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A space-time finite element method for the wave equation*

Donald A. French
Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221-0025, USA

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A numerical method is introduced which uses finite elements in time and space simultaneously to solve the wave equation. An error analysis for this scheme is presented and the results of several computations. This scheme is accurate for time slabs of arbitrary thickness without introducing extra least squares or stabilization terms.

1. Introduction

Consider the following problem: find $U = U(x, t)$ such that

$$\begin{aligned}
&\ddot{U} - \Delta U = f \quad \text{in } Q = \Omega \times (0, T]\,, \\
&U = 0 \quad \text{on } \Sigma = \partial\Omega \times (0, T]\,, \\
&U(\cdot\,, 0) = U_0 \quad \text{and} \quad \dot{U}(\cdot\,, 0) = U_1 \quad \text{on } \Omega\,,
\end{aligned} \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^d$ with $d = 1, 2$ and $T > 0$.

We have restricted our attention to a specific problem entirely to keep the presentation simple. Our results apply to considerably more general second-order hyperbolic problems.

Typically an approximation to (1) is found by first discretizing in space to obtain the semidiscrete problem that consists of ordinary differential equations that depend on t. A standard finite difference technique is applied to obtain the full discretization.

Lost in this approach is the ability to produce local mesh refinements in the space-time domain Q. For example, it would be difficult to accurately track a sharp wave front without taking small time steps.

Another option involves converting the second-order equation into a first-order system of equations. However, new unknowns are introduced which lead to a larger discrete problem. Thus it is reasonable to try to approximate U directly through the form (1). (Note that there are several papers on space-time methods for first-order hyperbolic systems. See [1] for discussion and references).

Correspondence to: Donald A. French, Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221-0025, USA.

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Hughes and Hulbert [2] introduced a space-time finite element method for second-order hyperbolic problems. This scheme has several least squares terms which they cleverly add to obtain a convergence theorem. Their discretization method retains several features expected of a finite element method; arbitrary order elements and unstructured meshes. Their analysis requires some uniformity in the mesh and the thickness of each time slab is proportional to the size of the element domains.

Our scheme is similar to the one introduced by Hughes and Hulbert and some of its key features are motivated by the work of Johnson and others on discontinuous Galerkin methods (see [1] and references therein). There are no extra terms in the formulation of our scheme and our convergence theorem holds without restricting the time slab thickness.

In the case where the element shape functions have degree two or higher, we use a weighted inner product. This device was first used for space-time methods by Axelsson and Maubach [3].

An outline of the paper is as follows. In Section 2, we describe the numerical method, give a stability theorem, and show that the solution of (1) satisfies a related weak problem. In Section 3, we present our convergence theorems. We show for the case when V_h consists of piecewise linear functions that the asymptotic rate of convergence of the approximation to the true solution in an H^1 norm consisting of space and time derivatives is $O(h^{1/2})$. When V_h has piecewise polynomials of degree r the rate in a similar norm is $O(h^{r-1})$. In Section 4, we give the results of our numerical computation in the case V_h consists of piecewise linear functions. In the range tested, the numerics indicate that the convergence rate may be $O(h)$.

2. The space-time method

In this section, we introduce our numerical method that uses finite elements in space and time simultaneously. First we describe the approximation spaces and specify our notation. Let

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T$$

be a partition of $[0, T]$ with $t_n = nk$ where $k > 0$ is the thickness of each time slab. Define

$$I_n = (t_{n-1}, t_n)\,, \qquad Q_n = \Omega \times I_n\,, \qquad \Sigma_n = \partial\Omega \times I_n \quad \text{and} \quad \Omega_n = \Omega \times \{t_n\}\,.$$

Let $\{\mathcal{T}_h^n\}_{h>0}$ represent a family of triangulations of Q_n depending on parameter $0 < h < 1$ with element domains τ. Typically h is proportional to the diameter of the τ. Let $V_h^n \subset C^0(Q_n)$ be a finite dimensional space such that $\chi \in V_h^n$ satisfies

$$\chi|_\tau \in C^\infty(\tau) \quad \forall \tau \in \mathcal{T}_h^n \quad \text{and} \quad \chi|_{\Sigma_n} = 0\,. \tag{2}$$

There are many examples of finite element spaces that fit the specified criterion for V_h^n. Typically this space will consist of piecewise polynomials. The element domains could be triangles, rectangles, tetrahedron, boxes, or polynomial mappings of any of these domains (see [4] for examples). We will obtain some specific results in the case where V_h^n consists of piecewise linears on triangles or tetrahedrons. We seek approximations in the space

$$V_h = \{v \in L^2(Q):\ v|_{Q_n} \in V_h^n\}\,.$$

In Section 3 where the convergence theorems are stated and proved we will require certain approximation properties which we discuss here. We suppose that the family of triangulations is *quasi-uniform*, there exists constants c_0 and c_1 independent of h such that for each element $\tau \in \mathcal{T}_h^n$,

$$c_0 h \leqslant \rho_\tau \leqslant \operatorname{diam} \tau \leqslant c_1 h\,, \tag{3}$$

where ρ_τ is the diameter of the largest inscribed disk or ball inside τ. We assume there exists an interpolation operator π_h that maps $v \in H^{r+1}(Q_n)$ where $v = 0$ on Σ_n to V_h^n and satisfies

