13 *Let $p \in [1, \infty]$ and Ω be C^1. Then the following continuous embeddings hold:*

1. *If $p < N$, $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ with $p^* = \frac{Np}{N-p}$,*
2. *If $p = N$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < \infty$,*
3. *If $p > N$, $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{N}{p}}(\Omega)$, the space of Hölder-continuous functions with exponent $1 - \frac{N}{p}$,*

where "$\hookrightarrow$" means "is continuously embedded in".

Remark 3.6.14 *The proof is based on obtaining estimates for functions in $C_c^1(\mathbb{R}^N)$ and using then the density in $W^{1,p}(\Omega)$ of their restriction on Ω to get similar estimates for any function in $W^{1,p}(\Omega)$. Given the definition of $W_0^{1,p}(\Omega)$, these embeddings also hold for $W_0^{1,p}(\Omega)$ even if Ω is not C^1 (provided it is bounded).*

Lemma 3.6.15 ***(Gagliardo-Niremberg)*** *Let $f_1, \dots, f_N$ be nonnegative functions on $\mathbb{R}^{N-1}$. For $x \in \mathbb{R}^N$ we define $\widehat{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$. Let*

$$f(x) = f_1(\widehat{x}_1) f_2(\widehat{x}_2) \cdots f_N(\widehat{x}_N).$$

Then

$$||f||_{L^1(\mathbb{R}^N)} \leq ||f_1||_{L^{N-1}(\mathbb{R}^{N-1})} \cdots ||f_N||_{L^{N-1}(\mathbb{R}^{N-1})}.$$

Proof of Theorem 3.6.13

Case $p < N$: We first consider $p = 1$ and $\Omega = \mathbb{R}^N$. We write, for all $i = 1, \dots, N$, if $u \in C_c^1(\mathbb{R}^N)$,

$$|u(x)| = \left| \int_{-\infty}^{x_i} \partial_i u(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_N)\, ds \right| \le \int_{\mathbb{R}} |\partial_i u(x)|\, dx_i =: f_i(\widehat{x_i})$$

with $\widetilde{x_i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$. Hence,

$$|u(x)|^{\frac{N}{N-1}} \le f_1(\widehat{x}_1)^{\frac{1}{N-1}} f_2(\widehat{x}_2)^{\frac{1}{N-1}} \cdots f_N(\widehat{x}_N)^{\frac{1}{N-1}}$$

and, by Gagliardo-Niremberg's inequality,

$$\int_{\mathbb{R}^N} |u|^{\frac{N}{N-1}} \le \prod_{i=1}^N ||f_i^{\frac{1}{N-1}}||_{L^{N-1}(\mathbb{R}^{N-1})} = \prod_{i=1}^N ||f_i||_{L^1(\mathbb{R}^{N-1})}^{\frac{1}{N-1}}.$$

Since $||f_i||_{L^1(\mathbb{R}^{N-1})} = ||\partial_i u||_{L^1(\mathbb{R}^N)}$, we deduce

$$||u||_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} \le \prod_{i=1}^N ||\,|\nabla u|\,||_{L^1(\mathbb{R}^N)}^{\frac{1}{N}} = ||\,|\nabla u|\,||_{L^1(\mathbb{R}^N)}. \tag{3.10}$$

This is the expected estimate when $\Omega = \mathbb{R}^N$ and $p = 1$.

If $p \in (1, N)$, we apply (3.10) to $v = |u|^s$ for some $s > 1$ to be chosen later (v also belongs to $C_c^1(\mathbb{R}^N)$). Since $|\nabla v| = s|u|^{s-1}|\nabla u|$, Hölder's inequality gives

$$||\,|u|^s||_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} \le s||\,|u|^{s-1}|\nabla u|\,||_{L^1(\mathbb{R}^N)} \le ||\,|u|^{s-1}||_{L^{p'}(\mathbb{R}^N)}||\,|\nabla u|\,||_{L^p(\mathbb{R}^N)}.$$

Take now s such that $s\frac{N}{N-1} = (s-1)p'$, that is $s = \frac{p(N-1)}{N-p} > 1$ (since $p > 1$). We see that $s\frac{N}{N-1} = p^*$ and the preceding estimate therefore reads

$$||u||_{L^{p^*}(\mathbb{R}^N)}^s \le s||u||_{L^{p^*}}^{s-1}||\,|\nabla u|\,||_{L^p(\mathbb{R}^N)}$$

which gives

$$||u||_{L^{p*}(\mathbb{R}^N)} \le s||\,|\nabla u|\,||_{L^p(\mathbb{R}^N)} \le \frac{p(N-1)}{N-p}||u||_{W^{1,p}(\mathbb{R}^N)} \tag{3.11}$$

This estimate also holds for $p = 1$ (it is (3.10)).

By density of $C_c^1(\mathbb{R}^N)$ functions in $W^{1,p}(\mathbb{R}^N)$, Estimate (3.11) also holds for functions in $W^{1,p}(\mathbb{R}^N)$. Consider now Ω a C^1 open set and take $u \in W^{1,p}(\Omega)$. Let $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ be an extension operator. We can apply (3.11) to Eu and, since $Eu = u$ on Ω and E is continuous, we get

$$||u||_{L^{p*}(\Omega)} \le ||Eu||_{L^{p^*}(\mathbb{R}^N)} \le \frac{p(N-1)}{N-p}||Eu||_{W^{1,p}(\mathbb{R}^N)} \le C||u||_{W^{1,p}(\Omega)},$$

with C not depending on u, and the proof is complete.

Case $p = N$: Exercise (remember that if $u \in W^{1,p}(\Omega)$ then $u \in W^{1,q}(\Omega)$ for all $q < p$).

Case $p > N$: cf references (Brezis, Adams & Fournier for example). ■

Proof of Lemma 3.6.15

By induction on N. The case $N=2$ is obvious: we have $f(x)=f_1(x_2)f_2(x_1)$ so that Fubini implies

$$\int_{\mathbb{R}^2} f_1(x_2)f_2(x_1)\,dx = \left(\int_{\mathbb{R}} f_1(x_2)\,dx_2\right)\left(\int_{\mathbb{R}} f_2(x_1)\,dx_1\right)$$

which is what we want since $N-1=1$.

Let us consider $N\geq 3$. Since $f_N(\widehat{x_N})$ does not depend on x_N, by Hölder's inequality with exponent $N-1$ and $\frac{N-1}{N-2}$,

$$\begin{aligned}\int_{\mathbb{R}^N} f(x)\,dx &= \int_{\mathbb{R}^{N-1}}\left(f_N(\widehat{x_N})\int_{\mathbb{R}} f_1(\widehat{x_1})\ldots f_{N-1}(\widehat{x_{N-1}})\,dx_N\right)d\widehat{x_N}\\ &\leq \left(\int_{\mathbb{R}^{N-1}} f_N(\widehat{x_N})^{N-1}\,d\widehat{x_N}\right)^{\frac{1}{N-1}}\left(\int_{\mathbb{R}^{N-1}} F(\widehat{x_N})^{\frac{N-1}{N-2}}\,d\widehat{x_N}\right)^{\frac{N-2}{N-1}} \quad (3.12)\end{aligned}$$

where $F(\widehat{x_N})=\int_{\mathbb{R}} f_1(\widehat{x_1})\ldots f_{N-1}(\widehat{x_{N-1}})\,dx_N$.

Using again Hölder's inequality with exponents $\frac{1}{N-1}+\cdots+\frac{1}{N-1}=1$ (see Exercise 3.6.16), we have

$$F(\widehat{x_N})\leq\left(\int_{\mathbb{R}} f_1(\widehat{x_1})^{N-1}\,dx_N\right)^{\frac{1}{N-1}}\cdots\left(\int_{\mathbb{R}} f_{N-1}(\widehat{x_{N-1}})^{N-1}\,dx_N\right)^{\frac{1}{N-1}}.$$

Let $g_i:\mathbb{R}^{N-2}\to[0,\infty]$ defined by $g_i(\widehat{x'_i})=\left(\int_{\mathbb{R}} f_i(\widehat{x_i})^{N-1}\,dx_N\right)^{\frac{1}{N-2}}$ with $x'=\widehat{x_N}=(x_1,\ldots,x_{N-1})$. We have

$$\int_{\mathbb{R}^{N-1}} F(\widehat{x_N})^{\frac{N-1}{N-2}}\,d\widehat{x_N}\leq\int_{\mathbb{R}^{N-1}} g_1(\widehat{x'_1})\cdots g_{N-1}(\widehat{x'_{N-1}})\,dx'$$

and thus, by the induction assumption,

$$\begin{aligned}&\int_{\mathbb{R}^{N-1}} F(\widehat{x_N})^{\frac{N-1}{N-2}}\,d\widehat{x_N}\\ &\qquad\leq\left(\int_{\mathbb{R}^{N-2}} g_1(\widehat{x'_1})^{N-2}\,d\widehat{x'_1}\right)^{\frac{1}{N-2}}\cdots\left(\int_{\mathbb{R}^{N-2}} g_{N-1}(\widehat{x'_{N-1}})^{N-2}\,d\widehat{x'_{N-1}}\right)^{\frac{1}{N-2}}.\end{aligned}$$

Since $\int_{\mathbb{R}^{N-2}} g_i(\widehat{x'_i})^{N-2}\,d\widehat{x'_i}=\int_{\mathbb{R}^{N-2}}\int_{\mathbb{R}} f_i(\widehat{x_i})^{N-1}\,dx_N d\widehat{x'_i}=\int_{\mathbb{R}^{N-1}} f_i(\widehat{x_i})^{N-1}\,d\widehat{x_i}$, the proof is concluded by injecting this estimate in (3.12). ■

Exercise 3.6.16 Prove that if $f_1,\ldots,f_r$ are nonnegative functions on some measurable set A and $\frac{1}{p_1}+\ldots+\frac{1}{p_r}=1$ then

$$||f_1\cdots f_r||_1\leq||f_1||_{p_1}\cdots||f_r||_{p_r}.$$

(*Hint: write* $f_1\cdots f_r=e^{\ln(f_1\cdots f_r)}=e^{\ln(f_1)+\cdots+\ln(f_r)}=e^{\frac{1}{p_1}\ln(f_1^{p_1})+\cdots+\frac{1}{p_r}\ln(f_r^{p_r})}$ *and use the convexity of the exponential to get* $f_1\cdots f_r\leq\frac{1}{p_1}e^{\ln(f_1^{p_1})}+\ldots+\frac{1}{p_r}e^{\ln(f_r^{p_r})}=\frac{1}{p_1}f_1^{p_1}+\ldots+$

$\frac{1}{p_r} f_r^{p_r}$. *Integrating gives the result when* $||f_1||_{p_1} = \ldots = ||f_r||_{p_r} = 1$. *Otherwise, start by normalising the function, replacing* f_i *by* $f_i/||f_i||_{p_i}$.)

