CHAPTER I

Topological Spaces and Metric Spaces

INTRODUCTION

General topology has formed a coherent doctrine only for the last half century; it is the outcome of a movement of ideas which goes back to antiquity.

The notions of limit and continuity intruded upon the Greek mathematicians as soon as they tried to make precise the notion of number. It was then necessary to await Cauchy (1821) and Abel (1823) for a clarification of the notions of convergent sequences and series, and the notion of a continuous function.

With Riemann (1851) the scope of the problem broadened; in his inaugural lecture, "On the hypotheses which serve as the foundations of geometry," he outlined a grandiose program: The study of "the general meaning of an entity obtained by a successive increase of dimensionality," by which he meant not only varieties of an arbitrary number of dimensions, but also spaces of functions and of sets.

But such a program could not be carried out without a better understanding of the real line (Dedekind) and of numerical functions (Riemann, Weierstrass), nor, above all, without a language both precise and general; it was Cantor (1873) who created this language and opened the door to a new world.

A heroic and fruitful period then commenced. Despite the opposition of mathematicians who were hostile to new ideas, discoveries followed one another, particularly in France (Poincaré, Hadamard, Borel, Baire, Lebesgue) and in Germany (Klein, Mittag-Leffler). This rapidly led to the study of functions of lines and the creation of a functional analysis (Ascoli, Volterra, Hilbert) which is a first step in the realization of Riemann's program.

But once again the need for a language and a framework adapted to these studies made itself felt: The metric spaces, defined by Fréchet, provided a tool which is essential for the study of uniform continuity and uniform convergence, and is also convenient for the study of topological structures. Finally, Hausdorff succeeded in extracting, from a jungle of axioms, a simple axiomatic system which is the cornerstone of present-day general topology.

We shall begin the study of general topology with an elementary study of the real line, whose importance has not been diminished by modern research. The definitions and statements of properties will be formulated in such a way as to be immediately generalizable to arbitrary topological spaces; it is in this framework that we shall then study most topological properties.

A topological space can be a curve or a surface as well as a space of curves or functions; thus each of the results which we shall formulate summarizes a host of particular results and can be applied to a great number of problems. But it is only gradually that we shall discover the great variety of applications to analysis and geometry.

Numerous examples will illustrate the definitions and theorems; nevertheless certain results will be motivated only subsequently. Thus, the study of general topology requires a certain act of faith from the outset, which however will be made easier by the internal beauty of this theory.

I. TOPOLOGY OF THE LINE R

1. OPEN SETS, CLOSED SETS, NEIGHBORHOODS, BOUNDS OF A SET

1.1. Definition. A SUBSET A OF **R** IS SAID TO BE *OPEN* IF IT IS EMPTY OR IF, FOR EVERY $x \in A$, THERE EXISTS AN OPEN INTERVAL CONTAINING x AND CONTAINED IN A.

In other words an open set in **R** is a set which is the union of open intervals.

The following assertions are an almost immediate consequence of this definition.

O_1 : Every union (finite or infinite) of open sets is open;
O_2 : Every *finite* intersection of open sets is open;
O_3 : The line **R** and the empty set Ø are open sets.

Property O_1 results from the fact that every union of sets, each of which is a union of open intervals, is itself a union of open intervals.

To prove property O_2, it is sufficient to prove it for the intersection of two open sets A, B:

By hypothesis

$$A = \bigcup_i A_i \qquad \text{and} \qquad B = \bigcup_j B_j ,$$

where the A_i and B_j are open intervals.

Therefore

$$A \cap B = \left(\bigcup_i A_i\right) \cap \left(\bigcup_j B_j\right) = \bigcup_{i,j} (A_i \cap B_j).$$

Since each of the sets $A_i \cap B_j$ is either empty or an open interval, $A \cap B$ is open.

Finally, property O_3 is obvious.

EXAMPLE 1. Every open interval is an open set.

EXAMPLE 2. The union of the open intervals $(n, n+1)$ (where $n \in \mathbf{Z}$) is an open set.

On the other hand, a closed interval $[a, b]$ is not an open set.

It is false that the intersection of an infinite number of open sets is always open. For example, the intersection of the open intervals $(-1/n, 1/n)$ $(n = 1, 2, ...)$ is the set consisting of the point 0, which is not open.

1.2. Definition. A SUBSET A OF **R** IS SAID TO BE *CLOSED* WHEN ITS COMPLEMENT $\complement_{\mathbf{R}} A$ IS OPEN.

Each of the properties O_1, O_2, O_3 at once implies a dual property for closed sets. We shall simply state these, as their proof is immediate.

F_1 : Every intersection of closed sets is closed;
F_2 : Every finite union of closed sets is closed;
F_3 : The line **R** and the empty set $\varnothing$ are closed sets.

EXAMPLE. Every closed interval $[a, b]$ (where $a \leqslant b$) is a closed set. Indeed, the complement of $[a, b]$ is the union of the two open intervals $(-\infty, a)$ and (b, ∞), and is therefore an open set.

It should be observed that a set can be neither open nor closed. This is, for example, the case for **Q**.

1.3. Definition. ANY SUBSET V OF **R** WHICH CONTAINS AN OPEN SET CONTAINING A POINT x OF **R** IS CALLED A *NEIGHBORHOOD* OF x.

In other words, V is a neighborhood of x if V contains an open interval containing x.

For example, every open set A is a neighborhood of each of its points. Conversely, every set A which is a neighborhood of each of its points is a union of open intervals, and therefore open.

If x and y are two arbitrary distinct points such that $x < y$, there exists a neighborhood V_x of x and a neighborhood V_y of y such that

$V_x \cap V_y = \emptyset$; indeed, if z is an arbitrary point between x and y, it suffices to take

$$V_x = (-\infty, z) \quad \text{and} \quad V_y = (z, \infty).$$

The meaning which we have just given to the word "neighborhood" appears different from that which it has in ordinary usage, since for us a point x of **R** has many neighborhoods, and one of them is the space **R** itself.

Indeed, we have rather enriched a notion which has up to now been unprecise, since we can henceforth say that a given point y belongs to a particular neighborhood of x; this neighborhood V, as it were, makes the degree of proximity of y to x more precise.

Accumulation Points of a Set. *If* A *is a subset of* **R**, *a point* x_0 *of* **R** *is called an* ***accumulation point*** *of* A *if, in every neighborhood of* x_0, *there exists at least one point of* A *different from* x_0.

There thus exists an infinity of points of A, different from x_0, in every neighborhood of x_0; otherwise there would exist an open interval (a, b) containing x_0 and containing only a finite number of points (x_i) of A.

There would thus exist an interval (a', b') which intersects A at most at the point x_0 (take for a' the largest of the x_i smaller than x_0, if such exist, or otherwise the point a; choose the point b' similarly). But this is excluded by hypothesis.

An accumulation point of a set does not necessarily belong to the set. For example, the point 0 is an accumulation point of the set of points $x_n = 1/n$ (n an integer > 0), but does not belong to this set. Again, the points 0 and 1 are accumulation points of (0, 1) without belonging to this interval.

1.4. Proposition. *Every closed set contains its accumulation points. Conversely, every set which contains its accumulation points is closed.*

Let A be a closed set; if $x \in \complement A$, the open set $\complement A$ is a neighborhood of x and does not contain any point of A. Thus x cannot be an accumulation point of A.

Conversely, if A is such that no point of $\complement A$ is an accumulation point of A, there exists for each $x \in \complement A$ a neighborhood of x not containing any point of A, and therefore contained in $\complement A$; the set $\complement A$ is thus a neighborhood of each of its points, i.e., it is open; in other words, A is closed.

Isolated Points. *An* ***isolated*** *point of a set* A *is a point* x *of* A *which* *is not an accumulation point of* A. *In other words, it is a point x of* A *which has a neighborhood* V *such that* $A \cap V = \{x\}$.

EXAMPLE. Set $A = [0, 1] \cup \mathbf{N}$; the isolated points of A are the integers $n \geqslant 2$.

Existence of the Supremum and Infimum. We have defined in Volume 1, Chapter I what is called the supremum of a subset A of an ordered set E. This supremum does not always exist, even if A is a subset of E which is bounded from above.

For example, if E is the totally ordered set of rationals $\geqslant 0$, the subset A of elements x of E such that $x^2 < 2$ does not have a supremum in E although it is evidently bounded from above.

The definition of **R** given in Volume 1, Chapter III ensures, on the other hand, that this cannot happen in **R**. Because of the great importance of this property, we shall repeat this statement here:

Fundamental Property of **R**. *Every nonempty subset of* **R** *which is bounded from above* (*from below*) *has a supremum* (*infimum*).

Let A be a nonempty subset of **R** which is bounded from above, and let b be its supremum. The halfline $(-\infty, b]$ contains A and is clearly the smallest closed negative halfline containing A.

For every $x < b$, $[x, b]$ intersects A; therefore either $b \in A$ or b is an accumulation point of A.

In particular, if A is closed, it contains its supremum b; this point is then the greatest element of A.

Similar properties clearly hold for the infimum.

REMARK. When a set A is not bounded form below (from above), one often says that $-\infty$, $(+\infty)$ is its infimum (supremum). Later we shall give a precise justification of this language.

Bounded Sets. *A nonempty subset* A *of* **R** *is said to be* ***bounded*** *if it is bounded from above and from below*; *in other words, if* A *is contained in a closed interval* $[a, b]$.

By the preceding theorem, in order that A be bounded it is necessary and sufficient that it have an infimum and a supremum; if these are a_0 and b_0, the closed interval $[a_0, b_0]$ is the smallest closed interval containing A.

If A is bounded and closed, a_0 and b_0 belong to A and are the smallest and largest elements of A.

Diameter. *For every* $A \subset \mathbf{R}$, *we call the supremum* $\delta(A)$ (*finite or* $+\infty$) *of the distances between two points of* A *the* ***diameter*** *of* A.

If $\delta(A)$ is finite, then for every $x \in A$ the set A is bounded from above and from below by $(x + \delta(a))$ and $(x - \delta(A))$, respectively, and is thus bounded. Conversely, if A is bounded and has a and b $(a \leqslant b)$ as its respective bounds, the diameter of A is finite and equals $(b - a)$.

2. LIMIT OF A SEQUENCE. THE CAUCHY CRITERION FOR CONVERGENCE

Let (a_i) $(i = 1, 2, ..., n, ...)$ be an infinite sequence of points of **R**. We say that this sequence converges to l, or that l is the limit of this sequence, if for every neighborhood V of l we have $a_i \in V$ except for at most a finite number of values of i (one can obviously take for the neighborhoods V only open intervals containing l).

This limit l is unique; for, let l_1 and l_2 be two distinct points of **R**, and V_1, V_2 disjoint neighborhoods of l_1 and l_2; if, for every i except at most finitely many, one has $a_i \in V_1$, then $a_i \in V_2$ is possible for only a finite number of the a_i; therefore if l_1 is the limit of the sequence, l_2 is not.

2.1. Theorem. *Every increasing (decreasing) sequence which is bounded from above (from below) has a limit.*

Indeed, let us suppose, for example, that the given sequence (a_i) is increasing, and let A be the set of points a_i. This set is nonempty and bounded from above, and therefore has a supremum l. But every open interval V containing l contains at least one point a_{i_0}, and thus also all the a_i with index $i > i_0$. Therefore l is the limit of the sequence.

The Cauchy criterion for convergence

Up to now we have used only the order structure of **R**; we are now, for the first time, going to use its group structure.

To say that the sequence (a_i) converges to l amounts to saying that for every $\epsilon > 0$ there exists an integer n such that

$$(i \geqslant n) \Rightarrow (| a_i - l | \leqslant \epsilon).$$

It follows that

$$(i \text{ and } j \geqslant n) \Rightarrow (| a_i - a_j | \leqslant 2\epsilon).$$

The remarkable thing about this inequality is that l does not enter into it; we shall see, conversely, that every sequence having this property is convergent. To be precise:

2.2. Definition. A SEQUENCE (a_i) $(i = 1, 2, ..., n, ...)$ IS CALLED A *CAUCHY SEQUENCE* IF $| a_i - a_j |$ TENDS TO 0 WHEN i AND j TEND TO $+\infty$, IN OTHER WORDS, IF FOR EVERY $\epsilon > 0$ THERE EXISTS AN INTEGER n SUCH THAT

$$(i \text{ and } j \geqslant n) \Rightarrow (| a_i - a_j | \leqslant \epsilon).$$

It amounts to the same thing to say that if A_n denotes the set of points (a_i) with index $i \geqslant n$, the decreasing sequence of diameters $\delta(A_n)$ has limit 0.

2.3. Theorem (the Cauchy criterion). *Every Cauchy sequence of points of* **R** *is convergent.*

PROOF. We already know that every convergent sequence of points of **R** is a Cauchy sequence; it is the converse which we want to establish.

Let (a_i) be a Cauchy sequence. With the above notation, the sequence $(\delta(A_n))$ tends to 0. Now let α_n and β_n $(\alpha_n \leqslant \beta_n)$ be the infimum and supremum of A_n. Since the sequence (A_n) is decreasing, the sequence of the α_n is increasing and that of the β_n is decreasing.

The increasing sequence of the α_n is bounded from above by β_1; therefore it has a limit α; similarly the β_n have a limit β.

Since α and β belong to every interval $[\alpha_n, \beta_n]$, we have

$$| \beta - \alpha | \leqslant | \beta_n - \alpha_n |$$

and thus

$$\beta - \alpha = 0.$$

Let us put

$$= \alpha = \beta.$$

For every n we have

$$l \in [\alpha_n, \beta_n] \qquad \text{and} \qquad a_n \in A_n \subset [\alpha_n, \beta_n],$$

therefore

$$| l - a_n | \leqslant \max (| l - \alpha_n |, | l - \beta_n |).$$

Thus $(l - a_n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore the sequence (a_n) converges to l.

We shall frequently use the Cauchy criterion to prove that a sequence is convergent. And we shall extend it to more general spaces than the real line.

3. COMPACTNESS OF CLOSED BOUNDED INTERVALS

One of the most important properties of closed bounded intervals is expressed by the Heine-Borel-Lebesgue theorem which allows the study of open coverings of such an interval to be reduced to the study of finite subcoverings. From this theorem follows immediately another property known as the Bolzano-Weierstrass theorem.

3.1. Definition. A COVERING OF A SET A ON THE LINE BY OPEN SETS IN **R** IS CALLED AN *OPEN COVERING* OF A.

3.2. Theorem (of Heine-Borel-Lebesgue). *Every open covering of a bounded closed interval* $[a, b]$ *has a finite subcovering.*

Explicitly, this means that for every family $(\omega_i)_{i \in I}$ of open sets in **R** such that

$$[a, b] \subset \bigcup_{i \in I} \omega_i ,$$

there exists a finite subset $J \subset I$ such that

$$[a, b] \subset \bigcup_{i \in J} \omega_i .$$

PROOF. Let $(\omega_i)_{i \in I}$ be the family of open sets which covers $[a, b]$ (where $a < b$, as the theorem is obvious if $a = b$).

Let us denote by A the set of points x of $[a, b]$ such that the interval $[a, x]$ can be covered by a finite number of the open sets ω_i; to prove the theorem amounts to showing that $b \in A$. But A is nonempty since it contains a, and it is bounded from above by b. Therefore it has a supremum m belonging to $[a, b]$.

There exists an $i_0 \in I$ such that $m \in \omega_{i_0}$; but ω_{i_0} is a neighborhood of m and there exist points x in A, in this neighborhood, which lie to the left of m and such that $[x, m] \subset \omega_{i_0}$. For such an x, $[a, x]$ has a finite covering by the ω_i; therefore $[a, m] = [a, x] \cup [x, m]$ also has such a covering. But every finite subfamily of the ω_i which covers $[a, m]$ also covers some interval $[a, m']$, where $m' > m$. This is compatible with the fact that m is the supremum of A only if $m = b$.

Z It is essential to note at this point that the assertion of Theorem 3.2 cannot be extended to unbounded intervals, or to intervals which are bounded but not closed. For example, the sequence of open sets $(1/n, 2)$ (where $n \geqslant 2$) covers the semi-open interval $(0, 1]$, but no finite subsequence of this sequence has this property.

3.3. Theorem (of Bolzano-Weierstrass). *Every infinite subset* X *of a closed bounded interval* $[a, b]$ *has an accumulation point in* $[a, b]$.

Equivalent Statement. *Every subset* X *of* $[a, b]$ *which has no accumulation point in* $[a, b]$ *is finite.*

Proof. If no point x of $[a, b]$ is an accumulation point of X, every x has an open neighborhood V_x containing at most one point of X, namely, x itself. These V_x form an open covering of $[a, b]$; by the preceding theorem, there exists a finite number of these V_x, say V_{x_i} $(i = 1, 2, ..., n)$, which cover $[a, b]$. Thus X contains at most the points x_i $(i = 1, 2, ..., n)$.

Z Here again let us note that the assertion of Theorem 3.3 does not extend either to bounded intervals which are not closed or to unbounded intervals. For example, the infinite sequence of points $1/n$ of the semi-open interval $(0, 1]$ has no accumulation point in $(0, 1]$; in fact its only accumulation point in $\mathbf{R}$ is the point 0, which does not belong to $(0, 1]$.

We shall not for the moment go further into the study of the topology of the line. Actually, many topological properties of the line are valid for spaces which are much more general than the line, and whose introduction into analysis, far from being artificial, has become indispensable for the proof and the discovery of many properties.

We shall similarly be content to give several definitions concerning the topology of Euclidean spaces and shall then begin the study of general topological spaces. When we begin this study, it would be well to keep in mind the most concrete special cases, which consist of the line $\mathbf{R}$, the spaces $\mathbf{R}^n$ and their subsets. In the sequel, the metric spaces will constitute another rather intuitive subject, from which we shall be able to draw examples and counterexamples.

4. TOPOLOGY OF THE SPACE $\mathbf{R}^n$

Let us recall that the space $\mathbf{R}^n$ is the product of n spaces identical with $\mathbf{R}$, and is therefore the set of ordered sequences $(x_1, x_2, ..., x_n)$ of n real numbers; but up to now we have introduced on this set only an algebraic structure (of a vector space). We are now going to put a topological structure on it.

