CHAPTER III

Topological Vector Spaces

We have emphasized in Chapter I, Section 22, the interest in putting topologies on spaces of functions, and have given several examples of such topologies.

We are now going to systematically study the most useful of these spaces, namely those which have the structure of a vector space over **R** or **C**.

I. GENERAL TOPOLOGICAL VECTOR SPACES. EXAMPLES

1. DEFINITION AND ELEMENTARY PROPERTIES OF TOPOLOGICAL VECTOR SPACES

In Chapter I, Section 14, we defined the notions of a topological group, ring, and field by a condition of compatibility between the topology and the algebraic structure. A vector space over **R** or **C** has not only an addition, but also a multiplication by the elements of **R** or **C**; we are thus led, if we wish to define a useful topology on such a space, to also bring in the topology of the underlying field.

To simplify the statements, we shall denote the underlying field by **K** whenever no properties particular to **R** or **C** are involved.

1.1. Definition. LET E BE A SET HAVING BOTH THE STRUCTURE OF A VECTOR SPACE OVER THE FIELD **K** AND AT THE SAME TIME A TOPOLOGY; WE SAY THAT THESE TWO STRUCTURES ARE *COMPATIBLE* IF:

1. THE TOPOLOGY OF E IS COMPATIBLE WITH THE ADDITIVE GROUP STRUCTURE OF E.

2. THE MAPPING $(\lambda, x) \rightarrow \lambda x$ OF THE TOPOLOGICAL SPACE $\mathbf{K} \times E$ INTO E IS CONTINUOUS.

THE SET E WITH THESE TWO COMPATIBLE STRUCTURES IS CALLED A *TOPOLOGICAL VECTOR SPACE* (ABBREVIATED TVS), *REAL* OR *COMPLEX* ACCORDING AS **K** IS **R** OR **C**.

We note at this point that if E is a TVS over **C**, the fact that **R** is a subfield of **C** implies that the topology of E is also compatible with its TVS structure over **R**. This remark will enable us, when it is useful, and regardless of what **K** is, to use the properties of E regarded as a vector space over **R**, for example, the properties of its convex subsets.

EXAMPLE 1. We mention without proof, since it will be a consequence of later results, that the product $\mathbf{K}^n$, and the space $\mathscr{C}([0, 1], \mathbf{K})$ taken with the topology of uniform convergence, are topological vector spaces.

EXAMPLE 2. On the other hand, the vector space $\mathscr{C}(\mathbf{R}, \mathbf{R})$ of continuous numerical functions on **R**, with the topology of uniform convergence (associated with the ecart $d(f, g) = \sup |f(x) - g(x)|$), is not a topological vector space. In fact, its topology is indeed compatible with its group structure, but λf is not a continuous function of the pair (λ, f) since, for example, if f is an unbounded function, λf does not converge uniformly to the function O as $\lambda \to 0$.

1.2. Proposition. *The compatibility of a topology and a vector space structure on a set* E *can be expressed by the following conditions*:

1 *The mapping* $(x, y) \to (x + y)$ *of* $E \times E$ *into* E is *continuous.*

2. *For every* $a \in E$, *the mapping* $\lambda \to \lambda a$ *of* **K** *into* E *is continuous at the point* $\lambda = 0$.

3. *For every* $\alpha \in \mathbf{K}$, *the mapping* $x \to \alpha x$ *of* E *into* E *is continuous at the point* $x = O$.

4. *The mapping* $(\lambda, x) \to \lambda x$ *of* $\mathbf{K} \times E$ *into* E *is continuous at the point* $(0, O)$.

PROOF. By Definition 1.1, if the structures are compatible, then conditions 1–4 are clearly satisfied.

Conversely, suppose these conditions are satisfied; the relation

$$\lambda x = \alpha a + (\lambda - \alpha)a + \alpha(x - a) + (\lambda - \alpha)(x - a)$$

shows that when $\lambda \to \alpha$ and $x \to a$ (which implies $(\lambda - \alpha) \to 0$ and $(x - a) \to O$), then

$$\lambda x \to \alpha a + 0a + \alpha O + 0O = \alpha a.$$

Thus the mapping $(\lambda, x) \to \lambda x$ is continuous.

In particular, the mapping $x \to (-1)x = -x$ is continuous; and since by hypothesis addition is continuous, the topology is indeed compatible with the additive group structure of E.

1.3. Proposition. *Let* E *be a topological vector space. For every scalar* $\alpha \neq 0$ *and every* $b \in \mathrm{E}$, *the dilation* $x \to \alpha x + b$ *is a homeomorphism of* E *with itself.*

Proof. Every dilation is bijective and the mapping inverse to a dilation is itself a dilation. But it follows from Definition 1.1 that every dilation is continuous; hence it is a bicontinuous bijection of E to E.

1.4. Corollary. 1. *Every dilation of* E *carries every open (closed) set in* E *onto an open (closed) set in* E.

2. *The family* $\mathscr{V}_a$ *of neighborhoods of a point* $a \in \mathrm{E}$ *is the image, under the translation* $x \to x + a$, *of the family* $\mathscr{V}$ *of neighborhoods of the point* O.

This is a direct consequence of the fact that every dilation, hence in particular every translation, is a homeomorphism of E with itself.

We recall that the second property is true in every topological group; it shows that the topology of E is known whenever one knows the family of neighborhoods of the point O.

1.5. Proposition. *Let P be a subset of a topological vector space* E.

If P *is a vector subspace of* E (*respectively, a convex set or a cone*), *the same is true of its closure* $\bar{\mathrm{P}}$ *in* E.

Proof. We recall first that if f is a continuous mapping of one topological space A into another B, then for every $\mathrm{X} \subset \mathrm{A}$ one has $f(\bar{\mathrm{X}}) \subset \overline{f(\mathrm{X})}$; indeed, $\mathrm{X} \subset f^{-1}(\overline{f(\mathrm{X})})$ which is closed, hence also $\bar{\mathrm{X}} \subset f^{-1}(\overline{f(\mathrm{X})})$, from which the desired inclusion follows.

Now let P be a vector subspace of E; let $\lambda, \mu \in \mathbf{K}$ and let f be the mapping $(x, y) \to \lambda x + \mu y$ of $\mathrm{E} \times \mathrm{E}$ into E; f carries $\mathrm{P} \times \mathrm{P}$ into P. But since it is continuous, it carries $\bar{\mathrm{P}} \times \bar{\mathrm{P}} = \overline{\mathrm{P} \times \mathrm{P}}$ into $\bar{\mathrm{P}}$; in other words, for all $x, y \in \bar{\mathrm{P}}$ we have $\lambda x + \mu y \in \bar{\mathrm{P}}$, that is, $\bar{\mathrm{P}}$ is a vector subspace of E.

The analogous result concerning convex subsets of E is obtained by using those mappings f for which $\lambda, \mu \geqslant 0$ and $\lambda + \mu = 1$.

The result concerning cones with vertex O is obtained by using the mappings f of E into E of the form $x \to \lambda x$, where $\lambda > 0$.

1.6. Corollary. *In a topological vector space, every hyperplane* H *is either closed or everywhere dense.*

Indeed, by Proposition 1.5, and taking account of the fact that every vector subspace of E containing H is either E or H, either $\bar{\mathrm{H}} = \mathrm{H}$ or $\bar{\mathrm{H}} = \mathrm{E}$.

Continuous linear mappings

1.7. Proposition. *Let* E *and* F *be topological vector spaces over the same field* **K**.

1. *In order that a linear mapping of* E *into* F *be continuous, it is sufficient that it be continuous at the point* O *of* E.

2. *The set* $\mathscr{L}(E, F)$ *of continuous linear mappings of* E *into* F *is a vector subspace of the vector space* $\mathscr{F}(E, F)$ *of mappings of* E *into* F.

PROOF. The first statement is a special case of a characterization of continuous representations in topological groups (see Chapter I, Section 14).

To prove the second statement, let $f, g \in \mathscr{L}(E, F)$, and let $\lambda, \mu \in \mathbf{K}$; the linear mapping $x \to \lambda f(x) + \mu g(x)$ of E into F is continuous since f and g are continuous and since the mapping $(u, v) \to \lambda u + \mu v$ of $F \times F$ into F is continuous; in other words, $\lambda f + \mu g \in \mathscr{L}(E, F)$.

In general, the space $\mathscr{L}(E, E)$ of continuous linear mappings of E into E is denoted by $\mathscr{L}(E)$, and its elements are called continuous linear *operators* in E.

Continuous linear functionals

1.8. The most important continuous linear mappings are those with values in **K**; in other words, the continuous linear functionals. The space $\mathscr{L}(E, \mathbf{K})$ is denoted by E' and is called the *topological dual* of E.

Clearly $E' \subset E^*$, where E^* denotes the algebraic dual of E, that is, the set of linear functionals on E.

In general $E' \neq E^*$, that is, in general there exist discontinuous linear functionals on E (see Problem 69).

For every linear functional f, the hyperplane $f^{-1}(0)$ is the inverse image under f of the closed set $\{0\} \subset \mathbf{K}$; therefore if f is continuous, this hyperplane is closed in E. One can prove, conversely, that if $f^{-1}(0)$ is closed, then f is continuous (see Problem 8). Here is another convenient characterization:

1.9. Proposition. *To say that a linear functional f on* E *is continuous is equivalent to saying that there exists a nonempty open set in* E *on which f is bounded.*

PROOF. If f is continuous, the continuity of f at O implies the existence of a neighborhood of O, hence also a nonempty open set, for all elements x of which $|f(x)| \leqslant 1$.

Conversely, let X be a nonempty open set in E on which $|f(x)| \leqslant k$, and let a be a point of X. The translate $X - a$ of X contains O and is therefore a neighborhood of O, and for every $x \in X - a$ we have

$$f(x) \in f(X-a) = f(X) - f(a), \qquad \text{whence} \qquad |f(x)| \leqslant k + f(a).$$

Thus for every $\epsilon > 0$ there exists a neighborhood V of O on which $|f(x)| \leqslant \epsilon$, namely the image of $X - a$ under the dilation

$$x \to \epsilon(k + (f(a)|)^{-1}x.$$

In other words, f is continuous at the point O, hence everywhere by Proposition 1.7.

1.10. *Total subsets*

We recall that if X is a subset of a vector space E, the smallest vector subspace of E containing X is the set F of linear combinations of elements of X; F is called the subspace generated by X.

If, now, E is a TVS, Proposition 1.5 shows that the closure $\bar{F}$ of F is also a vector subspace of E; $\bar{F}$ is called the *closed subspace of* E *generated by* X.

When $\bar{F} = E$, that is, when F is everywhere dense on E, one says that X is *total* in E. In other words, X is total in E if for every $x \in E$ there exist, in every neighborhood of x, points of E which are linear combinations of elements of X.

For example, every basis of E is a total subset of E; but it is essential to note that a set X can be total in E without generating E. For example, in $\mathscr{C}([0, 1], \mathbf{R})$ the set X of monomials generates the vector space F of polynomials, which is distinct from $\mathscr{C}([0, 1], \mathbf{R})$ but everywhere dense on it by virtue of the Stone-Weierstrass theorem. We shall encounter other examples of total sets, in particular in the study of Hilbert spaces.

2. TOPOLOGY ASSOCIATED WITH A FAMILY OF SEMINORMS

The topologies on the function spaces studied in Chapter I, Section 22, were defined by metrics or more generally by ecarts. When dealing with spaces having a vector space structure, we are led to require of these metrics and ecarts that they be compatible, in a sense to be specified, with the vectorial structure of these spaces. We are thus led to single out the important notion of a seminorm and to study the topologies defined by a family of seminorms.

2.1. Definition. LET E BE A VECTOR SPACE OVER $\mathbf{K}$. BY A *SEMINORM* ON E IS MEANT A MAPPING p OF E INTO $\mathbf{R}$ SUCH THAT:

S_1: $p(x) \geqslant 0$ FOR EVERY $x \in E$;

S_2: $p(\lambda x) = |\lambda| p(x)$ FOR EVERY $x \in E$ AND EVERY $\lambda \in \mathbf{K}$;

S_3: $p(x + y) \leqslant p(x) + p(y)$ FOR ALL $x, y \in E$.

WE CALL p A *NORM* IF IN ADDITION $p(x) \neq 0$ FOR EVERY $x \neq O$.

In studying a well-specified seminorm on a space E, we generally denote $p(x)$ by $\| x \|$.

2.2. EXAMPLE. If f denotes a linear functional on E, $|f|$ is a seminorm on E; indeed,

$$|f| \geqslant 0; \qquad |f(\lambda x)| = |\lambda f(x)| = |\lambda| \, |f(x)|$$

and

$$|f(x + y)| = |f(x) + f(y)| \leqslant |f(x)| + |f(y)|.$$

In order that this seminorm be a norm, it is necessary that $f(x)$ vanish only for $x = O$, hence that E be one-dimensional; conversely, if E is one-dimensional and if $f \neq O$, then $|f|$ is a norm.

2.3. ***Immediate properties and operations***

1. For every seminorm p on E, $p(O) = 0$.

2. Every seminorm p on a vector space E over $\mathbf{C}$ is also a seminorm on E regarded as a vector space over $\mathbf{R}$. In particular, p being subadditive and positive-homogeneous, it is convex in E (see Proposition 19.6 of Chapter II).

We now indicate various operations which enable one to construct seminorms from other seminorms; these operations are very similar to those which we have studied in connection with ecarts (Chapter I, Section 15).

3. Every *linear positive* (that is, with coefficients $\geqslant 0$) combination of seminorms is again a seminorm. In particular, every finite sum of norms is a norm.

For example, in $\mathbf{C}^n$, the function $x \to$ (coordinate x_i) is a linear functional, hence $|x_i|$ is a seminorm; the same is thus true of $\Sigma_i |x_i|$, and since the latter vanishes only if $x = O$, it is a norm.

4. Every (everywhere finite) limit of seminorms is a seminorm.

5. For every family (p_i) of seminorms whose upper envelope p is everywhere finite, p is a seminorm.

Indeed, p clearly satisfies S_1 and S_2; moreover, for every i we have

$$p_i(x + y) \leqslant p_i(x) + p_i(y) \leqslant p(x) + p(y),$$

whence

$$p(x + y) = \sup p_i(x + y) \leqslant p(x) + p(y).$$

For example, let $\mathscr{B}(\mathrm{A}, \mathbf{K})$ be the vector space of bounded mappings of a set A into $\mathbf{K}$. For every $a \in \mathrm{A}$ the mapping $f \to f(a)$ of $\mathscr{B}(\mathrm{A}, \mathbf{K})$ into $\mathbf{K}$ is linear; therefore by Example 2.2 the function $p_a : f \to |f(a)|$ is a seminorm.

Now let X be an arbitrary nonempty subset of A; since every $f \in \mathscr{B}(\mathrm{A}, \mathbf{K})$ is bounded, the function $\sup_{a \in \mathrm{X}} p_a$ is finite, and therefore a seminorm; it is called the seminorm of uniform convergence on X.

6. Let E and F be vector spaces over the same field $\mathbf{K}$, φ a linear mapping of E into F, and p a seminorm on F; then $p \circ \varphi$ is a seminorm on E; the verification of this is immediate.

Example 2.2 rests on this procedure; here is another example. Let F_1 and F_2 be vector spaces over $\mathbf{K}$, and p_1 a seminorm on F_1; then the function $(x_1, x_2) \to p_1(x_1)$ defined on $\mathrm{F}_1 \times \mathrm{F}_2$ is a seminorm on $\mathrm{F}_1 \times \mathrm{F}_2$.

7. Finally, here is an operation analogous to that on ecarts studied in Chapter I; 15.2.6:

Let $(p_1, p_2, ..., p_n)$ be a finite sequence of seminorms on a vector space E, and let φ be an increasing, convex, and positive-homogeneous mapping of $(\mathbf{R}_+)^n$ into $\mathbf{R}_+$.

Then the function $\varphi(p_1, p_2, ..., p_n)$ is a seminorm on E: Property S_1 is evident; S_2 follows from the positive-homogeneity of φ; finally S_3 follows from the fact that φ is increasing and subadditive.

For example, for every number $\alpha \geqslant 1$, $(\Sigma p_i^{\alpha})^{1/\alpha}$ is a seminorm.

2.4. *Ecart associated with a seminorm*

Let E be a vector space with a seminorm p; for all $x, y \in \mathrm{E}$ we put

$$d(x, y) = p(x - y).$$

It is immediate that

1. $d(x, y) \geqslant 0$ and $d(x, x) = 0$;
2. $d(x, y) = d(y, x)$ since $p(x - y) = p(y - x)$;
3. $d(x, y) \leqslant d(x, z) + d(z, y)$;

indeed, this inequality can be written as

$$p(x-y) \leqslant p(x-z) + p(z-y);$$

but

$$(x-y) = (x-z) + (z-y),$$

hence the last inequality follows from S_3.

These three properties show that d is an ecart on E; it is immediate that this ecart is invariant under translation and that, more generally, it is multiplied by $|\alpha|$ under a dilation $x \to \alpha x + b$.

This ecart is finite; in order that it be a metric, it is therefore sufficient that $p(x-y) = 0$ imply $x - y = 0$; in other words, that the seminorm on E be a norm.

2.5. *Balls associated with a seminorm*

We maintain the preceding notation. The set $B(a, \rho)$ of points $x \in E$ such that $d(a, x) < \rho$ is called the *open p-ball* with center a and radius ρ (where $a \in E$ and $\rho > 0$); the closed p-balls are defined in the same way, by replacing $<$ by $\leqslant$. We will call the ball $B(O, 1)$ the *unit p-ball.*

It is immediate that the dilation $x \to \alpha x + b$ transforms $B(O, \rho)$ into $B(b, |\alpha|\rho)$; it is therefore sufficient to study the balls with center O.

2.6. Proposition 1. *Every open p-ball* B *with center* O *is a convex set which is invariant under each of the isometries $x \to \alpha x$, where $|\alpha| = 1$.*

2. *In order that p be a norm, it is necessary and sufficient that* $B(O, \rho)$ *contain no vector subspace of dimension* 1.

PROOF. 1. Since p is convex, the set $B(O, \rho)$ of x such that $p(x) < \rho$ is convex (Proposition 19.3, Chapter II).

The relation $p(\alpha x) = |\alpha| p(x)$ shows that if $|\alpha| = 1$, the dilation $x \to \alpha x$ is a bijective isometry of $B(O, \rho)$ with itself.

2. If there exists $a \neq O$ in E such that $p(a) = 0$, then all the points λa belong to $B(O, \rho)$; while if $p(a) \neq 0$, there exists a λ such that $\lambda a \notin B(O, \rho)$, for example $\lambda = 2\rho/p(a)$.

A similar assertion holds for the closed balls with radius $\rho > 0$.

Finally, we note that the closed ball $B(O, 0)$ is a vector subspace of E, which reduces to $\{O\}$ only if p is a norm.

2.7. *Balls associated with a family of seminorms*

In Chapter I, Section 16, we defined and studied the topology associated with a metric on an arbitrary set E. Therefore if E is a vector space with a norm, the metric associated with this norm defines a topology on E. We shall make a detailed study of such topologies in Sections 4–6; for the moment, more generally, we are going to define the topology associated with a family of seminorms on a vector space.

Let E be a vector space, and let $\mathscr{P} = (p_i)_{i\in I}$ be a family, finite or infinite, of seminorms on E. We denote by $B_i(a, \rho)$ the open p_i-ball with center a and radius ρ.

We will call every finite intersection of open p_i-balls with center a an *open $\mathscr{P}$-ball with center a* (the intersection of two open $\mathscr{P}$-balls with center a is thus also a $\mathscr{P}$-ball with center a).

The consideration of the particular case in which the elements of $\mathscr{P}$ are the multiples λp (where $\lambda > 0$) of a seminorm p shows that one cannot speak of the radius of a $\mathscr{P}$-ball.

Definition of a $\mathscr{P}$-topology

Let us now say that a subset X of E is "open" if either $X = \emptyset$, or for every $x \in X$ there exists an open $\mathscr{P}$-ball, with center x, contained in X.

It is immediate that the collection of these "open sets" satisfies axioms O_1, O_2, O_3 of a topological space; we can therefore give the following definition:

2.8. Definition. LET E BE A VECTOR SPACE, AND LET $\mathscr{P}$ BE AN ARBITRARY FAMILY OF SEMINORMS ON E. THEN THE *TOPOLOGY ASSOCIATED WITH THE FAMILY* $\mathscr{P}$, OR THE *$\mathscr{P}$-TOPOLOGY*, IS DEFINED AS THE TOPOLOGY ON E WHOSE OPEN SETS ARE THE SETS X EACH OF WHOSE POINTS IS THE CENTER OF AN OPEN $\mathscr{P}$-BALL CONTAINED IN X.

Let us show that for this topology, every open p_i-ball is an open set; indeed, let $x \in B_i(a, \rho)$; the open ball $B_i(x, \epsilon)$, where $\epsilon = \rho - p_i(a - x)$, is contained in $B_i(a, \rho)$, for the relation

$$p_i(x - y) < \rho - p_i(a - x)$$

implies

$$p_i(a - y) \leqslant p_i(a - x) + p_i(x - y) < \rho.$$

It follows from this, by axiom O_2, that every open $\mathscr{P}$-ball is an open set; by axiom O_1, every union of open $\mathscr{P}$-balls is therefore an open set.

Conversely, the definition of an open set implies that every open set is the union of open $\mathscr{P}$-balls. Hence *the open sets in* E *and the unions of open $\mathscr{P}$-balls are identical.*

The definition of the $\mathscr{P}$-topology shows that every neighborhood of a point a contains an open $\mathscr{P}$-ball with center a; hence every point

a of E has a neighborhood base of open $\mathscr{P}$-balls with center a, that is, sets of the form $a + \mathrm{B}$, where B is an open $\mathscr{P}$-ball with center O.

2.9. Proposition. *Every $\mathscr{P}$-topology on a vector space* E *is compatible with the vector space structure of* E.

PROOF. We will verify conditions 1–4 of Proposition 1.2; to do this, let us denote by B an open arbitrary $\mathscr{P}$-ball with center O.

1. For all $a, b \in \mathrm{E}$, the balls $a + \frac{1}{2}\mathrm{B}$ and $b + \frac{1}{2}\mathrm{B}$ are neighborhoods of a and b and, taking into account the convexity of B, we have

$$(a + \tfrac{1}{2}\mathrm{B}) + (b + \tfrac{1}{2}\mathrm{B}) = (a + b) + \tfrac{1}{2}(\mathrm{B} + \mathrm{B}) \subset (a + b) + \mathrm{B},$$

whence the continuity of the mapping $(x, y) \to (x + y)$ at the point (a, b).

2. If $\mathrm{B} = \bigcap_{i \in \mathrm{J}} \mathrm{B}_i(\mathrm{O}, \rho_i)$, the condition $\lambda a \in \mathrm{B}$ takes the form $|\lambda|\, p_i(a) \leqslant \rho_i$ for all $i \in \mathrm{J}$, or $|\lambda| \leqslant \inf_{i \in \mathrm{J}} (\rho_i p_i(a)^{-1})$, that is, $|\lambda| \leqslant k$ where $k > 0$.

3. The condition $\alpha x \in \mathrm{B}$ is satisfied for every x if $\alpha = 0$; otherwise it can be written as $x \in \alpha^{-1}\mathrm{B}$, and $\alpha^{-1}\mathrm{B}$ is a neighborhood of O.

4. The condition $\lambda x \in \mathrm{B}$ is satisfied whenever $|\lambda| \leqslant 1$ and $x \in \mathrm{B}$.

Convergence criterion for a filter base

In a metric space E, to say that a sequence (x_n) of points of E converges to a point a of E is equivalent to saying that

$$\lim_{n \to \infty} d(a, x_n) = 0.$$

In particular, if E is a normed space taken with the metric associated with the norm p, this condition assumes the form

$$\lim_{n \to \infty} p(a - x_n) = 0.$$

We are going to see that this condition extends in a simple way to $\mathscr{P}$-topologies.

2.10. Proposition. *Let $\mathscr{B}$ be a filter base on a vector space* E *with a $\mathscr{P}$-topology. To say that $\mathscr{B}$ converges to a point a of* E *is equivalent to saying that, for every $p \in \mathscr{P}$, $\lim_{\mathscr{B}} p(x - a) = 0$.*

PROOF. The open $\mathscr{P}$-balls with center a constitute a neighborhood base of a; therefore the convergence of $\mathscr{B}$ to a is equivalent to the assertion that for every open $\mathscr{P}$-ball B with center a, there exists an $X \in \mathscr{B}$ contained in B.

On the other hand, to say that $\lim_{\mathscr{B}} p(x - a) = 0$ is equivalent to saying that for every $\epsilon > 0$, there exists an $X \in \mathscr{B}$ contained in the ball $\{x : p(x - a) < \epsilon\}$.

Therefore to say that $\lim_{\mathscr{B}} p(x - a) = 0$ for every $p \in \mathscr{P}$ is equivalent to saying that for every finite family of balls of the form

$$\{x : p_i(x - a) < \epsilon_i\},$$

there exists an $X \in \mathscr{B}$ contained in each of them, and hence in their intersection. Since every open $\mathscr{P}$-ball with center a is such an intersection, we have established the stated equivalence.

2.11. EXAMPLE. Let (f_i) be a family of linear functionals on a vector space E; the family of seminorms $|f_i|$ defines a $\mathscr{P}$-topology on E which is called the *weak topology associated with the family of linear functionals* f_i. If $\mathscr{B}$ is a filter base on E, it amounts to the same thing to say that

$$\lim_{\mathscr{B}} f_i(x - a) = 0 \qquad \text{or that} \qquad \lim_{\mathscr{B}} |f_i(x - a)| = 0.$$

Therefore, by Proposition 2.10, to say that $\mathscr{B}$ converges to a in the weak topology associated with the linear functionals f_i is equivalent to saying that, for every f_i,

$$\lim_{\mathscr{B}} f_i(x - a) = 0, \qquad \text{or again that} \qquad \lim_{\mathscr{B}} f_i(x) = f(a).$$

Criterion for the continuity of a linear functional

2.12. Proposition. *Let* E *be a vector space with a $\mathscr{P}$-topology. Then the continuity of a linear functional f on* E *is equivalent to the existence of a finite subfamily $(p_i)_{i \in J}$ of $\mathscr{P}$ and a constant $k > 0$ such that*

$$|f| \leqslant k \sup_{i \in J}(p_i).$$

PROOF. The above relation implies that $|f(x)| \leqslant k$ on the intersection of the open balls $\{x : p_i(x) < 1\}_{i \in J}$. Hence by Proposition 1.9, f is continuous.

2. Conversely, if f is continuous, there exists an open $\mathscr{P}$-ball B with center O on which $|f(x)| \leqslant 1$.

For every $x \in$ E, there exists a $\lambda > 0$ such that $x \in \lambda$B, and for such a λ we have $|f(x)| \leqslant \lambda$. But B is of the form

$$B = \bigcap_{i \in J} \{x : p_i(x) < \epsilon_i\}.$$

We therefore have $|f(x)| \leqslant \lambda$ for every $\lambda > 0$ such that $p_i(x) < \lambda\epsilon_i$, i.e., such that

$$\epsilon_i^{-1} p_i(x) < \lambda \qquad \text{(for every } i \in \mathrm{J}).$$

It follows that

$$|f(x)| \leqslant \sup_{i \in \mathrm{J}} (\epsilon_i^{-1} p_i(x)),$$

whence

$$|f| \leqslant k \sup_{i \in \mathrm{J}} p_i \qquad \text{(where } k = \sup_{i \in \mathrm{J}} \epsilon_i^{-1}).$$

Subspace and product space

2.13. Let E be a vector space, and let F be a vector subspace of E. For every seminorm p on E, the trace of p on F is a seminorm q on F; and for every $a \in \mathrm{F}$, the trace on F of the open p-ball with center a and radius ρ is the open q-ball with center a and radius ρ.

Therefore for every family (p_i) of seminorms on E, the trace on F of the topology associated with the family (p_i) is identical with the topology associated with the family (q_i) of the traces of the p_i on F.

2.14. Let E and F be vector spaces over the same field **K**, and let (p_i), (q_j) be families of seminorms on E and F, respectively. The spaces E and F have corresponding $\mathscr{P}$-topologies; therefore the product $\mathrm{E} \times \mathrm{F}$ has both a vector space structure and a topology which is the product of the topologies of E and F (Chapter I, Section 20).

On the other hand, the functions $(x, y) \to p_i(x)$ and $(x, y) \to q_j(y)$ constitute (see 2.3.6) a family of seminorms on $\mathrm{E} \times \mathrm{F}$ with which is associated a $\mathscr{P}$-topology. We shall see that this $\mathscr{P}$-topology is identical with the product of the topologies of E and F.

Indeed, the open balls with center O associated with the seminorms on $\mathrm{E} \times \mathrm{F}$ are finite intersections of open balls of the form

$$\{(x, y) : p_i(x) < \epsilon_i\} \text{ or } \{(x, y) : q_j(y) < \epsilon_j\};$$

in other words, these balls are simply the products of an open ball with center O in E, and an open ball with center O in F. Since these products constitute a neighborhood base of O for the product topology, the stated identity is proved.

This result clearly extends to every finite product of vector spaces.

Criterion for separation

2.15. Proposition. *In order that a $\mathscr{P}$-topology on a vector space* E *be separated, it is necessary and sufficient that for every $x \neq \mathrm{O}$ of* E, *there exist a seminorm $p \in \mathscr{P}$ such that $p(x) \neq 0$.*

PROOF. 1. If there exists an $x \neq O$ such that $p(x) = 0$ for every $p \in \mathscr{P}$, every open $\mathscr{P}$-ball with center O contains x; therefore x and O cannot be separated by disjoint open neighborhoods.

2. Suppose that for every $x \neq O$ of E, there exists $p \in \mathscr{P}$ such that $p(x) \neq 0$; the open p-balls $B(O, \rho)$ and $B(x, \rho)$, where $\rho = \frac{1}{2}p(x)$, are disjoint. Hence O and x can be separated by disjoint open sets.

More generally, any two distinct points are of the form y, $y + x$, where $x \neq O$, and with the preceding notation they can be separated by the open p-balls $B(y, \rho)$ and $B((y + x), \rho)$.

Metrizable $\mathscr{P}$-topologies

2.16. Proposition. *Let $\mathscr{P}$ be a finite or countable family of seminorms on* E.

1. *If $\mathscr{P}$ is finite, the topology associated with each of the seminorms*

$$\sup p_i, \qquad \left(\sum p_i^2\right)^{1/2}, \qquad \sum p_i,$$

is identical with the $\mathscr{P}$-topology.

2. *If $\mathscr{P} = (p_n)_{n \in N}$, the $\mathscr{P}$-topology is identical with the topology on* E *associated with the ecart*

$$d(x, y) = \sum_n 2^{-n} p_n'(x - y), \qquad \textit{where} \qquad p_n' = \inf(1, p_n).$$

PROOF. 1. If $\mathscr{P}$ has r elements, then

$$\sup p_i \leqslant \left(\sum p_i^2\right)^{1/2} \leqslant \sum p_i \leqslant r \sup p_i .$$

Hence the ecarts associated with the three seminorms in question define the same topology on E (Proposition 16.7, Chapter I).

On the other hand, the balls $\bigcap_i B_i(O, \epsilon)$ of E constitute a neighborhood base of O; but such a ball is simply the ball with radius ϵ associated with the seminorm $p = \sup p_i$. Hence the p-topology and the $\mathscr{P}$-topology are identical.

2. Put

$$d_r(x, y) = \sum_{n \leqslant r} 2^{-n} p_n'(x - y).$$

The relation $d(x, y) < \epsilon$ implies $d_r(x, y) < \epsilon$; therefore the d_r-ball with center O and radius ϵ contains the d-ball with the same center and radius.

Conversely, the relation $d_r(x, y) < \frac{1}{2}\epsilon$ implies

$$d(x, y) < \tfrac{1}{2}\epsilon + \sum_{r+1}^{\infty} 2^{-n};$$

therefore the d-ball with radius ϵ contains the d_r-ball with radius $\frac{1}{2}\epsilon$ for every r such that $\epsilon 2^{r-1} \geqslant 1$.

But the open d_r-balls with center O constitute a neighborhood base of O in the $\mathscr{P}$-topology, since for every r the topologies associated with d_r and with $\Sigma_{n \leqslant r} p_n$ are identical; whence the desired identity.

Note finally that d is not the ecart associated with a seminorm, since d is bounded; this is not related to the method of proof adopted, for there exist $\mathscr{P}$-topologies defined by a countable family of seminorms which are not definable by a single seminorm (see Problem 17).

2.17. Corollary. *If $\mathscr{P}$ is finite or countable, and if the $\mathscr{P}$-topology is separated, then this topology is metrizable.*

Indeed, we have just seen that in these two cases the $\mathscr{P}$-topology is definable by an ecart; if it is separated, this ecart is a metric.

2.18. *The role of $\mathscr{P}$-topologies*

There exist topological vector spaces whose topology is not a $\mathscr{P}$-topology (see Problem 3), but such spaces have not, to the present time, played any role in Analysis.

Indeed, on the one hand, the class of $\mathscr{P}$-topologies suffices for most of the needs of Analysis, and on the other hand, it is the only known class of topological vector spaces for which one is able to establish worthwhile and useful theorems.

Vector spaces with a $\mathscr{P}$-topology are often called *locally convex spaces* because in such a space the point O has a neighborhood base consisting of convex neighborhoods, and because one can show, conversely, that the topology of every TVS having the latter property is a $\mathscr{P}$-topology.

3. CLASSICAL EXAMPLES OF TOPOLOGICAL VECTOR SPACES

We shall now study several classical types of topological vector spaces; they will all be spaces with $\mathscr{P}$-topologies. In other words, in each case it will be a question of defining a family of seminorms adapted to the phenomena which it is desired to bring out.

3.1. Let us recall the vector space $\mathscr{B}(X, \mathbf{K})$ of *bounded* mappings of an arbitrary set X into **K**, with the norm of uniform convergence:

$$\|f\| = \sup_{x \in X} |f(x)|.$$

When X is a compact topological space, the space $\mathscr{C}(X, \mathbf{K})$ of continuous functions is a particularly important subspace of $\mathscr{B}(X, \mathbf{K})$.

3.2. Let $\mathscr{C}^r([0, 1], \mathbf{K})$ be the space of mappings of $[0, 1]$ into **K** which have continuous derivatives up to order r inclusive (where $r \in \mathbf{N}$). We define on this space the topology of uniform convergence for each derivative of order $\leqslant r$, by defining the family of seminorms p_i according to

$$p_i(f) = \sup_{t \in [0,1]} |f^{(i)}(t)| \qquad (\text{where } i = 0, 1, \ldots, r).$$

By Proposition 2.16, this topology can also be defined by the single norm

$$p(f) = \sum_0^r p_i(f).$$

One can similarly define a topology on the space $\mathscr{C}^r([0, 1]^n, \mathbf{K})$ of mappings of the interval $[0, 1]^n$ into **K** which have continuous partial derivatives up to order r inclusive, by the following seminorms:

$$p_i(f) = \sup |\, \mathrm{D}^i f(t)|$$

where $i = (i_1, i_2, \ldots, i_n)$, $i_j \geqslant 0$, $\Sigma_j\, i_j \leqslant r$, and

$$\mathrm{D}^i = \frac{\partial^{i_1 + i_2 + \cdots + i_n}}{\partial x_1^{i_1}\, \partial x_2^{i_2} \cdots \partial x_n^{i_n}}.$$

Note that for $r = 0$, the spaces $\mathscr{C}^r([0, 1]^n, \mathbf{K})$ and $\mathscr{C}([0, 1]^n, \mathbf{K})$ are identical.

3.3. We denote by $\mathscr{C}^\infty([0, 1]^n, \mathbf{K})$ the space of infinitely differentiable functions on $[0, 1]^n$ with values in **K**, with the seminorms p_i defined in 3.2.

By Proposition 2.16, the topology associated with this family can be defined by a metric; however, to verify the convergence of a filter base, it is much more simple and intuitive to apply the criterion of Proposition 2.10 than to use this metric.

3.4. Here is a family of spaces analogous to the preceding: Let K be a compact set in $\mathbf{R}^n$; we denote by $\mathscr{D}_K{}^r(\mathbf{R}^n, \mathbf{K})$, or more concisely by $\mathscr{D}_K{}^r$

when there is no chance for ambiguity, the space of mappings of $\mathbf{R}^n$ into $\mathbf{K}$ which have continuous partial derivatives up to order r inclusive, and which are zero off the compact set K.

As in Section 3.2, we give this space the topology associated with the seminorms

$$p_i(f) = \sup_{t \in \mathbf{R}^n} |\mathrm{D}^i f(t)|.$$

The space $\mathscr{D}^r([0, 1]^n, \mathbf{K})$ is similarly defined.

3.5. Let p be a real number $\geqslant 1$; for every infinite sequence $x = (x_i)$ of elements of $\mathbf{K}$, we define the positive, finite or infinite number $\| x \|$ by

$$\| x \|^p = \sum_i | x_i |^p.$$

From Minkowski's inequality (Chapter II, Section 20.7) we deduce that

$$\left(\sum_0^n | x_i + y_i |^p\right)^{1/p} \leqslant \left(\sum_0^n | x_i |^p\right)^{1/p} + \left(\sum_0^n | y_i |^p\right)^{1/p} \leqslant \| x \| + \| y \|,$$

whence

$$\| x + y \| \leqslant \| x \| + \| y \|.$$

It follows from this inequality and the obvious relation $\| \lambda x \| = | \lambda | \, \| x \|$ that the set of x such that $\| x \| < \infty$ is a vector space, and that $\| x \|$ is a seminorm on this space. Moreover $\| x \|$ vanishes only if $x_i = 0$ for all i; hence $\| x \|$ is a norm on this space.

This space (real or complex) with this norm is denoted by l^p. In particular, the space l^1 can be identified with the space of absolutely convergent series; as for l^2, we shall meet it again in the study of Hilbert spaces.

3.6. Here now is an important space which we will only be able to define in full generality when we are familiar with the theory of integration:

Let p again be a real number $\geqslant 1$; for every $f \in \mathscr{C}([0, 1], \mathbf{K})$ we define the positive real number $\| f \|$ by

$$\| f \|^p = \int_0^1 | f(t) |^p \, dt.$$

Let us associate, with every finite increasing sequence $\sigma = (t_1, t_2, \ldots, t_n)$

of points of $[0, 1]$, where $t_1 = 0$ and $t_n = 1$, the expression $\|f\|_\sigma$ defined by

$$\|f\|_\sigma{}^p = \sum_i (t_{i+1} - t_i)|f(t_i)|^p.$$

Minkowski's inequality shows that $\|f+g\|_\sigma \leqslant \|f\|_\sigma + \|g\|_\sigma$.

But when the modulus of σ (Chapter I, Section 24.6) tends to 0, $\|f\|_\sigma$ tends to $\|f\|$; we deduce from this that $\|f+g\| \leqslant \|f\| + \|g\|$.

This inequality, coupled with the fact that $\|\lambda f\| = |\lambda| \|f\|$ and that $\|f\|$ is zero only if $f = O$, shows that $\|f\|$ is a norm on $\mathscr{C}([0, 1], \mathbf{K})$; for $p = 1$ it is called the norm of convergence in the mean; for $p = 2$ it is called the norm of convergence in quadratic mean (or, of mean square convergence).

In the following two examples we deal with the topology of uniform convergence on every compact set.

3.7. Let X be a separated topological space; for every compact $K \subset X$, every $f \in \mathscr{C}(X, \mathbf{K})$ is bounded on K; therefore, if we put

$$p_K(f) = \sup_{x \in K} |f(x)|,$$

p_K is a seminorm on $\mathscr{C}(X, \mathbf{K})$. The $\mathscr{P}$-topology defined on $\mathscr{C}(X, \mathbf{K})$ by the family (p_K) is called the topology of uniform convergence on every compact set (or, more concisely, *compact uniform convergence*). For every $f \neq O$, there exists a compact set K such that $p_K(f) \neq 0$; hence this topology is separated.

By Proposition 2.10, if $(f_i)_{i \in I}$ is a family of elements of $\mathscr{C}(X, \mathbf{K})$ and if $\mathscr{B}$ is a filter base on I, to say that the f_i converge in this topology to f along $\mathscr{B}$ means that, for every compact $K \subset X$, the f_i converge uniformly to f on K.

It is often convenient to replace the family (p_K) by a subfamily which defines the same topology: If (K_j) denotes a family of compact sets in X which is absorbing in the sense that every compact set K is contained in some K_j, one can verify that the family (p_{K_j}) defines the same topology as does the family of all the p_K.

For every vector subspace E of $\mathscr{C}(X, \mathbf{K})$, the trace on E of the topology of $\mathscr{C}(X, \mathbf{K})$ is also called the topology of compact uniform convergence.

For example, let D be a domain (open connected set) in $\mathbf{C}^n$, and let $\mathscr{H}(D)$ be the vector space of functions which are holomorphic in D. The most useful topology on $\mathscr{H}(D)$ is the topology of uniform convergence on every compact set; from the fact that there exists a countable and absorbing family of compact sets in D (verify this), it follows that the topology on $\mathscr{H}(D)$ can be defined by a countable family of ecarts;

since moreover this topology is separated, it is metrizable (see Proposition 2.16).

3.8. Let A be an open set in $\mathbf{R}^n$, and let r be an integer $\geqslant 0$; we denote by $\mathscr{E}^r(\mathrm{A}, \mathbf{K})$ the vector subspace of $\mathscr{C}(\mathrm{A}, \mathbf{K})$ consisting of the functions which have continuous partial derivatives up to order r inclusive.

For every compact $\mathrm{K} \subset \mathrm{A}$ and every D^i (see Section 3.2) of order $|i| = i_1 + i_2 + \cdots + i_n \leqslant r$, we put

$$p_{\mathrm{K},i}(f) = \sup_{t \in \mathrm{K}} |\mathrm{D}^i f(t)|.$$

The topology defined on $\mathscr{E}^r(\mathrm{A}, \mathbf{K})$ by the family of seminorms $p_{\mathrm{K},i}$ is called the topology of compact uniform convergence for all the derivatives of order $\leqslant r$.

As in the preceding example, one can require that the compact sets K belong to a countable absorbing family; hence the topology of $\mathscr{E}^r(\mathrm{A}, \mathbf{K})$ is metrizable.

The space $\mathscr{E}^\infty(\mathrm{A}, \mathbf{K})$ is similarly defined; its topology is also metrizable.

3.9. We denote by $\mathscr{D}^r(\mathbf{R}^n, \mathbf{K})$ the subspace of $\mathscr{C}(\mathbf{R}^n, \mathbf{K})$ consisting of the functions which vanish off some compact set, and which have continuous partial derivatives up to order r inclusive.

The topology of this space is defined by the family of seminorms $p_{\varphi,i}$ given by

$$p_{\varphi,i}(f) = \sup_{t \in \mathbf{R}^n} |\varphi(t)\mathrm{D}^i f(t)|,$$

where D^i is as previously, and φ is any element of $\mathscr{C}(\mathbf{R}^n, \mathbf{K})$.

It can be shown that, in contrast with the preceding topologies, the topology thus defined is not metrizable.

One could similarly define a topology on $\mathscr{D}^\infty(\mathbf{R}^n, \mathbf{K})$, but in fact the topology on this space which is useful is defined by other seminorms.

Here now are two examples of weak topologies.

3.10. Once again let $\mathscr{F}(\mathrm{X}, \mathbf{K})$ be the space of mappings of a set X into $\mathbf{K}$. For every $a \in \mathrm{X}$, the function $f \to f(a)$ is a linear functional; the topology on $\mathscr{F}(\mathrm{X}, \mathbf{K})$ defined by the family of seminorms $f \to |f(a)|$ is called the *topology of pointwise convergence on* X. This is a weak topology; hence (see Example 2.11) to say that a family of functions f_i converges pointwise to a function f (along a filter base) is equivalent to saying that, for every $x \in \mathrm{X}$, the $f_i(x)$ converge to $f(x)$ in $\mathbf{K}$.