$$\|(I - \pi_h)v\|_{H^l(\tau)} \leqslant Ch^{r-l+1}\|v\|_{H^{r+1}(\tau)}\,, \quad l = 0, 1, 2 \tag{4}$$

and

$$\|(I - \pi_h)v\|_{W^{l,\infty}(\tau)} \leqslant Ch^{r-l+1}\|v\|_{W^{r+1,\infty}(\tau)}\,, \quad l = 0, 1, 2 \tag{5}$$

for all $\tau \in \mathcal{T}_h^n$ where C is a constant independent of h, τ and v. The exponent r is identified with the space V_h^n; typically representing the degree of the polynomials used. For example, $r = 1$ if V_h^n consists of piecewise linear functions. $H^s(A)$ denotes the usual Sobolev space of functions with s derivatives in $L^2(A)$ and $W^{s,\infty}(A)$ denotes the space of functions with s derivatives in $L^\infty(A)$. We also assume the inverse property

$$\|\chi\|_{H^s(\tau)} \leqslant Ch^{s-l}\|\chi\|_{H^l(\tau)}\,, \quad 0 \leqslant s \leqslant l \leqslant r\,, \tag{6}$$

where C is independent of h and χ.

Note that functions in V_h are not necessarily continuous across the time levels t_n and their derivatives jump across the boundaries of the element domains τ. Let function v have a jump discontinuity at (x, t). Along the line normal to $\Omega \times \{t\}$ define

$$v_-(x, t) = \lim_{s \to t, s < t} v(x, s)\,, \qquad v_+(x, t) = \lim_{s \to t, s > t} v(x, s)$$

and

$$[v] = v_+ - v_-\,.$$

Let $n = (n_1, \ldots, n_d, n_t)^{\mathrm{t}}$ represent the outward pointing normal to the surface of a given region in Q with space coordinates $n_1, \ldots, n_d$ and n_t is the t-coordinate. Define

$$\partial\tau_- = \{(x, t) \in \partial\tau:\ n_t < 0\} \quad \text{and} \quad \partial\tau_+ = \partial\tau \backslash \partial\tau_- = \{(x, t) \in \partial\tau:\ n_t \geqslant 0\}\,.$$

Let

$$\lambda_n(t) = \mathrm{e}^{-\alpha(t - t_{n-1})}\,, \quad \alpha \geqslant 0$$

be the weight function. Define the inner products

$$(v, w)_A = \int_A vw \,\mathrm{d}x\,\mathrm{d}t\,, \quad A \subseteq Q$$

and

$$\langle v, w\rangle_\gamma = \int_\gamma vw \, ds , \quad \gamma \subseteq \partial A , \; A \subset Q .$$

We use the following notation for L^2 norms:

$$\|z\|_A^2 = \int_A z^2 \, dx \, dt \quad \text{and} \quad |||z|||_A^2 = \|\dot{z}\|_A^2 + \|\nabla z\|_A^2.$$

Let

$$M_n(v, w) = \sum_{\tau \in \mathcal{T}_h^n} [(\ddot{v}, \dot{w}\lambda_n)_\tau - <[\dot{v}], \dot{w}_+ \lambda_n n_t>_{\partial\tau_- \setminus \Omega_{n-1}}] + \langle \dot{v}_+, \dot{w}_+ \rangle_{\Omega_{n-1}} ,$$

$$K_n(v, w) = \sum_{\tau \in \mathcal{T}_h^n} [-(\nabla\dot{v}, \nabla w \lambda_n)_\tau - (\nabla v, \nabla w \dot{\lambda}_n)_\tau + \langle [\nabla v], \nabla w_- \lambda_n n_t \rangle_{\partial\tau_- \setminus \Omega_{n-1}}]$$
$$+ e^{-\alpha k} \langle \nabla v_-, \nabla w_- \rangle_{\Omega_n}$$

and

$$L_n(v) = \int_{Q_n} f\dot{v}\lambda_n \, dx \, dt + \langle \dot{u}_-, \dot{v}_+ \rangle_{\Omega_{n-1}} + \langle \nabla u_-, \nabla v_+ \rangle_{\Omega_{n-1}} ,$$

where

$$L_1(v) = \int_{Q_1} f\dot{v}\lambda_1 \, dx \, dt + \langle U_1, \dot{v}_+ \rangle_{\Omega_0} + \langle \nabla U_0, \nabla v_+ \rangle_{\Omega_0} .$$

The bilinear forms M_n, K_n and L_n are variational definitions of the time derivative, space derivative, and forcing terms in (1). The specific terms in M_n, K_n and L_n were chosen to provide some positivity as well as consistency to the problem from which stability and convergence theorems could be proved. We are now in a position to define the numerical method: find $u \in V_h$ such that for $n = 1, 2, \ldots, N$,

$$M_n(u, v) + K_n(u, v) = L_n(v) \quad \forall v \in V_h^n . \tag{7}$$

We will see in Lemma 1 that the approximate solution of (7) is consistent with (1) by demonstrating that the true solution U of (1) can be substituted for u in (7). The positivity in this approach is exposed in the proof of Theorem 1.

Problem (7) leads to a square linear system of equations on each time slab Q_n since u and v belong to the same finite dimensional space for each n.