Example 3.6.17 Consider the domain with cusp $\Omega = \{(x, y) \,|\, 0 < x < 1\,,\ 0 < y < x^r\}$ for some $r \geq 1$. Let $u(x, y) = x^{-\alpha}$ for $\alpha > 0$.
Then $u \in W^{1,p}(\Omega)$ if and only if $\int_0^1 \int_0^{x^r} x^{-p\alpha}\, dy dx = \int_0^1 x^{-p\alpha} x^r < \infty$ and $\int_0^1 x^{-p(\alpha+1)} x^r < \infty$, that is to say if and only if $-p(\alpha + 1) + r > -1$, which gives

$$p < \frac{r + 1}{\alpha + 1} \tag{3.13}$$

(in order for this to be satisfied for some $p \geq 1$, we also need to take $\alpha < r$). We have $u \notin L^q(\Omega)$ if and only if $-q\alpha + r \leq -1$, that is

$$q \geq \frac{r + 1}{\alpha}. \tag{3.14}$$

Take $r = 1$. If p satisfies (3.13) and q satisfies (3.14) then, since $z \to z^* = \frac{Nz}{N-z}$ is increasing,

$$p^* < \frac{2\frac{r+1}{\alpha+1}}{2 - \frac{r+1}{\alpha+1}} = \frac{4}{2\alpha} \leq q$$

(note that $(r + 1)/(\alpha + 1) = 2/(\alpha + 1) < 2$ in this case) and there is no contradiction since we do not expect $W^{1,p}$ to be embedded in L^q for $q > p^*$. In this case, Ω does not have any cusp and, although not C^1, is regular enough to satisfy Theorem 3.6.11 and thus Theorem 3.6.13.
Take now $r > 1$ (Ω has a cusp). Then, when p is close to the upper limit in (3.13), p^* is close to

$$\frac{2\frac{r+1}{\alpha+1}}{2 - \frac{r+1}{\alpha+1}} = \frac{2(r + 1)}{2\alpha + 1 - r} > \frac{r + 1}{\alpha},$$

(we need to take α such that $\frac{r+1}{\alpha+1} < 2$, that is $r < 2\alpha + 1$, which is compatible with the requirement $\alpha < r$ by taking, for example, $\alpha = r/2$). Hence we can take p satisfying (3.13) in such a way that $q = p^*$ satisfies (3.14). The corresponding function u then belongs to $W^{1,p}(\Omega)$ but not to $L^{p*}(\Omega)$.
In particular, this ensures that, in such a set Ω, the extension result given in Theorem 3.6.10 does not hold.

3.6.5 Generalised chain rule

Lemma 3.6.18 *Let* $u \in W^{1,p}(\Omega)$ *and* $k \in \mathbb{R}$. *Then* $\nabla u = 0$ *a.e. on the level set* $\{x \in \Omega \,|\, u(x) = k\}$.

Proof

Since $W^{1,p}(\Omega) \subset W^{1,1}(\Omega)$ and the result does not depend on p, we can assume $p = 1$. We can also assume $k = 0$ (otherwise work with $u - k$ instead of u). Let us take $(\varphi_n), (\psi_n) \in C^\infty(\mathbb{R})$ such that $\varphi_n(s)$ and $\psi_n(s)$ converge for all $s \in \mathbb{R}$ to $s^+ = \max(s, 0)$, (φ'_n) and (ψ'_n) are bounded in $L^\infty(\mathbb{R})$ and $\varphi'_n \to \mathbf{1}_{(0,\infty)}$, $\psi'_n \to \mathbf{1}_{[0,\infty)}$ everywhere on $\mathbb{R}$. Possible choices are

$$\varphi_n(s) = \begin{cases} 0 & \text{if } s \le 0, \\ \frac{1}{2} n s^2 & \text{if } s \in \left(0, \frac{1}{n}\right), \\ s - \frac{1}{2n} & \text{if } s \ge \frac{1}{n} \end{cases} \qquad \psi_n(s) = \begin{cases} 0 & \text{if } s \le -\frac{1}{n}, \\ \frac{1}{2} n \left(s + \frac{1}{n}\right)^2 & \text{if } s \in \left(-\frac{1}{n}, 0\right), \\ s + \frac{1}{2n} & \text{if } s \ge 0. \end{cases}$$

Then, by the classical chain rule (Theorem 3.5.1 and Remark 3.5.2), $\varphi_n(u)$ and $\psi_n(u)$ belong to $W^{1,p}(\Omega)$ and

$$\nabla(\varphi_n(u)) = \varphi_n'(u)\nabla u\,, \quad \nabla(\psi_n(u)) = \psi_n'(u)\nabla u. \tag{3.15}$$

Since φ_n and ψ_n converge everywhere to s^+ and satisfy $|\varphi_n(s)| \le C + C|s|$, $|\psi_n(s)| \le C + Cs$ with C not depending on n, the Dominated Convergence Theorem shows that $\varphi_n(u)$ and $\psi_n(u)$ converge to u^+ in $L^1(\Omega)$. Their gradient therefore have the same limit in $\mathcal{D}'$, that is $\nabla(u^+)$. But (3.15) and the convergences *everywhere* of φ_n' and ψ_n' show that $\nabla(\varphi_n(u)) \to \mathbf{1}_{(0,\infty)}(u)\nabla u$ and $\nabla(\psi_n(u)) \to \mathbf{1}_{[0,\infty)}(u)\nabla u$ in $L^1(\Omega)^N$.

We therefore have $\mathbf{1}_{(0,\infty)}(u)\nabla u = \mathbf{1}_{[0,\infty)}(u)\nabla u$ a.e., which shows that $\nabla u = 0$ a.e. on $\{u = 0\}$. ■

Theorem 3.6.19 ***(Stampacchia)*** *Let $T \in C(\mathbb{R})$ be piecewise C^1 with a bounded derivative. If $u \in W^{1,p}$ then $T(u) \in W^{1,p}$ and $\nabla(T(u)) = T'(u)\nabla u$.*
If we assume that $T(0) = 0$ and $u \in W_0^{1,p}$ (with $p < \infty$) then $T(u) \in W_0^{1,p}$.

Proof

We can approximate T by $T_n \in C^1(\mathbb{R})$ in the following way: $T_n \to T$ everywhere, $T_n(s) \to T'(s)$ for all s such that T' is continuous at s and (T_n') remains bounded in $L^\infty(\mathbb{R})$.

We therefore have $|T_n(s)| \le C + C|s|$ with C not depending on n and the Dominated Convergence Theorem shows that $T_n(u) \to T(u)$ in L^1. Hence, $\nabla(T_n(u)) \to \nabla(T(u))$ in $\mathcal{D}'$. But $\nabla(T_n(u)) = T_n'(u)\nabla u$ converges to $T'(u)\nabla u$ except perhaps at some points of $A = \cup_{i=1}^r \{x \in \Omega \mid u(x) = k_i\}$ where k_i are the discontinuity points of T'. A.e. on A we have $T_n'(u)\nabla u = T'(u)\nabla u = 0$ (whatever the definition of T' in this case) and therefore the convergence of $\nabla(T_n(u)) = T_n'(u)\nabla u$ to $T'(u)\nabla u$ holds a.e. on Ω. The Dominated Convergence Theorem thus shows that this convergence holds in $L^1(\Omega)$ and therefore in $\mathcal{D}'$, which implies $\nabla(T(u)) = T'(u)\nabla u$. This formula clearly shows that if $u \in W^{1,p}$ then $\nabla(T(u)) \in L^p$. Since we already know that $T(u) \in L^p$ (because, T' being bounded, $|T(s)| \le C + C|s|$), the proof is complete in the general case.

If we assume that $T(0) = 0$, then we can choose T_n such that $T_n(0) = 0$ (just apply an offset $T_n - T_n(0)$ to any sequence (T_n) you might have previously chosen). From Theorem 3.5.1 we get $T_n(u) \in W_0^{1,p}(\Omega)$. The preceding reasoning, based on the convergence of T_n, show that $T_n(u) \to T(u)$ in $W^{1,p}(\Omega)$, which gives that $T(u) \in W_0^{1,p}(\Omega)$. ■

3.7 [PDE] L^∞ estimates for elliptic PDEs

Ref: G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) 15 (1965), 189–258.

Theorem 3.7.1 ***(Stampacchia, 65)*** *Let $p > N$, $f \in L^p(\Omega)$ and u be the weak solution to*

$$\begin{cases} -\mathrm{div}(A\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

Then $u \in L^\infty(\Omega)$ and there exists C only depending on Ω, p, A such that $||u||_\infty \le C||f||_p$.

Proof

We take $S_k(s) = 0$ if $|s| \le k$, $S_k(s) = s - k$ if $s \ge k$ and $S_k(s) = s + k$ if $s \le -k$. Then, by Stampacchia's lemma, $S_k(u)$ is a valid test function. Plugging this in the weak formulation of (3.16) gives

$$\int fS_k(u) = \int A\nabla u \cdot \nabla(S_k(u)) = \int S_k'(u)A\nabla u \cdot \nabla u \ge \alpha \int S_k'(u)|\nabla u|^2.$$

But, since $\nabla(S_k(u)) = \mathbf{1}_{\{|u|\le k\}}\nabla u$, we have $S_k'(u)|\nabla u|^2 = (S_k'(u))^2|\nabla u|^2 = |\nabla(S_k(u))|^2$ and thus

$$||\nabla S_k(u)||_2^2 \le \frac{1}{\alpha}||f||_p||S_k(u)||_{p'}. \tag{3.17}$$

We now use Poincaré's inequality, the fact that $\nabla(S_k(u)) = S_k'(u)\nabla u = 0$ on $E_k = \{|u| \le k\}$ and Hölder's inequality with exponent $\frac{2}{p'}$ (notice that $p' \le N/(N-1) \le 2$) to get

$$\begin{aligned} ||S_k(u)||_{p'}^{p'} &\le C\int |\nabla(S_k(u))|^{p'} \\ &= C\int \mathbf{1}_{E_k}|\nabla(S_k(u))|^{p'} \\ &\le C\mathrm{mes}(E_k)^{1-\frac{p'}{2}}||\nabla(S_k(u))||_2^{p'}. \end{aligned}$$

Plugging this in (3.17) leads to

$$||\nabla S_k(u)||_2 \le C||f||_p\mathrm{mes}(E_k)^{\frac{1}{p'}-\frac{1}{2}}.$$

We use again Poincaré's inequality, in conjunction with Sobolev's embedding, to write

$$||S_k(u)||_{\frac{N}{N-1}} \le C||\nabla S_k(u)||_1 \le C\mathrm{mes}(E_k)^{\frac{1}{2}}||\nabla S_k(u)||_2,$$

which gives

$$||S_k(u)||_{\frac{N}{N-1}} \le C||f||_p\mathrm{mes}(E_k)^{1-\frac{1}{p}}.$$

Now, on $E_h = \{|u| \ge h\}$ with $h > k$ we have $|S_k(u)| \ge h - k$. Hence $||S_k(u)||_{\frac{N}{N-1}} \ge (h-k)\mathrm{mes}(E_h)^{\frac{N-1}{N}}$ and we obtain

$$\mathrm{mes}(E_h) \le \frac{C||f||_p^\beta}{(h-k)^\beta}\mathrm{mes}(E_k)^\gamma$$

with $\beta = \frac{N}{N-1} > 0$ and $\gamma = (1-\frac{1}{p})\frac{N}{N-1} > 1$ (because $p > N$). Lemma 3.7.2 shows that $\mathrm{mes}(E_H) = 0$ (i.e. $|u| \le H$ a.e.) for $H = C||f||_p$, which proves the L^∞ estimate. ■

Lemma 3.7.2 *Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be non-increasing and such that, whenever $h > k$, $F(h) \leq \frac{M}{(h-k)^\beta} F(k)^\gamma$ with M, $\beta > 0$ and $\gamma > 1$ not depending on h, k. Then $F(H) = 0$ for $H = C(2M)^{\frac{1}{\beta}} F(0)^{\frac{\gamma-1}{\beta}}$, where C only depends on β, γ.*

Proof

We construct a sequence (h_n) such that $F(h_n) \leq \frac{F(h_{n-1})}{2}$ and we prove that this sequence is bounded by H. Then, since F is non-increasing, $F(H) \leq F(h_n) \leq \frac{F(0)}{2^n}$ for all n, which shows that $F(H) = 0$.

