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A characteristic finite element method for optimal control problems governed by convection–diffusion equations

Hongfei Fu

School of Mathematics and Computational Science, China University of Petroleum, Dongying 257061, China

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ABSTRACT

In this paper we analyze a characteristic finite element approximation of convex optimal control problems governed by linear convection-dominated diffusion equations with pointwise inequality constraints on the control variable, where the state and co-state variables are discretized by piecewise linear continuous functions and the control variable is approximated by either piecewise constant functions or piecewise linear discontinuous functions. A priori error estimates are derived for the state, co-state and the control. Numerical examples are given to show the efficiency of the characteristic finite element method.

1. Introduction

Optimal control problems governed by convection–diffusion equations arise in many scientific and engineering applications, such as atmospheric pollution control problems [1,2]. Efficient numerical methods are essential to successful applications of such optimal control problems. To the best of my knowledge, there are only a few published results on optimal control problems governed by steady convection–diffusion equations; see [3] of SUPG method, [4] of standard finite element discretizations with stabilization based on local projection method, [5] of symmetric stabilization method; [6] of edge-stabilization method and [7] of the application of RT mixed DG scheme. For the approximation of constrained optimal control problems governed by time-dependent convection–diffusion equations, it is much more complicated and there are nearly no related papers published so far. Systematic introductions of the finite element method for PDEs and optimal control problems can be found in, for example, [8–13].

In this paper we consider the following linear-quadratic optimal control problems for the state variable y and the control variable u:

$$\min_{u\in K} \frac{1}{2}\int_0^T \left(\|y-z_d\|_{0,\Omega}^2 + \alpha\|u\|_{0,\Omega_U}^2\right) \mathrm{d}t, \tag{1.1}$$

subject to

$$\begin{cases} y_t + \mathbf{v}\cdot\nabla y - \operatorname{div}(A\nabla y) = f + Bu, & (x,t)\in\Omega\times(0,T],\\ y(x,t) = 0, & (x,t)\in\partial\Omega\times[0,T],\\ y(x,0) = y_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

E-mail address: hongfeifu@upc.edu.cn.

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and

$$\xi_1 \leq u(x,t) \leq \xi_2, \quad (x,t) \in \Omega_U \times (0,T], \tag{1.3}$$

where $\mathbf{v} = \mathbf{v}(x,t)$ denotes a velocity field in the flow control, $A = A(x,t)$ is a diffusion coefficient, $f = f(x,t)$ accounts for external sources and sinks, B is a linear continuous operator, and $y_0(x)$ is a prescribed initial data. In our case, we assume that the convection term dominates the diffusion term. A precise formulation of this problem including a functional analytic setting is given in the next section.

The methods of characteristics [14–16] combine the convection and capacity terms in the governing equations to carry out the temporal discretization in a Lagrange coordinate. These methods symmetrize the governing equation and stabilize their numerical approximations. They generate accurate numerical solutions and significantly reduce the numerical diffusion and grid-orientation effect present in upwind methods, even if large time steps and coarse spatial meshes are used. The goal of the present paper is to apply the methods of characteristics to the quadratic optimal control problems governed by linear convection-dominated diffusion equations, and we obtain a priori error estimates for both the control and state approximations. The present paper extends [17] in two aspects: First, it deals with either piecewise linear elements or piecewise constant elements for the control approximation. Second, the error estimates are obtained in the framework of L^2-error and bilateral pointwise inequality control constraints. The results obtained and the techniques used here are also different from that of [17].

The rest of the paper is organized as follows: In Section 2, we first refine the statement of the model problem and then derive a generic weak formulation and optimality conditions. In Section 3, we construct a characteristic finite element approximation scheme for the optimal control problems. In Section 4, the main error estimates are derived for the control problems with obstacle constraints. In Section 5, we conduct some numerical experiments to observe the convergence behavior of the numerical scheme. Section 6 contains concluding remarks.

In this paper, we denote C and δ be a generic constant and small positive number which are independent of the discrete parameters and may have different values in different circumstances, respectively.

2. Optimal control problems and optimality conditions

Let Ω and Ω_U be bounded open sets in $\mathbb{R}^2$, with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$. Just for simplicity of presentation, we assume that Ω and Ω_U are convex polygon. We employ the usual notion for Lebesgue and Sobolev spaces; see [8,9] for details.

Now we give a description of the mathematical model of the optimal control problems governed by convection–diffusion equations. To fix the idea, let $I = (0,T]$ and we shall take the state space $W = H^1(I;V)$ with $V = H_0^1(\Omega)$, the control space $X = L^2(I;U)$ with $U = L^2(\Omega_U)$, and the observation space $Y = L^2(I;H)$ with $H = L^2(\Omega)$. B is a linear continuous operator from U to H, and K is a closed convex set in X.

Let $0 = t_0 < t_1 < t_2 < \cdots < t_{N_T} = T$ be a subdivision of I, with corresponding time intervals $I_n = (t_{n-1}, t_n]$ and time steps $k_n = t_n - t_{n-1}$, $n = 1, 2, \ldots, N_T$. Denote $k = \max_{1\leq n\leq N_T} k_n$ and $f^n = f(t_n)$. We define, for $1 \leq q < \infty$, the discrete time-dependent norms

$$\|f\|_{l^q(I;X)} = \left(\sum_{n=1}^{N_T} k_n \|f^n\|_X^q\right)^{\frac{1}{q}}$$

and the standard modification for $q = \infty$. Let

$$l^q(I;X) := \left\{f : \|f\|_{l^q(I;X)} < \infty\right\}, \quad 1 \leq q \leq \infty.$$

In problems (1.1)–(1.3), α is a positive constant, the bounds ξ_1, ξ_2 are two real numbers that fulfill $\xi_1 < \xi_2, f \in L^2(I;L^2(\Omega))$, $z_d \in H^1(I;L^2(\Omega))$, $y_0 \in V = H_0^1(\Omega)$, and

$$A(x) = (a_{i,j}(x))_{2\times 2} \in (W^{1,\infty}(\bar{\Omega}))^{2\times 2},$$

such that there is a positive constant c satisfying

$$\sum_{i,j=1}^{2} a_{i,j}(x)\xi_i\xi_j \geq c|\xi|^2, \quad \forall \xi \in \mathbb{R}^2.$$

The velocity field vector $\mathbf{v} = (V_1(x,t), V_2(x,t))$ lies in the function space $L^\infty(I; W^{1,\infty}(\bar{\Omega})^2)$ and is divergence-free, i.e.,

$$\nabla \cdot \mathbf{v} = 0, \quad \forall x \in \Omega,\ t \in I.$$

To avoid technical boundary difficulties associate with the methods of characteristics, we assume that Ω is a rectangle and the state equation is Ω-periodic, i.e., we assume that all functions in Eq. (1.2) are spatially Ω-periodic; see, [14,15] for example.