4.1. Definition. LET ω_i $(i = 1, 2, \ldots, n)$ BE AN OPEN SET IN $\mathbf{R}$ WHICH IS EITHER AN OPEN INTERVAL (a_i, b_i) OR THE EMPTY SET $\varnothing$. THE SUBSET $\omega_1 \times \omega_2 \times \cdots \times \omega_n$ OF $\mathbf{R}^n$ IS CALLED AN *OPEN INTERVAL* OF $\mathbf{R}^n$ WITH BASES ω_i (IT IS EMPTY IF ONE OF ITS BASES IS EMPTY); WHEN IT IS NON-EMPTY, ITS CENTER IS THE POINT WITH COORDINATES $x_i = (a_i + b_i)/2$.

The intersection of two open intervals with bases (ω_i), (ω_i') is the open interval with bases $(\omega_i \cap \omega_i')$.

One could similarly define the *closed intervals* by replacing the open intervals ω_i by closed intervals.

4.2. Definition. EVERY UNION OF OPEN INTERVALS IN $\mathbf{R}^n$ IS CALLED AN OPEN SET IN $\mathbf{R}^n$.

Thus, to say that a set A in $\mathbf{R}^n$ is open is equivalent to saying that for every $x \in \mathrm{A}$ there exists an open interval containing x and contained in A (if it is useful, one can require that this interval have x as its center).

EXAMPLES. Every open interval in $\mathbf{R}^n$ is an open set in $\mathbf{R}^n$. On the other hand, a line in $\mathbf{R}^2$ is not an open set in $\mathbf{R}^2$.

Let us verify that the family of open sets in $\mathbf{R}^n$ satisfies properties O_1, O_2, O_3 of Section 1. (In O_3 replace $\mathbf{R}$ by $\mathbf{R}^n$.) This is immediate for O_1 and O_3; let us show it for O_2:

If A and A′ are open sets, we have

$$\mathrm{A} = \bigcup_{i \in \mathrm{I}} p_i \qquad \text{and} \qquad \mathrm{A}' = \bigcup_{j \in \mathrm{J}} p_j'$$

where the p_i and p_j' are open intervals.

Thus

$$\mathrm{A} \cap \mathrm{A}' = \bigcup_{(i,j) \in \mathrm{I} \times \mathrm{J}} (p_i \cap p_j').$$

But $(p_i \cap p_j')$ is an open interval; therefore $\mathrm{A} \cap \mathrm{A}'$ is open. This result extends by induction to every finite intersection of open sets.

Closed Sets, Neighborhoods, Accumulation points, etc. In $\mathbf{R}^n$, a set is said to be *closed* if its complement is open; a set is a *neighborhood* of a point x if it contains an open set containing x; a point x is called an *accumulation point* of a set A if, in every neighborhood of x, there exists at least one point of A which is different from x.

We could study the consequences of these definitions in detail, as we have done for **R**. But at this point it is more instructive to carry out this study in a more general context.

II. TOPOLOGICAL SPACES

In the elementary study of **R** and $\mathbf{R}^n$ which we have just carried out, almost all of the concepts have been defined in terms of open sets, and most of the properties have been obtained by using only properties O_1, O_2, O_3 of the open sets. Hence the idea of basing topology on the notion of open set; we shall try to express all of the classical topological notions, such as limit and continuity, in terms of open sets, and to obtain as many as possible of the classical theorems, starting from some simple hypotheses concerning the collection of open sets.

5. OPEN SETS, CLOSED SETS, NEIGHBORHOODS

5.1. Definition. A *TOPOLOGICAL SPACE* IS A PAIR CONSISTING OF A SET E AND A COLLECTION $\mathcal{O}$ OF SUBSETS OF E CALLED *OPEN* SETS SATISFYING THE FOLLOWING THREE PROPERTIES:

O_1 : EVERY UNION (FINITE OR NOT) OF OPEN SETS IS OPEN;
O_2 : EVERY *FINITE* INTERSECTION OF OPEN SETS IS OPEN;
O_3 : THE SET E AND THE EMPTY SET $\emptyset$ ARE OPEN.

One also says that the collection $\mathcal{O}$ of subsets of E defines a topology on E.

One can define several topologies on every set E, except when E contains at most one point. One of these is the *discrete topology*; this is the one for which $\mathcal{O}$ is the collection of all subsets of E. It is the topology which admits the largest possible number of open sets.

Another is the *coarse topology*; this is the one for which $\mathcal{O}$ contains only two elements: $\emptyset$ and E. It is the topology which admits the fewest possible open sets.

However, the interesting topologies are, in general, neither the discrete nor the coarse.

One should observe that the properties O_1, O_2, O_3 are those which we put forward in the study of the topology of the line. It is remarkable that they are sufficient to obtain very rich results. We shall have to supplement them only when we study separated topological spaces and compact spaces.

EXAMPLE. *Topology associated with a totally ordered set.* Let E be an arbitrary totally ordered set; we shall call every union of open intervals in E an *open* set in E; in other words, A is an open set in E if A is empty or if, for every $x \in A$, there exists an open interval containing x and contained in A.

This definition, obviously, simply repeats the procedure used in Section 1 for the case of $\mathbf{R}$.

One can easily verify that properties O_1, O_2, O_3 are satisfied.

The topology thus defined on E is called the order topology.

SPECIAL CASE. Let $\bar{\mathbf{R}}$ be the totally ordered set defined as follows: The points of $\bar{\mathbf{R}}$ are the points of $\mathbf{R}$ plus two additional points denoted by $-\infty$ and $+\infty$; we shall say that $x \leqslant y$ in $\bar{\mathbf{R}}$ if $x, y \in \mathbf{R}$ with $(y - x)$ positive, or if $x = -\infty$, or if $y = +\infty$.

It is easy to verify that this definition defines a total order on $\bar{\mathbf{R}}$; $-\infty$ is the smallest element, and $+\infty$ is the largest element.

The set $\bar{\mathbf{R}}$ with this order and the associated topology is called the *extended line*.

5.2. Definition. A SUBSET A OF E IS SAID TO BE *CLOSED* WHEN ITS COMPLEMENT $\complement_E A$ IS OPEN.

As in the case of the line, the statements O_1, O_2, O_3 immediately imply three statements F_1, F_2, F_3 which are equivalent to them by duality, and which concern the closed sets in E:

F_1 : Every intersection (finite or not) of closed sets is closed;
F_2 : Every *finite* union of closed sets is closed;
F_3 : The empty set and the space E are closed.

For example, in the discrete topology on E, every subset of E is both open and closed; in the coarse topology on E, the only closed sets are $\emptyset$ and E; if E is a totally ordered set, every closed interval of E is a closed set in E in the order topology.

Neighborhoods

5.3. Definition. A *NEIGHBORHOOD* OF A POINT x OF E IS A SUBSET OF E CONTAINING AN OPEN SET CONTAINING x.

GENERALLY THE FAMILY OF NEIGHBORHOODS V OF A POINT x IS DENOTED BY $\mathscr{V}(x)$.

A *NEIGHBORHOOD OF A SET* A IN E IS A SUBSET OF E CONTAINING AN OPEN SET CONTAINING A.

Characterization of open sets

It follows from the preceding definition that an open set is a neighborhood of each of its points. *Conversely*, if a set A is a neighborhood of each of its points, it is an open set. In fact, for every $x \in \mathrm{A}$ there exists an open set ω_x containing x and contained in A. Therefore

$$\mathrm{A} = \bigcup_{x \in \mathrm{A}} \omega_x .$$

This is a union of open sets, and is therefore an open set.

CONSEQUENCE. It follows from this that the open sets in a space are known whenever the neighborhoods of x are known for every x. In other words, two topologies on the same set which admit the same neighborhoods are identical.

Here are several essential properties of neighborhoods which are often taken as a starting point in defining a topological space.

V_0 : Every point x has at least one neighborhood;
V_1 : Every neighborhood of x contains x;
V_2 : Every set containing a neighborhood of x is a neighborhood of x;
V_3 : The intersection of two neighborhoods of x is a neighborhood of x;
V_4 : If V is a neighborhood of x, there exists a subneighborhood W of x (that is, $\mathrm{W} \subset \mathrm{V}$) such that V is a neighborhood of each point in W.

The first three properties are obvious. The fourth follows from the fact that the intersection of two open sets is open.

The fifth is more subtle; it expresses the following vague idea: Every point which is quite close to a point which is quite close to x is close to x. By hypothesis there exists an open set ω such that $x \in \omega$ and $\omega \subset \mathrm{V}$. But ω is a neighborhood of each of its points, and *a fortiori* V is a neighborhood of each point in ω. It therefore suffices to take $\mathrm{W} = \omega$;

Neighborhood base of a point

To know $\mathscr{V}(x)$, it suffices to know sufficiently many elements of $\mathscr{V}(x)$.

5.4. Definition. WE SAY THAT A SUBSET $\mathscr{B}$ OF $\mathscr{V}(x)$ CONSTITUTES A BASE FOR $\mathscr{V}(x)$ IF EVERY $\mathrm{V} \in \mathscr{V}(x)$ CONTAINS AN ELEMENT $\mathrm{W} \in \mathscr{B}$.

Knowing $\mathscr{B}$, $\mathscr{V}(x)$ is obtained as the collection of all sets V which have a subset W belonging to $\mathscr{B}$.

EXAMPLE 1. If E is an arbitrary space, for every $x \in \mathrm{E}$ the open sets containing x constitute a base for $\mathscr{V}(x)$.

EXAMPLE 2. If E is the real line **R** (or $\mathbf{R}^n$), every point $x \in E$ has a neighborhood base consisting of the open intervals with center x and halflength $1/n$ (n an integer > 0). Thus every point x of E has a countable neighborhood base.

6. CLOSURE, INTERIOR, BOUNDARY

Adherent point, accumulation point, isolated point

6.1. Definition. LET A BE A SET IN E AND LET $x \in E$.

WE CALL x AN *ADHERENT POINT* OF A IF EVERY NEIGHBORHOOD OF x CONTAINS A POINT OF A.

WE CALL x AN *ACCUMULATION POINT* OF A IF EVERY NEIGHBORHOOD OF x CONTAINS AT LEAST ONE POINT OF A OTHER THAN x (x ITSELF NEED NOT BELONG TO A).

WE CALL x AN *ISOLATED POINT* OF A IF IT BELONGS TO A BUT IS NOT AN ACCUMULATION POINT OF A, IN OTHER WORDS IF THERE EXISTS A NEIGHBORHOOD OF x WHICH CONTAINS NO POINT OF A OTHER THAN x.

Thus, to say that x is an adherent point of A is equivalent to saying that either x is an accumulation point of A, or x is an isolated point of A.

The set of points of E which are adherent points of A is called the *adherence* of A.

For example, in **R**, the adherence of **Q** is **R** itself; the adherence of an interval (a, b) with distinct endpoints is $[a, b]$; the adherence of the set of points $1/n$ ($n = 1, 2, \ldots$) is this set with the addition of the point 0.

Closure of a set

For every $A \subset E$ there exist closed sets containing A (for example E itself). By property F_1, the intersection of these closed sets is a closed set containing A, and is the smallest such. Hence the definition:

6.2. Definition. THE SMALLEST CLOSED SET CONTAINING A SET A IS CALLED THE *CLOSURE* OF A, AND IS DENOTED BY $\bar{A}$.

6.3. Proposition. *For every set* A, *the adherence and the closure of* A *are identical.*

Indeed, if A is a subset of E and if x denotes an arbitrary point of E, each of the properties

$$(x \notin \bar{A}) \qquad \text{and} \qquad (x \text{ not adherent to A})$$

implies the existence of an open neighborhood ω of x which does not meet A.

Corollary 1. *The relation* $A = \bar{A}$ *characterizes the closed sets.*

Corollary 2. *In order that* A *be closed, it is necessary and sufficient that it contain its accumulation points.*

PROOF. 1. If A is closed, it is clearly identical with its closure. Conversely, it follows from $A = \bar{A}$ that A is closed since every closure is by definition a closed set.

Let us note here that $\bar{\bar{A}} = \bar{A}$ for every A.

2. Let A' be the set of accumulation points of A. By the definition of the adherence we have $\bar{A} = A \cup A'$. Therefore to say that $A = \bar{A}$ is equivalent to saying that $A = A \cup A'$ or that $A' \subset A$.

Interior of a set

The notion which is dual to closure is that of interior.

6.4 Definition. THE (POSSIBLY EMPTY) UNION OF ALL OPEN SETS CONTAINED IN A SUBSET A OF E IS CALLED THE *INTERIOR* OF A. IT IS THUS THE LARGEST OPEN SET CONTAINED IN A; WE DENOTE IT BY $\mathring{A}$.

It is immediate that the relation $A = \mathring{A}$ characterizes the open sets.

Relations between the topological operations $\mathring{A}$, $\bar{A}$ *and the elementary operations*

1. Duality between closure and interior:

$$\complement \mathring{A} = \overline{\complement A}. \tag{1}$$

Indeed, by definition

$$\mathring{A} = \bigcup_{i \in I} \omega_i ,$$

where $(\omega_i)_{i \in I}$ denotes the family of all open sets contained in A; therefore

$$\complement \mathring{A} = \bigcap_{i \in I} \complement \omega_i = \bigcap_{i \in I} \varphi_i ,$$

where $(\varphi_i)_{i \in I}$ denotes the family of all closed sets containing $\complement A$; this is therefore its closure.

$$\complement \bar{A} = \mathring{\complement} A. \tag{2}$$

This formula is derived from the preceding one by replacing A by $\complement A$.

2. Properties of the closure:

$$(1)\quad \bar{\emptyset} = \emptyset; \qquad (2)\quad A \subset \bar{A};$$

$$(3)\quad \bar{\bar{A}} = \bar{A}; \qquad (4)\quad \overline{A \cup B} = \bar{A} \cup \bar{B}.$$

The first two relations are immediate; the third follows from the fact that the closure of every closed set A is identical with A.

To prove the fourth relation, we observe first that

$$(X \subset Y) \Rightarrow (X \subset \bar{Y}) \Rightarrow (\bar{X} \subset \bar{Y});$$

from this we deduce that $\bar{A}$, $\bar{B} \subset \overline{A \cup B}$, hence

$$\bar{A} \cup \bar{B} \subset \overline{A \cup B};$$

conversely, $\bar{A} \cup \bar{B}$ is a closed set containing A and B, therefore also $A \cup B$, from which follows

$$\overline{A \cup B} \subset \bar{A} \cup \bar{B}.$$

This important relation evidently carries over to every finite union. On the other hand it does not carry over to infinite unions due to the fact that an arbitrary union of closed sets is not always closed.

There is no analogous relation for intersection, even finite intersection. For example, if E is the line **R** and if A and B denote, respectively, the set of rationals and the set of irrationals, then $\bar{A} \cap \bar{B} = \mathbf{R}$, while $\overline{A \cap B} = \emptyset$. One only has the inclusion $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

Similarly, for every family (A_i) of sets in E one has the inclusions:

$$\bigcup \bar{A}_i \subset \overline{\bigcup A_i} \quad \text{and} \quad \overline{\bigcap A_i} \subset \bigcap \bar{A}_i\,.$$

3. Properties of the interior. These properties are the duals of those of the closure:

$$(1)\quad \mathring{E} = E; \qquad 2)\quad \mathring{A} \subset A;$$

$$(3)\quad \mathring{\mathring{A}} = \mathring{A}; \qquad (4)\quad \overset{\circ}{\overline{A \cap B}} = \mathring{A} \cap \mathring{B}.$$

Boundary of a set

6.5. Definition. THE BOUNDARY A^* OF A SUBSET A OF E IS THE SET OF POINTS x, EACH OF WHOSE NEIGHBORHOODS V CONTAINS AT LEAST ONE POINT OF A AND ONE POINT OF $\complement$ A.

Thus

$$A^* = \bar{A} \cap \overline{\complement A}.$$

From this formula it is seen that the boundary of every set is closed and that two complementary sets have the same boundary.

6.6. Proposition. *For every set* A *in* E, $A^* = \bar{A} \dot{-} \mathring{A}$.

In fact, we have the relations

$$A^* = \bar{A} \cap \overline{\complement A} \qquad \text{and} \qquad \overline{\complement A} = \complement \mathring{A},$$

from which follows

$$A^* = \bar{A} \cap \complement \mathring{A},$$

which is just the desired relation.

Corollary. *The following equivalences hold for every closed subset* A of E:

$$(A = A^*) \Leftrightarrow (\mathring{A} = \emptyset) \Leftrightarrow (\overline{\complement A} = E).$$

Everywhere dense, dense, and nondense sets

On the real line, there are rational points in every nonempty open set. On the other hand, every nonempty open set in $\mathbf{R}$ contains a nonempty open subset which does not contain any integer. To make more precise the vague notions suggested by these differences between the distributions of $\mathbf{Q}$ and of $\mathbf{Z}$ over $\mathbf{R}$, we introduce the following definition.

6.7. Definition. LET A BE A SUBSET OF A SPACE E.

THE SET A IS SAID TO BE *EVERYWHERE DENSE* ON E, *DENSE* ON E, OR *NONDENSE* ON E ACCORDING AS

$\bar{A} = E$; $\bar{A}$ HAS A NONEMPTY INTERIOR; $\bar{A}$ HAS AN EMPTY INTERIOR.

For example, on $\mathbf{R}$, the set $\mathbf{Q}$ is everywhere dense; the set $\mathbf{Q} \cap [0, 1]$ is dense; the set $\mathbf{Z}$ is nondense, as is the set of numbers $1/n$ (where $n = 1, 2, \ldots$).

IMMEDIATE PROPERTIES (*from the corollary of Proposition* 6.6). 1. If A is everywhere dense on E and if $A \subset B \subset E$, B is also everywhere dense on E.

2. (A everywhere dense) $\Leftrightarrow$ (Every nonempty open set in E meets A).
3. (A nondense) $\Leftrightarrow$ ($\bar{A}$ nondense) $\Leftrightarrow$ ($\complement \bar{A}$ everywhere dense) $\Leftrightarrow$ (Every nonempty open set in E contains a nonempty open subset which does not intersect A).
4. If A and B are nondense on E, so is the set $A \cup B$.

This last property carries over to arbitrary finite unions, but not to infinite unions (take the case of $\mathbf{Q}$ on $\mathbf{R}$).

Z If A is nondense, $\complement A$ is everywhere dense, but it can happen that both A and $\complement A$ are everywhere dense; this is the case for the subset $\mathbf{Q}$ of $\mathbf{R}$.