3.11. Let E be a topological vector space, and E′ its topological dual. The weak topology on E associated with the family of (continuous) linear functionals $l \in \mathrm{E}'$ is called the *weak topology of* E. This topology is separated whenever there exists, for every $x \in \mathrm{E}$, $x \neq \mathrm{O}$, an $l \in \mathrm{E}'$

such that $l(x) \neq 0$; in other words, whenever, in the sense of Definition 12.3 of Chapter II, the continuous linear functionals on E separate the points of E.

To say that (x_i) converges weakly to $x \in E$ is equivalent to saying that, for every $l \in E'$, the $l(x_i)$ converge to $l(x)$.

We shall meet this weak topology again, for example in the study of Hilbert spaces.

Similarly, the dual E′ of a TVS E has a very useful weak topology, namely, that associated with the family of linear functionals $\varphi_a : l \to l(a)$, where a is an arbitrary point of E; it is often denoted by $\sigma(E', E)$.

Less classical examples of TVS's will be found in Problems 5, 30, 32, 59, etc.

Use of these topological vector spaces

A well-chosen topology on a vector space E of functions, in addition to furnishing a convenient language, leads to the possibility of utilizing all the notions and results of the theory of TVS's.

It leads in addition to the definition of new mathematical entities consisting of the elements of the dual E′. Here are two important examples:

3.12. Let X be a compact topological space; the elements of the dual of $\mathscr{C}(X, \mathbf{R})$ taken with the uniform topology are called *real Radon measures*.

The best known of these measures is the Lebesgue measure on [0, 1] defined (for X = [0, 1]) by the continuous linear functional

$$f \to \int_0^1 f(t)\, dt.$$

We shall come back to the Radon measures in the study of integration.

3.13. The elements of the dual of $\mathscr{D}^r(\mathbf{R}^n, \mathbf{R})$ are called *real distributions of order r*. They are a special case of the distributions of L. Schwartz.

The distributions of order 0 are the Radon measures on $\mathbf{R}^n$.

II. NORMED SPACES

Normed spaces were introduced into analysis after Hilbert spaces and have been much studied, in particular by Banach, even before the development of a general theory of topological vector spaces.

Although their importance diminished after the discovery of topologies associated with a family of seminorms, they still constitute a powerful tool, and their study is relatively simple.

4. TOPOLOGY ASSOCIATED WITH A NORM; CONTINUOUS LINEAR MAPPINGS

4.1. Definition. A VECTOR SPACE E WITH A NORM IS CALLED A *NORMED* SPACE. A NORMED SPACE E IS CALLED A *BANACH SPACE* IF IT IS COMPLETE FOR THE METRIC ASSOCIATED WITH THE NORM.

For example, the space $\mathscr{C}([0, 1], \mathbf{K})$ with the norm of uniform convergence is a Banach space since it is complete (Chapter I, 22.7); on the other hand, the same space with the norm of mean convergence is not complete (argue as in 14.7 below).

Let us recall that the sum and the upper envelope of every finite family of norms is also a norm.

Recall also that the metric associated with a norm p on E is defined by

$$d(x, y) = p(x - y).$$

With this metric is associated a topology which is called the topology of the normed space E; by Proposition 2.9 this topology is compatible with the vector space structure of E. We are going to reprove this result while at the same time making it more precise.

4.2. Proposition. *Let* E *be a normed space over* $\mathbf{K}$.

1. *The mapping* $(\lambda, x) \to \lambda x$ of $\mathbf{K} \times$ E into E *is continuous.*

2. *The mapping* $(x, y) \to x + y$ *of* E $\times$ E *into* E *is of Lipschitz class with ratio* 2.

3. *The mapping* $x \to \| x \|$ *of* E *into* $\mathbf{R}$ *is of Lipschitz class with ratio* 1.

PROOF. 1. We have

$$\Delta(\lambda x) = (\lambda + \Delta\lambda)(x + \Delta x) - \lambda x = \Delta\lambda \cdot x + \lambda\, \Delta x + \Delta\lambda \cdot \Delta x\,,$$

from which

$$\| \Delta(\lambda x)\| \leqslant | \Delta\lambda | \cdot \| x \| + | \lambda | \cdot \| \Delta x \| + | \Delta\lambda | \cdot \| \Delta x \|.$$

Thus $\| \Delta(\lambda x) \|$ tends to 0 as $| \Delta\lambda |$ and $\| \Delta x \|$ tend to 0, which implies the continuity of the mapping $(\lambda, x) \to \lambda x$.

However, this mapping is not uniformly continuous when E has points $\neq$ O; indeed, let $a \neq$ O in E, and take $x = \lambda a$. The mapping

$(\lambda, \lambda) \to \lambda^2$ is not uniformly continuous, hence neither is the mapping $(\lambda, x) \to \lambda \cdot \lambda a = \lambda^2 a$.

2. The relations

$$\Delta(x + y) = (x + \Delta x) + (y + \Delta y) - (x + y) = \Delta x + \Delta y;$$
$$\| \Delta x + \Delta y \| \leqslant \| \Delta x \| + \| \Delta y \|$$

show that the mapping $(x, y) \to x + y$ is of Lipschitz class with ratio 2 (for the usual norms on $\mathrm{E} \times \mathrm{E}$).

3. The relation $\big| \| x + \Delta x \| - \| x \| \big| \leqslant \| \Delta x \|$ shows that the function $\| x \|$ is of Lipschitz class with ratio 1.

Continuous linear mappings

The fact that in every normed space the origin has a neighborhood base consisting of the homothetic images of a given ball makes possible a simple characterization of the continuous linear mappings of one normed space into another.

4.3. Proposition. *Let* E *and* F *be normed spaces, and f a linear mapping of* E *into* F. *The following three assertions are equivalent*:

1. *f is continuous.*
2. *f is bounded on every bounded subset of* E.
3. *There exists a constant $k > 0$ such that, for every $x \in \mathrm{E}$,*

$$\|f(x)\| \leqslant k \| x \|.$$

When one of these conditions is satisfied, f is of Lipschitz class.

Proof. We first recall that a subset X of a metric space is said to be bounded if its diameter is finite; in a normed space, this condition can be expressed more simply by the statement that X is contained in some ball with center O.

$1 \Rightarrow 2$. The continuity of f at O implies the existence of an open ball $\mathrm{B}(\mathrm{O}, \rho)$ in E whose image under f is contained in the unit ball of F. By the homogeneity of f, the image under f of every ball in E with center O is contained in a ball of F; the same is therefore true of every bounded subset X of E.

$2 \Rightarrow 3$. The unit sphere $\mathrm{S} = \{x : \| x \| = 1\}$ of E is bounded; therefore $f(\mathrm{S})$ is bounded, in other words, there exists a constant $k \geqslant 0$ such that

$$\|f(x)\| \leqslant k = k \| x \| \qquad \text{for every} \quad x \in \mathrm{S}.$$

But every point $y \in \mathrm{E}$ is of the form λx, where $x \in \mathrm{S}$ and $\lambda \geqslant 0$; therefore the relation $\lambda \| f(x) \| \leqslant k\lambda \| x \|$ can be written as

$$\| f(y) \| \leqslant k \| y \| \qquad \text{for every} \quad y \in \mathrm{E}.$$

$3 \Rightarrow 1$. Indeed, $\| f(x) \| \leqslant k \| x \|$ implies

$$\| f(u) - f(v) \| = \| f(u - v) \| \leqslant k \| u - v \|.$$

Thus f is of Lipschitz class with ratio k, and hence continuous.

Equivalent norms

We have seen in Chapter I, Section 16 that the same topology can be defined by inequivalent metrics d_1 and d_2, that is, such that $d_1(x, y)$ and $d_2(x, y)$ do not tend to 0 simultaneously. We shall see that this singular occurrence cannot take place for metrics associated with norms.

4.4. Definition. TWO NORMS ON A VECTOR SPACE E ARE SAID TO BE *EQUIVALENT* IF THE TOPOLOGIES ASSOCIATED WITH THESE NORMS ARE IDENTICAL.

This is evidently an equivalence relation; we denote it by $\sim$.

4.5. Proposition. *Let p_1 and p_2 be norms on a vector space* E.

$$(p_1 \sim p_2) \Leftrightarrow (\exists k_1, k_2 \in \mathbf{R}_+ \quad \text{such that} \quad p_2 \leqslant k_1 p_1 \quad \text{and} \quad p_1 \leqslant k_2 p_2).$$

PROOF. This is a consequence of Proposition 4.3. Indeed, to say that $p_1 \sim p_2$ is the same as saying that the linear mapping $x \to x$ of E taken with the norm p_1 into E taken with the norm p_2 is continuous, and similarly with p_1, p_2 interchanged; that is, there exist two numbers $k_1, k_2 \geqslant 0$ such that

$$p_2(x) \leqslant k_1 p_1(x) \quad \text{and} \quad p_1(x) \leqslant k_2 p_2(x) \qquad \text{for every} \quad x \in \mathrm{E}.$$

These numbers k_1, k_2 are clearly > 0.

We shall show in Section 7 that all the norms on a finite-dimensional vector space are equivalent. On the other hand, here is an example of an infinite-dimensional vector space on which there exist two inequivalent norms.

The space in question is $\mathscr{C}([0, 1], \mathbf{K})$ with the norm p_1 of uniform convergence, and the norm p_2 defined by

$$p_2(f) = \int_0^1 |f(t)| \, dt.$$

Clearly $p_2 \leqslant p_1$; on the other hand, if we denote by f_n the monomial $t \to t^n$, then

$$p_1(f_n) = 1 \qquad \text{whereas} \qquad p_2(f_n) = (n+1)^{-1}.$$

Therefore there exists no constant k such that $p_1 \leqslant kp_2$.

Choice of a norm on $\mathscr{L}$(E, F)

Let us try to find a norm on the vector space $\mathscr{L}$(E, F) of continuous linear mappings of E into F which characterizes the size of an element f of $\mathscr{L}$(E, F). It is out of the question to use the norm of uniform convergence on all of E, since, apart from the mapping O, no $f \in \mathscr{L}$(E, F) is bounded on E. On the other hand, we have characterized the continuous linear mappings f by the property of being bounded on the unit ball B of E; moreover, two elements of $\mathscr{L}$(E, F) which coincide on this ball are identical. We are thus led to use the norm of uniform convergence on the ball B, in other words, to put for every $f \in \mathscr{L}$(E, F),

$$\|f\| = \sup_{\|x\| \leqslant 1} \|f(x)\|.$$

This is evidently a seminorm (see 2.3, Nos. 5 and 6); and since $f = 0$ if $f(x) = 0$ for every $x \in$ B, it is a norm.

The proof of the implication ($2 \Rightarrow 3$) in Proposition 4.3 shows that $\|f\|$ is the smallest constant $k \geqslant 0$ such that $\|f(x)\| \leqslant k\|x\|$ for every $x \in$ E; in particular, one can therefore write $\|f(x)\| \leqslant \|f\|\,\|x\|$. To sum up:

4.6. Proposition. *The function* $f \to \|f\|$ *is a norm on* $\mathscr{L}$(E, F). *For every* $f \in \mathscr{L}$(E, F), $\|f\|$ *is the smallest number* $k \geqslant 0$ *such that* $\|f(x)\| \leqslant k\|x\|$ *for every* $x \in$ E.

Z If the norms of E and F are replaced by equivalent norms, it is immediate that $\|f\|$ is replaced by an equivalent norm. Thus the topology of $\mathscr{L}$(E, F) depends only on the topology of E and of F.

When the norms of E and F are specified, the norm which we have defined on $\mathscr{L}$(E, F) is the usual norm. It is especially convenient by virtue of the simplicity of its definition and by virtue also of the following property:

Let E, F, G be normed spaces, and let $f \in \mathscr{L}$(E, F), $g \in \mathscr{L}$(F, G); the relation

$$\|g(f(x))\| \leqslant \|g\|\,\|f(x)\| \leqslant \|g\|\,\|f\|\,\|x\|$$

shows that

$$\|g \circ f\| \leqslant \|g\|\,\|f\|.$$

Thus the norm chosen is well suited to the composition of linear mappings.

Nevertheless it might be convenient in certain cases to replace this norm by other equivalent norms. For example, let us associate, with every linear mapping f of $\mathbf{K}^n$ into itself defined by a matrix (a_{ij}) for the canonical basis in $\mathbf{K}^n$, the sum

$$\|f\| = \sum_{i,j} |a_{ij}|.$$

This is a norm which is sometimes used on $\mathscr{L}(\mathbf{K}^n)$; incidentally, it also satisfies the inequality $\|g \circ f\| \leqslant \|g\| \|f\|$.

4.7. Proposition. *When* F *is complete, so is* $\mathscr{L}(\mathrm{E}, \mathrm{F})$.

PROOF. Let B be the closed unit ball of E. We have defined the norm of an element f of $\mathscr{L}(\mathrm{E}, \mathrm{F})$ as the norm (for uniform convergence) of the trace of f on B.

But the trace of f on B is a bounded affine mapping g of B into F such that $g(\mathrm{O}) = \mathrm{O}$, and conversely one can verify that every bounded affine mapping g of B into F such that $g(\mathrm{O}) = \mathrm{O}$ can be extended in a unique way to a linear mapping of E into F which, being bounded on B, is continuous.

In other words, if $\mathscr{A}(\mathrm{B}, \mathrm{F})$ denotes the vector space of bounded affine mappings g of B into F such that $g(\mathrm{O}) = \mathrm{O}$, the mapping $f \rightarrow$ (trace of f on B) is an isomorphism of $\mathscr{L}(\mathrm{E}, \mathrm{F})$ onto $\mathscr{A}(\mathrm{B}, \mathrm{F})$ which preserves the norm.

Since F is complete, so is $\mathscr{C}(\mathrm{B}, \mathrm{F})$ (see Chapter I, 22.7). Since $\mathscr{A}(\mathrm{B}, \mathrm{F})$ is closed in $\mathscr{C}(\mathrm{B}, \mathrm{F})$ (as every uniform limit of bounded affine mappings is affine and bounded), $\mathscr{A}(\mathrm{B}, \mathrm{F})$ is complete; hence $\mathscr{L}(\mathrm{E}, \mathrm{F})$ is complete.

In particular, since $\mathbf{K}$ is complete, we can state:

4.8. Corollary. *The dual* E′ *of every normed space is complete.*

Extension of a continuous linear mapping

We proved in Chapter I (Theorem 20.14) that a uniformly continuous mapping of an everywhere dense subset X of a metric space E into a *complete* metric space F can be extended in a unique way to a continuous mapping of E into F.

This general result leads to the following assertion:

4.9. Proposition. *Let* X *be an everywhere dense vector subspace of a normed space* E, *and let* f *be a continuous linear mapping of* X *into a* **complete** *normed space* F.

Then there exists a unique continuous mapping g of E *into* F *whose restriction to* X *is f; this mapping g is linear, and* $\|g\| = \|f\|$.

PROOF. The inequality $\|f(u) - f(v)\| \leqslant \|f\| \|u - v\|$ shows that f is of Lipschitz class with ratio $\|f\|$; hence by Theorem 20.14 of Chapter I, f has a unique continuous extension g to E, and g is of Lipschitz class with ratio $\|f\|$.

Let us verify that g is linear. Let $x, y \in$ E; they are the respective limits of two sequences (x_n) and (y_n) of points of X. For every $n \in \mathbf{N}$ and all scalars λ, μ we have

$$g(\lambda x_n + \mu y_n) = \lambda g(x_n) + \mu g(y_n).$$

But addition and multiplication by scalars are continuous in E and in F, and g is moreover continuous; thus each term in this relation tends to a limit as $n \to \infty$, and we have

$$g(\lambda x + \mu y) = \lambda g(x) + \mu g(y).$$

In other words, g is linear.

The fact that g is of Lipschitz class with ratio $\|f\|$ allows us to write

$$\|g(x)\| \leqslant \|f\| \|x\|.$$

In other words, $\|g\| \leqslant \|f\|$; but on the other hand, g is an extension of f. Hence $\|f\| \leqslant \|g\|$, which gives the desired equality.

4.10. Corollary. *Every continuous linear mapping f of a vector subspace* X *of a normed space* E *into a complete normed space* F *can be extended in a unique way to a continuous linear mapping* of $\bar{X}$ *into* F.

Indeed, by Proposition 1.5, $\bar{X}$ is a vector subspace of E; since X is everywhere dense on $\bar{X}$, one can apply Proposition 4.9.

EXAMPLE. Since $\mathbf{K}$ is complete, assertions 4.9 and 4.10 apply to the extension of continuous linear functionals.

4.11. REMARK. The following example will show why it is essential that F be complete.

Let E be an infinite-dimensional complete normed space, and let X = F be a vector subspace of E which is everywhere dense on, but distinct from E (for example E = $\mathscr{C}([0, 1], \mathbf{K})$ and X is the subspace consisting of all the polynomials). We take for f the identity mapping of X into F. One can verify that f has no continuous extension to all of E.

4.12. *Examples of continuous linear mappings*

1. Let $\varphi \in \mathscr{C}([0, 1], \mathbf{K})$; the linear mapping $f \to \varphi f$ of $\mathscr{C}([0, 1], \mathbf{K})$ into itself is continuous and of norm $\| \varphi \|$. In order that it be an isomorphism of $\mathscr{C}([0, 1], \mathbf{K})$ onto itself, it is necessary and sufficient that φ not vanish at any point of $[0, 1]$.

2. For every $f \in \mathscr{C}([0, 1], \mathbf{K})$, let $\mathrm{P}f$ denote the primitive of f which vanishes at the point 0. The linear mapping $f \to \mathrm{P}f$ of $\mathscr{C}([0, 1], \mathbf{K})$ into itself is injective and has norm 1.

3. Let A and B be compact spaces, and let φ be a continuous mapping of A into B. The mapping $f \to f \circ \varphi$ of $\mathscr{C}(\mathrm{B}, \mathbf{K})$ into $\mathscr{C}(\mathrm{A}, \mathbf{K})$ is linear and has norm 1.

4. Let (k_n) be a sequence of elements of $\mathbf{K}$ such that $| k_n | \leqslant k < \infty$; the mapping $(x_n) \to (k_n x_n)$ of l^p into itself is linear and has norm $\leqslant k$.

5. The mapping $(x_n) \to (x_n')$, where $x_n' = x_{n+1}$, of l^p into itself is linear and has norm 1.

Here, now, are two examples of linear mappings which are not continuous.

6. Let E be the normed subspace of $\mathscr{C}([0, 1], \mathbf{K})$ (taken with its usual norm) consisting of the functions f which have a continuous derivative f'.

The linear mapping $f \to f'$ of E into $\mathscr{C}([0, 1], \mathbf{K})$ is not continuous since, for example, if we put $f_n(x) = n^{-1} \sin nx$, the sequence (f_n) converges to O whereas the sequence (f_n') does not converge to O.

7. Let p_1 be the norm of mean convergence on $\mathscr{C}([0, 1], \mathbf{K})$ and let p_2 be its usual norm. The identity mapping of this space taken with the norm p_1 into the same space taken with the norm p_2 is not continuous.

Z One observes that in Examples 6 and 7 of discontinuous linear mappings, the spaces on which they are defined are incomplete. This is not accidental; one can in fact show (see Problem 62) that when a linear mapping of a Banach space into a normed space possesses a certain degree of regularity, it is necessarily continuous. To be sure, one can by using the axiom of choice define discontinuous linear functionals on every infinite-dimensional Banach space, but this involves the construction of a family of functionals rather than the effective construction of one functional.

In short, one can expect that every linear mapping of a Banach space into a normed space, constructed by the procedures currently used in Analysis, will be continuous.

5. STABILITY OF ISOMORPHISMS

Let us first give a precise statement of what we mean by an isomorphism.

5.1. Definition. LET E AND F BE NORMED SPACES, AND LET f BE A LINEAR MAPPING OF E INTO F. THEN f IS CALLED AN *ISOMORPHISM* IF IT IS INJECTIVE, AND IF BOTH f AND THE INVERSE MAPPING f^{-1} OF $f(\mathrm{E})$ ONTO E ARE CONTINUOUS.

Thus, we do not require that $f(\mathrm{E}) = \mathrm{F}$.

5.2. Proposition. *To say that a linear mapping f of* E *into* F *is an isomorphism is equivalent to saying that there exist constants k_1, $k_2 > 0$ such that*

$$k_1\| x \| \leqslant \|f(x)\| \leqslant k_2\| x \|$$

for every $x \in \mathrm{E}$.

PROOF. Suppose f is an isomorphism; the continuity of f implies the existence of k_2; that of f^{-1} implies the existence of k_1 (the best constant k_1 being $\| f^{-1} \|^{-1}$).

Conversely, if $k_1 \| x \| \leqslant \| f(x) \|$, then f is injective and f^{-1} is continuous; if moreover $\| f(x) \| \leqslant k_2 \| x \|$, then f is continuous.

EXAMPLE. One can verify in Example 2 of 4.12 that the mapping $f \to \mathrm{P}f$ is continuous but is not an isomorphism.

5.3. Corollary. *The subset* I *of $\mathscr{L}(\mathrm{E}, \mathrm{F})$ consisting of the isomorphisms is open.*

PROOF. Suppose f is an isomorphism, and therefore satisfies a relation of the kind in Proposition 5.2; then for every $g \in \mathscr{L}(\mathrm{E}, \mathrm{F})$ such that $\| g \| < k_1$ we have

$$(k_1 - \| g \|)\| x \| \leqslant \|f(x) + g(x)\| \leqslant (k_1 + k_2)\| x \|.$$

Since $k_1 - \| g \| > 0$, $f + g$ is an isomorphism.

Corollary 5.3 exhibits a certain stability of isomorphisms, since it shows that a linear mapping f which is an isomorphism remains an isomorphism after the addition of a "small" continuous linear mapping. This result can be generalized a bit as follows:

5.4. Proposition. *Let* A *be an arbitrary metric space, let f be an injection of* A *into a normed space* F *which multiplies distances by at least a factor* $\mathrm{K} > 0$ (*in other words, f^{-1} is of Lipschitz class with ratio* K^{-1}), *and let*

g be another mapping of A *into* F *which is of Lipschitz class with ratio* $k < \mathrm{K}$.

Then the mapping $f + g$ of A *into* F *is injective and multiplies distances by at least a factor* $(\mathrm{K} - k)$.

This is an immediate consequence of the relation

$$\|(f+g)(x) - (f+g)(y)\| = \|(f(x) + f(y)) + (g(x) - g(y))\|$$
$$\geqslant (\mathrm{K} - k)\, d(x, y).$$

Let us add that if f is of Lipschitz class, so is $f + g$.

The results which follow will assume that the space F is complete, and will lead later to important applications.

We remark, to begin with, that if f denotes the identity mapping of a complete normed space F into itself, and if g denotes a linear mapping of F into F of norm < 1, not only is $f + g$, by Proposition 5.2, an isomorphism, it also carries F onto F. Indeed, for every $a \in \mathrm{F}$ the equation $f(x) + g(x) = a$ can also be written as $x = a - g(x)$, and since the mapping $x \to a - g(x)$ is contractive, this equation has a solution (by Theorem 21.1, Chapter I).

What we are now going to do is replace f by an isomorphism of a complete space E onto another such space F, and replace g by a mapping of Lipschitz class with sufficiently small ratio, and then "localize" these mappings f and g.

5.5. Lemma. *Let ω be an open set in a Banach space* F, *and let g' be a mapping of ω into* F *which is of Lipschitz class with ratio $k' < 1$. Then the image of ω under the mapping $\gamma : y \to y + g'(y)$ is open in* F.

More precisely, the image under γ of every closed ball $\mathrm{B}(b, \rho) \subset \omega$ *contains the closed ball* $\mathrm{B}(\gamma(b), (1 - k')\rho)$.

Proof. It clearly suffices to prove the second part of the assertion. To simplify the notation, we assume by a translation of ω (respectively, $\gamma(\omega)$) that $b = \mathrm{O}$ (respectively, $\gamma(b) = \mathrm{O}$).

We want to show that for every $c \in \mathrm{F}$ such that $\|c\| \leqslant (1 - k')\rho$, the equation $x + g'(x) = c$ has a solution such that $\|x\| \leqslant \rho$. But this equation can be written as $x = c - g'(x)$; the mapping $x \to c - g'(x)$ is of Lipschitz class with ratio $k' < 1$ and maps $\mathrm{B}(\mathrm{O}, \rho)$ into itself, since if $\|x\| \leqslant \rho$, then

$$\|c - g'(x)\| \leqslant \|c\| + \|g'(x)\| \leqslant (1 - k')\rho + k'\rho = \rho.$$

Since $\mathrm{B}(\mathrm{O}, \rho)$ is a complete metric space, Theorem 21.1 of Chapter I (the idea of successive approximations) shows that the equation has a solution.

5.6. Lemma. *Let* A *be a metric space*; *let* f *be an injection of* A *into a Banach space* F *which multiplies distances by at least a number* $\mathrm{K} > 0$, *and such that* $f(\mathrm{A})$ *is open in* F, *and let* g *be another mapping of* A *into* F *which is of Lipschitz class with ratio* $k < \mathrm{K}$. *Then the image of* A *under* $f + g$ *is an open set in* F.

Proof. We put $g' = g \circ f^{-1}$; then g' is a mapping of the open set $\omega = f(\mathrm{A})$ in F into F, and is of Lipschitz class with ratio $k' = k/K$, which ratio is by hypothesis < 1.

If we denote by f' the identity mapping in F, then

$$(f+g)(\mathrm{A}) = (f+g)(f^{-1}(\omega)) = (f'+g')(\omega).$$

Lemma 5.5 now shows that $(f' + g')(\omega)$ is an open set in F, which is the desired result.

5.7. Theorem. *Let* E *and* F *be Banach spaces, and let* f *be a bijective isomorphism of* E *onto* F. *Let* A *be an open set in* E, *and* g *a mapping of* A *into* F *which is of Lipschitz class with ratio* k.

When $k < \|f^{-1}\|^{-1}$, *then* $f + g$ *is injective, both it and its inverse are of Lipschitz class, and* $(f + g)(A)$ *is open in* F.

This is a special case of Lemma 5.6; indeed, since f is an isomorphism of E onto F, $f(\mathrm{A})$ is open in F, and f multiplies distances by at least $K = \|f^{-1}\|^{-1}$. Proposition 5.4 shows that $f + g$, which is clearly of Lipschitz class, multiplies distances by at least $\mathrm{K} - k$; therefore its inverse is of Lipschitz class with ratio $(\mathrm{K} - k)^{-1}$.

This theorem will be extremely valuable to us in the study of continuously differentiable functions, for the proof of theorems concerning implicit functions.

Z A simple example will show us that the completeness of F is essential in the assertions 5.5–5.7.

Let F be the vector subspace of $\mathscr{C}([0, 1], \mathbf{R})$ consisting of the restrictions to $[0, 1]$ of real polynomials; let g be the mapping of F into itself which associates with every polynomial $x \in \mathrm{F}$ the polynomial $t \to x(t^2)$. Then $\|g\| = 1$; therefore for every k such that $0 < k < 1$, kg is of Lipschitz class with ratio $k < 1$. The image of F under the mapping $x \to x + kg(x)$ is a vector subspace of F which is distinct from F since it contains only polynomials of even degree; hence it contains no open subset of F.

6. PRODUCT OF NORMED SPACES; CONTINUOUS MULTILINEAR MAPPINGS

Let $(E_i)_{i \in I}$ be a *finite* family of normed spaces over the same field **K**, whose norms we shall uniformly denote by $\| x \|$, and let E be the product vector space of the E_i.

We have seen in 2.14 that the topology on E associated with the family of seminorms $\| x_i \|$ is identical with the product of the topologies of the E_i.

The seminorms $\sup \| x_i \|$, $\Sigma \| x_i \|$, and $(\Sigma \| x_i \|^2)^{1/2}$ are equivalent and each of them defines the same topology on E as does the family of $\| x_i \|$ (see 2.16); in the present context they are norms on E since the relations $\| x_i \| = 0$ together imply $x = O$.

Continuity of a multilinear mapping

Let us recall that if (E_i) denotes a finite family of vector spaces, and f is a mapping of the product ΠE_i into another vector space F, then f is said to be *multilinear* if it is separately linear with respect to each of its variables. When the E_i and F are normed, the product of the E_i is also normed and one can seek to extend Proposition 4.3, characterizing the continuous linear mappings, to multilinear mappings.

6.1. Proposition. *Let* $E_1, E_2, \ldots, E_n$ *and* F *be normed spaces and let* f *be a multilinear mapping of the product* E *of the* E_i *into* F; *then the following three assertions are equivalent*:

1. f *is continuous.*
2. f *is bounded on every bounded subset of* E.
3. *There exists a constant* $k \geqslant 0$ *such that, for every* $x = (x_i) \in E$,

$$\|f(x_1, x_2, \ldots, x_n)\| \leqslant k\| x_1 \| \cdot \| x_2 \| \cdots \| x_n \|.$$

Proof. For the norm on E we shall take $\| x \| = \sup \| x_i \|$.

$1 \Rightarrow 2$. If f is continuous, its continuity at O implies the existence of a ball with center O in E on which $\| f(x) \| \leqslant 1$; by a homothetic transformation and from the relation $f(\lambda x) = \lambda^n f(x)$, we deduce that f is bounded on every ball with center O, hence also on every bounded subset of E.

$2 \Rightarrow 3$. If f is bounded on every bounded subset of E, it is in particular bounded on the ball $\| x \| \leqslant 1$; we then put

$$\| f \| = \sup_{\|x\| \leqslant 1} \|f(x)\|.$$

For every $x = (x_i)$ such that $x_i \neq O$ for every i, the point $(\| x_i \|^{-1} x_i)$ belongs to the unit ball of E; therefore

$$\left\| f\left(\frac{x_1}{\| x_1 \|}, \frac{x_2}{\| x_2 \|}, \ldots, \frac{x_n}{\| x_n \|}\right) \right\| \leqslant \| f \|,$$

from which $\| f(x_1, x_2, \ldots, x_n) \| \leqslant \| f \| \prod_i \| x_i \|$. When one of the x_i is O, this inequality still holds since both sides are then zero.

$3 \Rightarrow 1$. To simplify the notation, we shall carry out the proof only for $n = 2$. Suppose therefore that $\| f(x_1, x_2) \| \leqslant k \| x_1 \| \| x_2 \|$.

This relation clearly implies continuity at the point O; to prove continuity at a point (a_1, a_2), we put

$$x_1 = a_1 + u_1, \qquad x_2 = a_2 + u_2 .$$

Then

$$f(a_1 + u_1, a_2 + u_2) - f(a_1, a_2) = f(a_1, u_2) + f(u_1, a_2) + f(u_1, u_2),$$

from which, if $\| u_1 \| \leqslant \epsilon$ and $\| u_2 \| \leqslant \epsilon$,

$$\| f(x_1, x_2) - f(a_1, a_2) \| \leqslant k(\| a_1 \| \epsilon + \| a_2 \| \epsilon + \epsilon^2).$$

The term on the right tends to 0 with ϵ, which gives the desired continuity.

6.2. *The normed space $\mathscr{L}(E_1, \ldots, E_n; F)$*

The continuous multilinear mappings of the product E of the E_i into F clearly form a vector space; we shall denote it by $\mathscr{L}(E_1, \ldots, E_n; F)$.

One can verify that

$$\| f \| = \sup_{\| x \| \leqslant 1} \| f(x) \| \qquad (\text{where } \| x \| = \sup \| x_i \|)$$

is a norm on this vector space, and that $\| f \|$ is the smallest number $k \geqslant 0$ satisfying the inequality of Proposition 6.1.

A proof modeled on that of Proposition 4.7 shows that when F is complete, the space $\mathscr{L}(E_1, \ldots, E_n; F)$ is also complete.

EXAMPLE. Let E and E′ denote a normed space and its topological dual, respectively. The mapping $(x, l) \to l(x)$ of $E \times E'$ into **K** is a bilinear form on $E \times E'$; the inequality $| l(x) | \leqslant \| l \| \| x \|$ shows that this bilinear form is continuous and of norm $\leqslant 1$.

More generally, let E and F be normed spaces; the mapping $(x, l) \to l(x)$ of $E \times \mathscr{L}(E, F)$ into F is bilinear and of norm $\leqslant 1$.

7. FINITE-DIMENSIONAL NORMED SPACES

We are already acquainted with a normed space of dimension n over $\mathbf{K}$, namely $\mathbf{K}^n$ with the equivalent norms $\sup |x_i|$, $(\Sigma |x_i|^2)^{1/2}$, $\Sigma |x_i|$. We are going to see that this is the only one, up to isomorphism.

7.1. Lemma. *Every norm p on $\mathbf{K}^n$ is continuous.*

Proof. Let $\{a_1, a_2, \ldots, a_n\}$ be the canonical basis of $\mathbf{K}^n$. For every point $x = (x_i)$ of $\mathbf{K}^n$ we have

$$p(x) = p\left(\sum x_i a_i\right) \leqslant \sum |x_i| p(a_i).$$

Therefore p is continuous at the point O; finally, the relation $| p(a + u) - p(a) | \leqslant p(u)$ shows that p is continuous at every point a.

7.2. Proposition. *All norms on a finite-dimensional vector space are equivalent.*

Proof. Since every n-dimensional vector space over $\mathbf{K}$ is vectorially isomorphic to $\mathbf{K}^n$, it is sufficient to consider $\mathbf{K}^n$.

Let p_1 and p_2 be arbitrary norms on $\mathbf{K}^n$; by Lemma 7.1, they are continuous on $\mathbf{K}^n$, and since they do not vanish anywhere on the set $S = \{x : \Sigma |x_i| = 1\}$, the quotients $p_1 p_2^{-1}$ and $p_2 p_1^{-1}$ are defined and continuous on S. But the set S is bounded and closed in $\mathbf{K}^n$, hence compact; therefore these two quotients are bounded on S, and consequently on all of $\mathbf{K}^n$ by homogeneity.

7.3. Corollary. *For every n-dimensional normed space* E *and every basis $(b_1, b_2, \ldots, b_n)$ of* E, *the vectorial isomorphism $(x_i) \to \Sigma x_i b_i$ of $\mathbf{K}^n$ onto* E *is bicontinuous.*

One can say, more concisely, that every n-dimensional normed space is isomorphic to $\mathbf{K}^n$.

7.4. Corollary. *Every finite-dimensional subspace of a normed space* E *is complete, and hence closed in* E.

Indeed, since $\mathbf{K}$ is complete, so is $\mathbf{K}^n$ and therefore every finite-dimensional normed space, by Corollary 7.3.

7.5. Corollary. *Every multilinear mapping of a product of finite-dimensional normed spaces into an arbitrary topological vector space* F *is continuous.*

PROOF. To simplify the notation, we restrict ourselves to the case of a bilinear mapping f of a product $X \times Y$ into F. Corollary 7.3 shows that for X and Y we can take $\mathbf{K}^p$ and $\mathbf{K}^q$; if (a_i) and (b_j) denote the canonical bases of $\mathbf{K}^p$ and $\mathbf{K}^q$, respectively, then

$$f(x,y) = f\left(\sum x_i a_i, \sum y_j b_j\right) = \sum x_i y_j f(a_i, b_j).$$

For every i, j, the mapping $(x, y) \to x_i y_j$ is continuous; by the axioms of a TVS, the same is therefore true of f.

Topological characterization of finite-dimensional normed spaces

Since the field $\mathbf{K}$ is locally compact, the same is true of every space $\mathbf{K}^n$, hence also of every finite-dimensional normed space.

We are going to see that this statement has a converse, which is extremely important in the study of integral equations.

7.6. Theorem (of Frederic Riesz). *Every locally compact normed space is finite-dimensional.*

PROOF. Suppose the normed space E is locally compact. Then the origin O has a compact neighborhood V, and since the closed balls with center O and nonzero radius form a neighborhood base of O, one of these is contained in V, and is thus compact. By a homothetic transformation the closed unit ball B is thus compact.

The theorem will be proved if we show that the closed unit ball B of an infinite-dimensional normed space E is not compact. For this it suffices to show that B contains an infinite sequence (a_n) of points whose mutual distances are $> \frac{1}{2}$ (see Proposition 18.2, Chapter I).

Suppose the points a_i are defined for every $i \leqslant n$, and let F_n be the vector subspace of E spanned by these points. Since F_n is finite-dimensional, it is closed (Corollary 7.4) and there exists a point a of E such that $a \notin F_n$; therefore $d(a, F_n) \neq 0$, and there exists a point $b \in F_n$ such that $d(a, b) < 2\, d(a, F_n)$. This inequality is preserved by the translation $x \to (x - b)$ which leaves F_n invariant; in other words,

$$\| a - b \| = d(a - b), O) < 2\, d((a - b, F_n).$$

We put $a_{n+1} = \| a - b \|^{-1}(a - b)$; by a homothetic transformation with scale $\| a - b \|^{-1}$, this inequality becomes

$$1 = d(a_{n+1}, O) < 2\, d(a_{n+1}, F_n).$$

Thus $a_{n+1} \in \mathrm{B}$, and the distance from a_{n+1} to F_n, and hence also to the a_i with index $i \leqslant n$, is $> \frac{1}{2}$. The desired sequence (a_n) can therefore be constructed recursively.

Example. The normed space $\mathscr{C}([0, 1], \mathbf{K})$ is infinite-dimensional; therefore its closed unit ball is not compact.

An immediate verification of this fact consists in observing that the distance between any two monomials of the form t^{2^n} is $\geqslant \frac{1}{4}$.

III. SUMMABLE FAMILIES; SERIES; INFINITE PRODUCTS; NORMED ALGEBRAS

When one seeks to define the sum of an infinite family of real numbers, one is led to use a notion of limit, therefore also the topology of $\mathbf{R}$.

But once this topology is specified, the definition of the sum is still not unique. The first definition used, and which for a long time was the only one, assumed that the numbers to be added were arranged in a sequence a_0, a_1, ...; the convergence of this "series" is then defined in terms of the finite sums $s_n = a_0 + a_1 + \cdots + a_n$. When the sequence (s_n) converges to a number s, the series is said to converge and have sum s.

The choice of this definition was unavoidable when it was a question of the sums of infinite families having a natural order relation, such as the families (k^n), (n^{-2}), $((-1)^n n^{-1})$; it was then maintained by habit even when people began to consider arbitrary sequences (a_n), or "multiple series" for which the set of indices did not have a natural order.

But hardly any role was played in this definition by the algebraic properties of addition such as commutativity and associativity; thus, when these algebraic properties were better elucidated, and more general topological vector spaces began to be used, the need for a definition of the summability of a family $(a_i)_{i \in \mathrm{I}}$ which was both valid in a more general setting and independent of an order on the index set I made itself felt.

We are going to study such a definition here.

8. SUMMABLE FAMILIES OF REAL NUMBERS

Let $(a_i)_{i \in \mathrm{I}}$ be a finite or infinite family of real numbers; we denote by $\mathscr{F}$ the set, ordered by inclusion, of finite subsets of I, and for every $\mathrm{J} \in \mathscr{F}$ we denote by A_J the finite sum $\sum_{i \in \mathrm{J}} a_i$.

We shall say, in language which is still quite vague, that the given family is summable and has sum A, if A_J tends to A when J becomes larger and larger.

The first way of making this vague idea more precise consists in observing that since the ordered set $\mathscr{F}$ is an increasing directed set, the subsets of $\mathscr{F}$ of the form $\{J : J \supset J_0\}$ form a filter base $\mathscr{B}$ on $\mathscr{F}$. We then say that the family (a_i) has sum A if $A = \lim_{\mathscr{B}} A_J$.

But to avoid recourse to filter bases, we shall give a more direct equivalent definition of summability:

8.1. Definition. A FAMILY $(a_i)_{i \in I}$ OF REAL NUMBERS IS SAID TO BE *SUMMABLE* IF THERE EXISTS A REAL NUMBER A HAVING THE FOLLOWING PROPERTY:

FOR EVERY $\epsilon > 0$ THERE EXISTS A $J_0 \in \mathscr{F}$ SUCH THAT, FOR EVERY $J \in \mathscr{F}$ CONTAINING J_0,

$$| A - A_J | \leqslant \epsilon.$$

Let us show at once that if such a number A exists, it is unique. Indeed, let A and A′ be two such numbers; the relations

$$| A - A_J | \leqslant \epsilon \quad \text{for} \quad J_0 \subset J, \qquad \text{and} \qquad | A' - A_J | \leqslant \epsilon \quad \text{for} \quad J_0' \subset J$$

hold simultaneously if $J = J_0 \cup J_0'$; it follows that

$$| A - A' | \leqslant | A - A_J | + | A' - A_J | \leqslant 2\epsilon.$$

Since ϵ is arbitrary, $A = A'$.

Definition 8.1 can now be completed as follows:

THE NUMBER A WHOSE UNIQUENESS HAS JUST BEEN SHOWN IS CALLED THE *SUM* OF THE FAMILY $(a_i)_{i \in I}$, AND IS DENOTED BY $\sum_{i \in I} a_i$ OR $\sum_i a_i$ OR $\sum a_i$ WHEN NO CONFUSION OVER I IS POSSIBLE.

Families of positive numbers

Definition 8.1 will be usable only when we have a convenient criterion for summability; the study of families of positive numbers will give us such a criterion.

8.2. Proposition. *To say that a family $(a_i)_{i \in I}$ of numbers $\geqslant 0$ is summable is equivalent to saying that the set of finite sums A_K is bounded from above; we then have*

$$\sum_{i \in I} a_i = \sup_K A_K .$$

PROOF. 1. Suppose the family (a_i) is summable; then with the notation of Definition 8.1 we have

$$|A - A_J| \leqslant \epsilon, \qquad \text{from which} \qquad A_J \leqslant A + \epsilon \quad \text{for every} \quad J \supset J_0 .$$

Since the a_i are $\geqslant 0$, for every $K \in \mathscr{F}$ we have

$$A_K \leqslant A_{K \cup J_0} \leqslant A + \epsilon.$$

Therefore the set of numbers A_K is bounded from above by $A + \epsilon$, and hence by A since ϵ is arbitrary.

2. Conversely, suppose the set of A_K's is bounded from above, and let A be its supremum. By the definition of the supremum, for every $\epsilon > 0$ there exists a $J_0 \in \mathscr{F}$ such that

$$A - \epsilon \leqslant A_{J_0} \leqslant A.$$

For every $J \in \mathscr{F}$ such that $J_0 \subset J$ we therefore have

$$A - \epsilon \leqslant A_{J_0} \leqslant A_J \leqslant A, \qquad \text{hence} \qquad |A - A_J| \leqslant \epsilon.$$

This relation shows that the family (a_i) is summable and has sum A.

EXAMPLE. Let $(\alpha_i)_{i \in I}$ be a family of pairwise disjoint open subintervals of the interval [0, 1], and let a_i be the length of α_i. For every finite subset K of I we clearly have $A_K \leqslant 1$; therefore the family (a_i) is summable and has sum $\leqslant 1$.

8.3. Corollary. *Every subfamily of a summable family of numbers $\geqslant 0$ is summable.*

Indeed, if $I' \subset I$, the set of finite sums A_J such that $J \subset I$ contains the analogous set defined relative to I'; therefore if the first is bounded from above, so is the second.

8.4. Corollary (principle of comparison). *Let $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ be families of numbers $\geqslant 0$ with the same index set, and such that $a_i \leqslant b_i$ for every $i \in I$.*

If the family $(b_i)_{i \in I}$ is summable, so is the family $(a_i)_{i \in I}$, and the sums A and B satisfy the relation $A \leqslant B$.

PROOF. Let B be the sum of the family (b_i); for every $J \in \mathscr{F}$ we have

$$A_J \leqslant B_J \leqslant B.$$

Therefore Proposition 8.2 shows that the family (a_i) is summable with sum $A \leqslant B$.

Note that these conclusions extend to the case where there is a constant $k \geqslant 0$ such that $a_i \leqslant kb_i$ for every i; we then have $A \leqslant kB$.

EXAMPLES. This comparison principle is a powerful tool which enables one, starting from given summable families, to find others.

8.5. The most used basic families are those with index set **N** or **N***; they are thus sequences; we cite the most useful:

The geometric sequence (k^n), summable when $0 \leqslant k < 1$.

