Chapter 2
Solution of Homogeneous and Inhomogeneous Linear Equations

Although we are concerned primarily with equations (differential and difference) of first and second order, the analysis of these equations applies equally to equations of higher order. The methods for dealing with these equations is in fact best elucidated by considering the nth order equations and then giving the results for the first and second order equations as specific examples. We first present the analysis for differential equations and then follow with the analysis for difference equations.

The idea essential to most of the methods for solving an inhomogeneous equation of nth order is that if one knows a number, m, $(1 \leq m \leq n)$ of linearly independent solutions of the homogeneous equation, then one can derive a linear equation of order $n - m$. For the case in which one knows one solution of the homogeneous equation $(m = 1)$, the method is referred to as "reduction of order". This is particularly useful if one starts with a second order homogeneous or inhomogeneous equation, resulting in a first order equation which can then be solved directly in closed form. For the case in which one knows all n linearly independent solutions of the nth order homogeneous equation, the method (discussed earlier by Lagrange) is referred to as "variation of parameters" or "variation of constants"—a characterization that will become clear shortly. One then obtains the solution to the inhomogeneous equation in terms of the n linearly independent solutions of the homogeneous equation. The case of an intermediate m, $1 < m < n$, has been treated succinctly in a short article by Locke [31], linking the particular cases $m = 1$ (reduction of order) and $m = n$ (variation of parameters). An alternative approach is given in [18]. An analysis for arbitrary m, $1 \leq m \leq n$, in which the nth order equation is transformed into a first order matrix equation, is given in [20].

The nth order linear homogeneous differential equation has the form

$$\begin{aligned} Ly(x) &\equiv a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) \\ &= \sum_{i=0}^{n} a_i(x)y^{(i)}(x) = 0 \end{aligned} \tag{2.1}$$

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and the corresponding inhomogeneous equation is

$$\sum_{i=0}^{n} a_i(x)y^{(i)}(x) = f(x) \tag{2.2}$$

where

$$y^{(i)}(x) \equiv \frac{d^i y(x)}{dx^i}. \tag{2.3}$$

The corresponding Nth order linear homogeneous and inhomogeneous difference equations are

$$\begin{aligned} Ly(n) &\equiv p_N(n)y(n+N) + p_{N-1}(n)y(n+N-1) + \cdots + p_0(n)y(n) \\ &= \sum_{i=0}^{N} p_i(n)y(n+i) = 0 \end{aligned} \tag{2.4}$$

and

$$\sum_{i=0}^{N} p_i(n)y(n+i) = q_n \tag{2.5}$$

respectively. Here the coefficients $p_i(n)$ are often also functions of independent parameters. We note that these two Nth order difference equations can be written in a form similar to that for the differential equations using the difference operator $\Delta y(n) = y(n+1) - y(n)$. From (1.9) we have

$$\begin{aligned} \sum_{i=0}^{N} p_i(n)y(n+i) &= \sum_{i=0}^{N} p_i(n) \sum_{j=0}^{i} \binom{i}{j} \Delta^j y(n) \\ &= \sum_{j=0}^{N} \Delta^j y(n) \sum_{i=j}^{N} \binom{i}{j} p_i(n). \end{aligned} \tag{2.6}$$

We can then write the homogeneous and inhomogeneous difference equations in the form

$$\sum_{j=0}^{N} r_j(n)\Delta^j y(n) = 0 \tag{2.7}$$

and

$$\sum_{j=0}^{N} r_j(n)\Delta^j y(n) = q_n \tag{2.8}$$

respectively, where

$$r_j(n) = \sum_{i=j}^{N} \binom{i}{j} p_i(n). \tag{2.9}$$

2.1 Variation of Constants

We start with an analysis of the method of variation of constants since it provides the clearest understanding of the essential aspects of the method.

2.1.1 Inhomogeneous Differential Equations

As given above, the nth order homogeneous differential equation is

$$\sum_{j=0}^{n} a_j(x) y^{(j)}(x) = 0 \tag{2.10}$$

and the corresponding inhomogeneous equation is

$$\sum_{j=0}^{n} a_j(x) y^{(j)}(x) = f(x). \tag{2.11}$$

We assume that the solution to the inhomogeneous equation, $y(x)$, and its $n-1$ derivatives, can be given in terms of the n linearly independent solutions of the homogeneous equation, $u_k(x)$, $(k = 1, 2, \ldots, n)$ by

$$y^{(j)}(x) = \sum_{k=1}^{n} c_k(x) u_k^{(j)}(x) \qquad j = 0, 1, \ldots, n-1. \tag{2.12}$$