The same example shows that A and B may be everywhere dense on E, and $A \cap B$ empty.

7. CONTINUOUS FUNCTIONS. HOMEOMORPHISMS

In order to be able to speak of the continuity of a mapping f of a set X into a set Y, it is necessary that there be defined on X and Y a notion of neighboring points, in other words, that X and Y be topological spaces.

An analysis of the classical definition of the continuity of a numerical function of a real variable leads to the following definition:

Continuity at a point

7.1. Definition. A MAPPING f OF A TOPOLOGICAL SPACE X INTO ANOTHER TOPOLOGICAL SPACE Y IS SAID TO BE *CONTINUOUS* AT A POINT x_0 OF X IF, FOR EVERY NEIGHBORHOOD V OF $f(x_0)$, THERE EXISTS A NEIGHBORHOOD v OF x_0 WHOSE IMAGE UNDER f IS IN V, THAT IS, SUCH THAT $f(v) \subset V$.

In the notation of symbolic logic, this relation is written as

$$(f \text{ continuous at } x_0) \overset{\text{def}}{\Leftrightarrow} (\forall V, V \in \mathscr{V}(f(x_0)))(\exists v, v \in \mathscr{V}(x_0)) : (f(v) \subset V).$$

Clearly an equivalent definition is obtained by requiring that V and v belong to given neighborhood bases of $f(x_0)$ and x_0, respectively.

Here is another convenient form of the continuity of f at x_0: *For every neighborhood* V *of* $f(x_0)$, $f^{-1}(V)$ *is a neighborhood of* x_0. In fact, let V be a neighborhood of $f(x_0)$; if v is a neighborhood of x_0 such that $f(v) \subset V$, then $v \subset f^{-1}(V)$; therefore $f^{-1}(V)$, which contains a neighborhood of x_0, is also a neighborhood of x_0. Conversely, if $f^{-1}(V)$ is a neighborhood of x_0, we set $v = f^{-1}(V)$; then surely $f(v) \subset V$. Here again we can require only that V belong to a given neighborhood base of $f(x_0)$.

EXAMPLE 1. If f is a constant mapping of X into Y, then $f^{-1}(V) = X$ for every neighborhood V of $f(x_0)$; thus every constant mapping of X into Y is continuous at every point of X.

EXAMPLE 2. The identity mapping $x \to x$ of X into X is continuous at every point of X.

Continuity in the entire space

7.2. Definition. A MAPPING f OF X INTO Y IS SAID TO BE *CONTINUOUS* ON X (OR IN X) IF IT IS CONTINUOUS AT EVERY POINT OF X.

7.3. Theorem. *To say that f is continuous on* X *is equivalent to saying that for every open set* B *in* Y, $f^{-1}(B)$ *is an open set in* X.

Indeed, suppose f is continuous on X; if B is open in Y, then since B is a neighborhood of each of its points, its inverse image is a neighborhood of each of its points, and is therefore an open set.

Conversely, suppose that $f^{-1}(B)$ is open for every open set B in Y; then for every x_0 and every neighborhood V of $f(x_0)$, the set $f^{-1}(\mathring{V})$ is an open set containing x_0, therefore *a fortiori* $f^{-1}(V)$ is a neighborhood of x_0. Thus f is continuous at every point x_0 of X.

7.4. Theorem. *To say that f is continuous on* X *is equivalent to saying that for every closed set* B *in* Y, $f^{-1}(B)$ *is a closed set in* X.

This theorem follows from the preceding one by applying the relation

$$f^{-1}(\complement B) = \complement f^{-1}(B).$$

Corollary. *If f is a continuous mapping of* E *into* $\mathbf{R}$, *the set of points x in* E *such that* $f(x) = 0$ *is a closed set in* E.

Analogous statements hold for the solutions of the relations of the form

$$f(x) \geqslant 0, \qquad f(x) \leqslant 0, \qquad f(x) > 0, \qquad f(x) < 0.$$

In particular, if f is a polynomial in n real variables with real coefficients, the corollary shows that the real algebraic variety of solutions of $f(x) = 0$ is a closed set in $\mathbf{R}^n$; the same result holds in $\mathbf{C}^n$ for polynomials in n complex variables.

Z It is essential to note that the two characterizations of the continuity of f stated in Theorems 7.3 and 7.4 make use of the *inverse* images, and not the direct images, under f.

In fact, the image of an open set in X under a continuous mapping is an open set in Y only in exceptional cases. For example, for the constant mapping $x \to 0$ of **R** into **R** no nonempty open set in X has an open image.

Similarly, the image of a closed set in X under a continuous mapping may be nonclosed in Y: for example, the mapping $x \to 1/(x^2 + 1)$ of **R** into **R** carries the closed set **R** onto $(0, 1]$, which is not closed in **R**.

Transitivity of continuous mappings

Let X, Y, Z, be topological spaces, f a mapping of X into Y, g a mapping of Y into Z, and $h = g \circ f$.

Let $x_0 \in X$; we put $y_0 = f(x_0)$ and $z_0 = g(y_0) = g(f(x_0)) = h(x_0)$.

7.5. Proposition. *If f is continuous at x_0 and if g is continuous at y_0, then the composite $h = g \circ f$ is continuous at x_0.*

Indeed, let V be an arbitrary neighborhood of z_0. The continuity of g at y_0 implies that $g^{-1}(V)$ is a neighborhood of y_0; the continuity of f at x_0 implies, therefore, that $f^{-1}(g^{-1}(V)) = h^{-1}(V)$ is a neighborhood of x_0.

In particular, if f is continuous on X and g is continuous on Y, $h = g \circ f$ is continuous on X.

EXAMPLE. The mapping $u \to |u|$ of **R** into **R** is continuous; therefore for every continuous mapping f of X into **R**, the mapping $|f|$ of X into **R** defined by $x \to |f(x)|$ is also continuous.

Homeomorphisms

It is natural to say that two topological spaces X and Y are isomorphic if there exists a bijection f of X to Y which interchanges their open sets, that is, such that for every open set A in X, $f(A)$ is an open set in Y, and for every open set B in Y, $f^{-1}(B)$ is an open set in X. Such an isomorphism is called a *homeomorphism.*

It is clear that the inverse of a homeomorphism is a homeomorphism, and that the product of two homeomorphisms is a homeomorphism. In particular, the collection of all homeomorphims of a space with itself forms a group.

REMARKS. The notion of a homeomorphism is fundamental in topology, because a homeomorphism is nothing more than an isomorphism of topological structures; it is the basic equivalence relation in topology.

When two topological spaces are homeomorphic, every property which is true for one is true for the other; they can be considered as two representations of the same geometric entity.

When a set E is provided with various structures, one of which is a topological structure (the other structures can be algebraic, metric, etc.), a property of E is said to be topological if it is true for every topological space which is homeomorphic to E: for example, the fact that **R** contains a countable everywhere dense subset is topological. With a little practice one can in general easily recognize whether a property is topological or not; a property is in any case always topological, when it is stated in terms of open sets and derived notions such as closed sets, neighborhoods, accumulation point, everywhere dense, etc.

Here, in Fig. 1, without the trouble of being precise and without proof, is a series of examples intended to convey the intuitive content of homeomorphness; the letters and figures appearing on each line are homeomorphic to one another.

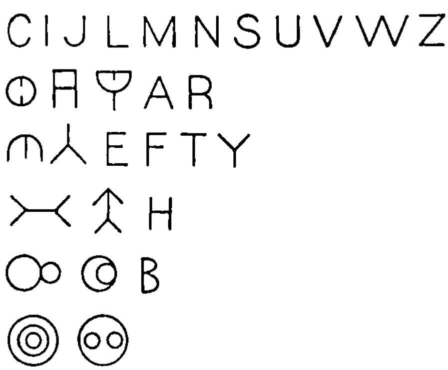

FIG. 1

The lateral surface of a truncated cone is homeomorphic to a circular ring.

A hemisphere is homeomorphic to a closed interval in $\mathbf{R}^2$.

One can get a good idea of homeomorphism, as it applies to surfaces in the space $\mathbf{R}^3$, by imagining that these surfaces are made of rubber; a homeomorphism is then a deformation of these surfaces by stretching and shrinking, without tearing, folding, or gluing. Nevertheless, such recourse to intuition can lead to error, and should never replace an exact argument.

Example of a proof of homeomorphness. Let X and Y be two isomorphic totally ordered sets, and let f be a bijection of X to Y which is an isomorphism for the order structure (in other words, f is an increasing bijection).

Then f is a homeomorphism for the topologies on X *and* Y *associated with their order.*

Indeed, f interchanges the open intervals of X and Y, and therefore their open sets also, since the latter are defined in terms of open intervals.

Particular Case. The mapping $f : x \to x/(1 + |x|)$ is an increasing bijection of $\mathbf{R}$ to $(-1, 1)$; we complete it to an increasing bijection of $\bar{\mathbf{R}}$ to $[-1, 1]$ by setting

$$f(-\infty) = -1 \qquad \text{and} \qquad f(+\infty) = 1.$$

Therefore $\bar{\mathbf{R}}$ and $[-1, 1]$, taken with their order topologies, are homeomorphic.

We shall now prove a convenient general criterion for homeomorphness.

7.6. Theorem. *In order that a bijection f of a space* X *to a space* Y *be a homeomorphism, it is necessary and sufficient that it be bicontinuous, that is, that f and f^{-1} be continuous.*

Indeed, to say that f and f^{-1} are continuous is equivalent to saying that the inverse image of every open set in Y is open and that the direct image of every open set in X is open.

Example. The translations of $\mathbf{R}^n$ are homeomorphisms of $\mathbf{R}^n$ to $\mathbf{R}^n$; more generally, the same is true of the *dilations*

$$x \to \lambda x + a \qquad (\lambda \neq 0).$$

Z A hurried study of various special cases might lead one to believe that every continuous bijection of a space X to a space Y is also bicontinuous. We shall see later that this is indeed so in certain cases (see the study of compact spaces). But it is not a general fact. Here are some examples:

Example 1. Y is the real line $\mathbf{R}$; the elements of X are those of $\mathbf{R}$, but its topology is the discrete topology. The mapping f of X onto Y is the identity mapping $x \to x$. It is evidently continuous, but not bicontinuous.

EXAMPLE 2. X and Y are two subsets of **R**, with the topologies induced by the topology of **R** (see Section 9); X consists of the interval $[0, 1)$ and the point 2; Y is the interval $[0, 1]$. Finally, f is defined by $f(x) = x$ if $x \in [0, 1)$, and $f(2) = 1$.

It is immediate that this mapping is one-to-one and continuous, but that f^{-1} is not continuous at the point 1.

A study of these two examples is enough to show that the lack of continuity of f^{-1} results from the fact that f, while being one-to-one and continuous, brings points "closer together," and that, more precisely, there can exist a set A in X which is not a neighborhood of a point a, while $f(\mathrm{A})$ is a neighborhood of $f(a)$.

8. NOTION OF A LIMIT

Limit of a sequence

8.1. Definition. LET (a_i) $(i = 1, 2, ..., n, ...)$ BE A SEQUENCE OF POINTS OF A SPACE E. WE SAY THAT THIS SEQUENCE *CONVERGES* TO A POINT a OF E, OR THAT a IS THE *LIMIT* OF THIS SEQUENCE, IF FOR EVERY NEIGHBORHOOD V OF a THERE EXISTS AN INTEGER i_0 SUCH THAT $a_i \in \mathrm{V}$ FOR EVERY $i \geqslant i_0$.

In symbolic notation,

$$(\forall \mathrm{V}, \mathrm{V} \in \mathscr{V}(a))(\exists i_0, i_0 \in \mathbf{N})(\forall i, i \geqslant i_0): (a_i \in \mathrm{V}).$$

This condition can also be expressed as follows: For every neighborhood V of a we have $a_i \in \mathrm{V}$ for all except at most finitely many values of i.

It is immediate that if the sequence (a_i) converges to a, every infinite subsequence also converges to a.

If E is an arbitrary space, a sequence can have several limit points; for example, when E has the coarse topology, every point of E is a limit of every sequence $(a_i)_{i \in \mathrm{I}}$.

But one can assert that the limit is unique when E is a *separated* space,* in the following sense:

8.2. Definition. A SPACE E IS SAID TO BE *SEPARATED* WHEN ANY TWO DISTINCT POINTS HAVE DISJOINT NEIGHBORHOODS.

The most useful spaces are always separated; for example the line **R** is separated (see Section 1, pp. 3-4).

In a separated space E, every one-point set is closed; in fact, to say that $\{a\}$ is closed is equivalent to saying that $\mathrm{E} - \{a\}$ is open, that is,

* An equivalent terminology is *Hausdorff* space.

is a neighborhood of each of its points. But this last assertion is true since for every $x \neq a$ there exists by hypothesis a neighborhood of x which does not contain a, and which therefore is contained in $E - \{a\}$.

8.3. Proposition. *Every sequence of points in a separated space has at most one limit.*

The proof is an exact repetition of the proof given for **R**.

Adherent points of a sequence

The sequence of real numbers $a_n = (-1)^n$ is not convergent; nevertheless the numbers 1 and -1 appear as limits in a larger sense. The analysis of this notion leads to the following definition:

8.4. Definition. LET (a_i) $(i = 1, 2, ...)$ BE A SEQUENCE OF POINTS OF A SPACE E. WE SAY THAT A POINT a OF E IS AN *ADHERENT POINT* OF THIS SEQUENCE IF FOR EVERY NEIGHBORHOOD V OF a THERE EXIST ARBITRARILY LARGE INDICES i SUCH THAT $a_i \in V$.

If we denote by A_n the set of points a_i with index $i \geqslant n$, we can say that a is an adherent point when, for every n, the point a belongs to the adherence of A_n.

The set of adherent points of the sequence $(a_i)_{i \in I}$ is thus

$$A = \bigcap_1^{\infty} \bar{A}_n .$$

This set is closed; it may be empty; for example, the sequence $(a_n = n)$ in **R** has no adherent points.

If all the points a_i of the sequence belong to a closed subset F of E, then $A \subset F$; in fact $A_n \subset F$, hence $\bar{A}_n \subset F$, and $A \subset F$.

In a separated space, if a sequence (a_i) converges to a, this point is the only adherent point of the sequence. In fact, suppose $b \neq a$ and let V_b, V_a be disjoint neighborhoods of these two points. There exists n such that $A_n \subset V_a$; since $A_n \cap V_b = \emptyset$, the point b is not adherent to the sequence.

Z It is, however, false in general, even in a separated space, that if a sequence has a single adherent point, then it converges to this point. For example, the sequence $(1/2, 2, 1/3, 3, ..., 1/n, n, ...)$ in **R** has a single adherent point 0, although it does not converge to 0.

Limit and adherent point along a filter base

Classical analysis does not define the notion of a limit for sequences only: for example, the function $x^2/(x^2 + 1)$ tends to 1 as x tends to ∞; $(1/x^2 + 1/y^2)$ tends to 0 as x and y tend to ∞; $\sin x/x$ tends to 1 as x tends to 0 through values $\neq 0$.

We shall see that all of the limit notions which enter into these examples are special cases of a general concept.

8.5. Definition. LET E BE AN ARBITRARY SET. BY A *FILTER BASE* ON E WE MEAN ANY COLLECTION $\mathscr{B}$ OF SUBSETS OF E SUCH THAT:

1. FOR EVERY B_1, $B_2 \in \mathscr{B}$ THERE EXISTS $B_3 \in \mathscr{B}$ SUCH THAT $B_3 \subset B_1 \cap B_2$;
2. NO ELEMENT B OF $\mathscr{B}$ IS EMPTY.

It follows from this definition that the intersection of a finite number of elements of $\mathscr{B}$ is never empty.

EXAMPLE 1. $E = \mathbf{N}$, and the elements of $\mathscr{B}$ are the subsets of the form $\{n, n + 1, ...\}$.

EXAMPLE 2. $E = \mathbf{N}^2$, and the elements of $\mathscr{B}$ are the subsets B_n of E, where B_n is the family of pairs (p, q) such that $p \geqslant n$ and $q \geqslant n$.

EXAMPLE 3. $E = \mathbf{R}$, and the elements of $\mathscr{B}$ are the intervals $[a, \infty)$ of $\mathbf{R}$.

EXAMPLE 4. E is a topological space, and $\mathscr{B}$ is a neighborhood base of a point a of E.

EXAMPLE 5. E is a topological space; we denote by A a nonempty subset of E, and by a an adherent point of A.

The elements of $\mathscr{B}$ are the sets of the form $A \cap V$, where V is an arbitrary neighborhood of a (we could even require that V belong only to a neighborhood base of a).

8.6. Definition. LET f BE A MAPPING OF A SET X INTO A TOPOLOGICAL SPACE Y; LET $\mathscr{B}$ BE A FILTER BASE ON X, AND LET b BE A POINT OF Y.

WE SAY THAT f CONVERGES TO b (OR HAS LIMIT b) ALONG $\mathscr{B}$ IF FOR EVERY NEIGHBORHOOD V OF b THERE EXISTS A $B \in \mathscr{B}$ SUCH THAT $f(B) \subset V$. WE THEN WRITE $\lim_{\mathscr{B}} f = b$.

IN PARTICULAR, WHEN $X = Y$ AND f IS THE IDENTITY MAPPING OF X INTO X, WE SAY SIMPLY THAT THE FILTER BASE $\mathscr{B}$ CONVERGES TO b.

EXAMPLE 1. Let (a_n) be a sequence of points of Y; we denote by f the mapping $n \to a_n$ of $\mathbf{N}$ into Y, and by $\mathscr{B}$ the collection of subsets of $\mathbf{N}$ whose complements are finite. Then Definition 8.6 coincides with Definition 8.1.

EXAMPLE 2. If X is a topological space and $\mathscr{B}$ denotes the collection of neighborhoods of a point a of X, to say that $f(a)$ is the limit of f along $\mathscr{B}$ is equivalent to saying that f is continuous at the point a.

We remark that the notation $\lim_{\mathscr{B}} f$ is frequently replaced by $\lim_{\mathscr{B}} f(x)$, in particular when $f(x)$ has a simple expression and a classical notation for f is not available; this is the case for the elementary functions $x \to x^n$ and $x \to |x|$.