The sequence $(n^{-\alpha})$, summable when $\alpha > 1$.

A convenient procedure for constructing summable sequences (a_n) of numbers $\geqslant 0$ consists of starting with a decreasing numerical function f on **N** such that $\lim_{n\to\infty} f(n) = 0$, and putting

$$a_n = f(n) - f(n+1).$$

The relation $a_1 + a_2 + \cdots + a_n = f(1) - f(n+1) \leqslant f(1)$ shows that the family (a_n) is summable with sum $f(1)$.

For example, taking successively $f(n) = k^n$ and $f(n) = n^{-p}$ (where p is > 0), we obtain the summability of the sequences mentioned above.

The integration of decreasing functions defined on $[0, \infty)$ constitutes another procedure for constructing decreasing summable sequences; we shall examine this procedure in the study of integration under the heading "comparison of series and integrals."

8.6. Let us recall two classical sufficient conditions for the summability of positive sequences, which are deduced by comparison with a sequence (k^n):

(a) If $\limsup(a_{n+1}/a_n) < 1$, the sequence (a_n) is summable; indeed, there then exists a positive number $k < 1$ and an integer n_0 such that $a_{n+1}/a_n \leqslant k$ for $n \geqslant n_0$. We deduce from this that $a_{n_0+p} \leqslant a_{n_0}k^p$, which implies summability.

If $\limsup(a_{n+1}/a_n) \geqslant 1$, nothing can be concluded; for example, if $a_n = 2^{-n}$ or 3^{-n} according as n is even or odd, the limit superior above equals $+\infty$, yet the sequence (a_n) is summable.

(b) If $\limsup(a_n)^{1/n} < 1$, the sequence is summable, since for all n sufficiently large

$$a_n < k^n \qquad \text{where} \quad k < 1.$$

If this limit superior equals 1, nothing can be concluded (consider the case where $a_n = n^{-\alpha}$); if it is > 1, then $\limsup a_n = +\infty$; hence the sequence is not summable.

8.7. The families whose index sets are $\mathbf{N}^2$, $\mathbf{N}^3$, or more generally $\mathbf{N}^n$, or a rather simple subset of $\mathbf{N}^n$ are often called (improperly, by the way, since there exists no natural order relation on these index sets) *double series*, *triple series*, *or n-uple series*.

Here again integration theory will enable us to establish the summability of many "multiple series."

For example, the n-uple series with general term $(p_1 + p_2 + \cdots + p_n)^{-\alpha}$ (where $p_i \in \mathbf{N}^*$) is summable if $\alpha > n$, and not summable if $\alpha \leqslant n$.

The double series with general term $(a^m + b^n)^{-1}$ (where $a, b > 1$) is summable.

Limit of families of positive numbers

8.8. Proposition. *Let $(a_i(\lambda))_{i\in \mathrm{I}}$ be a summable family of positive numbers, depending on a parameter $\lambda \in \mathrm{L}$, and let $\mathscr{B}$ be a filter base on* L *such that, for every $i \in \mathrm{I}$, $a_i(\lambda)$ has a limit a_i along $\mathscr{B}$.*

If there exists a constant k such that $\sum_i a_i(\lambda) \leqslant k$ for every $\lambda \in \mathrm{L}$, then the family (a_i) is summable, and $\sum_i a_i \leqslant k$.

PROOF. For every finite $\mathrm{J} \subset \mathrm{I}$ we have

$$\sum_{i\in \mathrm{J}} a_i(\lambda) \leqslant k.$$

Holding J fixed and passing to the limit along $\mathscr{B}$ in this inequality, we obtain

$$\sum_{i\in \mathrm{J}} a_i \leqslant k, \qquad \text{from which} \qquad \sum_{i\in \mathrm{I}} a_i \leqslant k.$$

Here is an example showing that the sums $s_\lambda = \sum_i a_i(\lambda)$ need not converge (along $\mathscr{B}$) to $s = \sum_i a_i$.

We take $\mathrm{I} = \mathrm{L} = \mathbf{N}$ and put

$$a_i(n) = 0 \quad \text{if} \quad i \neq n; \qquad a_n(n) = 1.$$

One can verify that

$$a_i = \lim_{n\to\infty} a_i(n) = 0, \qquad s_n = 1 \qquad \text{and} \qquad s = 0;$$

hence

$$s \neq \lim_{n\to\infty} s_n .$$

A sufficient condition for the equality

$$s = \lim_{\mathscr{B}} s_\lambda$$

will be found in Problem 83.

Application to the space l_{I}^p

Let I be an arbitrary finite or infinite index set, and let p be a real number $\geqslant 1$.

As in Example 3.5, one can show that the subset of $\mathscr{F}(\mathrm{I}, \mathbf{K})$ consisting of the families $(x_i)_{i\in\mathrm{I}}$ of elements of $\mathbf{K}$ such that $\Sigma \mid x_i \mid^p < \infty$ is a vector subspace of $\mathscr{F}(\mathrm{I}, \mathbf{K})$, and that $\| x \| = (\Sigma \mid x_i \mid^p)^{1/p}$ is a norm on this subspace.

We denote this vector space (real or complex) with this norm by l_{I}^p; when $\mathrm{I} = \mathbf{N}$ we obtain the space l^p of Example 3.5.

8.9. Proposition. *For any* I, *the space l_{I}^p is complete.*

Proof. Let $(x(n))$ be a Cauchy sequence in l_{I}^p; for every $i \in \mathrm{I}$, the relation

$$| x_i(r) - x_i(s)| \leqslant \| x(r) - x(s)\|$$

shows that the sequence of numbers $x_i(n)$ is a Cauchy sequence in $\mathbf{K}$; let its limit be x_i.

Since the Cauchy sequence $(x(n))$ is bounded in l_{I}^p, there exists a positive number k such that

$$\sum_i | x_i(n)|^p \leqslant k \qquad \text{for every } n.$$

Then Proposition 8.8 shows that $\Sigma_i \mid x_i \mid^p \leqslant k$; hence the point $x = (x_i)$ of $\mathscr{F}(\mathrm{I}, \mathbf{K})$ belongs to l_{I}^p.

Finally, let us show that the point x is the limit in l_{I}^p of the sequence $(x(n))$. For every $\epsilon > 0$ there exists an integer $n(\epsilon)$ such that for all integers r, $s \geqslant n(\epsilon)$,

$$\sum_i | x_i(r) - x_i(s)|^p \leqslant \epsilon.$$

We hold r fixed and let s tend to ∞; Proposition 8.8 shows that

$$\sum | x_i(r) - x_i |^p \leqslant \epsilon.$$

In other words, $\| x(r) - x \|^p \leqslant \epsilon$ for every $r \geqslant n(\epsilon)$; therefore

$$x = \lim_{n \to \infty} x(n).$$

Families of real numbers

To study a family of real numbers, we are going to write it as the difference of two positive families.

8.10. Lemma. *Let $(a_i)_{i \in \mathrm{I}}$ and $(b_i)_{i \in \mathrm{I}}$ be summable families of real numbers with the same index set, with respective sums* A *and* B.

Then the family $(c_i)_{i \in \mathrm{I}}$, where $c_i = a_i + b_i$, is summable with sum $\mathrm{A} + \mathrm{B}$.

PROOF. Let $\epsilon > 0$; there exist corresponding J_0, $\mathrm{J}_0' \in \mathscr{F}$ such that

$$|\mathrm{A} - \mathrm{A_J}| \leqslant \epsilon \quad \text{for} \quad \mathrm{J} \supset \mathrm{J}_0 \qquad \text{and} \qquad |\mathrm{B} - \mathrm{B_J}| \leqslant \epsilon \quad \text{for} \quad \mathrm{J} \supset \mathrm{J}_0'.$$

These inequalities hold simultaneously if $\mathrm{J} \supset \mathrm{J}_0 \cup \mathrm{J}_0'$; hence

$$|\mathrm{A} + \mathrm{B} - (\mathrm{A_J} + \mathrm{B_J})| \leqslant |\mathrm{A} - \mathrm{A_J}| + |\mathrm{B} - \mathrm{B_J}| \leqslant 2\epsilon$$

or, putting $\mathrm{C} = \mathrm{A} + \mathrm{B}$,

$$|\mathrm{C} - \mathrm{C_J}| \leqslant 2\epsilon \qquad \text{for every J such that} \qquad \mathrm{J}_0 \cup \mathrm{J}_0' \subset \mathrm{J},$$

which shows that the family (c_i) is summable with sum C.

8.11. Proposition. *Let* I *be an index set. The collection of summable families of real numbers with index set* I *is a vector subspace of* $\mathscr{F}(\mathrm{I}, \mathbf{R})$, *and the mapping $(a_i) \to \Sigma\, a_i$ is a linear functional on this space.*

This is immediate, from Lemma 8.10, if we observe that when the family (a_i) is summable with sum A, the family (λa_i) is summable with sum λA.

Since $\Sigma\, a_i \geqslant 0$ when each a_i is $\geqslant 0$, we also have $\Sigma\, a_i \leqslant \Sigma\, b_i$ when $a_i \leqslant b_i$ for every i.

REMARK. It is sometimes useful to note that if I′ is a subset of I such that $a_i = 0$ for every $i \notin \mathrm{I}'$, the families $(a_i)_{i \in \mathrm{I}}$ and $(a_i)_{i \in \mathrm{I}'}$ are simultaneously summable (or not summable) and have equal sums.

8.12. Definition. A FAMILY (a_i) OF REAL NUMBERS IS SAID TO BE *ABSOLUTELY SUMMABLE* IF THE FAMILY $(|a_i|)$ OF THEIR ABSOLUTE VALUES IS SUMMABLE.

The word *absolutely* in this definition does not at all mean that the family (a_i) is summable in some absolute sense, which would in any case be meaningless; it refers simply to the fact that it is the *absolute value* of the a_i which enters in the definition.

8.13. Proposition. *Let (a_i) be a family of real numbers; then the following three statements are equivalent:*

1. *This family is absolutely summable.*
2. *This family is summable.*
3. *The set of finite sums A_K is bounded.*

PROOF. $1 \Rightarrow 2$. The members of each of the positive families (a_i^+) and (a_i^-) are bounded from above by the corresponding members of the family $(|a_i|)$; therefore if $(|a_i|)$ is summable, the comparison principle shows that the families (a_i^+) and (a_i^-) are summable. Hence, by Proposition 8.11, the family (a_i) which is their difference is also summable.

$2 \Rightarrow 3$. If the family (a_i) is summable, there exists $J_0 \in \mathscr{F}$ such that

$$|A - A_J| \leqslant 1 \qquad \text{for every finite} \quad J \supset J_0 .$$

But for every $K \in \mathscr{F}$ we have

$$|A_{K \cup J_0} - A_K| \leqslant \sum_{i \in J_0} |a_i|.$$

Therefore, comparing these inequalities for $J = K \cup J_0$ gives

$$|A - A_K| \leqslant 1 + \sum_{i \in J_0} |a_i|.$$

Thus, the set of numbers A_K is bounded.

$3 \Rightarrow 1$. The hypothesis implies the existence of two positive numbers k_1, k_2 such that $A_K \in [-k_1, k_2]$ for every $K \in \mathscr{F}$.

For every $K \in \mathscr{F}$, the sum of the $|a_i|$ such that $i \in K$ and $a_i \geqslant 0$ belongs to $[0, k_2]$, and the sum of the $|a_i|$ such that $i \in K$ and $a_i \leqslant 0$ belongs to $[0, k_1]$; therefore the sum of the $|a_i|$ such that $i \in K$ is $\leqslant k_1 + k_2$.

These sums are thus bounded from above; hence the family (a_i) is absolutely summable.

8.14. Corollary. *Every subfamily of a summable family of real numbers is summable.*

Indeed, the summability of the family $(a_i)_{i \in I}$ implies that of the family $(|a_i|)_{i \in I}$, hence that of the subfamily $(|a_i|)_{i \in I'}$, therefore, finally, that of the subfamily $(a_i)_{i \in I'}$.

Z The equivalence which has just been established between summability and absolute summability does not have an analogue in the theory of series; we recall for example that the alternating series with general term $(-1)^n n^{-1}$ is convergent although the series with general term n^{-1} is divergent.

We shall return later to the comparison of series and summable families.

9. SUMMABLE FAMILIES IN TOPOLOGICAL GROUPS AND NORMED SPACES

A number of the results of the preceding section rest solely on the existence of a group structure and a separated topology on **R**; thus, one might expect that the theory which we have expounded is valid in every commutative and separated topological group. We shall consider primarily the case of normed spaces, but it is interesting, in view of multipliable families, to give the definitions in the framework of topological groups.

We shall continue to use the notation $\mathscr{F}$, A_J introduced in the previous section.

9.1. Definition. LET G BE A SEPARATED COMMUTATIVE TOPOLOGICAL GROUP, WRITTEN ADDITIVELY, AND LET $(a_i)_{i\in I}$ BE A FAMILY OF ELEMENTS OF G.

THIS FAMILY IS SAID TO BE *SUMMABLE* IF THERE EXISTS AN ELEMENT A OF G HAVING THE FOLLOWING PROPERTY:

FOR EVERY NEIGHBORHOOD V OF O THERE EXISTS A $J_0 \in \mathscr{F}$ SUCH THAT

$$A - A_J \in V \qquad \text{for every} \quad J \in \mathscr{F}, \quad J \supset J_0 .$$

WHEN SUCH AN ELEMENT EXISTS, IT IS UNIQUE; IT IS CALLED THE *SUM* OF THE FAMILY AND DENOTED BY

$$\sum_{i\in I} a_i \quad \text{or} \quad \sum_i a_i \quad \text{or} \quad \sum a_i .$$

Let us prove that A is unique when it exists. It suffices to slightly modify the proof given following Definition 8.1.

For every neighborhood U of O there exists a symmetric neighborhood V of O such that $V + V \subset U$. We conclude from the relations $A - A_J \in V$ and $A' - A_J \in V$ that

$$A - A' \in V + V, \qquad \text{hence} \qquad A - A' \in U.$$

Thus the element $A - A'$ belongs to every neighborhood of O; since G is separated, $A - A' = O$ or $A = A'$.

REMARK 1. When the group operation in G is written multiplicatively, it is preferable to say that the family (a_i) is *multipliable* and has product $A = \prod a_i$.

This is, for example, the case for the topological groups $\mathbf{R}^*$ and $\mathbf{C}^*$.

REMARK 2. When I is finite, the family (a_i) is always summable and its sum is equal to the sum in the ordinary sense.

REMARK 3. If I′ denotes a subset of I such that $a_\iota = 0$ for every $i \notin I'$, the families $(a_i)_{\iota \in I}$ and $(a_i)_{i \in I'}$ are simultaneously summable (or not summable) and their sums are equal.

9.2. Proposition. *Let $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ be summable families of elements of* G *with the same index set* I, *and let* A *and* B *be their sums.*

Then the family $(c_i)_{i \in I}$, where $c_i = a_i + b_i$, is summable with sum $A + B$.

The proof is an adaptation of that of Lemma 8.9.

9.3. Proposition. *Let* G *and* G′ *be commutative and separated topological groups, and let φ be a continuous representation of* G *in* G′.

If (a_i) is a summable family in G *with sum* A, *the family (a_i'), where $a_i' = \varphi(a_i)$, is summable in* G′ *with sum* $A' = \varphi(A)$.

PROOF. For every neighborhood V′ of O in G′, we put $V = \varphi^{-1}(V')$; the relation

$$A - A_J \in V \qquad \text{when} \quad J_0 \subset J$$

implies

$$\varphi(A) - \varphi(A_J) \in \varphi(V) \qquad \text{or} \qquad A' - A_J' \in V' \qquad \text{when } J_0 \subset J.$$

In other words, the family (a_i') has sum A′.

Cauchy criterion

9.4. Definition. A FAMILY (a_i) IN G IS SAID TO SATISFY THE *CAUCHY CRITERION* IF FOR EVERY NEIGHBORHOOD V OF O THERE EXISTS A $J_0 \in \mathscr{F}$ SUCH THAT $A_K \in V$ FOR EVERY $K \in \mathscr{F}$ SUCH THAT $K \cap J_0 = \emptyset$.

In other words, after removing from the family a finite number of elements which are "too large," all the finite sums are small.

We now denote the set of numbers A_J for which $J_0 \subset J$ by $\mathscr{A}(J_0)$.

9.5. Proposition. *To say that the family (a_i) satisfies the Cauchy criterion is equivalent to saying that for every neighborhood* V *of* O *there exists a* $J_0 \in \mathscr{F}$ *and a translate $a + V$ of* V *such that $\mathscr{A}(J_0) \subset a + V$.*

PROOF. 1. Suppose (a_i) satisfies the Cauchy criterion; every J containing J_0 is of the form $J_0 \cup K$, where $K \cap J_0 = \emptyset$; therefore $A_J = A_{J_0} + A_K$.

It follows from $A_K \in V$ that

$$A_J \in A_{J_0} + V, \qquad \text{hence} \qquad \mathscr{A}(J_0) \subset A_{J_0} + V.$$

2. Given V, there exists a symmetric neighborhood U of O such that $U + U \subset V$. If by hypothesis there exists a $J_0 \in \mathscr{F}$ and $a \in G$ such that $\mathscr{A}(J_0) \subset a + U$, then for every K with $K \cap J_0 = \emptyset$ we have

$$A_{J_0} \in a + U, \qquad A_{J_0} + A_K \in a + U, \qquad \text{from which} \qquad A_K \in U + U \subset V.$$

9.6. Corollary. *Every summable family satisfies the Cauchy criterion.*

Indeed, the condition $\mathscr{A}(J_0) \subset a + V$ holds on taking for a the sum of the family, and using Definition 9.1.

Z The converse of this corollary is in general false. But we shall see that it is true in complete normed spaces.

9.7. Proposition. *Every subfamily of a family satisfying the Cauchy criterion also satisfies it.*

Indeed, since the index sets of these families are I and I′, with $I' \subset I$, it suffices in the statement of 9.4 to take, for the set J_0 relative to the subfamily, the set $I' \cap J_0$.

9.8. Proposition. *Let (a_i) be a family satisfying the Cauchy criterion. For every neighborhood V of O we have*

$$a_i \in V \quad \textit{except for at most finitely many of the} \quad i.$$

To see this, it suffices to apply the Cauchy criterion to the sets K of the form $\{i\}$, where $i \notin J_0$.

9.9. Corollary. *If the group G is such that O has a countable neighborhood base, and if the family (a_i) satisfies the Cauchy criterion, then the set of indices i for which $a_i \neq O$ is finite or countable.*

Indeed, let (V_n) be a neighborhood base of O; for every n, the set I_n of indices i such that $a_i \notin V_n$ is finite. Since the set of i such that $a_i \neq O$ is simply the union of the I_n, this set is finite or countable.

Z This corollary might lead one to believe that the study of summable families in the usual groups, in particular in normed spaces, reduces to the study of families with countable index sets, that is, to the study of sequences (a_n).

This is not so for two reasons: On the one hand, the artificial introduction of an order on the set of indices introduces the risk of masking the commutativity and additivity properties of the sum; on the other hand, it can happen that one has to study *all* the summable families with a given nondenumerable index set I.

9.10. *Commutativity*

Since the definition of the summability of a family $(a_i)_{i\in \mathrm{I}}$ does not involve any order structure on I, one can say, in a vague sense, that this notion is commutative. This assertion can be made precise in the following way: Let $(a_i)_{i\in \mathrm{I}}$ and $(b_j)_{j\in \mathrm{I}'}$ be two families of elements of G; if there exists a bijection φ of I to I′ such that $b_{\varphi(i)} = a_i$ for every i, then the summability of one of these families implies that of the other, and their sums are equal.

Associativity

Associativity is deeper, and consists of the following proposition:

9.11. Proposition. *Let* $(\mathrm{I}_\lambda)_{\lambda\in \mathrm{L}}$ *be an arbitrary partition of an index set* I, *and let* $(a_i)_{i\in \mathrm{I}}$ *be a family of elements of the group* G *indexed by* I.

If this family, as well as each of the subfamilies associated with the I_λ, *is summable, and if we denote their sums by* A *and* s_λ, *respectively, then the family* $(s_\lambda)_{\lambda\in \mathrm{L}}$ *is summable and has sum* A.

Proof. We borrow the notation $\mathscr{A}(\mathrm{J}_0)$ used in Proposition 9.5; similarly, for every finite subset M_0 of L, let $\mathscr{L}(\mathrm{M}_0)$ denote the set of finite sums $\sum_{\lambda\in \mathrm{M}} s_\lambda$, where $\mathrm{M}_0 \subset \mathrm{M}$.

Given J_0, if we let M_0 be the finite set of those λ for which $\mathrm{J}_0 \cap \mathrm{I}_\lambda \neq \emptyset$, every sum $\sum_{\lambda\in \mathrm{M}} s_\lambda$ for which $\mathrm{M}_0 \subset \mathrm{M}$ is the limit of sums A_J where $\mathrm{J}_0 \subset \mathrm{J}$; it follows that

$$\mathscr{L}(\mathrm{M}_0) \subset \overline{\mathscr{A}(\mathrm{J}_0)}, \qquad \text{from which} \qquad \overline{\mathscr{L}(\mathrm{M}_0)} \subset \overline{\mathscr{A}(\mathrm{J}_0)}.$$

For every neighborhood V of O, there exists a J_0 such that

$$\mathscr{A}(\mathrm{J}_0) \subset \mathrm{A} + \mathrm{V}.$$

Therefore if V is closed, we also have

$$\overline{\mathscr{A}(\mathrm{J}_0)} \subset \mathrm{A} + \mathrm{V}, \qquad \text{from which} \qquad \overline{\mathscr{L}(\mathrm{M}_0)} \subset \mathrm{A} + \mathrm{V}.$$

Since the closed neighborhoods of O constitute a neighborhood base of O (see Problem 81), this relation shows that the family (s_λ) is summable with sum A.

EXAMPLE. Let $(a_{p,q})$ be a summable double sequence of real numbers; every subfamily is then summable (Corollary 8.12), and

$$\sum_{p,q} a_{p,q} = \sum_p \Big(\sum_q a_{p,q}\Big) = \sum_q \Big(\sum_p a_{p,q}\Big).$$

Z It is not correct that if each of the subfamilies $(a_i)_{i\in I_\lambda}$ is summable and the family (s_λ) is summable, then $(a_i)_{i\in I}$ is always summable. To see this it suffices to take L infinite, with each of the subfamilies consisting of two elements, 1 and -1; then each s_λ equals 0. Hence the family (s_λ) is summable, but the family (a_i) is not.

On the other hand, here are two important cases in which this assertion is correct.

9.12. L is finite. The family $(a_i)_{i\in I}$ is then the finite sum of the families $(a_i^\lambda)_{i\in I}$ defined as follows:

$$a_i^\lambda = a_i \quad \text{if} \quad i\in I_\lambda ; \qquad a_i^\lambda = O \quad \text{if} \quad i \notin I_\lambda .$$

But if one of the two families $(a_i^\lambda)_{i\in I}$ and $(a_i)_{i\in I}$ is summable, the other is also, and they have the same sums; the desired result now follows from Proposition 9.2.

9.13. The a_i are numbers $\geqslant 0$. We start by slightly extending the notion of the sum: By the sum of a family $(a_i)_{i\in I}$ of elements of $\bar{\mathbf{R}}_+ = [0, +\infty]$ we shall mean the supremum, finite or infinite, of the sums of all finite subfamilies of elements a_i.

Let then $(I_\lambda)_{\lambda\in L}$ be a partition of I; we shall show that for these generalized sums we again have

$$\sum_{i\in I} a_i = \sum_\lambda s_\lambda , \qquad \text{where} \qquad s_\lambda = \sum_{i\in I_\lambda} a_i .$$

First of all, $\Sigma_i\, a_i \geqslant$ (every finite sum of elements s_λ), by an obvious comparison principle; hence we obtain the desired relation with $\geqslant$ instead of $=$. The same relation, but with $\leqslant$ instead of $=$, follows from the fact that every finite sum of elements a_i is bounded from above by a finite sum of elements s_λ.

In particular, if each of the sums s_λ is finite and if $\Sigma_\lambda\, s_\lambda$ is finite, the same is true of $\Sigma_i\, a_i$ which, by Proposition 8.2, gives the desired result.

9.14. EXAMPLE. Let $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ be two summable families of real numbers; then the families $(|a_i|)_{i\in I}$ and $(|b_j|)_{j\in J}$ are summable. The family $(|a_ib_j|)$, whose index set is $I \times J$, can be partitioned into subfamilies indexed by the sets $i \times J$; but then, by Proposition 9.13,

$$\sum_{i,j} |a_ib_j| = \sum_i \Big(\sum_j |a_ib_j|\Big) = \sum_i \Big(|a_i| \sum_j |b_j|\Big)$$
$$= \Big(\sum_i |a_i|\Big)\Big(\sum_j |b_j|\Big).$$

Therefore the family (a_ib_j) is absolutely summable, hence summable; by Proposition 9.11 we can write

$$\sum_{i,j} a_ib_j = \sum_i \Big(\sum_j a_ib_j\Big) = \sum_i \Big(a_i \sum_j b_j\Big) = \Big(\sum_i a_i\Big)\Big(\sum_j b_j\Big).$$

In short, the family (a_ib_j), which is called the *product of the families* (a_i) and (b_j), is summable and its sum is the product of their sums.

Summability in normed spaces

The existence of a norm will enable us to rederive part of the results obtained in the case of **R**.

9.15. Theorem. *In a complete normed space* E, *every family of elements of* E *which satisfies the Cauchy criterion is summable.*

PROOF. Let $(a_i)_{i\in I}$ be a family of elements of E. It is immediate that the sets $\mathscr{A}(J_0)$ used in Proposition 9.5 form a filter base $\mathscr{B}$ on E; in addition this proposition shows that when the family (a_i) satisfies the Cauchy criterion, $\mathscr{B}$ is a Cauchy filter base (see Definition 20.7, Chapter I). Since E is complete, this filter base converges to a point $A \in E$ (see Proposition 20.8, Chapter I); therefore the family (a_i) is summable with sum A.

9.16. Corollary. *In a complete normed space, every subfamily of a summable family is summable.*

This is a consequence of the results 9.6, 9.7, and 9.15.

REMARK. If F denotes a vector subspace of a complete normed space E, with $\bar{F} = E$, and if (a_i) is a family of elements of F satisfying the Cauchy criterion, we have just seen that (a_i) is summable in E; but clearly if its sum does not belong to F, then (a_i) is not summable in F. This example shows the reason why a family can satisfy the Cauchy criterion without being summable.

Absolutely summable families in normed spaces

It is not always easy to recognize whether a family of elements of a normed space is summable. It is thus useful to have sufficient criteria for summability; such a criterion is furnished by the notion of absolute summability.

9.17. Definition. LET E BE A NORMED SPACE. A FAMILY (a_i) OF ELEMENTS OF E IS SAID TO BE *ABSOLUTELY SUMMABLE* IF THE FAMILY OF THEIR NORMS $\| a_i \|$ IS SUMMABLE.

This definition appears to depend upon the norm of E; in fact, it only depends on the topology of E, for if p and p' are equivalent norms on E, there exist constants k, $k' > 0$ such that $p' \leqslant kp$ and $p \leqslant k'p'$; thus if one of the families $(p(a_i))$, $(p'(a_i))$ is summable, the other is also.

9.18. Proposition. *In a complete normed space, every absolutely summable family is summable.*

PROOF. By Theorem 9.15, it suffices to show that if the family (a_i) is absolutely summable, it satisfies the Cauchy criterion.

But if the family $(\| a_i \|)$ is summable, it satisfies the Cauchy criterion; therefore for every $\epsilon > 0$ there exists a $J_0 \in \mathscr{F}$ such that

$$\sum_{i \in K} \| a_i \| \leqslant \epsilon \qquad \text{for every} \qquad K \in \mathscr{F}, \quad K \cap J_0 = \emptyset.$$

But

$$\| A_K \| = \Big\| \sum_{i \in K} a_i \Big\| \leqslant \sum_{i \in K} \| a_i \|, \qquad \text{from which} \qquad \| A_K \| \leqslant \epsilon.$$

Therefore the family (a_i) satisfies the Cauchy criterion.

Z It is essential to note that in a normed space, a family can be summable without being absolutely summable; here are two examples.

EXAMPLE 1. Let (f_n) be the sequence of elements of $\mathscr{C}([0, 1], \mathbf{R})$ defined by

$$f_n(t) = t \sin^2 \frac{\pi}{t} \qquad \text{on} \quad \left[\frac{1}{n+1}, \frac{1}{n}\right]; \qquad f_n(t) = 0 \quad \text{off this interval.}$$

Then

$$\| f_n \| \geqslant f_n\left(\frac{1}{n+\frac{1}{2}}\right) = \frac{2}{2n+1};$$

thus the family (f_n) is not absolutely summable; but it is summable and its sum is the function f defined by

$$f(0) = 0; \qquad f(t) = t \sin^2 \frac{\pi}{t} \qquad \text{if} \quad t \neq 0.$$

Example 2. Let a_n be the vector in the normed space l^2 all of whose coordinates are zero, except for the nth, which equals n^{-1}. The family (a_n) is not absolutely summable since $\| a_n \| = n^{-1}$; but this family is summable, and its sum is the vector

$$x = (1, 2^{-1}, ..., n^{-1}, ...).$$

These two examples are simply special cases of the following general result:

In every infinite-dimensional normed space, there exist summable sequences which are not absolutely summable (see Problem 84).

We are going to see that the situation is simpler in finite-dimensional spaces.

9.19. Proposition. *In a finite-dimensional space, summability and absolute summability are equivalent.*

Proof. Every finite-dimensional normed space over **K** is a finite-dimensional normed space over **R**; on the other hand, Corollary 7.3 permits us to assume that the latter space is $\mathbf{R}^n$, with the norm

$$\| x \| = \sum | x_p |.$$

Since $\mathbf{R}^n$ is complete, every absolutely summable family in $\mathbf{R}^n$ is summable. Conversely, if the family (a_i) is summable in $\mathbf{R}^n$, Proposition 9.3 shows that for every $p \leqslant n$, the family (a_i^p) of the pth coordinates of the a_i is summable; therefore the family $(| a_i^p |)$ is summable. Since $\| a_i \| = \Sigma_p \, | a_i^p |$, the family $(\| a_i \|)$, which is the sum of n summable families, is summable.

9.20. Corollary. *For every family of complex numbers, summability and absolute summability are equivalent.*

Multilinear mappings of summable families

We are going to extend the result obtained in 9.14 concerning the product of two families of real numbers.

9.21. Proposition. *Let* E *and* F *be normed spaces, and let* f *be a continuous bilinear mapping of* $\mathrm{E} \times \mathrm{F}$ *into a complete normed space* G.

Let $(a_i)_{i \in \mathrm{I}}$ $((b_j)_{j \in \mathrm{J}})$ *be a summable family of elements of* E (F) *with sum* A (B). *When these two families are* **absolutely summable,** *then the family* $(f(a_i, b_j))$ *is absolutely summable, and has sum* $f(\mathrm{A}, \mathrm{B})$.

PROOF. Since f is continuous, there exists (see Proposition 6.1) a constant k such that

$$\|f(x, y)\| \leqslant k \, \|x\| \, \|y\| \qquad \text{for all} \quad x, y.$$

Therefore

$$\|f(a_i, b_j)\| \leqslant k \, \|a_i\| \, \|b_j\|;$$

but since the families $(\|a_i\|)$ and $(\|b_j\|)$ are summable, the family of products $\|a_i\| \, \|b_j\|$ is summable (see 9.14). Therefore the family $(f(a_i, b_j))$ is absolutely summable, and since G is complete, it is summable.

We now observe that for every $a \in \mathrm{E}$ we have $\Sigma_j f(a, b_j) = f(a, \mathrm{B})$, since the mapping $y \to f(a, y)$ is linear and continuous (see Proposition 9.3); similarly for the $f(a_i, b)$.

Therefore, by the associativity of the sum we have

$$\sum_{i,j} f(a_i, b_j) = \sum_i \Big(\sum_j f(a_i, b_j) \Big) = \sum_i f(a_i, \mathrm{B}) = f(\mathrm{A}, \mathrm{B}).$$

9.22. Corollary. *If* (a_i) *and* (b_j) *are two summable families of complex numbers with respective sums* A *and* B, *the family* $(a_i b_j)$ *of products is summable with sum* AB.

Indeed, the mapping $(x, y) \to xy$ of $\mathbf{C} \times \mathbf{C}$ into $\mathbf{C}$ is bilinear and continuous.

REMARK 1. Proposition 9.21 evidently extends to every continuous multilinear mapping.

REMARK 2. In the statement of Proposition 9.21, the hypothesis that the families (a_i) and (b_j) are absolutely summable is essential; however, the assumption that G is complete can be omitted, for it suffices to observe that if G is not complete, we can assume it to be imbedded in a complete normed space G′ (see Problem 48).

10. SERIES; COMPARISON OF SERIES AND SUMMABLE FAMILIES

10.1. Definition. LET $(a_n)_{n \in \mathrm{N}}$ BE A SEQUENCE OF ELEMENTS OF A SEPARATED COMMUTATIVE TOPOLOGICAL GROUP G.

THE *SERIES* DEFINED BY THE SEQUENCE (a_n) IS THE PAIR CONSISTING OF THE TWO SEQUENCES (a_n) AND (s_n), WHERE s_n IS THE PARTIAL SUM $\sum_{i \leqslant n} a_i$.

THIS SERIES IS SAID TO *CONVERGE* IF THE SEQUENCE (s_n) IS CONVERGENT; ITS LIMIT s (WHICH IS UNIQUE, AS G IS SEPARATED) IS CALLED THE *SUM* OF THE SERIES AND IS DENOTED BY $\sum_0^\infty a_n$ OR SIMPLY $\sum a_n$.

WHEN THE SERIES DOES NOT CONVERGE, IT IS SAID TO *DIVERGE*.

REMARK 1. It is convenient to denote the series defined by the sequence (a_n) by the symbol $a_0 + a_1 + \cdots + a_n \ldots$; this notation is *a priori* meaningless, but it is rather suggestive.

One also says, more simply, "the series with general term a_n" or "the series (a_n)."

REMARK 2. Sometimes the index set $\mathbf{N}$ is replaced by $\mathbf{N}^*$; more generally, one can replace $\mathbf{N}$ by a sequence of indices with an order isomorphic to that of $\mathbf{N}$.

REMARK 3. If the operation in G is written multiplicatively, the term "series" is replaced by "infinite product," and the limit of the partial products $p_n = \prod_0^n a_i$ is denoted by $\prod_0^\infty a_i$.

REMARK 4. When the a_n are real numbers, one also says that the series (a_n) converges in $\bar{\mathbf{R}}$ and has sum $+\infty$ (or $-\infty$) if $\lim s_n = +\infty$ (or $-\infty$).

Let us point out, without giving either statements or proofs, that Propositions 9.2 and 9.3 have immediate analogues for series.

Cauchy criterion

10.2. Definition. A *SERIES* WITH GENERAL TERM a_n IN G IS SAID TO SATISFY THE *CAUCHY CRITERION* IF THE SEQUENCE (s_n) OF PARTIAL SUMS IS A CAUCHY SEQUENCE IN G; IN OTHER WORDS, IF FOR EVERY NEIGHBORHOOD V OF O THERE EXISTS AN INTEGER n_0 SUCH THAT THE RELATIONS $n_0 \leqslant p \leqslant q$ IMPLY

$$s_q - s_p \in \mathrm{V}, \qquad \text{i.e.,} \qquad (a_{p+1} + a_{p+2} + \cdots + a_q) \in \mathrm{V}.$$

When G is a normed space, it is more convenient to replace the neighborhood V in this statement by a number $\epsilon > 0$, and the condition $s_q - s_p \in \mathrm{V}$ by $\| s_q - s_p \| \leqslant \epsilon$.

10.3. Proposition. 1. *Every convergent series satisfies the Cauchy criterion.*

2. *Conversely, when* G *is a complete normed space, every series satisfying the Cauchy criterion is convergent.*

This is immediate, since the convergence of a series (a_n) is defined in terms of the convergence of the sequence (s_n).

10.4. Proposition. *For every series satisfying the Cauchy criterion (thus, in particular, for every convergent series), the sequence* (a_n) *tends to* O.

Indeed, with the notation of Definition 10.2, we have

$$a_{n+1} = s_{n+1} - s_n \in V \qquad \text{for every} \quad n \geqslant n_0 .$$

Z Let us recall the well-known fact that the converse of this proposition is false, even in **R**. For example, the series with general term $a_n = n^{-1}$ does not satisfy the Cauchy criterion since $s_{2p} - s_p \geqslant \frac{1}{2}$ for every p; nevertheless $\lim a_n = 0$.

Commutativity and associativity

We first study *associativity*, which is easier to study than commutativity. If one does not change the order of the elements a_n of a series, the only grouping of terms which is possible consists of grouping them in segments of the form $(a_{p+1} + a_{p+2} + \cdots + a_q)$; we can then state:

10.5. Proposition. *Let* (a_n) *be a sequence of elements of* G, *and let* (α_n) *be a strictly increasing sequence of integers* $\geqslant 0$; *we put* $b_n = \sum a_i$, *where the summation is over those* i *satisfying* $\alpha_{n-1} \leqslant i < \alpha_n$.

If the series (a_n) *satisfies the Cauchy criterion, so does the series* (b_n). *If the first converges, so does the second, and their sums are equal.*

Indeed, the sequence of partial sums of the second series is a subsequence of the sequence of partial sums of the first series.

Z It can happen that the second series converges and the first series does not; for example, if $a_n = (-1)^n$ and $b_n = a_{2n} + a_{2n+1}$.

10.6. Definition. A SERIES (a_n) IS SAID TO BE *COMMUTATIVELY CONVERGENT** IF THE SERIES $(a_{\pi(n)})$ IS CONVERGENT FOR EVERY PERMUTATION π OF **N**.

Observe that this definition does not require *a priori* that the series (a_n) and $(a_{\pi(n)})$ have the same sum.

* An equivalent terminology is *unconditionally convergent.*

10.7. Theorem. *To say that a series (a_n) is commutatively convergent is equivalent to saying that the family (a_n) is summable.*

The sum of the family (a_n) is then equal to the sum of each of the series $(a_{\pi(n)})$.

Proof. 1. Suppose the family (a_n) is summable, and let A be its sum. For every neighborhood V of O there exists a finite subset J_0 of **N** such that $A - A_J \in V$ for every finite $J \supset J_0$.

If n_0 is the largest element of J_0, the segment $[0, p]$ of **N** contains J_0 for every $p \geqslant n_0$, hence $A - s_p \in V$; in other words, the series (a_n) has sum A. This argument is clearly applicable to each of the series $(a_{\pi(n)})$.

2. Suppose, on the contrary, that the family (a_n) is not summable. Then there exists a neighborhood V of O such that for every finite $J \subset \mathbf{N}$ there exists a finite $K(J) \subset \mathbf{N} \dot{-} J$ for which $A_{K(J)} \notin V$.

Put $J_0 = \emptyset$, and $J_{n+1} = J_n \cup K(J_n)$ for $n \geqslant 0$. The sets $K(J_n)$ are finite subsets of **N** which are pairwise disjoint, and for each of them

$$A_{K(J_n)} \notin V.$$

One can easily construct a permutation π of **N** such that in the sequence $(a_{\pi(n)})$ the elements a_n for which $n \in K(J_p)$ (for given p) are consecutive.

We now put $b_n = a_{\pi(n)}$; the series (b_n) does not satisfy the Cauchy criterion because it contains an infinite number of disjoint "segments" $(b_{p+1}, \ldots, b_q)$ each of whose sum lies outside V.

Therefore the series (a_n) is not commutatively convergent.

Remark. In fact the proof shows more: It shows that if the family (a_n) does not satisfy the Cauchy criterion, there exists a series $(a_{\pi(n)})$ which does not satisfy it either.

Absolutely convergent series in normed spaces

10.8. Definition. Let E be a normed space; a series (a_n) in E is said to be *absolutely convergent* if the series with general term $\| a_n \|$ is convergent.

This definition leads us to the study of series with positive terms.

10.9. Lemma. *Let (a_n) be a sequence of positive numbers; the following three statements are equivalent:*

1. *The series (a_n) is convergent.*
2. *The set of sums s_n is bounded from above.*
3. *The family (a_n) is summable.*

PROOF. 1 ⇒ 2. For, the sequence (s_n) is increasing; thus if $\lim s_n = s$, we have $s_n \leqslant s$ for every n.

2 ⇒ 3. For, every sum A_K is bounded from above by a sum $A_{[0,n]} = s_n$; thus if $s_n \leqslant k$, the sums A_K are bounded from above, which implies the summability of the family (a_n).

3 ⇒ 1. For by Theorem 10.7, summability implies the convergence of the series (and even commutative convergence).

10.10. Corollary. *To say that a series (a_n) in a normed space is absolutely convergent is equivalent to saying that the family (a_n) is absolutely summable.*

10.11. Corollary. *In a complete normed space, every absolutely convergent series is commutatively convergent.*

This last corollary follows from 9.18, 10.7, and 10.10.

10.12. Proposition (case of finite-dimensional spaces). *Let (a_n) be a sequence of elements of a finite-dimensional normed space.*
The following statements are equivalent:

1. *The series (a_n) is commutatively convergent.*
2. *The series (a_n) is absolutely convergent.*
3. *The family (a_n) is summable.*
4. *The family (a_n) is absolutely summable.*

This is a consequence of 9.19, 10.10, and 10.11.
This proposition applies in particular to series of complex numbers.

10.13. Conditionally convergent series. The alternating harmonic series with general term $(-1)^n n^{-1}$ is an example of a convergent series which is not commutatively convergent; such a series is sometimes said to be *conditionally convergent*. This is a rather evocative terminology which should not however be abused: For example, if a series (a_n) is conditionally convergent, so is the series $(-a_n)$, but the sum of these two series is commutatively convergent, hence not conditionally convergent. Similarly there do not exist, properly speaking, criteria for conditional convergence; there exist only criteria (sufficient) for convergence in which one is not concerned with knowing whether or not the series (a_n) is commutatively convergent.

The most useful criteria of this type are furnished by a simple inequality due to Abel.

10.14. Lemma. *Let (a_n) be a sequence of elements of a normed space* E, *and let (λ_n) be a decreasing sequence of positive numbers*; *we put*

$$k_n = \sup_{p \geqslant 0} \| a_n + a_{n+1} + \cdots + a_{n+p} \|.$$

Then for every n, p we have

$$\| \lambda_n a_n + \lambda_{n+1} a_{n+1} + \cdots + \lambda_{n+p} a_{n+p} \| \leqslant \lambda_n k_n .$$

Proof. Put $b_{n,p} = a_n + a_{n+1} + \cdots + a_{n+p}$; by a simple transformation called *Abel's transformation* (similar to integration by parts) one can write

$$\begin{aligned} &\| \lambda_n a_n + \cdots + \lambda_{n+p} a_{n+p} \| \\ &\quad = \| \lambda_n b_{n,0} + \lambda_{n+1}(b_{n,1} - b_{n,0}) + \cdots + \lambda_{n+p}(b_{n,p} - b_{n,p-1})\| \\ &\quad = \|(\lambda_n - \lambda_{n+1}) b_{n,0} + \cdots + (\lambda_{n+p-1} - \lambda_{n+p}) b_{n,p-1} + \lambda_{n+p} b_{n,p} \| \\ &\quad \leqslant k_n((\lambda_n - \lambda_{n+1}) + \cdots + (\lambda_{n+p-1} - \lambda_{n+p}) + \lambda_{n+p}) = \lambda_n k_n . \end{aligned}$$

It follows from this lemma that if the sequence $(\lambda_n k_n)$ tends to 0, the series $(\lambda_n a_n)$ satisfies the Cauchy criterion. Hence, using the fact that the sequence (λ_n) is decreasing, we obtain two cases where the series $(\lambda_n a_n)$ will converge:

10.15. Proposition (Abel's rule). *Let (a_n) be a sequence of elements of a complete normed space* E, *and let (λ_n) be a decreasing sequence of positive numbers.*

In each of the following two cases, the series $(\lambda_n a_n)$ is convergent:

(a) *The series (a_n) is convergent.*

(b) *The set of sums $s_n = a_0 + a_1 + \cdots + a_n$ is bounded in* E, *and* $\lim_{n\to\infty} \lambda_n = 0$.