We then have n linear equations for the n functions $c_k(x)$. They do not define the functions $c_k(x)$, but merely relate them to the derivatives $y^{(j)}(x)$, which is possible given the linear independence of the functions $u_k(x)$. We note that if the $c_k(x)$ are constants, then $y(x)$ as defined by this equation is a solution of the *homogeneous* equation. By allowing the $c_k(x)$ to vary (i.e., to be functions of x), we can determine them so that $y(x)$ is a solution of the *inhomogeneous* equation, whence the name variation of constants, or variation of parameters. An equation for the c_k is obtained by substituting (2.12) into (2.2); still required is $y^{(n)}(x)$. Differentiating (2.12) for $j = n-1$ we have

$$y^{(n)}(x) = \sum_{k=1}^{n} c_k(x) u_k^{(n)}(x) + \sum_{k=1}^{n} c_k'(x) u_k^{(n-1)}(x). \tag{2.13}$$

Substituting (2.12) and (2.13) in (2.2) then gives

$$\begin{aligned} f(x) &= \sum_{j=0}^{n-1} a_j(x) \sum_{k=1}^{n} c_k(x) u_k^{(j)}(x) + a_n(x) \left[\sum_{k=1}^{n} c_k(x) u_k^{(n)}(x) + \sum_{k=1}^{n} c_k'(x) u_k^{(n-1)}(x) \right] \\ &= \sum_{j=0}^{n} a_j(x) \sum_{k=1}^{n} c_k(x) u_k^{(j)}(x) + a_n(x) \sum_{k=1}^{n} c_k'(x) u_k^{(n-1)}(x). \end{aligned} \tag{2.14}$$

Interchanging the order of summation in the first term here we have

$$\sum_{j=0}^{n} a_j(x) \sum_{k=1}^{n} c_k(x) u_k^{(j)}(x) = \sum_{k=1}^{n} c_k(x) \sum_{j=0}^{n} a_j(x) u_k^{(j)}(x) = 0 \tag{2.15}$$

since the $u_k(x)$ are solutions of the homogeneous equation (2.1). We now have one equation for the first derivative of the n functions $c_k(x)$:

$$\sum_{k=1}^{n} c_k'(x) u_k^{(n-1)}(x) = \frac{f(x)}{a_n(x)} \equiv g_n(x). \tag{2.16}$$

The remaining $n-1$ equations defining the functions c_k' follow from (2.12): Differentiation of (2.12) gives

$$y^{(j+1)}(x) = \sum_{k=1}^{n} c_k(x) u_k^{(j+1)}(x) + \sum_{k=1}^{n} c_k'(x) u_k^{(j)}(x). \tag{2.17}$$

Here, from (2.12), for $j = 0, 1, \ldots, n-2$, the first sum on the right hand side is $y^{(j+1)}(x)$, from which

$$\sum_{k=1}^{n} c_k'(x) u_k^{(j)}(x) = 0, \qquad j = 0, 1, \ldots, n-2. \tag{2.18}$$

Equations (2.16) and (2.18) now give n equations for the n functions $c_k'(x)$ which can then be integrated to give $c_k(x)$.

All of the results just given can be obtained more succinctly by formulating the matrix equivalent of these equations, giving a first order matrix differential equation. To that end, we define the Wronskian matrix, $\mathbf{W}(x)$,

$$\mathbf{W}(x) = \begin{pmatrix} u_1(x) & u_2(x) & \cdots & u_n(x) \\ u_1^{(1)}(x) & u_2^{(1)}(x) & \cdots & u_n^{(1)}(x) \\ \vdots & \vdots & \vdots & \vdots \\ u_1^{(n-1)}(x) & u_2^{(n-1)}(x) & \cdots & u_n^{(n-1)}(x) \end{pmatrix} \tag{2.19}$$

and the column vectors $\mathbf{c}(x)$, $\boldsymbol{g}(x)$

$$\mathbf{c}(x) = \begin{pmatrix} c_1(x) \\ c_2(x) \\ \vdots \\ c_n(x) \end{pmatrix} \tag{2.20}$$

$$\boldsymbol{g}(x) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g_n(x) \end{pmatrix} \tag{2.21}$$

and

$$\mathbf{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix} \tag{2.22}$$

in which the components $y_j(x)$ are given in terms of the derivatives of the solution to the inhomogeneous equation (2.11) by

$$y_1(x) = y(x), \quad y_2(x) = y^{(1)}(x), \ldots, y_n(x) = y^{(n-1)}(x) \tag{2.23}$$

from which

$$y_j'(x) = y_{j+1}(x), \qquad j = 1, 2, \ldots, n-1 \tag{2.24}$$

The inhomogeneous equation (2.11) may then be written in the form

$$a_n(x)y_n'(x) + \sum_{j=0}^{n-1} a_j(x)y_{j+1}(x) = f(x) \tag{2.25}$$

or

$$y_n'(x) = -\sum_{j=0}^{n-1} b_j y_{j+1}(x) + g_n(x) \tag{2.26}$$

where

$$b_j = b_j(x) = a_j(x)/a_n(x), \qquad g_n(x) = f(x)/a_n(x). \tag{2.27}$$

Taken together, Eqs. (2.24) and (2.26) then give $y'_j(x)$ in terms of $y_j(x)$ for $j = 1, 2, \ldots n$ and can be written in matrix form:

$$\mathbf{y}' = \mathbf{B}\mathbf{y} + \boldsymbol{g} \tag{2.28}$$

where

$$\mathbf{B}(x) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & \cdots & -b_{n-1} \end{pmatrix} \tag{2.29}$$