Let

$$\phi(x,t) := \left(|\mathbf{v}|^2+1\right)^{1/2} = \left(V_1(x,t)^2+V_2(x,t)^2+1\right)^{1/2},$$

and let the characteristic direction associate with the material derivative term $y_t+\mathbf{v}\cdot\nabla y$ be denoted by $s=s(x,t)$, where

$$\phi\frac{\partial y}{\partial s} = y_t+\mathbf{v}\cdot\nabla y. \tag{2.1}$$

To formulate the optimal control problem we introduce the admissible set K collecting the inequality constraints (1.3) as

$$K=\{v\in X:\xi_1\le v(x,t)\le\xi_2,\,(x,t)\in\Omega_U\times I\}.$$

Let $(\cdot,\cdot)$ and $(\cdot,\cdot)_U$ denote the inner product of Ω and Ω_U, respectively, and denote $a(v,w)=(A\nabla v,\nabla w)$. Then the above-mentioned convex optimal control problem can be restated as follows: (QCP)

$$\min_{u\in K}\frac{1}{2}\int_0^T\left(\|y-z_d\|_{0,\Omega}^2+\alpha\|u\|_{0,\Omega_U}^2\right)\mathrm{d}t, \tag{2.2}$$

where $y=y(u)\in W$ satisfying the following standard weak formulation:

$$\begin{cases}\left(\phi\dfrac{\partial y(u)}{\partial s},w\right)+a(y(u),w)=(f+Bu,w), & \forall w\in V,\ t\in I,\\ y(u)(x,0)=y_0(x).\end{cases}$$

It is well known (see, e.g., [10]) that the control problem (QCP) has a unique solution (y,u), and that a pair (y,u) is the solution of (QCP) if and only if there is a co-state $p\in W$ such that the triplet (y,p,u) satisfies the following optimality conditions: (QCP–OPT)

$$\begin{cases}\left(\phi\dfrac{\partial y}{\partial s},w\right)+a(y,w)=(f+Bu,w), & \forall w\in V=H_0^1(\Omega),\\ y(0)=y_0,\end{cases} \tag{2.3}$$

$$\begin{cases}-\left(\phi\dfrac{\partial p}{\partial s},q\right)+a(q,p)=(y-z_d,q), & \forall q\in V=H_0^1(\Omega),\\ p(T)=0,\end{cases} \tag{2.4}$$

$$\int_0^T(\alpha u+B^*p,v-u)_U\,\mathrm{d}t\ge 0,\quad \forall v\in K\subset X=L^2(I;U), \tag{2.5}$$

where B^* is the adjoint operator of B.

Inequality (2.5) is equivalent to

$$u=\max\left(\xi_1,\min\left(\xi_2,-\frac{1}{\alpha}B^*p\right)\right). \tag{2.6}$$

3. Characteristic finite element discretizations

In this section, the characteristic finite element approximation scheme for the control problem (QCP) is presented. The approximation scheme is also applicable to the control problem with more general convex objective functionals. Here we consider the n-simplex conforming Lagrange elements which are most widely used in practical computations.

Let $G(x^*,t^*;t)$ be an approximate characteristic curve passing through point x^* at time t^*, which is defined by

$$G(x^*,t^*;t):=x^*-\mathbf{v}(x^*,t^*)(t^*-t). \tag{3.1}$$

We denote by $\bar{x}=G(x,t_n;t_{n-1})$ be the foot at time t_{n-1} of the characteristic curve with head x at time t_n, and $\bar{f}(x)=f(\bar{x})$. Approximate $(\partial y^n/\partial s)(x)=(\partial y/\partial s)(x,t_n)$ by a backward difference quotient in the s-direction,

$$\phi^n\frac{\partial y^n}{\partial s}\simeq\frac{y^n-\bar{y}^{n-1}}{k_n}. \tag{3.2}$$

We remark that, since the problem is Ω-periodic, $\bar{y}^{n-1}$ is always defined and the tangent to the characteristics (i.e., the s-segment) cannot cross a boundary to an undefined location.

The time difference (3.2) will be combined with a standard Galerkin procedure in the space variables. Let $\mathcal{T}^h$ and $\mathcal{T}_U^h$ be regular triangulations of Ω and Ω_U, respectively, so that $\bar{\Omega}=\cup_{\tau\in\mathcal{T}^h}\bar{\tau}$, $\bar{\Omega}_U=\cup_{\tau_U\in\mathcal{T}_U^h}\bar{\tau}_U$. Let $h=\max_{\tau\in\mathcal{T}^h}h_\tau$, $h_U=\max_{\tau_U\in\mathcal{T}_U^h}h_{\tau_U}$, where h_τ and h_{τ_U} denote the diameter of the element τ and τ_U, respectively.

Associate with $\mathcal{T}^h$ is a finite-dimensional subspace S^h of $C(\bar{\Omega})$, such that $\chi|_\tau$ are of first-order polynomials for all $\chi \in S^h$ and $\tau \in \mathcal{T}^h$. Let $V^h = \{v_h \in S^h : v_h|_{\partial\Omega} = 0\}$. It is clear that $V^h \subset V$.

Associate with $\mathcal{T}_U^h$ is another finite-dimensional subspace U^h of $L^2(\Omega_U)$, such that $\chi|_{\tau_U}$ are polynomials of r-order $(r = 0, 1)$ for all $\chi \in U^h$ and $\tau_U \in \mathcal{T}_U^h$. Here there is no requirement for the continuity. From the computation point of view, in general the sizes of the elements in $\mathcal{T}_U^h$ are smaller than those in $\mathcal{T}^h$. So we assume here $h_U/h \leq C$.

A possible fully discrete approximate scheme of (QCP), which will be labeled as $(\text{QCP})^{hk}$, is to find $\left(y_h^n, u_h^n\right) \in V^h \times K^h$, $n = 1, 2, \ldots, N_T$, such that

$$\min_{u_h^n \in K^h} \frac{1}{2} \sum_{n=1}^{N_T} k_n (\|y_h^n - z_d(x, t_n)\|_{0,\Omega}^2 + \alpha \|u_h^n\|_{0,\Omega_U}^2), \tag{3.3}$$

subject to

$$\begin{cases} \left(\dfrac{y_h^n - \bar{y}_h^{n-1}}{k_n}, w_h\right) + a\left(y_h^n, w_h\right) = (f(x, t_n) + Bu_h^n, w_h), \quad \forall w_h \in V^h \\ y_h^0(x) = y_0^h(x), \quad x \in \Omega, \end{cases}$$

where K^h is a closed convex set in $K \cap U^h$, and $y_0^h \in V^h$ is an approximation of y_0 which will be specified later on.

It follows that the control problem $(\text{QCP})^{hk}$ has a unique solution $\left(Y_h^n, U_h^n\right)$, and that a pair $\left(Y_h^n, U_h^n\right) \in V^h \times K^h$, is the solution of $(\text{QCP})^{hk}$ if and only if there is a co-state $P_h^{n-1} \in V^h$, such that the triplet $\left(Y_h^n, P_h^{n-1}, U_h^n\right) \in V^h \times V^h \times K^h$ satisfies the following optimality conditions: $(\text{QCP–OPT})^{hk}$

$$\begin{cases} \left(\dfrac{Y_h^n - \bar{Y}_h^{n-1}}{k_n}, w_h\right) + a\left(Y_h^n, w_h\right) = \left(f^n + BU_h^n, w_h\right), \quad \forall w_h \in V^h,\ n = 1, 2, \ldots, N_T, \\ Y_h^0(x) = y_0^h(x), \quad x \in \Omega, \end{cases} \tag{3.4}$$

$$\begin{cases} \left(\dfrac{P_h^{n-1} - \bar{\bar{P}}_h^n \cdot J^n}{k_n}, q_h\right) + a\left(q_h, P_h^{n-1}\right) = \left(Y_h^n - z_d^n, q_h\right), \quad \forall q_h \in V^h,\ n = N_T, \ldots, 2, 1, \\ P_h^{N_T}(x) = 0, \quad x \in \Omega, \end{cases} \tag{3.5}$$