Assume that h_n is constructed. If we take $h_{n+1} > h_n$ such that

$$\frac{M}{(h_{n+1} - h_n)^\beta} F(h_n)^{\gamma - 1} = \frac{1}{2}, \tag{3.18}$$

then we have $F(h_{n+1}) \leq \frac{F(h_n)}{2}$. The choice (3.18) gives, by induction,

$$h_{n+1} = h_n + (2M)^{\frac{1}{\beta}} F(h_n)^{\frac{\gamma-1}{\beta}} \leq h_n + (2M)^{\frac{1}{\beta}} \frac{F(0)^{\frac{\gamma-1}{\beta}}}{(2^n)^{\frac{\gamma-1}{\beta}}}$$

This shows that, for all n,

$$h_n \leq (2M)^{\frac{1}{\beta}} F(0)^{\frac{\gamma-1}{\beta}} \sum_{i=0}^{\infty} \frac{1}{(2^{\frac{\gamma-1}{\beta}})^n} =: H$$

which is finite since $2^{\frac{\gamma-1}{\beta}} > 1$. The proof is complete. ∎

Theorem 3.7.3 ***(Stampacchia, full version)*** *Under the assumptions of Theorem 3.7.1, there exists $\kappa \in (0, 1 - N/p]$ only depending on Ω, A, p such that the weak solution u to* (3.16) *is κ-Hölder continuous.*

3.8 [L] Duality in Lebesgue spaces

Let p, p' be conjugate. The function $T : L^{p'}(\Omega) \to (L^p(\Omega))'$ defined by

$$\langle T(f), g \rangle = \int fg$$

is well defined, linear and continuous (thanks to Hölder's inequality).

Theorem 3.8.1 *If $p \in [1, \infty)$, T is an isomorphism. Put it another way: $(L^p(\Omega))' = L^{p'}(\Omega)$.*

Remark 3.8.2 *If S is dense in L^p and $T : S \to \mathbb{R}$ is continuous for the L^p norm, then it can be extended to L^p and thus identified with an element of $L^{p'}$.*
Example: $S = C_c^\infty$, distributions continuous for some L^p norms...

3.9 [PDE] Interior regularity

Lemma 3.9.1 *Let $p > 1$, $u \in L^p(\mathbb{R}^N)$ and e_i a basis vector of $\mathbb{R}^N$. Then $\partial_i u \in L^p(\mathbb{R}^N)$ if and only if there exists $C > 0$ such that, for all $h \in \mathbb{R}$,*

$$\int_{\mathbb{R}^N} |u(x+he_i) - u(x)|^p \leq C|h|^p.$$

We can then take $C = ||\partial_i u||_p$.

Proof

Step 1: Using truncation and convolution, it is clear that any function $u \in L^p(\mathbb{R}^N)$ such that $\partial_i u \in L^p(\mathbb{R}^N)$ can be approximated in $L^p(\mathbb{R}^N)$ by functions $u_n \in C_c^\infty(\mathbb{R}^N)$ such that $\partial_i u_n \to \partial_i u$ in $L^p(\mathbb{R}^N)$. Hence, if we prove that, for $u \in C_c^\infty(\mathbb{R}^N)$, we have $\int_{\mathbb{R}^N} |u(x+he_i) - u(x)|^p \leq ||\partial_i u||_p |h|^p$, this density will prove that this estimate holds also for $u \in L^p(\mathbb{R}^N)$ such that $\partial_i u \in L^p(\mathbb{R}^N)$.

Assuming that u is smooth and compactly supported, we just write Taylor's formula $u(x+he_i) - u(x) = \int_0^1 \partial_i u(x + she_i) h\, ds$ and bound

$$|u(x+he_i) - u(x)| \leq |h| \int_0^1 |\partial_i u(x+she_i)|\, ds.$$

Taking the power p of this inequality, using Hölder's inequality (to get $(\int_0^1 |\partial u|)^p \leq \int_0^1 |\partial u|^p$) and using a change of variable $x + she_i \to x$ leads to the required estimate.

Step 2: the converse.

Let $\varphi \in C_c^\infty(\omega)$. By the Dominated Convergence Theorem, since $\frac{\varphi(x+he_i)-\varphi(x)}{h} \to \partial_i \varphi$ on $\mathbb{R}^N$ and remains bounded by $||\varphi'||_\infty$,

$$\int u \partial_i \varphi = \lim_{h\to 0} \int u(x) \frac{\varphi(x+he_i) - \varphi(x)}{h}.$$

A linear change of variable gives

$$\begin{aligned} \int u\partial_i\varphi &= \lim_{h\to 0} \frac{1}{h}\left(\int u(x-he_i)\varphi(x) - \int u(x)\varphi(x)\right) \\ &= \lim_{h\to 0} \int \frac{u(x-he_i) - u(x)}{h}\varphi(x). \end{aligned}$$

We now use Hölder's inequality and the assumption on u to get

$$\left|\int u\partial_i\varphi\right| \leq C||\varphi||_{p'}.$$

This shows that the linear form $C_c^\infty(\mathbb{R}^N) \to \int u\partial_i\varphi$ is continuous for the $L^{p'}(\mathbb{R}^N)$ norm and can therefore be extended as a linear form on $L^{p'}(\mathbb{R}^N)$ (because $p' < \infty$). But $(L^{p'})' = L^p$ and there exists therefore $g_i \in L^p(\mathbb{R}^N)$ such that, for all φ smooth compactly supported, $\int u\partial_i\varphi = \int g_i\varphi$, which precisely shows that $\partial_i u = -g_i \in L^p(\mathbb{R}^N)$ and concludes the proof. ■

We define $H^2(\Omega) = \{u \in L^2(\Omega) \mid \forall i,j\,,\ \partial_i\partial_j u \in L^2(\Omega)\}$ or, similarly, $H^2(\Omega) = \{u \in H^1(\Omega) \mid \forall i\,,\ \partial_i u \in H^1(\Omega)\}$. $H^2_{\mathrm{loc}}(\Omega)$ is the set of functions on Ω which are $H^2(\omega)$ for any $\omega \subset\subset \Omega$ or, similarly, the set of functions u such that, for all $\varphi \in C_c^\infty(\Omega)$, $\varphi u \in H^2(\mathbb{R}^N)$.

Theorem 3.9.2 ***(H^2 regularity in $\mathbb{R}^N$)*** *Assume that $A : \mathbb{R}^N \to M_N(\mathbb{R})$ is in $C_b^1(\mathbb{R}^N)$ and uniformly coercive and let $f \in L^2(\mathbb{R}^N)$. If $u \in H^1(\mathbb{R}^N)$ satisfy $-\mathrm{div}(A\nabla u) = f$ in $\mathcal{D}'(\mathbb{R}^N)$, then $u \in H^2(\mathbb{R}^N)$.*

Corollary 3.9.3 ***(Interior H^2 regularity for Ω bounded)*** *Let $\Omega \subset \mathbb{R}^N$ and $A : \Omega \to M_N(\mathbb{R})$ in C_b^1 and uniformly coercive. Let $f \in L^2(\Omega)$. If $u \in H_0^1(\Omega)$ is the weak solution to (3.16) then $u \in H^2_{\mathrm{loc}}(\Omega)$.*

Proof of Corollary 3.9.3

Let $\gamma \in C_c^\infty(\Omega)$. Let $v = \gamma u$, extended by 0 outside Ω. We also let $\widetilde{A} : \mathbb{R}^N \to M_N(\mathbb{R}^N)$ be a C_b^1 uniformly coercive matrix transformation such that $\widetilde{A} = A$ on a neighborhood of $\mathrm{supp}(\gamma)$ ([2]).

We can check ([3]), thanks to the fact that γ has a compact support in Ω, that $v \in H^1(\mathbb{R}^N)$ and that, in the sense of distributions *on* $\mathbb{R}^N$, $-\mathrm{div}(\widetilde{A}\nabla v) = -\widetilde{A}\nabla u \cdot \nabla\gamma - \gamma f - \widetilde{A}\nabla\gamma \cdot \nabla u - u\mathrm{div}(\widetilde{A}\nabla\gamma) =: F \in L^2(\mathbb{R}^N)$, where f, u and ∇u have been extended by 0 outside Ω.

Theorem 3.9.2 then shows that $v \in H^2(\mathbb{R}^N)$, and thus that $\gamma u \in H^2(\mathbb{R}^N)$. Taking γ equal to 1 on a neighborhood of some $\omega \subset\subset \Omega$ shows that $u \in H^2(\omega)$ and concludes the proof. ■

Proof of Theorem 3.9.2

We have, for all $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\int A\nabla u \cdot \nabla\varphi = \int f\varphi. \tag{3.19}$$

But $C_c^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$ (Theorem 3.6.1) and both sides of the previous equation are continuous with respect to φ for the H^1 norm. Hence (3.19) also holds for $\varphi \in H^1(\mathbb{R}^N)$.

Let, for some $i = 1, \ldots, N$, $\tau_h v(x) = v(x + he_i)$ and $D_h v = \frac{\tau_h v - v}{h}$. We have, by trivial changes of variables, whenever $v, w \in L^2$,

$$\int (D_h v) w = -\int v D_{-h} w. \tag{3.20}$$

Apply (3.19) with $\varphi = -D_{-h}D_h u \in H^1(\mathbb{R}^N)$. Since D_h and ∇ are commutative, this gives

$$-\int A\nabla u \cdot (D_{-h}D_h\nabla u) = -\int f D_{-h}D_h u \le ||f||_2 ||D_{-h}D_h u||_2.$$

[2] A possible construction of $\widetilde{A}$ is the following: take $\theta \in C_c^\infty(\Omega)$ with values in $[0,1]$ and which is equal to 1 on a neighborhood of $\mathrm{supp}(\gamma)$ and let $\widetilde{A} = \theta A + (1-\theta)\mathrm{Id}$ (where θA has been naturally extended by 0 outside Ω).

[3] Remember that if f is a scalar-valued function and V is a vector-valued function, then $\mathrm{div}(fV) = f\mathrm{div}(V) + V \cdot \nabla f$.

Using then (3.20) and Lemma 3.9.1, we get

$$\int D_h(A\nabla u)\cdot(D_h\nabla u) \leq ||f||_2||\,|\nabla D_h u|\,||_2. \tag{3.21}$$

But

$$\begin{aligned} D_h(A\nabla u)(x) &= \frac{A(x+he_i)\nabla u(x+he_i) - A(x)\nabla u(x)}{h} \\ &= A(x+he_i)\frac{\nabla u(x+he_i)-\nabla u(x)}{h} + \frac{A(x+he_i)-A(x)}{h}\nabla u(x) \\ &= A(x+he_i)D_h u(x) + \frac{A(x+he_i)-A(x)}{h}\nabla u(x) \end{aligned}$$

and so, by coercivity of A and the fact that $A \in C_b^1$,

$$\begin{aligned} \int D_h(A\nabla u)\cdot(D_h\nabla u) &= \int A(\cdot+he_i)(D_h\nabla u)\cdot(D_h\nabla u) \\ &\quad + \int \frac{A(\cdot+he_i)-A}{h}\nabla u\cdot(D_h\nabla u) \\ &\geq \alpha||\,|D_h\nabla u|\,||_2^2 - ||A'||_\infty||\,|\nabla u|\,||_2||\,|D_h\nabla u|\,||_2. \end{aligned}$$

Plugged in (3.21), this leads to

$$||\,|D_h\nabla u|\,||_2^2 \leq C|||D_h\nabla u|\,||_2$$

where C does not depend on h. Hence, $(D_h\nabla u)$ remains bounded in L^2 independently on h. Recalling the definition of D_h, Lemma 3.9.1 then shows that $\partial_i\nabla u \in L^2(\mathbb{R}^N)$. Since this is true for all direction i, this shows that $u \in H^2(\mathbb{R}^N)$. ■

Corollary 3.9.4 ***(Interior C^∞ smoothness)*** *Let $\Omega \subset \mathbb{R}^N$ and $A : \Omega \to M_N(\mathbb{R})$ in C_b^∞ and uniformly coercive. Let $f \in C_b^\infty(\Omega)$. If $u \in H_0^1(\Omega)$ is the weak solution to* (3.16) *then $u \in C^\infty(\Omega)$.*

Proof

The proof is made by showing, by induction on k, that $u \in H^k_{\text{loc}}(\Omega)$ and using the Sobolev embeddings.