Proof. Since E is complete it suffices, by Lemma 10.14, to verify that the sequence $(\lambda_n k_n)$ tends to 0.

In the first case, the convergence of the series (a_n) implies $\lim_{n\to\infty} k_n = 0$; but $\lambda_n \leqslant \lambda_0$, from which $\lambda_n k_n \leqslant \lambda_0 k_n$, and the result follows.

In the second case, the relation $a_n + a_{n+1} + \cdots + a_{n+p} = s_{n+p} - s_{n-1}$ shows that the set of $b_{n,p}$ is bounded; therefore there exists a number $k \geqslant 0$ such that $k_n \leqslant k$. We therefore have $\lambda_n k_n \leqslant \lambda_n k$, from which $\lim_{n\to\infty} \lambda_n k_n = 0$.

10.16. EXAMPLES. Let us call a series of real numbers whose general term can be written as $\lambda_n a_n$, where $a_n = (-1)^n$ and $\lambda_n \geqslant 0$ (or $\lambda_n \leqslant 0$, but a change of sign throughout leads back to the preceding case) an *alternating series*.

The condition $\| s_n \| \leqslant k$ is satisfied since $s_n = 0$ or 1. Therefore if λ_n *decreases* to 0, the alternating series with general term $(-1)^n \lambda_n$ is convergent.

10.17. Let us take for a_n the complex number k^n, where $| k | = 1$ and $k \neq 1$. The relation $s_n = (1 - k^{n+1})/(1 - k)$ shows that

$$| s_n | \leqslant 2 | 1 - k |^{-1}.$$

Therefore, with the preceding notation, the series $(\lambda_n k^n)$ is convergent when (λ_n) is a decreasing positive sequence with limit 0.

The series of real (imaginary) parts also converges; in other words, if we put $k = e^{it}$, the real series with general term $\lambda_n \cos nt$ ($\lambda_n \sin nt$) converges for every $t \not\equiv 0 \bmod 2\pi$.

10.18. Method of study of a numerical series. The central idea consists in studying the behavior of the general term a_n; if it does not tend to 0, the series diverges; if it tends to 0, one tries to determine whether the series of absolute values $| a_n |$ converges.

To do this, if $| a_n |$ is given by a complicated expression, one tries to bound $| a_n |$ from above by the general term of a positive series which is easier to study; for example, if $a_n = n^{-2} \cos n$, we have $| a_n | \leqslant n^{-2}$, which implies absolute convergence. If $| a_n |$ is given by a simple and regular algorithm, one can try to apply the criteria using lim sup of $| a_{n+1} |/| a_n |$ or of $| a_n |^{1/n}$ (see 8.6). But the truly general rule consists in studying the rapidity of convergence of $| a_n |$ to 0.

If the series $(| a_n |)$ diverges, one can try to apply Abel's rule; it is in theory always applicable (see Problems 107 and 108), but apart from the cases pointed out in 10.16 and 10.17, the cases where it is of service are rather rare.

Finally, it should not be forgotten that the convergence of a series (a_n) is simply the convergence of the sequence (s_n). For example, in the study of the expansion of functions in series, when one has the identity

$$f(t) = a_0(t) + a_1(t) + \cdots + a_n(t) + r_n(t),$$

and when, for a given t, $\lim_{n\to\infty} r_n(t) = 0$, the series with general term $a_n(t)$ converges and has sum $f(t)$.

11. SERIES AND SUMMABLE FAMILIES OF FUNCTIONS

We are going to study here series and summable families whose elements depend on a parameter; in other words, series and summable families of functions; we shall give only the statements of the results for summable families; a simple adaptation will give the corresponding results for series.

11.1. Definition. LET X BE AN ARBITRARY SET; LET G BE A SEPARATED COMMUTATIVE TOPOLOGICAL GROUP, AND LET $(a_i)_{i\in I}$ BE A FAMILY OF MAPPINGS OF X INTO G.

WE SAY THAT THE FAMILY $(a_i(x))$ IS *UNIFORMLY SUMMABLE* ON X IF IT IS SUMMABLE (WITH SUM $s(x)$) FOR EVERY $x \in X$, AND IF THE FINITE PARTIAL SUMS $s_J = \sum_{i\in J} a_i$ CONVERGE UNIFORMLY TO s, THAT IS, IF FOR EVERY NEIGHBORHOOD V OF O THERE EXISTS A FINITE $J_0 \subset I$ SUCH THAT FOR EVERY FINITE J CONTAINING J_0, AND FOR EVERY $x \in X$,

$$s(x) - s_J(x) \in V.$$

For series $(a_n(x))$, uniform convergence is expressed by the uniform convergence of the sequence $s_n = a_0 + a_1 + \cdots + a_n$ to s.

11.2. Proposition. *If the family $(a_i(x))$ is summable for every $x \in X$, the following statements are equivalent*:

1. *The family $(a_i(x))$ is uniformly summable on* X.

2. (*Uniform Cauchy criterion*). *For every neighborhood* V *of* O, *there exists a finite $J_0 \subset I$ such that $s_K(x) \in V$ for every finite* K *disjoint from J_0 and every $x \in X$.*

PROOF. $1 \Rightarrow 2$. Let U be a symmetric neighborhood of O such that $U + U \subset V$. By hypothesis there exists a finite $J_0 \subset I$ such that, for every finite K disjoint from J_0 and every $x \in X$,

$$s(x) - s_{J_0}(x) \in U; \qquad s(x) - s_{J_0\cup K}(x) \in U;$$

hence

$$s_K(x) \in U + U \subset V.$$

$2 \Rightarrow 1$. The hypothesis implies that for every neighborhood V of O there exists a finite $J_0 \subset I$ such that for any finite J and J′ containing J_0, and for every $x \in X$,

$$s_{J'}(x) - s_J(x) \in V.$$

Let us assume V closed and fix J; for every $x \in X$, $s(x)$ is the limit of the sums $s_{J'}(x)$; we therefore have

$$s(x) - s_J(x) \in V.$$

Since the closed V form a neighborhood base of O, we have proved the uniform summability of the family $(a_i(x))$.

REMARK. When G is a normed space, the condition $s_K(x) \in V$ can, if preferable, be expressed in the form $\| s_K(x) \| \leqslant \epsilon$.

11.3. Corollary. *Let (a_i) be a family of mappings of a set X into a complete normed space E.*

In order that the family $(a_i(x))$ be uniformly summable on X, it is necessary and sufficient that it satisfy the uniform Cauchy criterion.

PROOF. If the family is uniformly summable, by Proposition 11.2 it satisfies the uniform Cauchy criterion. Conversely, if this criterion is satisfied, the family $(a_i(x))$ is summable for every x since E is complete (see Theorem 9.15); therefore by 11.2 it is uniformly summable.

11.4. Proposition. *Let (a_i) be a family of mappings of a topological space X into a normed space E.*

If each a_i is continuous and if the family (a_i) is uniformly summable, then its sum is continuous.

Indeed, for every $\epsilon > 0$ there exists a finite $J_0 \subset I$ such that

$$\| s(x) - s_{J_0}(x) \| \leqslant \epsilon$$

for every $x \in X$. Therefore s is the uniform limit of functions s_{J_0}; since the s_{J_0} are continuous, so is s.

Normal convergence

It is not always easy to prove the uniform summability of a family of functions; it is well to have available convenient sufficient conditions for this. Here is one such condition which is nothing but absolute summability in the space $\mathscr{B}(X, E)$.

11.5. Definition. LET $(a_i)_{i \in I}$ BE A FAMILY OF MAPPINGS OF A SET X INTO A NORMED SPACE; THIS FAMILY IS SAID TO BE *NORMWISE SUMMABLE* IN X IF THERE EXISTS A SUMMABLE FAMILY $(k_i)_{i \in I}$ OF NUMBERS $\geqslant 0$ SUCH THAT $\| a_i(x) \| \leqslant k_i$ FOR EVERY $i \in I$ AND $x \in X$.

This condition can also be expressed, more concisely, by the condition

$$\sum_i \| a_i \| < \infty, \qquad \text{where} \quad \| a_i \| = \sup_{x \in X} \| a_i(x) \|.$$

Note that the word "normwise" in this definition is chosen to remind one of the use of the "norm" of uniform convergence.

11.6. Proposition. *Every normwise summable family of mappings of a set* X *into a* **complete** *normed space is uniformly summable.*

Proof. The inequality

$$\left\| \sum_{i \in K} a_i(x) \right\| \leqslant \sum_{i \in K} k_i$$

shows, since the family (k_i) is summable, that the family (a_i) satisfies the uniform Cauchy criterion, which by Corollary 11.3 implies uniform summability.

Comparison of the various modes of convergence

11.7. With every family (a_i) of mappings of a set X into a normed space E are associated the notions of pointwise, uniform, absolute, and normwise summability, with similar distinctions for the convergence of series.

The relations among these various modes of summability or convergence are a source of confusion for the beginner; we are therefore going to dwell a bit on these relations.

We shall examine here only the case of series; and to eliminate difficulties of another order which exist when E is not complete, we shall assume that E *is complete.*

The propositions established above lead to the following diagram:

$$\begin{tikzcd} & \text{absolute conv.} \arrow[dr, Rightarrow] & \\ \text{normwise conv.} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{pointwise conv.} \\ & \text{uniform conv.} \arrow[ur, Rightarrow] & \end{tikzcd}$$

It clearly follows that normwise convergence implies pointwise convergence, but we shall see from several examples that there exist no implications other than the five just written down, even when the functions in question are continuous numerical functions on [0, 1].

11.8. Absolute convergence (for every x) does not imply uniform convergence.

Indeed, put

$$a_n(x) = (x^n - x^{n+1});$$

the series $(a_n(x))$ is positive and converges for every x, and therefore converges absolutely. But

$$s(x) = 1 \quad \text{if} \quad x \neq 1, \qquad \text{and} \qquad s(x) = 0 \quad \text{if} \quad x = 1.$$

Since the sums s_n are continuous, the limit s would be continuous if the convergence were uniform; since s is not continuous, the convergence is not uniform.

11.9. Conversely, uniform convergence does not imply absolute convergence.

Indeed, put

$$a_n(x) = (-1)^n n^{-1} \qquad \text{for every} \quad x \in [0, 1] \quad \text{and every} \quad n \geqslant 1.$$

The series $(a_n(x))$ converges uniformly, but does not converge absolutely for any x.

11.10. Finally, we shall show that uniform convergence and absolute convergence together do not imply normwise convergence: It suffices to take the series (f_n) used in Example 1 which follows Proposition 9.18.

11.11. Application. Let α and β be nonzero complex numbers whose quotient is not real, and let P be the subset of $\mathbf{C}$ consisting of the complex numbers of the form

$$w_{p,q} = p\alpha + q\beta, \qquad \text{where} \quad p, q \in \mathbf{Z};$$

this is a closed additive subgroup of $\mathbf{C}$, each of whose points is isolated.

Let $a_{p,q}$ denote the function defined on $\mathbf{C} \dot{-} \mathrm{P}$ by

$$a_{0,0}(z) = z^{-2}; \qquad a_{p,q} = (z - w_{p,q})^{-2} - w_{p,q}^{-2} \qquad \text{if} \quad (p, q) \neq (0, 0).$$

We shall show that for every compact set K in $\mathbf{C}$, the family obtained from the family $(a_{p,q})$ by removing a suitable finite subfamily is uniformly summable on K.

Indeed, let $\rho > 0$; we discard from $(a_{p,q})$ those terms $a_{p,q}$ such that $|w_{p,q}| \leqslant 2\rho$.

If $|w| > 2\rho$, and if $|z| \leqslant \rho$, then

$$\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| = \left| \frac{z(2w - z)}{w^2(z-w)^2} \right| \leqslant |\rho| \frac{3|w|}{|w|^2 |\tfrac{1}{2}w|^2} = \frac{12\rho}{|w|^3}.$$

But the family of numbers $|w_{p,q}|^{-3}$ (where $(p, q) \neq (0, 0)$) is summable; indeed, we observe that $|w_{p,q}| \geqslant k(|p| + |q|)$ where k is a number > 0 (use the fact that $|x\alpha + y\beta|$ and $|x| + |y|$ are two norms on $\mathbf{C}$). Therefore

$$\sum |w_{p,q}|^{-3} \leqslant \sum k^{-3}(|p| + |q|)^{-3} \leqslant 4k^{-3} \sum_{p,q \geqslant 0} (p+q)^{-3}.$$

Since the family of numbers $(p + q)^{-3}$, where p, $q \in \mathbf{N}$ and $(p, q) \neq (0, 0)$, is summable (see 8.7), the family of numbers $|w_{p,q}|^{-3}$ is summable.

Thus the family $(a_{p,q})$, after the removal of a finite number of terms, is normwise convergent on the disk $|z| < \rho$; since the $a_{p,q}$ under consideration are continuous on this disk (and even holomorphic*), their sum is continuous (and even holomorphic) on this disk.

The sum f of all the $a_{p,q}$ is therefore holomorphic in $\mathbf{C} \dot{-} \mathrm{P}$, and more precisely it is meromorphic* in $\mathbf{C}$, with P the set of poles, and the singular part of f at the pole $w_{p,q}$ is $(z - w_{p,q})^{-2}$.

This function f plays an important role in the theory of doubly periodic meromorphic functions.

12. MULTIPLIABLE FAMILIES AND INFINITE PRODUCTS OF COMPLEX NUMBERS

Summable families and series in a separated and commutative topological group have been defined in Sections 9 and 10; and we remarked that when the group operation is written multiplicatively, they are referred to as multipliable families and infinite products.

This is the case, in particular, when the group G is the multiplicative group $\mathbf{C}^*$ of nonzero complex numbers, taken with the topology induced by that of $\mathbf{C}$.

Since the theory which was developed for general groups G applies to $\mathbf{C}^*$, these general results have immediate counterparts in $\mathbf{C}^*$. Nevertheless, the fact that the essential applications of multipliable families of complex numbers will be in the theory of holomorphic functions leads us to slightly modify the definition of multipliable families and infinite products by no longer excluding the value 0 as a possible product. This extension will in addition simplify the discussion by permitting us to use the fact that $\mathbf{C}$ is a complete metric space.

Let $(a_i)_{i \in \mathrm{I}}$ be a family of elements of $\mathbf{C}$; we shall use the notations $\mathscr{F}$ and $\mathscr{B}$ which appear in Definition 8.1, and for every $\mathrm{J} \in \mathscr{F}$ we shall write p_J for the finite product $\prod_{i \in \mathrm{J}} a_i$.

12.1. Definition. A FAMILY $(a_i)_{i \in \mathrm{I}}$ OF ELEMENTS OF $\mathbf{C}$ IS SAID TO BE *MULTIPLIABLE IN* $\mathbf{C}$, WITH PRODUCT p, IF FOR EVERY $\epsilon > 0$ THERE EXISTS $\mathrm{J}_0 \in \mathscr{F}$ SUCH THAT $|p - p_\mathrm{J}| \leqslant \epsilon$ FOR EVERY $\mathrm{J} \in \mathscr{F}$ CONTAINING J_0.

THE PRODUCT p IS WRITTEN AS $\prod_{i \in \mathrm{I}} a_i$, OR $\prod_i a_i$, OR $\prod a_i$.

IF IN ADDITION $p \neq 0$, THE FAMILY $(a_i)_{i \in \mathrm{I}}$ IS SAID TO BE MULTIPLIABLE IN $\mathbf{C}^*$.

* This allusion to holomorphic and meromorphic functions is intended for students who have had a course in the theory of functions of a complex variable.

In other words, the family (a_i) is multipliable with product p if the p_J converge to p along the filter base $\mathscr{B}$.

The uniqueness of p follows from the fact that the topology of $\mathbf{C}$ is separated.

We note that if $p \neq 0$, then $a_i \neq 0$ for every i; Definition 12.1 then coincides with the definition of multipliability in the topological group $\mathbf{C}^*$ (whence the terminology adopted in 12.1).

We can similarly define the convergence of an infinite product with general term a_n (where $n \in \mathbf{N}$ or $\mathbf{N}^*$) by the convergence of the sequence of partial products $p_n = \prod_{k \leqslant n} a_k$ to p.

12.2. Definition. A FAMILY $(a_i)_{i \in I}$ OF ELEMENTS OF $\mathbf{C}$ IS SAID TO SATISFY THE *CAUCHY CRITERION* (FOR *MULTIPLICATION*) IF FOR EVERY $\epsilon > 0$ THERE EXISTS $J_0 \in \mathscr{F}$ SUCH THAT $|1 - p_K| \leqslant \epsilon$ FOR EVERY $K \in \mathscr{F}$ DISJOINT FROM J_0.

We remark that when all the a_i are $\neq 0$, this criterion is identical with that of 9.4 for the multiplicative group $\mathbf{C}^*$.

When a family (a_i) satisfies the Cauchy criterion, for every $\epsilon > 0$ the number of i such that $|1 - a_i| > \epsilon$ is finite; in particular, $a_i = 0$ for only a finite number of indices i.

It is immediate that every subfamily of a family satisfying the Cauchy criterion also satisfies it.

12.3. Lemma. *When a family (a_i) satisfies the Cauchy criterion, the set of finite products p_J is bounded in $\mathbf{C}$.*

PROOF. With the notation of Definition 12.2, every $J \in \mathscr{F}$ can be written as $J = J_0' \cup K$, where $J_0' \subset J_0$ and $K \cap J_0 = \emptyset$. Put $h = \sup |p_{J_0'}|$ where $J_0' \subset J_0$; since J_0 is finite, there are finitely many J_0', and so $h < \infty$.

We therefore have

$$|p_J| = |p_{J_0'}|\,|p_K| \leqslant h(1 + \epsilon).$$

12.4. Proposition. *Let (a_i) be a family of elements of $\mathbf{C}$.*

1. *If this family is multipliable in $\mathbf{C}^*$, it satisfies the Cauchy criterion (for multiplication).*

2. *Conversely, if it satisfies the Cauchy criterion, it is multipliable in $\mathbf{C}$; if in addition $a_i \neq 0$ for every i, it is multipliable in $\mathbf{C}^*$.*

PROOF. 1. If the family (a_i) is multipliable in $\mathbf{C}^*$, the general theory of Section 9 is applicable; therefore Corollary 9.6 shows that (a_i) satisfies the Cauchy criterion.

2. Put $k = \sup |p_J|$ (where $J \in \mathscr{F}$); Lemma 12.3 shows that $k < \infty$. Now put for every $J_0 \in \mathscr{F}$,

$$\epsilon(J_0) = \sup_{J_0 \subset J} |1 - p_{J-J_0}|.$$

Then for every $J \in \mathscr{F}$ containing J_0 we have, with $K = J \dot{-} J_0$,

$$|p_{J_0} - p_J| = |p_{J_0}(1 - p_K)| \leqslant k\epsilon(J_0).$$

Therefore the set $\mathscr{A}(J_0)$ of p_J such that $J_0 \subset J$ has diameter in $\mathbf{C}$ not greater than $2k\epsilon(J_0)$. But the Cauchy criterion implies that $\epsilon(J_0)$ tends to 0 along the filter base $\mathscr{B}$; therefore the filter base consisting of the $\mathscr{A}(J_0)$ is a Cauchy filter base in $\mathbf{C}$. Since $\mathbf{C}$ is complete, this filter base converges to a point p of $\mathbf{C}$. Therefore the family (a_i) is multipliable with product p.

If one of the a_i equals 0, the family (a_i) is multipliable with product 0.

If no one of the a_i equals 0, the relation $p_{J_0 \cup K} = p_{J_0} p_K$ (for $J_0 \cap K = \emptyset$) shows that if we take $\epsilon < \frac{1}{2}$ (and if J_0 and K have the meaning adopted in 12.1 and 12.2), then

$$|p_J| \geqslant \tfrac{1}{2} |p_{J_0}| \neq 0.$$

Therefore, since p is the limit of the p_J,

$$|p| \geqslant \tfrac{1}{2} |p_{J_0}|, \qquad \text{or} \qquad p \neq 0.$$

Z A multipliable family which has product 0 does not always satisfy the Cauchy criterion. Example: Every infinite family such that $|a_i| < \frac{1}{2}$ for every i.

12.5. Corollary. *Every subfamily of a family which is multipliable in* $\mathbf{C}^*$ *is multipliable in* $\mathbf{C}^*$.

Indeed, if (a_i) is multipliable in $\mathbf{C}^*$, it satisfies the Cauchy criterion, as does therefore every subfamily. Hence every subfamily is multipliable in $\mathbf{C}$, and indeed in $\mathbf{C}^*$ since no a_i equals 0.

Relation between the multipliability of $(1 - u_i)$ *and the summability of* (u_i)

The fact that for every family (a_i) multipliable in $\mathbf{C}^*$, all the a_i, with the exception of a finite number, are ϵ-close to 1, leads us to put $a_i = 1 + u_i$.

Not only is this form suggestive, but it leads to the following theorem, which forms a bridge between the notions of multipliability and summability.

12.6. Theorem. *Let $(1 + u_i)$ be a family of elements of* $\mathbf{C}$.

1. *If the family (u_i) is summable in $\mathbf{C}$, the family $(1 + u_i)$ is multipliable in $\mathbf{C}$; if in addition $1 + u_i \neq 0$ for every i, its product is in $\mathbf{C}^*$.*

2. *If the family $(1 + u_i)$ is multipliable in $\mathbf{C}^*$, the family (u_i) is summable in $\mathbf{C}$.*

Proof. We put, for every $z \in \mathbf{C}$,

$$f(z) = e^z = \sum_0^\infty z^n/n!\,.$$

The elementary theory of holomorphic functions enables one to establish the following properties, which we shall assume here (see also Problem 144):

(a) The function f (which is called the "exponential") is a continuous representation of the additive group $\mathbf{C}$ in the multiplicative group $\mathbf{C}^*$.

(b) If we put $V = \{z : |z| \leqslant \frac{1}{2}\}$, the set $W = f(V)$ is a closed neighborhood of 1, and the restriction of f to V is a homeomorphism of V with $f(V)$.

We shall also use the following relation, whose verification is elementary:

$$\tfrac{1}{2}|\alpha| \leqslant |f(\alpha) - 1| \leqslant 2\alpha \qquad \text{for every} \quad \alpha \in V.$$

Taking account of Corollary 9.20, this relation proves the following property:

(c) For every family (α_i) of elements of V, the summability of (α_i) and that of $(f(\alpha_i) - 1)$ are equivalent.

1. Suppose all the $1 + u_i$ belong to W, and denote by α_i the element of V such that $f(\alpha_i) = 1 + u_i$; property (c) shows that if the family (u_i) is summable, so is the family (α_i). But then, since f is a continuous representation of $\mathbf{C}$ into $\mathbf{C}^*$, Proposition 9.3 shows that the family of numbers $f(\alpha_i) = 1 + u_i$ is multipliable and that its product is $\neq 0$.

When the u_i form an arbitrary summable family, all the u_i with the exception of a finite number belong to the neighborhood $-1 + W$ of 0; therefore, by what we have just seen, the family of the corresponding $1 + u_i$ is multipliable and its product is $\neq 0$. Thus the original family is multipliable and its product is $\neq 0$ if all the numbers $1 + u_i$ are $\neq 0$.

2. Conversely, let $(1 + u_i)_{i \in I}$ be a family, multipliable in $\mathbf{C}^*$, such that all the finite products p_J belong to W; if p is its product, we then have $p \in W$ since W is closed.

Let α_i be the point of V such that $f(\alpha_i) = 1 + u_i$; let s be the point of V such that $f(s) = p$, and put $s_J = \sum_{i \in J} \alpha_i$ for every $J \in \mathscr{F}$

Property (a) shows that $f(s_J) = p_J$; hence by property (b), the convergence of the p_J to p implies the convergence of the s_J to s. Therefore the family (α_i) is summable with sum s. Finally, property (c) shows that the family of numbers $u_i = f(\alpha_i) - 1$ is summable.

Now suppose that $(1 + u_i)$ is an arbitrary family, multipliable in $\mathbf{C}^*$. It therefore satisfies the Cauchy criterion (see Proposition 12.4); therefore after the removal of a finite number of terms, the remaining family will have all its partial products in W. We have just seen that the corresponding family of u_i is then summable, from which follows the summability of the initial family (u_i).

Commutatively convergent infinite products

Theorem 10.7 shows that in the topological group $\mathbf{C}^*$, to say that an infinite product with general term a_n is commutatively convergent is equivalent to saying that the family (a_n) is multipliable. We are going to extend this result to products in $\mathbf{C}$.

12.7. Proposition. *Let (a_n) be a sequence of elements of* $\mathbf{C}$.

To say that the infinite product of the a_n is commutatively convergent is equivalent to saying that the family (a_n) is multipliable in $\mathbf{C}$. The product of (a_n) is then equal to each of the infinite products $\prod a_{\pi(n)}$.

Proof. 1. If the family (a_n) is multipliable in $\mathbf{C}$ with product p, the same argument as was used in Theorem 10.7 shows that each of the infinite products $\prod a_{\pi(n)}$ converges to p.

2. Suppose, on the other hand, that the family (a_n) is not multipliable in $\mathbf{C}$; this clearly implies that $a_n \neq 0$ for every n. Then there are two possible cases:

(a) The set of partial products p_J is not bounded; an argument analogous to that used for Theorem 10.7 enables us to construct a permutation π of $\mathbf{N}$ such that the infinite product $\prod a_{\pi(n)}$ diverges.

(b) The set of partial products p_J is bounded. Then let k be the supremum of the set $(|p_J|)$.

The point 0 cannot be an adherent point of the set of p_J, as otherwise there would exist, for every $\epsilon > 0$, a $J_0 \in \mathscr{F}$ such that $|p_{J_0}| \leqslant \epsilon$, and we would have $|p_J| \leqslant k\epsilon$ for every $J \in \mathscr{F}$ containing J_0; therefore the family (a_n) would be multipliable with product 0.

But, since (a_n) is not multipliable in $\mathbf{C}^*$, by Theorem 10.7 there exists a permutation π of $\mathbf{N}$ such that the infinite product $\prod a_{\pi(n)}$

diverges in $\mathbf{C}^*$; nor can it converge to 0 since 0 is not an adherent point of the set of p_J. Therefore this infinite product diverges in $\mathbf{C}$.

12.8. Corollary. *Let $(1 + u_n)$ be a sequence of elements of $\mathbf{C}^*$. The following statements are equivalent:*

1. *The series with general term $|u_n|$ is convergent.*

2. *The infinite product with general term $1 + u_n$ is commutatively convergent and its product (unique by 12.7) is $\neq 0$.*

This is a consequence of the statements 12.6, 12.7, and 10.12.

When the sequence $(1 + u_n)$ satisfies one of these equivalent conditions, the infinite product of the numbers $1 + u_n$ is said to be *absolutely convergent*, or *commutatively convergent*.

Conditionally convergent infinite products

Just as for conditionally convergent series, we shall say that an infinite product with general term $1 + u_n$ is conditionally convergent if it is convergent without being commutatively convergent.

One might think, in analogy with Theorem 12.6, that the conditional convergence of the infinite product of the numbers $1 + u_n$ is equivalent to the conditional convergence of the series (u_n); we shall see that this equivalence is approximately fulfilled when the u_n are real, but that this is not so when they are complex.

12.9. Proposition. *Let (u_n) be a sequence of* **real** *numbers with limit 0, and such that $1 + u_n \neq 0$ for every n.*

1. *If the series (u_n) is convergent, the product of the $1 + u_n$ converges to a number $p \in \mathbf{R}$; $p \neq 0$ if $\Sigma u_n^2 < \infty$, and $p = 0$ if $\Sigma u_n^2 = \infty$.*

2. *Conversely, if the product of the $1 + u_n$ converges to a number $p \neq 0$, the series (u_n) converges if $\Sigma u_n^2 < \infty$, and has sum $+\infty$ if $\Sigma u_n^2 = \infty$.*

PROOF. We can assume, after discarding a finite number of terms, that $1 + u_n > 0$ for every n. We are then led to use the isomorphism $x \to \log x$ of the multiplicative group $\mathbf{R}_+^*$ onto the additive group $\mathbf{R}$. If we put $\alpha_n = \log(1 + u_n)$, the convergence of the product of the $1 + u_n$ to p is equivalent to the convergence of the series (α_n) to $\log p$. But

$$\alpha_n = \log(1 + u_n) = u_n - \tfrac{1}{2} u_n^2(1 + \epsilon_n), \qquad \text{where} \quad \lim_{n\to\infty} \epsilon_n = 0.$$

Since the series (u_n^2) is positive, this relation shows that when $\Sigma u_n^2 < \infty$, the convergence of one of the series (u_n), (α_n) implies that of the other.

When the series (u_n) converges and $\Sigma\, u_n^2 = \infty$, then $\Sigma\, \alpha_n = -\infty$; therefore $p = 0$.

Conversely, when the series (α_n) converges and $\Sigma\, u_n^2 = \infty$, we have $\Sigma\, u_n = \infty$.

REMARK. When $p = 0$, which is equivalent to $\Sigma\, \alpha_n = -\infty$, the situation is more complex; it can happen that the series (u_n) converges, diverges to $+\infty$ or to $-\infty$, or diverges in the most complete sense.

12.10. General case. The preceding proof has already shown how the logarithm function comes in; we are going to use it again when the u_n are complex.

Let $1 + u_n$ be the general term of an infinite product, with $\lim u_n = 0$. With the notation used in connection with Theorem 12.6, we have $1 + u_n \in \mathrm{W}$ for every sufficiently large n; therefore $1 + u_n$ is of the form $\exp \alpha_n$, where $\alpha_n \in \mathrm{V}$. Thus if we denote the inverse f^{-1} of f in W by log, then

$$\alpha_n = \log(1 + u_n),$$

and we can assert that *the convergence of the infinite product of the* $1 + u_n$ *in* $\mathbf{C}^*$ *is equivalent to the convergence of the series with general term* $\alpha_n = \log(1 + u_n)$.

The elementary theory of holomorphic functions shows that

$$\alpha = \log(1 + u) = u - \frac{u^2}{2} + \cdots + \frac{(-1)^{n+1}u^n}{n} + \cdots \qquad \text{for} \quad 1 + u \in \mathrm{W}\,.$$

In practice it is this expansion in an infinite series which allows one to conclude whether or not an infinite product converges.

EXAMPLE 1. If the series $(|\, u_n\,|^2)$ converges, the relation

$$\alpha = \log(1 + u) = u - \frac{u^2}{2}(1 + \epsilon(u)), \qquad \text{where} \ \lim_{u \to 0} \epsilon(u) = 0,$$

shows that the series (u_n) and (α_n) converge or diverge simultaneously.

EXAMPLE 2. Here, on the other hand, is an example showing that if the series (u_n^2) is not absolutely convergent, the series (u_n) can converge without the product of the $1 + u_n$ converging in $\mathbf{C}$, or even in $\mathbf{C}$ taken with a point at infinity.

We put $u_n = (-1)^n n^{-1/2}$ or $i(-1)^n n^{-1/2}$ according as the integer part of $\log n$ is even or odd.

One can verify that the series (u_n) is convergent and that the series (u_n^3) is absolutely convergent, and, finally, that the series (u_n^2) diverges and that the sequence of its partial sums s_n has as adherent points all the points of a closed interval (not reducing to a single point of $\mathbf{R}$). Then it follows from the relation

$$\alpha = \log(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3}(1 + \epsilon(u)), \qquad \text{where } \lim_{u \to 0} \epsilon(u) = 0,$$

that the sequence of partial products $\prod_1^n (1 + u_p)$ has as its adherent points in $\mathbf{C}$ the points of a bounded interval belonging to a line passing through 0.

Products and multipliable families of functions

Definition 11.1 for the uniform summability of a family of functions with values in a group G is immediately applicable, together with its consequences, to the case where G is the group $\mathbf{C}^*$.

We are now going to study a sufficient condition for uniform multipliability by formulating it in such a way as to include applications with values in $\mathbf{C}$ rather than in $\mathbf{C}^*$ only.

12.11. Definition. LET X BE AN ARBITRARY SET, AND LET $(u_i)_{i \in \mathrm{I}}$ BE A FAMILY OF MAPPINGS OF X INTO $\mathbf{C}$.

THE FAMILY OF FUNCTIONS $1 + u_i$ IS SAID TO BE *NORMWISE MULTIPLIABLE* IF ALL THE FUNCTIONS u_i ARE BOUNDED AND IF THE FAMILY (u_i) IS NORMWISE SUMMABLE (THAT IS, PUTTING $\| u_i \| = \sup_x | u_i(x)|$, THE FAMILY $(\| u_i \|)$ IS SUMMABLE).

12.12. Proposition. *Let* $(1 + u_i)_{i \in \mathrm{I}}$ *be a normwise multipliable family of mappings of a set* X *into* $\mathbf{C}$.

1. *For every* $x \in \mathrm{X}$ *the family* $(1 + u_i(x))$ *is multipliable in* $\mathbf{C}$, *and its product* $p(x)$ *is zero only if one of the* $1 + u_i(x)$ *is zero.*

2. *The product* p *is the uniform limit on* X *of the finite products* p_{J} *(along the filter base* $\mathscr{B}$ *associated with* I).

PROOF. 1. The first statement is a direct result of Theorem 12.6.

2. Put $k_i = \| u_i \|$; by hypothesis the family (k_i) is summable, hence the family $(1 + k_i)$ is multipliable; let k be its product.

For every $\mathrm{J} \in \mathscr{F}$ and every $x \in \mathrm{X}$ we have

$$| p_{\mathrm{J}}(x)| \leqslant \prod_{i \in \mathrm{J}} (1 + k_i) \leqslant k.$$

On the other hand, for every $\epsilon > 0$ there exists $J_0 \in \mathscr{F}$ such that, for every $K \in \mathscr{F}$ disjoint from J_0,

$$\prod_{i \in K} (1 + k_i) - 1 \leqslant \epsilon, \qquad \text{from which} \qquad \left| \prod_{i \in K} (1 + u_i(x)) - 1 \right| \leqslant \epsilon.$$

This relation implies, by passing to the limit, that for every $J \in \mathscr{F}$ containing J_0,

$$\left| \prod_{i \notin J} (1 + u_i(x)) - 1 \right| \leqslant \epsilon.$$

But we also have

$$p - p_J = p_J(p_{I-J} - 1);$$

therefore $| p(x) - p_J(x) | \leqslant k\epsilon$ for every $J \in \mathscr{F}$ containing J_0, from which follows the uniform convergence of the p_J to p.

12.13. Corollary. *If $(1 + u_i)_{i \in I}$ is a normwise multipliable family of continuous mappings of a topological space X into **C**, the product p of this family is bounded and continuous, and the set of its zeros is the union of the zeros of the factors.*

Example. Let a_n be the mapping of **C** into **C** defined by

$$a_0(z) = z; \qquad a_n(z) = 1 - z^2/n^2 \qquad \text{for every} \quad n > 0.$$

On each disk $\{z : | z | < \rho\}$ we have $| n^{-2}z^2 | \leqslant n^{-2}\rho^2$; therefore the family (a_n) is normwise multipliable on each of these disks. Its product p is the uniform limit on each of these disks of finite products of the a_n, therefore of polynomials in z. Therefore p is a holomorphic function on **C**; the set of its zeros is **Z**, and each of these zeros is simple.

13. NORMED ALGEBRAS

Let us recall that an *algebra* over the field **K** is a ring A together with a rule for associating, with every $(\lambda, x) \in \mathbf{K} \times A$, an element λx of A in such a way that:

1. A is a vector space over **K** when taken with this rule.
2. $\lambda(xy) = (\lambda x)y = x(\lambda y)$.

The algebra A is said to be commutative when the multiplication in A is commutative.

We are now going to study algebras over **K** which have a norm compatible with the multiplication in A, in a sense which we are going to specify.

13.1. Definition. AN ALGEBRA A OVER THE FIELD **K** IS CALLED A *NORMED ALGEBRA* OVER **K** IF IT HAS A NORM (ON A CONSIDERED AS A VECTOR SPACE) SUCH THAT

$$\| xy \| \leqslant \| x \| \| y \| \qquad \text{FOR ALL} \quad x, y \in A.$$

A NORMED ALGEBRA A IS SAID TO BE *COMPLETE* (OR IS CALLED A *BANACH ALGEBRA*) IF THE NORMED VECTOR SPACE A IS COMPLETE.

Examples of normed algebras

13.2. **C** taken with the norm $\| x \| = | x |$ is a Banach algebra.

13.3. The algebra $\mathscr{M}^{(n)}$ of nth order square matrices (x_{ij}) over **K** is a normed algebra for the norm $\| \mathrm{M} \| = \Sigma_{i,j} | x_{ij} |$, where the x_{ij} are the elements of the matrix M.

It is easily verified that this norm is compatible with multiplication. This algebra is complete since the dimension of the vector space $\mathscr{M}^{(n)}$ is finite (in fact n^2).

13.4. Let E be a normed vector space; the vector space $\mathscr{L}(\mathrm{E})$ of endomorphisms of E (that is, continuous linear mappings of E into E) becomes an algebra when it is provided with the product $f \circ g$ defined by $f \circ g(x) = f(g(x))$.

The norm which we have taken on $\mathscr{L}(\mathrm{E})$ (Proposition 4.6) satisfies the relation

$$\| f \circ g \| \leqslant \| f \| \| g \|.$$

Therefore $\mathscr{L}(\mathrm{E})$, taken with this norm, is a normed algebra. When E is complete, this algebra is complete (see Proposition 4.7).

13.5. The vector space $\mathscr{B}(\mathrm{X}, \mathbf{C})$ of bounded mappings of a set X into **C** is an algebra under the ordinary multiplication defined by

$$fg(x) = f(x)g(x).$$

The norm of uniform convergence on this space satisfies the condition $\| fg \| \leqslant \| f \| \| g \|$.

This normed algebra is complete.

13.6. When X is a topological space, the subset of $\mathscr{B}(\mathrm{X}, \mathbf{C})$ consisting of the continuous functions is a subalgebra of $\mathscr{B}(\mathrm{X}, \mathbf{C})$; it is closed in the latter, and hence complete.

With every subset Y of X one can associate the subset of $\mathscr{B}(\mathrm{X}, \mathbf{C})$ consisting of the continuous functions which vanish on Y; this is a complete subalgebra of $\mathscr{B}(\mathrm{X}, \mathbf{C})$.

13.7. The vector space $\mathscr{D}^0(\mathbf{R}, \mathbf{R})$ of continuous numerical functions with compact support can be taken as an algebra with the ordinary product (defined by $fg(x) = f(x)g(x)$). But it can also be taken as an algebra in another way, thanks to the *convolution product* $f * g$ defined by

$$f * g(x) = \int_{\mathbf{R}} f(x - t)g(t)\, dt.$$

One can verify that $f * g$ belongs to $\mathscr{D}^0(\mathbf{R}, \mathbf{R})$, that the convolution product is associative and commutative, and that the mapping $(f, g) \rightarrow f * g$ is bilinear.

If we put

$$\| f \| = \int_{\mathbf{R}} | f(t) |\, dt,$$

the vector space $\mathscr{D}^0(\mathbf{R}, \mathbf{R})$ becomes a normed space, which is, incidentally, incomplete; let us verify that this norm is compatible with the convolution product, that is, that $\| f * g \| \leqslant \| f \| \, \| g \|$:

$$\| f * g \| \leqslant \int_{\mathbf{R}^2} | f(x - t)g(t) |\, dt\, dx = \int_{\mathbf{R}^2} | f(u)g(v) |\, du\, dv = \| f \| \, \| g \|.$$

13.8. Here is a similar example, in which $\mathbf{R}$ is replaced by $\mathbf{Z}$ and the Lebesgue measure on $\mathbf{R}$ is replaced by the measure m on $\mathbf{Z}$ for which $m(\mathrm{X}) =$ number of points of X (for every finite $\mathrm{X} \subset \mathbf{Z}$):

Let A be the vector space of mappings $a : n \rightarrow a_n$ of $\mathbf{Z}$ into $\mathbf{R}$ such that $\Sigma | a_n | < \infty$; if we put $\| a \| = \Sigma | a_n |$, A becomes a normed space, which is complete by Proposition 9.24.

For every $a, b \in \mathrm{A}$ we define the convolution product $c = a * b$ by the relation

$$c_n = \sum_{i+j=n} a_i b_j .$$

Since the families (a_i) and (b_j) are absolutely summable, the same is true of the family of products $a_i b_j$, and therefore of each of its subfamilies. This shows, on the one hand, that c_n is well defined for every n, and, on the other hand, that

$$\sum |c_n| \leqslant \left(\sum |a_n|\right)\left(\sum |b_n|\right).$$

In other words, $a * b$ is indeed an element of A, and $\| a * b \| \leqslant \| a \| \| b \|$. Finally, the convolution product is associative (and commutative), and the mapping $(a, b) \to a * b$ is bilinear. We have therefore defined the structure of a complete normed algebra on A.

This algebra A contains, as a closed subalgebra, the space l^1 of summable families (a_n) such that $a_n = 0$ for every $n < 0$.

13.9. Let Δ be the closed disk $\{z : |z| \leqslant 1\}$ of $\mathbf{C}$, and let $\mathscr{H}(\Delta)$ be the subspace of $\mathscr{C}(\Delta, \mathbf{C})$ consisting of the functions which are holomorphic in the interior of Δ. This is an important subalgebra of the Banach algebra $\mathscr{C}(\Delta, \mathbf{C})$; the fact that every uniform limit of holomorphic functions is holomorphic implies that this algebra is closed in $\mathscr{C}(\Delta, \mathbf{C})$ and therefore complete.

13.10. The vector space $\mathscr{C}^1([0, 1], \mathbf{R})$ of numerical functions on $[0, 1]$ which have a continuous first derivative is an algebra when taken with the ordinary product.

An elementary calculation shows that the norm

$$\|f\| = \sup_x |f(x)| + \sup_x |f'(x)|$$

(for which this space is complete) is compatible with the product defined on this algebra.

13.11. Remark. Every normed algebra over $\mathbf{C}$ is also a normed algebra over $\mathbf{R}$; these two algebras are simultaneously complete or incomplete.

13.12. Remark. (a) The relation $\| xy \| \leqslant \| x \| \| y \|$ implies, by induction, $\| x^n \| \leqslant \| x \|^n$.

(b) Since multiplication in A is continuous, the mapping $x \to xx = x^2$ of A into A is continuous; more generally, an induction argument shows that the mapping $x \to x^n$ is continuous for every integer $n \geqslant 1$.

13.13. Remark. Let A be a normed vector space, and suppose that A is provided with a multiplication which makes A an algebra. It is not necessarily a normed algebra; but if the multiplication is continuous, that is, if the mapping $(x, y) \to xy$ of $A \times A$ into A is continuous, it follows from Proposition 6.1 that there exists a constant $k > 0$ such that

$$\| xy \| \leqslant k \| x \| \| y \|.$$

If we put $p(x) = k \| x \|$, then p is a norm on A which is equivalent to the original norm, and which moreover satisfies

$$p(xy) = k\| xy \| \leqslant k^2 \| x \| \| y \| = p(x)p(y).$$

Therefore the new norm is compatible with multiplication.

For example, if in the algebra $\mathscr{M}^{(n)}$ of square matrices of order n we put $\| \mathrm{M} \| = \sup_{i,j} | x_{ij} |$, the norm obtained is not compatible with multiplication, but its product by n is.

Norm of the unit

13.14. Recall that the *unit* of an algebra A is any element e of A such that

$$ex = xe = x$$

for every $x \in \mathrm{A}$.

If one such element e exists, it is unique; but such an element need not exist. One can determine, in the preceding examples, those cases in which a unit exists.

When a normed algebra A has a unit e, the relation $e = ee$ implies $\| e \| \leqslant \| e \|^2$, from which $\| e \| \geqslant 1$. In most of the preceding examples, $\| e \| = 1$; nevertheless it can happen, as in Example 13.3, that $\| e \| \neq 1$. But we shall see that one can then replace the norm on A by an equivalent (in the sense of the equivalence of norms on a vector space) norm p which is still compatible with multiplication, and such that $p(e) = 1$.