$$\left(\alpha U_h^n + B^* P_h^{n-1}, v_h - U_h^n\right)_U \geq 0, \quad \forall v_h \in K^h \subset K \cap U^h, n = 1, 2, \ldots, N_T, \tag{3.6}$$

where $\bar{\bar{P}}_h^n = P_h^n(\bar{\bar{x}})$, and $\bar{\bar{x}}$ represents the head of the characteristic curve with foot x at time t_{n-1}, namely,

$$x = G(\bar{\bar{x}}, t_n; t_{n-1}). \tag{3.7}$$

We denote by $J^n := |\det \mathcal{D}G(x, t_n; t_{n-1})^{-1}|$ the determinant of the Jacobian transformation from G to x. It is clear that for the incompressible flow, the determinant can be expressed as

$$\det \mathcal{D}G(x, t_n; t_{n-1}) = 1 - (\nabla \cdot \mathbf{v}^n) k_n + \mathcal{O}(k_n^2) = 1 + \mathcal{O}(k_n^2).$$

Let

$$\Omega_U^*(t) = \{\cup \tau_U : \tau_U \subset \Omega_U, \xi_1 < u(\cdot, t)|_{\tau_U} < \xi_2\},$$
$$\Omega_U^c(t) = \{\cup \tau_U : \tau_U \subset \Omega_U, u(\cdot, t)|_{\tau_U} \equiv \xi_1, \text{ or } u(\cdot, t)|_{\tau_U} \equiv \xi_2\},$$
$$\Omega_U^b(t) = \Omega_U \setminus (\Omega_U^*(t) \cup \Omega_U^c(t)).$$

It is easy to check that the three parts do not intersect on each other, and $\Omega_U = \Omega_U^*(t) \cup \Omega_U^c(t) \cup \Omega_U^b(t)$. In this paper we assume that u and $\mathcal{T}_U^h$ are regular such that $\text{meas}(\Omega_U^b(t)) \leq Ch_U$ (see [18,19]). Moreover, set

$$\Omega_U^{**}(t) = \{x \in \Omega_U, \xi_1 < u(x, t) < \xi_2\}.$$

Then it is easy to see that $\Omega_U^*(t) \subset \Omega_U^{**}(t)$.

In the rest of the paper, we shall use two auxiliary variables $\left(Y_h^n(u), P_h^n(u)\right) \in V^h \times V^h,\ n = 1, 2, \ldots, N_T$, associate with the control variable u:

$$\begin{cases} \left(\dfrac{Y_h^n(u) - \bar{Y}_h^{n-1}(u)}{k_n}, w_h\right) + a\left(Y_h^n(u), w_h\right) = \left(f^n + Bu^n, w_h\right), \quad \forall w_h \in V^h, \\ Y_h^0(u) = y_0^h(x), \quad x \in \Omega, \end{cases} \tag{3.8}$$

$$\begin{cases} \left(\dfrac{P_h^{n-1}(u) - \bar{\bar{P}}_h^n(u) \cdot J^n}{k_n}, q_h\right) + a\left(q_h, P_h^{n-1}(u)\right) = \left(Y_h^n(u) - z_d^n, q_h\right), \quad \forall q_h \in V^h, \\ P_h^{N_T}(u) = 0, \quad x \in \Omega. \end{cases} \tag{3.9}$$

Set

$$\theta^n = Y_h^n - Y_h^n(u), \qquad \eta^n = y^n - Y_h^n(u), \quad n = 0, 1, \ldots, N_T,$$
$$\zeta^n = P_h^n - P_h^n(u), \qquad \xi^n = p^n - P_h^n(u), \quad n = N_T, \ldots, 1, 0.$$

It is obvious that $\theta^0 = 0$ and $\zeta^{N_T} = 0$.

4. A priori error estimates

In this section, we expect the following main a priori error estimates for the optimal control problem (QCP–OPT) and its characteristic finite element approximation $(\text{QCP–OPT})^{hk}$.

Theorem 4.1. *Suppose that $\{y, p, u\}$ and $\{Y_h, P_h, U_h\}$ are the solutions of (2.3)–(2.5) and (3.4)–(3.6), respectively. Then for $m = 0, 1$, we have*

$$\|y - Y_h\|_{l^\infty(I;L^2(\Omega))} + \|p - P_h\|_{l^\infty(I;L^2(\Omega))} + \|u - U_h\|_{l^2(I;L^2(\Omega_U))} \le C\left(h_U^{1+\frac{m}{2}} + h^2 + k\right), \tag{4.1}$$

where C depends on some spatial and temporal derivatives of y, p, z_d and u.

To derive the above main result, some useful lemmas are needed.

Lemma 4.2 ([14]). *Suppose that $f \in L^2(\Omega)$, and $\bar{f}(x) = f(x - g(x)k)$, where we assume that $g, \nabla g$ are bounded on $\bar{\Omega}$, then for sufficiently small k, we have that*

$$\|f(x) - \bar{f}(x)\|_{-1} \le Ck\|f\|,$$

here the constant C depends only on $\|g\|_{L^\infty(\Omega)}$ and $\|\nabla g\|_{L^\infty(\Omega)}$, and the negative-norm $\|\cdot\|_{-1}$ is defined as follows:

$$\|v\|_{-1} = \sup_{0\neq\phi\in H^1} \frac{(v, \phi)}{\|\phi\|_1}.$$

Lemma 4.3. *Let (Y_h, P_h) and $(Y_h(u), P_h(u))$ be the solutions of (3.4)–(3.5) and (3.8)–(3.9), respectively. Then the following estimate holds*

$$\|Y_h - Y_h(u)\|_{l^\infty(I;L^2(\Omega))} + \|P_h - P_h(u)\|_{l^\infty(I;L^2(\Omega))} \le C\|u - U_h\|_{l^2(I;L^2(\Omega_U))}. \tag{4.2}$$

Proof. Firstly, it follows from Eqs. (3.4) and (3.8) that

$$\left(\frac{\theta^n - \bar{\theta}^{n-1}}{k_n}, w_h\right) + a\left(\theta^n, w_h\right) = (B(U_h^n - u^n), w_h), \quad \forall w_h \in V^h. \tag{4.3}$$

Selecting $w_h = \theta^n$ as a test function. The inequality $a(a - b) \ge \frac{1}{2}(a^2 - b^2)$ and a direct calculation show that

$$\left(\frac{\theta^n - \bar{\theta}^{n-1}}{k_n}, \theta^n\right) \ge \frac{1}{2k_n}\left(\|\theta^n\|^2 - \|\bar{\theta}^{n-1}\|^2\right), \tag{4.4}$$

$$\|\bar{\theta}^{n-1}\|^2 \le (1 + Ck_n)\|\theta^{n-1}\|^2. \tag{4.5}$$

Inserting Eqs. (4.4)–(4.5) into (4.3), and multiplying both sides of (4.3) by $2k_n$ and summing over n from 1 to N, we obtain

$$\|\theta^N\|^2 + 2\sum_{n=1}^{N} k_n\|\theta^n\|_a^2 \le C\sum_{n=1}^{N} k_n(\|\theta^n\|^2 + \|\theta^{n-1}\|^2) + C\sum_{n=1}^{N} k_n\|u^n - U_h^n\|^2. \tag{4.6}$$