8.7. Proposition. *If* Y *is a separated space, f can have at most one limit along* $\mathscr{B}$.

In fact, suppose that b and b' are limits of f along $\mathscr{B}$. For any neighborhood V, V′ of b, b' there exist B, B′ $\in \mathscr{B}$ such that

$$f(\mathrm{B}) \subset \mathrm{V} \qquad \text{and} \qquad f(\mathrm{B}') \subset \mathrm{V}'.$$

Since $\mathrm{B} \cap \mathrm{B}'$ is nonempty, the same is true of $f(\mathrm{B}) \cap f(\mathrm{B}')$, therefore *a fortiori* of $\mathrm{V} \cap \mathrm{V}'$. Since Y is separated, this is possible only if $b = b'$.

8.8. Definition. LET f BE A MAPPING OF X INTO A TOPOLOGICAL SPACE Y, AND LET $\mathscr{B}$ BE A FILTER BASE ON X.

WE SAY THAT A POINT b OF Y IS AN ADHERENT VALUE OF f ALONG $\mathscr{B}$ IF, FOR EVERY B $\in \mathscr{B}$ AND EVERY NEIGHBORHOOD V OF b, f(B) INTERSECTS V.

This is equivalent to saying that $b \in \overline{f(\mathrm{B})}$ for every B $\in \mathscr{B}$, or again that

$$b \in \bigcap_{\mathrm{B}\in\mathscr{B}} \overline{f(\mathrm{B})}.$$

The set of these b (called the *adherence* of f along $\mathscr{B}$) is therefore the set

$$\bigcap_{\mathrm{B}\in\mathscr{B}} \overline{f(\mathrm{B})};$$

it is clearly closed; we denote it by $\bar{f}(\mathscr{B})$.

As in the case of sequences, one can verify that if Y is separated, and if f converges to b along $\mathscr{B}$, then b is the only adherent value of f along $\mathscr{B}$.

SPECIAL CASE. *Limit and adherent values of a function at a point.*

We now assume that X is a topological space; let A be a nonempty subset of X, and $a \in \bar{\mathrm{A}}$; we denote by $\mathscr{B}$ the collection of subsets of X of the form $\mathrm{A} \cap \mathrm{V}$, where V is an arbitrary neighborhood of a.

We then say that f has limit b as x tends to a while staying in A, if

$$b = \lim_{\mathscr{B}} f.$$

EXAMPLE. If $X = \mathbf{R}$ and $A = (a, \infty)$, we denote by $f(a_+)$ the possible limit of f along $\mathscr{B}$ (this is the limit *from the right*). The limit from the left $f(a_-)$ is similarly defined.

When $a \in A$ and $\lim_{\mathscr{B}} f$ exists, this limit can only be $f(a)$ when Y is separated.

In the same way, we can define the adherent values of f as x tends to a while staying in A.

9. SUBSPACES OF A TOPOLOGICAL SPACE

Let E be a topological space and A a subset of E.

Among all the topologies which can be defined on A, let us study those which render the identity mapping f of A into E continuous. The inverse image of an open set in E under this mapping is simply the intersection of this open set with A. Therefore, in order than the canonical mapping f be continuous, it is necessary and sufficient that the family of open sets of the topology of A contain all these intersections.

But it is immediate that the traces on A of the open sets in E satisfy the axioms O_1, O_2, O_3. The topology that these traces define on A is the one we shall take.

9.1. Definition. FOR EVERY SUBSET A OF A TOPOLOGICAL SPACE E, THE SET A WITH THE TOPOLOGY WHOSE OPEN SETS ARE THE TRACES ON A OF THE OPEN SETS IN E WILL BE CALLED THE *SUBSPACE* A OF E.

WE ALSO SAY THAT THE TOPOLOGY OF A IS INDUCED BY THAT OF E, OR THAT IT IS THE TRACE OF THAT OF E.

The identity mapping of A into E is then continuous.

Closed sets and neighborhoods in a subspace

The formula

$$A \dot{-} A \cap \omega = A \bigcap \complement\omega,$$

where ω is an open set in E, shows that the closed sets in the subspace A are simply the traces on A of the closed sets in E.

Similarly, in a subspace A the neighborhoods of a point a of A are the sets $A \cap V$, where V is a neighborhood of a in E.

Z One should carefully note that an open (closed) set in a subspace A of E is not necessarily an open (closed) set in E. This remark is made precise by the following proposition:

9.2. Proposition. *In order that every open (closed) set in the subspace* A *of* E *be an open (closed) set in* E, *it is necessary and sufficient that* A *be open (closed) in* E.

In fact, if A is open in E, the trace on A of every open set in E is open in E. Conversely, if the trace on A of every open set in E is open in E, this is true of the trace of E on A, that is of A itself.

One has a parallel argument upon replacing the word "open" by "closed."

Transitivity of subspaces

9.3. Proposition. *Let* X *be a topological space,* Y *a subspace of* X, *and* Z *a subspace of* Y. *The topologies on* Z *induced by those of* X *and* Y *are identical.*

In fact, the open sets in the subspace Y are the sets of the form $Y \cap \omega$, where ω is an open set in X. Therefore the open sets in Z, in the topology induced by that of Y, are the sets of the form $Z \cap (Y \cap \omega)$; but such a set is simply $Z \cap \omega$; therefore the open sets of the topologies on Z induced by those of X and Y are identical.

9.4. Proposition. *Let f be a mapping of a topological space* X *into a subspace* Y *of a topological space* Z.

To say that f is continuous at a point a of X *is equivalent to saying that f, regarded as a mapping of* X *into* Z, *is continuous at a.*

In fact, the neighborhoods of $f(a)$ in Y are the sets $V \cap Y$, where V is a neighborhood of $f(a)$ in Z; and since $f(X) \subset Y$, we have

$$f^{-1}(V) = f^{-1}(V \cap Y).$$

9.5. Proposition. *Let f be a mapping of a space* X *into a space* Y. *If f is continuous at the point a, its restriction to every subspace* A *containing a is continuous at a.*

In fact, let g be the restriction of f to A; for every neighborhood V of $f(a)$ in Y, we have

$$g^{-1}(V) = A \cap f^{-1}(V).$$

Thus if $f^{-1}(V)$ is a neighborhood of a in X, $g^{-1}(V)$ is a neighborhood of a in A.

Z On the other hand, it can happen that g is continuous at a without f being so; for example, if f is the mapping of $\mathbf{R}$ into $\mathbf{R}$ which equals 0 on $\mathbf{Q}$ and 1 on $\complement\mathbf{Q}$, f is discontinuous at every point of $\mathbf{R}$, while the restriction of f to $\mathbf{Q}$ is continuous (as is also the restriction of f to $\complement\mathbf{Q}$).

There is nevertheless an important case in which the continuity of g at a is equivalent to that of f at a; this is the case in which A is a neighborhood of a. In fact in this case, if $A \cap f^{-1}(V)$ is a neighborhood of a in A, it is also a neighborhood of a in X, therefore *a fortiori* $f^{-1}(V)$ is a neighborhood of a in X.

We state this equivalence by saying that the continuity of a mapping at a point is a *local* property.

9.6. Proposition. *Every subspace of a separated space is separated.*

In fact, let A be a subspace of a separated space X. For any $x, y \in A$ such that $x \neq y$, there exist disjoint neighborhoods V_x, V_y in X of these points; the sets $A \cap V_x$ and $A \cap V_y$ are neighborhoods of x, y in the subspace A, and they are disjoint.

Applications. The notion of a subspace is a convenient way of defining and studying new topological spaces. Thus every subset A of $\mathbf{R}$ or $\mathbf{R}^n$, taken with the induced topology, constitutes a topological space.

For example, the *sphere* S_{n-1} in $\mathbf{R}^n$ defined by $\Sigma x_i^2 = 1$ constitutes a highly interesting topological space.

Every space homeomorphic to S_{n-1} is called a *topological sphere* of dimension $(n - 1)$; in particular, every space homeomorphic to S_1 is called a *simple closed curve.*

Similarly, every space homeomorphic to $(0, 1)$ ($[0, 1]$) is called a *simple open* (closed) *arc.*

Z All the open intervals (a, b) (where $a < b$) in $\mathbf{R}$ are homeomorphic, since we can pass from one to another by a dilation of the form $x \to \alpha x + \beta$; and each of them is homeomorphic to $\mathbf{R}$ since, for example, the bijection $x \to x/(1 + |x|)$ of $\mathbf{R}$ to $(-1, 1)$ is a homeomorphism.

It also turns out that the topology induced on each of the intervals (a, b) by that of $\mathbf{R}$ is identical with the order topology on (a, b). But it should not be assumed that this identity extends to every subset of $\mathbf{R}$; for example, these two topologies are distinct on $A = [0, 1) \cup \{2\}$. We shall specify a class of sets for which this identity holds by the following:

9.7. Proposition. *Let* X *be a totally ordered set; for every generalized interval* A *of* X, *the order topology on* A *is identical with the trace on* A *of the order topology on* X.

In fact, every open interval of the ordered set A is of the form (a, b), or $(\leftarrow, a)$, or $(a, \rightarrow)$, or A, where a, b denote points of A. But such a set is the intersection of A with an open interval of X; therefore a union of such sets is simply the intersection of A with an arbitrary open set in X, from which follows the identity of the two topologies.

For example, in **R**, the order topology on $[0, 1]$ or on $(0, 1)$ is identical with the topology induced by that of **R**.

10. FINITE PRODUCTS OF SPACES

We have earlier defined a topology on $\mathbf{R}^n$ derived from that of **R** by using products of open intervals of **R**. This procedure can be extended to the case of arbitrary products of spaces. Here we shall study only finite products of spaces.

Let E_i $(i = 1, 2, ..., n)$ be a finite family of topological spaces and let $E = \prod E_i$ $(i = 1, 2, ..., n)$ be the collection of sequences $x = (x_1, x_2, ..., x_n)$ where $x_i \in E_i$. Among all the possible topologies on E we shall take only those for which each of the projections $x \rightarrow x_i = f_i(x)$ of E onto E_i is continuous; this condition amounts to saying that for every open set $\omega_i \subset E_i$, the set $f_i^{-1}(\omega_i)$, which is simply the product

$$E_1 \times \cdots \times E_{i-1} \times \omega_i \times E_{i+1} \times \cdots \times E_n,$$

should be open in E. The same must therefore also be true of every finite intersection of such sets, that is, of every set of the form $\prod \omega_i$ $(i = 1, 2, ..., n)$ and of every union of such sets. If one could show that these sets satisfy axioms O_1, O_2, O_3, it would be natural to take, as the topology on E, the topology which they define.

We are thus led to the following definitions:

10.1. Definition. ANY SET IN $E = \prod E_i$ OF THE FORM $p = \prod \omega_i$ $(i = 1, 2, ..., n)$, WHERE ω_i IS AN ARBITRARY OPEN SET IN E_i, IS CALLED AN *ELEMENTARY OPEN SET* IN E.

EVERY UNION OF ELEMENTARY OPEN SETS IN E IS CALLED AN OPEN SET IN E.

The family of these open sets clearly satisfies axioms O_1 and O_3. It also satisfies O_2, for if

$$A = \bigcup_{j \in J} p_j \quad \text{and} \quad A' = \bigcup_{k \in K} p_k',$$

then

$$A \cap A' = \bigcup_{(j,k)\in J\times K} (p_j \cap p_k')$$

and each $p_j \cap p_k'$ is easily seen to be an elementary open set.

These open sets therefore define a topology on E. By construction, this topology is that for which the projection of E onto E_i is a continuous mapping for every i. With this topology, E is called the *topological product* of the spaces E_i .

EXAMPLE 1. We have defined a topology on $\mathbf{R}^n$ in Section 4; we there took for the open sets the unions of open intervals. But since every open set in $\mathbf{R}$ is a union of open intervals, the topology on $\mathbf{R}^n$ defined by means of open intervals is identical with the product topology.

EXAMPLE 2. The product $S_1 \times \mathbf{R}$ is a topological space called a *cylinder*.

EXAMPLE 3. The product $(S_1)^n$ is called the n-dimensional *torus*.

Product of subspaces

If A_i denotes a subspace of E_i , one can verify that the product topology on $A = \prod A_i$ is identical with the topology induced by that of $\prod E_i$ on its subset A.

In particular, for every $a_i \in E_i$ $(i = 2, 3, ..., n)$ the space E_1 is homeomorphic to the subspace $E_1 \times \{a_2\} \times \cdots \times \{a_n\}$ of E under the mapping $x_1 \to (x_1 , a_2 , ..., a_n)$.

Associativity of the topological product. *If* A, B, and C *are topological spaces, the one-to-one canonical correspondence between the spaces* $(A \times B) \times C$ *(respectively,* $A \times (B \times C)$) *and* $A \times B \times C$ *is a homeomorphism.*

It suffices (see Section 5) to show that this correspondence preserves neighborhoods. But every point (a, b, c) of $(A \times B) \times C$ has a neighborhood base consisting of the sets

$$(\omega_a \times \omega_b) \times \omega_c ;$$

and every point (a, b, c) of $A \times B \times C$ has a neighborhood base consisting of the sets $\omega_a \times \omega_b \times \omega_c$ (where $\omega_a , \omega_b , \omega_c$ denote open neighborhoods of a, b, c in A, B, C). But the sets $(\omega_a \times \omega_b) \times \omega_c$ and $\omega_a \times \omega_b \times \omega_c$ are homologous (that is, are carried into each other) by the canonical one-to-one correspondence, which implies the desired property.

This associativity will enable us to simplify certain proofs by carrying them out only for products of two spaces.

Commutativity. It is easily verified that the topological product is commutative in the sense, for example, that the canonical bijection $(x, y) \to (y, x)$ of $A \times B$ to $B \times A$ is a homeomorphism.

Continuous mappings in a product space. In general, a mapping $x \to f(x)$ of a space E into a product space $F = \prod F_i$ is denoted by the coordinate mappings $x \to f_i(x)$ of E into each F_i. The close connection between the continuity of f and that of the f_i is given by the following proposition:

10.2. Proposition. *In order that the mapping f of* E *into the finite product* $F = \prod F_i$ *be continuous at a, it is necessary and sufficient that each of the coordinate mappings f_i of* E *into* F_i *be continuous at a.*

In fact, if f is continuous at a, then since $f_i = \mathrm{pr}_i \circ f$, where pr_i denotes the operation of projection of F onto F_i, the mapping f_i is continuous at a. Conversely, suppose that each f_i is continuous at a; for every elementary open set $\omega = \prod \omega_i$ in F containing $f(a)$, $f_i^{-1}(\omega_i)$ is a neighborhood of a in E, and therefore $f^{-1}(\omega)$, which is equal to

$$\bigcap_i f_i^{-1}(\omega_i),$$

is also a neighborhood of a. Since every neighborhood V of $f(a)$ in F contains an elementary open set ω containing $f(a)$, then $f^{-1}(V)$, which contains $f^{-1}(\omega)$, is *a fortiori* a neighborhood of a.

Example. Let u_1 and u_2 be continuous mappings of E into F_1 and F_2, respectively, and let $g : (y_1, y_2) \to g(y_1, y_2)$ be a continuous mapping of $F_1 \times F_2$ into a space G.

Then the mapping $x \to g(u_1(x), u_2(x))$ of E into G is continuous.

Mappings of a product space into another space

Let $f : (x, y) \to f(x, y)$ be a mapping of a product $X \times Y$ into a space F.

If f is continuous, its restriction to every subspace of $X \times Y$ is continuous; in particular, for every $a \in X$ the restriction of f to $(a \times Y)$ is continuous; in other words, since the mapping $y \to (a, y)$ of Y onto $(a \times Y)$ is a homeomorphism, the mapping $y \to f(a, y)$ of Y into F is continuous.

In other words, the continuity of f implies that of the partial mappings $x \to f(x, b)$ and $y \to f(a, y)$.

Z But *the converse is false*; in fact, let f be the mapping of $\mathbf{R}^2$ into $\mathbf{R}$ defined by

$$f(0,0) = 0 \quad \text{and} \quad f(x,y) = xy/(x^2+y^2) \quad \text{if} \quad (x,y) \neq (0,0).$$

It is immediate that all the partial mappings are continuous; at the same time, f is not continuous at (0, 0) since, for example,

$$f(x,x) = 1/2$$

for every $x \neq 0$, while

$$f(0,0) = 0.$$

Similar examples can be constructed in which the set of points of discontinuity of f are everywhere dense on $\mathbf{R}^2$.

Limits in a product space

The following result can be proved in the same way as was Proposition 10.2.

10.3. Proposition. *In order that a sequence* (a_n) *of points of* $\mathrm{F} = \Pi \mathrm{F}_i$ *converge to the point* $l = (l_i)$ *of* F, *it is necessary and sufficient that for every* i, *the sequence* $(a_n)_i$ *converge to* l_i .

More generally, let $\mathscr{B}$ be a filter base on a set E, let $f = (f_i)$ be a mapping of E into the product topological space $\mathrm{F} = \Pi \mathrm{F}_i$, and let $l = (l_i)$ be a point of F.

One can verify that

$$(l = \lim_{\mathscr{B}} f) \Leftrightarrow (\text{for every } i, \quad l_i = \lim_{\mathscr{B}} f_i).$$

Product of separated spaces

10.4. Proposition. *If the spaces* E_i *are separated, their product* E *is separated.*

It clearly suffices to prove this for the product of two spaces E_1 and E_2 .

But if a and b are distinct points of E, with coordinates (a_1, a_2) and (b_1, b_2), then either $a_1 \neq b_1$ or $a_2 \neq b_2$. If for example $a_1 \neq b_1$, the points a_1 and b_1 of E_1 have neighborhoods V_1 and W_1 which are disjoint; the points a and b thus have disjoint neighborhoods $\mathrm{V}_1 \times \mathrm{E}_2$ and $\mathrm{W}_1 \times \mathrm{E}_2$.

Example. Since $\mathbf{R}$ is separated, so is every space $\mathbf{R}^n$.

Every subspace of $\mathbf{R}^n$ is therefore separated; in particular S_{n-1} is separated.

10.5. Proposition. *For every separated space* E, *the diagonal* Δ *of* $E \times E$ *is closed in* $E \times E$.

Indeed, let $(a, b) \notin \Delta$; the points a, b of E being distinct, they have disjoint neighborhoods V_a, V_b; the product $V_a \times V_b$ is a neighborhood of (a, b) in $E \times E$ and does not intersect Δ; therefore $(E \times E) - \Delta$ is a neighborhood of (a, b).

