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TWO-GRID METHOD FOR CHARACTERISTICS FINITE-ELEMENT SOLUTION OF 2D NONLINEAR CONVECTION-DOMINATED DIFFUSION PROBLEM *

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Abstract: For two-dimension nonlinear convection diffusion equation, a two-grid method of characteristics finite-element solution was constructed. In this method the nonlinear iterations is only to execute on the coarse grid and the fine-grid solution can be obtained in a single linear step. For the nonlinear convection-dominated diffusion equation, this method can not only stabilize the numerical oscillation but also accelerate the convergence and improve the computational efficiency. The error analysis demonstrates if the mesh sizes between coarse-grid and fine-grid satisfy the certain relationship, the two-grid solution and the characteristics finite-element solution have the same order of accuracy. The numerical example confirms that the two-grid method is more efficient than that of characteristics finite-element method.

Key words: convection-diffusion equation; characteristics finite-element; two-grid method; convergence

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Introduction

Convection-diffusion equation governs such phenomena as the flow of heat within a moving fluid, the transport of dissolved nutrients or contaminants within the groundwater, and the transport of a surfactant or tracer within an incompressible oil in a petroleum reservoir. We consider the two-dimension nonlinear equation

$$c(\boldsymbol{x},t)\frac{\partial u}{\partial t} + \boldsymbol{b}\cdot\nabla u - \nabla\cdot(a(\boldsymbol{x},t)\nabla u) = f(u), \qquad (\boldsymbol{x},t)\in\Omega\times J, \tag{1a}$$

$$u(\boldsymbol{x},t) = 0, \qquad (\boldsymbol{x},t)\in\partial\Omega\times J, \tag{1b}$$

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$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{1c}$$

where Ω is bounded in R^2. $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}, J = [0,T], \boldsymbol{b}(\boldsymbol{x},t) = (b_1(\boldsymbol{x},t), b_2(\boldsymbol{x},t))^{\mathrm{T}}, f(u) = f(u,\boldsymbol{x},t)$. For $(\boldsymbol{x},t) \in \Omega \times J$, we assume

(a) $0 < a_* \leqslant a(\boldsymbol{x},t) \leqslant a^*,\ 0 < c_* \leqslant c(\boldsymbol{x},t) \leqslant c^*,$

$|\boldsymbol{b}(\boldsymbol{x},t)| = \sqrt{b_1^2 + b_2^2} \leqslant b^* < \infty;$

(b) $\left|\frac{\boldsymbol{b}}{c}\right| + \left|\frac{\partial}{\partial x_i}\left(\frac{\boldsymbol{b}}{c}\right)\right| \leqslant C_1, \qquad i = 1,2;$

(c) $\left|\frac{\partial f}{\partial x_i}\right| + \left|\frac{\partial f}{\partial u}\right| + \left|\frac{\partial^2 f}{\partial u^2}\right| \leqslant C_2, \qquad i = 1,2,$

where a_*, a^*, c_*, c^*, C_1 and C_2 are positive constants. And we also assume that the solution of the problem (1) satisfies

(d) $u \in L^\infty(0,T;H^{r+1}) \cap H^1(0,T;H^{r+1}) \cap H^2(0,T;H^1);$

(e) $\frac{\partial u}{\partial t} \in L^2(0,T;H^{r+1});$

(f) $\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;L^2(\Omega)),$

where $r \geqslant 1$. Let $W^{m,p}(\Omega)$ denote the Sobolev space on Ω, and $W^{m,2}(\Omega) = H^m(\Omega)$, $\|v\|_{H^m(\Omega)} = \|v\|_m$, $\|v\|_{L^2(\Omega)} = \|v\|$.

We use the modified method of characteristics finite-element developed by Douglas[1] and Russell[2], to discrete in space with equal-order accuracy in u. To linearize the resulting discrete equations, we use a two-grid scheme, which allows us to iterate on a grid much coarser than that used for the final solution.

We owe the impetus for using a two-grid approach to XU[3,4]. One solves the nonlinear problem via Newton-like iterations. After convergence on the coarse grid, one then extrapolates back to the fine grid using a Taylor expansion. This method has been extended to nonlinear reaction-diffusion equations[5,6] and N-S equations[7].