We apply the discrete Gronwall's lemma to conclude

$$\|Y_h - Y_h(u)\|_{l^\infty(I;L^2(\Omega))} + \|Y_h - Y_h(u)\|_{l^2(I;H_0^1(\Omega))} \le C\|u - U_h\|_{l^2(I;L^2(\Omega_U))}. \tag{4.7}$$

Similarly, we derive from Eqs. (3.5) and (3.9) that

$$\|\zeta\|_{l^\infty(I;L^2(\Omega))} + \|\zeta\|_{l^2(I;H_0^1(\Omega))} \le C\|Y_h - Y_h(u)\|_{l^2(I;L^2(\Omega))}. \tag{4.8}$$

Therefore Lemma 4.3 is proved from (4.7)–(4.8). □

Lemma 4.4. *Let (y, p, u) and (Y_h, P_h, U_h) be the solutions of (QCP–OPT) and $(QCP–OPT)^{hk}$, respectively. Assume that $u \in l^2(I; H^1(\Omega_U))$, $p \in l^2(I; H_0^1(\Omega)) \cap H^1(I; L^2(\Omega))$ and $K^h \subset K$. Let U^h be the piecewise constant element space ($r = 0$). Then we have*

$$\|u - U_h\|_{l^2(I;L^2(\Omega_U))} \le C \left(h_U + k + \|p - P_h(u)\|_{l^2(I;L^2(\Omega))}\right), \tag{4.9}$$

where $P_h(u)$ is defined in (3.9).

Furthermore, let U^h be the piecewise linear element space ($r = 1$), $u \in l^2(I; W^{1,\infty}(\Omega_U))$, $p \in l^2(I; W^{1,\infty}(\Omega))$, $u(t) \in H^2(\Omega_U^{**}(t))$, where $\Omega_U^{**}(t)$ is defined in the last section. Then we have

$$\|u - U_h\|_{l^2(I;L^2(\Omega_U))} \le C \left(h_U^{\frac{3}{2}} + k + \|p - P_h(u)\|_{l^2(I;L^2(\Omega))}\right). \tag{4.10}$$

Proof. Let $\Pi_h u^n \in K^h$ be an approximation of $u(t_n)$. Then we have

$$\begin{aligned}
\alpha\|u - U_h\|^2_{l^2(I;L^2(\Omega_U))} &= \sum_{n=1}^{N_T} k_n \left(\alpha u^n, u^n - U_h^n\right)_U - \sum_{n=1}^{N_T} k_n \left(\alpha U_h^n, u^n - U_h^n\right)_U \\
&= \sum_{n=1}^{N_T} k_n \left(\alpha u^n + B^* p^n, u^n - U_h^n\right)_U + \sum_{n=1}^{N_T} k_n \left(\alpha U_h^n + B^* P_h^{n-1}, U_h^n - \Pi_h u^n\right)_U \\
&\quad + \sum_{n=1}^{N_T} k_n \left(\alpha(U_h^n - u^n), \Pi_h u^n - u^n\right)_U + \sum_{n=1}^{N_T} k_n \left(\alpha u^n + B^* p^n, \Pi_h u^n - u^n\right)_U \\
&\quad + \sum_{n=1}^{N_T} k_n \left(B^*(p^{n-1} - p^n), \Pi_h u^n - u^n\right)_U + \sum_{n=1}^{N_T} k_n \left(B^*(p^{n-1} - P_h^{n-1}(u)), u^n - \Pi_h u^n\right)_U \\
&\quad + \sum_{n=1}^{N_T} k_n \left(B^*(P_h^{n-1}(u) - P_h^{n-1}), u^n - \Pi_h u^n\right)_U + \sum_{n=1}^{N_T} k_n \left(B^*(P_h^{n-1} - P_h^{n-1}(u)), u^n - U_h^n\right)_U \\
&\quad + \sum_{n=1}^{N_T} k_n \left(B^*(P_h^{n-1}(u) - p^{n-1}), u^n - U_h^n\right)_U + \sum_{n=1}^{N_T} k_n \left(B^*(p^{n-1} - p^n), u^n - U_h^n\right)_U .
\end{aligned} \tag{4.11}$$

Recalling the inequalities (2.5) and (3.6), we know that the first and second terms on the right-hand side of (4.11) are less than or equal to zero. Besides, Eqs. (3.4)–(3.5) and (3.8)–(3.9) show that

$$\begin{aligned}
\text{the eighth term on the right-hand side of (4.11)} &= \sum_{n=1}^{N_T} k_n \left(P_h^{n-1} - P_h^{n-1}(u), B(u^n - U_h^n)\right) \\
&= -\sum_{n=1}^{N_T} k_n \left(\frac{\theta^n - \bar{\theta}^{n-1}}{k_n}, \zeta^{n-1}\right) - \sum_{n=1}^{N_T} k_n a\left(\theta^n, \zeta^{n-1}\right) \\
&= -\sum_{n=1}^{N_T} k_n \left(\frac{\zeta^{n-1} - \bar{\bar{\zeta}}^n \cdot J^n}{k_n}, \theta^n\right) - \sum_{n=1}^{N_T} k_n a\left(\theta^n, \zeta^{n-1}\right) \\
&= -\sum_{n=1}^{N_T} k_n (\theta^n, \theta^n) = -\|\theta\|^2_{l^2(I;L^2(\Omega))} \le 0.
\end{aligned} \tag{4.12}$$

Then we obtain from Lemma 4.3, Cauchy–Schwarz inequality, and the above inequality that

$$\begin{aligned}
\alpha\|u - U_h\|^2_{l^2(I;L^2(\Omega_U))} &\le \sum_{n=1}^{N_T} k_n (\alpha u^n + B^* p^n, \Pi_h u^n - u^n)_U + C\|u - \Pi_h u\|^2_{l^2(I;L^2(\Omega_U))} \\
&\quad + C\|p - P_h(u)\|^2_{l^2(I;L^2(\Omega))} + Ck^2\|p_t\|^2_{L^2(I;L^2(\Omega))} + \frac{\alpha}{2}\|u - U_h\|^2_{l^2(I;L^2(\Omega_U))}.
\end{aligned} \tag{4.13}$$

First let us consider the case that U^h is the piecewise constant element space. Let Π_h be the L^2-projection from $U = L^2(\Omega_U)$ to U^h such that for any $v \in U$

$$(v - \Pi_h v, \phi) = 0, \quad \forall \phi \in U^h.$$

It is easy to prove that $\Pi_h u^n \in K^h$, and it follows from [8] that for $u \in l^2(I; H^1(\Omega_U))$

$$\|u - \Pi_h u\|_{l^2(I;L^2(\Omega_U))} \le Ch_U \|u\|_{l^2(I;H^1(\Omega_U))}. \tag{4.14}$$

Moreover, if $u \in l^2(I; H^1(\Omega_U))$ and $p \in l^2(I; H^1(\Omega))$, we have

$$\begin{aligned}\sum_{n=1}^{N_T} k_n(\alpha u^n + B^*p^n, \Pi_h u^n - u^n)_U &= \sum_{n=1}^{N_T} k_n \sum_{\tau_U \in \mathcal{T}_U^h} \int_{\tau_U} \left(\alpha u^n + B^*p^n - \Pi_h(\alpha u^n + B^*p^n)\right)(\Pi_h u^n - u^n)\\ &\le \|\alpha u + B^*p - \Pi_h(\alpha u + B^*p)\|_{l^2(I;L^2(\Omega_U))} \|\Pi_h u - u\|_{l^2(I;L^2(\Omega_U))}\\ &\le Ch_U^2 \left(\|u\|^2_{l^2(I;H^1(\Omega_U))} + \|p\|^2_{l^2(I;H^1(\Omega))}\right) \le Ch_U^2. \end{aligned} \tag{4.15}$$

Then (4.9) follows from (4.13)–(4.15).