Since the complement of Δ is a neighborhood of each of its points and is therefore open, Δ is closed.

Corollary 1. *If f and g are continuous mappings of a space* X *into a separated space* E, *the set of points x of* X *such that $f(x) = g(x)$ is closed.*

Indeed, let h be the mapping $x \to (f(x), g(x))$ of X into $E \times E$; h is continuous, and the set in question is simply $h^{-1}(\Delta)$; this set is closed since Δ is closed.

Corollary 2. *If f is a continuous mapping of a space* E *into a separated space* F, *the graph Φ of f is closed in* $E \times F$.

Indeed, Φ is the set of points (x, y) of $E \times F$ such that $y = f(x)$; but the mappings $(x, y) \to y$ and $(x, y) \to f(x)$ of $E \times F$ into F are continuous; therefore Φ is closed by the preceding corollary.

It should be noted that the converse of Corollary 2 is false: Even if E and F are separated, the graph of f can be closed without f being continuous.

For example, let f be the mapping of **R** into **R** defined by

$$f(0) = 0, \qquad f(x) = 1/x \qquad \text{if} \qquad x \neq 0.$$

The graph of f is the union of the curve $xy - 1 = 0$ and the set $\{0\}$, and is thus closed; nevertheless f is not continuous at the point 0.

11. COMPACT SPACES

In Section 3 we established an important property of closed bounded intervals of **R**, which was called the theorem of Heine-Borel-Lebesgue; its importance lies in that it allows one to replace certain global studies by a local study.

We have also studied other topological spaces having an analogous property; we shall see, for example, that this is the case for all the closed bounded sets in $\mathbf{R}^n$.

We shall undertake here a general study of spaces which have this property.

11.1. Definition. A SPACE E IS SAID TO BE *COMPACT* IF IT IS SEPARATED AND IF FROM EVERY OPEN COVERING OF E ONE CAN SELECT A FINITE SUBCOVERING OF E.

The sole purpose, in this definition, of the condition that E be separated is to exclude spaces which have little usefulness, such as spaces with the coarse topology.

EXAMPLES. Every finite separated space is compact. On the other hand, **R** is not a compact space (see Section 3). The theorems which follow will furnish us with huge classes of compact spaces.

It is well to be acquainted with several equivalent forms of the condition of compactness. Here are two others:

11.2. Formulation. E IS SEPARATED, AND FROM EVERY FAMILY OF CLOSED SETS IN E WHOSE INTERSECTION IS EMPTY ONE CAN SELECT A FINITE SUBFAMILY HAVING THE SAME PROPERTY.

This condition is the dual of the preceding one; indeed, if $(G_i)_{i\in I}$ is a family of open sets in E, the formula

$$E = \bigcup_{i\in I} G_i$$

is equivalent to

$$\emptyset = \bigcap_{i\in I} F_i \qquad \text{where} \qquad F_i = \complement G_i\,.$$

Let us now say that a family $\mathscr{F}$ of subsets of E has the *finite intersection property* if every finite subfamily of $\mathscr{F}$ has nonempty intersection. Wiht this convention the preceding statement is evidently equivalent to the following:

11.3. Formulation. E IS SEPARATED, AND EVERY FAMILY OF CLOSED SETS IN E WHICH HAS THE FINITE INTERSECTION PROPERTY HAS NONEMPTY INTERSECTION.

In particular we can state:

11.4. Proposition. *In a compact space* E, *every family of nonempty closed sets which is totally ordered by inclusion has a nonempty intersection. For example, every decreasing sequence of nonempty closed sets has a nonempty intersection.*

In fact, every finite subfamily of a totally ordered family of sets has a smallest element, which is therefore the intersection—in the present case nonempty—of this subfamily.

The real line does not have this last property since, for example, the sequence of closed intervals $[n, +\infty)$ $(n = 1, 2, \ldots)$ has empty intersection.

11.5. Proposition. (1) *In a compact space, every sequence of points has at least one adherent point.*

(2) *If it has a single adherent point, the sequence converges to this point.*

(1) Indeed, with the notation used following Definition 8.4, the set of adherent points is the intersection of the decreasing sequence of nonempty closed sets $\overline{A_n}$.

(2) Let a be the unique element of

$$A = \bigcap_1^\infty \overline{A_n}.$$

For every open neighborhood V of a, the closed sets $\overline{A}_n \cap \complement V$ clearly have empty intersection. Therefore, since they form a decreasing sequence, one of them is empty; in other words, from some n_0 on we have $\overline{A_n} \subset V$ and *a fortiori* $a_n \in V$.

More generally, one can verify that every mapping f of a set X with a filter basis $\mathscr{B}$ into a compact space E has at least one adherent value along $\mathscr{B}$, and that if f has a single adherent value, then f converges along $\mathscr{B}$ to this value.

Z It is not always true, in a compact space, that the *set* A of points of a sequence (a_n) has an accumulation point; it can happen that A is either finite or consists even of a single point, which is the case if all the a_i are identical. In other words, a point can be adherent to a sequence (a_n) either because it is an accumulation point of the set A of the a_n, or because it coincides with infinitely many of the a_n.

11.6. Proposition (analog of the Bolzano-Weierstrass theorem).

(1) *Every infinite subset* A *of a compact space* E *has at least one accumulation point in* E.

(2) *Every subset* A *of* E *which does not have any accumulation point in* E *is finite.*

It is clear that these two properties are equivalent; the second is proved exactly as was Theorem 3.3.

Z It is not true that, conversely, every separated space E for which every infinite subset has at least one accumulation point is compact; nevertheless, we shall prove later that this converse is true for metric spaces E.

11.7. Proposition. *Let* A *be a separated subspace of a space* E:

(A *is compact*) $\Leftrightarrow$ (*Every family of open sets in* E *which covers* A *contains a finite subfamily which covers* A).

PROOF. 1. If A is compact, and if $(\omega_i)_{i\in I}$ is a family of open sets in E which covers A, the sets $(A \cap \omega_i)_{i\in I}$ constitute an open covering of the compact space A; thus there exists a finite subset J of I such that the sets $(A \cap \omega_i)_{i\in J}$ cover A; *a fortiori* the family $(\omega_i)_{i\in J}$ covers A.

2. Conversely, suppose that A has the second property. Let $(\omega_i')_{i\in I}$ be a family of open sets in A which covers A; every ω_i' is of the form $\omega_i' = A \cap \omega_i$, where ω_i is an open set in E.

The ω_i cover A; therefore there exists a finite subset J of I such that the sets $(\omega_i)_{i\in J}$ also cover A.

But

$$\bigcup_{i\in J} (A \cap \omega_i) = A \cap \Big(\bigcup_{i\in J} \omega_i\Big) = A,$$

and therefore the finite family $(\omega_i')_{i\in J}$ covers A.

It is therefore true that every covering of A by open sets ω_i' in A has a finite subcovering; therefore A is compact.

11.8. Corollary. *For every* $a, b \in \mathbf{R}$, *the closed interval* $[a, b]$ *is compact.*

This is an immediate consequence of Theorem 3.2, and of Proposition 11.7 which was just proved.

11.9. Corollary. *Let* E *be a separated space and let* (a_n) *be a sequence of points of* E *which converges to a point* a *of* E.

Then the set $A = \{a, a_1, a_2, \ldots\}$ *is compact.*

Indeed, the set A is, first of all, separated; next, let (ω_i) be a covering of A by open sets in E. There exists an ω_{i_0} which contains a; this open set is a neighborhood of a and therefore contains all the a_n except at most a finite number, say $a_{n_1}, \ldots, a_{n_p}$. These points are contained, respectively, in certain of the ω_i, say $\omega_{i_1}, \ldots, \omega_{i_p}$. The open sets $\omega_{i_0}, \omega_{i_1}, \ldots, \omega_{i_p}$ constitute a finite covering of A. Therefore A is compact.

The next theorem is of fundamental importance.

11.10. Theorem. *In a compact space* E, *every closed subset is a compact subspace.*

PROOF. For a simple proof, we shall use the criterion for compactness which is best suited to the problem, for example criterion 11.3 which is stated in terms of closed sets.

Let A be closed in E, and let $(X_i)_{i \in I}$ be a family of closed sets in the space A, having the finite intersection property. Since A is closed, the X_i are also closed sets in E; therefore since E is compact, the intersection of the X_i is nonempty. Hence A is compact.

11.11. Theorem. *In every separated space* E, *every compact subspace of* E *is closed in* E.

Let A be a compact subspace of E. We shall show that $\complement A$ is open. Suppose $x_0 \in \complement A$; for every $y \in A$, let V_y and W_y be two *open disjoint* neighborhoods in E of x_0 and y, respectively. There exists a finite subfamily $(W_{y_i})_{i \in I}$ of the open sets W_y which covers A. The open set

$$V = \bigcap_{i \in I} V_{y_i}$$

is a neighborhood of x_0 and is disjoint from each of the W_{y_i}, hence also from A; in other words, $V \subset \complement A$. Thus $\complement A$ is a neighborhood of each of its points, and is therefore open.

Let us remark that the hypothesis that E is separated is essential. For example, if E has the coarse topology and contains more than one point, every one-point subset of E is a compact subspace, and yet is not closed.

This theorem shows that every compact space can be called "absolutely closed" since it is closed in every space E which contains it (at least if E is separated).

11.12. Corollary (of Theorems 11.10 and 11.11). *In every compact space* E, *the classes of closed subsets and compact subsets are identical.*

11.13. Corollary. *The compact subspaces of* **R** *are the subsets of* **R** *which are closed and bounded.*

Indeed, if X is a closed bounded subset of **R**, there exists an interval $[a, b]$ containing X; since X is closed in **R**, it is also closed in the subspace $[a, b]$, and since $[a, b]$ is compact, X is compact.

Conversely if X is a compact subset of **R**, X is closed in **R** since **R** is separated. On the other hand, X is bounded since we can select, from the sequence of open intervals $(-n, n)$ (where $n = 1, 2, \ldots$), which forms an open covering of X, a finite sequence of intervals which covers X, and the largest among them will contain X.

11.14. Theorem. *In every separated space, the union of two compact sets is compact; every intersection of compact sets is compact.*

PROOF. Let A and B be compact sets in E.

1. Since E is separated, so is $A \cup B$.

On the other hand, every open covering of $A \cup B$ is also an open covering of A and of B. We can therefore find two finite coverings, one of A, the other of B; together they constitute a finite covering of $A \cup B$; therefore $A \cup B$ is compact.

2. If the A_i $(i \in I)$ are compact, each of them is closed in E, and therefore their intersection is closed in E, and *a fortiori* in any member A_{i_0} of $(A_i)_{i \in I}$. This intersection is thus a compact set (Theorem 11.10).

Z Clearly the union of an infinite family of compact sets is not in general a compact set.

11.15. Theorem. *For every continuous mapping f of a compact space* E *into a separated space* F, *the subspace* $f(E)$ *of* F *is compact.*

PROOF. First of all, $f(E)$ is separated because F is. Next, let $(\omega_i)_{i \in I}$ be a covering of $f(E)$ by open sets in $f(E)$. The sets $f^{-1}(\omega_i)$ constitute an open covering of E; one can find a finite covering $(f^{-1}(\omega_i))_{i \in J}$. Since $f(f^{-1}(\omega_i)) = \omega_i$, the $(\omega_i)_{i \in J}$ cover $f(E)$. Therefore $f(E)$ is compact.

11.16. Corollary. *Every continuous bijection of a compact space* E *to a separated space* F *is a homeomorphism.*

It suffices to show that f^{-1} is continuous, hence that for every closed set X in E, $f(X)$ is closed in F.

But X is closed in the compact space E and thus compact; its image $f(X)$ in the separated space F is thus compact, and hence also closed.

This corollary thus gives us an important case in which the continuity of a one-to-one mapping implies its bicontinuity (that is, the continuity of f and f^{-1}).

11.17. Corollary. *Every numerical function which is continuous on a compact space* E *is bounded on* E *and attains its supremum and infimum on* E.

Let f be a continuous mapping of E into $\mathbf{R}$; $f(E)$ is compact, therefore closed and bounded in $\mathbf{R}$, and hence contains its infimum b_1 and its supremum b_2; thus there exists $x_1 \in E$ such that $f(x_1) = b_1$ and $x_2 \in E$ such that $f(x_2) = b_2$.

In particular, if $f(x) > 0$ on E, there exists $b > 0$ such that $f(x) \geqslant b$ on E.

Z If E is not compact, a continuous numerical function on E need not be bounded, and when it is bounded, need not attain its bounds.

For example, the function $x \to x$ is not bounded on **R**.

The function $x \to x/(1 + |x|)$ is bounded on **R** but does not attain either of its bounds.

The function $x \to 1/x$ is not bounded on $(0, 1]$.

The function $x \to x$ is bounded on $(0, 1)$ but does not attain either of its bounds on this interval.

11.18. Corollary. *If* E *is a product of separated spaces* E_i, *the projection of every compact set in* E *onto each of the* E_i *is compact.*

Indeed, the projection onto each E_i is a continuous mapping.

Product of compact spaces

Theorems 11.10 and 11.15 give us a powerful method for constructing compact spaces. Here is another method, which is particularly convenient for the study of functions of several variables.

11.19. Theorem. *Every finite product of compact spaces is compact.*

Proof. By the associativity of the product topology, it suffices to prove the theorem for the product of two spaces.

Let $E = X \times Y$ be the product of compact spaces X and Y. Since X and Y are separated, E is separated (Proposition 10.4).

Now let $(\omega_i)_{i \in I}$ be an open covering of E. For every $m = (x, y) \in E$ there exists an $i_m \in I$ such that $m \in \omega_{i_m}$. Therefore there exist open neighborhoods V_m and W_m of x and y in X and Y such that $V_m \times W_m \subset \omega_{i_m}$; we set $U_m = V_m \times W_m$.

But for every $x_0 \in X$, the subset $Y_0 = x_0 \times Y$ of $X \times Y$ is homeomorphic to Y, hence compact.

The U_m, $m \in Y_0$, constitute an open covering of Y_0; we can find a finite subcovering

$$(U_{m_j})_{j \in J}, \qquad \text{where} \qquad m_j = (x_0, y_j).$$

We set

$$V_{x_0} = \bigcap_{j \in J} V_{m_j};$$

this is an open neighborhood of x_0 and it is clear that

$$\bigcup_{j \in J} \omega_{i_{m_j}} \supset V_{x_0} \times Y.$$

The V_{x_0} form an open covering of X; we can find a finite subcovering. With each of the (finitely many) corresponding points x_0 there is associated a (finite) subfamily $(\omega_{i_{m_j}})$ of open set ω_i; the union of these families is a finite family which covers E.

11.20. Corollary. *The compact subspaces of* $\mathbf{R}^n$ *are the closed and bounded subsets of* $\mathbf{R}^n$ (A *is said to be bounded in* $\mathbf{R}^n$ *if it is contained in an interval with bounded sides*).

Indeed, if A is a compact set in $\mathbf{R}^n$, it is closed in $\mathbf{R}^n$; on the other hand, the projection of A onto each factor $\mathbf{R}$ is compact, and therefore contained in a bounded interval. Hence A is contained in an interval with bounded sides.

Conversely, if A is closed and bounded, it is a closed subset of a finite product of compact intervals $[a_i, b_i]$; such a product is compact, hence so is A.

EXAMPLE. The sphere S_{n-1} of $\mathbf{R}^n$ is closed and bounded, therefore compact. It follows that the torus $(S_1)^p$ is also compact.

12. LOCALLY COMPACT SPACES; COMPACTIFICATION

There exist many spaces which, without being compact, behave locally like a compact space; this is the case with $\mathbf{R}$ for example. Precisely:

12.1. Definition. A SPACE E IS SAID TO BE *LOCALLY COMPACT* IF IT IS SEPARATED AND IF EACH OF ITS POINTS HAS AT LEAST ONE COMPACT NEIGHBORHOOD.

EXAMPLE 1. Every compact space is locally compact.

EXAMPLE 2. Every discrete topological space is locally compact (example: $\mathbf{Z}$).

EXAMPLE 3. The line $\mathbf{R}$ is locally compact; indeed, $\mathbf{R}$ is first of all separated. Next, for every $x \in \mathbf{R}$ there exist $a, b \in \mathbf{R}$ such that $a < x < b$ and $[a, b]$ is a compact neighborhood of x.

We know, by the way, that $\mathbf{R}$ is not compact.

EXAMPLE 4. The subspace $\mathbf{Q}$ of $\mathbf{R}$ is neither compact nor locally compact; indeed, suppose for example that 0 has, in $\mathbf{Q}$, a compact neighborhood V. The neighborhood V of 0 contains a subneighborhood of the form $\mathbf{Q} \cap [-a, a]$; since this subneighborhood is closed in $\mathbf{Q}$, the set $A = \mathbf{Q} \cap [-a, a]$ would then be compact. But this is clearly false since, for every irrational $x \in [-a, a]$, the decreasing sequence of closed sets $A \cap [x - 1/n, x + 1/n]$ in A has empty intersection.

We now give several results which yield powerful methods for constructing locally compact spaces.

12.2. Proposition. *Every closed subset of a locally compact space is locally compact.*

Proof. Let E be a locally compact space, and A a closed subset of E. Every $x \in A$ has, in E, a compact neighborhood V. The set $V \cap A$ is closed in V, hence compact; since it is a neighborhood of x in A, x thus has a compact neighborhood (in A). Finally, A is separated since it is contained in the separated space E; hence it is locally compact.

Example. Every algebraic variety in $\mathbf{R}^n$ is locally compact.

12.3. Proposition. *If* A *and* B *are locally compact subspaces of a separated space, their intersection is also locally compact.*

Proof. To begin with, $A \cap B$ is separated; next, for every $x \in A \cap B$ there exist compact neighborhoods V and W of x, in A and B, respectively. The set $V \cap W$ is a neighborhood of x in $A \cap B$, and is compact.

Z On the other hand, the *union* of A and B need not be locally compact. For example, let A be the subset of $\mathbf{R}^2$ consisting of the points (x, y) such that $x > 0$, and let $B = \{(0, 0)\}$; the set $A \cup B$ is not locally compact because the point $(0, 0)$ does not have any compact neighborhood in $A \cup B$.

12.4. Proposition. *Every finite product of locally compact spaces is locally compact.*

Proof. It clearly suffices to prove this for two spaces A, B.

