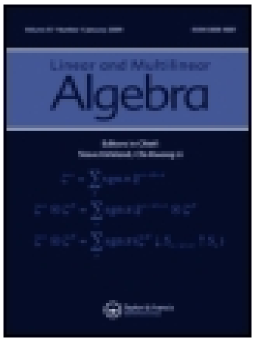

Linear and Multilinear Algebra

ISSN: 0308-1087 (Print) 1563-5139 (Online) Journal homepage: https://www.tandfonline.com/loi/glma20

Eigenvalues of the Laplacian of a graph

William N. Anderson Jr. & Thomas D. Morley

To cite this article: William N. Anderson Jr. & Thomas D. Morley (1985) Eigenvalues of the Laplacian of a graph , Linear and Multilinear Algebra, 18:2, 141-145, DOI: 10.1080/03081088508817681

To link to this article: https://doi.org/10.1080/03081088508817681

Published online: 30 May 2007.

Linear and Multilinear Algebra, 1985, Vol. 18, pp. 141–145
0308–1087/85/1802–0141 \$10.00/0

Eigenvalues of the Laplacian of a Graph*

WILLIAM N. ANDERSON, JR. AND THOMAS D. MORLEY
Fairleigh Dickinson University, Teaneck, NJ 07666

(Received July 10, 1984)

Let G be a finite undirected graph with no loops or multiple edges. We define the Laplacian matrix of G, $\Delta(G)$, by Δ_{ii} = degree of vertex i and $\Delta_{ij} = -1$ if there is an edge between vertex i and vertex j. In this paper we relate the structure of the graph G to the eigenvalues of $\Delta(G)$; in particular we prove that all the eigenvalues of $\Delta(G)$ are non-negative, less than or equal to the number of vertices, and less than or equal to twice the maximum vertex degree. Precise conditions for equality are given.

1. INTRODUCTION

Let G be a finite undirected graph with no loops or multiple edges. We define the *Laplacian* matrix of $G, \Delta(G)$, by Δ_{ii} = the degree of vertex i and $\Delta_{ij} = -1$ if there is an edge between vertex i and vertex j. This matrix is discussed by Harary [5]. Our name for Δ is chosen because Δ arises in numerical analysis as a discrete analog of the Laplacian operator [3]. In this paper we relate the structure of the graph G to the eigenvalues of $\Delta(G)$; in particular, we prove that all the eigenvalues of Δ are non-negative, less than or equal to the number of vertices, and less than or equal to twice the maximum vertex degree.

There is a considerable body of literature relating the eigenvalues of the adjacency matrix of a graph to its structure [6]; except for Fisher's paper [2], little seems to be known about the Laplacian.

The authors wish to thank Professor Henry P. McKean, Jr. for suggesting this study.

* This is the original of a paper which has been widely circulated in preprint form, as University of Maryland technical report TR-71-45, October 1971.

2. PRELIMINARIES

Our basic graph theory reference is Harary [5]. The definitions of Δ and E, as well as Lemma 1, are taken from Chapter 13 of Harary. To define E, the *vertex-edge incidence matrix*, we first orient G. Then $E_{ij} = 1$ if edge j points toward vertex i, $E_{ij} = -1$ if edge j points away from vertex i, and $E_{ij} = 0$ otherwise. Let E^* denote the transpose of E.

LEMMA 1 $\Delta = EE^*$.

Proof Two distinct rows of E will have a non-zero entry in the same column if and only if an edge joins the corresponding vertices; the corresponding entry will be 1 in one row and -1 in the other, giving a product of -1. Q.E.D.

We will also need to consider the matrix N defined by $N = E^*E$. The important property of N is that if λ is a non-zero eigenvalue of Δ, then it is also an eigenvalue of N, and conversely. In fact, if $\Delta x = \lambda x$ with $\lambda \neq 0$, then $NE^*x = E^*\Delta x = \lambda E^*x$ so that λ is an eigenvalue of N with eigenvector E^*x. The matrix N of course depends on the choice of orientation; we will vary the orientation as needed. In particular, if G is a bipartite graph, we may point all edges toward vertices of one class. Then all entries of N are non-negative; in fact $N = 2I + A$, where A is the adjacency matrix of the line graph of G. Results about line graphs of bipartite graphs thus translate directly into the present context [1].