Next we consider the case that U^h is the piecewise linear element space. Let $\Pi_h u^n \in U^h$ be the standard Lagrange interpolation of u such that $\Pi_h u^n(z) = u(z, t_n)$ for any vertex z. It is clear that $\Pi_h u^n \in K^h$, and for $u \in l^2(I; W^{1,\infty}(\Omega_U))$, $u(t) \in H^2(\Omega_U^{**}(t))$ we have

$$\|u^n - \Pi_h u^n\|_{0,\Omega_U^*(t_n)} \le Ch_U^2 \|u^n\|_{2,\Omega_U^*(t_n)}, \qquad \|u^n - \Pi_h u^n\|_{0,\infty,\Omega_U^b(t_n)} \le Ch_U \|u^n\|_{1,\infty,\Omega_U^b(t_n)}.$$

Noting that $\Pi_h u = u$ on $\Omega_U^c(t)$, then it follows that

$$\begin{aligned}\|u - \Pi_h u\|^2_{l^2(I;L^2(\Omega_U))} &= \sum_{n=1}^{N_T} k_n \int_{\Omega_U} (u^n - \Pi_h u^n)^2\\ &= \sum_{n=1}^{N_T} k_n \left(\int_{\Omega_U^*(t_n)} (u^n - \Pi_h u^n)^2 + \int_{\Omega_U^c(t_n)} (u^n - \Pi_h u^n)^2 + \int_{\Omega_U^b(t_n)} (u^n - \Pi_h u^n)^2\right)\\ &\le Ch_U^4 \sum_{n=1}^{N_T} k_n \|u^n\|^2_{2,\Omega_U^*(t_n)} + 0 + Ch_U^2 \sum_{n=1}^{N_T} k_n \|u^n\|^2_{1,\infty,\Omega_U^b(t_n)} \mathrm{meas}(\Omega_U^b(t_n))\\ &\le Ch_U^4 \sum_{n=1}^{N_T} k_n \|u^n\|^2_{2,\Omega_U^*(t_n)} + Ch_U^3 \sum_{n=1}^{N_T} k_n \|u^n\|^2_{1,\infty,\Omega_U^b(t_n)}\\ &\le Ch_U^3 \left(\|u\|^2_{l^2(I;H^2(\Omega_U^{**}(t)))} + \|u\|^2_{l^2(I;W^{1,\infty}(\Omega_U))}\right) \le Ch_U^3. \end{aligned} \tag{4.16}$$

Moreover, it follows from (2.5) or (2.6) that $\alpha u + B^*p = 0$ on $\Omega_U^*(t)$, and we conclude from the definition of $\Omega_U^b(t)$ that for any element $\tau_U \subset \Omega_U^b(t)$, there is an x_0 such that $\xi_1 < u(x_0, t) < \xi_2$, and hence $(\alpha u + B^*p)(x_0) = 0$. Therefore for any $\tau_U \subset \Omega_U^b(t)$ we have

$$\|\alpha u + B^*p\|_{0,\infty,\tau_U} = \|\alpha u + B^*p - (\alpha u + B^*p)(x_0)\|_{0,\infty,\tau_U} \le Ch_U \|\alpha u + B^*p\|_{1,\infty,\tau_U}.$$

Then

$$\begin{aligned}\sum_{n=1}^{N_T} k_n(\alpha u^n + B^*p^n, \Pi_h u^n - u^n)_U &= \sum_{n=1}^{N_T} k_n \int_{\Omega_U^*(t_n)} (\alpha u^n + B^*p^n)(\Pi_h u^n - u^n)\\ &\quad + \sum_{n=1}^{N_T} k_n \int_{\Omega_U^c(t_n)} (\alpha u^n + B^*p^n)(\Pi_h u^n - u^n) + \sum_{n=1}^{N_T} k_n \int_{\Omega_U^b(t_n)} (\alpha u^n + B^*p^n)(\Pi_h u^n - u^n)\\ &= 0 + 0 + \sum_{n=1}^{N_T} k_n \int_{\Omega_U^b(t_n)} (\alpha u^n + B^*p^n)(\Pi_h u^n - u^n)\\ &\le \sum_{n=1}^{N_T} k_n \|\alpha u^n + B^*p^n\|_{0,\infty,\Omega_U^b(t_n)} \|\Pi_h u^n - u^n\|_{0,\infty,\Omega_U^b(t_n)} \mathrm{meas}(\Omega_U^b(t_n))\\ &\le Ch_U^3 \left(\|u\|^2_{l^2(I;W^{1,\infty}(\Omega_U))} + \|p\|^2_{l^2(I;W^{1,\infty}(\Omega))}\right) \le Ch_U^3. \end{aligned} \tag{4.17}$$

Thus (4.10) is proved by inserting (4.16)–(4.17) into (4.13). □

Lemma 4.5. *Let (y, p) and $(Y_h(u), P_h(u))$ be the solutions of (2.3)–(2.4) and (3.8)–(3.9), respectively. Assume that $y, p \in l^\infty(I; H_0^1(\Omega) \cap H^2(\Omega)) \cap H^1(I; H^2(\Omega)) \cap H^2(I; L^2(\Omega))$, $z_d \in H^1(I; L^2(\Omega))$. Then the following estimate holds*

$$\|y - Y_h(u)\|_{l^\infty(I;L^2(\Omega))} + \|p - P_h(u)\|_{l^\infty(I;L^2(\Omega))} \leq C(h^2 + k), \tag{4.18}$$

where C depends on some spatial and temporal derivatives of y, p and z_d.

Proof. First, we give an estimate for the difference η between the exact solution y and the intermediate solution $Y_h(u)$. Thus, we subtract Eq. (3.8) from Eq. (2.3) to obtain an error equation on $\eta = y - Y_h(u)$:

$$\left(\frac{\eta^n - \bar{\eta}^{n-1}}{k_n}, w_h\right) + a\left(\eta^n, w_h\right) = -(\sigma^n, w_h), \quad \forall w_h \in V^h,\ n \geq 1, \tag{4.19}$$

where

$$\sigma^n = \phi\frac{\partial y^n}{\partial s} - \frac{y^n - \bar{y}^{n-1}}{k_n}.$$

We decompose the error $\eta = y - Y_h(u)$ as $\eta = (y - \Theta y) + (\Theta y - Y_h(u)) = \mu + \nu$, where $\Theta y(t) \in V^h$ is defined to be the Ritz projection of $y(t) \in V$ which satisfies

$$a(y(t) - \Theta y(t), w_h) = 0, \quad \forall w_h \in V^h,\ t \in I. \tag{4.20}$$