1 Description of Method

1.1 Characteristics finite-element method

We begin by briefly reviewing the characteristics finite-element discretization of the problem (1). Let

$$\psi(\boldsymbol{x},t) = [c(\boldsymbol{x},t)^2 + |\boldsymbol{b}(\boldsymbol{x},t)|^2]^{1/2}, \tag{2}$$

and let the characteristic direction associated with the operator $cu_t + \boldsymbol{b} \cdot \nabla u$ be denoted by $\tau = \tau(\boldsymbol{x},t)$, where

$$\frac{\partial}{\partial \tau} = \frac{c(\boldsymbol{x},t)}{\psi(\boldsymbol{x},t)}\frac{\partial}{\partial t} + \frac{\boldsymbol{b}(\boldsymbol{x},t)}{\psi(\boldsymbol{x},t)} \cdot \nabla. \tag{3}$$

Then, Eq. (1) can be put in form

$$\begin{cases} \psi(\boldsymbol{x},t)\dfrac{\partial u}{\partial \tau} - \nabla \cdot (a(\boldsymbol{x})\nabla u) = f(u), & (\boldsymbol{x},t) \in \Omega \times J, \\ u(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \partial\Omega \times J, \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{4}$$

Let $V = H_0^1(\Omega)$, and introduce $(u,v) = \int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$, $v, w \in V$. Then applying Green formula, Eq. (4) can be written in the equivalent variation form

$$\left(\psi \frac{\partial u}{\partial \tau}, v\right) + (a \nabla u, \nabla v) = (f(u), v), \qquad v \in V, \tag{5a}$$

$$(u(0), v) = (u_0, v), \qquad v \in V. \tag{5b}$$

We consider a time step Δt and approximate the solution at times $t^n = n\Delta t, n = 0, 1, \cdots, N; \Delta t = T/N$. The characteristics derivative is approximated in the following way at $t = t^n$:

$$\left(\psi \frac{\partial u}{\partial \tau}\right)^n \approx \psi(x, t^n) \frac{u(x, t^n) - u(\bar{x}, t^{n-1})}{(|x - \bar{x}|^2 + (\Delta t^n)^2)^{1/2}} = c(x, t^n) \frac{u(x, t^n) - u(\bar{x}, t^{n-1})}{\Delta t}. \tag{6}$$

Namely, a backtracking algorithm is used to approximate the characteristic derivative. $\bar{x} = x - \frac{b(x,t)}{c(x,t)}\Delta t$ is the foot (at level t^{n-1}) of the characteristic corresponding to x at the head (at level t^n).

We consider the quasi-uniform triangulations or rectangles of Ω. A coarse partition with mesh size H denoted by Δ_h, and a refinement of this partition with mesh size h denoted by Δ_h. Let V_h be a finite-element subspace on Δ_h. The characteristics finite-element method for Eq. (1) is defined: For $n = 0, 1, \cdots, N$, find $u_h^n \in V_h$ such that

$$\left(c^n \frac{u_h^n - \bar{u}_h^{n-1}}{\Delta t}, v\right) + (a^n \nabla u_h^n, \nabla v) = (f(u_h^n), v), \qquad v \in V_h, \tag{7a}$$

$$(u_h^0 - u_0, v) = 0, \qquad v \in V_h, \tag{7b}$$

where

$$u_h^n = u_h(t^n), \ \bar{u}_h^{n-1}(x) = u_h(\bar{x}, t^{n-1}) = u_h\left(x - \frac{b(x,t^n)}{c(x,t^n)}\Delta t, t^{n-1}\right). \tag{8}$$

The initial approximation u_h^0 can be defined as any reasonable approximation of u_0 in V_h such as the interpolation of u_0 in V_h. Eq. (7) determines $\{u_h^n\}$ uniquely in terms of the data u_0.

1.2 Two-grid method

To solve the system (7), we use the Newton-like iterations. The idea is to devote all of the effort of nonlinear iteration to coarse-grid problems. Thus the iterations yield approximate finite-element solutions u_H^n on a coarse subgrid $\Delta_H \subset \Delta_h$ having mesh size $H > h$ and corresponding space V_H. The two-grid method has two stages.