For every $a \in \mathrm{A}$, let $\bar{a}$ denote the linear mapping $x \to ax$ of A into A; $\bar{a}$ is an element of $\mathscr{L}(\mathrm{A})$ since $\| ax \| \leqslant \| a \| \| x \|$, and the mapping $a \to \bar{a}$ of A into $\mathscr{L}(\mathrm{A})$ is linear; therefore if we put $p(a) = \| \bar{a} \|$, p is a seminorm on A.

The relation $\| ax \| \leqslant \| a \| \| x \|$ shows that

$$p(a) = \| \bar{a} \| \leqslant \| a \|;$$

but on the other hand, since $\bar{a}(e) = ae$, we have

$$p(a) \geqslant \| e \|^{-1} \| ae \| = \| e \|^{-1} \| a \|,$$

from which

$$\| e \|^{-1} \| a \| \leqslant p(a) \leqslant \| a \|.$$

This relation shows that p is a norm on A which is equivalent to the original norm. The algebra A taken with this norm is isomorphic to the subalgebra of $\mathscr{L}(A)$ consisting of the elements $\bar{a}$; therefore $p(ab) \leqslant p(a)p(b)$, and $p(e) = 1$.

We note, finally, that there can exist, on a given algebra, several equivalent norms which are compatible with multiplication and such that $\| e \| = 1$; for example, if E denotes a finite-dimensional vector space, the norm which we have taken on the algebra $A = \mathscr{L}(E)$ satisfies the required conditions, and it varies with the norm chosen on E.

Product of two absolutely summable families

13.15. Proposition. *If $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ are two absolutely summable families of elements of a Banach algebra, with respective sums α and β, the family of products $(a_i b_j)$ is also absolutely summable, and has sum $\alpha\beta$.*

This result, which generalizes Corollary 9.22, follows as does the latter from Proposition 9.21, since the mapping $(x, y) \to xy$ of $A \times A$ into A is on the one hand bilinear, and on the other hand continuous since $\| xy \| \leqslant \| x \| \| y \|$.

Power series in a Banach algebra

13.16. One can add and multiply in an algebra A over **K**; thus, with every formal polynomial P(X) with coefficients in **K** one can associate the mapping $x \to P(x)$ of A into A. If A is a normed algebra, one can carry out passage to the limit, and therefore one can expect to be able to define the sum of a power series with coefficients in **K** (and even with coefficients in A when A is commutative).

If $a_0 + a_1X + \cdots + a_nX^n + \cdots$ is a formal series in X, with coefficients in **K**, we shall define the *radius of convergence* of this formal series as the element ρ of $\bar{\mathbf{R}}_+$ defined by

$$1/\rho = \overline{\lim_{n \to \infty}} \mid a_n \mid^{1/n}.$$

This definition is justified by the well-known fact that if x denotes an element of **K**, the series (a_nx^n) converges when $\mid x \mid < \rho$, and diverges when $\mid x \mid > \rho$.

We are going to rederive various of these conclusions for Banach algebras. All the algebras with which we shall be concerned from now on will be assumed to have units, denoted by e, and we will identify the element λ of **K** with the element λe of A.

13.17. Proposition. *Let* A *be a Banach algebra over* **K**, *and let* $(a_n X^n)$ *be a formal power series (over* **K**), *with radius of convergence* $\rho > 0$.

For every positive number $r < \rho$, *the series* $(a_n x^n)$ *is convergent and normwise convergent in the ball* $\{x : \| x \| \leqslant r\}$. *Its sum* $f(x)$ *is continuous on the open ball* $\{x : \| x \| < \rho\}$.

PROOF. Let r' be any number such that $r < r' < \rho$. For every n sufficiently large,

$$| a_n |^{1/n} \leqslant 1/r';$$

therefore if $\| x \| \leqslant r$, we have

$$\| a_n x^n \| \leqslant | a_n | r^n \leqslant (r/r')^n.$$

Since the geometric series with general term $(r/r')^n$ is convergent, the series $(a_n x^n)$ is normwise convergent in the ball $\{x : \| x \| \leqslant r\}$, and since A is complete, the series is convergent.

Each of the terms $a_n x^n$ of the series is continuous; therefore the normwise convergence implies that the sum f of this series is continuous in each open ball $\{x : \| x \| < r\}$, hence also in the open ball

$$\{x : \| x \| < \rho\}.$$

13.18. REMARK. Contrary to the situation in the classical case where A is the field **C** itself, we cannot assert that when $\| x \| > \rho$, the series $(a_n x^n)$ diverges; this is essentially tied up with the fact that one can have $\| x^n \| < \| x \|^n$, while in **C** one has $\| x^n \| = \| x \|^n$.

For example, if there exists a $u \neq 0$ such that $u^n = 0$ for every $n \geqslant 2$ (which, in particular, can happen in the algebra $\mathscr{M}^{(2)}$ of square matrices of order 2), the series $(a_n x^n)$ clearly converges when $x = ku$, even when $\| ku \| > \rho$.

13.19. Proposition. *Let* $(a_n X^n)$ *and* $(b_n X^n)$ *be two formal power series in* X, *with radii of convergence* α *and* β, *and let* $(c_n X^n)$ *be their formal product (defined by* $c_n = \sum_{p+q=n} a_p b_q$).

If A *is a Banach algebra, the series* $(c_n x^n)$ *is absolutely convergent for every* $x \in A$ *such that* $\| x \| < \inf(\alpha, \beta)$, *and the sums* A$(x)$, B$(x)$, C$(x)$ *of these series satisfy the relation*

$$C(x) = A(x)B(x) = B(x)A(x).$$

PROOF. If $\| x \| < \alpha$ and β, each of the series $(a_n x^n)$ and $(b_n x^n)$ is absolutely convergent; therefore (see Proposition 13.15) the family of products $a_p b_q x^{p+q}$ is absolutely summable and its sum is equal to $A(x)B(x)$; since $a_p b_q x^{p+q} = b_q a_p x^{q+p}$, this sum is also equal to $B(x)A(x)$. The associativity of the sum then permits us to group the set of terms $a_p b_q x^{p+q}$ of degree $p + q = n$, whose sum is simply $c_n x^n$, which is the desired result.

13.20. Corollary. *Let* A *be a Banach algebra. For every* $x \in A$ *such that* $\| x \| < 1$, *the element* $e - x$ *is invertible and its inverse is the element* $e + x + \cdots + x^n + \cdots$.

Indeed, let us take for the series $(a_n X^n)$ the polynomial $1 - X$, and for the series $(b_n X^n)$ the series $1 + X + \cdots + X^n + \cdots$; the formal product of these two series is the constant 1.

On the other hand, $\alpha = +\infty$ and $\beta = 1$; hence the corollary follows from Proposition 13.19.

The exponential on a Banach algebra

The formal series $(X^n/n!)$ has radius of convergence $+\infty$. Therefore if A is a Banach algebra, the series $(x^n/n!)$ is absolutely convergent on all of A; its sum is denoted by $\exp x$ or e^x, and the mapping $x \to \exp x$ is called the *exponential*; it is continuous.

13.21. Proposition. *If* x *and* y *are any two* **permutable** *elements of* A, *then*

$$\exp(x + y) = (\exp x)(\exp y) = (\exp y)(\exp x).$$

PROOF. Since the families $(x^p/p!)$ and $(y^q/q!)$ are absolutely summable, the same is true of the family $(x^p y^q/p!q!)$, and its sum is $(\exp x)(\exp y)$; since $x^p y^q = y^q x^p$, this sum is also equal to $(\exp y)(\exp x)$.

Moreover, the commutativity of x and y implies, for every $n \geqslant 0$,

$$\sum_{p+q=n} \frac{x^p y^q}{p!q!} = \frac{1}{n!} \sum_{p+q=n} \frac{(p+q)!}{p!q!} x^p y^q = \frac{(x+y)^n}{n!}.$$

The associativity of the sum of the $x^p y^q/p!q!$ thus allows us to write

$$(\exp x)(\exp y) = \sum_n \frac{(x+y)^n}{n!} = \exp(x + y).$$

13.22. Corollary. *For every* $x \in A$ *and for all* $\lambda, \mu \in \mathbf{K}$, *we have*

$$\exp((\lambda + \mu)x) = (\exp \lambda x)(\exp \mu x).$$

13.23. Corollary. *For every* $x \in \mathrm{A}$, $\exp x$ *is invertible and its inverse is* $\exp(-x)$.

The group of invertible elements of a Banach algebra

As in every ring, the set of invertible elements of an algebra A forms a group G under multiplication, whose identity element is the unit in A. When A is a Banach algebra, the structure of G can be further specified.

13.24. Proposition. *Let* A *be a Banach algebra.*

1. *The group* G *of invertible elements of* A *is an open set in* A.

2. *The topology on* G *induced by that of* A *is compatible with the group structure of* G.

Proof. 1. By Corollary 13.20, G contains the open ball V with center e and radius 1. Now for every $a \in \mathrm{G}$, the mapping $x \to ax$ is a homeomorphism of A with itself; therefore $a\mathrm{V}$ is an open set containing a. But all the elements of $a\mathrm{V}$ are invertible, hence $a\mathrm{V} \subset \mathrm{G}$. Thus G is the union of the open sets $a\mathrm{V}$, and is therefore open.

2. Multiplication is continuous in A; it therefore remains to show that the mapping $x \to x^{-1}$ is continuous in G.

Corollary 13.20 shows that this mapping is continuous at the point $x = e$; to show its continuity at every point $a \in \mathrm{G}$, we write x^{-1} in the form $x^{-1} = a^{-1}v$, where $v = u^{-1}$ and $u = xa^{-1}$. The mapping $x \to u$ is continuous and $u(a) = e$; the mapping $u \to v$ is continuous at $u = e$, and the mapping $v \to a^{-1}v$ is continuous; therefore the mapping $x \to x^{-1}$ is continuous at the point $x = a$.

Example. Let E be a complete normed space; then the algebra $\mathscr{L}(\mathrm{E})$ is complete. Therefore the set of invertible elements of $\mathscr{L}(\mathrm{E})$ is open. We have thus rederived a result pointed out during the study of the stability of isomorphisms of E onto itself.

IV. HILBERT SPACES

The first normed vector spaces which came to the attention of mathematicians, apart from the Euclidean spaces $\mathbf{R}^n$, were those whose norm resembles the Euclidean norm of $\mathbf{R}^n$. These are the Hilbert spaces, named after the man who gave the first examples of these spaces and obtained important applications of them to Analysis.

The first mathematicians who used these spaces appreciated above all the ease of calculation and the great analogy between the geometry of these spaces and that of the finite-dimensional Euclidean spaces. But interest in these spaces has not diminished, and at present they are still the spaces most used in functional analysis and theoretical physics.

14. DEFINITION AND ELEMENTARY PROPERTIES OF PREHILBERT SPACES

Hermitean forms

14.1. The Euclidean norm of $\mathbf{R}^3$ is related to the traditional scalar product $(x \mid y) = \Sigma x_i y_i$ by the equality $\| x \|^2 = (x \mid x)$. If one wishes to introduce analogous notions in $\mathbf{C}^3$ by putting $(x \mid y) = \Sigma x_i y_i$, one runs into two difficulties: On the one hand, there exist $x \neq 0$ in $\mathbf{C}^3$ for which $(x \mid x) = 0$; and on the other hand, $(x \mid x)$ is not positive. Therefore the bilinear form $(x \mid y)$ on $\mathbf{C}^3$ cannot be used to define a norm on $\mathbf{C}^3$.

But if we observe that the classical norm of $\mathbf{C}$ has a square which can be written as $\| x \|^2 = x\bar{x}$, it becomes natural to define a norm on $\mathbf{C}^3$ by putting $\| x \|^2 = \Sigma x_i \bar{x}_i$; if we agree to call the function $(x \mid y) = \Sigma x_i \bar{y}_i$ the scalar product on $\mathbf{C}^3$, we obtain the convenient relation $\| x \|^2 = (x \mid x)$.

The new scalar product $(x \mid y)$ is still linear with respect to x, but it is conjugate-linear with respect to y (a mapping f of one vector space into another is said to be *conjugate-linear* if it is additive and if $f(\lambda x) = \bar{\lambda} f(x)$ for every $\lambda \in \mathbf{K}$).

We are going to systematically study such scalar products.

14.2. Definition. Let E be a vector space over K. By a *Hermitean form* on E is meant a mapping φ of $E \times E$ into $\mathbf{K}$ such that:

1. For every $y \in E$, the mapping $x \to \varphi(x, y)$ is a linear functional on E;

2. For all $x, y \in E$ we have $\varphi(y, x) = \overline{\varphi(x, y)}$.

Condition 2 is called *Hermitean symmetry*; it implies that $\varphi(x, x)$ is real; when $\mathbf{K} = \mathbf{R}$, it is simply ordinary symmetry.

Properties 1 and 2 imply

$$\begin{aligned} \varphi(x, y_1 + y_2) = \overline{\varphi(y_1 + y_2, x)} &= \overline{\varphi(y_1, x)} + \overline{\varphi(y_2, x)} \\ &= \varphi(x, y_1) + \varphi(x, y_2); \end{aligned}$$

$$\varphi(x, \lambda y) = \overline{\varphi(\lambda y, x)} = \bar{\lambda}\, \overline{\varphi(y, x)} = \bar{\lambda}\, \varphi(x, y).$$

Thus φ is conjugate-linear with respect to y.

EXAMPLE 1. We are going to determine every Hermitean form on an n-dimensional space E over **K**: Let (a_i) be a basis of E, and let (x_i), (y_i) denote the coordinates of two points x, y of E with respect to this basis.

If φ is a Hermitean form on E, then

$$\varphi(x, y) = \varphi\left(\sum x_i a_i, \sum y_i a_i\right) = \sum x_i \bar{y}_j \varphi(a_i, a_j).$$

If we put $\alpha_{ij} = \varphi(a_i, a_j)$, the Hermitean symmetry of φ implies $\alpha_{ij} = \overline{\alpha_{ji}}$. Conversely, for every system of n^2 complex numbers α_{ij} such that $\alpha_{ij} = \overline{\alpha_{ji}}$, the function $\varphi(x, y) = \sum \alpha_{ij} x_i \bar{y}_j$ is a Hermitean form on E.

EXAMPLE 2. Let E be the vector space $\mathscr{C}^1([0, 1], \mathbf{K})$ and let α and β be continuous *real-valued* functions on $[0, 1]$.

For every $x, y \in$ E we put

$$\varphi(x, y) = \int_0^1 (\alpha(t)x(t)\overline{y(t)} + \beta(t)x'(t)\overline{y'(t)})\,dt.$$

The function φ is clearly linear with respect to x and is Hermitean symmetric; therefore φ is a Hermitean form on E.

14.3. Definition. A HERMITEAN FORM φ ON E IS SAID TO BE *POSITIVE* IF $\varphi(x, x) \geqslant 0$ FOR EVERY $x \in$ E; IT IS SAID TO BE *POSITIVE DEFINITE* OR *POSITIVE AND NONDEGENERATE* IF MOREOVER $\varphi(x, x) > 0$ FOR EVERY $x \neq$ O.

It is evident that every linear combination with positive coefficients of positive Hermitean forms is itself positive and Hermitean.

EXAMPLES. In Example 1 above, if $\alpha_{ij} = 0$ when $i \neq j$, and $\alpha_{ii} \geqslant 0$ for every i, the form φ is positive. If in addition $\alpha_{ii} > 0$ for every i, it is positive definite.

In Example 2, if $\alpha \geqslant 0$ and $\beta \geqslant 0$, the form φ is positive. If in addition $\alpha > 0$ it is positive definite; on the other hand, if $\alpha = 0$ and $\beta > 0$, it is degenerate.

14.4. Proposition. *Let φ be a positive Hermitean form on* E.

1. *For all x, $y \in$ E we have*

$$|\varphi(x, y)|^2 \leqslant \varphi(x, x)\,\varphi(y, y) \qquad \text{(Cauchy–Schwartz inequality).}$$

If φ is positive definite, the equality holds only when x and y are linearly dependent.

2. *The mapping $x \to (\varphi(x, x))^{1/2}$ is a seminorm on* E; *it is a norm when φ is positive definite.*

PROOF. 1. We shall make use of the fact that $\varphi(\lambda x + y, \lambda x + y) \geqslant 0$ for every $\lambda \in \mathbf{K}$; then

$$f(\lambda) = \varphi(\lambda x + y, \lambda x + y) = a\lambda\bar{\lambda} + b\lambda + \overline{b\lambda} + c \geqslant 0, \tag{1}$$

where we have put

$$a = \varphi(x, x), \qquad b = \varphi(x, y), \qquad c = \varphi(y, y).$$

If $a = c = 0$, then setting $\lambda = -\bar{b}$ in (1), we obtain

$$-2b\bar{b} \geqslant 0,$$

from which $b = 0$; therefore $|b|^2 = ac$.

If for example $a \neq 0$, then setting $\lambda = -\bar{b}a^{-1}$ in (1), we obtain

$$\frac{ac - b\bar{b}}{a} \geqslant 0,$$

which yields the desired relation since $a > 0$.

Now suppose that the form φ is positive definite; the relation $f(\lambda) = 0$ implies $\lambda x + y = O$. Now if $|b|^2 = ac$ and $a \neq 0$, the preceding calculation shows that for $\lambda = -\bar{b}a^{-1}$ we have $f(\lambda) = 0$, which implies $\lambda x + y = O$; if $c \neq 0$ we have a similar relation $x + \mu y = O$, while if $a = c = 0$, then $x = y = O$.

Conversely, when x and y are linearly dependent, it is immediate that $|b|^2 = ac$.

2. Put $p(x) = (\varphi(x, x))^{1/2}$; the relation $p(x + y) \leqslant p(x) + p(y)$ can be written, after squaring both sides, as

$$\varphi(x + y, x + y) \leqslant \varphi(x, x) + \varphi(y, y) + 2(\varphi(x, x)\,\varphi(y, y))^{1/2}$$

or

$$2\mathscr{R}\,\varphi(x, y) = \varphi(x, y) + \overline{\varphi(x, y)} \leqslant 2(\varphi(x, x)\,\varphi(y, y))^{1/2},$$

which follows from the Cauchy-Schwartz inequality.

Moreover, the relation $\varphi(\lambda x, \lambda x) = |\lambda|^2 \varphi(x, x)$ shows that

$$p(\lambda x) = |\lambda| p(x);$$

therefore p is a seminorm. It is a norm when $p^2(x) = \varphi(x, x)$ vanishes only for $x = O$, that is, when φ is positive definite.

14.5. *Definition.* A VECTOR SPACE E, TOGETHER WITH A POSITIVE DEFINITE HERMITEAN FORM φ DEFINED ON E AND THE NORM ASSOCIATED WITH φ BY THE RELATION $\| x \|^2 = \varphi(x, x)$, IS CALLED A *PREHILBERT SPACE*.

THE PREHILBERT SPACE E IS CALLED A *HILBERT SPACE* WHEN THE NORMED SPACE E IS COMPLETE.

In general $\varphi(x, y)$ is denoted by $(x \mid y)$, and $(x \mid y)$ is called the *scalar product* of x and y; in the course of a calculation we shall often simplify the notation by writing xy for $(x \mid y)$ and x^2 for $(x \mid x)$.

If F is a vector subspace of E, the trace on F of the scalar product on E is a scalar product on F.

14.6. REMARK. If E is a prehilbert space over **C**, the scalar product $(x \mid y)$ takes nonreal values as well as real values (unless E consists of the point O); therefore $(x \mid y)$ is not a scalar product on E regarded as a vector space over **R**. On the other hand, it is clear that $\mathscr{R}(x \mid y)$ is symmetric in x and y, and linear with respect to x when E is regarded as a vector space over **R**. Moreover $(x \mid x) = \mathscr{R}(x \mid x)$; therefore $\mathscr{R}(x \mid y)$ is a scalar product on the real space E, and the norms associated with $(x \mid y)$ and $\mathscr{R}(x \mid y)$ are identical.

We shall have occasion at times to use the scalar product $\mathscr{R}(x \mid y)$, which we shall call the *real scalar product associated with* $(x \mid y)$, or more simply the real scalar product of E.

Examples of prehilbert spaces

14.7. The vector space $E = \mathscr{C}([0, 1], \mathbf{R})$ taken with the scalar product defined by

$$(x \mid y) = \int_0^1 x(t)y(t)\, dt$$

is a real prehilbert space; we show that it is incomplete.

Put $x_n(t) = \inf(n, t^{-1/3})$; then

$$\| x_n - x_{n+p} \|^2 = \int_0^1 (x_n(t) - x_{n+p}(t))^2\, dt \leqslant \int_0^{\epsilon_n} t^{-2/3}\, dt,$$

where $\epsilon_n = 1/n^3$.

Since $\| x_n - x_{n+p} \|$ tends to 0 as $n \to \infty$, the sequence (x_n) is a Cauchy sequence in E. But it does not converge, as for every $x \in \mathrm{E}$ we have

$$\| x - x_n \|^2 \geqslant \int_{\epsilon_n}^1 (x(t) - x_n(t))^2 \, dt = \int_{\epsilon_n}^1 (x(t) - t^{-1/3})^2 \, dt.$$

Therefore

$$\liminf_{n\to\infty} \| x - x_n \|^2 \geqslant \int_0^1 (x(t) - t^{-1/3})^2 \, dt.$$

But since the function $t^{-1/3}$ is not bounded on $(0, 1]$, there exists a compact interval on which $(x(t) - t^{-1/3})^2 > 0$; therefore the integral of this function is > 0, and the sequence (x_n) cannot have x as a limit.

In fact, the actual limit of the sequence (x_n) would be the function $t^{-1/3}$, which belongs to a Hilbert space containing E, namely the space of square summable functions on $[0, 1]$, which we shall study with the theory of integration.

14.8. We are going to see that the complete normed space l_{I}^2 defined earlier (see 8.9) is a Hilbert space; to show this, it suffices to prove that its norm is associated with a scalar product.

But for all $x, y \in l_{\mathrm{I}}^2$, the inequality $2 \sum | x_i \overline{y_i} | \leqslant \sum (| x_i |^2 + | y_i |^2)$ shows that the family $(x_i \overline{y_i})$ is summable.

We then put $(x \mid y) = \sum x_i \overline{y_i}$; this is evidently a positive Hermitean form on l_{I}^2, and the relation $(x \mid x) = \sum | x_i |^2$ shows that the seminorm associated with this form is simply the norm on l_{I}^2.

The importance of the spaces l_{I}^2 stems from the simple form of the expression defining their scalar product and from the fact, which we will establish later (see 16.13), that every Hilbert space is isomorphic to a space l_{I}^2. When $\mathrm{I} = \{1, 2, ..., n\}$, l_{I}^2 is simply the space $\mathbf{K}^n$, taken with the Euclidean norm if $\mathbf{K} = \mathbf{R}$, and the Hermitean norm if $\mathbf{K} = \mathbf{C}$; when $\mathrm{I} = \mathbf{N}$, l_{I}^2 is simply the Hilbert space l^2.

14.9. Useful inequalities and identities. 1. With the simplified notation introduced in 14.5, the Cauchy-Schwartz inequality is written as

$$| xy |^2 \leqslant x^2 y^2 \quad \text{or} \quad | xy | \leqslant \| x \| \| y \|.$$

From this inequality and the inequality $2ab \leqslant a^2 + b^2$ (where $a, b \in \mathbf{R}$) follows also

$$2| xy | \leqslant x^2 + y^2. \tag{1}$$

Finally, we note the inequality

$$(x + y)^2 \leqslant 2(x^2 + y^2),$$

which is equivalent to the relation $0 \leqslant (x - y)^2$.

2. From the relations

$$(x + y)^2 = x^2 + y^2 + xy + yx \qquad \text{and} \qquad (x - y)^2 = x^2 + y^2 - xy - yx$$

we obtain by addition the following important identity, which involves only the norm:

$$\| x + y \|^2 + \| x - y \|^2 = 2\| x \|^2 + 2\| y \|^2. \tag{2}$$

This result is sometimes expressed by the statement that for any parallelogram the sum of the squares of the diagonals equals the sum of the squares of the sides.

3. Expanding $(x + y)^2$ and $(x - y)^2$, we obtain by subtraction

$$2(xy + yx) = \| x + y \|^2 - \| x - y \|^2. \tag{3}$$

When $\mathbf{K} = \mathbf{R}$, then $xy = yx$, from which

$$4xy = \| x + y \|^2 - \| x - y \|^2.$$

When $\mathbf{K} = \mathbf{C}$, replacing y by iy in (3) yields

$$2(xy - yx) = i(\| x + iy \|^2 - \| x - iy \|^2) \tag{3'}$$

from which, on adding (3) and (3′), we obtain

$$4xy = \| x + y \|^2 - \| x - y \|^2 + i(\| x + iy \|^2 - \| x - iy \|^2). \tag{4}$$

This relation shows that the scalar product of a prehilbert space is determined by its norm.

14.10. Isomorphism of two prehilbert spaces. Let E and F be prehilbert spaces over the same field $\mathbf{K}$. By an *isomorphism of* E *onto* F is meant a vectorial isomorphism f of E onto F which preserves the scalar product, in the sense that

$$(f(x) \mid f(y)) = (x \mid y) \qquad \text{for all} \quad x, y \in \mathrm{E}.$$

Then this isomorphism also preserves the norm; therefore E and F are simultaneously complete or incomplete.

Conversely, if f is a vectorial isomorphism which preserves the norm, then by identity (4) of 14.9 f also preserves the scalar product.

Orthogonal vectors

14.11. Definition. Two *elements* x and y of a prehilbert space E are said to be *orthogonal* if their scalar product $(x \mid y)$ is zero.

Two *subsets* X and Y of E are said to be *orthogonal* if every element of X is orthogonal to every element of Y; this relation is written $X \perp Y$.

If $(x \mid y) = 0$, then $(y \mid x) = \overline{(x \mid y)} = 0$; thus the relation of orthogonality is symmetric.

Since the relation $(x \mid x) = 0$ implies $x = O$, the only vector which is orthogonal to itself is O; this vector is also orthogonal to every $x \in E$.

It follows, for example, that the relation of orthogonality between vector subspaces of E is symmetric, and that the only common point of two orthogonal subspaces is the point O. Therefore the sum of the dimensions of two orthogonal subspaces in $\mathbf{K}^n$ is $\leqslant n$; thus two planes in $\mathbf{R}^3$ are never orthogonal (although they can be in the weaker sense used in elementary geometry).

More generally, two *affine varieties* X and Y in E are said to be *orthogonal* if the vector subspaces in E which are parallel to X and Y, respectively, are orthogonal.

14.12. Proposition (Pythagorean theorem). *Let x and y be vectors in a prehilbert space over* $\mathbf{K}$.

If x and y are orthogonal, then $\| x + y \|^2 = \| x \|^2 + \| y \|^2$. When $\mathbf{K} = \mathbf{R}$, the converse is true.

Proof. The relation $(x + y)^2 = x^2 + y^2 + xy + yx$ shows that if $xy = 0$, which implies $yx = 0$, then $(x + y)^2 = x^2 + y^2$.

Conversely, this last relation implies $xy + yx = 0$; if $\mathbf{K} = \mathbf{R}$, we have in addition $xy = yx$, hence $2xy = 0$, or $xy = 0$.

On the other hand, if $\mathbf{K} = \mathbf{C}$ the relation $(x + y)^2 = x^2 + y^2$ implies only that $2\mathscr{R}(xy) = xy + yx = 0$, from which follows the orthogonality of x and y relative to the *real* scalar product of E.

For example, in $\mathbf{C}$ regarded as a vector space over $\mathbf{C}$, there exists no pair of orthogonal nonzero vectors; still, we have

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2$$

whenever the vectors $x = x_1 + ix_2$, $y = y_1 + iy_2$ satisfy the relation

$$\mathscr{R}(xy) = x_1 y_1 + x_2 y_2 = 0.$$

14.13. Lemma. *In every prehilbert space* E, *the mapping* $(x, y) \to (x \mid y)$ *of* $E \times E$ *into* **R** *is continuous.*

PROOF. Since we have $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, $(x \mid y)$ is always a bilinear form on E regarded as a vector space over **R**. The inequality $|(x \mid y)| \leqslant \|x\| \|y\|$ therefore shows, by Proposition 6.1, that this function is continuous on $E \times E$.

14.14. Proposition. *The set of vectors of* E *orthogonal to a given vector* $a \neq O$ *is a closed hyperplane.*

PROOF. The set in question is the set of zeros of the linear functional $f : x \to (x \mid a)$, and is therefore a hyperplane. On the other hand, f is continuous by Lemma 14.13; therefore the hyperplane $f^{-1}(0)$ is closed.

14.15. Corollary. *The set* X^0 *of vectors in* E *orthogonal to a set* $X \subset E$ *is a closed vector subspace of* E (*possibly the set* $\{O\}$).

This is immediate since every intersection of closed vector subspaces of E is itself a closed vector subspace.

It is evident that $X \subset Y$ implies $X^0 \supset Y^0$; therefore if we denote the set $(X^0)^0$ by X^{00}, we also have $X^{00} \subset Y^{00}$.

We also observe that $X \subset X^{00}$ since $X \perp X^0$.

14.16. Median hyperplane of two points. Let E be a prehilbert space over **R**, and let a, b be distinct points of E. The set X of points $x \in E$ which are equidistant from a and b is defined by the relation

$$(a - x)^2 = (b - x)^2, \qquad \text{or} \qquad x(b - a) = \tfrac{1}{2}(b^2 - a^2).$$

If we put $x = y + \frac{1}{2}(b + a)$, this relation becomes $y(b - a) = 0$. Therefore the set in question is the affine hyperplane orthogonal to $b - a$ and passing through the midpoint of the segment joining a and b.

When E is a prehilbert space over **C**, we can reduce the problem to the preceding case by taking E with its real scalar product; the real hyperplane X then contains the complex affine hyperplane orthogonal to $b - a$ and passing through the midpoint of the segment joining a and b, but is not identical with it.

Angle between two nonnull vectors

It is proved in elementary geometry that if x and y are nonnull vectors in $\mathbf{R}^2$, then

$$(x \mid y) = \|x\| \|y\| \cos \theta,$$

where θ denotes the angle between the halflines passing from the origin O through x and y, respectively.

Now if E is a real prehilbert space, the Cauchy-Schwartz inequality shows that if x and y are nonnull vectors in E, then

$$-1 \leqslant \frac{(x \mid y)}{\| x \| \| y \|} \leqslant 1.$$

On the other hand, the theory of the cosine function, defined as the sum of the series with general term $(-1)^n t^{2n}/(2n)!$, shows that for every $k \in [-1, 1]$ there exists a single real number $\theta \in [0, \pi]$ such that $\cos \theta = k$; we are thus led to the following definition:

14.17. Definition. BY THE *ANGLE* BETWEEN TWO NONNULL VECTORS x, y OF A REAL PREHILBERT SPACE E IS MEANT THE REAL NUMBER θ DEFINED BY THE RELATIONS

$$0 \leqslant \theta \leqslant \pi; \qquad (x \mid y) = \| x \| \| y \| \cos \theta.$$

To say that θ is an acute (right, obtuse) angle is equivalent to saying that $(x \mid y) > 0$ $(= 0, < 0)$.

Proposition 14.4 shows that $| \cos \theta | = 1$ only when x and y are linearly dependent; when $y = \lambda x$ with $\lambda > 0$, $\theta = 0$; when $y = \lambda x$ with $\lambda < 0$, $\theta = \pi$.

In a prehilbert space over $\mathbf{C}$, the only way of introducing the notion of angle is to go back to the real case by using the real scalar product $\mathcal{R}(x \mid y)$; in other words, we shall put

$$\mathcal{R}(x \mid y) = \| x \| \| y \| \cos \theta.$$

By means of this definition we can write, for all $x, y \neq 0$,

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2 + 2\| x \| \| y \| \cos \theta.$$

Z In using angles in a *complex* prehilbert space, it should not be forgotten that the condition $\theta = \pi/2$ does not imply the orthogonality of x and y.

15. ORTHOGONAL PROJECTION. STUDY OF THE DUAL

One is often led, both in theoretical research and applied mathematics, to problems of the following kind:

Let E be a normed space, X a subset of E, and x a point of E; to find a point x' of X which best approximates x, in other words, whose distance from x is equal to the distance $d(x, X)$ from x to X.

Such a point x' will be called the *projection* of x on X.

When X is compact, this problem has at least one solution (see Chapter I, Proposition 17.4); otherwise it may have no solution.

The following theorem answers the question when X is a complete convex subset of a prehilbert space.

15.1. Theorem. *Let* E *be a prehilbert space over* **K**, *and let* X *be a complete convex subset of* E.

1. *Every point* $x \in \mathrm{E}$ *has a unique projection* x' (*which we shall sometimes denote by* $\mathrm{P_X}(x)$) *on* X.

2. *The projection* x' *of* x *is characterized by the condition*

$$\mathscr{R}((x - x') \mid (u - x')) \leqslant 0 \qquad \text{for every} \quad u \in \mathrm{X}.$$

3. For all $x, y \in \mathrm{E}$ with projections x', y' on X, we have

$$\| x' - y' \| \leqslant \| x - y \|.$$

Proof. 1. Put $d = d(x, \mathrm{X})$; for every $\epsilon > 0$ let B_ϵ denote the closed ball with center x and radius $d + \epsilon$, and put $\mathrm{P}_\epsilon = \mathrm{X} \cap \mathrm{B}_\epsilon$. The set of projections of x on X is simply the intersection of the P_ϵ; it is a question of showing that this intersection contains a unique point.

Every P_ϵ is nonempty, and since X and B_ϵ are convex, P_ϵ is convex; hence for all $a, b \in \mathrm{P}_\epsilon$ the midpoint of the segment joining a and b belongs to P_ϵ. But the identity (2) of 14.9 allows us to write

$$\| x - a \|^2 + \| x - b \|^2 = 2\| x - m \|^2 + \tfrac{1}{2}\| a - b \|^2.$$

Since $\| x - a \|$, $\| x - b \|$, and $\| x - m \|$ lie between d and $d + \epsilon$, we thus have

$$\| a - b \|^2 \leqslant 2(2(d + \epsilon)^2 - 2d^2) = 4\epsilon(2d + \epsilon).$$

This inequality gives an upper bound for the diameter of P_ϵ; we see that this diameter tends to 0 with ϵ.

On the other hand, P_ϵ is an increasing function of ϵ, hence the intersection of the P_ϵ is equal to the intersection of the decreasing sequence $(\mathrm{P}_{\epsilon_n})$, where $\epsilon_n = n^{-1}$; but the P_{ϵ_n} are closed in X which is assumed complete. Therefore Proposition 20.6 of Chapter I shows that the intersection of the P_{ϵ_n} contains exactly one point.

2. To simplify the calculations, we reduce the problem to the case $x' = \mathrm{O}$ by a translation. For every $u \in \mathrm{X}$, and every $\lambda \in (0, 1]$, we have $\lambda u \in \mathrm{X}$; therefore if O is the projection of x on X,

$$\| x \|^2 \leqslant \| x - \lambda u \|^2 \qquad \text{or} \qquad 2\mathscr{R}(x \mid u) \leqslant \lambda \| u \|^2.$$

Since λ is arbitrarily small, we have $\mathscr{R}(x \mid u) \leqslant 0$; this is just the desired relation when we take $x' = O$.

Conversely, if $\mathscr{R}(x \mid u) \leqslant 0$ for every $u \in X$, then

$$\| x - u \|^2 = (x - u)^2 = x^2 - 2\mathscr{R}(x \mid u) + u^2 \geqslant x^2 = \| x \|^2;$$

thus O is the projection of x on X.

3. Put $x - y = (x - x') + (x' - y') + (y' - y) = (x' - y') + u$. Then

$$(x - y)^2 = (x' - y')^2 + u^2 + 2\mathscr{R}(u \mid (x' - y')).$$

But

$$u(x' - y') = -(x - x')(y' - x') - (y - y')(x' - y').$$

By the second part of the theorem, the real part of the right side is $\geqslant 0$; therefore $(x - y)^2 \geqslant (x' - y')^2$.

This inequality can be expressed by the statement that the mapping P_X of E onto X is of Lipschitz class with ratio 1.

REMARK. Part 2 of the theorem can be expressed in terms of angles; indeed, the relation $\mathscr{R}((x - x') \mid (u - x')) \leqslant 0$ asserts, when $x \neq x'$ and $u \neq x'$, that the angle between the vectors $x - x'$ and $u - x'$ is $\geqslant \pi/2$.

15.2. Corollary. *Let* X *be a complete convex cone in a prehilbert space* E. *If x' denotes the projection of a point x on* X, *then*

$$\| x \|^2 = \| x' \|^2 + \| x - x' \|^2 \qquad \text{and} \qquad \mathscr{R}(x \mid x') = (x' \mid x').$$

PROOF. Each of these relations is equivalent to the relation $\mathscr{R}(x' \mid x - x') = 0$, which we are going to prove.

Since X is a cone, we have $\lambda x' \in X$ for every $\lambda > 0$; we can therefore put $u = \lambda x'$ in the inequality of part 2 of Theorem 15.1. According as $\lambda < 1$ or $\lambda > 1$, we obtain

$$\mathscr{R}(x - x' \mid x') \geqslant 0 \qquad \text{or} \qquad \mathscr{R}(x - x' \mid x') \leqslant 0,$$

from which the desired equality follows.

15.3. Corollary. *Let* X *be a complete vector subspace of a prehilbert space* E, *and let* $x \in$ E.

The projection x' of x on X *is the only point of* X *such that $x - x'$ is orthogonal to* X.

Proof. We reduce the proof by translation to the case where $x' = O$ (which leaves X invariant).

1. If O is the projection of x on X, then $\mathscr{R}(x \mid u) \leqslant 0$ for every $u \in X$. If v is any point of X, the points $-v$, iv, $-iv$ also belong to X; therefore the relation $\mathscr{R}(x \mid u) \leqslant 0$ is satisfied when $u = v$, $-v$, iv, or $-iv$. It follows that the real part and the imaginary part of $(x \mid v)$ are zero, whence $(x \mid v) = 0$.

2. Conversely, if $(x \mid v) = 0$ for every $v \in X$, we have

$$\| x - v \|^2 = \| x \|^2 + \| v \|^2 \geqslant \| x \|^2;$$

hence O is the projection of x on X.

15.4. Corollary (Transitivity of projections). *Let* E *be a prehilbert space,* F *a complete vector subspace of* E, X *a convex subset of* F, *x a point of* E *and x' its projection on* F.

If x' has a projection x'' on X, *then x'' is also the projection of x on* X, *and conversely.*

Proof. Since $x - x'$ is orthogonal to F, for every $y \in X$ we have

$$\| x - y \|^2 = \| x - x' \|^2 + \| x' - y \|^2.$$

Therefore the functions $\| x - y \|$ and $\| x' - y \|$ of the variable y attain their minimum values simultaneously on X, which proves the corollary.

15.5. Remark. Corollary 15.4 extends by translation to the case where F is a complete affine variety of E.

Therefore if F_1, F_2, ..., F_n is a finite decreasing sequence of complete affine varieties of E, if x' denotes the projection of x on F_1, and if x_p is defined recursively as the projection of x_{p-1} on F_p, the point x_n coincides with the projection of x on F_n. This is an extension of a classical result of elementary geometry in $\mathbf{R}^3$.

Z The statement of 15.4 is not correct if the subspace F of E is replaced by an arbitrary complete convex subset of E.

Orthogonal subspaces

15.6. Proposition. *Let* X *be a complete vector subspace of a prehilbert space* E.

1. *The mapping* P_X *of* E *onto* X *is a continuous linear mapping of norm* 1 (if $X \neq \{O\}$);

2. *The kernel* $P_X^{-1}(O)$ *of this mapping is the subspace* X^0 *orthogonal to* X, *and* $X^{00} = X$;

3. E *is the direct sum of* X *and* X^0, *and for every* $x \in X$ we have

$$\| x \|^2 = \| P_X(x) \|^2 + \| P_{X^0}(x) \|^2.$$

PROOF. 1. For all $x, y \in E$ and for every $u \in X$ we have, by Corollary 15.3,

$$(x - x')u = 0 \qquad \text{and} \qquad (y - y\ )u = 0.$$

Hence for all $\lambda, \mu \in \mathbf{K}$,

$$((\lambda x + \mu y) - (\lambda x' + \mu y'))u = 0.$$

By 15.3, $\lambda x' + \mu y'$ is thus the projection of $\lambda x + \mu y$ on X; in other words, $P_X(\lambda x + \mu y) = \lambda P_X(x) + \mu P_X(y)$, which shows the linearity of P_X.

The relation $\| x \|^2 = \| x' \|^2 + \| x - x' \|^2$ shows that P_X is norm decreasing; on the other hand, the restriction of P_X to X is the identity operator; therefore if $X \neq \{O\}$, $\| P_X \| = 1$.

2. To say that $P_X(x) = O$ is equivalent to saying that $x - O$ is orthogonal to X (by 15.3); therefore $P_X^{-1}(O) = X^0$.

For every x orthogonal to X^0, we have simultaneously

$$x'(x - x') = 0 \qquad \text{and} \qquad x(x - x') = 0,$$

from which $(x - x')^2 = 0$. Hence $x = x'$; in other words, $x \in X$.

This shows that $X^{00} \subset X$; but we already had the opposite inclusion, hence $X^{00} = X$.

3. The relation $x = x' + (x - x')$, where $x - x' \in X^0$, shows that $E = X + X^0$; since the only element common to X and X^0 is O, E is indeed the direct sum of X and X^0.

Finally, the relation $\| x \|^2 = \| P_X(x) \|^2 + \| P_{X^0}(x) \|^2$ follows at once from $x = x' + (x - x')$.

15.7. Corollary. *For every vector subspace* X *of a Hilbert space* E, X^{00} *is the closure of* X.

PROOF. Since X^{00} is closed and $X \subset X^{00}$, we also have $\bar{X} \subset X^{00}$; on the other hand, the relation $X \subset \bar{X}$ implies $X^{00} \subset (\bar{X})^{00} = \bar{X}$, which yields the desired equality.

The dual of a Hilbert space

We have seen in 4.8 that the topological dual $E' = \mathscr{L}(E, \mathbf{K})$ of a normed space E is a complete normed space. We are going to see that when E is a Hilbert space, the normed spaces E and E′ are isomorphic under a remarkable conjugate-linear isomorphism. This is one of the properties which explains the important role of Hilbert spaces.

15.8. Proposition. *Let* E *be a Hilbert space and* E′ *its topological dual.*

1. *For every* $a \in E$ *the linear functional* $\varphi_a : x \to (x \mid a)$ *has norm* $\| a \|$.

2. *The mapping* $a \to \varphi_a$ *of* E *into* E′ *is a conjugate-linear isomorphism of the normed spaces* E *and* E′.

PROOF. 1. The Cauchy-Schwartz inequality shows that $\| \varphi_a \| \leqslant \| a \|$; on the other hand, for $a \neq O$, if $x = \| a \|^{-1} a$, then $(x \mid a) = \| a \|$, hence $\| \varphi_a \| \geqslant \| a \|$; in short, $\| \varphi_a \| = \| a \|$.

2. The mapping $a \to \varphi_a$ of E into E′ is conjugate-linear, that is,

$$\varphi_{a+b} = \varphi_a + \varphi_b\,; \qquad \varphi_{\lambda a} = \bar{\lambda}\varphi_a\,;$$

this is a consequence of the fact that $(x \mid a)$ is conjugate-linear with respect to a.

On the other hand, the mapping φ is norm preserving, therefore injective. It remains for us to show that it is surjective, that is, every $u \in E'$ is of the form φ_a.

If $u = O$, this is obvious, with $a = O$. If $u \neq O$, let X be the hyperplane $u^{-1}(0)$; since $E = X + X^0$ and since $X \neq E$, there exists $b \in X^0$, $b \neq O$. The two linear functionals u and φ_b vanish on the hyperplane X and are not identically zero, hence are proportional; therefore there exists a scalar λ such that $u(x) = \lambda(x \mid b)$ for every x. In other words, $u = \varphi_a$ where $a = \bar{\lambda} b$.

