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JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER

Journal of Computational and Applied Mathematics 168 (2004) 31–39

www.elsevier.com/locate/cam

Solution of convection–diffusion equation by the method of characteristics

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Received 30 September 2002; received in revised form 6 May 2003

Abstract

We solve a nonlinear convection–diffusion problem by the method of characteristics. The velocity field depends on the unknown solution and is generally not bounded. The convergence of the semi-discrete scheme is proved. Finally, on a one-dimensional numerical experiment computed by the ELLAM method we demonstrate some features of the scheme.

MSC: 62M25; 65M12; 35K60

Keywords: Method of characteristics; Nonlinear convection–diffusion equation; Convergence of numerical methods; Eulerian–Lagrangian methods

1. Introduction

We consider a nonlinear problem of the type

$$\partial_t u + F(u)\nabla u - D\,\Delta u = 0 \tag{1}$$

with an homogeneous Dirichlet boundary condition and with the initial condition

$$u(0,x) = u_0(x) \quad \text{in } \Omega. \tag{2}$$

Setting $F(u) \equiv u$ in (1) we get in the one-dimensional case the so-called Burgers equation

$$u_t + (0.5u^2 - Du_x)_x = 0. \tag{3}$$

If the case of $D = 0$, Eq. (1) represents a hyperbolic conservation law.

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2003.12.003

We solve our problem on a finite time interval $I \equiv (0,T)$. We divide I into n subintervals (t_{i-1}, t_i) with $t_i = i\tau$, where $\tau = T/n$ is the time step. We use the notation $u_i = u(t_i)$, $\Delta u_i = (u_i - u_{i-1})/\tau$. At the time interval (t_{i-1}, t_i) we approximate the characteristics by

$$\varphi_i(x) = x - \tau\omega_h * F(u_{i-1}), \tag{4}$$

where $\omega_h * f$ is the convolution of $f \in L_2(\Omega)$ with the mollifier ω_h (cf. [8]).

Then, at each time level we solve an elliptic problem

$$\frac{u_i - u_{i-1} \circ \varphi_i}{\tau} - D\,\Delta u_i = 0, \quad i = 1, \dots, n. \tag{5}$$

The approximation of the convective term can be improved by iterations

$$\frac{u_i^{(k)} - u_{i-1} \circ \varphi_i^{(k-1)}}{\tau} - D\,\Delta u_i^{(k)} = 0 \tag{6}$$

with $\varphi_i^{(k-1)}(x) = x - \tau\omega_h * F(u_i^{(k-1)})$, where $u_i^{(0)} = u_{i-1}$.

In the following text we use the notation $\|.\|$ and $\|.\|_\infty$ for the standard norms in the functional spaces $L_2(\Omega)$ and $L_\infty(\Omega)$. We denote $(u,v) \equiv \int_\Omega uv\,\mathrm{d}x$. Moreover, we put $V \equiv H_0^1$, where $H_0^1(\Omega)$ is the standard first-order Sobolev space with zero trace on the boundary Γ.

2. Variational formulation, assumptions

In related works, which deal with the convergence of the method of characteristics [2,7,8], the authors assume boundedness of the velocity field. Here we use a weaker assumption, namely that the velocity field $F(s)$ is continuous in s.

We also assume that Ω is a bounded domain with Lipschitz continuous boundary and that D is a symmetric positive definite matrix:

$$\xi^{\mathrm{T}}(D\xi) \geqslant D_0|\xi^2| \quad \forall x \in \Omega \quad \forall \xi \in \mathbb{R}^N, \tag{7}$$

where $D_0 > 0$ is a constant. Further we take $u_0 \in L_\infty(\Omega) \bigcap V$.

The variational formulation of problem (5) reads as follows: Find $u_i \in V$ such that

$$\left(\frac{u_i - u_{i-1}}{\tau}, v\right) + (D\nabla u_i, \nabla v) = -\left(\frac{u_{i-1} - u_{i-1} \circ \varphi_i}{\tau}, v\right) \quad \forall v \in V \tag{8}$$

for $i = 1, \dots, n$.