Step 1: We prove that $u \in H^k_{\text{loc}}(\Omega)$ for all $k \geq 1$.

The proof is made by induction on k (the cases $k = 1$ and $k = 2$ are known). Let $\gamma \in C_c^\infty(\Omega)$ and $v = \gamma u \in H^{k-1}(\mathbb{R}^N)$. Using the same notations as in the proof of Corollary 3.9.3, we have, in $\mathcal{D}'(\mathbb{R}^N)$,

$$-\text{div}(\widetilde{A}\nabla v) = -\widetilde{A}\nabla u\cdot\nabla\gamma - \gamma f - \widetilde{A}\nabla\gamma\cdot\nabla u - u\text{div}(\widetilde{A}\nabla\gamma) := F \in H^{k-2}(\mathbb{R}^N)$$

(we have extended f, u and ∇u by 0 outside Ω, but since these functions are multiplied by γ or derivatives of γ, which have compact support in Ω, this extension by 0 do not create any issue for the regularity of F).

Taking a $(k-2)$-th derivative ∂^{k-2} of this equality, we see that $\partial^{k-2}v \in H^1(\mathbb{R}^N)$ satisfies, in $\mathcal{D}'(\mathbb{R}^N)$,

$$-\mathrm{div}(\widetilde{A}\nabla(\partial^{k-2}v)) = \partial^{k-2}F + G \in L^2(\mathbb{R}^N)$$

where G is a polynomial in derivatives of $\widetilde{A}$ (up to order $k-2$) and derivatives of v (up to order $k-3$). Theorem 3.9.2 then shows, since $\partial^{k-2}v \in H^1$, that $\partial^{k-2}v \in H^2(\mathbb{R}^N)$. Since this is true for any derivative of order $k-2$ of v, this shows that $\gamma u = v \in H^k(\mathbb{R}^N)$. The function γ being arbitrary in $C_c^\infty(\Omega)$, we deduce that $u \in H^k_{\mathrm{loc}}(\mathbb{R}^N)$ as required.

Step 2: $u \in C^\infty(\Omega)$.

We prove in fact that any function $\varphi \in H^k_{\mathrm{loc}}(\Omega)$ for k such that $k > N/2$ belongs to $C(\Omega)$. Since all derivatives, of any order, of u belong to such a $H^k_{\mathrm{loc}}(\Omega)$, this shows that all these derivatives are continuous and thus that u is smooth.

Let B be a ball whose closure is compact in Ω. We have $\varphi \in H^k(B)$ so the Sobolev embeddings show that $D^{k-1}\varphi \in H^1(B) \subset L^{2^*}(B)$, where D^{k-1} denotes the derivatives of order $k-1$ of φ and $p^* = \frac{Np}{N-p}$ when $p < N$. This shows that $D^{k-2}\varphi \in W^{1,2^*}(B)$ and, by the Sobolev embeddings, that $D^{k-2}\varphi \in L^{(2^*)^*}(B)$, that is $D^{k-3}\varphi \in W^{1,(2^*)^*}(B)$.

Interating this we get $\varphi \in W^{1,2^{(k-1)*}}(B)$ where $2^{(p)*} = (\cdots(2^*)^*\cdots)^*$ with p stars. But induction readily gives $2^{(p)*} = \frac{2N}{N-2p}$. The assumption on φ shows that we can k such that $k = [N/2]+1$, in which case $2^{(k-1)*} = \frac{2N}{N-2[N/2]} > N$ (because $[N/2] > (N/2)-1$). Hence $\varphi \in W^{1,r}(B)$ for some $r > N$ and Sobolev's embedding shows that $\varphi \in C(B)$. Since this is true for any ball in Ω, this concludes the proof. ■

A brief word on boundary regularity:

1. Transport the PDE to B_+,
2. Use the translations parallel to $x_N = 0$, they ensure that $u(x+he_i)$ remains in $H^1_0(B_+)$ for h small. They give $\partial_i(\nabla u) \in L^2(B_+)$ for $i < N$.
3. Get $\partial^2_N u \in L^2(B_+)$ from the PDE in distribution formulation.

3.10 [S] Trace theorems in Sobolev spaces

There is no "value on the boundary" for generic functions in $L^p(\Omega)$. More precisely, there is no linear continuous operator $\gamma : L^p(\Omega) \to L^p(\partial\Omega)$ that would extend the natural notion of value on the boundary of continuous functions, i.e. such that, for all $u \in C(\overline{\Omega})$, $\gamma(u) = u_{|\partial\Omega}$.

Exercise 3.10.1 Let $\Omega = (0,1)$. Find a sequence $(u_n) \subset C(\overline{\Omega})$ and $u \in C(\overline{\Omega})$ such that $u_n \to u$ in $L^p(\Omega)$ (with $p < \infty$) but $u_n(0) \not\to u(0)$.
Can you do the same for a sequence (u_n) that converges in $W^{1,p}(\Omega)$.

Theorem 3.10.2 ***(Trace theorem)*** *Let Ω be C^1. Then there exists a (unique) linear continuous operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that $\gamma(u) = u_{|\partial\Omega}$ whenever $u \in C^\infty(\overline{\Omega})$. γ is called the trace operator.*

Proof

Let $W^{1,p}_{B_N}(B_+) = \{u \in W^{1,p}(B_+)\,,\ \mathrm{supp}(u) \text{ is compact in } B_N\}$. We prove the existence of a trace $\gamma_0 : W^{1,p}_{B_N}(B_+) \to L^p(B_{N-1})$ and the trace for a generic C^1 open set is obtained by transporting this trace, in a similar way as when we define the extension operator.

Let us start from smooth functions. We take $\varphi \in C_c^\infty(B_N)$ and write

$$\varphi(x',0) = -\int_0^1 \partial_N \varphi(x',t)\,dt$$

(because $\varphi(x',1) = 0$ since $(x',1) \notin B_N$). Thus, from Hölder's inequality,

$$\int_{B_{N-1}} |\varphi(x',0)|^p\,dx' \le \int_{B_{N-1}} \left(\int_0^1 |\partial_N \varphi(x',t)|\,dt\right)^p dx' \le \int_{B_+} |\nabla\varphi|^p.$$

In other words,

$$||\varphi_{|B_{N-1}}||_{L^p(B_{N-1})} \le ||\varphi||_{W^{1,p}(B_+)}.$$

Letting $S = \{\varphi_{|B_+}\,,\ \varphi \in C_c^\infty(B_N)\}$, this estimate shows that the natural trace (for smooth functions) $\bar\gamma_0 : S \to L^p(B_{N-1})$ is linear continuous.

Extending $u \in W^{1,p}_{B_N}(B_+)$ to $\mathbb{R}^N_+$ by 0 outside B_+, using the extension operator $E_0 : W^{1,p}(\mathbb{R}^N_+) \to W^{1,p}(\mathbb{R}^N)$ and convolving the resulting function by smoothing kernels, we see that u can be approximated in $W^{1,p}(B_+)$ by functions $\varphi \in C_c^\infty(B_N)$. In other words, S is dense in $W^{1,p}_{B_N}(B_+)$ and, thus, the linear continuous operator $\bar\gamma_0$ can be extended in a unique linear continuous $\gamma_0 : W^{1,p}_{B_N}(B_+) \to L^p(B_{N-1})$, which concludes the proof. ■

Exercise 3.10.3 Prove the following integration-by-parts formula: if Ω is C^1, (p,p') are conjugate exponents and $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$ then, for all $i = 1,\dots,N$,

$$\int_\Omega u\partial_i v = \int_{\partial\Omega} \gamma(u)\gamma(v)\mathbf{n}_i - \int_\Omega v\partial_i u$$

where $\mathbf{n} = (\mathbf{n}_1,\dots,\mathbf{n}_N)$ is the unit normal to $\partial\Omega$ in the outward direction of Ω.

Exercise 3.10.4 Prove that $W^{1,p}_0(\Omega) = \ker(\gamma)$ (*Hint: only consider the case* $W^{1,p}_{B_N}(B_+)$ *and prove that* $\ker(\gamma_0) = W^{1,p}_0(B_+) = \overline{C_c^\infty(B_+)}^{W^{1,p}}$. *The inclusion* $\supset$ *is easy, since* γ_0 *is continuous and equal to* 0 *on* $C_c^\infty(B_+)$. *To prove* $\subset$, *use Exercise 3.10.3 to prove that if* $\gamma_0(u) = 0$ *then its extension to* $\mathbb{R}^N$ *by* 0 *outside* B_+ *belongs to* $W^{1,p}(\mathbb{R}^N)$, *and use then a shifted convolution kernel to approximate* u *by functions in* $C_c^\infty(B_+)$ *(remember that* u *has a compact support in* B_N*))*.

3.11 [PDE] Other boundary conditions in elliptic PDEs

The trick to consider elliptic PDEs with non-homogeneous Dirichlet boundary conditions

$$\begin{cases} -\mathrm{div}(A\nabla u) = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

is to take $w : \Omega \to \mathbb{R}$ such that $w_{|\partial\Omega} = g$ and to consider $v = u - w$, which satisfies

$$\begin{cases} -\mathrm{div}(A\nabla v) = f + \mathrm{div}(A\nabla w) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

This of course requires some regularity assumptions on g, to ensure the existence of w regular enough so that $\mathrm{div}(A\nabla w)$ is at least in L^2.

Other boundary conditions are of particular physical interest:

- Neumann boundary condition, which is a condition on the derivative of u on $\partial\Omega$: $A\nabla u \cdot \mathbf{n} = g$.

  When $g = 0$, this models isolated systems. Think about the heat equation: setting $A\nabla u \cdot \mathbf{n} = 0$ on $\partial\Omega$ consists in setting the heat flux on the boundary of Ω equal to 0, i.e. no heat is entering or leaving Ω.

- Fourier (or Robin) boundary condition, a mix of Dirichlet and Neumann together: $A\nabla u \cdot \mathbf{n} + \lambda u = g$, with $\lambda \geq 0$ most of the time.

  If $g = 0$, we say that the outer flux $-A\nabla u \cdot \mathbf{n}$ through $\partial\Omega$ should be positively proportional to u, which makes sense if you consider again the heat equation with an exterior temperature equal to 0 (if $u > 0$ on $\partial\Omega$, then heat should flow from the system and the outer flux should be positive).

  Of course, Neumann's conditions is a special case of Fourier condition, with $\lambda = 0$.

