Addendum to "A coarse space for heterogeneous Helmholtz problems based on the Dirichlet-to-Neumann operator" [J. Comput. Appl. Math. 271 (2014) 83–99]

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Abstract

This communication gives an addendum to the paper "A coarse space for heterogeneous Helmholtz problems based on the Dirichlet-to-Neumann operator" [J. Comput. Appl. Math. 271 (2014) 83–99].

Keywords: Helmholtz equation, domain decomposition, coarse space, Dirichlet-to-Neumann operator

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The preconditioner

$$P_{\mathrm{BNN}} = QM^{-1}P + ZE^{-1}Y^{\dagger} \tag{1}$$

from [1, Equation (7)] might be singular for general non-singular matrices A, M and $E = Y^TAZ$, and full ranked matrices Z and Y. Consider

$$A = \begin{pmatrix} 2 & 5 & 2 \\ 0 & 6 & 0 \\ 0 & 1 & 4 \end{pmatrix}, \qquad Z = Y = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrices A, M, and E are clearly non-singular, but $\begin{pmatrix} 15 & -4 & 7 \end{pmatrix}^T$ is an eigenvector of P_BA with eigenvalue 0. This is in contradiction to a result of Erlangga and Nabben [2], on which our work was based. Their consequently wrong theorem reads

Theorem 0.1 ([2, Theorem 2.9])**.** *Let Z and Y be full ranked. Let M be non-singular. Then $P_{\mathrm{BNN}}A$ is non-singular. In addition, any zero eigenvalue of $M^{-1}P_DA$ is shifted to one in $P_{\mathrm{BNN}}A$.*

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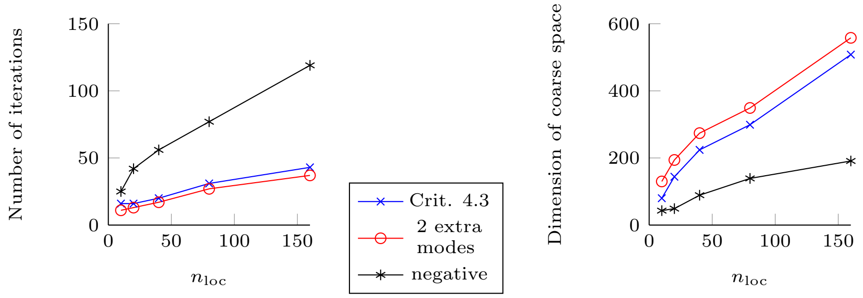


(a) Number of iterations (b) Dimension of the coarse space

Figure 5: Comparison of different criteria of how many DtN modes to choose.

Choice	# iterations	
	$m_i = 12$	$m_i = 24$
no coarse space	115	115
Re(λ) minimal	17	11
$\lvert\lambda\rvert$ minimal	27	17
$\lvert\lambda - k\rvert$ minimal	49	21
$\lvert\lambda\rvert$ maximal	155	145

Table 1: Iteration numbers for different choices of DtN eigenfunctions.

The solutions of the preconditioned of the original system might hence differ and the GMRES solver employed in [1] is not adapted to solve systems with singularities. For that reason, in this corrigendum the results of [1] are reproduced using a non-singular preconditioner. Numbering and notation are identitical to the original paper. The new results use the provably non-singular preconditioner [3]

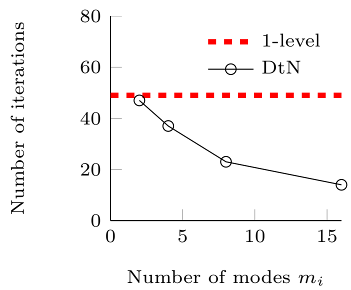
$$P_{\text{new}} = I - Z\left(Z^\dagger M^{-1} A Z\right)^{-1} Z^\dagger M^{-1} A + Z\left(Z^\dagger M^{-1} A Z\right)^{-1} Z^\dagger \qquad (2)$$

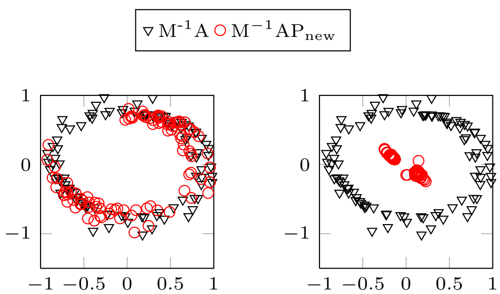
and solve the preconditioned problem $M^{-1}AP_{\text{new}} = M^{-1}b$. The coarse matrix is now $Z^\dagger M^{-1}AZ$ instead of $Z^\dagger AZ$ in Equation (1). Its sparsity structure hence changes; it has blocks not only for neighboring subdomains but also for neighbors of neighbors, which constitutes a drawback for parallel implemetation.