Lemma 1. *Let us take in (4) $h = \tau^\omega$ where $\omega \in (0,1)$. If $\|u_i\|_\infty \leqslant C$, then we have*

$$\tfrac{1}{2}|x_1 - x_2| \leqslant |\varphi_i(x_1) - \varphi_i(x_2)| \leqslant 2|x_1 - x_2| \quad \forall x_1, x_2 \in \bar{\Omega} \tag{9}$$

for $i = 1, \dots, n$, when $\tau \leqslant \tau_0$ is taken sufficiently small.

Proof. This is similar to the proof of Lemma 2.2 in [8]. □

Remark 2. The previous lemma guarantees that the characteristics do not intersect with each other.

3. Convergence

In what follows C is a generic positive constant, not depending on n.

We begin with four lemma's. The proof of the first lemma is skipped as it proceeds similar to the proof of a result in [8].

Lemma 3. *Let the assumptions of Lemma* 1 *hold. If* $u_{i-1} \in V$, *then also* $u_{i-1} \circ \varphi_i \in V$, *where* φ_i *is defined in* (4). *Moreover, we have*

$$\|u_{i-1} \circ \varphi_i\|_{H^1(\Omega)} \leqslant C\|u_{i-1}\|_{H^1(\Omega)}. \tag{10}$$

Lemma 4. *If* $u_0 \in L_\infty(\Omega)$ *then*

$$\|u_i\|_\infty \leqslant C, \quad i = 1, \ldots, n. \tag{11}$$

Proof. Let us set $v = u_i^p$ in (8), where p is an odd number. Then, we get

$$\int_\Omega (u_i)^{p+1}\,\mathrm{d}x + pD_0\tau \int_\Omega (u_i)^{p-1}(\nabla u_i)^2\,\mathrm{d}x \leqslant \int_\Omega (u_i)^p u_{i-1} \circ \varphi_i\,\mathrm{d}x. \tag{12}$$

Using the inequality

$$xy \leqslant \frac{1}{p+1}x^{p+1} + \frac{p}{p+1}y^{(p+1)/p}, \tag{13}$$

we deduce from (12) that

$$\begin{aligned}\int_\Omega (u_i)^{p+1}\,\mathrm{d}x &\leqslant \int_\Omega (u_i)^p u_{i-1} \circ \varphi_i\,\mathrm{d}x \\ &\leqslant \frac{1}{p+1}\int_\Omega (p(u_i)^{p+1} + (u_{i-1} \circ \varphi)^{p+1})\,\mathrm{d}x. \end{aligned} \tag{14}$$

This leads to

$$\int_\Omega (u_i)^{p+1}\,\mathrm{d}x \leqslant \int_\Omega (u_{i-1} \circ \varphi)^{p+1}\,\mathrm{d}x. \tag{15}$$

Passing to the limit $p \to \infty$, we see that $\|u_i\|_\infty \leqslant \|u_{i-1} \circ \varphi_i\|_\infty$. Further, we recurrently get $\|u_{i-1} \circ \varphi_i\|_\infty \leqslant \|u_{i-1}\|_\infty \leqslant \|u_0\|_\infty$. Therefore, $\|u_i\|_\infty \leqslant \|u_0\|_\infty$, from which we get the assertion of our lemma. □

Lemma 5. *Retain the assumption of Lemmas* 4 *and* 1. *The a priori estimates*

$$\sum_{i=1}^{j} \|\nabla u_i\|^2\tau \leqslant C, \tag{16}$$

$$\|u_j\| \leqslant C \tag{17}$$

hold for $j = 1, \ldots, n$.

Proof. On account of the assumptions mentioned and of Lemma 1, problem (8) makes sense for $i=1,\dots,n$. The existence of a unique solution $u_i \in V$ is guaranteed by the Lax–Milgram lemma.