Let us consider the Fourier problem

$$\begin{cases} -\mathrm{div}(A\nabla u) = f & \text{in } \Omega, \\ A\nabla u \cdot \mathbf{n} + \lambda u = g & \text{on } \partial\Omega. \end{cases} \tag{3.22}$$

Multiplying the equation by φ and using (formal) integration by part, we arrive at

$$\int_\Omega A\nabla u \cdot \nabla\varphi + \int_{\partial\Omega} \lambda u\varphi = \int_\Omega f\varphi + \int_{\partial\Omega} g\varphi.$$

This motivates the weak form of (3.22):

$$\begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ \forall \varphi \in H^1(\Omega)\,, \ \displaystyle\int_\Omega A\nabla u \cdot \nabla\varphi + \int_{\partial\Omega} \lambda\gamma(u)\gamma(\varphi) = \int_\Omega f\varphi + \int_{\partial\Omega} g\gamma(\varphi). \end{cases} \tag{3.23}$$

If A is bounded uniformly elliptic and $\lambda \in L^\infty(\partial\Omega)$ is non-negative then, since $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ is continuous, the bilinear form

$$a(u, \varphi) = \int_\Omega A\nabla u \cdot \nabla\varphi + \int_{\partial\Omega} \lambda\gamma(u)\gamma(\varphi)$$

is continuous on $H^1(\Omega) \times H^1(\Omega)$. Assuming that $f \in L^2(\Omega)$ and that $g \in L^2(\partial\Omega)$, and still using the continuity of the trace γ, the linear form

$$l(\varphi) = \int_\Omega f\varphi + \int_{\partial\Omega} g\gamma(\varphi)$$

is continuous on $H^1(\Omega)$.
The only assumptions which remains to check, in order to apply Lax-Milgram's theorem and get existence and uniqueness to a solution of (3.23), is the coercivity of a on $H^1(\Omega)$. This requires the (uniform) positivity of λ or, in the case of the pure Neumann boundary conditions ($\lambda = 0$), the reduction of the space. In either case, we also need some additional inequalities in Sobolev spaces.

- Either we assume that $\lambda(x) \geq \lambda_0 > 0$ for all $x \in \partial\Omega$ and we use the following inequality, which controls the norm of u in Ω using its gradient and boundary value: there exists $C > 0$ such that

$$\text{for any } u \in H^1(\Omega)\,,\ ||u||_{L^2(\Omega)} \leq C||\nabla u||_{L^2(\Omega)} + C||\gamma(u)||_{L^2(\Omega)}.$$

- Or, if $\lambda = 0$ (pure Neumann boundary condition), noticing that $u + C$ is a solution whenever u is a solution, we see that we need to restrict the space of solutions to "kill" the constants in the kernel of the operator.

  We usually replace $H^1(\Omega)$ by $H^1_\star(\Omega) = \{u \in H^1(\Omega) : \int_\Omega u = 0\}$ in (3.23) and the coercivity of a on $H^1_\star(\Omega)$ is ensured by the Poincaré-Wirtinger inequality: there exists $C > 0$ such that

$$\text{for any } u \in H^1_\star(\Omega)\,,\ ||u||_{L^2(\Omega)} \leq C||\nabla u||_{L^2(\Omega)}.$$

4 Singular elliptic PDEs

4.1 [S] Weak compactness in Lebesgue and Sobolev spaces

The density of smooth functions in L^q for $q < \infty$ and the fact that any continuous function can be uniformly approximated by a polynomial on a compact set allow to prove that $L^q(\Omega)$ is separable.
Hence, if $p \in (1, \infty]$, we have $p' \in [1, \infty)$ and thus $L^p(\Omega) = (L^{p'}(\Omega))'$ is the dual space of a separable Banach space. As a consequence:

Corollary 4.1.1 *If $p > 1$ and (u_n) is a bounded sequence in $L^p(\Omega)$, then up to a subsequence we have $u_n \to u$ in L^p weak-$*$, meaning: for all $\varphi \in L^{p'}(\Omega)$,*

$$\int u_n \varphi \to \int u \varphi$$

Exercise 4.1.2 Example of a weakly converging sequence which does not strongly converges: oscillating function.

As a consequence, we also have weak compactness of bounded sequences in Sobolev spaces.

Corollary 4.1.3 *If $p > 1$ and (u_n) is a bounded sequence in $W^{1,p}(\Omega)$ (resp. $W^{1,p}_0(\Omega)$), then there exists $u \in W^{1,p}(\Omega)$ (resp. $u \in W^{1,p}_0(\Omega)$) such that, up to a subsequence, $u_n \to u$ in $W^{1,p}(\Omega)$ weak-$*$, that is to say: $u_n \to u$ and $\nabla u_n \to \nabla u$ in $L^p(\Omega)$ weak-$*$.*

4.2 [PDE] Linear elliptic PDEs with L^1 or measure right-hand side

References:

- G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) 15 (1965), 189–258.
- L. Boccardo, T. Gallouët, *Nonlinear elliptic equations with right-hand side measures*, Communications in Partial Differential Equations, 17 (1992), no. 3-4, 189–258.
- A. Prignet, Remarks on existence and uniqueness of solutions of elliptic problems with right-hand side measures, Rendiconti di Matematica 15 (1995), 321–337.

We are interested here in the usual elliptic problem

$$\begin{cases} -\mathrm{div}(A\nabla u) = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

with A a uniformly elliptic matrix transformation and $f \in L^1(\Omega)$ or even f a measure on Ω.
Multiplying the equation by $\varphi \in C_c^\infty(\Omega)$ and integrating by parts, we get the basics for the weak formulation:

$$\int A\nabla u \cdot \nabla\varphi = \int \varphi\, df.$$

The right-hand side makes sense even if f is a measure. To give a sense to the left-hand side, we need at least $u \in W_0^{1,1}(\Omega)$, the index 0 being here to take into account the boundary condition $u = 0$ on $\partial\Omega$. In fact, we will get a solution in a better space.

Theorem 4.2.1 ***(Stampacchia 65, Boccardo-Gallouët 92)*** *Let $A : \Omega \to M_N(\mathbb{R})$ be uniformly elliptic and f be a bounded Radon measure on Ω. Then there exists a unique solution to* (4.1) *in the sense*

$$\begin{cases} u \in W_0^{1,q}(\Omega) \text{ for all } q < \frac{N}{N-1}\,, \\ \forall \varphi \in C_c^\infty(\Omega)\,,\ \displaystyle\int A\nabla u \cdot \nabla\varphi = \int \varphi\, df. \end{cases} \tag{4.2}$$

Remark 4.2.2 *The solution is not unique in general (cf Prignet 95). But we can define another notion of solution that would give existence and uniqueness (Stampacchia 65). However, this notion of solution (and the technique used to prove its existence and uniqueness) does not easily extend to non-linear equations, whereas* (4.2) *does.*

This existence of a solution to (4.2) is obtained by passing to the limit in regularised problems. This is a classical way of proving existence of solution to singular PDEs. Here, the regularisation consists in approximating (in a proper sense) f by $f_n \in L^2(\Omega)$, in taking u_n the usual weak solution to (4.1) with $f = f_n$ and proving some estimates on (u_n) which allow to show that this sequence converge to some u solution to (4.2).
As for the L^∞ estimate (Theorem 3.7.1), the trick to get estimates is to use functions $\varphi(u)$ in the weak formulation of the problem, for well-chosen φ. Here are some generic tips on how to choose φ:

- *φ should be non-decreasing.*

  This allows to write $A\nabla u \cdot \nabla(\varphi(u) =)\varphi'(u)A\nabla u \cdot \nabla u \geq \alpha\varphi'(u)|\nabla u|^2$.

- *The choice of $\varphi(u)$ as test function will give estimate on $\psi(u)$ in H^1, where $\psi = \int_0^s \sqrt{\varphi'(t)}\,dt$.*

  Indeed, $\varphi'(u)|\nabla u|^2 = \psi'(u)^2|\nabla u|^2 = |\nabla(\psi(u))|^2$.

- *φ should be taken so that $\varphi(u)$ makes sense with respect to the expected regularity for all the terms in the formulation, and/or such that it allows to easily bound some of these terms.*

  This is the hardest part, for which there is no generic trick, just experience that helps understand how to adjust φ with respect to the given equation.

Proof

Step 1: approximating f. We prove that there exists $f_n \in L^\infty(\Omega)$ such that $f_n \to f$ in the weak-$*$ sense of measures, i.e. for all $\varphi \in C_c(\Omega)$, $\int f_n\varphi\,dx \to \int \varphi\,df$.

Let $n \geq 1$ and $(A_k^n)_{k=1,\dots,N_n}$ be a partition of Ω in sets of diameter $\leq 1/n$. We define $f_n : \Omega \to \mathbb{R}$ by $f_n|A_k^n = f(A_k^n)/\mathrm{meas}(A_k^n)$ where meas is the classical Lebesgue measure. Then f_n is a bounded function and, for any continuous φ,

$$\begin{aligned}
\int f_n\varphi\,dx - \int \varphi\,df &= \sum_{k=1}^{N_n}\left(\frac{1}{\mathrm{meas}(A_k^n)}\int_{A_k^n}\varphi\right)f(A_k^n) - \sum_{k=1}^{N_n}\int_{A_k^n}\varphi\,df \\
&= \sum_{k=1}^{N_n}\int_{A_k^n}(\varphi_n - \varphi)\,df
\end{aligned}$$

where $\varphi_n : \Omega \to \mathbb{R}$ is piecewise constant equal to $\frac{1}{\mathrm{meas}(A_k^n)}\int_{A_k^n}\varphi$ on A_k^n. Denoting by $\omega_\varphi(h) = \sup_{|x-y|\leq h}|\varphi(x) - \varphi(y)|$ the modulus of continuity of φ (which tends to 0 as $h \to 0$, because φ is uniformly continuous), we have, since $\mathrm{diam}(A_k^n) \leq 1/n$, for all $x \in A_k^n$,

$$|\varphi_n(x) - \varphi(x)| = \left|\frac{1}{\mathrm{meas}(A_k^n)}\int_{A_k^n}(\varphi(y) - \varphi(x))\,dy\right| \leq \omega_\varphi(1/n).$$

Hence, as $n \to \infty$,

$$\left|\int f_n\varphi\,dx - \int \varphi\,df\right| \leq \omega_\varphi(1/n)\sum_{k=1}^{N_n}\int_{A_k^n} d|f| \leq |f|(\Omega)\omega_\varphi(1/n) \to 0.$$

Let us also remark that the sequence (f_n) is bounded in L^1:

$$\int |f_n| = \sum_{k=1}^{N_n}|f(A_k^n)| \leq \sum_{k=1}^{N_n}|f|(A_k^n) = |f|(\Omega).$$

We have $f_n \in L^\infty(\Omega) \subset L^2(\Omega)$ and there exists therefore a solution to the variational elliptic equation (3.7) with $f = f_n$. Let u_n be this solution.

Step 2: estimates on u_n.

Given the discussion before this proof, if we take $\varphi(u_n)$ as test function in the equation satisfied by u_n, we get

$$\alpha \int \varphi'(u_n)|\nabla(u_n)|^2 \leq \int f_n \varphi(u_n).$$

with $\psi(s) = \int_0^s \sqrt{\varphi'}$. Since (f_n) is only bounded in L^1, to get an estimate on the right-hand side we need to have $(\varphi(u_n))$ bounded in L^∞, i.e. φ bounded. Since the estimate involves φ', the choice of φ should be made through its derivative and getting φ bounded means that φ' must be integrable on $\mathbb{R}$.

Let $\varphi'(s) = \frac{1}{(1+|s|)^\nu}$ with $\nu > 1$, one of the simplest type of integrable functions. We then take $\varphi(s) = \int_0^s \frac{dt}{(1+|t|)^\nu}$ (which is a valid function for the chain-rule) and we therefore get

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\nu} \leq C \tag{4.3}$$

with C not depending on n.