We make a few observations, refraining however from giving a detailed interpretation of the new results to save space. The eigenvalue distribution in Figure 7a is more favorable than the one for $P_{\text{BNN}}A$. This is also reflected in the iteration counts for small coarse size, see e.g. Figure 6 or the last line of Table 14 for $\text{PW}\left(10^{-2}\right)$. Moreover, the convergence problems for the plane wave coarse space were not caused by the singularity of the preconditioner P_{BNN}. In fact, e.g. in Table 3, convergence for $\text{PW}\left(10^{-2}\right)$ is even worse. That is why

L	k	kL	# iterations	coarse space dimension
1	30	30	20	224
5	6	30	20	224
10	3	30	19	224

Table 2: Dependence on the size L of the domain $\Omega = [0, L]^2$.





(a) $m_i = 2$ (b) $m_i = 16$

Figure 6: Number of iterations in dependence of m_i.

Figure 7: 100 largest eigenvalues for $I - M^{-1}A$ and $I - M^{-1}AP_{\text{new}}$ in the complex plane.

n_{loc}	k	1-lev	DtN		PW(10^{-2})		PW(10^{-1})	
20	18.5	80	16	(144)	–	(352)	9	(293)
40	29.3	116	19	(224)	–	(467)	13	(382)
80	46.5	156	30	(299)	–	(577)	16	(505)
160	73.8	217	40	(508)	–	(609)	25	(597)

Table 3: Number of iterations (dimension of coarse space).

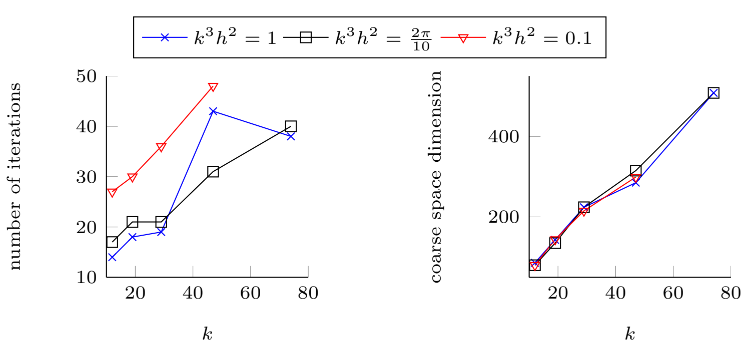


Figure 12: Testing different values of k^3h^2. Problem 1, 5×5 subdomains.

		m_i from DtN coarse space						m_i from PW coarse space					
n_{loc}	k	m_i	DtN	PW(10^{-2})		PW(10^{-1})		m_i	DtN	PW(10^{-2})		PW(10^{-1})	
10	11.6	4	15	17	(100)	17	(100)	12	8	7	(288)	7	(244)
20	18.5	6	19	19	(150)	19	(146)	15	9	–	(355)	9	(305)
40	29.3	9	23	22	(225)	22	(225)	17	13	–	(409)	13	(373)
80	46.5	12	35	30	(296)	29	(292)	24	19	–	(556)	16	(496)
160	73.8	21	42	–	(521)	31	(513)	25	39	–	(609)	25	(597)

Table 4: Comparison of number of iterations with identical coarse space size for DtN and PW.

k	1-level	DtN	
5	106	79	(25)
10	115	58	(70)
15	117	57	(90)
30	133	33	(224)
45	169	39	(299)

Table 5: Dependence on wave number for fixed mesh width.

we additionally give results for $\mathrm{PW}(10^{-1})$. In total, the results do not change substantially and the conclusions drawn in [1] remain valid.