It follows from [20] that for $q = 2$ or ∞ the following estimates hold:

$$\begin{aligned} &\|y - \Theta y\|_{l^q(I;L^2(\Omega))} + h\|y - \Theta y\|_{l^q(I;H^1(\Omega))} \leq Ch^r\|y\|_{l^q(I;H^r(\Omega))}, \\ &\left\|\frac{\partial(y - \Theta y)}{\partial t}\right\|_{L^2(I;L^2(\Omega))} \leq Ch^r\|y\|_{H^1(I;H^r(\Omega))}, \quad \text{for } 1 \leq r \leq 2. \end{aligned} \tag{4.21}$$

Since the estimate for μ is known, we need only to derive an estimate for ν. We choose $w_h = \nu^n$ and make use of (4.20) to rewrite Eq. (4.19) in terms of μ and ν:

$$\left(\frac{\nu^n - \bar{\nu}^{n-1}}{k_n}, \nu^n\right) + a\left(\nu^n, \nu^n\right) = -(\sigma^n, \nu^n) - \left(\frac{\mu^n - \bar{\mu}^{n-1}}{k_n}, \nu^n\right). \tag{4.22}$$

Firstly, by standard backward difference error analysis [14,15], we have

$$\|\sigma^n\|^2 \leq Ck_n\left\|\frac{\partial^2 y}{\partial s^2}\right\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}. \tag{4.23}$$

Secondly, it follows from Lemma 4.2 and the well-known estimates (4.21) that

$$\begin{aligned} \left|(\mu^n - \bar{\mu}^{n-1}, \nu^n)\right| &\leq \left|(\mu^n - \mu^{n-1}, \nu^n)\right| + \left|(\mu^{n-1} - \bar{\mu}^{n-1}, \nu^n)\right| \\ &\leq \|\nu^n\|\int_{t_{n-1}}^{t_n}\|\mu_t\|\mathrm{d}t + \|\mu^{n-1} - \bar{\mu}^{n-1}\|_{-1}\|\nu^n\|_1 \\ &\leq k_n^{1/2}\|\nu^n\|\,\|\mu_t\|_{L^2(I_n;L^2(\Omega))} + Ck_n\|\mu^{n-1}\|\,\|\nu^n\|_1 \\ &\leq C\|\mu_t\|^2_{L^2(I_n;L^2(\Omega))} + C(\delta)k_n\|\mu^{n-1}\|^2 + Ck_n\|\nu^n\|^2 + C\delta k_n\|\nu^n\|_1^2 \\ &\leq Ch^4\|y\|^2_{H^1(I_n;H^2(\Omega))} + Ck_nh^4\|y\|^2_{l^\infty(I;H^2(\Omega))} + Ck_n\|\nu^n\|^2 + C\delta k_n\|\nu^n\|_a^2. \end{aligned} \tag{4.24}$$

Multiplying both sides of (4.22) by k_n and summing over $1 \leq n \leq N$. We then conclude by Eqs. (4.23)–(4.24) and the same estimates as (4.5)–(4.6) that

$$\begin{aligned} \frac{1}{2}\|\nu^N\|^2 + \sum_{n=1}^N k_n\|\nu^n\|_a^2 \leq{}& Ck^2\left\|\frac{\partial^2 y}{\partial s^2}\right\|^2_{L^2(I;L^2(\Omega))} + Ch^4\|y\|^2_{H^1(I;H^2(\Omega))} + Ch^4\|y\|^2_{l^\infty(I;H^2(\Omega))} \\ &+ C\sum_{n=1}^N k_n(\|\nu^n\|^2 + \|\nu^{n-1}\|^2) + C\delta\sum_{n=1}^N k_n\|\nu^n\|_a^2, \end{aligned} \tag{4.25}$$

where we choose y_0^h to be the Ritz projection of y_0 which satisfies (4.20), i.e., $\nu^0 = 0$.

Applying the discrete Gronwall's lemma to (4.25) yields that

$$\|\nu\|_{l^\infty(I;L^2(\Omega))} \leq Ck\left\|\frac{\partial^2 y}{\partial s^2}\right\|_{L^2(I;L^2(\Omega))} + Ch^2\|y\|_{H^1(I;H^2(\Omega))} + Ch^2\|y\|_{l^\infty(I;H^2(\Omega))}. \tag{4.26}$$

Combining (4.26) with the well-known estimate (4.21) for μ finishes the proof of one part of (4.18).

Next, we consider the estimate for the difference ξ between the exact solution p and the intermediate solution $P_h(u)$.

Similarly, the error $\xi = p - P_h(u)$ can be split as $\xi = (p - \Theta p) + (\Theta p - P_h(u)) = \rho + \pi$, where $\Theta p \in V^h$ is defined to be the Ritz projection of $p \in V$ which satisfies for each $t \in I$,

$$a(q_h, p(t) - \Theta p(t)) = 0, \quad \forall q_h \in V^h. \tag{4.27}$$

It is clear that (4.21) is still valid with p instead of y. Eqs. (3.9) and (2.4) can be differenced with $q = q_h = \pi^{n-1}$ to give an error equation in terms of ρ and π:

$$\begin{aligned}
&\left(\frac{\pi^{n-1} - \bar{\bar{\pi}}^n}{k_n}, \pi^{n-1}\right) + a\left(\pi^{n-1}, \pi^{n-1}\right) \\
&\quad = -(\chi^{n-1}, \pi^{n-1}) - \left(\frac{\rho^{n-1} - \bar{\bar{\rho}}^n}{k_n}, \pi^{n-1}\right) - \left(\frac{\bar{\bar{\xi}}^n - \bar{\bar{\xi}}^n \cdot J^n}{k_n}, \pi^{n-1}\right) + \left(\frac{\bar{\bar{p}}^n - \bar{\bar{p}}^n \cdot J^n}{k_n}, \pi^{n-1}\right) \\
&\qquad + \left(y^n - Y_h^n(u), \pi^{n-1}\right) + \left(y^{n-1} - y^n, \pi^{n-1}\right) + \left(z_d^n - z_d^{n-1}, \pi^{n-1}\right),
\end{aligned} \tag{4.28}$$

where

$$\chi^{n-1} = -\phi \frac{\partial p^{n-1}}{\partial s} + \frac{\bar{\bar{p}}^n - p^{n-1}}{k_n}.$$

Multiplying both sides of (4.28) by k_n and summing over n from N_T to $M + 1$. Similar to the estimate of ν, and using the fact that $J^n = 1 + \mathcal{O}(k_n^2)$, we obtain

$$\begin{aligned}
\|\pi\|_{l^\infty(I;L^2(\Omega))} &\le Ck\left(\sum_{v=y,p}\left\|\frac{\partial^2 v}{\partial s^2}\right\|_{L^2(I;L^2(\Omega))} + \sum_{v=y,z_d}\left\|\frac{\partial v}{\partial t}\right\|_{L^2(I;L^2(\Omega))} + \|p\|_{l^2(I;L^2(\Omega))}\right) \\
&\quad + Ch^2 \sum_{v=y,p}\left(\|v\|_{H^1(I;H^2(\Omega))} + \|v\|_{l^\infty(I;H^2(\Omega))}\right).
\end{aligned} \tag{4.29}$$

Incorporating (4.29) with the well-known estimate (4.21) for ρ, we finish the proof of the other part of (4.18). Thus Lemma 4.5 is derived. □

Combining the bounds given by Lemmas 4.3–4.5, we can easily establish the main result of Theorem 4.1 by the triangle inequality.

5. Numerical experiments

In this section we carry out two numerical examples to demonstrate the theoretical results showed in Theorem 4.1.