We now show problem (7) is well posed.

THEOREM 1. The following two statements are valid:
(1) *If $\alpha > 0$, then problem (7) has a unique solution.*
(2) *If $\alpha = 0$ and V_h^n consists of piecewise linear functions ($d = 2$ or 3) for all n, then problem (7) has a unique solution.*

PROOF. The crucial steps in the proof involve showing that M_n and K_n are positive definite. Let $v \in V_h^n$. Then since

$$\lambda_n \ddot{v}\dot{v} = \frac{1}{2} \frac{d}{dt} (\lambda_n \dot{v}^2) - \frac{1}{2} \dot{\lambda}_n \dot{v}^2$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(\lambda_n \dot{v}^2), 1\right)_\tau = \langle \dot{v}_-, \dot{v}_- \lambda_n n_t \rangle_{\partial\tau_+} + \langle \dot{v}_+, \dot{v}_+ \lambda_n n_t \rangle_{\partial\tau_-},$$

we have

$$M_n(v, v) = \sum_{\tau\in\mathcal{T}_h^n} [\tfrac{1}{2}(\dot{v}, \dot{v}(-\dot{\lambda}_n))_\tau + \tfrac{1}{2}\langle \dot{v}_-, \dot{v}_- \lambda_n n_t \rangle_{\partial\tau_+} + \tfrac{1}{2}\langle \dot{v}_+, \dot{v}_+ \lambda_n n_t \rangle_{\partial\tau_-}$$
$$- \langle [\dot{v}], \dot{v}_+ \lambda_n n_t \rangle_{\partial\tau_- \setminus \Omega_{n-1}}] + \langle \dot{v}_+, \dot{v}_+ \lambda_n \rangle_{\Omega_{n-1}}.$$

Separate $[\dot{v}] = \dot{v}_+ - \dot{v}_-$ in the fourth term inside the braces. Let Q_n^0 denote the *interior* of Q_n and e represent an *edge* $(d = 1)$ or *face* $(d = 2)$ between the element domains τ in the triangulation of Q_n. Recall $n_t < 0$ on $\partial\tau_-$ and $n_t \geq 0$ on $\partial\tau_+$.

$$M_n(v, v) = \frac{1}{2}\Big[\|\dot{v}(-\dot{\lambda}_n)^{1/2}\|_{Q_n}^2 + \mathrm{e}^{-\alpha k}\|\dot{v}_-\|_{\Omega_n}^2 + \|\dot{v}_+\|_{\Omega_{n-1}}^2$$
$$+ \sum_{e\subset Q_n^0} \langle \dot{v}_-^2 + \dot{v}_+^2 - 2\dot{v}_-\dot{v}_+, \lambda_n |n_t| \rangle_e\Big].$$

Therefore

$$M_n(v, v) = \frac{1}{2}\Big(\|\dot{v}(-\dot{\lambda}_n)^{1/2}\|_{Q_n}^2 + \mathrm{e}^{-\alpha k}\|\dot{v}_-\|_{\Omega_n}^2 + \|\dot{v}_+\|_{\Omega_{n-1}}^2 + \sum_{e\subset Q_n^0} \|[\dot{v}]\sqrt{\lambda_n |n_t|}\|_e^2\Big). \tag{8}$$

We now show how to obtain a similar identity for the bilinear form K_n. Since

$$-\nabla\dot{v}\cdot\nabla v\lambda_n = \frac{1}{2}\nabla v\cdot\nabla v\dot{\lambda}_n - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nabla v\cdot\nabla v\lambda_n)$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(\nabla v\cdot\nabla v\lambda_n), 1\right)_\tau = \langle \nabla v_+, \nabla v_+ \lambda_n n_t \rangle_{\partial\tau_-} + \langle \nabla v_-, \nabla v_- \lambda_n n_t \rangle_{\partial\tau_+},$$

we have

$$K_n(v, v) = \frac{1}{2}\sum_{\tau\in\mathcal{T}_h^n} [-(\nabla v, \nabla v\dot{\lambda}_n)_\tau - \langle \nabla v_+, \nabla v_+ \lambda_n n_t \rangle_{\partial\tau_-} - \langle \nabla v_-, \nabla v_- \lambda_n n_t \rangle_{\partial\tau_+}$$
$$+ 2\langle [\nabla v], \nabla v_- \lambda_n n_t \rangle_{\partial\tau_- \setminus \Omega_{n-1}}] + \mathrm{e}^{-\alpha k}\|\nabla v_-\|_{\Omega_n}^2.$$

Combining the second, third and fourth terms inside the braces this becomes

$$K_n(v, v) = \frac{1}{2}\|\nabla v(-\dot{\lambda}_n)^{1/2}\|_{Q_n}^2 + \frac{1}{2}\sum_{\tau\in\mathcal{T}_h^n} [-\langle \nabla v_+, \nabla v_+ \lambda_n n_t \rangle_{\partial\tau_- \setminus \Omega_{n-1}} + \langle \nabla v_+, \nabla v_+ \rangle_{\Omega_{n-1}}$$
$$- \langle \nabla v_-, \nabla v_- \lambda_n n_t \rangle_{\partial\tau_+ \setminus \Omega_n} - \mathrm{e}^{-\alpha k}\langle \nabla v_-, \nabla v_- \rangle_{\Omega_n} + 2\langle [\nabla v], \nabla v_- \lambda_n n_t \rangle_{\partial\tau_- \setminus \Omega_{n-1}}]$$
$$+ \mathrm{e}^{-\alpha k}\|\nabla v_-\|_{\Omega_n}^2.$$