When $\mathbf{K} = \mathbf{R}$, φ is evidently an isomorphism in the ordinary sense.

EXAMPLE 1. Let us take $\mathbf{C}^n$ with the canonical scalar product

$$(x \mid y) = \sum x_i \bar{y}_i\,.$$

Every linear functional u on $\mathbf{C}^n$ is of the form $u(x) = \Sigma\, \alpha_i x_i$, that is, $u(x) = (x \mid a)$, where a is the point with coordinates $\overline{\alpha_i}$.

EXAMPLE 2. Proposition 15.8 shows that every continuous linear functional on the Hilbert space l_{I}^2 can be written in a unique way in the form $x \to \sum x_i \bar{a}_i$, where $\sum |a_i|^2 < \infty$.

15.9. Remark. Proposition 15.8 cannot be extended to any incomplete prehilbert space E since E′, being complete, cannot be isomorphic to an incomplete space E.

For example, let E be the subspace of l^2 consisting of the sequences (x_n) (where $n \geqslant 1$) such that $x_n = 0$ except for a finite number of indices. Let a be the point of l^2 with coordinates $a_n = n^{-1}$; the linear functional on E defined by

$$x \to (x \mid a) = \sum n^{-1} x_n$$

is continuous, but clearly cannot be written in the form $(x \mid b)$, where $b \in \mathrm{E}$.

Weak topology on a Hilbert space

In Example 3.11 we defined the weak topology of a topological vector space E. When E is a Hilbert space, for every $x \in \mathrm{E}$ such that $x \neq \mathrm{O}$ there exists a linear functional $u \in \mathrm{E}'$ such that $u(x) \neq 0$ (for example $u = \varphi_x$); therefore the weak topology of E is separated.

From Proposition 15.8 and the convergence criterion stated in 3.11, we have the following criterion:

15.10. Proposition. *Let* E *be a Hilbert space,* x_0 *a point of* E, *and* $\mathscr{B}$ *a filter base on* E.

To say that x_0 *is the weak limit of* $\mathscr{B}$ *is equivalent to saying that for every* $a \in \mathrm{E}$,

$$(x_0 \mid a) = \lim_{\mathscr{B}} (x \mid a).$$

Let us call the topology on E associated with the norm the *strong topology*.

Since each of the linear functionals $x \to (x \mid a)$ is continuous, the strong convergence of $\mathscr{B}$ to x_0 implies the weak convergence of $\mathscr{B}$ to x_0. It follows that if $\mathscr{B}$ converges weakly to x_0, $\mathscr{B}$ cannot converge strongly to any point $x_0' \neq x_0$; but it can happen that $\mathscr{B}$ does not converge strongly to x_0. Here is an example in l^2:

Let u_n be the point of l^2 all of whose coordinates are zero except for that with index n, which equals 1. For every $a \in l^2$, the sequence $((u_n \mid a))$ tends to 0; therefore the sequence (u_n) tends weakly to O. But this sequence does not converge strongly to O since $\| u_n \| = 1$ for every n.

15.11. Proposition. *Let* E *be a Hilbert space,* x_0 *a point of* E *and* $\mathscr{B}$ *a filter base on* E; *the following statements are equivalent*:

1. $$x_0 = \text{strong}\lim_{\mathscr{B}} x.$$

2. $$x_0 = \text{weak}\lim_{\mathscr{B}} x \quad \textit{and} \quad \| x_0 \| = \lim_{\mathscr{B}} \| x \|.$$

PROOF. The implication $1 \Rightarrow 2$ is obvious; let us prove the converse.

The hypothesis implies that $x_0{}^2$ is the limit of xx_0 (hence also of x_0x) and of x^2. Thus $x_0{}^2 + x^2$ and $x_0x + xx_0$ have limit $2x_0{}^2$; we therefore have

$$\lim_{\mathscr{B}}(x_0 - x)^2 = \lim_{\mathscr{B}}(x_0{}^2 + x^2 - (x_0x + xx_0)) = 0.$$

In other words, $\mathscr{B}$ converges strongly to x_0.

16. ORTHOGONAL SYSTEMS

The convenience of rectangular coordinate systems in the Euclidean spaces $\mathbf{R}^n$ impels us to ask whether one cannot use analogous systems in prehilbert spaces; in fact, the total orthogonal systems will play this role.

16.1. Definition. LET E BE A PREHILBERT SPACE. BY AN *ORTHOGONAL SYSTEM* (OR FAMILY) IN E WE MEAN A FAMILY $(a_i)_{i\in \mathrm{I}}$ OF ELEMENTS OF E WHICH ARE NONZERO AND PAIRWISE ORTHOGONAL.

AN ORTHOGONAL SYSTEM (a_i) IS CALLED AN *ORTHONORMAL SYSTEM* IF THE NORM OF EVERY a_i EQUALS 1.

An orthonormal system (a_i) is thus characterized by the relations

$$(a_i \mid a_j) = 0 \quad \text{if} \quad i \neq j, \qquad \text{and} \qquad = 1 \quad \text{if} \quad i = j.$$

It is immediate that if (a_i) is an orthogonal system, the system $(\| a_i \|^{-1} a_i)$ is orthonormal.

Every orthogonal system is linearly independent; indeed, if $\sum_{i\in \mathrm{J}} \lambda_i a_i = 0$, where J is a finite subset of I, then

$$\left(\sum \lambda_i a_i\right)^2 = \sum | \lambda_i |^2 a_i{}^2 = 0,$$

which implies $\lambda_i = 0$ for every $i \in \mathrm{J}$.

16.2. Definition. LET (a_i) BE AN ORTHONORMAL SYSTEM IN E. FOR EVERY $x \in \mathrm{E}$, THE SCALAR PRODUCT $\xi_i = (x \mid a_i)$ IS CALLED THE ith *COORDINATE* OF x WITH RESPECT TO THE SYSTEM (a_i).

The coordinate ξ_i of x therefore depends only on the element a_i and not on the other elements of the system.

16.3. Proposition. *Let* E *be a prehilbert space. For every orthonormal system* (a_i) *in* E *and for every* $x \in \mathrm{E}$, *the family* $(|\xi_i|^2)$ *is summable, and*

$$\sum_i |\xi_i|^2 \leqslant \|x\|^2 \qquad (\textit{Bessel's inequality}).$$

PROOF. It is a question of proving that for every finite subset J of I,

$$\sum_{i \in \mathrm{J}} |\xi_i|^2 \leqslant \|x\|^2.$$

But taking into account that $(\xi_i a_i \mid x) = (x \mid \xi_i a_i) = |\xi_i|^2$, we have

$$0 \leqslant \Big(x - \sum_{i \in \mathrm{J}} \xi_i a_i\Big)^2 = x^2 - \sum_{i \in \mathrm{J}} |\xi_i|^2,$$

which gives the desired relation.

We are now going to characterize the orthonormal systems for which the sum of the $|\xi_i|^2$ is equal to $\|x\|^2$.

Recall that a family (a_i) of elements of E is said to be *total* if the vector subspace of E which it generates is everywhere dense on E, or equivalently if every element of E is the limit of linear combinations of elements a_i.

16.4. Theorem. *Let* E *be a prehilbert space, and let* (a_i) *be an orthonormal system in* E. *The following statements are equivalent*:

1. *For all* $x, y \in \mathrm{E}$, *the family* $(\xi_i \bar{\eta}_i)$ *is summable with sum* $(x \mid y)$ (*where* ξ_i *and* η_i *are the* ith *coordinates of* x *and* y, *respectively*).
2. *For every* $x \in \mathrm{E}$ *we have* $\|x\|^2 = \Sigma |\xi_i|^2$ (*Parseval relation*).
3. *For every* $x \in \mathrm{E}$, *the family* $(\xi_i a_i)$ *is summable with sum* x.
4. *The family* (a_i) *is total in* E.

PROOF. $1 \Rightarrow 2$. Indeed, (2) is a special case of (1).

$2 \Rightarrow 3$. Since the sum of the family $(|\xi_i|^2)$ is $\|x\|^2$, for every $\epsilon > 0$ there exists a finite subset J_0 of I such that

$$\|x\|^2 - \sum_{i \in \mathrm{J}} |\xi_i|^2 \leqslant \epsilon^2$$

for every finite subset J of I containing J_0. Hence

$$\Big(x - \sum_{i \in J} \xi_i a_i\Big)^2 = \| x \|^2 - \sum_{i \in J} | \xi_i |^2 \leqslant \epsilon^2, \quad \text{from which} \quad \Big\| x - \sum_{i \in J} \xi_i a_i \Big\| \leqslant \epsilon,$$

which proves the desired property.

$3 \Rightarrow 4$. Since $x = \Sigma \xi_i a_i$ for every $x \in E$, every x is the limit of finite sums of elements $\xi_i a_i$; therefore the family (a_i) is total.

$4 \Rightarrow 1$. Let $x \in E$ and $\epsilon > 0$; by hypothesis there exists a finite linear combination $y = \Sigma_{i \in J} \alpha_i a_i$ such that $\| x - y \| \leqslant \epsilon$. Therefore *a fortiori* the projection x' of x on the space generated by the finite family $(a_i)_{i \in J}$ satisfies the relation $\| x - x' \| \leqslant \epsilon$; but the vector $(x - \Sigma_{i \in J} \xi_i a_i)$ is orthogonal to each of the a_i (where $i \in J$), hence $x' = \Sigma_{i \in J} \xi_i a_i$.

We therefore have

$$0 \leqslant \Big(x - \sum_{i \in J} \xi_i a_i\Big)^2 = x^2 - \sum_{i \in J} | \xi_i |^2 \leqslant \epsilon^2.$$

This inequality holds for every ϵ; therefore, taking into account Bessel's inequality, we obtain

$$\| x \|^2 = \sum_i | \xi_i |^2 = \sum_i \xi_i \bar{\xi}_i . \tag{1}$$

If now x and y are arbitrary elements of E, with coordinates (ξ_i) and (η_i), the summability of the $| \xi_i |^2$ and the $| \eta_i |^2$ implies the summability of the $\xi_i \overline{\eta_i}$ and the $\overline{\xi_i} \eta_i$; this allows us to use relation (1) to express all the squares in the identity

$$4xy = (x + y)^2 - (x - y)^2 + i((x + iy)^2 - (x - iy)^2)$$

as functions of the ξ_i and η_i. We thus obtain the desired identity $xy = \Sigma \xi_i \overline{\eta_i}$.

16.5. Definition. EVERY ORTHONORMAL SYSTEM IN A PREHILBERT SPACE E WHICH SATISFIES ONE OF THE ASSERTIONS OF THEOREM 16.4 IS CALLED AN *ORTHONORMAL BASIS* OF E.

16.6. Extension of the formulas to an orthogonal basis. It often happens that the family (a_i) which appears naturally in a theory is not an orthonormal basis, but there exist scalars $\lambda_i \neq 0$ such that the family of the $b_i = \lambda_i a_i$ is an orthonormal basis.

We shall call such a family (a_i) an *orthogonal basis* of E. To say that a family (a_i) is of this type is clearly equivalent to saying that it is orthogonal and total.

The relations

$$x = \sum \xi_i b_i \qquad \text{and} \qquad \| x \|^2 = \sum | \xi_i |^2, \qquad \text{where} \quad \xi_i = (x \mid b_i),$$

then become

$$x = \sum \alpha_i a_i \qquad \text{and} \qquad \| x \|^2 = \sum \| a_i \|^2 | \alpha_i |^2, \qquad \text{where} \quad \alpha_i = \frac{(x \mid a_i)}{\| a_i \|^2}.$$

The scalar α_i is called the ith *Fourier coefficient* of x with respect to the basis (a_i); more generally, the scalars $\alpha_i = (x \mid a_i)/\| a_i \|^2$, where (a_i) is an orthogonal system, total or not, are called the Fourier coefficients of x with respect to (a_i).

In actual fact these formulas can be rederived without introducing the b_i; one writes formally

$$x = \sum \alpha_i a_i, \qquad \text{from which} \quad x^2 = \left(\sum \alpha_i a_i\right)^2 \quad \text{and} \quad x a_i = \left(\sum \alpha_i a_i\right) a_i.$$

The formal expansion of these two relations gives the equations written above.

It is essential to note that by Proposition 16.13, an "orthonormal (or orthogonal) basis" of a Hilbert space E is an algebraic basis of E only when E is finite dimensional (see also Problem 195).

Characterization of orthonormal bases in a Hilbert space

16.7. Definition. LET E BE A PREHILBERT SPACE. AN ORTHONORMAL SYSTEM $(a_i)_{i \in \mathrm{I}}$ IN E IS SAID TO BE *MAXIMAL* IF EVERY ORTHONORMAL SYSTEM CONTAINING IT IS IDENTICAL WITH IT.

This condition is equivalent to saying that every vector x which is orthogonal to all the a_i is zero; in other words, the condition $((x \mid a_i) = 0$ for every $i \in \mathrm{I})$ implies $(x = \mathrm{O})$, or equivalently, the only vector x all of whose coordinates are zero is $x = \mathrm{O}$.

16.8. Proposition. *When* E *is complete, the four equivalent assertions in Theorem* 16.4 *are equivalent to the following*:

5. *The orthonormal system* (a_i) *is maximal.*

PROOF. 1. If the system (a_i) satisfies assertion 3, the relation

$x = \sum \xi_i a_i$ shows that if all the ξ_i vanish, then $x = O$; hence the system is maximal.

2. Conversely, suppose the system (a_i) is maximal; let L be the vector subspace generated by the a_i. If we had $\overline{L} \neq E$, Proposition 15.6 would show the existence of a vector $x \neq O$ orthogonal to $\overline{L}$ and hence to each of the a_i, which is impossible since (a_i) is maximal.

Therefore $\overline{L} = E$; in other words, the system (a_i) is total.

Z Proposition 16.8 does not extend to incomplete prehilbert spaces (see Problems 197–199); this does not mean, however, that such a space cannot have an orthonormal basis.

Existence of orthonormal bases

The four assertions in Theorem 16.4 show the usefulness of orthonormal bases for the representation of elements of a prehilbert space E. It is therefore important to know conditions which are sufficient for the existence of such a basis.

We are going to show, by different methods, that such a basis exists in complete spaces E and in separable spaces E.

16.9. Lemma. *Every orthonormal system in a prehilbert space* E *is contained in a maximal orthonormal system.*

PROOF. Let us first recall that an ordered set A is said to be *inductive* if every totally ordered subset of A has an upper bound which belongs to A; let us also recall the statement of Zorn's lemma (which is equivalent to the axiom of choice):

"In an inductive ordered set A, every element of A has an upper bound which is a maximal element of A."

To apply Zorn's lemma, it will be convenient here to consider an orthonormal system in E as a subset of E (rather than an indexed subset of E).

If we order the set A of orthonormal systems s in E by inclusion, A becomes an inductive set. Indeed, let S be a totally ordered subset of A; the union $\hat{S}$ of the $s \in S$ is an orthonormal system, for if a_1 and a_2 are elements of $\hat{S}$ belonging, respectively, to s_1 and s_2 (where $s_1 \subset s_2$ for example), a_1 and a_2 belong to s_2 and are therefore orthogonal. Thus $\hat{S}$ is an element of A which is an upper bound for every $s \in S$; hence we have shown that A is inductive, from which the lemma follows.

16.10. Proposition. *Every Hilbert space* E *has an orthonormal basis.*

PROOF. Lemma 16.9 shows that E contains a maximal orthonormal system, and by Proposition 16.8 this system is an orthonormal basis.

More generally, it follows from 16.8 and 16.9 that every finite or infinite orthonormal system in a Hilbert space is contained in an orthonormal basis.

16.11. Lemma. *Let* E *be a prehilbert space, and let* $(a_0, a_1, \ldots)$ *be a finite or infinite sequence of linearly independent vectors in* E; *for every* n, *let* L_n *be the subspace of* E *generated by* $a_0, a_1, \ldots, a_n$.

If we put $b_0 = a_0$ *and* $b_{n+1} = a_{n+1} - P_{L_n}(a_{n+1})$, *then the sequence* (b_n) *is an orthogonal system, and for every* n *the vectors* $b_0, b_1, \ldots, b_n$ *generate* L_n.

PROOF. Using induction, let us assume that $(b_0, b_1, \ldots, b_n)$ is an orthogonal system which generates L_n.

The relation $b_{n+1} = a_{n+1} - P_{L_n}(a_{n+1})$ shows that $b_{n+1} \neq O$ since a_{n+1} does not belong to L_n, and b_{n+1} is orthogonal to L_n by Corollary 15.3. Therefore $(b_0, b_1, \ldots, b_{n+1})$ is an orthogonal system, and since $a_{n+1} = b_{n+1} +$ a vector in L_n, this system generates the same subspace as does $(a_0, a_1, \ldots, a_{n+1})$.

The orthogonal sequence (b_n) is called the *orthogonal system obtained from the sequence* (a_n) *by the Gram-Schmidt orthogonalization procedure.*

In practice, the sequence (b_n) is determined as follows: Assuming the b_i to be known for all $i \leqslant n$, we put

$$b_{n+1} = a_{n+1} + \sum_1^n \lambda_i b_i .$$

For every $i \leqslant n$ we therefore have

$$0 = (a_{n+1} \mid b_i) + \lambda_i (b_i \mid b_i), \qquad \text{which determines the } \lambda_i .$$

Finally, if we put $b_n' = \| b_n \|^{-1} b_n$, the sequence (b_n') is clearly orthonormal.

16.12. Proposition. *Every separable prehilbert space* E (*that is, containing an everywhere dense countable subset*) *has an orthonormal basis which is finite or countable.*

PROOF. By hypothesis there exists a countable subset D of E such that $\overline{D} = E$. Let us arrange the points of D in a sequence (α_n) and denote by (α_{i_n}) the subsequence consisting of the elements α_i which are not linear combinations of elements α_j with index $j < i$. By construction the α_{i_n} are linearly independent, and a simple induction argument shows that every α_i is a linear combination of elements α_{i_n}; hence the sequence (α_{i_n}) is total in E.

The sequence obtained from (α_{i_n}) by the Gram-Schmidt orthogonalization procedure is total since it generates the same vector space as does the sequence (α_{i_n}); hence it is an orthogonal basis of E, and an orthonormal basis is at once obtained from it.

The assertions of 16.11 and 16.12 are of great importance for the representation of functions; the study of Fourier series and orthogonal polynomials will furnish an illustration of this.

Isomorphism of Hilbert spaces

16.13. Proposition. *Let* E *be a prehilbert space with an orthonormal basis* $(a_i)_{i\in I}$. *If for every* $x \in$ E *we denote by* ξ_i *the* ith *coordinate of* x *with respect to this basis, then the mapping* $f : x \to (\xi_i)$ *of* E *into* l_I^2 *is an isomorphism of* E *onto an everywhere dense vector subspace of* l_I^2. *When* E *is complete, the mapping* f *is an isomorphism of* E *onto* l_I^2.

Proof. Theorem 16.4 shows that for every $x \in$ E, $\|x\|^2 = \Sigma \,|\xi_i|^2$; therefore (ξ_i) is a point of l_I^2 and the mapping f, which is clearly linear, is norm preserving. Hence by 14.10, f is an isomorphism of E onto f(E).

Since f(E) evidently contains the canonical basis of l_I^2, which is total in l_I^2, f(E) is everywhere dense on l_I^2.

In particular, if E is complete, so is f(E); therefore f(E) is both everywhere dense and closed in l_I^2; in other words, $f(\mathrm{E}) = l_I^2$.

16.14. Corollary. *Let* E *be a Hilbert space, and let* $(a_i)_{i\in I}$ *be an orthonormal basis of* E. *For every family* $(\xi_i)_{i\in I}$ *of elements of* **K** *such that* $\Sigma\,|\xi_i|^2 < \infty$, *there exists a point* $x \in$ E *having* (ξ_i) *for its coordinates with respect to the given basis.*

This result, which follows directly from 16.13, constitutes a converse to statement 2 of Theorem 16.4.

16.15. Corollary. *Let* E *be a separable prehilbert space. If* E *has finite dimension* n, E *is isomorphic to* $\mathbf{K}^n$.

If E *is infinite-dimensional,* E *is isomorphic to an everywhere dense vector subspace of* l^2, *and to* l^2 *itself if* E *is complete.*

This is an immediate consequence of 16.12 and 16.13.

16.16. Proposition. 1. *All orthonormal bases of a Hilbert space* E *have the same cardinality* (*called the* geometric dimension of E).

2. *To say that two Hilbert spaces are isomorphic is equivalent to saying that they have the same geometric dimension.*

PROOF. 1. If E has finite dimension n, every orthonormal basis of E is an algebraic basis; hence its cardinal number is n.

Suppose therefore that E is infinite-dimensional, and let $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ be two orthonormal bases of E.

The set A of finite linear combinations with rational coefficients (rational complex if $\mathbf{K} = \mathbf{C}$) of the a_i is everywhere dense on E; since I is infinite and $\mathbf{Q}$ is countable, the cardinals $\bar{A}$ and $\bar{I}$ of A and I are equal (use the fact, which we shall accept here, that for every integer p, I and I^p have the same cardinality).

But the open balls B_j of E with center b_j and radius $\frac{1}{2}$ are pairwise disjoint; hence, since each B_j contains at least one point of A, we have $\bar{J} \leqslant \bar{A} = \bar{I}$. Similarly $\bar{I} \leqslant \bar{J}$; hence $\bar{I} = \bar{J}$.

2. Every isomorphism of two Hilbert spaces carries every orthonormal basis of one into an orthonormal basis of the other; therefore two such spaces have the same geometric dimension.

Conversely, if two Hilbert spaces E and F have the same geometric dimension, they have orthonormal bases which can be indexed by the same set I. By Proposition 16.13, E and F are each isomorphic to l_I^2; hence they are isomorphic.

16.17. REMARK. The most used Hilbert spaces are those whose geometric dimension is $\aleph_0$. The only space of this type which we have encountered up to now is l^2; we shall see another important concrete example in the theory of integration.

17. FOURIER SERIES AND ORTHOGONAL POLYNOMIALS

Fourier series

Let E be the vector space of continuous mappings of $\mathbf{R}$ into $\mathbf{C}$ with period 2π.

For every $x, y \in E$ we put

$$(x \mid y) = \int_0^{2\pi} x(t)\overline{y(t)}\, dt = \int_\alpha^{\alpha+2\pi} x(t)\overline{y(t)}\, dt.$$

This positive Hermitean form is a scalar product on E, as the relation $(x \mid x) = 0$ implies $x(t) = 0$ for every $t \in [0, 2\pi]$ since x is continuous, hence also $x(t) = 0$ for every $t \in \mathbf{R}$ since x has period 2π.

The space E taken with this scalar product is a prehilbert space, but it is not complete (see Example 14.7), and it is only by using the Lebesgue integral that we shall later be able to substitute for E the space of square summable functions with period 2π, which is complete.

For every $n \in \mathbf{Z}$, let us denote by a_n the function $t \to e^{int}$, which clearly belongs to E.

17.1. Proposition. 1. *The family of functions* e^{int} *is an orthogonal basis of* E.

2. *For every* $x \in E$ *we have*

$$(x \mid x) = 2\pi \sum |\alpha_n|^2, \qquad \textit{where} \qquad \alpha_n = (x \mid a_n)/2\pi.$$

3. *The family of functions* $\alpha_n e^{int}$ *is summable in* E *and has sum* x.

PROOF. We have

$$(a_p \mid a_q) = \int_0^{2\pi} e^{i(p-q)t}\,dt = 0 \qquad (\text{if } p \neq q),$$
$$\text{or} \quad = 2\pi \qquad (\text{if } p = q);$$

therefore (a_n) is an orthogonal family.

On the other hand, by application no. 2, Chapter II, p. 149, every function $x \in E$ is the uniform limit of linear combinations of the a_n; but the relation

$$\int_0^{2\pi} |f(t)|^2\,dt \leqslant 2\pi(\sup |f(t)|)^2$$

shows that uniform convergence implies convergence in the norm of E. It follows that the family (a_n) is total in E; therefore it is an orthogonal basis of E.

Assertions 2 and 3 then follow from Theorem 16.4 and the formulas of 16.6.

17.2. REMARK. It is sometimes convenient, particularly when the functions x which one wishes to study are real valued, to replace the family of functions e^{int} by the family of functions $\cos nt$ (where $n \geqslant 0$) and $\sin nt$ (where $n \geqslant 1$); an elementary calculation shows that this family is also orthogonal.

On the other hand, it is total in E since the family (e^{int}) is total and the relation $e^{int} = \cos nt + i \sin nt$ shows that each e^{int} is a linear combination of $\cos nt$ and $\sin nt$; therefore the family of these latter functions is an orthogonal basis of E.

If we denote the elements $\cos nt$ and $\sin nt$ of E by c_n and s_n, respectively, the Fourier coefficients v_n and σ_n of a function $x \in E$ are given by the relations

$$v_0 = (x \mid c_0)/2\pi; \qquad v_n = (x \mid c_n)/\pi; \qquad \sigma_n = (x \mid s_n)/\pi \qquad \text{for} \quad n \geqslant 1.$$

The Parseval relation here assumes the form

$$(x \mid x) = 2\pi| v_0 |^2 + \pi \sum_{1}^{\infty} (| v_n |^2 + | \sigma_n |^2).$$

Z Proposition 17.1 shows that for every function $x \in \mathrm{E}$, the sequence of functions

$$x_n(t) = \sum_{-n}^{n} \alpha_p e^{ipt}$$

converges to x in quadratic mean. But it says nothing concerning pointwise convergence or uniform convergence.

We shall return to these questions during the study of integration. Let us simply note here that if the sequence of functions x_n converges uniformly to a function x', then $x' = x$; indeed, the sequence (x_n) will then also converge to x' in the sense of convergence in E, and since it can have only one limit, we have $x' = x$.

17.3. Orthogonal polynomials. Let I be a closed interval, bounded or not, of $\mathbf{R}$, and let p be a continuous numerical function which is > 0 on the interior of I, and such that

$$\int_{\mathrm{I}} | t |^n p(t)\, dt < \infty$$

for every integer $n \geqslant 0$. We shall call such a function a *weight on* I.

Let E_p be the subset of $\mathscr{C}(\mathrm{I}, \mathbf{K})$ consisting of the functions x such that

$$\int_{\mathrm{I}} | x(t)|^2 p(t)\, dt < \infty.$$

The relations $| x + y |^2 \leqslant 2 | x |^2 + 2 | y |^2$ and $2 | xy | \leqslant | x |^2 + | y |^2$ show that E_p is a vector space and that the function

$$(x \mid y) = \int_{\mathrm{I}} x(t)\overline{y(t)} p(t)\, dt$$

is a positive Hermitean form on E_p.

Since in addition p is continuous, and > 0 on the interior of I, the relation $(x \mid x) = 0$ implies $x(t) = 0$ for every $t \in \mathrm{I}$, hence $x = 0$. In other words, $(x \mid y)$ is a scalar product on E_p which makes E_p a prehilbert space.

The assumptions on p imply that every monomial t^n is an element of E_p; moreover, the t^n are linearly independent since every polynomial which is identically zero on I has vanishing coefficients. We can therefore apply the Gram-Schmidt orthogonalization procedure to the sequence (t^n).

Put $a_n(t) = t^n$; the recursion relations

$$P_0 = a_0\,; \qquad P_n = a_n - (\text{projection of } a_n \text{ on } L_{n-1})$$

show that P_n is a polynomial of degree n with real coefficients, and whose term of highest degree is t^n.

The P_n are called the *orthogonal polynomials associated with the weight* p *on* I.

One is sometimes led, in certain questions, to replace the P_n by the proportional polynomials $p_n = \lambda_n P_n$, where λ_n is a nonzero scalar (for example, it is sometimes assumed that p_n takes the value 1 at one of the end points of I); but in what follows, we shall only consider the P_n.

17.4. Classical examples.

1. $I = [-1, 1]$ and $p(t) = (1 - t)^{\alpha}(1 + t)^{\beta}$ (where α, $\beta > -1$).

The corresponding polynomials P_n are called the *Jacobi polynomials.* For $\alpha = \beta = 0$ (hence $p(t) = 1$) these are the *Legendre polynomials.* For $\alpha = \beta = -\frac{1}{2}$ these are the *Chebyshev polynomials.*

2. $I = [0, \infty)$ and $p(t) = e^{-t}$.

The polynomials P_n are then called the *Laguerre polynomials.*

3. $I = (-\infty, +\infty)$ and $p(t) = \exp(-t^2)$.

The polynomials P_n are then called the *Hermite polynomials.*

17.5. Proposition. *When the interval* I *is compact, the sequence* (P_n) *of polynomials associated with the weight* p *on* I *is an orthogonal basis of the prehilbert space* E_p.

PROOF. Every continuous function on I is the uniform limit of polynomials (see application no. 1, Chapter II, p. 149), hence of linear combinations of the P_n. Therefore the sequence (P_n) is total in E_p (same proof as for Proposition 17.1); since this sequence is moreover orthogonal, it is an orthogonal basis of E_p.

Z 1. When the interval I is unbounded, there exist weights p on I for which the sequence (P_n) is not an orthogonal basis of E_p. But one can prove that the Laguerre and Hermite polynomials do form orthogonal bases of the respective spaces E_p (see Problem 205).

2. Just as for Fourier series, the series with general term $\alpha_n P_n(t)$ associated with an element $x \in E_p$ does not always converge uniformly, or even pointwise, to the function $x(t)$; but if it converges uniformly, it is necessarily to $x(t)$.

General properties of sequences of orthogonal polynomials

We are going to show that the sequence (P_n) of orthogonal polynomials associated with a weight p on an interval I has various interesting properties independent of the weight chosen; we shall use for this the following two obvious remarks:

17.6. For every n, the polynomial P_n is orthogonal to all polynomials Q of degree $< n$, since such a Q is a linear combination of the P_i with $i < n$.

17.7. For every n, $P_n - tP_{n-1}$ is a polynomial of degree $< n$, hence

$$(tP_{n-1} \mid P_n) = (P_n \mid P_n).$$

17.8. The definition of the scalar product of E_p shows that for every $x, y, z \in E_p$ we have

$$(xy \mid z) = (xy\bar{z} \mid 1) = (x \mid \bar{y}z).$$

17.9. Proposition. *The roots of each of the polynomials* P_n *are real and distinct, and lie in the interior of* I.

PROOF. The orthogonality of P_n and P_0 implies that

$$\int_I P_n(t)p(t)\,dt = 0.$$

Therefore P_n changes sign at one interior point at least of I. More generally, let $(t_1, t_2, \ldots, t_r)$, where $t_i < t_{i+1}$, be the sequence of roots of P_n interior to I at which P_n changes sign. We want to show that $r = n$; but since obviously $r \leqslant n$, it suffices to rule out the case $r < n$.

If we put $Q(t) = (t - t_1)(t - t_2) \cdots (t - t_r)$, the polynomial $P_n Q$ has a fixed sign on I, which implies $(P_n \mid Q) \neq 0$; Remark 17.6 shows that this is incompatible with the inequality $r < n$.

17.10. Proposition (Recursion relation). *There exist two sequences of real numbers* λ_n, μ_n, *with* $\mu_n > 0$, *such that*

$$P_n = (t + \lambda_n)P_{n-1} - \mu_n P_{n-2} \qquad \textit{for every} \quad n \geqslant 2.$$

PROOF. Since $P_n - tP_{n-1}$ is of degree $\leqslant (n-1)$, we can write

$$P_n - tP_{n-1} = \sum_{i \leqslant n-1} c_i P_i .$$

We therefore have, for every $i \leqslant n-1$,

$$-(tP_{n-1} \mid P_i) = c_i(P_i \mid P_i). \tag{1}$$

But by Remark 17.8, $(tP_{n-1} \mid P_i) = (P_{n-1} \mid tP_i)$; therefore if $i + 1 < n - 1$, that is, $i < n - 2$, this scalar product is zero, hence $c_i = 0$ except for $i = n - 2$ and $n - 1$.

For $i = n - 2$, Remark 17.7 shows that

$$-(P_{n-1} \mid P_{n-1}) = c_{n-2}(P_{n-2} \mid P_{n-2}).$$

Thus $c_{n-2} < 0$; for $i = n - 1$, relation (1) shows only that c_{n-1} is real.

REMARK. The recursion relation for the polynomials $p_n = \lambda_n P_n$ clearly has a similar form, although less simple:

$$p_n = (u_n t + v_n)p_{n-1} + w_n p_{n-2} .$$

PROBLEMS

NOTE. The rather difficult problems are marked by an asterisk.

GENERAL TOPOLOGICAL VECTOR SPACES

1. Let E be a topological vector space, and let X, Y $\subset$ E; prove the following properties:

(a) If X is open, so is X + Y.

(b) If X and Y are compact, and if E is separated, then X + Y is compact.

Show by an example constructed in $\mathbf{R}^2$ that X and Y can be closed without X + Y being closed.

2. Let $\mathscr{C}(\mathbf{R}, \mathbf{R})$ be the space of continuous functions on $\mathbf{R}$, taken with the topology of uniform convergence defined by the ecart

$$d(f, g) = \sup |f(x) - g(x)|.$$

Show that this topology is not compatible with the vector space structure of $\mathscr{C}(\mathbf{R}, \mathbf{R})$, although it is compatible with its additive group structure.

***3.** For every number $\epsilon > 0$, and every $x \in \mathscr{C}([0, 1], \mathbf{R})$, we shall say that ϵ is associated with x if the set of points $t \in [0, 1]$ such that $|x(t)| \geqslant \epsilon$ can be covered by a finite family of intervals, the sum of whose lengths is $\leqslant \epsilon$; we then put

$$p(x) = \text{infimum of the set of all } \epsilon \text{ associated with } x.$$

(a) Show that p is not a norm on $\mathscr{C}([0, 1], \mathbf{R})$ but that nevertheless $p(x) = p(-x)$ and $p(x + y) \leqslant p(x) + p(y)$; deduce from this that if we put $d(x, y) = p(x - y)$, d is a metric on this space.

(b) Show that for every ball B with center O and radius $\rho > 0$ associated with this metric, the convex envelope of B is the entire space.

(c) Show that the topology associated with this metric is compatible with the vector space structure of $\mathscr{C}([0, 1], \mathbf{R})$, but is not locally convex.

(d) Show that the only continuous linear functional on this space is the linear functional O.

4. For every number $\epsilon > 0$ and every $x \in \mathscr{C}([0, 1], \mathbf{R})$, we shall say that ϵ is associated with x if the set of points $t \in [0, 1]$ such that $|x(t)| \geqslant \epsilon$ is contained in an interval of length $\leqslant \epsilon$; we then put

$$q(x) = \text{infimum of the set of all } \epsilon \text{ associated with } x.$$

Does the function q has properties analogous to those of the function p of Problem 3?

5. Let E be a vector space over $\mathbf{R}$, and let $\mathscr{V}$ be the set of convex subsets V of E containing O and such that, for every line D in E containing O, the point O is interior to the interval $D \cap V$.

We shall say that a subset X of E is *open* if for every $x \in X$ there exists a $V \in \mathscr{V}$ such that $x + V \subset X$.

(a) Show that the collection of these "open sets" defines a topology on E which is compatible with the vector space structure of E, and that $\mathscr{V}$ constitutes a neighborhood base of O in this topology.

(b) Given an algebraic basis $B = (a_i)_{i \in I}$ of E, show that for every family $(\alpha_i)_{i \in I}$ of numbers > 0, the convex envelope of the set of elements $\alpha_i a_i$ and $-\alpha_i a_i$ belongs to $\mathscr{V}$, and that these convex envelopes form a neighborhood base of O.

(c) Show that every linear functional on E is continuous in this topology.

6. Let E be a vector space over **R**; show that if E is infinite-dimensional, the weak topology on E associated with the family of all the linear functionals on E is different from the topology defined in Problem 5.

7. Let E be a vector space over **K**. Given two topologies on E which are compatible with the vector space structure of E, we denote by $\mathscr{V}_1$ and $\mathscr{V}_2$ the collection of neighborhoods of O for each of these topologies. Show that the collection $\mathscr{V}$ of subsets of E of the form $V_1 \cap V_2$, where $V_1 \in \mathscr{V}_1$ and $V_2 \in \mathscr{V}_2$, is the collection of neighborhoods of O for a topology which is compatible with the vector space structure of E.

8. Let E be a topological vector space over **K**, and let f be a linear functional (not identically zero) on E. Show that if the hyperplane $H = f^{-1}(0)$ is closed, then f is continuous. (Show first that there exists an a such that $f(a) = 1$, and use the fact that the complement of $f^{-1}(a)$ is a neighborhood of O.)

TOPOLOGY ASSOCIATED WITH A FAMILY OF SEMINORMS

9. We shall say that a subset X of a vector space E with a $\mathscr{P}$-topology is *bounded* if every seminorm $p \in \mathscr{P}$ is bounded on X. Prove that the class of bounded sets is closed under the following operations:

Closure, convex envelope, image under a continuous linear mapping, finite union, finite vector sum, finite product.

Show that every compact set is bounded, and that for every convergent sequence (x_n), the set of elements x_n is bounded.

10. Let E be a vector space over **R**, and let X be a convex subset of E such that

$$E = \bigcup_{k \in \mathbf{R}_+} kX.$$

For every $x \in E$ we put

$$p(x) = \text{infimum of those } k > 0 \text{ such that } x \in kX.$$

Show that p is positive-homogeneous and convex, and that when X is symmetric, p is a seminorm; when is it a norm?

Show that

$$\{x : p(x) < 1\} \subset X \subset \{x : p(x) \leqslant 1\}.$$

11. Let E be a vector space over **R**, F a vector subspace of E, and p a mapping of E into **R** such that for every x, $y \in \mathrm{E}$ and every $\lambda \in \mathbf{R}_+$,

$$p(\lambda x) = \lambda p(x), \qquad p(x + y) \leqslant p(x) + p(y).$$

(a) Let f be a linear functional on E such that $f(x) \leqslant p(x)$ for every $x \in \mathrm{F}$; show that for every $a \in \mathrm{E}$ and every u, $v \in \mathrm{F}$ we have

$$f(u) - p(u - a) \leqslant f(a) \leqslant p(v + a) - f(v).$$

(b) More generally, let f be a linear functional defined only on F and such that $f(x) \leqslant p(x)$ for every $x \in \mathrm{F}$. For every $a \in \mathrm{E}$ we put

$$k_a = \sup_{u \in \mathrm{F}}(f(u) - p(u - a)), \qquad \mathrm{K}_a = \inf_{v \in \mathrm{F}}(p(v + a) - f(v)).$$

Show that k_a and K_a are finite and $k_a \leqslant \mathrm{K}_a$.

(c) We maintain the hypotheses of (b), and assume $a \notin \mathrm{F}$; let k be any number in $[k_a, \mathrm{K}_a]$.
For every $x \in \mathrm{F}$ and every $\lambda \in \mathbf{R}$ we put

$$g(x + \lambda a) = f(x) + \lambda k.$$

Show that g is a linear functional on the vector subspace F_a of E generated by F and a, that g is identical with f on F, and that $g(y) \leqslant p(y)$ for every $y \in \mathrm{F}_a$.

12. Let E and p be as in the preceding problem. Let $\mathscr{F}$ be the set of pairs (F, f), where F is a vector subspace of E and f is a linear functional on F such that $f \leqslant p$ on F. We shall write

$$(\mathrm{F}_1, f_1) \leqslant (\mathrm{F}_2, f_2),$$

and say that (F_2, f_2) is an extension of (F_1, f_1), if $\mathrm{F}_1 \subset \mathrm{F}_2$ and if f_1 is the restriction of f_2 to F_1.

(a) Show that this relation is an order relation on $\mathscr{F}$, and that $\mathscr{F}$, with this order, is inductive (see 16.9).

(b) Deduce from the preceding problem that every maximal element of $\mathscr{F}$ is of the form (E, f).

13. Let E, p, $\mathscr{F}$ be defined as in Problem 12.

(a) Show that every $(\mathrm{F}, f) \in \mathscr{F}$ has an extension (E, g). Show, in particular, that there always exists a linear functional f on E such that $f \leqslant p$ on E.

(b) Show that a necessary and sufficient condition for the extension (E, g) of (F, f) to be unique is that, with the notation of Problem 11, $k_a = K_a$ for every $a \in E$. (The results of this problem constitute the Hahn-Banach theorem, whose applications in Analysis are manifold.)

14. Let E be a vector space over **R**, with a separated $\mathscr{P}$-topology. Show, using Problem 13, that for every $x \neq O$ in E there exists a continuous linear functional f on E such that $f(x) \neq 0$.

15. Show that the norm of every normed space E over **R** is the upper envelope of a family of seminorms of the form $| l |$, where $l \in E'$ (use Problem 13).

Deduce from this that if $\mathscr{P}$ is an infinite family of seminorms on a vector space E, and if $\bar{\mathscr{P}}$ denotes the family of seminorms on E which are upper envelopes of subfamilies of $\mathscr{P}$, the topologies on E associated with $\mathscr{P}$ and $\bar{\mathscr{P}}$ are not in general identical.

Indicate several cases in which these topologies are identical.

16. Let B be the closed unit ball in a normed space E over **R**. Show that for every $x \in E$ such that $\| x \| = 1$, there exists a continuous linear functional on E such that $\| f \| = 1$ and $f(x) = 1$ (use Problem 13).

17. Let E and F be vector spaces over the same field **K**, and let f be a bilinear form on E $\times$ F. We shall say that f puts E and F *in duality* if

(a) for every $x \neq O$ in E, there exists $y \in F$ such that $f(x, y) \neq 0$;

(b) for every $y \neq O$ in F, there exists $x \in E$ such that $f(x, y) \neq 0$.

The weak topology associated with the family of linear functionals $l_y : x \to f(x, y)$ (where $y \in F$) is called the weak topology on E associated with f.

Show that this topology is separated, and that if E is infinite-dimensional, this topology cannot be defined by a norm (show, to this end, that every neighborhood of O contains a vector subspace of E with finite codimension).

Show that every linear functional l on E which is continuous in the weak topology just defined is of the form l_y.

18. Let D be the open disk $\{z : | z | < 3\}$ of **C**, and let K be the compact set $\{z : | z | \leqslant 1\}$. For every $f \in \mathscr{H}(D)$ (see 3.7) we put

$$p(f) = \sup_{z \in K} |f(z)|.$$

(a) Show that p is a norm on $\mathscr{H}(D)$.

(b) Show that the topology defined by p on $\mathscr{H}(\mathrm{D})$ is not identical with the topology of uniform convergence on every compact set (use the sequence (f_n) of functions $f_n(z) = e^{n(z-2)}$).

19. Let A be a countable subset of [0, 1], and let α be a mapping of A into $(0, \infty)$ such that $\Sigma_{t \in \mathrm{A}} \alpha(t) < \infty$. For every $x \in \mathscr{C}([0, 1], \mathbf{R})$ we put

$$\|f\| = \sum_{t \in \mathrm{A}} \alpha(t)|f(t)|.$$

(a) Show that $\|f\|$ is a seminorm on $\mathscr{C}([0, 1], \mathbf{R})$; when is it a norm? Is it then equivalent to the norm of uniform convergence?

(b) Show that two such seminorms define the same topology on $\mathscr{C}([0, 1], \mathbf{R})$ only if the corresponding sets A and A′ are identical and the ratios α/α' and α'/α are bounded.

TOPOLOGY ASSOCIATED WITH A NORM

20. Show that every compact subset of a normed space is bounded.

21. Show that if X and Y are compact subsets of a normed space, the union of the line segments joining a point of X with a point of Y is compact (use the product $\mathrm{X} \times \mathrm{Y}$).

22. Let E be a normed space, and X, Y two subsets of E. Show that if X is compact and Y closed, then $\mathrm{X} + \mathrm{Y}$ is closed.

23. Show that in every normed space, the closure of the open unit ball is the closed unit ball, and its boundary is the unit sphere

$$\{x : \|x\| = 1\}.$$

24. Let E and F be normed spaces over $\mathbf{R}$, and let f be a mapping of E into F such that:

(a) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathrm{E}$;

(b) f is bounded on the unit ball of E.