Equation (2.12), defining the $n - 1$ derivatives of $y(x)$, then takes the simple form

$$\mathbf{y} = \mathbf{W}\mathbf{c} \tag{2.30}$$

when written in terms of the Wronskian matrix $\mathbf{W}(x)$. Substituting this in the matrix form of the inhomogeneous equation, (2.28), then gives

$$(\mathbf{W}\mathbf{c})' = \mathbf{W}'\mathbf{c} + \mathbf{W}\mathbf{c}' = \mathbf{B}\mathbf{W}\mathbf{c} + \boldsymbol{g} \tag{2.31}$$

From the homogeneous equation satisfied by $u_k(x)$:

$$u_k^{(n)} = -\sum_{j=0}^{n-1} b_j u_k^{(j)}, \qquad k = 1, 2, \ldots, n \tag{2.32}$$

we have

$$\mathbf{B}(x)\mathbf{W}(x) = \mathbf{W}'(x), \tag{2.33}$$

which, when substituted in (2.31), gives

$$\mathbf{W}\mathbf{c}' = \boldsymbol{g}, \tag{2.34}$$

which is equivalent to Eqs. (2.16) and (2.18). Integration of this equation gives

$$\mathbf{c}(x) = \int^x \mathbf{W}^{-1}(x')\boldsymbol{g}(x')dx', \tag{2.35}$$

and from (2.30),

$$\mathbf{y}(x) = \mathbf{W}(x) \int^x \mathbf{W}^{-1}(x')\boldsymbol{g}(x')dx' \tag{2.36}$$

If we have an initial value problem in which $y(x)$ and its n derivatives are specified at a point $x = x_0$, then integration of equation (2.34) gives

$$\mathbf{c}(x) = \mathbf{c}(x_0) + \int_{x_0}^{x} \mathbf{W}^{-1}(x')\boldsymbol{g}(x')dx' \tag{2.37}$$

Multiplying both sides of this equation by $\mathbf{W}(x)$ and using $\mathbf{c}(x_0) = \mathbf{W}^{-1}(x_0)\mathbf{y}(x_0)$ from (2.30) we have the solution to the inhomogeneous equation (2.11) in matrix form:

$$\mathbf{y}(x) = \mathbf{W}(x)\left(\mathbf{W}^{-1}(x_0)\mathbf{y}(x_0) + \int_{x_0}^{x} \mathbf{W}^{-1}(x')\boldsymbol{g}(x')dx'\right). \tag{2.38}$$

We note that the three essential equations in this analysis are (2.28), $\mathbf{y}' = \mathbf{B}\mathbf{y} + \boldsymbol{g}$, which defines the inhomogeneous equation and is equivalent to Eq. (2.2); (2.30), $\mathbf{y} = \mathbf{W}\mathbf{c}$, which relates the functions $c_k(x)$ to the derivatives of $y(x)$ and is equivalent to Eq. (2.12); and (2.33), $\mathbf{B}\mathbf{W} = \mathbf{W}'$, which gives the homogeneous equation satisfied by its solutions $u_k(x)$ and is equivalent to Eq. (2.32). With only minor notational modifications, these three equations form the basis of the analysis for difference equations (note Eqs. (2.64), (2.66) and (2.69)).

An alternate but completely equivalent approach to the solution of the nth order linear inhomogeneous equation, (2.28),

$$\mathbf{y}' = \mathbf{B}\mathbf{y} + \boldsymbol{g}, \tag{2.39}$$

is provided by consideration of the Wronskian (in the case of the differential equation) and the Casoratian (in the case of the difference equation). We start from Eq. (2.34),

$$\mathbf{W}\mathbf{c}' = \boldsymbol{g}, \tag{2.40}$$

The solution to this set of equations is given by Cramer's rule (see Appendix E), by which the column vector $\mathbf{g}$(x), (2.21), replaces the jth column in the Wronskian matrix $\mathbf{W}(x)$, (2.19). The elements $c_j'(x)$ are then given by

$$c_j'(x) = \frac{1}{W(x)}\begin{vmatrix} u_1 & \cdots & u_{j-1} & 0 & u_{j+1} & \cdots & u_n \\ u_1^{(1)} & \cdots & u_{j-1}^{(1)} & 0 & u_{j+1}^{(1)} & \cdots & u_n^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{(n-2)} & \cdots & u_{j-1}^{(n-2)} & 0 & u_{j+1}^{(n-2)} & \cdots & u_n^{(n-2)} \\ u_1^{(n-1)} & \cdots & u_{j-1}^{(n-1)} & g_n & u_{j+1}^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} \tag{2.41}$$