Removing the inner products over Ω_{n-1} and Ω_n from the middle term and combining the interior inner products over the edges or faces e, we now have

$$K_n(v,v)=\frac{1}{2}\Bigg[\|\nabla v(-\dot{\lambda}_n)^{1/2}\|^2_{Q_n}+e^{-\alpha k}\|v_-\|^2_{\Omega_n}+\|v_+\|^2_{\Omega_{n-1}}$$
$$+\sum_{e\subset Q_n^0}\langle|\nabla v_+|^2-|\nabla v_-|^2-2[\nabla v]\cdot\nabla v_-,\ \lambda_n|n_t|\rangle_e\Bigg].$$

So, finally,

$$K_n(v,v)=\frac{1}{2}\Bigg[\|\nabla v(-\dot{\lambda}_n)^{1/2}\|^2_{Q_n}+e^{-\alpha k}\|\nabla v_-\|^2_{\Omega_n}+\|\nabla v_+\|^2_{\Omega_{n-1}}$$
$$+\sum_{e\subset Q_n^0}\|[\nabla v]\sqrt{\lambda_n|n_t|}\|^2_e\Bigg]. \tag{9}$$

We are now prepared to prove the theorem. Since problem (7) consists of N square finite dimensional linear systems, one for each slab Q_n, it is sufficient to show that the only solution of each homogeneous system is the zero solution.

If $\alpha>0$ and v is a solution of the homogeneous problem, then

$$M_n(v,v)+K_n(v,v)=0 \tag{10}$$

and from (8) and (9) it is clear that

$$\dot{v}=0 \quad\text{and}\quad \nabla v=0\,.$$

Since $v=0$ on Σ we must have $v\equiv 0$ proving statement (1).

To show statement (2) observe that on each slab Q_n, $\dot{v}$ and ∇v are constant in t for $\alpha\geqslant 0$. This follows from (8)–(10) since

$$[\dot{v}]=0 \quad\text{and}\quad [\nabla v]=0$$

on each edge or face e where $n_t\neq 0$. From the terms involving norms on Ω_{n-1} and Ω_n in (8) and (9), it follows from (10) that

$$\dot{v}=0 \quad\text{and}\quad \nabla v=0$$

on Ω_{n-1} and Ω_n. Thus $\dot{v}$ and $\nabla v=0$ on each Q_n. So, once again, $v\equiv 0$ and statement (2) is proved. □

REMARK. Observe that the identities (8) and (9) would hold if v was the sum of an element of V_h^n and a smooth function. This observation is exploited in Theorem 2 when we estimate norms of $e=U-u$.

We now show that problem (7) is, in a sense that will be clear in the forthcoming lemma, consistent with problem (1). For simplicity, we will require U is smooth. In particular, we need to assume $\dot{U}$ and ∇U are continuous on Q and $\nabla\dot{U}\in L^2(Q)$; the former is to ensure that the terms in K_n and M_n arising from jump discontinuities are zero and the latter to guarantee that integration by parts on terms of the form $(\nabla U, \nabla \dot{v})$ can be applied.

LEMMA 1. Suppose the solution U of problem (1) is smooth, then

$$M_n(U, v) + K_n(U, v) = L_n(v) \quad \forall v \in V_h^n \tag{11}$$

and $n = 1, 2, \ldots, N$.

PROOF. From (1) we have for $v \in V_h^n$

$$(\ddot{U} - \Delta U, \dot{v}\lambda_n)_{Q_n} = (f, \dot{v}\lambda_n)_{Q_n} .$$

Integrating by parts in t, this becomes

$$\begin{aligned}&(\ddot{U}, \dot{v}\lambda_n)_{Q_n} + (\Delta\dot{U}, v\lambda_n)_{Q_n} + (\Delta U, v\dot{\lambda}_n)_{Q_n} - \langle \Delta U, v_- \lambda_n \rangle_{\Omega_n} \\ &\quad = (f, \dot{v}\lambda_n)_{Q_n} - \langle \Delta U, v_+ \lambda_n \rangle_{\Omega_{n-1}} .\end{aligned}$$

Now apply integration by parts in space. This yields

$$\begin{aligned}&(\ddot{U}, \dot{v}\lambda_n)_{Q_n} - (\nabla\dot{U}, \nabla v\lambda_n)_{Q_n} - (\nabla U, \nabla v\dot{\lambda}_n)_{Q_n} + e^{-\alpha k} \langle \nabla U, \nabla v_- \rangle_{\Omega_n} \\ &\quad = (f, \dot{v}\lambda_n)_{Q_n} + \langle \nabla U, \nabla v_+ \rangle_{\Omega_{n-1}} .\end{aligned}$$

Again using the fact that

$$[\dot{U}] = 0 \quad \text{and} \quad [\nabla U] = 0$$

on interior edges or faces, we obtain

$$(M_n(U, v) - \langle \dot{U}, \dot{v}_+ \rangle_{\Omega_{n-1}}) + K_n(U, v) = (f, \dot{v}\lambda_n)_{Q_n} + \langle \nabla U, \nabla v_+ \rangle_{\Omega_{n-1}} .$$

Equation (11) now follows by adding the boundary term on the left side to the right side and identifying $L_n(v)$. □

3. Error estimates

In this section, we analyze the numerical method and prove our main convergence theorem. All our results involve the energy norm $|||\cdot|||_A$. We use the symbol C to denote a generic constant that is independent of h, k and u. It may depend on U and T, however. For simplicity of presentation, we do not track carefully the precise dependence of the constants on the higher order derivatives of U. Instead we make the assumption that U is sufficiently smooth and collect the norms of the derivatives in the constants C.