From this estimate, we want to deduce an estimate on the sequence (u_n) in some Sobolev space $W_0^{1,q}(\Omega)$ with $q < 2$ (each u_n belongs to such a space). We write, with $a > 0$ to be chosen and using Hölder's inequality with exponent $2/q > 1$ and $(2/q)' = \frac{2}{2-q}$,

$$\int |\nabla u_n|^q = \int (1+|u_n|)^a \frac{|\nabla u_n|^q}{(1+|u_n|)^a} \leq ||(1+|u_n|)^a||_{L^{\frac{2}{2-q}}(\Omega)} \left(\int \frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{2a}{q}}} \right).$$

We choose a such that $\frac{2a}{q} = \nu$, i.e. $a = \frac{q\nu}{2}$. Owing to (4.3) and since $(1+|u_n|)^a \leq 2^a(1+|u_n|^a)$ (separate the cases $|u_n| \leq 1$ and $|u_n| \geq 1$ to prove this), this gives

$$\int |\nabla u_n|^q \leq C + C \left(\int |u_n|^{\frac{q\nu}{2-q}} \right)^{\frac{2-q}{2}} \tag{4.4}$$

where C does not depend on n. Let us now choose $\nu > 1$ such that $\frac{q\nu}{2-q} = q^* = \frac{Nq}{N-q}$. It is possible provided that $\nu = \frac{(2-q)N}{N-q} > 1$, that is

$$q < \frac{N}{N-1}. \tag{4.5}$$

With such a choice and thanks to Sobolev embeddings, (4.4) gives

$$||u_n||^q_{W_0^{1,q}} \leq C + C||u_n||^{q\frac{\nu}{2}}_{W_0^{1,q}}. \tag{4.6}$$

If $N > 2$ then $\frac{\nu}{2} = \frac{(2-q)N}{2(N-q)} < 1$ and the exponent q of $||u_n||_{W_0^{1,q}}$ in the left-hand side is larger than the exponent $q\frac{\nu}{2}$ in the right-hand side. This shows that $||u_n||_{W_0^{1,q}} \leq R$ with R only depending on C and q (there exists $R = R(C,q)$ such that the function $s \to s^q - C - Cs^{q\frac{\nu}{2}}$ is positive for $s \geq R = R(C,q)$).

If $N=2$, we have to modify a bit the preceding estimates. We take $\nu>1$ such that $\frac{q\nu}{2-q}<q^*=\frac{2q}{2-q}$ (take for example $\nu=3/2$). Then, for any $\varepsilon>0$ there exists $c(\varepsilon,q,\nu)$ such that $|s|^{\frac{q\nu}{2-q}}\leq c(\varepsilon,q,\nu)+\varepsilon|s|^{q^*}$ for all $s\in\mathbb{R}$ (indeed, the function $s\to|s|^{\frac{q\nu}{2-q}}-\varepsilon|s|^{q^*}$ is continuous and tends to $-\infty$ at $\pm\infty$, so it is bounded above on $\mathbb{R}$). Hence, (4.4) gives

$$\begin{aligned}\int|\nabla u_n|^q &\leq C+C\left(\int c(\varepsilon,q,\nu)+\varepsilon|u_n|^{q^*}\right)^{\frac{2-q}{2}}\\ &\leq C+C\left(\mathrm{meas}(\Omega)c(\varepsilon,q,\nu)+\varepsilon\int|u_n|^{q^*}\right)^{\frac{2-q}{2}}\\ &\leq C+2^{\frac{2-q}{2}}C\mathrm{meas}(\Omega)^{\frac{2-q}{2}}c(\varepsilon,q,\nu)^{\frac{2-q}{2}}+2^{\frac{2-q}{2}}C\varepsilon^{\frac{2-q}{2}}\left(\int|u_n|^{q^*}\right)^{\frac{2-q}{2}}.\end{aligned}$$

Using then Sobolev embedding, we deduce

$$||u_n||^q_{W_0^{1,q}}\leq C(\varepsilon,q,\nu)+C'(q,\nu)\varepsilon^{\frac{2-q}{2}}||u_n||^{q^*\frac{2-q}{2}}_{W_0^{1,q}}.$$

Since $N=2$ here, we have $q^*\frac{2-q}{2}=q$ and both exponents of $||u_n||_{W_0^{1,q}}$ in the preceding estimate are the same. But $q<2$ (this is (4.5)) and we can therefore choose $\varepsilon>0$ such that $C'(q,\nu)\varepsilon^{\frac{2-q}{2}}=\frac{1}{2}$, which gives $||u_n||^q_{W_0^{1,q}}\leq 2C(\varepsilon,q,\nu)$.

Hence, in any case, we obtain a bound on (u_n) in $W_0^{1,q}(\Omega)$ for any $q<\frac{N}{N-1}$.

Step 3: passing to the limit.

Since the equation is linear, passing to the limit is very easy. Using Corollary 4.1.3 and a diagonal extraction we can find $u\in\cap_{q<\frac{N}{N-1}}W_0^{1,q}(\Omega)$ such that $u_n\to u$ in $W^{1,q}(\Omega)$ weak-$*$ for any $q<\frac{N}{N-1}$.

For all $\varphi\in C_c^\infty(\Omega)$ we have

$$\int f_n\varphi=\int A\nabla u_n\cdot\nabla\varphi=\int\nabla u_n\cdot(A^T\nabla\varphi).$$

Using Step 1 we can pass to the limit in the left-hand side. Fixing $q\in(1,\frac{N}{N-1})$, we have $A^T\nabla\varphi\in L^{q'}(\Omega)$ and therefore the weak-$*$ convergence of ∇u_n to ∇u in $L^q(\Omega)$ allows to pass to the limit in the right-hand side, thus proving that u satisfies (4.2). ■

Remark 4.2.3 *Let $p>N$. Then any function $\varphi\in W_0^{1,p}(\Omega)$ is a limit, in this space, of functions $(\varphi_n)\subset C_c^\infty(\Omega)$. Moreover, the Sobolev embedding shows that $\varphi\in C(\overline{\Omega})$ and that $\varphi_n\to\varphi$ in this space. If u satisfies* (4.2) *then, since $A\nabla u\in L^{p'}(\Omega)$ (because $p'<\frac{N}{N-1}$), we can apply* (4.2) *with φ_n and pass to the limit $n\to\infty$ to see that u also satisfies*

$$\left\{\begin{array}{l}u\in W_0^{1,q}(\Omega)\ \text{for all}\ q<\frac{N}{N-1}\,,\\ \forall\varphi\in\cup_{p>N}W_0^{1,p}(\Omega)\,,\ \displaystyle\int A\nabla u\cdot\nabla\varphi=\int\varphi\,df.\end{array}\right.$$

Note that the regularity $u \in \cap_{q<N} W_0^{1,q}(\Omega)$ for the solution to (4.2) is optimal in general, in the sense that there exists some $f \in L^1$ such that the solution to (4.2) does not belong to $W_0^{1,\frac{N}{N-1}}(\Omega)$.

To see that, consider that $u \in W_0^{1,\frac{N}{N-1}}(\Omega)$ for some $f \in L^1$. Then $\varphi \to \int A\nabla u \cdot \nabla\varphi$ is continuous for the $W_0^{1,N}$ norm, so the left-hand side $\varphi \to \int f\varphi$ should also be continuous (and defined!) for $\varphi \in W_0^{1,N}(\Omega)$. The following exercise shows that there exists $\varphi \in W_0^{1,N}$ and $f \in L^1$ such that $f\varphi$ is not integrable, thus showing that for this f, the solution u to (4.2) cannot belong to $W^{1,\frac{N}{N-1}}$.

Exercise 4.2.4 Let B be the unit ball in $\mathbb{R}^2$ and let $\varphi(x) = |\ln(|x|)|^\gamma$ for some $\gamma \in \mathbb{R}$ ($|x|$ is the Euclidean norm of $x \in \mathbb{R}^2$).

1. Show that $\varphi \in L^2(B)$ *(Hint: remember the polar change of variable $\int_B f(x)\,dx = \int_0^1 \int_0^{2\pi} \rho f(\rho\cos(\theta), \rho\sin(\theta))\,d\rho\,d\theta$ and remember that $\rho \to \frac{1}{\rho^\alpha |\ln(\rho)|^\beta}$ is integrable on $(0,1)$ if and only if $\alpha < 1$ or $\alpha = 1$ and $\beta > 1$).*

2. φ is smooth on $B\backslash\{0\}$ with classical derivatives $\partial_i^c\varphi = -\gamma|\ln(|x|)|^{\gamma-1}\frac{x_i}{|x|^2}$. Prove that these classical derivatives belong to $L^2(B)$ if and only if $\gamma < 1/2$.

3. For such γ, show that these classical derivatives are the distribution derivatives on B, i.e. that for all $\psi \in C_c^\infty(B)$,
$$\int_B \varphi\partial_i\psi = -\int_B (\partial_i^c\varphi)\psi.$$
(Hint: to remove the singularity, let B_ε be the ball of center 0 and radius ε and write first $\int_B \varphi\partial_i^c\psi = \lim_{\varepsilon\to 0}\int_{B\backslash B_\varepsilon} \varphi\partial_i\psi$, which is valid because $\varphi\partial_i\psi$ is integrable on B. Use then Stokes formula to get $\int_{B\backslash B_\varepsilon}\varphi\partial_i\psi = \int_{S_\varepsilon}\varphi\psi n_i - \int_{B\backslash B_\varepsilon}(\partial_i^c\varphi)\psi$, where S_ε is the circle of radius ε, and pass then to the limit $\varepsilon \to 0$ in this equality, estimating the boundary thanks to the expression of φ.)

4. Show that a choice of γ gives $\varphi \in H_0^1(B)\backslash L^\infty(B)$ and construct $f \in L^1(B)$ such that $f\varphi \notin L^1(B)$.

5 Non-linear elliptic PDEs

5.1 [S] Strong compactness in Lebesgue and Sobolev spaces

Theorem 5.1.1 ***(Kolmogorov's theorem: strong compactness in L^p)*** *Let $p < \infty$ and $\mathcal{F} \subset L^p(\Omega)$ such that there exists some extension operator $E : L^p(\Omega) \to L^p(\mathbb{R}^N)$ satisfying*

1. *For all $f \in \mathcal{F}$, $Ef = f$ on Ω,*

2. *$\{Ef\ ,\ f \in \mathcal{F}\}$ is bounded in $L^p(\mathbb{R}^N)$,*

3. Denoting $\tau_h g(x) = g(x+h)$ the translation of a vector h,

$$\lim_{h\to 0}\sup_{f\in\mathcal{F}} ||\tau_h Ef - Ef||_{L^p(\mathbb{R}^N)} = 0.$$

Then $\mathcal{F}$ is relatively compact in $L^p(\Omega)$.

Classical usage: Ef = extension of f by 0 outside Ω. If (f_n) is bounded in $L^p(\Omega)$ and satisfies $\sup_n ||\tau_h f_n - f_n||_p \to 0$ as $h\to 0$, then we can extract a subsequence of (f_n) which converges in L^p.

Corollary 5.1.2 ***(Rellich embedding)*** *Let Ω be C^1 and $p\in[1,\infty]$. Then $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, meaning that the bounded sets in $W^{1,p}(\Omega)$ are relatively compact in $L^p(\Omega)$.*
In other words, if the sequence (u_n) is bounded in $W^{1,p}(\Omega)$, then up to a subsequence $u_n \to u$ strongly in $L^p(\Omega)$.