Show that f is linear and continuous.

25. Let X be a closed subset of a complete normed space E. Show that for X to be compact, it is necessary and sufficient that for every $\epsilon > 0$ there exist a covering of X by a finite family of balls of radius ϵ.

26. Let E be a normed space.

(a) Show that if every series (a_n) in E which is absolutely convergent is convergent, then E is complete.

(b) Show that if every series (a_n) in E such that $\| a_n \| \leqslant 2^{-n}$ is convergent, then E is complete.

How can these results be extended to every metric space E?

27. Let E be a vector space with a seminorm p. We shall write $x \sim y$ if $p(x - y) = 0$.

Show that this relation is an equivalence relation on E which is compatible with the vectorial structure of E and with the seminorm, in a sense which it is required to make precise, and that the equivalence class containing O is a vector subspace of E. Let $\tilde{E}$ be the quotient space of E by this equivalence relation; show that $\tilde{E}$ can be given a unique normed vector space structure such that the canonical mapping $x \rightarrow \tilde{x}$ of E onto $\tilde{E}$ is a linear mapping such that $\| \tilde{x} \| = p(x)$.

We shall say that $\tilde{E}$ is the normed space associated with E.

COMPARISON OF NORMS

28. For every $x \in \mathscr{C}^1([0, 1], \mathbf{R})$ we put

$$\| x \| = \sup(| x(t)| + | x'(t)|).$$

Show that $\| x \|$ is a norm; is it equivalent to the norm

$$(\sup | x(t)| + \sup | x'(t)|)\,?$$

29. Let E be the vector subspace of $\mathscr{C}^1([0, 1], \mathbf{R})$ consisting of the x such that $x(0) = 0$.

For every $x \in E$ we put

$$\| x \| = \sup | x(t) + x'(t)|.$$

Show that $\| x \|$ is a norm on E, equivalent to the norm

$$(\sup | x(t)| + \sup | x'(t)|).$$

30. Compare, on $\mathscr{C}^1([0, 1], \mathbf{R})$, the norms

$$\sup | x(t)|; \qquad \sup | x(t)| + \int_0^1 | x(t)|\, dt;$$

$$\sup | x(t)| + \sup | x'(t)|; \qquad \sup | x(t)| + \int_0^1 | x'(t)|\, dt.$$

31. Show that in the space l^1 over $\mathbf{K}$, the norms $\Sigma | x_n |$ and $\sup | x_n |$ are not equivalent.

32. For every $x \in \mathscr{C}^1([0, 1], \mathbf{R})$ we define a positive number $p(x)$ by

$$p^2(x) = x^2(0) + \int_0^1 x'^2(t)\,dt.$$

Show that p is a norm, and that convergence in this norm implies uniform convergence.

NORMS AND CONVEX FUNCTIONS

33. Let E be a normed space, and let S be the unit sphere $\{x : \| x \| = 1\}$ of E. Assume that for all $x, y \in \mathrm{S}$ one has

$$\left\| \frac{x+y}{2} \right\| \leqslant 1 - \varphi(\| x - y \|),$$

where φ is an increasing mapping of $[0, \infty)$ into $\mathbf{R}_+$, with $\varphi(u) > 0$ for every $u > 0$.

Show that for every closed convex set $\mathrm{X} \subset \mathrm{E}$, every point $x \in \mathrm{E}$ has a unique projection on X.

34. Let E be a vector space over $\mathbf{R}$. The space $\mathrm{E}_c = \mathrm{E} \times \mathrm{E}$ with the ordinary addition and the multiplication by elements $(\alpha + i\beta)$ of $\mathbf{C}$ defined by

$$(\alpha + i\beta)(u, v) = (\alpha u - \beta v, \beta u + \alpha v),$$

is called the *complexification* of E.

(a) Show that E_c is a vector space over $\mathbf{C}$, and that if E is imbedded in E_c by identifying the element $u \in \mathrm{E}$ with the element (u, O) of E_c, then the subset E of E_c generates E_c.

(b) If E is normed, show that one can extend the norm of E, in several ways in general, to a norm on the vector space E_c over $\mathbf{C}$; show that these various norms are equivalent. Show that E and E_c are simultaneously complete or incomplete.

(c) If E is a real prehilbert space, show that its scalar product can be extended in a unique way to a scalar product on E_c, and give an expression for this extension.

35. Let E be a vector space over $\mathbf{C}$; by a *subscalar product* on E we shall mean a mapping q of $\mathrm{E} \times \mathrm{E}$ into $\mathbf{R}_+$ such that

$$q(x, y) = q(y, x); \tag{1}$$

$$q(\lambda x, y) = |\lambda| q(x, y); \tag{2}$$

$$q(u + v, y) \leqslant q(u, y) + q(v, y); \tag{3}$$

$$q^2(x, y) \leqslant q(x, x) q(y, y). \tag{4}$$

(a) Show that $(q(x, x))^{1/2}$ is a seminorm on E.

(b) Show that every sum, every limit, and every upper envelope of subscalar products is a subscalar product.

(c) Show that if $(x \mid y)$ is a scalar product on E, its absolute value is a subscalar product.

36. Let E and F be vector spaces over **R**; let A be a convex subset of $E \times F$, and let f be a convex function on A. We denote by X the projection of A on E, and for every $x \in X$ we put

$$g(x) = \inf_{(x,y)\in A} f(x, y).$$

Show that when g is finite on X, g is convex.

37. Let E be a normed space over **R**.

(a) Show that the mapping $x \to \| x \|$ of E into **R** is convex and of Lipschitz class with ratio 1.

(b) Show that the mapping $(x, y) \to \| x - y \|$ of $E \times E$ into **R** is convex and of Lipschitz class.

For every nonempty subset A of E, and for every $x \in E$, we now put

$$d_A(x) = \inf_{y\in A} \| x - y \|; \qquad D_A(x) = \sup_{y\in A} \| x - y \|.$$

(c) Show that if A is bounded, D_A is a convex function of Lipschitz class with ratio 1.

(d) We assume henceforth that A is convex; show that d_A is then convex (use Problem 36).

(e) Put $A' = \complement A$, and let f be the function defined by

$$f(x) = d_A(x) \text{ for } x \in A'; \qquad f(x) = -d_{A'}(x) \text{ for } x \in A.$$

Show that f is convex in E.

38. We denote by E the vector space $\mathscr{C}([0, 1], \mathbf{R})$ taken with the norm of uniform convergence.

(a) Show that for all $a, b \in E$ such that $a \leqslant b$, the set of $x \in E$ such that $a \leqslant x \leqslant b$ is a bounded closed convex set in E.

(b) For every $\epsilon > 0$ we denote by X_n the set of $x \in E$ such that

$$x(0) = 1; \qquad 0 \leqslant x(t) \leqslant 1 \quad \text{on} \quad [0, 1/n]; \qquad x(t) = 0 \quad \text{on} \quad [1/n, 1].$$

Show that (X_n) is a decreasing sequence of bounded closed convex sets in E, and that the intersection of the X_n is empty.

39. Let E be as in the preceding problem, and let X be the set of $x \in \mathrm{E}$ which are $\geqslant 0$, decreasing and such that

$$1 \leqslant x(0) \leqslant 2 \quad \text{and} \quad \int_0^1 x(t)\,dt = x(0) - 1.$$

(a) Show that X is a bounded closed convex subset of E, and that $d(\mathrm{O}, \mathrm{X}) = 1$.

(b) Show that there exists no $x \in \mathrm{X}$ such that $d(\mathrm{O}, x) = 1$.

LINEAR FUNCTIONALS ON NORMED SPACES

40. Let E be the space $\mathscr{C}([0, 1], \mathbf{R})$ taken with the norm of mean convergence.

(a) Show that the linear functional $x \to \int_0^1 \varphi(t)x(t)\,dt$ on E, where $\varphi \in \mathrm{E}$, is continuous. Find its norm.

(b) Show, on the other hand, that if $\varphi(t) = t^{-1/2}$, this linear functional is not continuous.

(c) Show that the $x \in \mathrm{E}$ such that $\int_0^1 x(t)\,dt = 0$ form a closed hyperplane in E.

(d) Show that the set of monomials t^n is a total set in E.

(e) Show that the quadratic functional $x \to \int_0^1 x^2(t)\,dt$ is not continuous on E.

41. Show that the linear functional $x \to x'(0)$ is not continuous on $\mathscr{C}^1([0, 1], \mathbf{R})$ taken with the norm of uniform convergence.

42. In the space $\mathscr{C}([0, 1], \mathbf{R})$ taken with the norm of uniform convergence, let E be the set of x such that $\int_0^1 x(t)\,dt = 0$.

Show that every $u \in \mathscr{C}([0, 1], \mathbf{R})$ has a unique primitive belonging to E; we denote this primitive by T_u.

Show that T is linear and calculate its norm.

43. Let E be the vector space $\mathscr{C}([0, 1], \mathbf{R})$ taken with the norm of uniform convergence.

Let (t_n) be a sequence of points in [0, 1] which are everywhere dense on [0, 1]; for every $x \in \mathrm{E}$ we put

$$l(x) = \sum_n (-2)^{-n} x(t_n).$$

(a) Show that l is a continuous linear functional on E.

(b) Show that l does not attain its supremum over the closed unit ball of E at any point of this ball.

44. Let X be a compact space, and let $\mathscr{M}(X)$ be the topological dual of the vector space $\mathscr{C}(X, \mathbf{R})$ taken with the norm of uniform convergence.

(a) Show that every linear functional on $\mathscr{C}(X, \mathbf{R})$ which is positive on the set of functions $x \geqslant 0$ belongs to $\mathscr{M}(X)$.

(b) Let (a_n) be a sequence of points of X, and let (α_n) be a sequence of real numbers such that $\Sigma \mid \alpha_n \mid < \infty$; for every $x \in \mathscr{C}(X, \mathbf{R})$ we put

$$\mu(x) = \sum \alpha_n x(a_n).$$

Show that μ is an element of $\mathscr{M}(X)$, and that its norm is equal to $\Sigma \mid \alpha_n \mid$.

45. We now assume $X = [0, 1]$; let μ and μ_n be the linear functionals defined by

$$\mu(x) = \int_0^1 x(t)\,dt \qquad \text{and} \qquad \mu_n(x) = 1/n \sum_1^n x(p/n).$$

(a) Calculate the norms of μ, μ_n, and $\mu - \mu_n$.

(b) Show that the sequence (μ_n) does not converge to μ in the sense of the norm on $\mathscr{M}(X)$, but does converge to μ in the $\sigma(E', E)$ topology on $\mathscr{M}(X) = \mathscr{C}(X, \mathbf{R})'$ (see 3.11).

TOPOLOGICAL DUAL AND BIDUAL

46. Let E be a normed space over **R**, F a vector subspace of E, and f a continuous linear functional on F.

Using the relation $|f(x)| \leqslant \|f\|\,\|x\|$ and Problem 13, show that there exists a linear functional g on E which is an extension of f, and such that $\|g\| = \|f\|$.

47. Let E be a normed space over **R**. Deduce from the preceding problem that for every $x \in E$ there exists an $l \in E'$ such that

$$\|l\| = 1 \qquad \text{and} \qquad l(x) = \|x\|.$$

48. Let E be a normed space over **R**, E′ its topological dual, and E″ the topological dual of E′ (called the *bidual* of E).

(a) For every $x \in E$, we denote by φ_x the linear functional $l \to l(x)$ on E′; show that $\varphi_x \in E''$ and that $\|\varphi_x\| \leqslant \|x\|$.

(b) Using the preceding problem, show that $\|\varphi_x\| = \|x\|$, and that the linear mapping φ of E into E″ is an isometry.

(c) Deduce from this that every normed space over **R** can be imbedded in a complete normed space; extend this result to normed spaces over **C**.

(d) Show, by taking for E the space $\mathscr{C}([0, 1], \mathbf{R})$ with the topology of uniform convergence, that even when E is complete, E″ in general contains points which do not belong to E (identified here with $\varphi(E)$).

***49.** Let E be a normed space, and let A be a convex subset of E.

(a) Show that if A is everywhere dense on E in the weak topology of E, the same is true in the strong topology (use Problem 13).

(b) Show, on the other hand, that in every infinite-dimensional normed space E, the complement A of the unit ball is everywhere dense on E in the weak topology of E.

COMPACT LINEAR MAPPINGS

50. Let f be a linear mapping of a normed space E into a normed space F; f is said to be *compact* if $\overline{f(X)}$ is compact for every bounded set $X \subset E$.

(a) Show that for f to be compact, it is necessary and sufficient that $\overline{f(B)}$ be compact (where B denotes the closed unit ball of E).

(b) Show that every compact linear mapping is continuous, but that the converse is false.

(c) Show that every $f \in \mathscr{L}(E, F)$ such that $f(E)$ is finite-dimensional is compact.

(d) Show that if E is finite-dimensional, then f is compact.

(e) Show that if f and g are compact, so is $f + g$. Deduce from this that the set C(E, F) of compact linear mappings of E into F is a vector subspace of $\mathscr{L}(E, F)$.

*(*f*) Show, using Problem 25, that if F is complete, then C(E, F) is closed in $\mathscr{L}(E, F)$.

51. Let E, F, G be normed spaces, $f \in \mathscr{L}(E, F)$, $g \in \mathscr{L}(F, G)$.
Show that if one of the mappings f, g is compact, then $f \circ g$ is compact.

52. Let E be a normed space over **K**, let $\lambda \in \mathbf{K}$, and let $f \in \mathscr{L}(E)$. We denote by E_λ the set of $x \in E$ such that $f(x) = \lambda x$.
Show that E_λ is a closed vector subspace of E.
Show that when f is compact, E_λ is finite-dimensional (use Theorem 7.6).

53. Let E be the space $\mathscr{C}([0, 1], \mathbf{C})$ taken with the norm of uniform convergence; given $K \in \mathscr{C}([0, 1]^2, \mathbf{C})$, we denote by T the linear mapping of E into itself defined by

$$Tx(t) = \int_0^1 K(t, u)x(u)\, du.$$

(a) Verify that $Tx \in E$, and then show that T is continuous; calculate its norm, or at least an upper bound for the norm.

(b) Show that when K is a polynomial, T(E) is finite-dimensional. Deduce from this, using the Stone-Weierstrass theorem and Problem 50, that T is compact for every K.

54. Same as the preceding problem, but E taken with the norm of mean convergence (or mean convergence of order $p \geqslant 1$).

55. Let (k_n) be a bounded sequence of scalars, and let T be the mapping of the normed space l^p into itself defined as follows:

For every $x = (x_n)$, $T(x)$ is the point $(k_n x_n)$.

Show that a necessary and sufficient condition for T to be compact is that the sequence (k_n) have limit 0.

56. Let E be the space $\mathscr{C}([0, 1], \mathbf{R})$ taken with the norm of uniform convergence, and let T be the mapping of E into itself defined by

$$Tx(t) = \int_0^t x(u)\, du.$$

(a) Show that T^n $(n > 1)$ can be defined by a relation of the form

$$T^n x(t) = \int_0^1 K_n(t, u)x(u)\, du,$$

where $K_n(t, u)$ is a continuous function of (t, u) on $[0, 1]^2$.

(b) Calculate the norm of T^n in $\mathscr{L}(E)$; show that the family (T^n) is summable in $\mathscr{L}(E)$, and calculate its sum.

(c) Use these results to solve the equation

$$(I - T)x = g,$$

where g is a given element of E, I is the identity mapping in E, and x is the unknown.

57. More generally, let X be a compact space; let E be the space $\mathscr{C}(X, \mathbf{R})$ taken with the norm of uniform convergence, and let T be a linear mapping of E into E.

(a) Show that if $Tx \geqslant 0$ for every $x \geqslant 0$, then T is continuous. From now on we assume in addition that for every integer n we have

$$Tu + T^2u + \cdots + T^nu \leqslant ku,$$

where k is a constant $\geqslant 0$, and u is the unit element of E ($u(t) = 1$ for every $t \in X$).

(b) Show that

$$\lim_{n\to\infty}(T^nu(t)) = 0$$

for every t, and that $T^{p+q}u \leqslant kT^pu$ for all integers $p, q \geqslant 1$. Deduce from this that the sequence (T^nu) converges uniformly to O (use Problem 86, Chapter I).

(c) Show that for every $x \in E$, $\| T^nx \| \leqslant A \| x \| \lambda^n$, where A and λ are positive constants, with $\lambda < 1$.

(d) Deduce from this that the mapping $x \to x - Tx$ is an isomorphism of E onto E, and give an expression for the general solution of the inequality $x \geqslant Tx$.

COMPLETE NORMED SPACES

58. Let E be the subset of $\mathscr{C}([0, 1], \mathbf{R})$ consisting of the continuous functions x of bounded variation such that $x(0) = 0$; and for every $x \in E$, let $p(x)$ denote the total variation of x on [0, 1].

(a) Show that E is a vector space, and that p is a norm on this space.

(b) Show that p is not continuous on E taken with the norm of uniform convergence, but only lower semicontinuous.

(c) Show that E, taken with the norm p, is a complete normed space.

***59.** Given a number $p \geqslant 1$, for every $f \in \mathscr{C}(\mathbf{R}, \mathbf{C})$ we denote by $\| f \|$ the positive, finite or infinite number defined by

$$\| f \|^p = \limsup_{a\to\infty}(1/2a) \int_{-a}^{a} | f(t)|^p \, dt.$$

We then let B_p denote the subset of $\mathscr{C}(\mathbf{R}, \mathbf{C})$ consisting of the f such that $\| f \| < \infty$.

(a) Show that B_p is a vector subspace of $\mathscr{C}(\mathbf{R}, \mathbf{C})$ and that $\| f \|$ is a seminorm on B_p; show by an example that it is not a norm.

(b) For every $u \in \mathbf{R}$ and every $f \in B_p$, let f_u denote the function defined by $f_u(x) = f(x - u)$.
Show that $\| f_u \| = \| f \|$.

(c) Show that the set S_p of $f \in B_p$ such that $\| f \|^p$ is exactly the limit of

$$(1/2a) \int_{-a}^{a} |f(t)|^p \, dt$$

is closed in B_p.

(d) We denote by $\mathscr{B}_p$ the normed space associated with B_p (see Problem 27); $\mathscr{B}_p$ is called the *Besicovitch space* of index p. Show that this normed space is complete.

***60.** Let E and F be normed spaces, with E complete, and let A be a subset of $\mathscr{L}(E, F)$ such that, for every $x \in E$, the set of points $f(x)$, $f \in A$, is bounded (in F).

Let φ be the mapping of E into $\mathbf{R}_+$ defined by

$$\varphi(x) = \sup_{f \in A} \| f(x) \|.$$

(a) Show that φ is lower semicontinuous and convex.

(b) Deduce from the lower semicontinuity of φ and from the completeness of E that there exists a nonempty open set in E on which φ is bounded from above (use Problems 102, 103, 104 of Chapter I).

(c) Deduce from this that A is a bounded subset of $\mathscr{L}(E, F)$.

61. Let E and F be normed spaces, with E complete, and let (f_n) be a sequence of elements of $\mathscr{L}(E, F)$ such that, for every $x \in E$, the sequence $(f_n(x))$ has a limit $f(x)$.

Show, using the preceding problem, that $f \in \mathscr{L}(E, F)$.

***62.** Let E and F be normed spaces, with E complete, and let f be a linear mapping of E into F.

We assume that there exists a subset A of E whose complement is a countable union of nondense subsets of E, and such that the restriction of f to A is continuous (we point out that all the f which can be explicitly defined in Analysis have this property).

(a) Show that there exists a translate $A' = A + a$ of A containing O which has the same properties as A; show that there exists an open ball B with center O such that $\| f(x) \| \leqslant 1$ for every $x \in A' \cap B$.

(b) Using Problems 102, 103, and 104 of Chapter I, show that

$$B \subset \tfrac{1}{2}[(A' \cap B) + (A' \cap B)].$$

Deduce from this that f is continuous.

SEPARABLE NORMED SPACES

We recall that a metric space is said to be *separable* if it contains a finite or countable everywhere dense set.

63. (a) Show that for a normed space to be separable, it is necessary and sufficient that it contain a finite or countable total subset.

(b) Deduce from this that each of the spaces $\mathscr{C}^p([0, 1], \mathbf{K})$ is separable.

(c) Same question as (b), for the space $\mathscr{C}([0, 1], \mathbf{K})$ taken with the norm of mean convergence of order p (where $p \geqslant 1$).

(d) Show that a necessary and sufficient condition for l_I^p to be separable is that I be finite or countable.

64. Let E be a vector space with two norms p, q such that $p \leqslant kq$ (where $k > 0$).

Show that if E, taken with the norm q, is separable, then E taken with the norm p is also separable.

***65.** Let E be a normed space with a countable algebraic basis $(a_n)_{n \in \mathrm{N}}$ such that $\| a_n \| = 1$ for every n.

(a) Show that for every sequence (α_n) of numbers > 0 such that $\Sigma\, \alpha_n < \infty$, the sequence (b_n), where $b_n = \Sigma_0^n\, \alpha_i a_i$, is a Cauchy sequence.

(b) For every $n > 0$, we denote by d_n the distance from a_n to the vector space generated by $(a_0, a_1, \ldots, a_{n-1})$, and we define the sequence (α_n) recursively by the following conditions:

$$\alpha_0 = \alpha_1 = 1; \qquad \alpha_{n+1} = 3^{-1} \alpha_n d_n .$$

Show that the Cauchy sequence (b_n) associated with the sequence (α_n) does not converge in E.

(c) Deduce from this that no infinite-dimensional Banach space has a countable algebraic basis.

66. Let E denote the vector space of bounded sequences $x = (x_n)$ of complex numbers, taken with the norm $\| x \| = \sup_n | x_n |$.

(a) What is the adherence E_0 in E of the vector space of sequences $x = (x_n)$ which have only a finite number of terms $\neq 0$?

(b) What is the adherence in E of the vector space of sequences $x = (x_n)$ such that $\Sigma | x_n | < \infty$?

(c) Show that the set of sequences $x = (x_n)$ such that $\Sigma | x_n | \leqslant 1$ is closed in E.

(d) Show that E is not separable, while on the contrary E_0 is.

(e) Show that the vector subspace of E formed by the sequences $x = (x_n)$ such that $\lim_{n\to\infty} x_n$ exists is complete and separable.

67. (a) Show that for any distinct real numbers λ, μ, the element

$$f(t) = e^{i\lambda t} - e^{i\mu t}$$

of the space B_p defined in Problem 59 has norm $\geqslant 1$.

(b) Deduce from this that the normed space $\mathscr{B}_p$ associated with B_p is not separable (use the family of functions $e^{i\lambda t}$).

DISCONTINUOUS LINEAR MAPPINGS

68. Let E be a vector space over **K**, and let $\mathscr{L}$ be the collection of linearly independent subsets of E, ordered by inclusion.

(a) Show that $\mathscr{L}$ is an inductive set.

(b) Show that every maximal linearly independent subset of E generates E.

(c) Deduce from this that E contains an algebraic basis.

69. Let B be an algebraic basis of a vector space E over **K**, and let f be an arbitrary mapping of B into **K**.

(a) Show that there exists a unique linear functional g on E such that $g(x) = f(x)$ for every $x \in B$.

(b) We now assume that E is an infinite-dimensional normed space; let (a_n) be an infinite sequence of distinct points of B. We denote by f the function on B defined by

$$f(a_n) = n\| a_n \| \qquad \text{for every } n;$$

$$f(x) = 0 \qquad \text{if } x \text{ is not an } a_n .$$

Show that the linear functional g associated with f is not continuous.

(c) Using the same idea, show that the set of discontinuous linear functionals on an infinite-dimensional normed space has cardinality at least equal to that of the set of continuous linear functionals.

PRODUCTS OF NORMED SPACES AND DIRECT SUMS

70. Let E be a normed space, and let A and B be vector subspaces of E, of which E is the direct algebraic sum (in other words, $A + B = E$ and $A \cap B = \{O\}$).

For every $x \in E$, we denote by x_A and x_B the components of x in A and B.

(a) Show that if the mapping $x \to x_A$ is continuous, the same is true of the mapping $x \to x_B$, and that A and B are then closed in E.

(b) Show that the mapping $(u, v) \to u + v$ of the normed space $A \times B$ into E is then an isomorphism, and that if A and B are complete, so is E.

(c) Show that the continuity of x_A can be expressed by saying that the distance between the unit spheres of A and B is $\neq 0$.

***71.** Let E be the space $\mathscr{C}([0, \pi], \mathbf{R})$ taken with the norm of uniform convergence. Let E_1 (E_2) be the adherence of the subspace of E generated by the functions 1 and $\cos n!\, t$ ($x^n/n + \cos n!\, t$) (where $n \in \mathbf{N}^*$).

(a) Show that $E_1 \cap E_2 = \{O\}$.

(b) Show that $E_1 + E_2$ is distinct from E; deduce from this that an incomplete normed space can be the direct algebraic sum of two complete vector subspaces.

FINITE-DIMENSIONAL NORMED SPACES

72. Let E_p be the vector space of polynomials $P(x)$ with complex coefficients and degree $\leqslant p$.

Let $z_0, z_1, \ldots, z_p$ be distinct points of $\mathbf{C}$; put

$$\| P \| = | P(z_0) | + \cdots + | P(z_p) |.$$

Show that $\| P \|$ is a norm on E_p, and that the different norms obtained by varying the points z_i are equivalent. Is the mapping $P \to P'$, where $P'(z) = zP(z)$, of E_p into E_{p+1} continuous?

73. Show that if there exists, in a normed space E, a compact set with nonempty interior, then E is finite-dimensional.

***74.** Show that in every n-dimensional normed space E over $\mathbf{R}$, there exists a basis $(a_1, a_2, \ldots, a_n)$ of E such that if (x_i) denotes the coordinates of x in this basis, we have

$$\sum |x_i| \leqslant \|x\| \leqslant n \sum |x_i|.$$

(*Hint*: Use the function $\langle$determinant of $(b_1, b_2, \ldots, b_n)\rangle$ and seek to maximize it, subject to the condition that the b_i have norm 1).

Show, at least for $n = 2$, that the constant n in the inequality above cannot be replaced by a smaller constant.

75. Study the problem, analogous to the preceding one, obtained by replacing $\Sigma |x_i|$ by $(\Sigma |x_i|^2)^{1/2}$, and n by another constant.

****76.** Prove the property assumed without proof in Problem 84.

SUMMABLE FAMILIES OF REAL OR COMPLEX NUMBERS

77. Let (ω_i) be a family of subintervals of $[0, 1]$, and let a_i denote the length of ω_i.

Show that if $\Sigma a_i < 1$, the family (ω_i) cannot cover $[0, 1]$. More precisely, show that the union of the ω_i has a complement in $[0, 1]$ which cannot be finite or countable.

(First study carefully the case of a finite family; then pass to the general case by a suitable use of the theorem of Heine-Borel-Lebesgue.)

78. Construct a family $(a_{p,q})$ of real numbers (where $p, q \in \mathbf{N}$) such that

(a) $a_{p,q} = a_{q,p}$;
(b) for every p, the family $(a_{p,q})_{q\in\mathbf{N}}$ is summable with sum 0;
(c) the family $(a_{p,q})$ is not summable.

***79.** Let (a_n) be a sequence of complex numbers such that $\Sigma |a_n|^2 < \infty$; show that the family with general term $a_p a_q/(p + q)$ (where $p, q \geqslant 1$) is summable.

80. Let $x \in \mathbf{C}$, with $|x| < 1$; prove the identities

$$\sum_1^\infty \frac{x^{2n-1}}{1 - x^{2n-1}} = \sum_1^\infty \frac{x^n}{1 - x^{2n}}; \qquad \sum_1^\infty \frac{nx^n}{1 - x^n} = \sum_1^\infty \frac{x^n}{(1 - x^n)^2}.$$

SUMMABLE FAMILIES IN TOPOLOGICAL GROUPS AND NORMED SPACES

81. (a) Let G be an arbitrary topological group, and let A be a subset of G. Show that $\bar{A}$ is the intersection of the sets $A \cdot V$ (as well as $V \cdot A$), where V runs through the set $\mathscr{V}$ of neighborhoods of the identity e of G) (use the fact that the symmetric neighborhoods of e form a base for $\mathscr{V}$).

(b) Deduce from this that the closed neighborhoods of e form a base for $\mathscr{V}$.

82. Let E be a normed space, I an index set, and p a real number $\geqslant 1$. We denote by $l_I^p(E)$ the subset of $\mathscr{F}(I, E)$ consisting of the families $a = (a_i)_{i \in I}$ of elements of E such that $\Sigma_i \| a_i \|^p < \infty$.

(a) Show that $l_I^p(E)$ is a vector subspace of $\mathscr{F}(I, E)$ and that $(\Sigma_i \| a_i \|^p)^{1/p}$ is a norm on this space.

(b) Show that when E is complete, $l_I^p(E)$ is complete, and conversely.

(c) Suppose E is a normed algebra, and define a multiplication in $l_I^p(E)$ as follows:

If $a = (a_i)$ and $b = (b_i)$, the point $c = a \cdot b$ is the point $(a_i b_i)$.

Verify that this multiplication is meaningful, and that it makes $l_I^p(E)$ a normed algebra.

83. Let E be a complete normed space, and let $(a_i(n))_{i \in I}$ be a sequence of families of elements of E such that for every integer n, $\| a_i(n) \| \leqslant \lambda_i$, where the family of positive numbers λ_i is summable.

Show that if for every $i \in I$, $\lim_{n \to \infty} a_i(n) = a_i$, then the family (a_i) is summable and the sums $s_n = \Sigma_i a_i(n)$ converge to $s = \Sigma_i a_i$.

Extend this result to the case where the families $(a_i(n))$ are replaced by a family $(a_i(\lambda))$ depending on a parameter $\lambda \in L$, and where for every i, $a_i(\lambda)$ converges to a_i along a filter base $\mathscr{B}$ on L.

***84.** We shall use the following property:

There exists a sequence (α_n) of numbers > 0 which converges to 0, and such that for every $n \geqslant 1$ and for every n-dimensional normed space E over **R**, there exist n vectors $a_1, a_2, \ldots, a_n$ of norm 1 in E such that every partial sum of the a_i has norm $\leqslant n\alpha_n$.

Using this property and Problem 95, show that in every infinite-dimensional normed space E there exists a sequence (b_n) which is summable but not absolutely summable.

*85. We consider the mappings f of $\mathbf{R}$ into $\mathbf{R}$ such that for every summable family (a_i) of real numbers, the family $(f(a_i))$ is also summable.

(a) Show that these f are characterized by the fact that f is of Lipschitz class at the point 0, in a sense which it is required to make precise. (Use Problem 99.)

(b) Extend this result to the mappings f of one finite-dimensional vector space into another.

(c) Can it be extended to infinite-dimensional normed spaces?

86. Let $(a_i)_{i \in \mathrm{I}}$ be a finite family of elements of a vector space E over $\mathbf{R}$. We assume that the sum of every finite subfamily of the family (a_i) belongs to a convex subset C of E containing O.

Show that for every family $(\lambda_i)_{i \in \mathrm{I}}$ of elements of $[0, 1]$, the sum of every finite subfamily of the family $(\lambda_i a_i)$ also belongs to C.

One is advised to use one of the following methods:

(a) Use induction on the number of elements of I.

(b) First prove the property when $\mathrm{E} = \mathbf{R}$ and C is an affine halfline of $\mathbf{R}$; then prove it when E is finite-dimensional by using the fact that every closed convex set in E is an intersection of affine halfspaces.

87. Let $(a_i)_{i \in \mathrm{I}}$ be a family of elements of a normed space E over $\mathbf{C}$, and let $(\lambda_i)_{i \in \mathrm{I}}$ be a family of elements of $\mathbf{C}$ such that $|\lambda_i| \leqslant 1$ for every i.

Show that if the sum of every finite subfamily of $(a_i)_{i \in \mathrm{I}}$ is of norm $\leqslant k$, the sum of every finite subfamily of $(\lambda_i a_i)_{i \in \mathrm{I}}$ is of norm $\leqslant 4k$ (and even $\leqslant 2k$ if the λ_i are real).

88. Let E be a complete normed space over $\mathbf{K}$, and let $(a_i)_{i \in \mathrm{I}}$ be a summable family of elements of E.

(a) Show that for every bounded family $(\lambda_i)_{i \in \mathrm{I}}$ of elements of $\mathbf{K}$, the family $(\lambda_i a_i)$ is summable.

(b) Let $(\lambda_i)_{i \in \mathrm{I}}$ be a family of mappings of a set X into $\mathbf{K}$ such that for every $i \in \mathrm{I}$ and every $x \in \mathrm{X}$, $|\lambda_i(x)| \leqslant \mathrm{K}$ (where $\mathrm{K} < \infty$).
Show that the family $(\lambda_i(x) a_i)_{i \in \mathrm{I}}$ is uniformly summable on X; deduce from this that if X is a topological space and if the λ_i are continuous, the mapping $x \rightarrow \Sigma_i \lambda_i(x) a_i$ of X into E is continuous.

*(c) Use this result to prove that the set of sums $\Sigma_i \lambda_i a_i$, where the λ_i are restricted only by the condition $|\lambda_i| \leqslant 1$, is a convex compact set in E. (Use Problem 87 for this problem.)

89. Let (a_n) (where $n \geqslant 1$) be a sequence of elements of a normed space E, and put

$$b_{n,p} = \frac{na_n}{p(p+1)} \qquad \text{(where } 1 \leqslant n \leqslant p\text{)}.$$

Show that the families (a_n) and $(b_{n,p})$ are simultaneously summable (or nonsummable) and that their sums are equal. (Use Problem 86.)

SERIES; COMPARISON OF SERIES AND SUMMABLE FAMILIES

90. Let (a_n) be a sequence of numbers $\geqslant 0$ such that $\Sigma\, a_n = \infty$. What can one say about the convergence of the series with general term

$$a_n/(1 + a_n); \qquad a_n/(1 + a_n^2); \qquad a_n/(1 + na_n); \qquad a_n/(1 + n^2a_n)\,?$$

91. Let (a_n) and (b_n) be two positive convergent series; show that the series $(\sqrt{a_n b_n})$ is convergent.

92. Let (a_n) and (b_n) be two sequences of numbers > 0. Show that when $b_{n+1}/b_n \leqslant a_{n+1}/a_n$ for every n, the convergence of the series (a_n) implies that of the series (b_n).

93. Let (a_n) be a positive decreasing sequence; show that if the series (a_n) converges, the sequence (na_n) tends to 0 (consider the sums $\Sigma_n^{2n}\, a_i$). Is the converse true?

94. Let (a_n) be a sequence of numbers $\geqslant 0$, and put

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

If the series (a_n^2) converges, what can one say about the series (b_n^2)?

95. Let (α_n) be an increasing sequence of numbers > 0 which converges to $+\infty$. Show that there exists a positive convergent series (a_n) and a divergent series (A_n) such that $A_n \leqslant \alpha_n a_n$.

$$\left(\text{Take} \quad a_n = \frac{1}{\sqrt{\alpha_n - 1}} - \frac{1}{\sqrt{\alpha_n}} \quad \text{and} \quad A_n = \sqrt{\alpha_n} - \sqrt{\alpha_{n-1}}\,.\right)$$

96. Show that for every positive convergent series (a_n) there exists a positive increasing sequence (α_n) with limit $+\infty$ such that the series $(\alpha_n a_n)$ is still convergent.

97. Show that for every positive divergent series (a_n) there exists a positive decreasing sequence (α_n) with limit 0 such that the series $(\alpha_n a_n)$ is still divergent.

98. Let f_n be a sequence of increasing mappings of $\mathbf{R}_+$ into $\mathbf{R}_+$. Show that there exists an increasing mapping f of $\mathbf{R}_+$ into $\mathbf{R}_+$ such that, for every n,

$$\lim_{x\to\infty} f(x)/f_n(x) = +\infty.$$

Deduce from this that if the rapidity of convergence of a positive series is measured by the rapidity of growth of the quotient

$$\varphi(p) = 1/(a_{p+1} + a_{p+2} + \cdots),$$

then, for any given sequence of positive convergent series, there exists a positive series which converges faster than each series of this sequence.

99. Let (λ_n) be a sequence of numbers $\geqslant 0$ such that for every positive convergent series (a_n), the series $(\lambda_n a_n)$ is convergent. Prove that the sequence (λ_n) is bounded.

100. Let (a_n) be a decreasing sequence of numbers > 0, and let $k > 1$; for every integer $n \geqslant 1$, we denote by k_n the nearest integer to k^n.

Show that the series (a_n) and $(k_n a_{k_n})$ converge or diverge simultaneously.

Deduce from this a criterion for the convergence of the series $(n^{-\alpha})$ and similar series such as $(n^{-1}(\log n)^{-\alpha})$.

101. Let (z_n) be a sequence of complex numbers, with real part $\geqslant 0$. Show that if the series (z_n) and (z_n^2) converge, the series (z_n^2) is absolutely convergent.

102. Let f be a mapping of $\mathbf{R}$ into $\mathbf{R}$ with a continuous second derivative and such that $f(0) = 0$, and let (a_n) be a sequence of real numbers.

Show that if the series (a_n) and (a_n^2) converge, the same is true of the series $(f(a_n))$.

103. Let (a_n) be a sequence of real numbers with limit 0, and such that the series (a_n^+) and (a_n^-) are divergent.

Show that for every number $\lambda \in \mathbf{R}$, there exists a permutation π of $\mathbf{N}$ such that the series $(a_{\pi(n)})$ is conditionally convergent and has sum λ.

***104.** Find and prove a similar result concerning sequences (a_n) of elements of $\mathbf{R}^n$ (start with $n = 2$).

105. Let (a_n) be a sequence of elements of a normed space; show that if the series (na_n) converges, then the series (a_n) converges.

106. Study the convergence of the series with general term

$$(-1)^n/(\sqrt{n} + (-1)^n); \qquad (-1)^n/(2n + (-1)^n n); \qquad (-1)^n/(n + \cos n\pi).$$

***107.** Let (λ_n) be a sequence of elements of $\mathbf{K}$; prove the equivalence of the following properties:

(a) The series with general term $|\lambda_n - \lambda_{n+1}|$ is convergent.

(b) For every convergent series (a_n) of elements of $\mathbf{K}$, the series $(\lambda_n a_n)$ is convergent.

***108.** Let (a_n) be a sequence of elements of a normed space E. Show that if the series (a_n) is convergent, there exists a decreasing sequence (α_n) of positive numbers with limit 0 and a sequence (b_n) of elements of E such that $a_n = \alpha_n b_n$, and such that the set of sums $(b_1 + b_2 + \cdots + b_n)$ is bounded in E.

(The last two problems concern converses of Abel's rule.)

109. Let E and F be normed spaces, and f a continuous bilinear mapping of $\mathrm{E} \times \mathrm{F}$ into a complete normed space G. Let (a_n) $((b_n))$ be a sequence of elements of E (F).

Show that if the series (a_n) converges, and if the series $(\| b_n - b_{n+1} \|)$ converges, then the series $(f(a_n, b_n))$ converges.

110. Let (a_n) be a sequence of elements of a normed space; we put

$$s_n = a_1 + a_2 + \cdots + a_n .$$

Show that if the series (a_n) has sum s, then the sequence $((s_1 + s_2 + \cdots + s_n)/n)$ converges to s.

111. We consider the mappings f of $\mathbf{R}$ into $\mathbf{R}$ such that for every real convergent series (a_n), the series $(f(a_n))$ is convergent.

Show that these f are characterized by the fact that there exists a neighborhood of 0 in $\mathbf{R}$ on which f is linear.

***112.** Extend the preceding result to mappings f of $\mathbf{R}^2$ into $\mathbf{R}$, and then to mappings f of one topological vector space E into another F.

SUMMABLE SERIES AND FAMILIES OF FUNCTIONS

113. Let $(a_n z^n)$ be a power series with complex coefficients such that $\Sigma |a_n| < \infty$. Show that this series is uniformly convergent on the closed disk $\{z : |z| \leqslant 1\}$ of $\mathbf{C}$. What can be said about its sum?

114. For every $f \in \mathscr{C}(\mathbf{R}, \mathbf{R})$ we put

$$\|f\| = \int_{\mathbf{R}} |f(t)|\, dt;$$

let E be the subset of $\mathscr{C}(\mathbf{R}, \mathbf{R})$ consisting of the f such that $\|f\| < \infty$.

(a) Show that E is a vector space and that $\|f\|$ is a norm on E.

(b) Does the series with general term f_n $(n \geqslant 1)$, where

$$f_n(t) = (-1)^n e^{-n|t|},$$

converge in E? Does it converge uniformly on certain intervals of $\mathbf{R}$?

115. Show that the positive series with general term $x^2(1 + x^2)^{-(n+1)}$ is convergent for every $x \in \mathbf{R}$, and that it is uniformly convergent on every compact set not containing 0, but not on $[0, 1]$.

116. Show that for every compact set K in $\mathbf{C}$, the series with general term $(n^2 - z^2)^{-1}$, after the deletion of a finite number of terms, is uniformly convergent on K. What can be deduced concerning its sum?

117. For each of the series whose general term will be given below, determine the set of $x \in \mathbf{R}$ for which the series converges, then determine the compact sets in $\mathbf{R}$ on which these series converge uniformly. Next, study the same problem for $x \in \mathbf{C}$ (for the logarithm we assume $\log(1 + u)$ defined for $|u| < 1$ by its power series expansion):

$$n^{-3/2} \cos n^2 x; \qquad \sin x^n; \qquad \sin x/n^2; \qquad 2^n \sin x/3^n;$$

$$n^{3/2} \sin x/n^3; \qquad 1 - \cos x/n; \qquad 1/(n + x)^2; \qquad x \cos(2n + 1)x^2;$$

$$x^{(1+1/2+\cdots+1/n)}; \qquad \log(1 + x^n); \qquad \log(\cos x/n).$$

What can be said about the sums of these series?

***118.** Let (a_n) be a sequence of complex numbers, and (λ_n) an increasing sequence of real numbers with limit $+\infty$.

Show that if the series with general term $a_n e^{-\lambda_n z}$ (where $z \in \mathbf{C}$) converges at a point z_0, then it converges uniformly on every sector of the form $\{z : |\arg(z - z_0)| \leqslant k < \pi/2\}$. (First reduce it to the case where $z_0 = 0$, then imitate the proof of Abel's rule.)

119. Let f be a lower semicontinuous mapping of $[0, 1]$ into $\mathbf{R}_+$. Show, using Problem 9, Chapter II, that there exists a sequence (a_n) of continuous mappings of $[0, 1]$ into $\mathbf{R}_+$ such that, for every x, the series $(a_n(x))$ is convergent and has sum $f(x)$.

Show that if $f > 0$, one can require in addition that the a_n be polynomials.

120. Let (a_n) be a sequence of elements of $\mathscr{C}([0, 1], \mathbf{R})$ such that, for every $x \in [0, 1]$, the series with general term $|a_n(x)|$ is convergent.

Show that for every x the series with general term $a_n(x)$ is convergent, and that if $f(x)$ denotes its sum, then f is the difference of two lower semicontinuous functions.

Using the preceding problem, state and prove a converse of this property.

121. Let X be a set, E a complete normed space, (a_n) a sequence of mappings of X into E, and (λ_n) a decreasing sequence of mappings of X into $\mathbf{R}_+$.

(a) By slightly modifying the calculations involved in the proof of Proposition 10.14, show that if one puts

$$k_p(x) = \sup_{q \geqslant p} \| a_p(x) + \cdots + a_q(x) \|,$$

the series with general term $\lambda_n(x)a_n(x)$ is uniformly convergent on X when the sequence $(\lambda_p(x)k_p(x))$ tends to 0 uniformly on X.

(b) Show that this holds, in particular, in the following two cases:
  (1) The set of sums $a_0(x) + \cdots + a_n(x)$ is bounded on E, and the sequence $(\lambda_n(x))$ tends to 0 uniformly on X.
  (2) The series with general term $a_n(x)$ is uniformly convergent on X, and λ_0 is bounded.

122. Show that if a power series with general term $a_n x^n$ (where $a_n \in \mathbf{C}$) is convergent for $x = 1$, it is uniformly convergent on the interval $[0, 1]$.