Let us put in (8) $v=u_i$ and sum up for $i=1,\dots,j$, $(j \leqslant n)$. From (7) we get

$$\sum_{i=1}^{j}(u_i-u_{i-1},u_i)+D_0\tau\sum_{i=1}^{j}\|\nabla u_i\|^2 \leqslant -\sum_{i=1}^{j}((u_{i-1}-u_{i-1}\circ\varphi_i),u_i). \tag{18}$$

The first term on the left-hand side is rewritten by the Abel's summation formula:

$$\sum_{i=1}^{j}(u_i-u_{i-1},u_i)=\frac{1}{2}\|u_j\|^2-\frac{1}{2}\|u_0\|^2+\frac{1}{2}\sum_{i=1}^{j}\|u_i-u_{i-1}\|^2. \tag{19}$$

The right-hand side of (18) can be rewritten by the mean value theorem when recalling (4):

$$\begin{aligned}(u_{i-1}-u_{i-1}\circ\varphi_i)(x)&=(x-\varphi_i(x))\int_0^1 \nabla u_{i-1}(x-s(x-\varphi_i(x)))\,\mathrm{d}s\\ &=\tau\omega_h * F(u_{i-1})\int_0^1 \nabla u_{i-1}(x-\tau\omega_h * F(u_{i-1})s)\,\mathrm{d}s.\end{aligned} \tag{20}$$

Next, using the boundedness of $\omega_h * F(u_{i-1})$, Cauchy and Young's inequalities, we get, for $\eta>0$ arbitrarily small,

$$\left|\sum_{i-1}^{j}((u_{i-1}-u_{i-1}\circ\varphi_i,u_i))\right| \leqslant \frac{\eta\tau}{2}\sum_{i=1}^{j}\left\|\int_0^1 \nabla u_{i-1}\circ\varphi_{i,s}\,\mathrm{d}s\right\|^2+\frac{\tau}{2\eta}C\sum_{i=1}^{j}\|u_i\|^2, \tag{21}$$

where $\varphi_{i,s}(x)=x-\tau\omega_h * F(u_{i-1})s$, $0<s<1$. Property (9) holds also for $\varphi_{i,s}(x), 0<s<1$. Analogous to (10) we have

$$\left\|\int_0^1 \nabla u_{i-1}\circ\varphi^{i,s}\,\mathrm{d}s\right\|^2 \leqslant C\|\nabla u_{i-1}\|^2 \quad \forall i=1,\dots,n. \tag{22}$$

Combining (20)–(22) with (18) we find

$$\|u_j\|^2+(2D_0-\eta C)\tau\sum_{i=1}^{j}\|\nabla u_i\|^2 \leqslant \|u_0\|^2+\eta C\tau\|\nabla u_0\|^2+\frac{C}{\eta}\tau\sum_{i=1}^{j}\|u_i\|^2, \tag{23}$$

where C is the constant from (22). Taking η such that $2D_0-\eta C>0$ and using the discrete Gronwall lemma, the assertion of our lemma follows. □

Lemma 6. *The following a priori estimates hold under the same assumptions as in Lemma* 5:

$$\|\nabla u_j\| \leqslant C, \tag{24}$$

$$\sum_{i=1}^{j}\|\delta u_i\|^2\tau \leqslant C, \quad j=1,\dots,n. \tag{25}$$

Proof. We put $v=u_i-u_{i-1}$ in (8) and sum up for $i=1,\dots,j$. Then, by using similar arguments as in the previous lemma and by taking τ sufficiently small, we arrive at the estimates above. □

Let us define the standard Rothe functions

$$u^n(t) = u_i + (t - t_{i-1})\delta u_i \quad \text{for } t \in [t_{i-1}, t_i] \tag{26}$$

and

$$\bar{u}^n(t) = u_i \quad \text{for } t \in (t_{i-1}, t_i], \quad \bar{u}^n(0) = u_0. \tag{27}$$

We can rewrite the a priori estimates (16), (25) in the form

$$\int_I \int_\Omega |\nabla \bar{u}^n|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant C,$$

$$\int_I \int_\Omega |\partial_t u^n|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant C. \tag{28}$$

We denote by $C(I, L_2(\Omega))$ and $L_2(I, V)$ spaces of abstract functions $u : I \to L_2(\Omega)$ and $u : I \to V$ which are continuous and square integrable (in Bochner sense), respectively (see [9]). We can prove the following result.