Before presenting the main theorem, we give an auxiliary lemma.

LEMMA 2. Suppose A_n, B_n for $n=1,\ldots,N$, β and ϵ are positive numbers, $0<k\leqslant 1$, $B_0=0$, $Nk=T$, and

$$A_l + e^{-\beta k}B_l \leqslant \epsilon k + B_{l-1}\,, \quad l=1,2,\ldots,N\,. \tag{12}$$

Then

$$\sum_{l=1}^{n} e^{(n-l+1)\beta k}A_l + B_n \leqslant C\epsilon\,, \tag{13}$$

where $C=C(\beta, T)$ is a constant.

PROOF. Multiplying (12) by $e^{(n-l)\beta k}$ we obtain

$$e^{(n-l)\beta k}A_l = -e^{(n-l-1)\beta k}B_l + e^{(n-l)\beta k}B_{l-1} + \epsilon k\, e^{(n-l)\beta k}\,. \tag{14}$$

Summing (14) for $l=1,\ldots,n$, we have

$$\sum_{l=1}^{n} e^{(n-l)\beta k}A_l + e^{-\beta k}B_n = \epsilon k\, e^{n\beta k}\sum_{l=1}^{n} e^{-l\beta k}\,, \tag{15}$$

where we have used the assumption that $B_0=0$. But

$$\sum_{l=1}^{n} e^{-l\beta k} \leqslant \frac{1}{e^{\beta k}-1} \leqslant \frac{1}{\beta k}\,.$$

Substituting this in (15), multiplying the result by $e^{\beta k}$, using the fact that $k\leqslant 1$, and $nk\leqslant T$ we obtain the estimate (13). □

We are now in a position to state and prove the main theorem.

THEOREM 2. Suppose the triangulations $\mathcal{T}_h^n$ satisfy (3), the spaces V_h^n satisfy the approximation properties (4)–(6) for some integer $r>0$, and $k=Ch^\gamma\leqslant 1$ for some $\gamma\geqslant 0$. If u is the solution of (4) and U is a smooth solution of (1), then there exist constants C independent of h such that:
(a) If the spaces V_h^n consist of piecewise linear functions ($r=1$) and $\alpha=0$, then

$$|||U-u_-|||_{\Omega_n} \leqslant C\max\{h^{1/2}, h^{1-\gamma}\} \tag{16}$$

for $n=1,2,\ldots,N$.
(b) If $\alpha>0$, then there exists $\beta>0$ independent of h such that

$$\left(\sum_{l=1}^{n} e^{(n-l+1)\beta k}|||U-u|||_{Q_l}^2 + |||U-u_-|||_{\Omega_n}^2\right)^{1/2} \leqslant C\max\{h^{r-1}, h^{r-\gamma}\}\,,\quad n=1,2,\ldots,N \tag{17}$$

and if k is a fixed constant independent of h, this becomes

$$(|||U-u|||_Q^2 + |||U-u_-|||_{\Omega_T}^2)^{1/2} \leqslant Ch^{r-1}\,. \tag{18}$$

PROOF. Let $e = U - u$ and define $e_- = \dot{e}_- = \nabla e_- = 0$ on Ω_0. From Lemma 1, we have

$$M_n(e, v) + K_n(e, v) = \langle \dot{e}_-, \dot{v}_+ \rangle_{\Omega_{n-1}} + \langle \nabla e_-, \nabla v_+ \rangle_{\Omega_{n-1}} . \tag{19}$$

From identities (8) and (9) (see the Remark after the proof of Theorem 1) it follows that

$$M_n(e, e) + K_n(e, e) \geq \frac{1}{2}\Bigg[\sigma |||e|||^2_{Q_n} + \mathrm{e}^{-\alpha k} |||e_-|||^2_{\Omega_n} + |||e_+|||^2_{\Omega_{n-1}} + \sum_{e \in Q_n^0} (\|[\dot{u}]\sqrt{\lambda_n |n_t|}\|^2_e + \|[\nabla u]\sqrt{\lambda_n |n_t|}\|^2_e)\Bigg] , \tag{20}$$

since $[e] = [u]$ where $\sigma \equiv \alpha\, \mathrm{e}^{-\alpha} \leq -\dot{\lambda}_n(t)$ on Q_n. Also from (19),

$$M_n(e, e) + K_n(e, e) = M_n(e, \eta) + K_n(e, \eta) + \langle \dot{e}_-, \pi_h \dot{U}_+ - \dot{u}_+ \rangle_{\Omega_{n-1}} + \langle \nabla e_-, \nabla(\pi_h U - u)_+ \rangle_{\Omega_{n-1}} ,$$

where $\eta = (I - \pi_h)U$ and thus

$$\begin{aligned} M_n(e, e) + K_n(e, e) &= M_n(e, \eta) + K_n(e, \eta) + \langle \dot{e}_-, \dot{e}_+ - \dot{\eta}_+ \rangle_{\Omega_{n-1}} \\ &\quad + \langle \nabla e_-, \nabla(e - \eta)_+ \rangle_{\Omega_{n-1}} \\ &= I_1 + I_2 + I_3 + I_4 . \end{aligned} \tag{21}$$