Stage 1: On the coarse grid Δ_H, solve the nonlinear system for $u_H^n \in V_H$:

$$\left(c^n \frac{u_H^n - \bar{u}_H^{n-1}}{\Delta t}, v\right) + (a^n \nabla u_H^n, \nabla v) = (f(u_H^n), v), \qquad \forall v \in V_H, \tag{9a}$$

$$(u_H^0 - u_0, v) = 0, \qquad \forall v \in V_H. \tag{9b}$$

Stage 2: Then, on the fine grid Δ_h, solve the linear system for $\hat{u}_h^n \in V_h$:

$$\left(c^n \frac{\hat{u}_h^n - \bar{\hat{u}}_h^{n-1}}{\Delta t}, v\right) + (a^n \nabla \hat{u}_h^n, \nabla v)$$
$$= (f(u_H^n) + f'(u_H^n)(\hat{u}_h^n - u_H^n), v), \qquad \forall v \in V_h, \tag{10a}$$

$$(\hat{u}_h^0 - u_0, v) = 0, \qquad \forall v \in V_h. \tag{10b}$$

By the two stages, we can get the approximation $\hat{u}_h^n$ of the characteristic finite-element solution u_h^n.

2 Convergence Analysis

For the finite space V_h and every $v \in H^s$, the following approximation property holds:

$$\inf_{v_h \in V_h} \left(\| v - v_h \| + h \| v - v_h \|_1 + h(\| v - v_h \|_\infty + h \| v - v_h \|_{1,\infty}) \right)$$
$$\leqslant Ch^s \| v \|_s, \qquad 2 \leqslant s \leqslant r+1. \tag{11}$$
$$\| v_h \|_\infty \leqslant Ch^{-1} \| v_h \|, \ \| v_h \|_{1,\infty} \leqslant Ch^{-1} \| v_h \|_1, \qquad v_h \in V_h. \tag{12}$$

Let $w_h: [0,T] \to V_h$ satisfy

$$(a \nabla (u - w_h), \nabla v) = 0, \qquad \forall v \in V_h. \tag{13}$$

Set $\eta = u - w_h, \xi = u_h - w_h$. Then $u - u_h = \eta - \xi$. It is well-known that[8], for $p = 2$ or ∞, and $2 \leqslant s \leqslant r+1$,

$$\| \eta \|_{L^p(0,T;L^2(\Omega))} + h \| \eta \|_{L^p(0,T;H^1(\Omega))} \leqslant Ch^s \| u \|_{L^p(0,T;H^s(\Omega))}, \tag{14}$$
$$\left\| \frac{\partial \eta}{\partial t} \right\|_{L^p(0,T;L^2(\Omega))} + h \left\| \frac{\partial \eta}{\partial t} \right\|_{L^p(0,T;H^1(\Omega))} \leqslant Ch^s \left[\| u \|_{L^p(0,T;H^s)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^p(0,T;H^s(\Omega))} \right]. \tag{15}$$

2.1 Error estimates for characteristics finite-element

Before we get the error estimates of the two-grid method, the error estimates of the characteristics finite element method for nonlinear equations should be derived. The H^1 and L^2 error estimates will be given in Theorem 2.1 and Theorem 2.2. Throughout this paper, letter C denotes a generic positive constant which may have different values at its different occurrences.

Theorem 2.1 *Let u and u_h be the respective solutions of Eqs. (5) and (7). Under Assumptions (a) - (f), we have*

$$\max_{1 \leqslant n \leqslant N} \| u^n - u_h^n \|_1 \leqslant C \left[\Delta t \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_{L^2(0,T;L^2(\Omega))} + h^r \left(\| u \|_{L^\infty(0,T;H^{r+1}(\Omega))} \right.\right.$$
$$\left.\left. + \left\| \frac{\partial u}{\partial t} \right\|_{H^1(0,T;H^{r+1}(\Omega))} \right) \right] \leqslant C(\Delta t + h^r) \tag{16}$$

for Δt sufficiently small and $r \geqslant 2$.

Proof At $t = t^n$, subtract Eq. (5a) from Eq. (7a) yields

$$\left(c^n \frac{\xi^n - \xi^{n-1}}{\Delta t}, v \right) + (a^n \nabla \xi^n, \nabla v)$$
$$= \left(\psi^n \frac{\partial u^n}{\partial \tau} - c^n \frac{u^n - \bar{u}^{n-1}}{\Delta t}, v \right) + \left(c^n \frac{\eta^n - \bar{\eta}^{n-1}}{\Delta t}, v \right)$$
$$- \left(c^n \frac{\xi^{n-1} - \bar{\xi}^{n-1}}{\Delta t}, v \right) + (f(u^n) - f(u_h^n), v), \tag{17}$$