where $u_k = u_k(x)$, $(k = 1, 2, \ldots, n)$, are the n linearly independent solutions of (2.1) and W is the determinant of the Wronskian matrix (2.19). (See Appendix B for $j = 1$ and $j = n$.) Expanding the determinant (2.41) in the elements of the jth column, $c_j'(x)$ can be expressed in terms of an $(n-1) \times (n-1)$ determinant:

$$c'_j(x) = (-1)^{n+j} \frac{g_n(x)}{W(x)} \begin{vmatrix} u_1 & \cdots & u_{j-1} & u_{j+1} & \cdots & u_n \\ u_1^{(1)} & \cdots & u_{j-1}^{(1)} & u_{j+1}^{(1)} & \cdots & u_n^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{(n-2)} & \cdots & u_{j-1}^{(n-2)} & u_{j+1}^{(n-2)} & \cdots & u_n^{(n-2)} \end{vmatrix} \tag{2.42}$$

(See Appendix B for $j = 1$ and $j = n$.)

The matrix solution to the inhomogeneous equation may then be written if one constructs a column vector $\mathbf{c}'(x)$ whose elements are the $c'_j(x)$ for $j = 1, 2, \dots, n$. Equation (2.30), $\mathbf{y}(x) = \mathbf{W}(x)\mathbf{c}(x)$, then gives

$$\mathbf{y}(x) = \mathbf{W}(x) \int_{x_0}^{x} \mathbf{c}'(x')dx' \tag{2.43}$$

The first element in the column vector $\mathbf{y}(x)$ gives the function $y(x)$:

$$y(x) = \sum_{j=1}^{n} u_j(x) \int_{x_0}^{x} c'_j(x')dx' \tag{2.44}$$

with $c'_j(x)$ given in (2.42). This provides a particular solution, to which an arbitrary solution to the homogeneous equation may be added to satisfy boundary conditions.

2.1.2 Inhomogeneous Difference Equations

We now look at the equivalent analysis for the Nth order inhomogeneous difference equation

$$p_N(n)y(N+n) + p_{N-1}(n)y(N+n-1) + \cdots + p_0(n)y(n) = \sum_{j=0}^{N} p_j(n)y(n+j) = q_N(n) \tag{2.45}$$

The Nth order homogeneous equation is

$$\sum_{j=0}^{N} p_j(n)y(n+j) = 0, \tag{2.46}$$

for which the N linearly independent solutions are denoted by $u_k(n)$, $(k = 1, 2, \dots, N)$, that is,

$$\sum_{j=0}^{N} p_j(n)u_k(n+j) = 0, \quad k = 1, 2, \dots, N \tag{2.47}$$

As with the differential equation, we assume that the solution to the inhomogeneous equation, $y(n)$, and the succeeding $N-1$ terms $y(n+1),\ y(n+2),\ldots,y(n+N-1)$ can be given in terms of the N linearly independent solutions $u_k(n)$ of the homogeneous equation by

$$y(n+j)=\sum_{k=1}^{N}c_k(n)u_k(n+j)\qquad j=0,1,\ldots,N-1 \tag{2.48}$$

We thus have N linear equations determining the N functions, $c_k(n)$, which is possible given the linear independence of the functions $u_k(n)$. We note that if the c_k are constants, then $y(n)$ as defined by this equation is a solution of the *homogeneous* equation. By allowing the c_k to vary (i.e., to be functions of n), we can determine them so that $y(n)$ is a solution of the *inhomogeneous* equation. From (2.48) we can thus write

$$y(n+j+1)=\sum_{k=1}^{N}c_k(n)u_k(n+j+1)\quad \text{for}\quad j=0,1,\ldots,N-2 \tag{2.49}$$

as well as

$$y(n+j+1)=\sum_{k=1}^{N}c_k(n+1)u_k(n+j+1)\quad \text{for}\quad j=0,1,\ldots,N-2 \tag{2.50}$$

from which we have the $N-1$ equations

$$\sum_{k=1}^{N}\Delta c_k(n)u_k(n+j+1)=0,\qquad j=0,1,\ldots,N-2 \tag{2.51}$$

From (2.48) for $j=N-1$ we have

$$y(n+N-1)=\sum_{k=1}^{N}c_k(n)u_k(n+N-1) \tag{2.52}$$

from which

$$\begin{aligned} y(n+N)&=\sum_{k=1}^{N}c_k(n+1)u_k(n+N)\\ &=\sum_{k=1}^{N}c_k(n)u_k(n+N)+\sum_{k=1}^{N}\Delta c_k(n)u_k(n+N) \end{aligned} \tag{2.53}$$

Substituting (2.48) and (2.53) in (2.45) we then have

$$\sum_{j=0}^{N} p_j(n) \sum_{k=1}^{N} c_k(n) u_k(n+j) + p_N(n) \sum_{k=1}^{N} \Delta c_k(n) u_k(n+N) = q_N(n) \tag{2.54}$$

Inverting the order of summation in the first term here, we see that this term vanishes since the $u_k(n)$ satisfy the homogeneous equation (2.46). We thus have

$$\sum_{k=1}^{N} \Delta c_k(n) u_k(n+N) = \frac{q_N(n)}{p_N(n)} \equiv h_N(n) \tag{2.55}$$

Equation (2.55) together with (2.51) for $j = 0, 1, \ldots, N-2$ then give N equations for the N differences $\Delta c_k(n)$ which can then be summed to give the functions $c_k(n)$.