The second to last step followed from $\pi_h U - u = -\eta + e$. We use the Cauchy–Schwarz and arithmetic-geometric mean inequalities to estimate I_3 and I_4.

$$\begin{aligned} I_3 &\leq \tfrac{1}{2}(1 + k)\|\dot{e}_-\|^2_{\Omega_{n-1}} + \tfrac{1}{2}(1 + k)^{-1}\|\dot{e}_+\|^2_{\Omega_{n-1}} + \tfrac{1}{2}k\|\dot{e}_-\|^2_{\Omega_{n-1}} + (1/2k)\|\dot{\eta}_+\|^2_{\Omega_{n-1}} \\ &\leq \tfrac{1}{2}(1 + 2k)\|\dot{e}_-\|^2_{\Omega_{n-1}} + \tfrac{1}{2}(1 + k)^{-1}\|\dot{e}_+\|^2_{\Omega_{n-1}} + Ck^{-1}h^{2r} , \end{aligned}$$

where the last step followed from (5). The estimate for I_4 is similar.

$$I_4 \leq \tfrac{1}{2}(1 + 2k)\|\nabla e_-\|^2_{\Omega_{n-1}} + \tfrac{1}{2}(1 + k)^{-1}\|\nabla e_+\|^2_{\Omega_{n-1}} + Ck^{-1}h^{2r} .$$

Combining the bounds for I_3 and I_4, we have

$$I_3 + I_4 \leq \tfrac{1}{2}(1 + 2k)|||e_-|||^2_{\Omega_{n-1}} + \tfrac{1}{2}(1 + k)^{-1}|||e_+|||^2_{\Omega_{n-1}} + Ck^{-1}h^{2r} . \tag{22}$$

We turn to bounding I_1 and I_2.

$$\begin{aligned} I_1 &\leq \sum_{\tau \in \mathcal{T}_h^n} (\ddot{e}, \dot{\eta}\lambda_n)_\tau + \sum_{e \in Q_h^0} \tfrac{1}{4}\, \|[\dot{u}]\sqrt{\lambda_n |n_t|}\|^2_e + \sum_{e \in Q_h^0} \|\dot{\eta}_+\|^2_e \\ &\quad + \tfrac{1}{2}\delta k\|\dot{e}_+\|_{\Omega_{n-1}} + C_\delta k^{-1}\|\dot{\eta}_+\|_{\Omega_{n-1}} \\ &= A_1 + A_2 + A_3 + A_4 + A_5 . \end{aligned}$$

$$
\begin{aligned}
I_2 &\leq \sum_{\tau\in\mathcal{T}_h^n} (-(\nabla\dot{e}, \nabla\eta\lambda_n)_\tau - (\nabla e, \nabla\eta\dot{\lambda}_n)_\tau) + \sum_{e\in Q_h^0} \tfrac{1}{4}|||[\nabla u]\sqrt{\lambda_n|n_t|}|||_e^2 \\
&\quad + \sum_{e\in Q_h^0} |||\nabla\eta_+|||_e^2 + \tfrac{1}{2}\epsilon k\, \mathrm{e}^{-\alpha k}\|\nabla e_-\|_{\Omega_n}^2 + C_\epsilon k^{-1}\|\nabla\eta_-\|_{\Omega_n}^2 \\
&= B_1 + B_2 + B_3 + B_4 + B_5 ,
\end{aligned}
$$

where δ and ϵ are small positive constants to be chosen later. We estimate A_3 and B_3 in tandem.

$$
A_3 + B_3 = \sum_{e\in Q_h^0} |||\eta_+|||_e^2 \leq Ch^{2r} \sum_{e\in Q_h^0} m(e) .
$$

The last step followed from (5) and $m(e)$ is the measure of edge or face e. Since

$$
m(e) \leq Ch^d \quad \text{and} \quad \sum_{e\in Q_h^0} 1 \leq C\frac{k}{h^{d+1}} ,
$$

we obtain

$$
A_3 + B_3 \leq Ch^{2r-1}k . \tag{23}
$$

We can also estimate A_5 and B_5 together:

$$
A_5 + B_5 \leq (C_\delta + C_\epsilon)k^{-1}h^{2r} . \tag{24}
$$

The terms A_4 and B_4 will be handled with the terms on the right side of (22). The terms A_2 and B_2 will ultimately be subtracted from the right side of (20). The pair A_1 and B_1 are handled differently for each of the cases (a) and (b). For the moment, we restrict our attention to case (a) where $r = 1$, $\alpha = 0$ and, thus, $\lambda_n = 1$.