where $\bar{\eta}^{n-1} = \bar{u}^{n-1} - \bar{w}_h^{n-1}, \bar{\xi}^{n-1} = \bar{u}_h^{n-1} - \bar{w}_h^{n-1}$. Let $d_t \xi^n = (\xi^n - \xi^{n-1})/\Delta t$ and choose $v = \xi^n - \xi^{n-1} = d_t \xi^n \Delta t$, we get

$$(c^n d_t \xi^n, d_t \xi^n) \Delta t + (a^n \nabla \xi^n, \nabla \xi^n - \nabla \xi^{n-1})$$
$$= \left(\psi^n \frac{\partial u^n}{\partial \tau} - c^n \frac{u^n - \bar{u}^{n-1}}{\Delta t}, d_t \xi^n \right) \Delta t + \left(c^n \frac{\eta^n - \eta^{n-1}}{\Delta t}, d_t \xi^n \right) \Delta t$$
$$+ \left(c^n \frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}, d_t \xi^n \right) \Delta t + \left(c^n \frac{\bar{\xi}^{n-1} - \xi^{n-1}}{\Delta t}, d_t \xi^n \right) \Delta t$$
$$+ (f(u^n) - f(u_h^n), d_t \xi^n) \Delta t$$

$$\equiv T_1 + T_2 + T_3 + T_4 + T_5. \tag{18}$$

We now estimate terms $T_i, i = 1,2,\cdots,5$. Use the results given by Russell[2] to the right terms T_1, T_2, T_3, T_4, we have

$$|T_1| \leqslant C \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|^2_{L^2(t^{n-1},t^n;L^2)} (\Delta t)^2 + \varepsilon \| \mathrm{d}_t \xi^n \|^2 \Delta t, \tag{19}$$

$$|T_2| \leqslant C \left\| \frac{\partial \eta}{\partial t} \right\|^2_{L^2(t^{n-1},t^n;L^2)} + \varepsilon \| \mathrm{d}_t \xi^n \|^2 \Delta t, \tag{20}$$

$$|T_3| \leqslant C \| \nabla \eta^{n-1} \|^2 \Delta t + \varepsilon \| \mathrm{d}_t \xi^n \|^2 \Delta t, \tag{21}$$

$$|T_4| \leqslant C \| \nabla \xi^{n-1} \|^2 \Delta t + \varepsilon \| \mathrm{d}_t \xi^n \|^2 \Delta t. \tag{22}$$

For T_5, at any point $\boldsymbol{x} \in \Omega$, by Taylor theorem, we have

$$\begin{aligned}(f(u^n) - f(u_h^n), \mathrm{d}_t \xi^n)\Delta t &= (f'(\tilde{u}^n)(u^n - u_h^n), \mathrm{d}_t \xi^n)\Delta t \\ &= (f'(\tilde{u}^n)\eta^n, \mathrm{d}_t \xi^n)\Delta t - (f'(\tilde{u}^n)\xi^n, \mathrm{d}_t \xi^n)\Delta t\end{aligned}$$

for some value $\tilde{u}^n(\boldsymbol{x})$.

By Assumption (c), we get

$$| T_5 | \leqslant C(\| \eta^n \|^2 + \| \xi^n \|^2)\Delta t + \varepsilon \| \mathrm{d}_t \xi \|^2 \Delta t. \tag{23}$$

The inequalitie (19) – (23) can be combined with Eq. (18) to give the recursion relation

$$\begin{aligned}&(c^n \mathrm{d}_t \xi^n, \mathrm{d}_t \xi^n)\Delta t + (a^n \nabla \xi^n, \nabla \xi^n - \nabla \xi^{n-1}) \\ &\quad \leqslant C\Big[\left\| \frac{\partial^2 u}{\partial \tau^2} \right\|^2_{L^2(t^{n-1},t^n;L^2(\Omega))} (\Delta t)^2 + \left\| \frac{\partial \eta}{\partial t} \right\|_{L^2(t^{n-1},t^n;L^2(\Omega))} + (\| \eta^n \|^2 + \| \nabla \eta^{n-1} \|^2)\Delta t \\ &\qquad + (\| \xi^n \|^2 + \| \nabla \xi^{n-1} \|^2)\Delta t \Big] + 5\varepsilon \| \mathrm{d}_t \xi \|^2 \Delta t.\end{aligned} \tag{24}$$