Proof

Let us take a linear continuous extension operator $E: W^{1,p}(\Omega)\to W^{1,p}(\mathbb{R}^N)$ defined by Theorem 3.6.10 and let F be bounded in $W^{1,p}(\Omega)$. Then $E(F)$ is bounded in $W^{1,p}(\mathbb{R}^N)$ so Items 1 and 2 in Kolmogorov's theorem are satisfied.

We have, for $v\in W^{1,p}(\mathbb{R}^N)$ and $h=(h_1,\dots,h_N)\in\mathbb{R}^N$,

$$\tau_h v - v = \sum_{i=1}^N \tau_{(h_1,\dots,h_i,0,\dots,0)}v - \tau_{(h_1,\dots,h_{i-1},0,\dots,0)}v$$

and thus, by Lemma 3.9.1 and linear change of variables,

$$||\tau_h v - v||_{L^p(\mathbb{R}^N)} \le \sum_{i=1}^N ||\tau_{(h_1,\dots,h_{i-1},0,\dots,0)}\partial_i v||_{L^p(\mathbb{R}^N)}|h_i| = \sum_{i=1}^N ||\partial_i v||_{L^p(\mathbb{R}^N)}|h_i|$$

(in fact, we can the slighlty better estimate $||\tau_h v - v||_p \le ||\,|\nabla v|\,||_p|h|$ by re-doing the estimate in Step 1 of the proof of Lemma 3.9.1 with a generic vector $h\in\mathbb{R}^N$ instead of a vector along the i-th direction). Since $E(F)$ is bounded in $W^{1,p}(\mathbb{R}^N)$, this estimates shows that $||\tau_h Ef - Ef||_{L^p(\mathbb{R}^N)} \le C|h|$ with C not depending on f, which proves Item 3 in Kolmogorov's lemma and thus concludes the proof: any set F bounded in $W^{1,p}(\Omega)$ is relatively compact in $L^p(\Omega)$. ■

5.2 [PDE] Quasi-linear equations

We consider here the problem

$$\begin{cases} -\mathrm{div}(A(x,u(x))\nabla u(x)) = f(x) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{5.1}$$

where Ω is a C^1 open set in $\mathbb{R}^N$ and $f \in L^2(\Omega)$. The difference is that the matrix A now depends on the unknown u, and we assume that

$$\begin{gathered}(x,s) \in \Omega \times \mathbb{R} \to A(x,s) \in M_N(\mathbb{R}) \text{ is bounded, measurable w.r.t } x\text{, continuous w.r.t. } s \\ \text{and there exists } \alpha > 0 \text{ s.t. } A(x,s)\xi \cdot \xi \geq \alpha|\xi|^2 \text{ for all } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N.\end{gathered} \tag{5.2}$$

The problem (5.1) is non-linear and Lax-Milgram theorem therefore cannot be directly applied to get a solution. We will however be able to prove at least existence of a weak solution (uniqueness is another matter).

Theorem 5.2.1 ***(Existence of a solution to*** (5.1)***)*** *Let $f \in L^2(\Omega)$ and A satisfy* (5.2)*. Then there exists a weak solution to* (5.1) *in the sense:*

$$u \in H_0^1(\Omega) \text{ satisfies, for all } v \in H_0^1(\Omega), \quad \int_\Omega A(x,u(x))\nabla u(x) \cdot \nabla v(x) = \int_\Omega fv. \tag{5.3}$$

The trick to prove Theorem (5.2.1), a very classical one to handle non-linear problems, is to come back to a *linear* problem by "freezing" the annoying non-linearity.
If $v \in L^2(\Omega)$ is a fixed function, then $A(\cdot, v(\cdot))$ is a bounded coercive matrix so the problem

$$\begin{cases} -\text{div}(A(\cdot,v(\cdot))\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.4}$$

has a (unique) weak solution $u \in H_0^1(\Omega)$. The trick now is go from here to the case where $u = v$ (a weak solution to (5.4) with $u = v$ is precisely a solution to (5.3)).
The idea is to define the function $T : v \in L^2 \to u \in L^2$, with u the weak solution to (5.4) corresponding to v, and to use a fixed point theorem to prove that T has a fixed point. This gives u such that $T(u) = u$, which precisely means that u is the weak solution to (5.4) with $v = u$.
Here, T is defined on an infinite dimensional vector space so we would need to invoke a fixed point theorem in an infinite dimensional space. The classical one, the contraction mapping theorem, does not seem to apply easily here. There are other fixed points theorems in infinite dimensional vector spaces (such as the Schauder or Schaefer fixed points) – and in fact a much encompassing tool, the topological degree, which gives the existence of fixed points in a number of instances. But we will not need these fixed point theorems here and we can complete the proof just by using the more classical Brouwer's fixed point theorem (in finite dimensional spaces):

Theorem 5.2.2 ***(Brouwer)*** *Let E be a finite dimensional normed space, B_R^E be the closed ball of center 0 and radius R in E and $T : B_R^E \to B_R^E$ be continuous. Then T has a fixed point in B_R: there exists $x \in B_R$ such that $T(x) = x$.*

For a proof, see e.g. Evans (Chapter 8).

Proof of Theorem 5.2.1

Step 1: a finite dimensional problem.

Since H^1_0 is separable, we can write it as the closure of the non-decreasing union of a countable number of finite dimensional vector spaces: $H^1_0 = \overline{\cup_{n\geq 1} E_n}$ with $\dim(E_n) < \infty$ and $E_n \subset E_{n+1}$. Each E_n is endowed with the H^1_0 norm. Let us consider the projection of the weak form of Problem (5.4) on E_n: if $v \in E_n$ is fixed, we look for a solution to

$$u \in E_n \text{ such that, for all } \varphi \in E_n\,, \int A(x,v)\nabla u \cdot \nabla \varphi = \int f\varphi. \tag{5.5}$$

Since $a(u,\varphi) = \int A(x,v)\nabla u \cdot \nabla\varphi$ is a continuous bilinear form on E_n (any bilinear form on a finite dimensional vector space is continuous) and coercive (because $A(\cdot, v)$ is coercive), and $l(\varphi) = \int f\varphi$ is linear continuous on E_n, the Lax-Milgram theorem applied in E_n gives existence and uniqueness, for any $v \in E_n$, of u satisfying (5.5).

Let us denote T the transformation $E_n \to E_n$ defined by $T(v) = u$. We want to show that there exists $R > 0$ such that T is continuous $B^{E_n}_R \to B^{E_n}_R$.

First, taking $\varphi = u \in E_n$ in (5.5), we find

$$\alpha||u||^2_{H^1_0} \leq \int A(x,v)\nabla u \cdot \nabla u = \int fu \leq ||f||_2||u||_2 \leq \mathrm{diam}(\Omega)||f||_2||u||_{H^1_0},$$

hence

$$||u||_{H^1_0} \leq \frac{\mathrm{diam}(\Omega)}{\alpha}||f||_2 =: R. \tag{5.6}$$

T therefore sends the whole space E_n into $B^{E_n}_R$.

To prove the continuity of T, we take $v_k \to v$ in E_n, let $u_k = T(v_k)$, $u = T(v)$, and prove that $u_k \to u$ in E_n. As in Exercise 2.2.12, we will in fact prove the convergence of a subsequence of (u_k) towards u and deduce from this the convergence of the whole sequence (the reasoning being applicable starting from any subsequence of (u_k)).

From (5.6), also satisfied by u_k, we see that u_k is bounded for the H^1_0 in E_n and thus, this space being finite dimensional, up to a subsequence we can assume that $u_k \to w$ for the H^1_0 norm in E_n. It remains to prove that $w = u = T(v)$. Let us write the weak formulation of (5.5) for u_k:

$$\forall \varphi \in E_n\,, \int A(x,v_k)\nabla u_k \cdot \nabla\varphi = \int f\varphi.$$

(v_k) converging to v in H^1_0, up to a subsequence we can assume that $v_k \to v$ a.e., which shows that $A(x,v_k) \to A(x,v)$ a.e. since A is continuous w.r.t. its second variable. A being bounded, the dominated convergence theorem shows that $A(x,v_k)^T\nabla\varphi \to A(x,v)^T\nabla\varphi$ in L^2 for all $\varphi \in H^1_0$. Since $\nabla u_k \to \nabla w$ in L^2 (because $u_k \to w$ in H^1_0), we can pass to the limit in $\int A(x,v_k)\nabla u_k \cdot \nabla\varphi = \int \nabla u_k \cdot (A(x,v_k)^T\nabla\varphi)$ and get

$$\forall \varphi \in E_n\,, \int A(x,v)\nabla w \cdot \nabla\varphi = \int f\varphi.$$

This precisely proves that $w = T(v)$ and concludes the proof of the continuity of T.

By Brouwer's fixed point theorem, T has a fixed point u which satisfies (5.5) with $v = u$.

Step 2: passing to the limit.

The preceding steps shows the existence, for all n, of $u_n \in E_n$ such that

$$\forall \varphi \in E_n\,, \int A(x, u_n)\nabla u_n \cdot \nabla\varphi = \int f\varphi.$$

We also know that u_n satisfies (5.6), which shows that the sequence (u_n) is bounded in H^1_0. By Rellich's theorem and Corollary 4.1.3, up to a subsequence we have $u_n \to u$ strongly in L^2 and weakly-$*$ in H^1_0. The strong convergence in L^2 ensures that, taking another subsequence if need be, $u_n \to u$ a.e.

Let us fix $l \geq 1$ and $\varphi \in E_l$. For any $n \geq l$, we have $\varphi \in E_n$ and therefore

$$\int A(x, u_n)\nabla u_n \cdot \nabla\varphi = \int f\varphi.$$

Since $u_n \to u$ a.e., the same arguments as in the previous step show that $A(x, u_n)^T\nabla\varphi \to A(x, u)^T\nabla\varphi$ strongly in L^2. The convergence $\nabla u_n \to \nabla u$ weakly-$*$ in L^2 and Lemma 5.2.4 allows to pass to the limit in $\int A(x, u_n)\nabla u_n \cdot \nabla\varphi = \int \nabla u_n \cdot (A(x, u_n)^T\nabla\varphi)$ to prove that

$$\int A(x, u)\nabla u \cdot \nabla\varphi = \int f\varphi.$$

This equality being valid for any $\varphi \in E_l$ for any $l \geq 1$, it is valid for $\varphi \in \cup_{l\geq 1}E_l$. But the left- and right-hand sides of this equality are continuous w.r.t. φ for the H^1_0, and the equality therefore holds in fact for any $\varphi \in \overline{\cup_{l\geq 1}E_l} = H^1_0$, which proves that u is a solution to (5.3). The proof is complete. ■

Remark 5.2.3 *The trick which consists in writing $H^1_0 = \overline{\cup_{n\geq 1}E_n}$ for some finite-dimensional spaces E_n and then to "solve the PDE" (i.e. apply Lax-Milgram theorem) in E_n is the basic idea of a number of numerical methods for elliptic equations (the Galerkin methods).*

Lemma 5.2.4 ***(Weak-strong convergence)*** *Let $p > 1$ and $(f_n), (g_n)$ be sequences such that (f_n) is bounded in L^p, $f_n \to f$ in L^p weak-$*$ and $g_n \to g$ in $L^{p'}$ strong. Then $\int f_n g_n \to \int fg$.*

Remark 5.2.5 *We do not really need to assume that (f_n) is bounded in L^p, since the Banach-Steinhauß theorem ensures that any sequence (f_n) weakly-$*$ converging in L^p is bounded in this space.*

Remark 5.2.6 *The result is false for a product of weakly-$*$ converging sequences: think about two oscillating functions shifted so that their product is equal to* 0.