Since A and B are separated, so is $A \times B$. On the other hand, for every $(x, y) \in A \times B$, the points x and y have compact neighborhoods V and W in A and B, respectively. The product $V \times W$ is the desired compact neighborhood of (x, y).

Example. The spaces $\mathbf{R}^n$ and $S_1 \times \mathbf{R}$ are locally compact.

Z Theorem 11.15 does not have an equivalent for locally compact spaces; in other words, it is false that every separated space which is the image of a locally compact space under a continuous mapping is itself locally compact.

For example, since $\mathbf{Q}$ is denumerable, there is a surjection $n \to f(n)$ of $\mathbf{Z}$ to $\mathbf{Q}$; since $\mathbf{Z}$ is discrete, f is continuous. But $\mathbf{Q}$ is not locally compact, although $\mathbf{Z}$ is.

12.5. Proposition. *Let the space* E *be locally compact but not compact, and let f be a continuous numerical function on* E *such that*

$$\lim_{x\to\infty} f(x) = +\infty$$

(*in the sense that for every* $h > 0$, *there exists a compact set* $K \subset E$ *such that* $f(x) > h$ *outside* K).

Then f has a lower bound and attains its infimum.

PROOF. Let a be an arbitrary point of E, and let h be a number $> f(a)$. By hypothesis there exists a compact set $K \subset E$ such that $f(x) > h$ outside K. The restriction of f to K is continuous, hence has an infimum m which is attained at some point b of K.

For every $x \notin K$ we have

$$f(x) > h > f(a) \geqslant m.$$

For every $x \in K$ we have

$$f(x) \geqslant m.$$

Therefore m is the infimum of f on E; it is attained at the point b.

We remark that the set of x such that $f(x) = m$ is closed and contained in K; hence it is a compact set.

One can similarly show that every continuous numerical function f on E which tends to a finite limit l as $x \to \infty$ (in other words, along the filter base consisting of the complements of the compact sets in E) is bounded and attains each of its bounds which is different from l.

EXAMPLE. Let $P(z)$ be an arbitrary polynomial in z with complex coefficients. The numerical function $z \to |P(z)|$ is continuous in the complex plane **C**; on the other hand, if $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots$ (with $a_n \neq 0$), for every $z \neq 0$ we can write

$$|P(z)| = |a_n| \times |z|^n \times |1 + \alpha_1/z + \alpha_2/z^2 + \cdots + \alpha_n/z^n|.$$

Thus

$$|P(z)| \to +\infty \qquad \text{as} \quad z \to \infty.$$

The hypothesis of Proposition 12.5 is satisfied; we can therefore assert that $|P(z)|$ attains its infimum at some point.

This is the basis of one of the proofs of the d'Alembert-Gauss theorem (fundamental theorem of algebra).

Points at infinity and compactification

In the statement of Proposition 12.5, the expression "$x \to \infty$" is only a convenient way of speaking. In fact, we can make it precise by showing that for every space E which is locally compact but not compact, an additional point ω, called the point at infinity, can be adjoined to E and a topology defined on the set $E \cup \{\omega\}$ which makes it a compact space, and whose trace on E is the original topology.

More generally one could set oneself the task of finding the compact spaces $\hat{E}$ of which E is an everywhere dense subspace; the points of $\hat{E} - E$ could then be interpreted as the points at infinity of E. Such a space $\hat{E}$ is called a *compactification* of E.

The particular compactification of interest is dictated by the needs of analysis or of geometry; for example, the compactification of $\mathbf{R}^n$ which is useful in projective geometry is the compact space P^n, called the projective space of dimension n, which can be identified with the set of lines in $\mathbf{R}^{n+1}$ passing through O, taken with a suitable topology.

We shall study two simple examples here.

1. **Compactification of $\mathbf{R}^n$ by a point at infinity.** The inversion f about the pole O and of degree 1 in $\mathbf{R}^{n+1}$, defined by

$$x \to \frac{x}{\| x \|^2} = \frac{x}{\sum x_i^2},$$

is a homeomorphism of $\mathbf{R}^{n+1} - \{O\}$ with itself.

It transforms the hyperplane $x_{n+1} = 1$ of $\mathbf{R}^{n+1}$ into $S - \{O\}$, where S is the sphere of $\mathbf{R}^{n+1}$ with diameter (O, A), calling A the point $(0, 0, \ldots, 0, 1)$.

Since the canonical mapping $g: (x_1, x_2, \ldots, x_n) \to (x_1, x_2, \ldots, x_n, 1)$ of $\mathbf{R}^n$ onto the hyperplane $x_{n+1} = 1$ of $\mathbf{R}^{n+1}$ is a homeomorphism, the bijection $f \circ g$ of $\mathbf{R}^n$ to $S - \{O\}$ is a homeomorphism. If we identify the spaces $\mathbf{R}^n$ and $S - \{O\}$ by this homeomorphism, the sphere S is the desired compactification of $\mathbf{R}^n$; its only point at infinity is the point O.

For $n = 1$, S is a circle.

For $n = 2$, S is a two-dimensional sphere which is called the Riemann sphere when $\mathbf{R}^2$ is identified with $\mathbf{C}$; it is extremely useful in the study of the complex plane, in part due to the fact that the homeomorphism $f \circ g$ preserves angles (one says that it is "conformal").

2. **The extended real line $\bar{\mathbf{R}}$.** We have just compactified $\mathbf{R}$ by a point at infinity. It is also convenient in analysis to use another compactification. We have defined the extended line $\bar{\mathbf{R}}$ in Section 5, and shown in Section 7 that $\bar{\mathbf{R}}$ is homeomorphic to $[-1, 1]$ taken with the order topology.

On the other hand, we have shown (in Section 9.7) that the order topology on $[-1, 1]$ is identical with that induced by the topology of $\mathbf{R}$; thus $[-1, 1]$ is compact, and the same is true of $\bar{\mathbf{R}}$.

Moreover, the topology induced by $\bar{\mathbf{R}}$ on its subspace $\mathbf{R}$ is identical with the original topology of $\mathbf{R}$; thus $\bar{\mathbf{R}}$ constitutes a compactification of $\mathbf{R}$.

Every nonempty subset X of $\bar{\mathbf{R}}$ has a supremum and infimum, since this is true in $[-1, 1]$; when X is closed, its bounds belong to X.

One neighborhood base of $+\infty$ in $\bar{\mathbf{R}}$ consists of the intervals $[n, +\infty]$ or $(n, +\infty]$ (where $n \in \mathbf{N}$); a similar assertion holds for $-\infty$.

Relatively compact sets in a topological space

We have seen that the compact sets in $\mathbf{R}^n$ are simply the closed bounded subsets of $\mathbf{R}^n$; but every bounded subset of $\mathbf{R}^n$ has a bounded closure. Thus the bounded subsets of $\mathbf{R}^n$ and those subsets of $\mathbf{R}^n$ with compact closure are identical.

More generally, in an arbitrary topological space, the sets which play the role of the bounded sets are those with compact closure; these sets are given a special name.

12.6. Definition. A SUBSET A OF A TOPOLOGICAL SPACE E IS SAID TO BE *RELATIVELY COMPACT* IF ITS CLOSURE $\bar{\mathrm{A}}$ IS COMPACT.

It is evident that this definition is not topological for A, but only for the pair (A, E); for example, the interval (0, 1) is relatively compact in $\mathbf{R}$, but not in itself.

Here are some immediate properties:

1. If A is relatively compact in E, so is every subset of A.
2. Every subset of a compact space E is relatively compact in E.
3. If $A_1, A_2, \ldots, A_n$ are relatively compact in a separated space E, so is their union.
4. Every sequence of points of a relatively compact subset A of E has at least one adherent point in E.

13. CONNECTIVITY

We shall try to make precise the intuitive notion by which we say that a set such as $[0, 1] \cup [2, 3]$ consists of two pieces, while $[0, 1]$ consists of one piece.

It is rather natural to regard two subsets A, B of a topological space E as being clearly separated in E when they are contained in two disjoint closed sets in E; this remark leads us to the following precise definition:

13.1. Definition. A TOPOLOGICAL SPACE E IS SAID TO BE *CONNECTED* IF THERE DOES NOT EXIST ANY PARTITION OF E INTO TWO NONEMPTY CLOSED SETS.

This property is evidently equivalent (by duality) to each of the following:

13.2. THERE DOES NOT EXIST ANY PARTITION OF E INTO TWO NONEMPTY OPEN SETS.

13.3. THE ONLY SUBSETS OF E WHICH ARE BOTH OPEN AND CLOSED ARE E AND Ø.

A *subset* A of a space E is said to be *connected* if the subspace A of E is connected.

EXAMPLE 1. We shall prove later, in the study of metric spaces, that $\mathbf{R}$ (as well as every interval of $\mathbf{R}$) is connected; we shall for the moment assume this.

EXAMPLE 2. On the other hand, the set $\mathbf{Q}$ of rationals is not connected; more generally, we shall show that if a subset A of $\mathbf{R}$ is not an interval, it is not connected. In fact, there then exist two distinct points x, $y \in A$ such that $[x, y] \not\subset A$; therefore there exists a point $a \in [x, y]$ such that $a \notin A$. The nonempty sets $A \cap (-\infty, a)$ and $A \cap (a, \infty)$ are open in A and constitute a partition of A; therefore A is not connected.

To sum up, the only connected subsets of $\mathbf{R}$ *are the intervals.*

Here are several theorems which often enable one to prove that a set is connected.

13.4. Theorem. *Let* $(A_i)_{i \in I}$ *be a family of connected subsets of* E. *If the intersection of this family is nonempty, its union is connected.*

PROOF. Set

$$A = \bigcup_{i \in I} A_i .$$

We consider an arbitrary partition of A into two open sets O_1 and O_2. For every i, $A_i \cap O_1$ and $A_i \cap O_2$ are open relative to A_i; since A_i is connected, one of these sets is empty. Therefore A_i is contained either in O_1 or in O_2. But the A_i have at least one common point x, which belongs to O_1, say. Therefore O_1 contains all the A_i and O_2 is empty. Thus A is connected.

EXAMPLE. Every convex subset X of $\mathbf{R}^n$ is connected; in fact, for every $a \in X$, X is the union of segments containing a, and each of these segments is connected since it is homeomorphic to $[0, 1]$.

In particular $\mathbf{R}^n$, every open or closed interval of $\mathbf{R}^n$, and every open or closed ball of $\mathbf{R}^n$, is connected.

13.5. Theorem. *The closure of every connected set is connected.*

PROOF. Let A be a connected subset of the space E. To every partition of $\bar{A}$ into two sets O_1 and O_2 which are open in $\bar{A}$, there corresponds the partition of A into two sets $A \cap O_1$ and $A \cap O_2$ which are open in A. Since A is connected, one of these, say $A \cap O_1$, is empty; since A is everywhere dense on $\bar{A}$, the set O_1 is empty. Therefore A is connected.

A similar proof shows that every B such that $A \subset B \subset \bar{A}$ is also connected.

EXAMPLE. All the spheres of $\mathbf{R}^{n+1}$ are homeomorphic to the sphere S of $\mathbf{R}^{n+1}$ used in Section 12 for the compactification of $\mathbf{R}^n$; but S is the closure of a subspace homeomorphic to $\mathbf{R}^n$. Since $\mathbf{R}^n$ is connected, so is S. Thus every sphere is connected.

13.6. Theorem. *Every continuous image of a connected space is connected.*

PROOF. Let f be a continuous surjection of a connected space E to F. For every subset X of F which is both open and closed in F, $f^{-1}(X)$ is open and closed in E, and is thus either E or $\emptyset$. But $X = f(f^{-1}(X))$, therefore $X = F$ or $\emptyset$; in other words, F is connected.

13.7. Definition. A COMPACT AND CONNECTED SPACE IS CALLED A *CONTINUUM*.

By Theorems 11.15 and 13.6, every separated space which is the continuous image of a continuum is a continuum.

In studying metric spaces, we shall prove a property of metric continua which makes the notion of connectivity more intuitive.

EXAMPLE 1. The interval $[-1, 1]$ of $\mathbf{R}$ is a continuum; the same is therefore true of $\bar{\mathbf{R}}$.

EXAMPLE 2. For every bounded and connected subset X of $\mathbf{R}^n$, $\bar{X}$ is connected and compact, and therefore a continuum.

For example, the graph Γ of the mapping $x \to \sin 1/x$ of $(0, 1]$ into $\mathbf{R}$ is a continuous image of $(0, 1]$, hence a connected subset of $\mathbf{R}^2$; its closure, which is the union of Γ and the interval $0 \times [-1, 1]$, is a continuum.

This continuum is used quite often in topology to construct counterexamples.

13.8. Definition. AN OPEN AND CONNECTED SUBSET D OF A SPACE E IS CALLED A *DOMAIN* OF E.

13.9. Proposition. *In order that an open subset* D *of* $\mathbf{R}^n$ *be a domain, it is necessary and sufficient that any two points* p *and* q *belong to a polygonal line whose segments are parallel to a coordinate axis and which is contained in* D.

PROOF. Let $(a_1, a_2, \dots, a_p)$ be a sequence of points of $\mathbf{R}^n$ such that each of the segments $[a_i, a_{i+1}]$ (where $i < p$) is parallel to one of the axes of $\mathbf{R}^n$. Using induction, it is immediate from Theorem 13.4 that the polygonal line made up of the union of these segments is connected.

Thus, if for every $p, q \in \mathrm{D}$ these points belong to such a line contained in D, then fixing p and letting q range over D, we see that D is the union of polygonal lines containing p; therefore D is connected.

Conversely, suppose that D is connected. For every $x, y \in \mathrm{D}$ we write $x \sim y$ if these points are the endpoints of a polygonal line of the preceding type. It is immediate that the relation $\sim$ is an equivalence relation on D, and that each of its classes is open (since for every point y of an open interval containing x and contained in D we have $x \sim y$). There can only be one such class, since otherwise D would admit a partitioning into two nonempty open sets (for example, one such class, and the union of all the others); in other words, for every $x, y \in \mathrm{D}$ we have $x \sim y$.

EXAMPLE. In $\mathbf{R}^n$, every open ball, every open interval, the complement of every closed ball and of every closed interval, is a domain.

Connected components of a space

We are now able to study the structure of spaces which are not connected, by making precise the vague notion of a "piece" of such a space.

13.10. Definition. FOR EVERY POINT x OF A SPACE E, THE UNION $\mathrm{C}(x)$ OF ALL THE CONNECTED SUBSETS OF E CONTAINING x IS CALLED THE *CONNECTED COMPONENT* OF x.

By Theorem 13.4, $\mathrm{C}(x)$ is a connected set; on the other hand, since $\overline{\mathrm{C}(x)}$ is also connected and since by construction $\mathrm{C}(x)$ is the largest connected subset of E containing x, we have $\mathrm{C}(x) = \overline{\mathrm{C}(x)}$; in other words, $\mathrm{C}(x)$ is *closed*.

The binary relation on E defined by "$x_1 \sim x_2$ if there is a connected subset of E containing x_1 and x_2" is evidently an equivalence relation. But for every x, $\mathrm{C}(x)$ is the equivalence class containing x. Therefore the sets $\mathrm{C}(x)$ can also be defined as the equivalence classes associated with the preceding equivalence relation. This is why they are called *connected components* of E.

EXAMPLE 1. If E is connected, there is only a single connected component, namely E itself.

EXAMPLE 2. In the space **Q**, every connected component consists of one point; therefore $C(x) = \{x\}$ for every $x \in \mathbf{Q}$.

EXAMPLE 3. In $\mathbf{R} - \{x\}$, the connected components are $(-\infty, x)$ and (x, ∞).

EXAMPLE 4. In $\mathbf{R}^2$, the connected components of $\mathbf{Q} \times \mathbf{R}$ are the lines $x \times \mathbf{R}$ where $x \in \mathbf{Q}$.

EXAMPLE 5. In $\mathbf{R}^2$, the subspace consisting of the union of the hyperbola $xy = 1$ and its asymptotes has three connected components.

Locally connected spaces

13.11. Definition. A SPACE E IS SAID TO BE *LOCALLY CONNECTED* AT THE POINT x OF E IF x HAS A NEIGHBORHOOD BASE CONSISTING OF CONNECTED SETS.

WE SAY THAT E IS LOCALLY CONNECTED IF IT IS LOCALLY CONNECTED AT EVERY POINT.

EXAMPLE 1. **R** and every interval of **R** is locally connected.

EXAMPLE 2. More generally, every convex subset A of $\mathbf{R}^n$ is locally connected; in fact, every point $x \in A$ has a neighborhood base of convex sets consisting of the intersections of A with the open intervals of $\mathbf{R}^n$ containing x.

EXAMPLE 3. **Z** is locally connected.

EXAMPLE 4. **Q**, on the other hand, is not locally connected at any point.

13.12. Proposition. *To say that* E *is locally connected is equivalent to saying that for every open set* ω *in* E, *the connected components of* ω *are open.*

PROOF. 1. Let E be locally connected, let ω be an open set in E, and let C be a connected component of ω.

For every $x \in C$, there exists a connected neighborhood V of x contained in ω; evidently $V \subset C$, and therefore C is a neighborhood of x. Since C is a neighborhood of each of its points, C is open.

2. Conversely, suppose that every connected component of every open set in E is open. For every $x \in E$ and for every neighborhood V of x, the

connected component C of $\overset{\circ}{V}$ which contains x is open; thus C is the desired connected neighborhood of x contained in V.

EXAMPLE 1. Since **R** is locally connected, for every closed set $F \subset \mathbf{R}$ the connected components of $\complement F$ are open, and thus open intervals; each endpoint of such an interval belongs to F.

Since each of these intervals contains at least one rational number, and since they are disjoint, there are finitely or countably many of them.

EXAMPLE 2. The criterion furnished by Proposition 13.12 is often convenient for showing that a space is not locally connected.

For example, let us consider the continuum $\bar{\Gamma}$ defined in Example 2 of Definition 13.7. Let ω be the open set in $\bar{\Gamma}$ defined by $\omega = \bar{\Gamma} \cap (\mathbf{R} \times (-1/2, 1/2))$ (see Fig. 2). One of the connected components of ω is the interval $0 \times (-1/2, 1/2)$, which is not open in $\bar{\Gamma}$; therefore $\bar{\Gamma}$ is not locally connected (although it is at every point of Γ).