If M is a matrix, let $\rho(M)$ denote the spectral radius of M.

Let $\bar{G}$ denote the graph complementary to G. That is, $\bar{G}$ has the same set of vertices as G, and vertices v and w are joined in $\bar{G}$ if and only if they are not joined in G.

Let K_n denote the complete graph on n vertices.

3. THE GLOBAL STRUCTURE OF G

In this section we obtain bounds for the eigenvalues of $\Delta(G)$ in terms of the number of vertices and the number of components of G.

LEMMA 2 *The eigenvalues of* $\Delta(K_n)$ *are* 0, *with multiplicity* 1, *and* n, *with multiplicity* $n - 1$.

Proof Let u be the vector with all components equal to 1; then $\Delta(K_n)u = 0$. If x is any vector orthogonal to u, it may be easily verified that $\Delta(K_n)x = nx$. Q.E.D.

THEOREM 1 *If the graph G has n vertices, and λ is an eigenvalue of $\Delta(G)$ then $0 \leqslant \lambda \leqslant n$. The multiplicity of 0 equals the number of components of G; the multiplicity of n is equal to one less than the number of components of $\bar{G}$.*

Proof Suppose λ is an eigenvalue of Δ. Then for some vector x, with $\|x\| = 1$, $\Delta x = \lambda x$. Thus $\lambda = (\lambda x, x) = (\Delta x, x) = (EE^*x, x) = \|E^*x\|^2$. Therefore λ is real and non-negative.

Let the vertices $v_1, \ldots, v_K$ be the vertices of a connected component of G; then the sum of the corresponding rows of E is 0, and any $K - 1$ of these rows are independent. Therefore the nullity of E, and thus of EE^*, is equal to the number of components of G.

If G has n vertices, then $\Delta(G) + \Delta(\bar{G}) = \Delta(K_n)$. If u is the vector with all components 1, then $\Delta(G)u = \Delta(\bar{G})u = \Delta(K_n)u = 0$. If $\Delta(G)x = \lambda x$ for some vector x orthogonal to u, then using Lemma 2 we have $\Delta(\bar{G})x = \Delta(K_n)x - \Delta(G)x = (n - \lambda)x$. Since the eigenvalues of $\Delta(\bar{G})$ are also non-negative, we must have $\lambda \leqslant n$. Moreover $\lambda = n$ if and only if $\Delta(\bar{G})x = 0$, and the dimension of the space of such vectors is one less than the nullity of $\Delta(\bar{G})$ (since all such x are orthogonal to u). Q.E.D.

COROLLARY *If G has n vertices, and $\lambda = n$ is an eigenvalue of $\Delta(G)$, then G is connected.*

Proof If G were not connected, then $\bar{G}$ would be, and by the theorem n could not be an eigenvalue of $\Delta(G)$. Q.E.D.

4. THE LOCAL STRUCTURE OF G

In this section we obtain an upper bound for the eigenvalues of $\Delta(G)$ in terms of vertex degrees.

Before proceeding we need to recall a few facts from the theory of non-negative matrices; our basic reference is Chapter XIII of Gantmacher [4]. Briefly, a matrix M is said to be non-negative if $M_{ij} \geqslant 0$ for all i and j. If M is a matrix, denote by M^+ the matrix obtained by replacing each entry of M by its absolute value. If M is irreducible, and λ is an eigenvalue of M, then $|\lambda| \leqslant \rho(M^+)$, with equality if and only if $M = e^{i\phi}DM^+D^{-1}$ where $D^+ = I$. For an irreducible non-negative matrix M, $\rho(\mathrm{M}) \leqslant$ the maximum row sum with equality if and only if all row sums are equal.

THEOREM 2 *Let G be a graph. Then $\rho(\Delta(G)) \leqslant \mathrm{Max}(\deg v + \deg w)$ where the maximum is taken over all pairs of vertices (v, w) joined by an* *edge of G. If G is connected, then equality holds if and only if G is bipartite and the degree is constant on each class of vertices.*

Proof We will work with the matrix N rather than Δ.