By $a(a-b) \geqslant (a^2 - b^2)/2$, we have $(a^n \nabla \xi^n, \nabla \xi^n - \nabla \xi^{n-1}) \geqslant (a_*/2)[(\nabla \xi^n, \nabla \xi^n) - (\nabla \xi^{n-1}, \nabla \xi^{n-1})]$. With proper choice of the initial function u_h^0, we have $\xi^0 = 0$. Assumption (a) gives $(c^n \mathrm{d}_t \xi^n, \mathrm{d}_t \xi^n) \geqslant c_* \| \mathrm{d}_t \xi^n \|^2$. And $(\nabla \xi^n, \nabla \xi^n) \geqslant \| \nabla \xi^n \|^2$. Summing over $n = 1$ to $l (1 \leqslant l \leqslant N)$ at both sides of Eq. (24) yields

$$\begin{aligned}c_* \sum_{n=1}^{l} \| \mathrm{d}_t \xi^n \|^2 \Delta t + \frac{a_*}{2} \| \nabla \xi^l \|^2 &\leqslant C\Big[\sum_{n=1}^{l} \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|^2_{L^2(t^{n-1},t^n;L^2)} (\Delta t)^2 \\ &\quad + \sum_{n=1}^{l} \left\| \frac{\partial \eta}{\partial t} \right\|^2_{L^2(t^{n-1},t^n;L^2)} \Big] + \sum_{n=1}^{l} (\| \nabla \eta^{n-1} \|^2 + \| \eta^n \|)\Delta t \\ &\quad + \sum_{n=1}^{l} (\| \nabla \xi^{n-1} \|^2 + \| \xi^n \|^2)\Delta t + 5\varepsilon \sum_{n=1}^{l} \| \mathrm{d}_t \xi^n \|^2 \Delta t \\ &\leqslant C\Big[\left\| \frac{\partial^2 u}{\partial \tau^2} \right\|^2_{L^2(0,T;L^2)} (\Delta t)^2 + \left\| \frac{\partial \eta}{\partial t} \right\|^2_{L^2(0,T;L^2)} \\ &\quad + \sum_{n=1}^{l} \| \eta^n \|_1^2 \Delta t + \sum_{n=1}^{l} \| \xi^n \|_1^2 \Delta t \Big] + 5\varepsilon \sum_{n=1}^{l} \| \mathrm{d}_t \xi^n \|^2 \Delta t\end{aligned}$$

or

$$\begin{aligned}c_* \sum_{n=1}^{l} \| \mathrm{d}_t \xi^n \|^2 + \frac{a^*}{2} \| \nabla \xi^l \|^2 &\leqslant C \sum_{n=1}^{l} \| \xi^n \|_1^2 \Delta t + C\Big[\| \eta \|^2_{L^\infty(0,T;H^1)} \\ &\quad + \sum_{n=1}^{l} \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|^2_{L_2(t^{n-1},t^n,L^2)} (\Delta t)^2 + \sum_{n=1}^{l} \left\| \frac{\partial \eta}{\partial t} \right\|^2_{L^2(t^{n-1},t^n;L^2)} \Big] + 5\varepsilon \sum_{n=1}^{l} \| \mathrm{d}_t \xi^n \|^2 \Delta t.\end{aligned} \tag{25}$$

Our next step is to modify the left-hand side of Eq. (25), replacing the seminorm on ξ^l by a full H^1-norm. We note

$$\begin{aligned}\| \xi^n \|^2 - \| \xi^{n-1} \|^2 &= (\xi^n + \xi^{n-1}, \xi^n - \xi^{n-1}) \\ &= (\xi^n - \xi^{n-1}, \xi^n - \xi^{n-1}) + 2(\xi^{n-1}, \xi^n - \xi^{n-1}) \\ &= \| \mathrm{d}_t \xi^n \|^2 (\Delta t)^2 + 2(\xi^{n-1}, \mathrm{d}_t \xi^n)\Delta t.\end{aligned}$$