Proof of Lemma 5.2.4

We write

$$\int f_n g_n - \int fg = \int f_n(g_n - g) + \int (f_n - f)g.$$

Since $f_n \to f$ weakly-$*$ in L^p and $g \in L^{p'}$ we have $\int (f_n - f)g \to 0$. We then estimate the first term in the right-hand side by using Hölder's inequality

$$\left|\int f_n(g_n - g)\right| \le ||f_n||_p ||g_n - g||_{p'}$$

and this tends to 0 as the sequence (f_n) is bounded in L^p and $g_n \to g$ in $L^{p'}$. ■

5.3 [PDE] Leray-Lions operators

Reference:

- Leray J., Lions J.L., *Quelques résultats de Višik sur les problèmes elliptiques semi-linéaires par les méthodes de Minty et Browder*. Bull. Soc. Math. France 93 (1965), 97–107.
- Evans, chap 9.

Let us go for full non-linearity! We consider

$$\left\{\begin{array}{ll} -\mathrm{div}(a(\nabla u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{array}\right. \tag{5.7}$$

where, for some $p \in (1, \infty)$, $f \in L^{p'}(\Omega)$ and a satisfies

$$\text{Continuity: } a : \mathbb{R}^N \to \mathbb{R}^N \text{ is continuous,} \tag{5.8}$$

$$\text{Growth: there exists } C > 0 \text{ such that } |a(\xi)| \le C|\xi|^{p-1} \text{ for all } \xi \in \mathbb{R}^N, \tag{5.9}$$

$$\text{Coercivity: there exists } \alpha > 0 \text{ such that } a(\xi)\cdot\xi \ge \alpha|\xi|^p \text{ for all } \xi \in \mathbb{R}^N, \tag{5.10}$$

$$\text{Monotony: } (a(\xi) - a(\eta))\cdot(\xi - \eta) \ge 0 \text{ for all } \xi, \eta \in \mathbb{R}^N. \tag{5.11}$$

Exercise 5.3.1 Show that if a satisfies the continuity and growth assumptions then $F \in (L^p(\Omega))^N \to a(F) \in L^{p'}(\Omega)^N$ is well-defined and continuous (*Hint: reason as in Exercise 2.2.12).*

Note that the coercivity is not the usual one encountered for linear problem: we do not control the square of ξ but some p power. An archetypal operator which satistfies these assumptions is $a(\xi) = |\xi|^{p-2}\xi$, which gives rise to the p-Laplace equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f$$

(for $p = 2$, we get back the classical Laplace linear equation).

Remark 5.3.2 *We can generalise the study to operators of the kind* $-\mathrm{div}(a(x, u, \nabla u))$ *for some* $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ *satisfying generalisations of these three assumptions.*

As usual, we can find a weak formulation by multiplying the equation by some test function φ which vanishes on the boundary of Ω:

$$\int_\Omega a(\nabla u)\cdot\nabla\varphi = \int f\varphi.$$

In which space shall we look for u? If we mimmic what we do for linear equations when we look for estimates on the solution, we would take $\varphi = u$ and the coercivity would give

$$\alpha\int|\nabla u|^p \le \int a(\nabla u)\cdot\nabla u = \int fu.$$

Hence, a proper space appears to be $W_0^{1,p}(\Omega)$, the power p being related to the above estimate on ∇u and the index 0 translating the Dirichlet boundary condition from (5.7). The weak formulation of this equation is therefore

$$\text{Find } u\in W_0^{1,p}(\Omega) \text{ which satisfies, for all } \varphi\in C_c^\infty(\Omega)\,,\ \int_\Omega a(\nabla u)\cdot\nabla\varphi = \int f\varphi. \quad (5.12)$$

Remark 5.3.3 *By Exercise 5.3.1 and since $f\in L^{p'}(\Omega)$, all terms in* (5.12) *are well-defined. In fact, by density of $C_c^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$, we can even see that* (5.12) *holds for all $\varphi\in W_0^{1,p}(\Omega)$.*

We are now equipped for the existence result.

Theorem 5.3.4 (Existence for the fully non-linear elliptic equation) *Let Ω be a bounded open set in $\mathbb{R}^N$ and $p\in(1,\infty)$. If $f\in L^{p'}(\Omega)$ and a satisfies Assumptions* (5.8)–(5.11) *then there exists at least one weak solution to* (5.7) *in the sense* (5.12).

As for Theorem (5.2.1), the proof is done by coming back to a finite-dimensional vector space and passing to the limit. We will need the following consequence of Brouwer's fixed point theorem to solve the projection of (5.12) in finite-dimensional vector spaces.

Lemma 5.3.5 *Let E be an inner product finite-dimensional vector space and $f: E\to E$ be continuous such that $\frac{\langle f(x),x\rangle}{|x|}\to\infty$ as $|x|\to\infty$. Then f is onto: for any $y\in E$ there exists $x\in E$ such that $f(x)=y$.*

Proof of Lemma 5.3.5

Let $g(x)=f(x)-y$. We also have $\frac{\langle g(x),x\rangle}{|x|} = \frac{\langle g(x),x\rangle}{|x|} - \frac{y\cdot x}{|x|} \ge \frac{\langle g(x),x\rangle}{|x|} - |y| \to\infty$ as $|x|\to\infty$. We need to prove that there exists $x\in E$ such that $g(x)=0$.

Assume that such an x does not exist and let $R>0$ such that $\langle g(x),x\rangle>0$ for all $|x|\ge R$. Then the function $\phi(x)=\frac{-Rg(x)}{|g(x)|}$ is continuous $B_R\to B_R$ and should therefore have a fixed point x. But $|\phi(x)|=R$ so we have $|x|=R$ and taking the inner product of $x=\phi(x)$ gives

$$R^2 = \langle x,x\rangle = \langle x,\phi(x)\rangle = -R\frac{\langle g(x),x\rangle}{|g(x)|}.$$

But $|x|=R$ implies, by choice of R, $\langle g(x),x\rangle>0$ and therefore $R^2<0$, which is a contradiction. The proof is complete. ■

Proof of Theorem 5.3.4

Step 1: finite-dimensional framework.

We write $W_0^{1,p}=\overline{\cup_{n\geq 1}E_n}$ with $\dim(E_n)<\infty$ and $E_n\subset E_{n+1}$. We endow E_n with an inner product and define $T:E_n\to E_n$ as the function such that, for all $u,v\in E_n$,

$$\langle T(u),v\rangle=\int_\Omega a(\nabla u)\cdot\nabla v.$$

Such a definition is legitimate since, for a fixed u, the transformation $v\in E_n\to\int_\Omega a(\nabla u)\cdot\nabla v$ is linear continuous on E_n and can therefore, by Riesz' theorem, be represented through the inner product as an element $T(u)$ of E_n.

It is easy to see that T is continuous: indeed, if $u_k\to u$ in E_n then, all norms being equivalent in E_n, $u_k\to u$ in $W_0^{1,p}(\Omega)$ and therefore, by Exercise 5.3.1, we have $a(\nabla u_k)\to a(\nabla u)$ in $L^{p'}(\Omega)$. We can therefore pass to the limit in $\int_\Omega a(\nabla u_k)\cdot\nabla v$ and see that $\langle T(u_k),v\rangle\to\langle T(u),v\rangle$ for any $v\in E_n$. Since E_n is finite-dimensional, this shows that $T(u_k)\to T(u)$ in E_n (choose for v the vector in an orthonormal basis, then each components $T(u_k)_i$ of $T(u_k)$ converge to the corresponding component $T(u)_i$ of $T(u)$).

We have

$$\frac{\langle T(u),u\rangle}{|u|}=\frac{\int_\Omega a(\nabla u)\cdot\nabla u}{|u|}\geq\alpha\frac{||\nabla u||_p^p}{|u|}.$$

All norms being equivalent in E_n and $||\nabla\cdot||_p$ being a norm in $E_n\subset W_0^{1,p}$, we deduce

$$\frac{\langle T(u),u\rangle}{|u|}\geq C\alpha\frac{|u|^p}{|u|}=C\alpha|u|^{p-1}\to\infty \text{ as } |u|\to\infty.$$

By Lemma 5.3.5, we infer that T is onto. In particular, if we take $y\in E_n$ which represents, through the inner product $\langle\cdot,\cdot\rangle$ the linear form $v\in E_n\to\int_\Omega fv$, we find the existence of $u_n\in E_n$ such that $T(u_n)=y$ which means that

$$\text{for all } v\in E_n\,,\ \int_\Omega a(\nabla u_n)\cdot\nabla v=\int_\Omega fv. \tag{5.13}$$

Step 2: passing to the limit.

Choosing $v=u_n$ in (5.13) we easily see that the sequence (u_n) is bounded in $W_0^{1,p}$. Up to a subsequence, we can therefore assume that $u_n\to u$ weakly-$*$ in $W_0^{1,p}$ (and strongly in L^p). However, because $a(\nabla u_n)$ is not linear in ∇u_n, we cannot pass to the limit: the weak-$*$ convergence only allows to pass to the limit in expressions of the form $\int_\Omega\nabla u_n\cdot\Psi$ with $\Psi\in(L^{p'})^N$... we therefore need to use another argument to pass to the limit.

This is where the monotony assumption on a is of utmost importance. Let $v\in E_l$, for some fixed l, and let $n\geq l$. Then we have, since u_n satisfies (5.13)

$$0\leq\int_\Omega(a(\nabla v)-a(\nabla u_n)\cdot(\nabla v-\nabla u_n)=\int_\Omega a(\nabla v)\cdot(\nabla v-\nabla u_n)-\int_\Omega f(v-u_n). \tag{5.14}$$

We know that $u_n \to u$ weakly-$*$ in L^p and, since $f \in L^{p'}$, this shows that $\int f(v-u_n) \to \int f(v-u)$. We have $\nabla v \in L^p$ and $|a(\nabla v)| \le C|\nabla v|^{p-1}$, hence $a(\nabla v) \in L^{p/(p-1)} = L^{p'}$ and, since $\nabla u_n \to \nabla u$ weakly-$*$ in L^p, we get $\int a(\nabla v)\cdot(\nabla v - \nabla u_n) \to \int a(\nabla v)\cdot(\nabla v - \nabla u)$.

Equation (5.14) therefore gives, passing to the limit, for all $v \in \cup_{l\ge 1} E_l$,

$$\int_\Omega a(\nabla v)\cdot(\nabla v - \nabla u) - \int_\Omega f(v-u) \ge 0. \tag{5.15}$$

Since $\cup_{l\ge 1} E_l$ is dense in $W_0^{1,p}$ and the preceding expression is continuous w.r.t. v for the $W_0^{1,p}$ norm (use Exercise 5.3.1)), (5.15) is in fact valid for any $v \in W_0^{1,p}$.

Take $\varphi \in W_0^{1,p}$, $t \ne 0$ and apply (5.15) with $v = u + t\varphi$. Letting

$$A(t) = \int_\Omega a(\nabla u + t\nabla\varphi)\cdot\nabla\varphi - \int_\Omega f\varphi,$$

we find $tA(t) \ge 0$. From Exercise 5.3.1, as $t \to 0$ we have $a(\nabla u + t\nabla\varphi) \to a(\nabla u)$ in $L^{p'}$ and therefore $A(t) \to A(0) = \int_\Omega a(\nabla u)\cdot\nabla\varphi - \int_\Omega f\varphi$. Let $t > 0$, divide $tA(t) \ge 0$ by t and let $t \to 0$. This gives $A(0) \ge 0$. Doing the same with $t < 0$ gives $A(0) \le 0$. Hence, $A(0) = 0$, which precisely proves that u satisfies (5.12). ∎