CASE a:

$$
\begin{aligned}
A_1 + B_1 &= \sum_{\tau\in\mathcal{T}_h^n} (\ddot{U}, \dot{\eta})_\tau - (\nabla\dot{U}, \nabla\eta)_\tau \\
&\leq (\|\ddot{U}\|_{Q_n} + \|\nabla\dot{U}\|_{Q_n})|||\eta|||_{Q_n} \\
&\leq C(k^{1/2}\|U\|_{W^{2,\infty}(Q_n)})(hk^{1/2}\|U\|_{W^{2,\infty}(Q_n)}) \\
&\leq Ckh .
\end{aligned} \tag{25}
$$

We applied (5) on the second to last step. We now combine (21) with the estimates (22)–(25) to obtain

$$
\begin{aligned}
M_n(e,e) + K_n(e,e) &\leq Ckh + \frac{1}{4}\sum_{e\in Q_h^0} (|||[\dot{u}]\sqrt{|n_t|}|||_e^2 + |||[\nabla u]\sqrt{|n_t|}|||_e^2) + (C_\delta + C_\epsilon)k^{-1}h^2 \\
&\quad + \frac{1}{2}\left(\frac{1}{1+k} + \delta k\right)|||e_+|||_{\Omega_{n-1}}^2 + \tfrac{1}{2}\epsilon k\|\nabla e_-\|_{\Omega_n}^2 \\
&\quad + \frac{1}{2}(1+2k)|||e_-|||_{\Omega_{n-1}}^2 .
\end{aligned} \tag{26}
$$

Choose $\delta = \frac{1}{4}$ so

$$\frac{1}{1+k} + \delta k < 1 \quad \text{for } 0 < k \leq 1 . \tag{27}$$

We now use (20) to obtain a lower bound on the left side of (26). Subtract the second, fourth and fifth terms on the right side of (26) from the right side of (20). Thus we have

$$(1 - \epsilon k)|||e_-|||^2_{\Omega_n} \leq C_\epsilon(kh + k^{-1}h^2) + (1 + 2k)|||e_-|||^2_{\Omega_{n-1}} . \tag{28}$$

If $\gamma = 0$, then pick $\epsilon = \frac{1}{4}$ and iterate the inequality (28) over the N intervals to obtain the estimate (16). Here N is a fixed number independent of h.

If $\gamma > 0$, so k depends on h, then pick $\epsilon = \frac{1}{2}$ and divide (28) by $(1 - k/2)$ to give

$$|||e_-|||^2_{\Omega_n} \leq C(kh + k^{-1}h^2) + \mathrm{e}^{ck}|||e_-|||^2_{\Omega_{n-1}} ,$$

where c is a constant. Iterating this inequality and using the fact that

$$\sum_{j=0}^{n} \mathrm{e}^{cjk} \leq Ck^{-1}\,\mathrm{e}^{cT} = Ck^{-1} ,$$

we have

$$|||e_-|||^2_{\Omega_n} \leq C\left(h + \frac{h^2}{k^2}\right) \leq C \max\{h, h^{2(1-\gamma)}\} ,$$

completing the proof of (16).

CASE b: We first estimate $A_1 + B_1$.

$$\begin{aligned} A_1 + B_1 &\leq C\Bigg(\sum_{\tau \in \mathcal{T}_h^n} \|\ddot{e}\|^2_\tau + \|\nabla \dot{e}\|^2_\tau + \|\nabla e\|^2_\tau\Bigg)^{1/2} |||\eta|||_{Q_n} \\ &\leq C\Bigg(\sum_{\tau \in \mathcal{T}_h^n} \|e\|^2_{H^2(\tau)}\Bigg)^{1/2} h^r \|u\|_{H^{r+1}(Q_n)} \\ &\leq Ch^r k^{1/2}\Bigg(\sum_{\tau \in \mathcal{T}_n} \|e\|^2_{H^2(\tau)}\Bigg)^{1/2} . \end{aligned} \tag{29}$$

We estimate the term inside the summation using the triangle inequality and the inverse estimate (6).

$$\begin{aligned} \|e\|_{H^2(\tau)} &\leq \|u - \pi_h U\|_{H^2(\tau)} + \|\eta\|_{H^2(\tau)} \\ &\leq Ch^{-1}\|u - \pi_h U\|_{H^1(\tau)} + Ch^{r-1}\|U\|_{H^{r+1}(\tau)} \\ &\leq Ch^{-1}\|e\|_{H^1(\tau)} + Ch^{-1}\|\eta\|_{H^1(\tau)} + Ch^{r-1}\|U\|_{H^{r+1}(\tau)} \\ &\leq Ch^{-1}\|e\|_{H^1(\tau)} + Ch^{r-1}\|U\|_{H^{r+1}(\tau)} . \end{aligned}$$