Summing both sides from $n = 1$ to l, we obtain

$$\begin{aligned}\| \xi^l \|^2 &\leqslant \Delta t \sum_{n=1}^{l} \| \mathrm{d}_t \xi^n \|^2 \Delta t + C \sum_{n=1}^{l} \| \xi^{n-1} \|^2 \Delta t + \varepsilon \sum_{n=1}^{l} \| \mathrm{d}_t \xi^n \|^2 \Delta t \\ &\leqslant C \sum_{n=1}^{l} \| \xi^{n-1} \|_1^2 \Delta t + \varepsilon \sum_{n=1}^{l} \| \mathrm{d}_t \xi^n \|^2 \Delta t\end{aligned} \tag{26}$$

for Δt sufficicently small. Adding Eq. (26) to both sides of Eq. (25) and choosing proper ε, we have

$$\begin{aligned}\| \xi^l \|_1^2 \leqslant C \sum_{n=1}^{l} \| \xi^n \|_1^2 \Delta t + C[\| \eta \|_{L^\infty(0,T;H^1)}^2 \\ + \sum_{n=1}^{l} \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_{L^2(t^{n-1},t^n;L^2)}^2 (\Delta t)^2 + \sum_{n=1}^{l} \left\| \frac{\partial \eta}{\partial t} \right\|_{L^2(t^{n-1},t^n;L^2)}^2].\end{aligned} \tag{27}$$

Applying the discrete Gronwall lemma, it follows that

$$\| \xi^l \|_1 \leqslant C\Big[\Delta t \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_{L^2(0,T;L^2(\Omega))} + \left\| \frac{\partial \eta}{\partial t} \right\|_{L^2(0,T;L^2)} + \| \eta \|_{L^\infty(0,T;H^1)} \Big]. \tag{28}$$

Which together with $u^n - u_h^n = \eta^n - \xi^n$ and Eqs. (11), (14) and (15) yield Eq. (16).

An optimal order estimate for $u - u_h$ in $L^2(\Omega)$ can be derived in a similar fashion.

Theorem 2.2 *Let u and u_h be the respective solutions of Eqs. (5) and (7). Under Assumptions (a) - (f), we have*

$$\max_{1 \leqslant n \leqslant N} \| u^n - u_h^n \| \leqslant C(\Delta t + h^{r+1}) \tag{29}$$

for Δt sufficiently small and $r \geqslant 1$.

2.2 Error estimates for two-grid method

Theorem 2.3 *Let u and $\hat{u}_h$ be the respective solutions of Eqs. (5) and (10). Under Assumptions (a) - (f), we have*

$$\max_{1 \leqslant n \leqslant N} \| u^n - \hat{u}_h^n \|_1 \leqslant C(\Delta t + h^r + H^{2r-1}) \tag{30}$$

for Δt sufficiently small and $r \geqslant 2$.

Proof To estimate the error, set $\eta^n = u^n - w_h^n$, $\hat{\xi}^n = \hat{u}_h^n - w_h^n$. Subtracting Eq. (5a) from its respective counterpart (10a) and choosing $v = \mathrm{d}_t \hat{\xi}^n \Delta t$, we have

$$\begin{aligned}&(c^n \mathrm{d}_t \hat{\xi}^n, \mathrm{d}_t \hat{\xi}^n) + (a^n \nabla \hat{\xi}^n, \nabla \hat{\xi}^n - \nabla \hat{\xi}^{n-1}) \\ &= \left(\psi^n \frac{\partial u^n}{\partial \tau} - c^n \frac{u^n - \bar{u}^{n-1}}{\Delta t}, \mathrm{d}_t \hat{\xi}^n\right)\Delta t + \left(c^n \frac{\eta^n - \eta^{n-1}}{\Delta t}, \mathrm{d}_t \hat{\xi}^n\right)\Delta t \\ &\quad + \left(c^n \frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}, \mathrm{d}_t \hat{\xi}^n\right)\Delta t - \left(c^n \frac{\hat{\xi}^{n-1} - \bar{\hat{\xi}}^{n-1}}{\Delta t}, \mathrm{d}_t \hat{\xi}^n\right)\Delta t \\ &\quad + (f(u^n) - f(u_H^n) + f'(u_H^n)(\hat{u}_h^n - u_H^n)), \mathrm{d}_t \hat{\xi}^n)\Delta t \\ &\equiv T_1 + T_2 + T_3 + T_4' + T_5'.\end{aligned}$$