First consider a connected graph G, then N is irreducible, and thus $\rho(N) \leqslant \rho(N^+) \leqslant$ maximum row sum of N^+. But if e is an edge of G joining vertices v and w, then the row sum in the row corresponding to e is $\deg v + \deg w$. The inequality is thus established for connected graphs.

If G is bipartite, then we may orient G with all edges pointing toward the vertices in one of the two classes; thus $N(G) = N^+(G)$. Then $\rho(N) =$ max row sum if and only if all row sums are equal; i.e. if and only if the condition of the theorem holds. Equivalently, equality holds if and only if the line graph of G is regular.

If G is not bipartite, then we will show that $\rho(N) < \rho(N^+)$, so that equality cannot hold in the theorem. In fact, suppose $N = e^{i\phi}DN^+D^{-1}$. Then since $N_{ii} = 2$, we have $2 = e^{i\phi} \cdot D_{ii} \cdot 2 \cdot D_{ii}^{-1}$, so that $e^{i\phi} = 1$. Now suppose that the edges $1, \ldots, K$ form an odd cycle (if no odd cycle exists, then G is bipartite); we may orient G so that the corresponding entries of N are -1. Then $N_{12} = -1 = D_{11} \cdot 1 \cdot D_{22}^{-1}$, so that $D_{22} = -D_{11}$; continuing around the cycle we have $D_{11} = -D_{11}$, contradicting the requirement that $D^+ = I$. Therefore, if G is not bipartite, equality cannot hold in the theorem.

If G is not connected, the inequality, and the corresponding equality statement, follow by applying the theorem to each component separately. Q.E.D.

COROLLARY *Let G be a connected graph. Then* $\rho(\Delta(G)) \leqslant$ *twice the maximum vertex degree with equality if and only if G is a regular bipartite graph.*

Proof This is a special case of the theorem. Q.E.D.

5. EXPLICIT COMPUTATIONS

Theorems 1 and 2 were conjectured from explicit computations with eigenvalues; many of these were done on a digital computer. Some of these results are stated below; the reader may verify them without difficulty.

If G is the complete bipartite graph $K_{m,n}$, then the eigenvalues of $\Delta(G)$ are $m + n$, m, n, 0 with respective multiplicities 1, $n - 1$, $m - 1$, 1.

If G is the cycle with n vertices, then the eigenvalues of $\Delta(G)$ are $4\sin^2(\pi K/n)$, $K = 1, 2, \ldots, n$.

If G is the path with n vertices, the eigenvalues of $\Delta(G)$ are $4\sin^2(\pi K/2n)$, $K = 0, 1, \ldots, n-1$.

If G is the wheel with $n+1$ vertices, the eigenvalue of $\Delta(G)$ are $n+1, 1$, and $1 + 4\sin^2(\pi K/n)$, $K = 1, 2, \ldots, n-1$.

Acknowledgements

This research was partially supported by NSF Grant GY 8987. Computer time for this research was supported by NASA Grant NsG-398 to the Computer Science Center of the University of Maryland.

References

[1] M. Doob, On characterizing certain graphs with four eigenvalues by their spectra, *Linear Algebra and Appl.* **3** (1970), 461–482.
[2] M. E. Fisher, On hearing the shape of a drum, *J. Comb. Theory* **1** (1966), 105–125.
[3] G. I. Forsythe and W. R. Wasow, *Finite-Difference Methods for Partial Differential Equations*, Wiley, New York, 1960.
[4] F. R. Gantmacher, *The Theory of Matrices*, Volume II, Chelsea, New York, 1959.
[5] F. Harary, *Graph Theory*, Addison-Wesley, New York, 1969.
[6] A. J. Hoffman, Some recent results on spectral properties of graphs, in *Beiträge zur Graphentheorie*, Teubner, Leipzig, 1968.
[7] M. Fiedler, Algebraic connectivity of graphs, *Czech. Math. J.* **23** (1973), 298–305.