References

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	$n_{\mathrm{loc}}=20,\ L=2$			$n_{\mathrm{loc}}=80,\ L=2$			$n_{\mathrm{loc}}=80,\ L=8$		
k	1-level	DtN		1-level	DtN		1-level	DtN	
1	73	51	(25)	94	73	(25)	66	46	(25)
5	64	40	(25)	96	70	(25)	55	34	(25)
10	68	24	(74)	106	47	(74)	66	24	(74)
20	84	22	(139)	107	34	(144)	86	21	(139)

Table 6: Dependence of number of iterations (coarse space dimension) on overlap/mesh width.

n_{loc}	k	Number of subdomains							
		5×5		5×10		5×20		5×40	
10	11.6	16	(80)	18	(180)	21	(380)	24	(780)
20	18.5	16	(144)	18	(314)	19	(654)	21	(1334)
40	29.3	20	(224)	20	(484)	22	(1004)	24	(2044)
80	46.5	31	(299)	37	(644)	45	(1334)		

Table 7: Dependence on number of subdomains, DtN coarse space.

	DtN		PW(10^-2)		PW(10^-1)	
# subdomains	# it.	size	# it.	size	# it.	size
2×2	24	(68)	–	(96)	18	(88)
4×4	31	(200)	–	(368)	15	(320)
8×8	40	(416)	–	(1116)	14	(924)
16×16	60	(960)	–	(3256)	12	(2686)
32×32	48	(2944)	?	(9208)	?	(?)

Table 8: Second scaling test: Vary the number of subdomains.

n_{loc}	ω	$\rho=5$						$\rho=10$					
		DtN		PW(10^{-2})		PW(10^{-1})		DtN		PW(10^{-2})		PW(10^{-1})	
10	11.6	21	(69)	8	(229)	10	(179)	23	(69)	9	(214)	11	(169)
20	18.5	27	(111)	–	(274)	14	(218)	29	(111)	–	(263)	16	(207)
40	29.3	35	(159)	–	(339)	12	(279)	44	(159)	–	(326)	28	(263)
80	46.5	38	(242)	–	(442)	–	(363)	45	(236)	–	(414)	–	(346)
160	73.8	53	(388)	–	(519)	–	(481)	62	(378)	–	(494)	–	455

Table 9: Number of iterations (coarse space dimension) for heterogeneous open cavity problem.

ρ	1-level	DtN		PW(10^{-2})		PW(10^{-1})	
10^0	156	31	(299)	–	(577)	16	(505)
10^1	154	45	(236)	–	(414)	–	(346)
10^2	173	59	(236)	–	(388)	–	(320)
10^3	177	64	(236)	–	(379)	–	(315)

Table 10: Varying contrast for heterogeneous open cavity problem.

n_{loc}	ω	m_i	DtN	PW(10^{-2})		PW(10^{-1})	
10	11.6	3	21	22	(75)	22	(75)
20	18.5	5	23	25	(123)	25	(123)
40	29.3	7	38	40	(171)	41	(163)
80	46.5	10	42	–	(237)	45	(223)
160	73.8	16	59	–	(358)	63	(346)

Table 11: Fixed coarse space size for heterogeneous open cavity problem.

n_{glob}	k	1-level	DtN	
50	11.6	64	15	(116)
100	18.5	92	17	(168)
200	29.3	130	25	(257)
400	46.5	173	33	(381)
800	73.8	256	43	(645)

Table 12: Decomposition with Metis.

		5×5 subdomains						10×10 subdomains					
k	n_{glob}	DtN		PW(10^{-2})		PW(10^{-1})		DtN		PW(10^{-2})		PW(10^{-1})	
18.5	100	15	(144)	8	(355)	9	(293)	17	(364)	23	(1152)	8	(872)
29.3	200	18	(224)	–	(466)	13	(379)	22	(460)	–	(1288)	11	(1132)
46.5	400	27	(315)	–	(577)	16	(511)	35	(660)	–	(1712)	15	(1380)
73.8	800	33	(514)	–	(609)	25	(597)	57	(956)	–	(2346)	18	(1928)

Table 13: Number of iterations (coarse space dimension) for Problem 2.

		15 subdomains						60 subdomains					
ω	n	DtN		PW(10^{-2})		PW(10^{-1})		DtN		PW(10^{-2})		PW(10^{-1})	
90	150×250	14	(267)	12	(346)	12	(323)	21	(541)	10	(1038)	12	(877)
180	300×500	15	(514)	24	(375)	24	(373)	22	(1074)	15	(1426)	15	(1333)
360	600×1000	18	(968)	50	(375)	50	(375)	26	(2113)	42	(1500)	42	(1500)

Table 14: Number of iterations (coarse space dimension). Problem 3 decomposed with Metis

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