Theorem 7. *We recall the assumptions on F, Ω, D, u_0 from the beginning of Section 2. Then, if the variational solution of (1)–(2) is unique, the method of characteristics is convergent in the sense that*

$$u^n \to u \quad \text{in } C(I, L_2(\Omega)) \quad \text{and} \quad \bar{u}^n \to u \quad \text{in } L_2(I, V) \tag{29}$$

for $\tau \to 0$. Here, $u \in L_2(I, V) \bigcap C(I, L_2)$ is the variational solution of (1)–(2) and $\partial_t u \in L_2(I, L_2)$. Moreover, u^n and $\bar{u}^n$ are defined in (26), (27).

Proof. From (25) and the definition of the Rothe functions we have

$$\begin{aligned} \int_I \|\bar{u}^n - u^n\|^2 \, \mathrm{d}t &= \int_I \left\| u_i - \left(u_{i-1} + \frac{t - t_{i-1}}{\tau}(u_i - u_{i-1}) \right) \right\|^2 \mathrm{d}t \\ &\leqslant C \sum_{i=1}^n \|\delta u_i\|^2 \tau^2 \leqslant \frac{C}{n}. \end{aligned} \tag{30}$$

Lemma 5 implies the boundedness of $\bar{u}^n$ in the Hilbert space $L_2(I, V)$. Moreover, recalling that the embedding $H^1(\Omega \times I) \hookrightarrow L_2(\Omega \times I)$ is compact, from the sequence $\{\bar{u}^n\}$ we can select subsequences (for simplicity denoted again by $\bar{u}^n$) so that,

$$\bar{u}^n \to \chi \quad \text{in } L_2(\Omega \times I),$$

$$\bar{u}^n \rightharpoonup \chi_1 \quad \text{in } L_2(I, V). \tag{31}$$

According to (20), we can rewrite the variational formulation (8) in the form

$$(\partial_t u^n, v) + (D\nabla \bar{u}^n, \nabla v) = -\left(\omega_h * F(\bar{u}_\tau^n) \int_0^1 \bar{u}_\tau^n(x - s(x - \varphi_i)) \, \mathrm{d}s, v \right) \tag{32}$$

where $\bar{u}_\tau^n \equiv \bar{u}^n(t - \tau)$, taking $\bar{u}^n(t) = u^n(t) = 0$ if $t \notin I$.

Let us subtract (32) for $n=r$ from (32) for $n=s$. We put $v=\bar{u}^r-\bar{u}^s$ and integrate over the interval $(0,t)$. Then we get

$$\frac{1}{2}\|u^r(t)-u^s(t)\|^2+\int_0^t \|\nabla(\bar{u}^r-\bar{u}^s)\|^2\,\mathrm{d}t$$
$$\leqslant \int_0^t (\partial_t(u^r-u^s),u^r-\bar{u}^r-(u^s-\bar{u}^s))\,\mathrm{d}t$$
$$+C\int_0^t \left(\int_0^1 \nabla\bar{u}_\tau^r(x-z(x-\varphi_i^r))-\nabla\bar{u}_\tau^s(x-z(x-\varphi_i^s))\,\mathrm{d}z,\bar{u}^r-\bar{u}^s\right)\,\mathrm{d}t, \tag{33}$$

where $\varphi_i^r=x-\tau\omega_h * F(\bar{u}_\tau^r)$.