The optimal control problem in which we are interested is the following type:

$$\begin{aligned}
&\min \frac{1}{2}\int_0^T \left(\|y - z_d\|_{0,\Omega}^2 + \|u - u_0\|_{0,\Omega}^2\right) \mathrm{d}t, \\
&\text{s.t.} \quad y_t + \mathbf{v} \cdot \nabla y - \varepsilon \Delta y = f + u, \quad 0 \le u \le 0.5.
\end{aligned} \tag{5.1}$$

In computing these examples, we use the C++ software package: AFEPack; it is available at http://dsec.pku.edu.cn/~rli. Besides, for simplicity we use the same mesh for $\mathcal{T}^h$ and $\mathcal{T}_U^h$. For constrained optimal control problems governed by convection–diffusion equations, people pay more attention both on the state and the control. Therefore in the following numerical examples, we mostly center on the state variable y, which is approximated by piecewise linear elements; and the control variable u, which is discretized using piecewise constant elements for the first example and piecewise linear elements for the second example.

Example 1. For the first example, the spatial domain is $\Omega = [0, 1]^2$, the time interval is $I = (0, 1]$, the velocity field is imposed as $\mathbf{v} = (0.5, 0.5)$, f and z_d are chosen such that the analytical solutions for Eq. (5.1) are as follows:

$$\begin{aligned}
p(x, t) &= \sin(\pi t)\sin(2\pi x_1)\sin(2\pi x_2)\exp\left(\frac{-1 + \cos(t_x)}{\sqrt{\varepsilon}}\right), \\
u_0(x, t) &= 0, \\
u(x, t) &= \max(0, \min(u_0 - p, 0.5)), \\
y(x, t) &= p\left(\frac{1}{2\sqrt{\varepsilon}}\sin(t_x) + 8\varepsilon\pi^2 + \frac{\sqrt{\varepsilon}}{2}\cos(t_x) - \frac{1}{2}\sin(t_x)^2\right) \\
&\quad - \pi\cos(\pi t)\sin(2\pi x_1)\sin(2\pi x_2)\exp\left(\frac{-1 + \cos(t_x)}{\sqrt{\varepsilon}}\right),
\end{aligned}$$

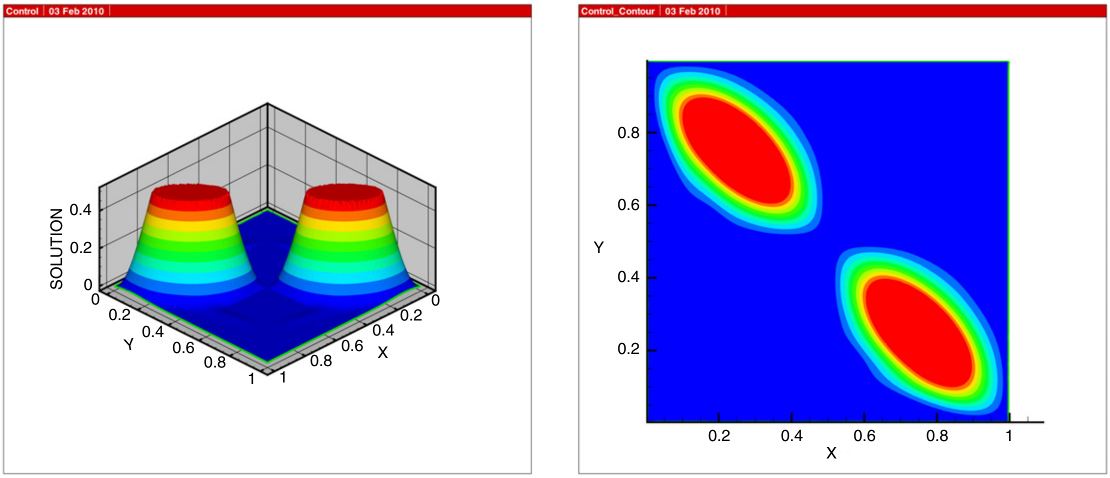

Fig. 1. The approximate control solution (left) and its contour line (right) for Example 1 with $h = 1/80$.

where

$$t_x = t - 0.5(x_1 + x_2), \quad \varepsilon = 1.0\text{e} - 5.$$

It is clear that for such chosen parameters both the state and control solutions are strictly dependent on the diffusion parameter ε, and for small ε the state equation is strong convection-dominated. However, the characteristic finite element method shows a good approximation to the control problem. It can be seen in Table 1 numerical convergence order is presented with $\varepsilon = 1.0\text{e} - 5$ for the control approximated by piecewise constant elements and the time step size $k = h$. Fig. 1 shows the approximate control solution and its contour line for the characteristic finite element approximation at $t = 0.5$. The elevation plot of the approximate state solution and its corresponding contour line at $t = 0.5$ are presented in Fig. 2.

Example 2. The second example considered is the transport of a two-dimensional rotating Gaussian pulse in $\Omega = [-0.5, 0.5]^2$ and $I = (0, 1]$. The corresponding analytical solutions for Eq. (5.1) with a rotating velocity field $\mathbf{v} = (-x_2, x_1)$, a constant diffusion coefficient $\varepsilon = 1.0\text{e} - 4$, $f = -u$, and $z_d = y$ are given by

$$\begin{aligned}
y(x,t) &= \frac{2\sigma^2}{2\sigma^2 + 4\varepsilon t} \exp\left(-\frac{(\bar{x}_1 - x_{1c})^2 + (\bar{x}_2 - x_{2c})^2}{2\sigma^2 + 4\varepsilon t}\right),\\
p(x,t) &= 0,\\
u_0(x,t) &= \sin(\pi t/2)\sin(\pi x_1)\sin(\pi x_2),\\
u(x,t) &= \max(0, \min(u_0 - p, 0.5)),
\end{aligned}$$

where x_{1c}, x_{2c}, and σ are the centered and standard deviations, respectively, and $\bar{x}_1 = x_1 \cos t + x_2 \sin t$, $\bar{x}_2 = x_2 \cos t - x_1 \sin t$.

In this numerical test, the data are chosen as follows: $x_{1c} = -0.25$, $x_{2c} = 0$, $\sigma = 0.0447$ which gives $2\sigma^2 = 0.0040$. This problem provides an example for a homogeneous two-dimensional advection–diffusion equation with a variable velocity field and a known analytical solution. It has been used widely to test for numerical artifacts of different schemes, such as numerical stability and numerical dispersion, spurious oscillations, and phase errors. To compute the convergence order for the piecewise linear element approximation for control, we take a small time step size $k = 1/100$, and spatial step sizes $h = 1/10, 1/15, 1/20, 1/25, 1/30$. The numerical results are presented in Table 2, which show that the characteristic finite element scheme maintains $3/2$th-order accuracy in space. In Figs. 3 and 4, we also show the approximate solutions and their corresponding contour lines for the control and state at $T = 1$ with $h = 1/80$, respectively.

6. Concluding remarks

In this paper, we derive a priori error estimates for a characteristic finite element discretization of optimal control problems governed by unsteady convection–diffusion equations subject to bilateral pointwise inequality control constraints. Numerical experiments are given to confirm the theoretical convergence order and show the efficiency of the present scheme.