For the estimate of T_5', a Taylor expansion of f about u_H^n yields

$$f(u^n) = f(u_H^n) + f'(u_H^n)(\hat{u}_h^n - u_H^n) + (1/2)f''(\tilde{u})(\hat{u}_h^n - u_H^n)^2$$

for some function $\tilde{u}$. Then

$$\begin{aligned} f(u^n) - [f(u_H^n) + f'(u_H^n)(\hat{u}_h^n - u_H^n)] &= f(u_H^n) + f'(u_H^n)(u^n - u_H^n) \\ &+ (1/2)f''(\tilde{u})(u^n - u_H^n)^2 - [f(u_H^n) + f'(u_H^n)(\hat{u}_h^n - u_H^n)] \\ &= f'(u_H^n)(u^n - \hat{u}_h^n) + (1/2)f''(\tilde{u})(u^n - u_H^n)^2 \\ &= f'(u_H^n)\eta^n - f'(u_H^n)\xi^n + (1/2)f''(\tilde{u})(u^n - u_H^n)^2. \end{aligned}$$

So that

$$\begin{aligned} &(f(u^n) - [(f(u_H^n) + f'(u_H^n)(\hat{u}_h^n - u_H^n)], \mathrm{d}_t \hat{\xi}) \\ &= (f'(u_H^n)\eta^n, \mathrm{d}_t \hat{\xi}^n) - (f'(u_H^n)\hat{\xi}^n, \mathrm{d}_t \hat{\xi}^n) + (1/2)(f''(\tilde{u})(u^n - u_H^n)^2, \mathrm{d}_t \hat{\xi}^n). \end{aligned} \tag{31}$$

From Assumption (c), we have

$$\begin{aligned} | T'_5 | \leqslant C_2 \| \eta^n \|^2 \Delta t + C_2 \| \hat{\xi}^n \|^2 \Delta t \\ + (C_2/2) \| (u^n - u_H^n)^2 \|^2 \Delta t + \varepsilon \| \mathrm{d}_t \hat{\xi}^n \|^2 \Delta t. \end{aligned}$$

For T_1, T_2, T_3, T'_4, we can estimate them as in Theorem 2.1. Then we obtain

$$\begin{aligned} c_* \sum_{n=1}^{l} \| \mathrm{d}_t \xi^n \|^2 \Delta t + \frac{a_*}{2} \| \nabla \xi^l \|^2 \leqslant C\Big[\sum_{n=1}^{l} \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|^2_{L^2(t^{n-1}, t^n; L^2)} (\Delta t)^2 \\ + \sum_{n=1}^{l} \left\| \frac{\partial \eta}{\partial t} \right\|^2_{L^2(t^{n-1}, t^n; L^2)} \Big] + \sum_{n=1}^{l} (\| \nabla \eta^{n-1} \|^2 + \| \eta^n \|^2) \Delta t \\ + \sum_{n=1}^{l} [\| \nabla \hat{\xi}^{n-1} \|^2 + \| \hat{\xi}^n \|^2] \Delta t + \sum_{n=1}^{l} \| (u^n - u_H^n)^2 \|^2 \Delta t + 5\varepsilon \sum_{n=1}^{l} \| \mathrm{d}_t \xi^n \|^2 \Delta t. \end{aligned} \tag{32}$$

Deriving as Eqs. (24) - (27), we get

$$\begin{aligned} \| \hat{\xi}^l \|_1 \leqslant C \Big[\Delta t \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_{L^2(0,T; L^2(\Omega))} + \left\| \frac{\partial \eta}{\partial t} \right\|_{L^2(0,T; L^2)} \\ + \| \eta \|_{L^\infty(0,T; H^1)} + \| (u^n - u_H^n)^2 \|_1 \Big]. \end{aligned} \tag{33}$$

Let $h = H$ in Eq. (16), and $u^n - \hat{u}_h^n = \eta^n - \xi^n$ together with Eqs. (11), (12), (14) and (15), yields Eq. (30).

An optimal order estimate for $u^n - \hat{u}_h^n$ in L^2 can be derived in the similar fashion.

Theorem 2.4 *Let u and $\hat{u}_h$ be the respective solutions of Eqs.* (5) *and* (10). *Under Assumptions* (*a*) - (*f*), *we have*

$$\max_{1 \leqslant n \leqslant N} \| u^n - \hat{u}_h^n \| \leqslant C(\Delta t + h^{r+1} + H^{2r+1}) \tag{34}$$

for Δt sufficiently small and $r \geqslant 1$.

Theorems 2.1 and 2.3 demonstrate a remarkable fact about two-grid scheme: when we iterate on a very coarse grid Δ_H and still get good approximations by taking one iteration on the fine grid Δ_h. Specifically, for $h^r = O(H^{2r-1})$, the two-grid scheme solution $\hat{u}_h$ has the same order of accuracy compared with the nonlinear iteration solution u_h. Theorems 2.2 and 2.4 tell us, when $h^{r+1} = O(H^{2r+1})$, we have the same conclusion.