We can now formulate the entire analysis for difference equations in terms of matrices, in a manner quite similar to that for differential equations, giving a first order matrix difference equation. To that end we define the Casoratian matrix:

$$\mathbf{K}(n) = \begin{pmatrix} u_1(n) & u_2(n) & \cdots\ u_N(n) \\ u_1(n+1) & u_2(n+1) & \cdots\ u_N(n+1) \\ \vdots & \vdots & \vdots\quad \vdots \\ u_1(n+N-1) & u_2(n+N-1) & \cdots\ u_N(n+N-1) \end{pmatrix} \tag{2.56}$$

and the column vectors $\mathbf{c}(n)$, $\mathbf{h}(n)$

$$\mathbf{c}(n) = \begin{pmatrix} c_1(n) \\ c_2(n) \\ \vdots \\ c_N(n) \end{pmatrix} \tag{2.57}$$

$$\mathbf{h}(n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ h_N(n) \end{pmatrix} \tag{2.58}$$

and

$$\mathbf{y}(n) = \begin{pmatrix} y_1(n) \\ y_2(n) \\ \vdots \\ y_N(n) \end{pmatrix} \tag{2.59}$$

in which the components $y_j(n)$ are given in terms of the solution of the inhomogeneous equation (2.45) for successive indices by

$$y_j(n) = y(j+n-1), \qquad j = 1, 2, \ldots, N \tag{2.60}$$

We can then write the inhomogeneous equation (2.45) in the form

$$y(N+n) + b_{N-1}y(N+n-1) + \cdots + b_0 y(n) = h_N(n) \tag{2.61}$$

or

$$y(N+n) = -\sum_{j=0}^{N-1} b_j y_{j+1}(n) + h_N(n) \tag{2.62}$$

where

$$b_j = b_j(n) = p_j(n)/p_N(n), \qquad h_N(n) = q_N(n)/p_N(n) \tag{2.63}$$

This may then be written in matrix form (cf. Eq. (2.28) for differential equations) as

$$\mathbf{y}(n+1) = \mathbf{B}(n)\mathbf{y}(n) + \mathbf{h}(n) \tag{2.64}$$

where

$$\mathbf{B}(n) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & \cdots & -b_{N-1} \end{pmatrix} \tag{2.65}$$

Equation (2.48) giving the N terms $y(n+j)$ for $j = 0, 1, \ldots, N-1$ then takes the simple matrix form (cf. Eq. (2.30) for differential equations)

$$\mathbf{y}(n) = \mathbf{K}(n)\mathbf{c}(n) \tag{2.66}$$

Substituting (2.66) in (2.64) we then have

$$\mathbf{y}(n+1) = \mathbf{B}(n)\mathbf{K}(n)\mathbf{c}(n) + \mathbf{h}(n) \tag{2.67}$$

Here

$$\begin{aligned}\mathbf{B}(n)\mathbf{K}(n) &= \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & \cdots & -b_{n-1} \end{pmatrix} \begin{pmatrix} u_1(n) & u_2(n) & \cdots & u_N(n) \\ u_1(n+1) & u_2(n+1) & \cdots & u_N(n+1) \\ \vdots & \vdots & \vdots & \vdots \\ u_1(n+N-1) & u_2(n+N-1) & \cdots & u_N(n+N-1) \end{pmatrix} \\ &= \begin{pmatrix} u_1(n+1) & u_2(n+1) & \cdots & u_N(n+1) \\ u_1(n+2) & u_2(n+2) & \cdots & u_N(n+2) \\ \vdots & \vdots & \vdots & \vdots \\ u_1(n+N) & u_2(n+N) & \cdots & u_N(n+N) \end{pmatrix}\end{aligned} \tag{2.68}$$

Thus

$$\mathbf{B}(n)\mathbf{K}(n) = \mathbf{K}(n+1) \tag{2.69}$$

(cf. Eq. (2.33) for differential equations.) The last line in $\mathbf{K}(n+1)$ follows from the homogeneous equation (2.47) satisfied by the functions $u_k(n)$:

$$u_k(N+n) = -b_0(n)u_k(n) - b_1(n)u_k(n+1) - \cdots - b_N(n)u_k(n+N-1) \tag{2.70}$$