Substituting this in (29) yields

$$A_1 + B_1 \leq Ch^r k^{1/2}(h^{-2}|||e|||^2_{Q_n} + h^{2r-2}\|U\|^2_{H^{r+1}(Q_n)})^{1/2},$$

where we use the fact that $\|e\|_{H^1(Q_n)} \leq C|||e|||_{Q_n}$ which follows from the Poincaré inequality since $e = 0$ on Σ_n. Since U is smooth $\|U\|_{H^{r+1}(Q_n)} \leq Ck^{1/2}$, so by the arithmetic-geometric mean inequality, we have

$$A_1 + B_1 \leq Ch^{2r-2}k + \frac{\sigma}{4}|||e|||^2_{Q_n}. \tag{30}$$

We again combine (21) with (22)–(24) and (30) yielding

$$\begin{aligned} M_n(e, e) + K_n(e, e) \leq{} & Ch^{2r-2}k + \frac{1}{4}\sum_{e\in Q_h^0}(\|[\dot{u}]\sqrt{\lambda_n|n_t|}\|^2_e + \|[\nabla u]\sqrt{\lambda_n|n_t|}\|^2_e) \\ & + (C_\delta + C_\epsilon)k^{-1}h^{2r} + \frac{1}{2}\left(\frac{1}{1+k} + \delta k\right)|||e_+|||^2_{\Omega_{n-1}} \\ & + \frac{1}{2}\epsilon k\,\mathrm{e}^{-\alpha k}\|\nabla e_-\|^2_{\Omega_n} + \frac{1}{2}(1+2k)|||e_-|||^2_{\Omega_{n-1}} + \frac{1}{4}\sigma|||e|||^2_{Q_n}. \end{aligned}$$

Use (20) for a lower bound of the left side, choose $\delta = \frac{1}{4}$ so (27) holds, subtract the second, fourth, fifth and seventh terms from the right side of (20) to obtain

$$\frac{\sigma}{2}|||e|||^2_{Q_n} + \mathrm{e}^{-\alpha k}(1 - \epsilon k)|||e_-|||^2_{\Omega_n} \leq C_\epsilon(h^{2r-2}k + h^{2r}k^{-1}) + (1+2k)|||e_-|||^2_{\Omega_{n-1}}. \tag{31}$$

If $\gamma = 0$, then pick $\epsilon = \frac{1}{4}$ and iterate the inequality (31) over the N intervals to obtain the estimate (18). Again, N is a fixed number independent of h.

If $\gamma > 0$ so k depends on h, then pick $\epsilon = \frac{1}{2}$ and divide (31) by $(1 - k/2)$ to give

$$\frac{\sigma}{2}|||e|||^2_{Q_n} + \mathrm{e}^{-\alpha k}|||e_-|||_2\Omega_n \leq C(h^{2r-2}k + h^{2r}k^{-1}) + \mathrm{e}^{ck}|||e_-|||^2_{\Omega_{n-1}}$$

and dividing by e^{ck} and noting that e^{-ck} is bounded above and below independent of k, we obtain

$$C|||e|||^2_{Q_n} + \mathrm{e}^{-(\alpha+c)k}|||e_-|||^2_{\Omega_n} \leq C(h^{2r-2}k + h^{2r}k^{-1}) + |||e_-|||^2_{\Omega_{n-1}}.$$

Applying Lemma 2 with $\beta = \alpha + c$, $\epsilon = C(h^{2r-2} + h^{2r}k^{-2})$, $A_n = C|||e|||^2_{Q_n}$ and $B_n = |||e_-|||^2_{\Omega_n}$, we obtain the estimate (17). □

REMARK. From the proof of Theorem 2, we can see that convergence estimates could also be obtained for

$$|||e_+|||_{\Omega_{n-1}} \quad \text{and} \quad \sum_{e\in Q_n^0}(\|[\dot{u}]\sqrt{|n_t|}\|^2_e + \|[\nabla u]\sqrt{|n_t|}\|^2_e).$$

Table 1

N	$\|u - U\|_Q$	Rate	$\|\|\|u - U\|\|\|_Q$	Rate	$\|\|\|u - U\|\|\|_{\Omega_1}$	Rate
10	6.07 (−2)	–	6.06 (−1)	–	1.43 (0)	–
20	3.46 (−2)	0.81	3.62 (−1)	0.74	8.84 (−1)	0.69
40	1.85 (−2)	0.90	2.01 (−1)	0.85	5.05 (−1)	0.81
80	8.61 (−3)	1.10	1.06 (−1)	0.92	2.53 (−1)	1.00

The latter appears to be somewhat undesirable, especially in the piecewise linear case since it implies there is some 'stiffness' or 'inflexibility' in the approximation. This feature seems necessary, however, to gain the 'positivity' which we exploited to prove this theorem. It is also typical of some discontinuous Galerkin methods (see [1, Chapter 9.8]).

4. Numerical results

In this section, we present the results of a simple computation using piecewise linear functions in one space and one time dimension.

We consider problem (1) on $Q = (0,1) \times (0,1)$ with $U_0(x) = \sin \pi x$, $f = 0$ and $U_1 = 0$ which has the solution

$$U(x,t) = \cos \pi t \sin \pi x .$$

We took $k = 1$. In our computation, problem (7) led to a nonsymmetric system of equations $Ax = b$ which we solved by first forming a symmetric problem $A^tAx = A^tb$ and then applying preconditioned conjugate gradients. The preconditioner was formed by an incomplete Cholesky decomposition. In Table 1 are given our results for various $N \times N$ meshes.

The rate was calculated by the formula

$$\text{Rate} = \frac{\ln(E_2/E_1)}{\ln(h_2/h_1)} ,$$

where h_1 and h_2, E_1 and E_2 are successive triangle diameters and errors, respectively. The actual L^2 integrals were approximated by the trapezoid rule on each triangle or edge.

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