3 Numerical Test

To illustrate the effectiveness of the two-grid linearization, we examine the following simple test problem:

$$\begin{cases} \frac{\partial u}{\partial t} + \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla u - \nabla \cdot (a(\boldsymbol{x}) \nabla u) = g(\boldsymbol{x},t), \\ \quad \boldsymbol{x} \in \Omega = [0,1]^2, \quad t \in (0,T], \\ u(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in \partial\Omega, t \in (0,T] \\ u(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \Omega, \end{cases} \tag{35}$$

where $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{b}(\boldsymbol{x}) = (1,1)$, $a(\boldsymbol{x}) = 0.001$, and $g(\boldsymbol{x},t) = -u^2 + G(\boldsymbol{x},t)$, $G(\boldsymbol{x},t)$ is determined by the exact solution $u(\boldsymbol{x},t) = tx_1x_2(1-x_1)(1-x_2)e^{x_1+x_2}$.

For $h = 2^{-3}$, we solve Eq. (35) by $\Delta t = 1.25 \times 10^{-4}$ from $t = 0$ to $t = 0.25$. Using the two-grid scheme presented in this paper, we give the coarse grid partition Δ_H with $H = h^{2/3} = 2^{-2}$ (Fig. 1) and get the nonlinear iteration solution u_H on Δ_H according to Eq. (9). Then we give the fine-grid partition Δ_h with $h = 2^{-3}$ (Fig. 2) and obtain the fine-grid solution $\hat{u}_h$ in one additional step on Δ_h by Eq. (10).

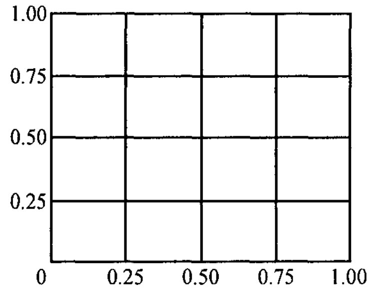

Fig. 1 The coarse-grid Δ_H with $H = 2^{-2}$

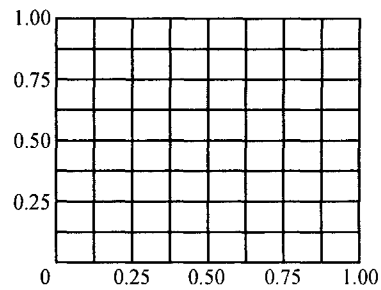

Fig. 2 The fine-gird Δ_h with $h = 2^{-3}$

To check up the numerical accuracy of the two-grid method, we obtain the nonlinear iteration solution u_h on the fine-grid Δ_h according to Eq. (7). Table 1 shows the L^2 errors in $\hat{u}_h$ and the CPU time for two-grid scheme when $H = h^{2/3}$. Table 2 shows the L^2 errors in u_h and the CPU time for nonlinear iteration method over the range of fine grids Δ_h.

Table 1 L^2-errors and CPU time of twogrid solution $\hat{u}_h$

h	H	$\frac{\Vert \hat{u}_h - u \Vert}{\Vert u \Vert}$	CUP time
2^{-3}	2^{-2}	3.1640×10^{-4}	30″

Table 2 L^2-errors and CPU time of nonlinear iteration solution u_h

h	$\frac{\Vert u_h - u \Vert}{\Vert u \Vert}$	CUP Time
2^{-3}	3.1085×10^{-4}	129″

Table 1 and Table 2 show that the two-grid solution $\hat{u}_h$ and the nonlinear iteration solution u_h have the same order of accuracy under different grid-partition. And the computational time of two-grid scheme is less than nonlinear iteration method. The convergence speed increases by about 4 times. The efficiency of the two-grid approach is obvious.

Two-grid linearization offers an attractive way to solve the nonlinear probems involving convection-diffusion equations. The key feature of the two-grid method is that it allows one to execute all of the nonlinear iterations on a system associated with a coarse spatial grid, without sacrificing the order of accuracy of the fine-grid solution. The two-grid scheme combined with the characteristics finite-element method, can not only stabilize the numerical oscillation caused by dominated convections, but also solve the nonlinear advection-dominated diffusion problems efficiently.

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