In this area there are still many important issues that need to be addressed. For example, we are going to study a mass-conservative characteristic FEM for optimal control problems governed by compressible convection–diffusion equations,

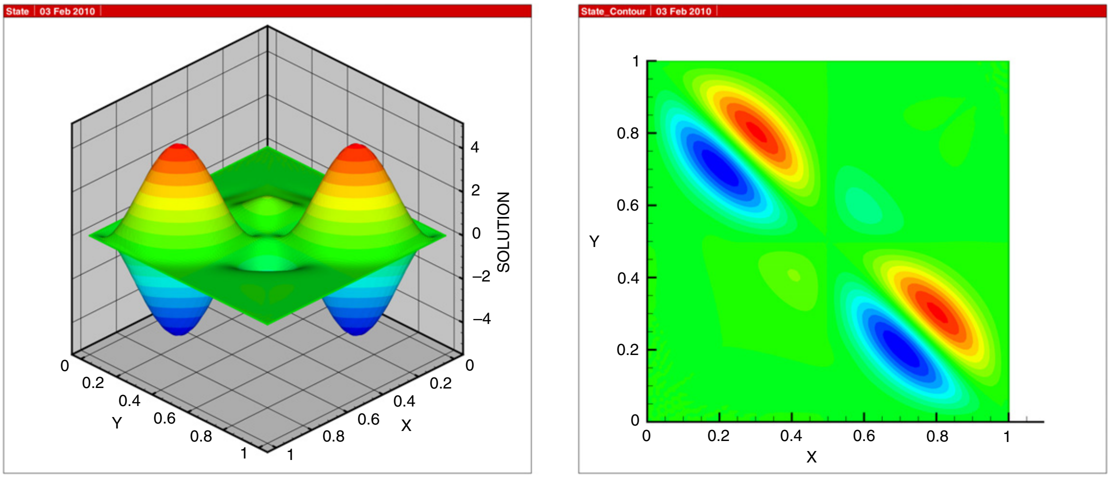

Fig. 2. The approximate state solution (left) and its contour line (right) for Example 1 with $h = 1/80$.

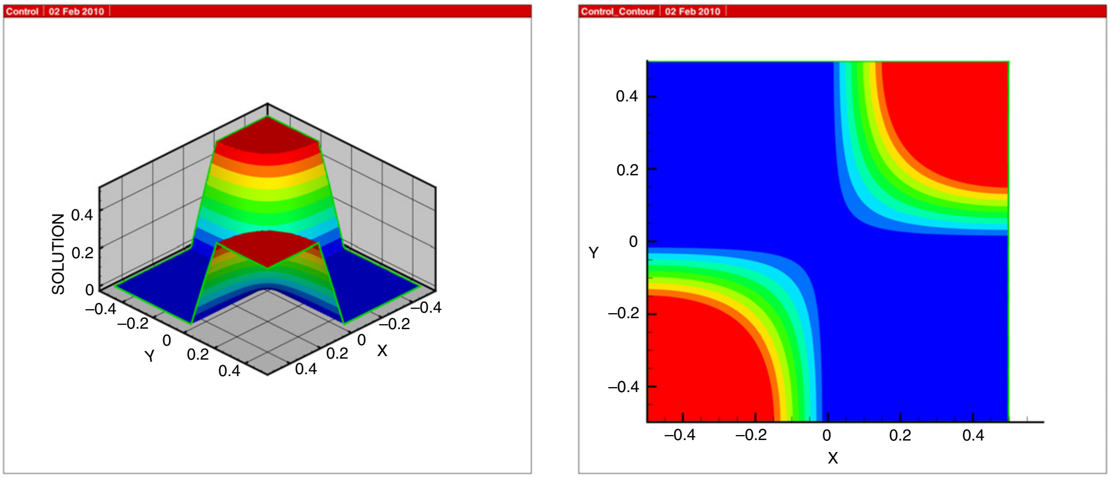

Fig. 3. The approximate control solution (left) and its contour line (right) for Example 2 with $h = 1/80$.

Table 1
Example 1 with the control approximated by piecewise constant elements.

h	$\|y - y_h\|$	Order	$\|p - p_h\|$	Order	$\|u - u_h\|$	Order
$\frac{1}{10}$	1.1568e+0	–	2.7331e−1	–	3.4787e−2	–
$\frac{1}{20}$	5.1605e−1	1.4092	1.4393e−1	0.9859	1.7204e−2	1.2129
$\frac{1}{40}$	2.5438e−1	1.2298	7.3791e−2	1.0942	9.4874e−3	1.0615
$\frac{1}{80}$	1.2672e−1	1.0512	3.7097e−2	1.0264	4.9471e−3	0.9594

Table 2
Example 2 with the control approximated by piecewise linear elements.

h	$\|y - y_h\|$	Order	$\|p - p_h\|$	Order	$\|u - u_h\|$	Order
$\frac{1}{10}$	4.2161e−2	–	2.3058e−2	–	1.3444e−2	–
$\frac{1}{15}$	2.0648e−2	1.7606	1.1315e−2	1.7558	6.8079e−3	1.6782
$\frac{1}{20}$	9.3726e−3	2.7456	6.4886e−3	1.9329	3.9697e−3	1.8749
$\frac{1}{25}$	5.3867e−3	2.4820	4.5622e−3	1.5785	2.6873e−3	1.7848
$\frac{1}{30}$	3.5462e−3	2.2805	3.2973e−3	1.7712	1.8940e−3	1.9084

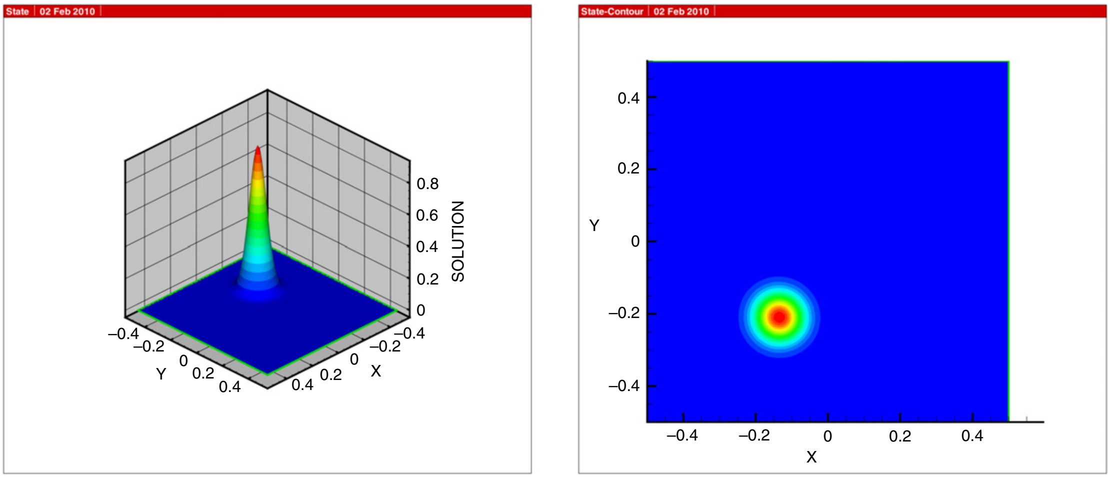

Fig. 4. The approximate state solution (left) and its contour line (right) for Example 2 with $h = 1/80$.

and try to lower the regularity assumptions on the optimal control problems in the coming work. Moreover, many computational issues have to be addressed, it is also important and challenging to investigate the adaptive computation for the time-dependent convection–diffusion optimal control problems.

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