Derivation of the 1D heat equation

Heat flow in an object occurs always in the direction of decreasing temperature, i.e. from hot to cool.

Consider the heat flow along a metal rod with

- specific heat σ,
- mass per unit length ρ,
- 1D position x,
- time variable t,
- temperature $u(x,t)$,
- initial data $f \equiv 0$.

The amount of heat H in a small element between x and $x+h_1$ at times t and $t+h_2$ can written as

$$H(t) = \sigma\rho\, h_1 u(x,t), \quad H(t+h_2) = \sigma\rho\, h_1 u(x,t+h_2),$$

so that the change in heat is given by

$$H(t+h_2) - H(t) = \sigma\rho\, h_1(u(x,t+h_2) - u(x,t)).$$

On the other hand, the change of heat must be equal to the heat flowing in at x minus the heat flowing out at $x+h_1$ during the time interval h_2. Heat flow is proportional to the temperature gradient

$$-K\frac{\partial u}{\partial x},$$

where K is a proportionality constant. Therefore, the change of heat can be also written as

$$H(t+h_2) - H(t) = \left(-K\frac{\partial u}{\partial x}\Big|_x + K\frac{\partial u}{\partial x}\Big|_{x+h_1}\right) h_2.$$

Equating theses expressions and dividing by h_1 and h_2 gives

$$\rho\sigma\frac{u(x,t+h_2) - u(x,t)}{h_2} = K\frac{\frac{\partial u}{\partial x}\Big|_{x+h_1} - \frac{\partial u}{\partial x}\Big|_x}{h_1}.$$

The RHS is independent of h_2, the LHS is independent of h_1 and taking the limits $h_1, h_2 \to 0$ finally yields

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad a = \frac{K}{\sigma\rho}.$$