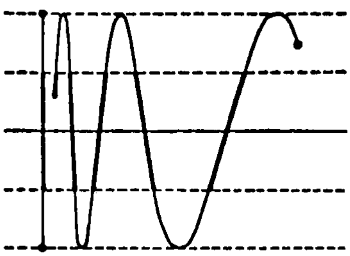

FIG. 2.

Arcwise connectivity

In many branches of mathematics, only connected spaces of a very regular type are used; for example, the spaces used in differential geometry are generally locally homeomorphic to $\mathbf{R}^n$ or to a closed halfspace in $\mathbf{R}^n$. It is then convenient to use the notion of arcwise connectedness.

13.13. Definition. A SPACE E IS SAID TO BE *ARCWISE CONNECTED* IF FOR EVERY $a, b \in E$ THERE EXISTS A CONTINUOUS MAPPING f OF AN INTERVAL $[\alpha, \beta]$ OF **R** INTO E SUCH THAT $f(\alpha) = a$ AND $f(\beta) = b$.

A SPACE E IS SAID TO BE *LOCALLY ARCWISE CONNECTED* IF EVERY POINT OF E HAS A NEIGHBORHOOD BASE OF ARCWISE CONNECTED SETS.

Since every continuous image of an interval $[\alpha, \beta]$ is connected, it is evident that every arcwise connected space is connected; but the converse is false (one can verify this on the continuum $\bar{\Gamma}$ of Example 2 above).

Similarly, every locally arcwise connected space is locally connected.

The expression "arcwise connected" comes from the fact that if there exists a continuous mapping f of an interval $[\alpha, \beta]$ of $\mathbf{R}$ into E such that $f(\alpha) = a$ and $f(\beta) = b(a \neq b)$, then there also exists a simple arc in $f([\alpha, \beta])$ with endpoints a, b. But this last property (which, by the way, is not evident) is rarely used, due to the following fact:

Let $\alpha, \beta, \gamma \in \mathbf{R}$ with $\alpha < \beta < \gamma$, and let f be a mapping of $[\alpha, \gamma]$ into E; if the restrictions of f to $[\alpha, \beta]$ and to $[\beta, \gamma]$ are continuous, then f is continuous. However, if we require in addition that f be one-to-one on $[\alpha, \beta]$ and on $[\beta, \gamma]$, this does not imply that f is one-to-one on $[\alpha, \gamma]$. Thus a definition using connectedness by simple arcs would be awkward.

Example 1. Every domain in $\mathbf{R}^n$ is arcwise connected and locally arcwise connected.

Example 2. The same is true of every convex subset of $\mathbf{R}^n$.

Example 3. Let E be a connected space; if E is locally arcwise connected, it is also arcwise connected (imitate the proof of Proposition 13.9).

This is the case for the locally Euclidean varieties studied in differential geometry.

One can verify the following properties as an exercise:

1. Every continuous image of an arcwise connected space is arcwise connected.

2. For every family (E_i) of arcwise connected sets having nonempty intersection, the union of the E_i is arcwise connected.

14. TOPOLOGICAL GROUPS, RINGS, AND FIELDS

The notion of a topological group arose from the study of special cases such as the additive group $\mathbf{R}$, or transformation groups depending on a finite number of parameters, such as the group of dilations of $\mathbf{R} : x \to \lambda x + a$ (where $\lambda \neq 0$). In these various examples, the set under study has both a group structure and a topological structure, and these two structures are compatible in the sense that the group operations are continuous.

More generally, if E is a set having both an algebraic structure defined by several operations, and a topological structure, E is said to be a topological algebraic structure if the algebraic operations in E are continuous for the given topology, in a sense which has to be made precise for each kind of structure.

Topological group

14.1. Definition. A TOPOLOGICAL GROUP G IS A GROUP WITH A TOPOLOGY FOR WHICH THE FUNCTIONS x^{-1} AND (xy) ARE CONTINUOUS.

More precisely, it is assumed that the mappings $x \to x^{-1}$ of G onto G and $(x, y) \to xy$ of $G \times G$ onto G are continuous.

When these conditions are satisfied, one also says that the given topology is *compatible with the group structure of* G.

EXAMPLE 1. The additive group **R** with its ordinary topology is a topological group.

EXAMPLE 2. Let $\mathbf{R}^*$ be the multiplicative group of real numbers > 0. The topology on $\mathbf{R}^*$ induced by that of **R** is compatible with its group structure.

EXAMPLE 3. Let T be the quotient group of **R** by the equivalence relation $\langle x_1 \sim x_2$ if $(x_1 - x_2)$ is an integer$\rangle$; in other words (see Volume 1, Chapter II) T is the one-dimensional torus.

For every $c \in T$, we set $|c|$ = the smallest of the absolute values of the representatives of c in **R**. If we then set, for every pair x, y of elements of T,

$$d(x, y) = |x - y|,$$

one can verify that d is a metric on T (see Section 15) which is incidentally invariant under translations of T, and that the topology associated with this metric is compatible with the group structure of T.

The group T, taken with this topology, is the *topological one-dimensional torus*. One can verify that the topological space T is homeomorphic to the circle S_1.

EXAMPLE 4. The multiplicative group of complex numbers $(a + ib)$ with absolute value 1, with the topology induced by that of the complex plane, is a topological group. One can prove that it is isomorphic to T.

EXAMPLE 5. If we identify the group of dilations of the line: $x \to \lambda x + a$ (where $\lambda \neq 0$) with the set D of points (λ, a) of the plane $\mathbf{R}^2$ with nonzero abscissa, and give D the topology induced by that of $\mathbf{R}^2$, D becomes a topological group.

In fact, if $s = (\lambda, a)$, then

$$s^{-1} = (\lambda, a)^{-1} = (1/\lambda, -a/\lambda);$$

thus s^{-1} is a continuous function of s on D. On the other hand, if $s = (\lambda, a)$ and $s' = (\lambda', a')$, then $s \circ s' = (\lambda\lambda', \lambda a' + a)$; thus $s \circ s'$ is a continuous function of the pair (s, s').

EXAMPLE 6. If G is an arbitrary group, then taken with the discrete topology, G becomes a topological group. It goes without saying that in general this topology is not very interesting.

CONSEQUENCE 1. The symmetry operation $x \to x^{-1}$ is continuous and is identical with its own inverse; therefore it is a homeomorphism of G with itself.

CONSEQUENCE 2. For every $a \in G$, the bijection $x \to ax$, as well as its inverse $y \to a^{-1}y$, is continuous.

Similarly, $x \to xa$, as well as its inverse, is continuous. In other words, every translation is a homeomorphism of G with itself. The same is true, more generally, of every transformation $x \to axb$.

These two consequences can also be expressed by saying that, for every open set $\omega \in G$, the symmetric set ω^{-1} is open, and every translate $a\omega$ or ωa of ω is open.

Neighborhoods of a point

The fact that every translation is a homeomorphism implies that the set of neighborhoods of a point x_0 can be obtained from the set of neighborhoods of the identity element e by translation by x_0, to the right or left. More precisely, the collection of all neighborhoods of x_0 is identical with the collection of all sets x_0V or all sets Vx_0, where V runs through the collection of all neighborhoods of e.

The study of the neighborhoods of a point x_0 of G thus reduces to that of the neighborhoods of e.

Neighborhoods of the identity element

1. *For every neighborhood* V *of e, the symmetric set* V^{-1} *is a neighborhood of e.*

2. *For every neighborhood* V *of e, there exists a neighborhood* $W \subset V$ *of e such that* $WW \subset V$.

The first property follows from the fact that the symmetry operation is a homeomorphism. It follows from this, since $V \cap V^{-1}$ is self-symmetric, that e has a neighborhood base of symmetric sets.

The second property simply expresses the fact that the mapping $(x, y) \to xy$ of $G \times G$ onto G is continuous at the point (e, e) of $G \times G$.

This property, by the way, is not a consequence of the fact that the symmetry operation and the translations of G are homeomorphisms. In other words, a topology on a group G, for which the symmetry operation and the translations are homeomorphims, does not necessarily define a topological group structure on G.

To define a topology on a group G which is compatible with the group structure of G, the following method is often used: One specifies a family $\mathscr{V}(e)$ of subsets of G containing the identity element e (these elements are intended as neighborhoods of e). For every $x_0 \in \mathrm{G}$, one denotes by $\mathscr{V}(x_0)$ the family $x_0\mathscr{V}(e)$ obtained from $\mathscr{V}(e)$ by translation from the left by x_0.

The open sets of G are then defined as those sets ω such that if $x \in \omega$, there exists an element of $\mathscr{V}(x)$ contained in ω.

If the collection of these ω satisfies axioms O_1, O_2, O_3 of a topological space, then a topology has been defined on G; if xy and x^{-1} are continuous in this topology, then it is compatible with the group structure of G. One says that this topology is generated by the family $\mathscr{V}(e)$.

Product of topological groups

Let G_1 and G_2 be topological groups. The product set $\mathrm{G} = \mathrm{G}_1 \times \mathrm{G}_2$ has both a product group structure and a product topology structure.

It is easily verified that the product topology on G is compatible with the product group structure on G; the set G with these two structures defined on it is called the product of the topological groups G_1 and G_2.

The product of any finite number of topological groups is similarly defined.

EXAMPLE 1. The product of n topological groups identical with the additive topological group $\mathbf{R}$ is called the topological group $\mathbf{R}^n$.

The product of n groups identical with the topological torus T is called the n-dimensional torus.

EXAMPLE 2. The operation $(n, \mathrm{G}) \to \mathrm{G}^n$ is a special case of a more general procedure for constructing topological groups:

Let E be an arbitrary set and G a topological group: let G^{E} be the set of all mappings of E into G. A group structure is defined on G^{E} by defining $h = gf$ by the equality $h(x) = g(x)f(x)$ for every $x \in \mathrm{E}$.

For every neighborhood v of e in G, we define V as the set of all elements f of G^{E} such that $f(x) \in v$ for every $x \in \mathrm{E}$. The family $\mathscr{V}$ of sets V generates a topology on G^{E} according to the procedure described above, and one can verify that this topology is compatible with the group structure of G^{E}.

EXAMPLE 3. $\mathbf{C}^*$ is isomorphic to the product of the multiplicative group $\mathbf{R}_+{}^*$ with T. Its subgroup $\mathbf{R}^*$ is isomorphic to the product of $\mathbf{R}_+{}^*$ with the multiplicative group $\{1, -1\}$.

Isomorphisms. Continuous representations

Let E and F be topological groups and f a representation of E in F (that is $f(xy) = f(x)f(y)$). This representation is said to be continuous if the mapping f is continuous on E.

The equality $f(x) = f(xx_0^{-1})f(x_0)$ shows that if a representation f is continuous at the point e of E, it is continuous at every point x_0 of E.

If f is an algebraic isomorphism of the group E onto the group F, and if f is a homeomorphism, the topological groups E and F are said to be isomorphic.

EXAMPLE 1. *Isomorphisms of the additive group* **R** *onto itself.* Let f be a continuous representation of the additive group **R** into itself. Set $f(1) = a$. We deduce that $f(p/q) = a(p/q)$ for all integers p and q, where $q \neq 0$; in other words, $f(x) = ax$ for every rational number x. Since the mappings f and $x \to ax$ are continuous, this equality extends to all of **R**.

Since the mapping $x \to ax$ is indeed a representation of **R** into **R**, the continuous representations of **R** in itself are the mappings $x \to ax$. Except for $a = 0$, such a representation is always an isomorphism.

We remark that there exist many discontinuous representations of **R** in itself; however, their existence is not evident, and it is not known how to construct them other than by using the axiom of choice.

EXAMPLE 2. It follows from a proposition of Volume 1, Chapter III, that the topological multiplicative group $\mathbf{R}_+{}^*$ is isomorphic to the topological additive group **R**. Every isomorphism of $\mathbf{R}_+{}^*$ onto **R** is by definition a logarithm; the isomorphism inverse to a logarithm is an exponential.

EXAMPLE 3. The method of Example 1 easily extends to $\mathbf{R}^n$ and enables one to show that every continuous representation of $\mathbf{R}^n$ in $\mathbf{R}^n$ is a linear mapping of $\mathbf{R}^n$ into $\mathbf{R}^n$; we have already studied these linear mappings.

EXAMPLE 4. *Continuous representations of* T *in itself.* We propose that the reader prove, as an exercise, that every continuous representation of T in itself is of the form $x \to nx$, where n is an arbitrary integer. It follows from this that there exist only two continuous automorphisms of T: the identity and the symmetry operation.

Continuous periodic functions on a topological group

14.2. Definition. LET G BE A COMMUTATIVE GROUP WRITTEN ADDITIVELY, AND LET f BE A MAPPING OF G INTO A SET E. EVERY ELEMENT a OF G SUCH THAT $f(x + a) = f(x)$ FOR EVERY $x \in$ G IS CALLED A *PERIOD* OF f.

It is immediate that the set P of periods of f forms a subgroup of G, called the group of periods of f.

The mapping f is said to be *periodic* if its group of periods does not consist solely of the element O of G.

14.3. Proposition. *The group* P *of periods of a continuous mapping of a topological group* G *into a separated topological space* E *is closed in* G.

In fact, for every $b \in G$ let G_b denote the set of $a \in G$ such that $f(b + a) = f(b)$; since the mapping $\varphi_b : a \to f(b + a)$ of G into E is continuous, and since $\{f(b)\}$ is closed in E, the set $\varphi_b^{-1}(\{f(b)\}) = G_b$ is closed. But by definition

$$P = \bigcap_{b \in G} G_b ;$$

therefore P is closed.

Closed subgroups of **R**

The preceding result shows the interest, for the study of periodic functions of a topological group, in studying the closed subgroups of G. We shall only carry out this study for the group **R**.

14.4. Proposition. *Every closed subgroup of* **R** *is either identical with* **R** *or* $\{0\}$, *or is a discrete group of the form* $a\mathbf{Z}$, *where* $a > 0$.

PROOF. Let P be a closed subgroup of **R**. If 0 is an accumulation point of P, for every $\epsilon > 0$ there exist elements x of P such that $x \neq 0$ and $|x| < \epsilon$; in every interval of **R** of length $> \epsilon$ there exists at least one integer multiple of such an x. In other words P is everywhere dense on **R**, and since it is closed, we have $P = \mathbf{R}$.

If 0 is an isolated point of P, every point of P is isolated; therefore since P is closed, each of the sets $P \cap [-l, l]$ is compact and discrete, therefore finite. Thus, either $P = \{0\}$, or the set of elements > 0 of P has a smallest member, say a.

The group $a\mathbf{Z}$ of integer multiples of a is contained in P; if $P - a\mathbf{Z}$ were not empty, there would exist an $x \in P$ and an integer n such that $a(n - 1) < x < an$; hence the element $(an - x)$ would satisfy $0 < (an - x) < a$, which is impossible by the choice of a. Therefore $P = a\mathbf{Z}$.

REMARK. Let f be a continuous periodic function on **R**, and let P be the group of periods of f.

If $P = \mathbf{R}$, then f is constant on **R**.

If $P = a\mathbf{Z}$, then f is known whenever its restriction to $[0, a)$ or to any one of the intervals $[x_0, x_0 + a)$ or $(x_0, x_0 + a]$ is known.

The period a is called the smallest period of f.

Uniform structure. Uniform continuity

In a topological group G, one can speak not only of points near to a given point, but more generally of the smallness of a set.

In fact, for every neighborhood V of the unit element e of G and for every subset A of G, we can say that A is *small of order* V if $xy^{-1} \in V$ for all $x, y \in A$.

This fundamental fact enables us to introduce the notion of the uniform continuity of a function, a notion which we shall meet with again in a closely related form in the study of metric spaces.

14.5. Definition. LET X AND Y BE TOPOLOGICAL GROUPS WHICH FOR SIMPLICITY WE SHALL ASSUME COMMUTATIVE, AND LET f BE A MAPPING OF X INTO Y. THEN f IS SAID TO BE *UNIFORMLY CONTINUOUS* IF FOR EVERY NEIGHBORHOOD W OF ZERO IN Y THERE EXISTS A NEIGHBORHOOD V OF ZERO IN X SUCH THAT EVERY $A \subset X$ WHICH IS SMALL OF ORDER V HAS AN IMAGE $f(A)$ WHICH IS SMALL OF ORDER W.

This condition can also be expressed by

$$((x_1 - x_2) \in V) \Rightarrow ((f(x_1) - f(x_2)) \in W).$$

EXAMPLES. Let G be a commutative topological group, and let V be an arbitrary symmetric neighborhood of O in G.

1. The relation $-(a + V) = -a - V = -a + V$ shows that every set symmetric to a set which is small of order V is small of order V; therefore the mapping $x \to -x$ of G onto G is uniformly continuous.

2. The continuity of the mapping $f : (x, y) \to (x + y)$ of $G \times G$ onto G at the point (O, O) implies the existence of a neighborhood W of O in G such that $W + W \subset V$.

But for all $a, b \in G$, the set of elements $x + y$ of G such that $x \in a + W$ and $y \in b + W$ is contained in $(a + b) + V$, and is therefore small of order V.

Therefore the mapping f is uniformly continuous.

Z The topology of the multiplicative group $\mathbf{R}^*$ is the trace on $\mathbf{R}^*$ of the topology of the additive group $\mathbf{R}$; therefore the continuous functions defined on the topological group $\mathbf{R}^*$ or with values in this group are the same as the continuous functions defined on the subspace $\mathbf{R}^*$ of $\mathbf{R}$ or with values in this subspace. For example, x^{-1} and xy are continuous on $\mathbf{R}^*$ in the topology of $\mathbf{R}$. But the situation changes completely when one is concerned with uniform continuity.

For example, the identity mapping $x \to x$ of the multiplicative group $\mathbf{R}^*$ into the additive group $\mathbf{R}$ is not uniformly continuous; in fact, for every neighborhood V of the unit element of $\mathbf{R}^*$, the translates of V are the sets λV and evidently there exists no neighborhood W of O in $\mathbf{R}$ such that all the λV are small of order W.

Similarly, the mapping $x \to x^{-1}$ of $\mathbf{R}^*$ into $\mathbf{R}$ is not uniformly continuous if we put on $\mathbf{R}^*$ the uniform structure associated with its group structure, or even the uniform structure induced by that of $\mathbf{R}$; these conclusions carry over to the mapping $(x, y) \to xy$.