We note that Eqs. (2.64), (2.66) and (2.69) for difference equations correspond respectively to Eqs. (2.28), (2.30) and (2.33) for differential equations. Substituting (2.69) in (2.64) then gives

$$\mathbf{y}(n+1) = \mathbf{K}(n+1)\mathbf{c}(n) + \mathbf{h}(n) \tag{2.71}$$

However, from (2.66) we can also write

$$\mathbf{y}(n+1) = \mathbf{K}(n+1)\mathbf{c}(n+1) \tag{2.72}$$

so that from the last two equations we have

$$\mathbf{K}(n+1)\Delta\mathbf{c}(n) = \mathbf{h}(n), \tag{2.73}$$

from which

$$\Delta\mathbf{c}(n) = \mathbf{K}^{-1}(n+1)\,\mathbf{h}(n) \tag{2.74}$$

and

$$\mathbf{c}(n+1) = \mathbf{c}(0) + \sum_{j=0}^{n} \mathbf{K}^{-1}(j+1)\,\mathbf{h}(j) \tag{2.75}$$

(cf. (2.37) for differential equations.) Here, the term $\mathbf{c}(0)$ adds an arbitrary solution of the homogeneous equation and is determined by the initial conditions. Equation (2.73) is the matrix form of Eq. (2.55) together with (2.51) for $j = 0, 1, \ldots, N-2$. Equations (2.73), (2.74) and (2.75) given above for difference equations correspond to Eqs. (2.34) and (2.37) for differential equations. Writing (2.75) with $n+1$ replaced by n we have

$$\mathbf{c}(n) = \mathbf{c}(0) + \sum_{j=1}^{n} \mathbf{K}^{-1}(j)\,\mathbf{h}(j-1) \tag{2.76}$$

Multiplying both sides of this equation by $\mathbf{K}(n)$ and using (2.66) (from which $\mathbf{c}(0) = \mathbf{K}^{-1}(0)\mathbf{y}(0)$) we have the solution to the inhomogeneous equation (2.64) in matrix form:

$$\mathbf{y}(n) = \mathbf{K}(n)\left(\mathbf{K}^{-1}(0)\mathbf{y}(0) + \sum_{j=1}^{n} \mathbf{K}^{-1}(j)\,\mathbf{h}(j-1)\right) \tag{2.77}$$

Similar to the case for differential equations, we note that the three essential equations in the analysis for difference equations are (2.64), $\mathbf{y}(n+1) = \mathbf{B}(n)\mathbf{y}(n) + \mathbf{h}(n)$, which defines the inhomogeneous equation and is equivalent to Eq. (2.45); (2.66), $\mathbf{y}(n) = \mathbf{K}(n)\mathbf{c}(n)$, which relates the N functions $c_k(n)$ to N successive terms $y(n+j)$ for $j = 0, 1, \ldots, N-1$ and is equivalent to Eq. (2.48); and (2.69), $\mathbf{B}(n)\mathbf{K}(n) = \mathbf{K}(n+1)$, which gives the homogeneous equation satisfied by its solutions $u_k(n)$ and is equivalent to Eq. (2.47).

An alternate but completely equivalent approach to the solution of the Nth order linear inhomogeneous equation, (2.64),

$$\mathbf{y}(n+1) = \mathbf{B}(n)\mathbf{y}(n) + \mathbf{h}(n) \tag{2.78}$$

is provided by consideration of the Casoratian in the case of the difference equation. We start from Eq. (2.73),

$$\mathbf{K}(n+1)\Delta\mathbf{c}(n) = \mathbf{h}(n), \tag{2.79}$$

Replacing $n+1$ by n we have

$$\mathbf{K}(n)\Delta\mathbf{c}(n-1) = \mathbf{h}(n-1), \tag{2.80}$$

The solution to this matrix equation is given by Cramer's rule, from which the elements $\Delta c_j(n-1)$ of the column vector $\Delta\mathbf{c}(n-1)$ for $j = 1, 2, \ldots, N$ are given by

$$\Delta c_j(n-1) = \frac{1}{K(n)} \begin{vmatrix} u_1(n) & \cdots u_{j-1}(n) & 0 & u_{j+1}(n) & \cdots u_N(n) \\ u_1(n+1) & \cdots u_{j-1}(n+1) & 0 & u_{j+1}(n+1) & \cdot, \; u_N(n+1) \\ \vdots & \vdots \;\; \vdots & \vdots & \vdots & \vdots \;\; \vdots \\ u_1(n+N-2) & \cdots u_{j-1}(n+N-2) & 0 & u_{j+1}(n+N-2) & \cdots u_N(n+N-2) \\ u_1(n+N-1) & \cdots u_{j-1}(n+N-1) & h_N(n-1) & u_{j+1}(n+N-1) & \cdots u_N(n+N-1), \end{vmatrix} \tag{2.81}$$

where $u_j = u_j(n),\ j = 1, 2, \ldots, N$, are the N linearly independent solutions of (2.4) and $K(n)$ is the determinant of the Casoratian matrix (2.56). Expanding the determinant (2.81) in the elements of the jth column, the elements $\Delta c_j(n-1)$ can be expressed in terms of an $(N-1) \times (N-1)$ determinant:

$$\Delta c_j(n-1) = (-1)^{N+j} \frac{h_N(n-1)}{K(n)} \begin{vmatrix} u_1(n) & \cdots u_{j-1}(n) & u_{j+1}(n) & \cdots u_N(n) \\ u_1(n+1) & \cdots u_{j-1}(n+1) & u_{j+1}(n+1) & \cdots u_N(n+1) \\ \vdots & \vdots \;\; \vdots & \vdots & \vdots \;\; \vdots \\ u_1(n+N-2) & \cdots u_{j-1}(n+N-2) & u_{j+1}(n+N-2) & \cdots u_N(n+N-2) \end{vmatrix} \tag{2.82}$$

It is clear that the determinant as written in (2.82) is valid for $j = 2, 3, \ldots, N-1$. For $j = 1$ and $j = N$ (as well as for all $1 \le j \le N$) one must simply omit the jth column. The matrix solution to the inhomogeneous equation may then be written from the column vector $\Delta\mathbf{c}(n-1)$ whose elements are the $\Delta c_j(n-1)$ given in (2.82) for $j = 1, 2, \ldots, N$. Equation (2.66), $\mathbf{y}(n) = \mathbf{K}(n)\mathbf{c}(n)$, then gives $\mathbf{y}(n)$, with

$$\mathbf{y}(n) = \mathbf{K}(n)\left(\mathbf{K}^{-1}(0)y(0) + \sum_{n'=1}^{n} \Delta\mathbf{c}(n'-1)\right) \tag{2.83}$$

The first term in parentheses gives a solution of the homogeneous equation. Therefore a particular matrix solution of the inhomogeneous equation is given by

$$\mathbf{y}(n) = \mathbf{K}(n) \sum_{n'=1}^{n} \Delta\mathbf{c}(n'-1) \tag{2.84}$$

The first element of this matrix equation gives the function $y(n)$:

$$y(n) = \sum_{j=1}^{N} u_j(n) \sum_{n'=1}^{n} \Delta c_j(n'-1) \tag{2.85}$$

with $\Delta c_j(n'-1)$ given by (2.82). This provides a particular solution to which an arbitrary solution to the homogeneous equation may be added to satisfy boundary conditions.

2.2 Reduction of the Order When One Solution to the Homogeneous Equation Is Known

The present method reduces the order of an nth order linear operator, giving an operator of order $n-1$ when one solution to the homogeneous equation is known. Thus, an nth order homogeneous equation $Ly = 0$ is transformed into a homogeneous equation $\mathscr{L}w = 0$ of order $n-1$ in w; an nth order inhomogeneous equation $Ly = f$ is transformed into an inhomogeneous equation $\mathscr{L}w = f$ of order $n-1$ in w. (In particular, for a second order equation we obtain a first order equation, which is then soluble in closed form.) The details in the analysis of differential and difference equations are quite similar, and the approach is the same as that given earlier in connection with the method of variation of constants (cf. (2.12)): By writing the dependent variable ($y(x)$ or $y(n)$) as the product of two functions,

$$y(x) = c(x)u(x) \tag{2.86}$$

or

$$y(n) = c(n)u(n), \tag{2.87}$$

one of which satisfies the homogeneous equation ($Lu(x) = 0$ or $Lu(n) = 0$, respectively), one can write the original equation in a form such that only derivatives (or differences) of the unknown function ($c(x)$ or $c(n)$) appear. Then, defining

$$w(x) = c'(x) \tag{2.88}$$

and

$$w(n) = \Delta c(n) = c(n+1) - c(n), \tag{2.89}$$

the order of the equation for $w(x)$ or $w(n)$ is less by one than that of the original equation.

We start by considering the nth order differential operator given in (2.1), viz.,

$$Ly(x) \equiv a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) = \sum_{j=0}^{n} a_j(x)y^{(j)}(x) \tag{2.90}$$

Writing

$$y(x) = c(x)u(x) \tag{2.91}$$

where $u(x)$ is assumed to be a known solution of

$$Lu(x) = \sum_{j=0}^{n} a_j(x)u^{(j)}(x) = 0, \tag{2.92}$$

we have

$$\begin{aligned} y^{(k)}(x) &= \frac{d^k(c(x)u(x))}{dx^k} \\ &= \sum_{j=0}^{k} \binom{k}{j} c^{(j)}(x)u^{(k-j)}(x) \end{aligned} \tag{2.93}$$

and from (2.1),

$$\begin{aligned} Ly(x) &= \sum_{k=0}^{n} a_k(x)y^{(k)}(x) \\ &= \sum_{k=0}^{n} a_k(x) \sum_{j=0}^{k} \binom{k}{j} c^{(j)}(x)u^{(k-j)}(x) \\ &= \sum_{j=0}^{n} c^{(j)}(x) \sum_{k=j}^{n} \binom{k}{j} a_k(x)u^{(k-j)}(x) \\ &= \sum_{j=1}^{n} c^{(j)}(x) \sum_{k=j}^{n} \binom{k}{j} a_k(x)u^{(k-j)}(x) + c(x)\sum_{k=0}^{n} a_k(x)u^{(k)}(x) \end{aligned} \tag{2.94}$$

