SDC: from preconditioned Richardson to the component wise iteration

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We want to prove that the preconditioned Picard iteration

$$\mathbf{U}^{k+1} = \mathbf{U}^k + P^{-1}(\mathbf{U}_0 - (I - \Delta t Q F(\mathbf{U}^k))) \tag{1}$$

with the preconditioned $P = (I - \Delta t Q_\Delta F)$ is equivalent to the component wise formula of the kth implicit SDC iteration[1]

$$U_m^{k+1} = U_{m-1}^{k+1} + \Delta t_m (f(U_m^{k+1}, t_m) - f(U_m^k, t_m)) + \Delta t S_m^{m+1} F(U^k). \tag{2}$$

Where

$$\mathbf{U}^k = [U_1^k, .., U_m^k, ..., U_M^k]^T,$$
$$F(\mathbf{U}^k) = [f(U_1^k, t_1), .., f(U_m^k, t_m), ..., f(U_M^k, t_M)]^T,$$

also Q is the matrix of the integration weights and S_m^{m+1} is the integration matrix for the interval $[t_m, t_{m+1}]$ s.t.

$$[Q\mathbf{U}]_m = \sum_{j=1}^{m} S_j^{j+1} \mathbf{U}, \tag{3}$$

or equivalently

$$[Q\mathbf{U}]_m \approx \int_{t_0}^{t_m} u(t) \mathrm{d}t, \qquad S_m^{m+1}\mathbf{U} \approx \int_{t_m}^{t_{m+1}} u(t) \mathrm{d}t.$$

Also we define Q_Δ as the left-hand side constant integration matrix

$$Q_\Delta = \frac{1}{\Delta t} \begin{pmatrix} 0 & & \cdots & & 0 \\ \Delta t_1 & 0 & & & \\ \Delta t_1 & \Delta t_2 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & 0 & \\ \Delta t_1 & \Delta t_2 & \cdots & \Delta t_{M-1} & 0 \end{pmatrix}.$$

[1] We consider the discretized domain $t_0 < ... < t_m < ... < t_M$ with $\Delta t = t_M - t_0$ and $\Delta t_m = t_m - t_{m-1}$.

Proof. Let's rewrite eq. (1) as

$$\begin{aligned}\mathbf{U}^{k+1} &= \mathbf{U}^k + (I - \Delta t Q_\Delta F)^{-1}(\mathbf{U}_0 - (I - \Delta t Q F(\mathbf{U}^k))) \\ (I - \Delta t Q_\Delta F)\mathbf{U}^{k+1} &= (I - \Delta t Q_\Delta F)\mathbf{U}^k + (\mathbf{U}_0 - (I - \Delta t Q F(\mathbf{U}^k))) \\ \mathbf{U}^{k+1} - \Delta t Q_\Delta F(\mathbf{U}^{k+1}) &= \mathbf{U}_0 + \Delta t (Q - Q_\Delta) F(\mathbf{U}^k). \qquad (4)\end{aligned}$$

Then we rewrite eq. (4) component wise for the node m

$$U_m^{k+1} - \sum_{j=1}^{m} \Delta t_j f(U_j^{k+1}, t_j) = U_0 + \left[\Delta t Q F(U^k)\right]_m - \sum_{j=1}^{m} \Delta t_j f(U_j^k, t_j). \qquad (5)$$

From the definition (3) we know

$$\left[\Delta t Q F(U^k)\right]_m = \sum_{j=1}^{m} S_j^{j+1} F(U^k),$$

and then we can rewrite eq. (5) as

$$U_m^{k+1} = U_0 + \sum_{j=1}^{m} S_j^{j+1} F(U^k) + \sum_{j=1}^{m} \Delta t_j \left(f(U_j^{k+1}, t_j) - f(U_j^k, t_j)\right). \qquad (6)$$

Similarly, the iteration of U on the previous node $(m-1)$ reads

$$U_{m-1}^{k+1} = U_0 + \sum_{j=1}^{m-1} S_j^{j+1} F(U^k) + \sum_{j=1}^{m-1} \Delta t_j \left(f(U_j^{k+1}, t_j) - f(U_j^k, t_j)\right). \qquad (7)$$

Subtracting (6) and (7) we get

$$U_m^{k+1} - U_{m-1}^{k+1} = S_m^{m+1} F(U^k) + \Delta t_m \left(f(U_m^{k+1}, t_m) - f(U_m^k, t_m)\right)$$

that, after reordering terms, we get eq. (2).

□

In the same way we get the correspondence between explicit SDC and right integration matrix.