We rewrite the first term on the right-hand side of the above inequality by using the Cauchy–Schwartz inequality. Then, recalling (28), (30) and (31) we find

$$\int_0^t (\partial_t(u^r-u^s),u^r-\bar{u}^r-(u^s-\bar{u}^s))\,\mathrm{d}t$$
$$\leqslant C\left(\int_0^t(\|\partial_t u^r\|^2+\|\partial_t u^s\|^2)\,\mathrm{d}t\int_0^t(\|u^r-\bar{u}^r\|^2+\|u^s-\bar{u}^s\|^2)\,\mathrm{d}t\right)^{1/2}$$
$$\leqslant C\left(\frac{1}{s}+\frac{1}{r}\right)^{1/2}\to 0 \quad \text{for } r\to\infty, s\to\infty. \tag{34}$$

The second term on the right-hand side of (33) is estimated by using (10) and the Cauchy–Schwartz inequality. From (28) and (31) we obtain

$$\int_0^t \left(\int_0^1 \nabla\bar{u}_\tau^r(x-z(x-\varphi_i^r))-\nabla\bar{u}_\tau^s(x-z(x-\varphi_i^s))\,\mathrm{d}z,\bar{u}^r-\bar{u}^s\right)\,\mathrm{d}t$$
$$\leqslant C\left(\int_0^t(\|\nabla\bar{u}^r\|^2+\|\nabla\bar{u}^s\|^2)\,\mathrm{d}t\right)^{1/2}\left(\int_0^t\|\bar{u}^r-\bar{u}^s\|^2\,\mathrm{d}t\right)^{1/2}$$
$$\leqslant C\left(\int_0^t\|\bar{u}^r-\bar{u}^s\|^2\,\mathrm{d}t\right)^{1/2}\to 0 \quad \text{for } r\to\infty,\ s\to\infty. \tag{35}$$

Consequently, by combining (34) and (35), we estimate (33) by

$$\|u^r(t)-u^s(t)\|^2+\int_0^t\|\nabla(\bar{u}^r-\bar{u}^s)\|^2\,\mathrm{d}t\leqslant C_{r,s}, \tag{36}$$

where $C_{r,s}\to 0$ for r, $s\to\infty$.

The previous result implies the strong convergence (in the sense of subsequence from (31)) $u^n\to u$ in $C(I,L_2)$ and also $\bar{u}^n\to u$ in $L_2(I,V)$ for $n\to\infty$. From this (see [6]) follows the existence of $\partial_t u\in L_2(I,L_2)$ and the weak convergence $\partial_t u^n\rightharpoonup\partial_t u$ in $L_2(I,L_2)$.

To complete the proof we need to show that $\int_0^1\nabla\bar{u}_\tau^n(x-s(x-\varphi_i))\,\mathrm{d}s\rightharpoonup\nabla u$ for $n\to\infty$.

We denote $w_\tau^n \equiv \int_0^1 \bar{u}_\tau^n(x - s(x-\varphi_i))\,\mathrm{d}s$. Then from (10) and (28) we have

$$\int_I \|\nabla w_\tau^n\|^2 \leqslant \int_I \|\nabla \bar{u}_\tau^n(x - s(x-\varphi_i))\|^2 \leqslant C. \tag{37}$$

Therefore, from $\{\nabla w_\tau^n\}$ we can select in $L_2(I, L_2)$ a subsequence (which we again denote ∇w_τ^n), such that

$$\nabla w_\tau^n \rightharpoonup \chi \quad \text{in } L_2(I, L_2). \tag{38}$$

From the boundedness of $\omega_h * F(\bar{u}_\tau^n)$ and by the mean value theorem we get

$$\begin{aligned}\int_I \|w_\tau^n - \bar{u}_\tau^n\|^2 &\leqslant \tau\|\omega_h * F(\bar{u}_\tau^n)\|_\infty \int_0^1\int_0^1\int_I \|\nabla \bar{u}_\tau^n(x - sr(x-\varphi_i(x)))\|^2\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s\\ &\leqslant \tau C\int_I \|\nabla \bar{u}_\tau^n\|^2\,\mathrm{d}t \to 0 \quad \text{for } \tau \to 0.\end{aligned} \tag{39}$$

Consequently, from the convergence $\bar{u}_\tau^n \to u$ in $L_2(I, L_2)$ for $n \to \infty$ we have, using Lebesgue theorem, that also $w_\tau^n \to u$ in $L_2(I, L_2)$. Because $\nabla w_\tau^n \rightharpoonup \chi$ for $n \to \infty$ we obtain

$$\int_I\int_\Omega u\nabla v\,\mathrm{d}x\,\mathrm{d}t = -\int_I\int_\Omega \chi v\,\mathrm{d}x\,\mathrm{d}t, \quad v \in H_0^1(\Omega) \tag{40}$$

from which we infer that $\chi = \nabla u$.