The last sum here is zero from (2.92). Then, defining

$$w(x) \equiv c^{(1)}(x), \tag{2.95}$$

we obtain a differential operator of order $n-1$ in $w(x)$:

$$\begin{aligned}
\sum_{j=1}^{n} c^{(j)}(x) \sum_{k=j}^{n} \binom{k}{j} a_k(x) u^{(k-j)}(x) &= \sum_{j=1}^{n} w^{(j-1)}(x) \sum_{k=j}^{n} \binom{k}{j} a_k(x) u^{(k-j)}(x) \\
&= \sum_{j=0}^{n-1} w^{(j)}(x) \sum_{k=j+1}^{n} \binom{k}{j+1} a_k(x) u^{(k-j-1)}(x) \\
&= \sum_{j=0}^{n-1} w^{(j)}(x) \sum_{k=j}^{n-1} \binom{k+1}{j+1} a_{k+1}(x) u^{(k-j)}(x) \\
&= \mathscr{L} w
\end{aligned} \tag{2.96}$$

We next look at the analogous procedure for an Nth order linear homogeneous difference operator, given in (2.4), viz.,

$$Ly(n) \equiv p_N(n)y(n+N) + p_{N-1}(n)y(n+N-1) + \cdots + p_0(n)y(n) \tag{2.97}$$

Again, we write the solution of this equation as the product of two functions:

$$y(n) = c(n)u(n) \tag{2.98}$$

where we assume $u(n)$ to be a known solution of the homogeneous equation

$$\begin{aligned}
Lu(n) &= p_N(n)u(n+N) + p_{N-1}(n)u(n+N-1) + \cdots + p_0(n)u(n) \\
&= \sum_{k=0}^{N} p_k(n)u(n+k) \\
&= 0
\end{aligned} \tag{2.99}$$

The operator (2.97) is then

$$\begin{aligned}
Ly(n) = {} & p_N(n)c(n+N)u(n+N) + p_{N-1}(n)c(n+N-1)u(n+N-1) \\
& + \cdots + p_0(n)c(n)u(n)
\end{aligned} \tag{2.100}$$

Applying (1.9) to the function $c(n+k)$, this operator can be written in the form

$$\sum_{k=0}^{N} p_k(n)u(n+k)\sum_{j=0}^{k}\binom{k}{j}\Delta^j c(n)$$
$$= \sum_{k=1}^{N} p_k(n)u(n+k)\sum_{j=1}^{k}\binom{k}{j}\Delta^j c(n) + c(n)\sum_{k=0}^{N} p_k(n)u(n+k) \tag{2.101}$$

As with the differential equation, the last sum in the above equation is zero in view of (2.99), giving

$$Ly(n) = \sum_{k=1}^{N} p_k(n)u(n+k)\sum_{j=1}^{k}\binom{k}{j}\Delta^j c(n) \tag{2.102}$$

Then, in analogy with (2.95), we define

$$w(n) = \Delta c(n), \tag{2.103}$$

giving

$$\begin{aligned}\sum_{k=1}^{N} p_k(n)u(n+k)\sum_{j=1}^{k}\binom{k}{j}\Delta^{j-1}w(n) &= \sum_{j=1}^{N}\Delta^{j-1}w(n)\sum_{k=j}^{N}\binom{k}{j}p_k(n)u(n+k)\\ &= \sum_{j=0}^{N-1}\Delta^{j}w(n)\sum_{k=j+1}^{N}\binom{k}{j+1}p_k(n)u(n+k)\\ &= \sum_{j=0}^{N-1}\Delta^{j}w(n)\sum_{k=j}^{N-1}\binom{k+1}{j+1}p_{k+1}(n)u(n+k+1)\\ &= \mathscr{L}w(n)\end{aligned} \tag{2.104}$$

which is a difference operator of order $N-1$ in $w(n)$.

2.2.1 Solution of *N*th Order Inhomogeneous Equations When *m* Linearly Independent Solutions of the Homogeneous Equation are Known, Where $1 < m < N$

The two methods—reduction of order and variation of parameters—have been presented separately, since that is how they are generally found in the literature. However, as has been shown in a succinct article by Phil Locke [31], each of these procedures can be viewed as particular limiting cases in the solution of an nth order linear nonhomogeneous equation when $m \leq n$ linearly independent solutions of the nth order homogeneous equation are known: $m = 1$ corresponds to reduction of order, $m = n$ corresponds to variation of parameters. Related treatments may be found in [18, Chap. IX, Sect. 3, pp. 319–322] and in [20, Chap. IV, Sect. 3, pp. 49–54].

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