Deduce from this, for example, using the classical power series expansion for $\log(1 + x)$, that

$$\log 2 = 1 - 1/2 + 1/3 - \cdots + (-1)^{n+1}/n + \cdots$$

123. Show that the sum of the series $((-1)^n/(2n^{1/2} + \cos x))$ (where $n \geqslant 1$) is uniformly convergent on $\mathbf{R}$ and that its sum is continuous.

124. Show that the series with general term

$$x \sin nx/(2n^{1/2} + \cos x) \qquad (\text{where } n \geqslant 1)$$

is uniformly convergent on every compact set in the open interval

$(-2\pi, 2\pi)$. What is the behavior of the sum in the neighborhood of the point 2π?

125. Let (a_n) be a convergent series of elements of a complete normed space E. Study the convergence and uniform convergence of the series with general terms

$$\left(\frac{x^n}{1+x^n}\right) a_n\,; \qquad \left(\frac{x^n}{1+x^{2n}}\right) a_n \qquad (\text{where } x \in [0, \infty)).$$

126. Study the summability and the uniform summability of the families with general terms

$$x^{p+q}; \qquad x^{pq}; \qquad (-1)^p x^{p(2q+1)} \qquad (\text{where } p, q \in \mathbf{N} \text{ and } x \in \mathbf{C}).$$

127. Let X and Y be topological spaces, $a \in \mathrm{X}$, $b \in \mathrm{Y}$, and let (f_n) be a sequence of mappings of X into Y such that

$$\lim_{n\to\infty} f_n(a) = b.$$

The sequence (f_n) is said to converge uniformly at the point a if, for every neighborhood V of b in Y, there exists a neighborhood U of a in X and an integer n_0 such that $f_n(\mathrm{U}) \subset \mathrm{V}$ for every $n \geqslant n_0$.

(a) Show that if the sequence (f_n) converges pointwise on X to a mapping f, and if b has a neighborhood base of closed sets, the uniform convergence of the sequence (f_n) at a and the continuity of the f_n at a implies the continuity of f at a.

(b) Suppose that Y is a normed space, and we are given a sequence (g_n) of mappings of X into Y, continuous at a, such that $\Sigma \| g_n(x) \|$ is everywhere finite, and continuous at a. Show that the series (g_n) converges uniformly at a.

*__128.__ Let (f_i) be a family of continuous mappings of a compact topological space X into a finite-dimensional normed space E. Show the equivalence of the following statements:

(a) The family (f_i) is uniformly summable.

(b) The family $(\| f_i(x) \|)$ is summable for every $x \in \mathrm{X}$, and its sum is a continuous function of x. When X is noncompact, show that the implication $a \Rightarrow b$ still holds, but one does not always have $b \Rightarrow a$ (give an example).

MULTIPLIABLE FAMILIES AND INFINITE PRODUCTS OF COMPLEX NUMBERS

129. Using the inequality $1 + x \leqslant e^x$, which is valid for all real x, prove by induction that for every finite family (a_i) of numbers $\geqslant 0$ (respectively, in $[-1, 0]$), one has

$$1 + \sum a_i \leqslant \prod (1 + a_i) \leqslant \exp\left(\sum a_i\right).$$

Deduce from this in a simple way that for every family $(a_i)_{i \in \mathrm{I}}$ of numbers $\geqslant 0$ (respectively, in $[-1, 0]$) the multipliability of the family $((1 + a_i))_{i \in \mathrm{I}}$ and the summability of the family $(a_i)_{i \in \mathrm{I}}$ are equivalent.

130. Let (a_n) be a sequence of numbers $\geqslant 0$ (with $a_0 > 0$); put

$$b_n = a_n/(a_0 + a_1 + \cdots + a_n).$$

Show that the series (a_n) and (b_n) converge and diverge simultaneously, and that when they converge, one has

$$a_0 \Big/ \sum a_n = \prod_1^{\infty} (1 - b_n).$$

131. Let (a_n) be a sequence of numbers $\geqslant 0$. Show that the series with general term $a_n/(1 + a_0)(1 + a_1) \cdots (1 + a_n)$ is always convergent and its sum is

$$1 - \left(1 \Big/ \prod_i (1 + a_i)\right).$$

132. Let $(a_i)_{i \in \mathrm{I}}$ be a summable family of elements of $\mathbf{C}$, and let $\mathscr{F}$ denote the collection of finite subsets of I. For every $\mathrm{K} \in \mathscr{F}$ we put

$$u_{\mathrm{K}} = \prod_{i \in \mathrm{K}} a_i \qquad (\text{and} \quad u_{\mathrm{K}} = 1 \quad \text{if} \quad \mathrm{K} = \emptyset).$$

Show that the family $(u_{\mathrm{K}})_{\mathrm{K} \in \mathscr{F}}$ is summable and that its sum is equal to

$$\prod_i (1 + a_i).$$

133. Let P denote the set of prime numbers $p \geqslant 2$. Deduce from the preceding problem that for every number $s > 1$, the family of numbers $(1 - p^{-s})^{-1}$ (where $p \in \mathrm{P}$) is multipliable and has product

$$\sum_0^{\infty} n^{-s}.$$

134. Show, by the same method, that for every $x \in \mathbf{C}$ such that $|x| < 1$, the family $((1 + x^{2^n}))_{n \in \mathrm{N}}$ is multipliable and has product $1/(1 - x)$.

135. Let x be an element of $\mathbf{C}$ such that $|x| < 1$; we put

$$Q_1 = \prod_1^\infty (1 + x^{2n}); \qquad Q_2 = \prod_1^\infty (1 + x^{2n-1}); \qquad Q_3 = \prod_1^\infty (1 - x^{2n-1}).$$

Show that $Q_1 Q_2 Q_3 = 1$.

136. Let k be a complex number such that $|k| > 1$. For every $z \in \mathbf{C}$ we put

$$P(z) = \prod_1^\infty (1 + z/k^n).$$

(a) Prove that this product is absolutely convergent and that

$$P(kz) = (1 + z)P(z).$$

(b) For every $z \neq 0$, put

$$S(z) = P(z)P(1/z)(1 + z).$$

Show that $S(kz) = kzS(z)$.

(c) Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two finite sequences of complex numbers $\neq 0$ such that $a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_n$; put

$$M(z) = \frac{S(a_1 z) \cdots S(a_n z)}{S(b_1 z) \cdots S(b_n z)}.$$

Show that $M(z) = M(kz)$; what does this imply for the function $u \to M(e^u)$?

(d) Show that P is the uniform limit, on every compact set, of polynomials in z. What can one conclude about P? Study S and M similarly.

137. Show that the infinite product $\prod_n (1 + i/n)$ is not convergent, but that the infinite product of the absolute values $|1 + i/n|$ is convergent.

138. Let (u_n) be a sequence of complex numbers such that

$$|u_n| \leqslant kn^{-\alpha} \qquad (\text{where } k > 0 \text{ and } \alpha > 0).$$

Show that there exists an integer p, depending only on α, such that the convergence of the infinite product of the numbers $1 + u_n$ is equivalent to the convergence of the series with general term

$$v_n = u_n - u_n^2/2 + \cdots + (-1)^{p+1}u_n^p/p.$$

139. Using the fact that the multiplicative topological group $\mathbf{R}_+^*$ is isomorphic to the additive group $\mathbf{R}$, define a metric on $\mathbf{R}_+^*$ compatible with the topology of $\mathbf{R}_+^*$ and invariant under the "translations" of this group.

Show that $\mathbf{R}_+^*$, taken with such a metric, is complete.

140. We denote by U the multiplicative group of complex numbers z such that $|z| = 1$, taken with the topology induced by that of $\mathbf{C}$.

(a) Using the fact that the topological group $\mathbf{C}^*$ is isomorphic to the product $\mathbf{R}_+^* \times \mathrm{U}$, show that there exists a metric on $\mathbf{C}^*$ compatible with its topology, and invariant under the "translations" of $\mathbf{C}^*$.

(b) Show that $\mathbf{C}^*$, taken with such a metric, is complete.

(c) Use this to obtain a new proof of the fact that every family of elements of $\mathbf{C}^*$ satisfying the Cauchy criterion (for multiplication) is multipliable in $\mathbf{C}^*$.

NORMED ALGEBRAS

141. Show that every norm on $\mathbf{C}$ (considered as a vector space over $\mathbf{R}$) which satisfies the relation $\| xy \| = \| x \| \| y \|$ is identical with the classical norm. Would the same conclusion hold if it were assumed only that $\| x^2 \| = \| x \|^2$ for every x?

142. Is the norm

$$\|f\| = \sup |f(x)| + \sup |f'(x)| + \cdots + \sup |f^{(n)}(x)|$$

on the vector space $\mathscr{C}^{(n)}([0, 1], \mathbf{R})$ compatible with its algebraic structure?

143. Let $(a_n \mathrm{X}^n)$ and $(b_n \mathrm{U}^n)$ be two formal series with complex coefficients, with $b_0 = 0$.

Let $(c_n \mathrm{U}^n)$ be the formal series obtained by substituting for X the formal sum $\Sigma\, b_n \mathrm{U}^n$.

Let A be a Banach algebra; we assume that the series $(a_n \mathrm{X}^n)$ and $(b_n \mathrm{U}^n)$ have radius of convergence equal to α and β, respectively, and we put

$$f(x) = \sum a_n x^n \qquad \text{for every } x \text{ such that } \| x \| < \alpha;$$

$$g(u) = \sum b_n u^n \qquad \text{for every } u \text{ such that } \| u \| < \beta.$$

Show that for every u such that $\| u \| < \beta$ and $\Sigma | b_n | \| u \|^n < \alpha$, the series $(c_n u^n)$ is absolutely convergent, and that its sum is $f(g(u))$.

144. Let A be a Banach algebra with a unit e.

(a) Show that for every $u \in A$ such that $\| u \| < 1$, the power series

$$u - u^2/2 + \cdots + (-1)^{n+1}u^n/n + \cdots$$

is convergent and that its sum is continuous on the open ball $B(O, 1)$; we shall denote this sum by $\log(e + u)$.

(*b*) Show, using the preceding problem, that

$$\exp(\log(e + u)) = e + u \qquad \text{for every} \quad u \in B(O, 1).$$

$$\log(\exp x) = x \qquad \text{for every } x \text{ such that} \quad \| x \| < \tfrac{1}{2}.$$

Deduce from this that the image of the open ball $B(O, \frac{1}{2})$ under the mapping $x \to \exp x$ is an open neighborhood of e, and that the restriction of this mapping to $B(O, \frac{1}{2})$ is a homeomorphism.

ELEMENTARY PROPERTIES OF PREHILBERT SPACES

145. Let f be an arbitrary bilinear functional on a vector space E over **C**. Show that there exist $x \neq O$ in E such that $f(x, x) = 0$.

146. Let E be a normed space over **K**; show that if, for every vector subspace F of dimension 2 in E, the norm of F is associated with a scalar product on F, then the norm of E is also associated with a scalar product on E.

147. Let E be a normed space over **K** such that, for all $x, y \in E$,

$$\| x + y \|^2 + \| x - y \|^2 = 2\| x \|^2 + 2\| y \|^2.$$

Show that the norm of E is associated with a scalar product.

148. By a quadratic form on a vector space E over **R** is meant a mapping f of E into **R** such that for all $x, y \in E$ and all $\lambda, \mu \in \mathbf{R}$,

$$f(\lambda x + \mu y) = a\lambda^2 + 2b\lambda\mu + c\mu^2,$$

where a, b, and c depend on x and y only.

Show that $b(x, y)$ is a bilinear functional on E, and that $b(x, x) = f(x)$.

Deduce from this that if a norm p on E is such that p^2 is a quadratic form, then $b(x, y)$ is a scalar product on E, and p is the norm associated with it.

149. Show that every vector space E over **K** can be provided with a scalar product (use an algebraic basis of E).

150. By a *sesquilinear* form on a vector space E (over **K**) is meant any mapping $(x, y) \to f(x, y)$ of $E \times E$ into **K** which is linear in x and conjugate-linear in y.

Let f be a continuous sesquilinear form on a normed space E. We denote by $b(f)$ (respectively, $q(f)$) the infimum of the positive numbers k such that

$$|f(x, y)| \leqslant k\|x\|\,\|y\| \qquad \text{(respectively, } |f(x, x)| \leqslant k\|x\|^2\text{)}.$$

(a) Show that $q \leqslant b \leqslant 4q$.

(b) Show that when $f(x, x)$ is real for every x, or when the norm of E is associated with a scalar product, one can replace 4 by 2; and show that when both conditions are satisfied, then $q = b$.

151. Let E be a prehilbert space over **C**, and let f be a sesquilinear form on E.

Show that the image of $E \dot{-} O$ under the mapping $x \to f(x, x)/\|x\|^2$ is a convex subset of **C**. From this, using Problem 149, deduce that for every sesquilinear form f on a vector space E, the image of $E \dot{-} O$ under the mapping $x \to f(x, x)$ is convex.

152. Let E and F be prehilbert spaces over **K**, and let f be a mapping of E into F such that $f(O) = O$, and such that for every $x, y \in E$

$$\|f(x) - f(y)\| = \|x - y\| \qquad \text{(in other words, } f \text{ is an isometry).}$$

Show that f is linear.

153. Show that the norm of the space $l_I^2(E)$ defined in Problem 82 can be defined by a scalar product only when E is itself a prehilbert space.

154. Let $(E_i)_{i \in I}$ be a family of prehilbert spaces over the same field **K**, whose scalar products and norms will be denoted by $(x \mid y)_i$ and $\|x\|_i$, respectively. We denote by F the vector space of families $x = (x_i)_{i \in I}$ such that $x_i \in E_i$, and for every $x \in F$ we put

$$f(x) = \sum_i (x_i \mid x_i)_i .$$

(a) Show that the set E of x such that $f(x) < \infty$ is a vector subspace of F.

(b) Show that for all $x, y \in$ E, the family of numbers $(x_i \mid y_i)_i$ is summable and that its sum, which we shall denote by $(x \mid y)$, is a scalar product on E.

(c) Show that the prehilbert space E (which is called the *direct sum* of the E_i) is complete when each of the E_i is complete (and conversely).

155. Let f be a continuous mapping of **R** into **C** such that

$$\int_{\mathbf{R}} |f(t)|^2 \, dt < \infty.$$

Show that

$$\left| \int_{\mathbf{R}} f(t)f(t-a) \, dt \right| \leqslant \int_{\mathbf{R}} |f(t)|^2 \, dt$$

for every $a \in \mathbf{R}$.

156. Let (a_n) be a sequence of elements of **C** such that $\Sigma \mid a_n \mid^2 < \infty$. Show that

$$\left| \sum a_n a_{n+1} \right| \leqslant \sum | a_n |^2.$$

157. We denote by P the subset of the Hilbert space l^2 over **R** consisting of the points $x = (x_n)$ such that $| x_n | \leqslant 1/n$.

Show that P is a compact subset of l^2 (one can use the criterion for the compactness of metric spaces and Proposition 9.23).

158. Let D be a domain of $\mathbf{R}^n$, with finite total volume, and let E be the set of real harmonic functions f in D such that the intergal of f^2 over D is finite.

(a) Show that E is a vector space, and that

$$(f \mid g) = \int_{\mathrm{D}} f(x)g(x) \, dx$$

is a scalar product on E. We denote the corresponding norm by $\| f \|$.

(b) Show that for every $f \in$ D,

$$\int_{\mathrm{D}} |f(x)| \, dx \leqslant k \| f \|,$$

where k denotes a constant to be determined.

(c) Using an elementary mean value property of harmonic functions, show that if $\lim_{n\to\infty} \| f_n \| = 0$, the sequence (f_n) converges uniformly to O on every compact set.

(d) Deduce from this that the prehilbert space E is complete.

159. Let F and G be vector subspaces of a real prehilbert space E. Show that if there exists a number $k \geqslant 0$ such that

$$|(x \mid y)| = k \| x \| \| y \|$$

for every $x \in F$ and $y \in G$, then either F and G have dimension 1, or $k = 0$, that is, F and G are orthogonal.

160. Let E be a prehilbert space over **K**, and let F and G be vector subspaces of E which do not coincide with {O}. By the *angle* between F and G is meant the angle α defined by

$$0 \leqslant \alpha \leqslant \pi; \qquad \cos\alpha = \sup \frac{|(u \mid v)|}{\| u \| \| v \|} \qquad \text{where} \qquad u \in F, \quad v \in G.$$

Now let φ denote the mapping $(u, v) \to u + v$ of the normed space $F \times G$ onto the subspace $F + G$ of E.

Show that a necessary and sufficient condition for φ to be a homeomorphism is that $\alpha \neq 0$.

161. Let X and Y be two normed spaces isomorphic to l^2, and let φ be the mapping of X into Y defined as follows:

If $x = (x_n)$, then $\varphi(x)$ is the point $(n^{-1}x_n)$.

Show that the graph Γ of φ in $X \times Y$ is a vector subspace of $X \times Y$ isomorphic to X.

Show that if we denote by X′ the subspace $X \times O$ of $X \times Y$, then $X' + \Gamma$ is everywhere dense on $X \times Y$ but is not identical with it.

162. Let E be a prehilbert space over **R**, and let $E \times E$ be provided with the structure of the direct sum of E with E (Problem 154). Let f be a linear bijective mapping of E to E which is norm preserving.

For every $k \in \mathbf{R}$, we denote by $\Gamma(k, f)$ the graph in $E \times E$ of the mapping $x \to kf(x)$ of E into E. Show that the angle α of every vector of $\Gamma(k, f)$ with $\Gamma(k', f)$ is a constant defined by the relation

$$\cos\alpha = \frac{| 1 + kk' |}{(1 + k^2)^{1/2}(1 + k'^2)^{1/2}}.$$

ORTHOGONAL PROJECTION. STUDY OF THE DUAL

163. Let X be a complete convex subset of a prehilbert space E over **R**. Using Theorem 15.1, show that X is an intersection of closed affine halfspaces of E (defined as sets of the form $\{x : f(x) \leqslant k\}$, where $f \in E'$ and $k \in \mathbf{R}$).

Show that when X is a cone, X is an intersection of closed halfspaces of the form $\{x : f(x) \leqslant 0\}$, where $f \in E'$.

164. Let (X_n) be a decreasing sequence of complete convex subsets of a prehilbert space E. For every $x \in E$ we denote by $d_n(x)$ the distance from x to X_n, and put

$$d(x) = \lim_{n\to\infty} d_n(x).$$

Show that if $d(x) < \infty$ for at least one x, the same is true for every x; we assume this to be the case from this point on. We then denote by $A(x, \epsilon, n)$ the intersection of X_n and the closed ball with center x and radius $(d(x) + \epsilon)$.

(a) Show that when ϵ tends to 0 and n tends to $+\infty$, the diameter of $A(x, \epsilon, n)$ tends to 0.

(b) Deduce from this that the intersection X of the X_n is nonempty, and that $d(x) = d(x, X)$.

165. Let Y be a bounded complete convex subset of a prehilbert space E, and let f be a numerical lower semicontinuous convex function on Y. Show, using the preceding problem, that f is bounded from below on Y, and that the set of points x of Y where f attains its infimum is a nonempty complete convex set.

Give an application of this property to the continuous linear functionals on E, and compare with the example of Problem 43.

166. Let A and B be complete convex subsets of a prehilbert space E, at least one of which is bounded. Show that there exists $a \in A$ and $b \in B$ such that

$$d(a, b) = d(A, B).$$

167. Extend the result of the preceding problem to the case where A and B are unbounded, but where $d(x, y)$ tends to infinity as $\| x \|$ and $\| y \|$ tend to infinity with $x \in A$ and $y \in B$. Show by an example in $\mathbf{R}^2$ that if this condition is not satisfied, the assertion can be false.

168. Let (C_n) be an increasing sequence of complete convex subsets of a prehilbert space E, such that $C = \overline{\bigcup C_n}$ is also complete.

For every $x \in E$, we denote by $P_n(x)$ ($P(x)$) the projection of x on C_n (C); show that

$$P(x) = \lim_{n \to \infty} P_n(x).$$

169. Let E be a prehilbert space over **R**, C a complete convex subset of E, and f a continuous linear functional on E.

Show that there exists a unique point of C at which $\| x \|^2 + f(x)$ attains its infimum.

170. Let E be a prehilbert space and E′ its topological dual. For every $a \in E$, we denote by φ_a the linear functional $x \to (x \mid a)$ on E. Show that the image $\varphi(E)$ of E in E′ under the conjugate-linear mapping φ is an everywhere dense vector subspace of E′.

Deduce from this that every prehilbert space E is isomorphic to an everywhere dense vector subspace of a Hilbert space.

171. Let V_1 and V_2 be affine varieties of a prehilbert space E, and let $x_1 \in V_1$, $x_2 \in V_2$.

Show the equivalence of the following properties:

(a) $d(x_1, x_2) = d(V_1, V_2)$.

(b) $x_1 - x_2$ is orthogonal to V_1 and V_2.

172. Let E be a real Hilbert space, and let P be a closed convex cone in E. We denote by P^* the set of $x \in E$ such that $(x \mid y) \leqslant 0$ for every $y \in P$. Show that P^* is also a closed convex cone and that $(P^*)^* = P$ (use Problem 163).

173. Let E be a real Hilbert space, and let P and Q be closed convex cones in E such that $P^* = Q$ (from which also $Q^* = P$ by the preceding problem).

Show the equivalence of the following properties:

(a) $z = x + y, \quad x \in P, \quad y \in Q; \quad (x \mid y) = 0.$

(b) $x =$ projection of z on P; $y =$ projection of z on Q.

174. Let E be a prehilbert space, and F a vector subspace of E such that $F = F^{00}$.

Show by an example that it is not always true that every point of E has a projection on F.

175. Let P be a mapping of a prehilbert space E into itself which satisfies the relations

$$(P(x) \mid y) = (x \mid P(y)); \qquad P(P(x)) = P(x) \qquad \text{for all } x, y \in E.$$

Show that P is linear and continuous; show next that if we put

$$F = \{x : P(x) = x\},$$

then $F = F^{00}$ and P is identical with the operator of projection of E on F.

176. Let E be a prehilbert space. Show that if P is a linear mapping of E into E such that $P^2 = P$ and $\| P \| \leqslant 1$, then P is a projection operator.

177. (a) Show that if $P_1, P_2, \ldots, P_n$ are projection operators (on a prehilbert space E), to say that $P_1 + \cdots + P_n$ is a projection operator is equivalent to saying that $P_i P_j = O$ for every $i \neq j$.

(b) Now assume E complete. Show that if (P_n) is an infinite sequence of projection operators such that $P_i P_j = O$ for $i \neq j$, then the family (P_n) is summable in $\mathscr{L}(E)$ and its sum is a projection operator.

178. Let E be a Hilbert space, and let A be an element of $\mathscr{L}(E)$.

(a) Show that for every $y \in E$, the mapping $x \to (Ax \mid y)$ is a linear functional on E; deduce from this that there exists a unique element $A_y{}^*$ of E such that $(Ax \mid y) = (x \mid A_y{}^*)$ for every $x \in E$.

(b) Show that the mapping $y \to A_y{}^*$ of E into E is linear, and that the norm of A^* is equal to that of A. The operator A^* is called the *adjoint* of the operator A; A is called *selfadjoint* if $A = A^*$, in other words, if

$$(Ax \mid y) = (x \mid Ay)$$

for all $x, y \in E$.

(c) More generally, let E be a prehilbert space, and let $A, B \in \mathscr{L}(E)$; we shall say that B is the *adjoint* of A if $(Ax \mid y) = (x \mid By)$ for all $x, y \in E$.

Show that every $A \in \mathscr{L}(E)$ has at most one adjoint, and that if B is the adjoint of A, then A is the adjoint of B.

179. Prove the following elementary properties:

(a) $(AB)^* = B^*A^*$.

(*b*) If A and B are selfadjoint, the selfadjointness of AB is equivalent to the relation $AB = BA$.

(c) If A is selfadjoint, then B^*AB is selfadjoint for any B.

(*d*) If $A = A^*$, then $((Ax \mid x) = 0$ for every $x)$ is equivalent to $A = O$.

(e) To say that $A = A^*$ is equivalent to saying that $(Ax \mid x)$ is real for every x.

180. Let $(k_i)_{i \in I}$ be a bounded family of elements of $\mathbf{C}$, and let A be the mapping of the complex space l_I^2 into itself defined as follows:

If $x = (x_i)_{i \in I}$, Ax is the element $(k_i x_i)_{i \in I}$.

What is the norm of A? What is its adjoint? When is A selfadjoint?

181. We denote by E the space $\mathscr{C}([0, 1], \mathbf{C})$ taken with the scalar product

$$(x \mid y) = \int_0^1 x(t)\bar{y}(t)\,dt;$$

let $k \in E$ and let $K \in \mathscr{C}([0, 1]^2, \mathbf{C})$. We define the elements A and B of $\mathscr{L}(E)$ by the relations

$$Ax(t) = \int_0^1 K(t, u)x(u)\,du; \qquad Bx(t) = k(t)x(t).$$

(a) Show that A and B have adjoints, although E is not complete; under what conditions are A and B selfadjoint?

(b) For every $x \in E$, put

$$Cx(t) = \int_0^1 k(t)u^{-1/4}x(u)\,du, \qquad \text{where} \quad k \in E.$$

Show that $C \in \mathscr{L}(E)$ and calculate its norm, and show that C does not have an adjoint which operates in E.

182. Let E be a prehilbert space, and let A be an element of $\mathscr{L}(E)$ which has an adjoint A^*; we say that A is *positive* if $A = A^*$ and if $(Ax \mid x) \geqslant 0$ for every $x \in E$. We say that A is *positive definite* if in addition $(Ax \mid x) \neq 0$ for every $x \neq O$.

Characterize those of the operators studied in the preceding problem which are positive or positive definite.

183. Let E be a prehilbert space, and let $A \in \mathscr{L}(E)$. Prove the equivalence of the following three properties:

(a) $A^*A = I$ (where I is the identity).

(b) $(Ax \mid Ay) = (x \mid y)$ for all $x, y \in E$.

(c) $\| Ax \| = \| x \|$ for every $x \in E$.

When A has these properties, A is called a *unitary* operator (or orthogonal operator, when **K** is the field **R**), or more simply an isometry.

184. Let E be a prehilbert space, and let $A \in \mathscr{L}(E)$; we shall say that A is a *symmetry* if $(I + A)/2$ is a projection operator.

Using the preceding problem and Problem 175, show that the symmetries are simply the operators which are both unitary and selfadjoint.

185. Let (x_n) be a sequence of points of a prehilbert space E; show that the strong convergence of (x_n) to O is equivalent to the convergence, uniformly on the set of a such that $\| a \| \leqslant 1$, of the sequence $((x_n \mid a))$ to 0. Extend this property to filter bases.

***186.** Let (x_n) be a sequence of points of a Hilbert space E such that, for every $a \in E$, the sequence $((x_n \mid a))$ has a limit. Show that the x_n converge weakly to a point $x \in E$, that the sequence $(\| x_n \|)$ is bounded, and that

$$\| x \| \leqslant \liminf_{n \to \infty} \| x_n \|.$$

(One can use Problems 60 and 61.)

***187.** Let (x_n) be a sequence of points of a Hilbert space E. Show that if this sequence converges weakly to a point x, there exists a subsequence (x_{α_n}) such that the sequence (b_n), where

$$b_n = 1/n \sum_1^n x_{\alpha_p},$$

converges strongly to x.

188. Show that in an infinite-dimensional prehilbert space E, the point O does not have a countable neighborhood base for the weak topology. Deduce from this that the weak topology of E cannot be defined by a metric.

189. Let E be a prehilbert space, (x_n) a *bounded* sequence of points of E, and X a total subset of E.

Show that for the sequence (x_n) to converge weakly to a point x, it is necessary and sufficient that for every $a \in X$ the sequence $((x_n \mid a))$ converge to $(x \mid a)$. Show, on the other hand, that one can find an unbounded sequence (x_n) in l^2 (which therefore cannot be weakly convergent) and a total subset X of l^2 (for example, the canonical orthonormal basis of l^2) such that for every $a \in X$, the sequence $((x_n \mid a))$ converges to 0.

190. Let E be the space $\mathscr{C}([0, 1], \mathbf{K})$ taken with the scalar product

$$(x \mid y) = \int_0^1 x(t)\bar{y}(t)\, dt.$$

Put

$$x_n(t) = \sin \pi n t;$$

show, using the total subset X of E consisting of the functions $\sin \pi p t$ and $\cos \pi p t$, and the preceding problem, that the sequence (x_n) converges weakly to O.

More generally, let (y_n) be a sequence of functions with continuous derivatives y_n' on [0, 1] and such that

$$\lim_{n\to\infty}(\sup_t \mid y_n(t)\mid) = 0;$$

show that the sequence (y_n') converges weakly to O in E.

ORTHOGONAL SYSTEMS

191. Let E be a prehilbert space; for every finite sequence $(x_1, x_2, ..., x_n)$ of points of E, the *Gramian* of the x_i is defined as the scalar $G(x_1, x_2, ..., x_n)$ equal to the determinant of the scalar products $(x_i \mid x_j)$.

(a) Show that $G(x_1, ..., x_n) \geqslant 0$ and that the relation $G(x_1, ..., x_n) > 0$ is equivalent to the linear independence of the x_i (make use of an orthonormal basis in the vector space generated by the x_i).

(b) Show that if the x_i are linearly independent, the square of the distance from any point $x \in E$ to the vector space L generated by the x_i is equal to $G(x, x_1, ..., x_n)/G(x_1, ..., x_n)$. (Use the projection of x on L.)

192. Show that every infinite-dimensional Hilbert space E is isomorphic to a vector subspace of E which is distinct from E.

193. Let E be a prehilbert space with an infinite orthonormal basis B. Show that $\bar{\bar{X}} \geqslant \bar{\bar{B}}$ for every everywhere dense subset X of E, and show that there exists an X for which $\bar{\bar{X}} = \bar{\bar{B}}$ (see 16.16).

194. Let E be an arbitrary topological vector space, and let $(a_i)_{i\in I}$ be a family of elements of E. We shall say that this family is *topologically free* if for every $i \in I$, a_i does not belong to the adherence of the vector subspace of E generated by the a_j with index $j \neq i$.

(a) Show that in a prehilbert space, every orthogonal family is topologically free.

(b) Show that every topologically free family is linearly independent, and show by an example that the converse is false.

195. Let E be a topological vector space, and let B be an algebraic basis of E which is topologically free.

Show that the vector subspace of E generated by an arbitrary subset of B is closed.

(a) Deduce from this, using Problem 65, that if E is a Banach space, E is necessarily finite dimensional.

(b) Give an example where B is infinite and E is normed (necessarily incomplete).

196. Let E be a topological vector space. We shall say that a subset A of E is *topologically very free* if it does not contain O, and if for every partition of A into nonempty subsets A_1 and A_2, the vector subspaces generated by A_1 and A_2 have adherences whose intersection is $\{O\}$.

(a) Verify that every orthogonal system in a prehilbert space is topologically very free.

(b) Verify that the canonical "basis" of every l_I^p space also has this property.

*(c) Show by an example that even in a prehilbert space, a family can be topologically free without being very free.

197. Let E be the space $\mathscr{C}([0, 1], \mathbf{R})$ taken with the scalar product

$$(x \mid y) = \int_0^1 x(t)y(t)\,dt.$$

Let K be a nonempty and nondense compact set in [0, 1], and let A_K denote the vector subspace of E formed by the functions which vanish at every point of K.

(a) Show that every $x \in E$ such that $x \perp A_K$ equals O.

(b) Show that if the measure of K is zero (in the sense that for every $\epsilon > 0$ there exists a covering of K by a finite family of intervals, the sum of whose lengths is $<\epsilon$), then $\overline{A_K} = E$. Show that otherwise $\overline{A_K} \neq E$.

(*c*) Deduce from this that there exists a maximal orthogonal system in E which is not total.

198. Let E be the vector subspace of the real space l^2 generated by the vectors of the canonical basis of l^2 with odd indices and by the vectors

$$x_k = (1/1^k, \ldots, 1/n^k, \ldots) \qquad (\text{where } k \in \mathbf{N}^*).$$

Show that the orthogonal system in E consisting of the vectors of the canonical basis of l^2 with odd indices is maximal.

199. Let E be an incomplete prehilbert space. Using Problem 170, show that there exists a closed hyperplane H in E such that there exists no $x \neq O$ in E orthogonal to H.

200. Let $\mathscr{B}_2$ be the Besicovitch space of order 2 defined in Problem 59. For every $x, y \in \mathscr{B}_2$ we put

$$[x, y] = \limsup_{a \to \infty} (1/2a) \int_{-a}^{+a} | x(t)y(t)| \, dt.$$

(a) Show that $[x, y] < \infty$, and more precisely that $[x, y] \leqslant \| x \| \| y \|$.
Show that $[x, y]$ is a subscalar product in the sense of Problem 35.

(b) Let A be a vector subspace of S_2 (see Problem 59). Show that for every $x, y \in A$,

$$(1/2a) \int_{-a}^{a} x(t)\bar{y}(t) \, dt$$

has a finite limit as $a \to \infty$, which we shall denote by $(x \mid y)$. Deduce from this that the canonical image of A in $\mathscr{B}_2$ is a prehilbert space.

(c) Let P denote the vector space of linear combinations of functions $e^{i\lambda t}$ (where $\lambda \in \mathbf{R}$).
Show that $P \subset S_2$ and that the family of functions $e^{i\lambda t}$ is an orthonormal system in the prehilbert space P. Deduce from this that the canonical image of P in $\mathscr{B}_2$ has closure $\mathscr{P}$ which is a Hilbert space whose geometric dimension is the cardinality of the continuum.

(d) Show that every element $x \in \mathscr{P}$ is the sum, in $\mathscr{P}$, of a sequence of functions $a_n e^{i\lambda_n t}$ (where the λ_n are distinct) such that $\Sigma | a_n |^2 < \infty$, and that, conversely, every such sequence is summable in $\mathscr{P}$.

(e) Study successively the sequences obtained by setting $a_n = n^{-2}$, λ_n arbitrary, and then $a_n = n^{-1}$, $\lambda_n = n^{-1}$.

How can the difficulty encountered in the second example be explained?

ORTHOGONAL POLYNOMIALS

201. Let E be the subset of $\mathscr{C}([0, 1], \mathbf{R})$ consisting of the functions x such that

$$\int_0^1 x^2(t)\, dt/t < \infty.$$

(a) Show that E is a vector subspace of $\mathscr{C}([0, 1], \mathbf{R})$ and that every function x which has a finite derivative at 0 and satisfies $x(0) = 0$ belongs to E.

(b) For every $x, y \in$ E, we put

$$(x \mid y) = \int_0^1 x(t)y(t)\, dt/t.$$

Verify that this integral is meaningful and that it is a scalar product on E.

(c) The Gram-Schmidt orthogonalization procedure applied to the sequence (t^n) (where $n \geqslant 1$) yields polynomials $P_1, \ldots, P_n, \ldots$. Calculate P_1, P_2, and P_3 explicitly.

(d) Show that the set of functions in the prehilbert space E which are zero on an interval of the form $[0, a]$ (where $a \neq 0$) is everywhere dense on E. Deduce from this that the orthogonal family (P_n) is total in E.

202. Same as Problem 201, with the weight t^{-1} replaced by $t^{-\alpha}$ (where $\alpha \in \mathbf{R}_+$).

203. Let p be a function $\geqslant 0$ and continuous on the open interval $I = (-1, 1)$, such that:

(a) The set of points $t \in I$ at which $p(t) \neq 0$ is everywhere dense on I.

(b) There exists a polynomial A, not identically zero, such that

$$\int_I A^2(t)p(t)\, dt < \infty.$$

Now let E be the vector subspace of $\mathscr{C}([0, 1], \mathbf{R})$ consisting of the functions x such that

$$\| x \|^2 = \int_{\mathrm{I}} x^2(t)p(t)\, dt < \infty.$$

Show, by a method analogous to that of Problem 201, that the set of polynomials of the form AP (where P is a polynomial) is total in the space E taken with the Hilbert norm $\| x \|$.

204. We put

$$Q_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n.$$

(a) Show, using integration by parts, that the sequence of polynomials Q_n is an orthogonal system on $[-1, 1]$ for the scalar product

$$(x \mid y) = \int_{-1}^{1} x(t)\bar{y}(t)\, dt,$$

and that

$$(Q_n \mid Q_n) = \frac{2}{2n + 1} .$$

(b) Verify that $Q_n(1) = 1$ and deduce from this that Q_n is related to the Legendre polynomial P_n defined in the text by

$$Q_n(t) = P_n(t)/P_n(1),$$

and calculate $P_n(1)$.

(c) Show that the Q_n satisfy the recursion relation

$$nQ_n = (2n - 1)Q_{n-1} - (n - 1)Q_{n-2}$$

and that every Q_n satisfies the differential equation

$$\frac{d}{dt}\left((1 - t^2)\frac{dQ_n}{dt}\right) + n(n + 1)Q_n = 0.$$

205. Let E be the prehilbert space E_p (see 17.3) obtained for $p(t) = e^{-t}$ on the interval $[0, \infty)$.

(a) We put

$$L_n(t) = \frac{1}{n!} e^t \frac{d^n}{dt^n} (e^{-t}t^n).$$

Verify that L_n is a polynomial of degree n, that the L_n form an orthogonal system in E, and calculate $(L_n \mid L_n)$.

(b) Verify that $L_n(0) = 1$, and deduce from this that L_n is related to the Laguerre polynomial P_n defined in the text by

$$L_n(t) = P_n(t)/P_n(0),$$

and calculate $P_n(0)$.

(c) Show that for every $\alpha \geqslant 0$, the function $e^{-\alpha t}$ belongs to E, and calculate its Fourier coefficients $\gamma_{n,\alpha}$ with respect to the orthogonal system (L_n).

(d) Show that the family $(\gamma_{n,\alpha} L_n)$ has sum $e^{-\alpha t}$ in E.

(e) Show, by means of a suitable change of variables and the Stone-Weierstrass theorem, that the functions e^{-nt} $(n \in \mathbf{N})$ form a total system in E.

(f) Deduce from questions (*e*) and (*d*) that the polynomials L_n constitute an orthogonal basis of E.

206. (a) Verify that the nth derivative of $\exp(-t^2)$ is of the form $(-1)^n H_n(t) \exp(-t^2)$, where H_n is a polynomial of degree n.

(b) Show that the polynomials H_n are orthogonal in the space E_p associated with the weight $p(t) = \exp(-t^2)$ on $[0, \infty)$. Deduce from this that they are proportional to the polynomials which were called the Hermite polynomials in the text.

(c) Verify the recursion relation

$$H_n = 2tH_{n-1} - 2(n-1)H_{n-2}\,; \qquad H_0 = 1; \qquad H_1(t) = 2t.$$

(d) Show that H_n satisfies the differential equation

$$H_n'' - 2tH_n' + 2nH_n = 0,$$

and that

$$H_n' = 2nH_{n-1}\,.$$

DEFINITIONS AND AXIOMS

We are concerned only with vector spaces over $\mathbf{K}$ (where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$).

Axioms of topological vector spaces. A topological vector space E is a set with a vector space structure and a topology such that:

TVS 1: The topology of E is compatible with the additive group structure of E;

TVS 2: The mapping $(\lambda, x) \to \lambda x$ of $\mathbf{K} \times E$ into E is continuous.

Total set. A subset X of a TVS E is said to be *total* if the set of linear combinations of elements of X is everywhere dense on E.

Seminorm. A *seminorm* on a vector space E is a mapping p of E into $\mathbf{R}_+$ such that, for all $x, y \in E$ and every $\lambda \in \mathbf{K}$,

$$p(\lambda x) = |\lambda| p(x); \tag{1}$$

$$p(x + y) \leqslant p(x) + p(y). \tag{2}$$

One says that p is a *norm* if $p(x) \neq 0$ for every $x \neq O$.

The open *p-ball* with center a and radius $\rho > 0$ in E is the set $\{x : p(x - a) < \rho\}$.

$\mathscr{P}$-topology on a vector space E. Let $\mathscr{P} = (p_i)$ be a family of seminorms on E. Every finite intersection of open p_i-balls with center a is called an *open $\mathscr{P}$-ball* with center a in E.

The *$\mathscr{P}$-topology* on E is the topology whose open sets are arbitrary unions of open $\mathscr{P}$-balls.

Normed space. A *normed space* is a vector space together with a norm p.

It is taken with the topology associated with the metric $d(x, y) = p(x - y)$.

When it is complete relative to this metric, it is called a *Banach space*.

Summable families in a separated commutative topological group G. A family $(a_i)_{i \in I}$ of elements of G is said to be *summable*, with sum A (where $A \in G$), if for every neighborhood V of A there exists a finite subset J_0 of I such that, for every finite subset J of I containing J_0, one has

$$\sum_{i \in J} a_i \in V.$$

Absolutely summable families in a normed space E. A family (a_i) of elements of E is said to be *absolutely summable* if the family of their norms $\| a_i \|$ is summable in $\mathbf{R}$.

Multipliable families in $\mathbf{C}$. A family $(a_i)_{i \in I}$ of elements of $\mathbf{C}$ is said to be multipliable in $\mathbf{C}$, with product p, if for every $\epsilon > 0$ there exists a finite subset J_0 of I such that, for every finite subset J of I containing J_0, one has

$$\left| p - \prod_{i \in J} a_i \right| \leqslant \epsilon.$$

Normed algebra. A *normed algebra* is an algebra over $\mathbf{K}$ together with a norm such that $\| xy \| \leqslant \| x \| \| y \|$.

When in addition it is complete in this norm, it is called a *Banach algebra.*

Hermitean forms. A *hermitean form* on a vector space E over $\mathbf{K}$ is a mapping φ of $E \times E$ into $\mathbf{K}$ such that:

(1) For every y, $\varphi(x, y)$ is linear in x.

(2) For every $x, y \in E$, $\varphi(y, x) = \overline{\varphi(x, y)}$.

It is said to be *positive* if $\varphi(x, x) \geqslant 0$ for every x, and *positive definite* if in addition $\varphi(x, x) \neq 0$ for every $x \neq O$; in this last case one can show that $\varphi(x, x)^{1/2}$ is a norm on E.

Prehilbert space. A *prehilbert space* is a vector space together with a positive-definite hermitean form (generally denoted by $(x \mid y)$) and the norm associated with this form.

When E is complete in this norm, it is called a *Hilbert space.*

Orthogonal basis. An *orthogonal basis* of a prehilbert space E is a family (a_i) of nonzero elements of E such that:

(1) This family is total in E.

(2) The a_i are pairwise orthogonal (that is, $(a_i \mid a_j) = 0$ if $i \neq j$).

When in addition $\| a_i \| = 1$ for every i, (a_i) is called an *orthonormal basis*.

NOTATION

$B(a, \rho)$	2.5	l_1^p	8.8
$\mathscr{B}(X, \mathbf{K})$	3.1	$\mathscr{L}(E)$, $\mathscr{L}(E, F)$	1.7, 4.6
$\mathscr{C}(X, \mathbf{K})$	3.1	$\mathscr{L}(E_1, ..., E_n ; F)$	6.2
$\mathscr{C}^r([0, 1]^n, \mathbf{K})$	3.2	$p(x)$	2.1
$\mathscr{C}^\infty([0, 1]^n, \mathbf{K})$	3.3	$P_X(x)$	15.1
$\mathscr{D}_K{}^r(\mathbf{R}^n, \mathbf{K})$	3.4	$(x \mid y)$, xy	14.5
$\mathscr{D}^r(\mathbf{R}^n, \mathbf{K})$	3.9	$X \perp Y$	14.11
$\mathscr{E}^r(A, \mathbf{K})$	3.8	X^0, X^{00}	14.15
$\mathscr{E}^\infty(A, \mathbf{K})$	3.8	$\sum_{i \in I} a_i$	9.1
E', E^*	1.8	$\prod_{i \in I} a_i$	12.1
$\mathscr{F}(X, \mathbf{K})$	3.10		
$\mathbf{K}$	1.1	$\| x \|$	2.1
l^p	3.5	$\| f \|$	4.6

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