Next, recalling the continuity of F we can pass to the limit for $n \to \infty$ and we obtain

$$\int_0^t\int_0^1 (\omega_h * F(\bar{u}_\tau^n)\nabla \bar{u}_\tau^n(x - s(x-\varphi_i)), v)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \to \int_0^t (F(u)\nabla u, v)\,\mathrm{d}t. \tag{41}$$

We integrate (32) over the interval $(0, t)$ to get

$$\int_0^t (\partial_t u^n, v) + \int_0^t (D\nabla \bar{u}^{\,n}, \nabla v) + \int_0^1 (\omega_h * F(\bar{u}_\tau^n)\int_0^t \nabla \bar{u}_\tau^n(x - s(x-\varphi_i))\,\mathrm{d}s, v) = 0. \tag{42}$$

Passing to the limit for $n \to \infty$ we arrive at

$$\int_0^t (\partial_t u, v) + \int_0^t (D\nabla u, \nabla v) + \int_0^t (F(u)\nabla u, v) = 0. \tag{43}$$

Taking the time derivative of (43) we deduce that u is the variational solution of (1).

According to the fact that the solution u of (1) is unique, we can select from every subsequence of the original sequences $\{\bar{u}^{\,n}\}$, $\{u^n\}$ a subsequence that converges to u. Thus the whole sequences converge to u, namely $u^n \to u$ in $C(I, L_2)$ and $\bar{u}^{\,n} \to u$ in $L_2(I, V)$ for $n \to \infty$. □

4. Numerical experiment

There is a variety of methods (see [2,3,5,7,10]) based on the original method of characteristics [4]. Here we used the ELLAM method with basis functions that are constant along the characteristics. The method used has the important mass conservation property. This method is closely related to

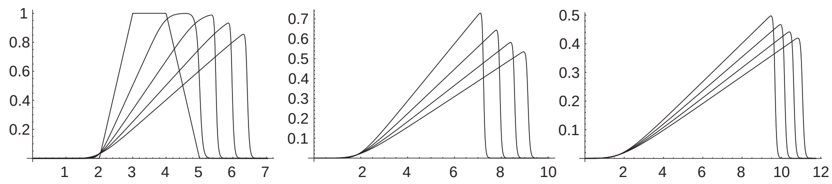

Fig. 1. Solution in time: $0, 1, 2, 3, 4, 6, 8, 10, 12, 14, 16, 18, 20$.

the method of characteristics in the case of (1). Similar arguments as in this paper can be used for its convergence analysis. For details on the method, see for instance [10].

We solved the nonlinear Burgers equation (3) with the following parameters: $D = 0.01$, $\tau = 0.1$, $h = 0.0014$. We dealt with the nonlinearity by using the iterative scheme (6) with less than eight iterations on every time step. In this example it was not necessary to use the smoothing of characteristics (4). For the space discretization we used linear finite elements. In Fig. 1, on the left we clearly see the formation of a shock, which is a typical property of the solution of (3). The method also maintains the mass balance. The depicted results are similar to results obtained by the method of characteristics combined with adaptive cubic finite elements. Moreover, we conclude that the proposed method does not suffer from extensive numerical dispersion or nonphysical oscillations, which is the main drawback of standard finite element methods when used for convection-dominated problems (cf. [1]).

Acknowledgements

The author would like to express his thanks to Jozef Kačur for his valuable advices concerning the preparation of this manuscript. He is also indebted to Roger Van Keer for reading the paper and constructive comments.

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