Topological rings

14.6. Definition. A TOPOLOGICAL RING A IS A RING WITH A TOPOLOGY FOR WHICH THE FUNCTIONS $(-x)$, $(x + y)$ AND xy ARE CONTINUOUS.

More precisely, we assume that the mapping $x \to -x$ of G onto G and the mappings $(x, y) \to x + y$ and xy of $G \times G$ into G are continuous. When these conditions are satisfied, the topology on A is said to be compatible with the ring structure of A. In particular, for every topological ring A the topology of A is compatible with the additive group structure of A.

Examples of topological rings

1. For every ring A, if A is given the discrete topology, one obtains a topological ring, which however is in general uninteresting.

2. Let G be a commutative topological group and A the ring of representations of G in itself (recall that $h = g + f$ is defined by $h(x) = f(x) + g(x)$ for every x, and fg by $f \circ g$).

Furthermore let S be a fixed subset of G.

Let 0 be the zero of A, that is, the mapping $x \to O$ of G into itself. For every neighborhood ν of zero of G, let $W(\nu)$ be the set of elements f of A such that $f(x) \in \nu$ for every $x \in S$. One can verify that the sets $W(\nu)$ constitute a neighborhood base of 0 for a topology on A which is compatible with the group structure of A. But it is not compatible, in general, with its ring structure. In order that it be so, one is led to consider only continuous representations, and to take for S some neighborhood of the zero of G with special properties.

3. Let A be a ring on which is defined a function φ with positive real values, and such that

$$\varphi(O) = 0, \qquad \varphi(x) > 0 \quad \text{if} \quad x \neq O$$

$$\varphi(-x) = \varphi(x); \qquad \varphi(x + y) \leqslant \varphi(x) + \varphi(y) \qquad \text{and} \qquad \varphi(xy) \leqslant \varphi(x)\varphi(y).$$

For every $x, y \in A$ we set $d(x, y) = \varphi(x - y)$. It is immediate that d is a metric on A. One can then verify that the topology on A associated with this metric (see Sections 15 and 16) is compatible with the ring structure of A.

Special cases

a. A is the ring **R** of real numbers with

$$\varphi(x) = |x|;$$

b. A is the ring of continuous real functions defined on $[0, 1]$ with $\varphi(f) = \sup|f(x)|$;

c. A is the ring of square matrices of order n with complex elements; if a_i^j are the elements of a matrix $m \in A$, we take

$$\varphi(m) = \sum_{i,j} |a_i^j|;$$

d. Let A be the ring of polynomials in one variable with complex coefficients. We take the topology on A for which the zero of A has as a neighborhood base the sets $V(\epsilon, n)$ ($\epsilon > 0$, n an integer $\geqslant 0$), where $V(\epsilon, n)$ denotes the set of polynomials $a_0 + a_1x + \cdots$ such that $|a_i| \leqslant \epsilon$ for every $i \leqslant n$.

Topological fields

14.7. Definition. A TOPOLOGICAL FIELD K IS A FIELD WITH A TOPOLOGY WHICH IS COMPATIBLE WITH THE RING STRUCTURE OF K, AND SUCH THAT THE MAPPING $x \to x^{-1}$ OF K^* INTO K IS CONTINUOUS (WHERE K^* DENOTES THE SET OF ELEMENTS OF K DIFFERENT FROM O).

When a topology on a field K satisfies these conditions, one says that it is compatible with the field structure of K.

EXAMPLES OF TOPOLOGICAL FIELDS. 1. The field **R** of real numbers with the topology of **R**.

2. The subfield **Q** of rational numbers of **R**.

3. The field **C** of complex numbers, with the topology obtained by carrying over the topology of $\mathbf{R}^2$ to **C** by the correspondence

$$(a + ib) \to (a, b).$$

The fact that this topology is compatible with the field structure of **C** follows from the properties of the absolute value:

$$|-z| = |z|, \qquad |z_1 + z_2| \leqslant |z_1| + |z_2|, \qquad |z_1z_2| = |z_1|\,|z_2|.$$

We will also give a more general result:

4. Let K be a field and $| x |$ a positive real function defined on K, not $\equiv 0$ and not $\equiv 1$, having the following properties:

$$| -x | = | x |, \qquad | x + y | \leqslant | x | + | y |, \qquad | xy | = | x | \, | y |.$$

It easily follows from these properties that $| e | = 1$, that $| x | \neq 0$ for $x \neq O$ and that $|O| = 0$.

Such a function is called an *absolute value* on K. A metric on K is associated with it by setting $d(x, y) = | x - y |$.

An elementary calculation shows that the topology associated with this metric is compatible with the field structure of K.

When K is the field **C**, the topology associated with the absolute value is identical with the topology defined in Example 3 above.

The quaternion field is another example of a field with an absolute value defined by

$$| a + bi + cj + dk | = (a^2 + b^2 + c^2 + d^2)^{1/2}.$$

Topological vector spaces

We shall study normed vector spaces, which constitute an important example of topological vector spaces, in Chapter III.

III. METRIC SPACES

In a topological space, the notion of neighborhood allows one to make precise the order of smallness of a set about each point; the notions of convergence and continuity follow from this.

In a topological group, the translations enable us to do better; namely, to specify the order of smallness of a set by comparing it, by translation, to the neighborhoods of the unit element; we can then speak of the uniform continuity of a function.

We shall encounter a similar possibility in metric spaces, whose general definition has been given by M. Fréchet; however in these spaces the proximity of two points is no longer defined by reference to a particular point of the space, but by a number depending on these two points.

15. METRICS AND ECARTS

15.1. Definition. A *METRIC SPACE* IS A PAIR CONSISTING OF A SET E AND A MAPPING $(x, y) \to d(x, y)$ OF $E \times E$ INTO $\mathbf{R}_+$, HAVING THE FOLLOWING PROPERTIES:

M_1 : $(x = y) \Leftrightarrow (d(x, y) = 0)$;
M_2 : $d(x, y) = d(y, x)$ (SYMMETRY);
M_3 : $d(x, y) \leqslant d(x, z) + d(z, y)$ (TRIANGLE INEQUALITY).

THE FUNCTION d IS CALLED A *METRIC* AND $d(x, y)$ IS CALLED THE DISTANCE BETWEEN THE POINTS x, y.

EXAMPLE 1. In **R**, the mapping $(x, y) \to |x - y|$ is the usual distance.

EXAMPLE 2. More generally, let G be a commutative group and let $x \to p(x)$ be a mapping of G into $\mathbf{R}_+$ such that:

$$(p(x) = 0) \Leftrightarrow (x = 0); \qquad p(-x) = p(x); \qquad p(x + y) \leqslant p(x) + p(y).$$

If for every $x, y \in G$ we put $d(x, y) = p(x - y)$, it is immediate that d satisfies axioms M_1, M_2; moreover, the relation

$$(x - y) = (x - z) + (z - y) \quad \text{implies} \quad p(x - y) \leqslant p(x - z) + p(z - y),$$

or

$$d(x, y) \leqslant d(x, z) + d(z, y).$$

Therefore d is a metric on G.

EXAMPLE 3. Let E be an arbitrary set, and put

$$d(x, y) = 0 \quad \text{if} \quad x = y; \qquad d(x, y) = 1 \quad \text{if} \quad x \neq y.$$

It is immediate that d is a metric on E; it is frequently convenient to use it for the construction of counterexamples.

Z Note that a metric on a set E is a function defined, not on E, but on E^2; one has to remember this when studying the properties of the metric.

Ecart on a set

It is often convenient, and not only in studying metrics, to use the notion of an ecart, which is less restrictive than that of a metric.

15.2. Definition. AN *ECART* ON A SET E IS A MAPPING f OF $E \times E$ INTO $\bar{\mathbf{R}}_+$ SUCH THAT:

E_1 : $(x = y) \Rightarrow (f(x, y) = 0)$;
E_2 : $f(x, y) = f(y, x)$;
E_3 : $f(x, y) \leqslant f(x, z) + f(z, y)$.

The only difference from the notion of a metric is therefore that f can assume the value $+\infty$, and that two distinct points can have ecart zero. Therefore to see whether an ecart f is a metric, it suffices to determine whether $f(x, y)$ is always finite and whether

$$(x \neq y) \Rightarrow (f(x, y) \neq 0).$$

EXAMPLE. Let α be a mapping of E into $\mathbf{R}$; the function f defined by

$$f(x, y) = | \alpha(x) - \alpha(y) |$$

is an ecart.

For example, if E is the set of numerical functions on [0, 1] and if $a \in [0, 1]$, the function f_a defined by

$$f_a(x, y) = | x(a) - y(a) |$$

is an ecart on E.

Operations on ecarts

The interest in ecarts lies in the great generality of the operations which preserve their properties; this flexibility renders their use very convenient.

1. The sum of every family of ecarts is an ecart.

In particular, every finite sum of metrics is a metric.

2. Every limit of ecarts is an ecart.
3. Every upper envelope of ecarts is an ecart.

Indeed, let $f(x, y) = \sup f_i(x, y)$, where the f_i are ecarts. The mapping of E^2 into $\bar{\mathbf{R}}_+$ clearly satisfies axioms E_1, E_2; moreover, for every i we have

$$f_i(x, y) \leqslant f_i(x, z) + f_i(z, y) \leqslant f(x, z) + f(z, y),$$

from which

$$f(x, y) \leqslant f(x, z) + f(z, y).$$

Therefore f is an ecart. In particular, if there are only finitely many f_i and they are metrics, then f is a metric.

4. Let α be a mapping of a set E into a set F having an ecart f. For every $x, y \in E$ we put

$$e(x, y) = f(\alpha(x), \alpha(y)).$$

It is immediate that e is an ecart on E; it is called the inverse image of f under the mapping α.

5. We shall say that a mapping φ of $\bar{\mathbf{R}}_+$ into $\bar{\mathbf{R}}_+$ is a *gauge*, if it is increasing and if

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi(x + y) \leqslant \varphi(x) + \varphi(y) \qquad \text{(subadditivity).}$$

One can verify that the family of gauges is invariant under the following operations: addition, ordinary passage to the limit, taking upper envelopes, composition: $(\varphi_1, \varphi_2) \to \varphi_1(\varphi_2)$.

Every φ which is increasing and concave and such that $\varphi(0) = 0$ is a gauge; indeed, the concavity of φ implies that

$$\varphi(u + v) - \varphi(0 + v) \leqslant \varphi(u) - \varphi(0),$$

whence $\varphi(u + v) \leqslant \varphi(u) + \varphi(v)$.

In particular, the functions $x/(1 + x)$ and $\inf(x, 1)$ are gauges.

For every gauge φ and every ecart f on a set E, $\varphi(f)$ is also an ecart on E: The properties E_1, E_2 are evident; moreover, let us put, for every $x, y, z \in E$,

$$a = f(x, y); \qquad b = f(x, z); \qquad c = f(y, z).$$

The relation $a \leqslant b + c$ implies $\varphi(a) \leqslant \varphi(b + c) \leqslant \varphi(b) + \varphi(c)$.

The relation $\varphi(a) \leqslant \varphi(b) + \varphi(c)$ proves property E_3.

6. More generally, one can define gauges on $(\bar{\mathbf{R}}_+)^n$. Let us order this set by putting $x \leqslant y$ if $x_i \leqslant y_i$ for every $i = 1, 2, \ldots, n$; we define $z = x + y$ by $z_i = x_i + y_i$ for every i.

By a *gauge* on $(\bar{\mathbf{R}}_+)^n$ we then mean any mapping φ of this set into $\bar{\mathbf{R}}_+$ which is increasing and such that

$$\varphi(O) = 0 \qquad \text{and} \qquad \varphi(x + y) \leqslant \varphi(x) + \varphi(y).$$

Every φ which is increasing, convex and positive-homogeneous of degree 1 in $(\bar{\mathbf{R}}_+)^n$ is a gauge of this sort; in fact, the convexity gives

$$\varphi((x + y)/2) \leqslant \tfrac{1}{2}(\varphi(x) + \varphi(y)),$$

from which

$$\varphi(x + y) = 2\varphi((x + y)/2) \leqslant \varphi(x) + \varphi(y).$$

For example, if (x_i) are the coordinates of x, the function

$$x \to \left(\sum x_i^2\right)^{1/2}$$

has these properties, and is therefore a gauge; the same holds, more generally, for

$$\Big(\sum |x_i|^p\Big)^{1/p} \qquad (p \geqslant 1).$$

The gauges on $(\bar{\mathbf{R}}_+)^n$ are a convenient means of constructing ecarts. Indeed, if $f_1, ..., f_n$ are ecarts on a set E, and if φ is a gauge on $(\bar{\mathbf{R}}_+)^n$, one can verify as for the case $n = 1$, that $\varphi(f_1, ..., f_n)$ is an ecart on E.

For example, $(\Sigma f_i^2)^{1/2}$ is an ecart.

Applications of operations on ecarts

1. **Classical metrics on $\mathbf{R}^n$.** Let (x_i) denote the coordinates of a point x of $\mathbf{R}^n$. For every i, we put

$$d_i(x, y) = |x_i - y_i|;$$

this is the inverse image of the distance in $\mathbf{R}$ under the mapping $x \to x_i$; therefore it is an ecart on $\mathbf{R}^n$.

By the foregoing, the functions

$$d(x, y) = \Big(\sum_i (x_i - y_i)^2\Big)^{1/2}; \qquad d'(x, y) = \sup_i |x_i - y_i|;$$
$$d''(x, y) = \sum_i |x_i - y_i|$$

are ecarts on $\mathbf{R}^n$; since they are finite and are zero only for $x = y$, they are metrics on $\mathbf{R}^n$. Each of them is invariant under the translations of $\mathbf{R}^n$.

It can be verified that the ratio of any two of these metrics is bounded, and that more precisely we have

$$d' \leqslant d \leqslant d'' \leqslant nd'.$$

2. **Product of metric spaces.** Let (E_i) be a finite family of metric spaces with metrics d_i.

Each of the functions $(x, y) \to d_i(x_i, y_i)$ is an ecart on the product E of the E_i ; therefore, as in the example above, the functions

$$d(x, y) = \Big(\sum_i d_i^2(x_i, y_i)\Big)^{1/2}; \qquad d'(x, y) = \sup_i d_i(x_i, y_i);$$
$$d''(x, y) = \sum_i d_i(x_i, y_i)$$

are ecarts on E; it is immediate that they are metrics, and that they are comparable, since here again

$$d' \leqslant d \leqslant d'' \leqslant nd'$$

(where n is the number of E_i).

Depending upon the circumstance, one or another of these metrics can be used; the first, d, enters only when the spaces E_i are vector spaces and the metrics d_i are derived from a scalar product; it is called the Cartesian or Euclidean metric, and the space $\mathbf{R}^n$ with this metric is called n-dimensional *Euclidean space.*

3. Let E be a metric space and d its metric.

The functions $d' = d/(1 + d)$ and $d'' = \inf(d, 1)$ are metrics on E.

We have $d' \leqslant d'' \leqslant 2d'$; thus these metrics are comparable. We shall see that they give E the same topology and the same "uniform structure" as does d, with the often-appreciable advantage that they are $\leqslant 1$.

4. Let E be the family of mappings of a set A into a metric space B with a metric d.

We know that for every $a \in A$, $d(x(a), y(a))$ is an ecart on E; this ecart measures the proximity of the functions x, y at the point a.

More generally, let us put, for every $X \subset A$,

$$d_X(x, y) = \sup_{a \in X} d(x(a), y(a)).$$

This is an ecart on E, which measures the proximity of the functions x, y on X; in particular d_A is an ecart on E which vanishes only for $x = y$. If in addition, therefore, $d \leqslant 1$, then $d_A \leqslant 1$; hence d_A is a metric.

We shall use these ecarts in the study of uniform convergence.

Most of the concepts which we are now going to introduce extend in an evident way to sets with an ecart; we shall do this explicitly only when it will be useful.

Metric subspaces of a metric space

15.3. Definition. LET E BE A METRIC SPACE DEFINED BY A METRIC d, AND LET A BE A SUBSET OF E. THE SET A WITH THE METRIC d_A DEFINED BY $d_A(x, y) = d(x, y)$ FOR $x, y \in A$ IS CALLED THE *METRIC SUBSPACE* A OF E.

In other words, d_A is the restriction of d to the subset A^2 of E^2. This definition gives us extensive examples of metric spaces: for example, every subset of $\mathbf{R}^n$ becomes a metric subspace of $\mathbf{R}^n$ when $\mathbf{R}^n$ is taken with one of the metrics defined above.

Isometries

15.4. Definition. LET E AND E′ BE TWO METRIC SPACES WITH THE METRICS d AND d'; LET $f : x \to x'$ BE A BIJECTION OF E TO E′.

THEN f IS CALLED AN ISOMETRY IF FOR ALL x, $y \in$ E,

$$d(x, y) = d'(x', y').$$

Thus, an isometry is simply an isomorphism for the metric space structures.

EXAMPLE 1. In a commutative group with a metric d of the form $d(x, y) = p(x - y)$, the symmetry operation and every translation is an isometry.

EXAMPLE 2. In the space $\mathbf{R}^n$ with one of its classical metrics, every line is a subspace isometric to the real line $\mathbf{R}$.

EXAMPLE 3. One can prove that the isometries of the Euclidean space $\mathbf{R}^n$ on itself which leave the origin fixed are simply the linear transformations of $\mathbf{R}^n$ which preserve the quadratic form $\sum x_i^2$.

Open and closed balls. Spheres

15.5. Definition. IN A METRIC SPACE E, THE SET B(x, ρ) OF POINTS x OF E SUCH THAT $d(a, x) < \rho$ $(\leqslant \rho)$ IS CALLED THE *OPEN* (*CLOSED*) BALL WITH CENTER a AND RADIUS ρ ($\rho \geqslant 0$ OR $+\infty$ AND $a \in$ E).

WHEN E IS THE EUCLIDEAN PLANE $\mathbf{R}^2$ THE TERM BALL IS OFTEN REPLACED BY THE TERM *DISK*.

THE SET S(a, ρ) OF POINTS OF E SUCH THAT $d(a, x) = \rho$ IS CALLED THE *SPHERE* WITH CENTER a AND RADIUS $\rho \geqslant 0$.

WHEN E IS THE EUCLIDEAN PLANE $\mathbf{R}^2$ THE TERM SPHERE IS OFTEN REPLACED BY THE TERM *CIRCLE* OR CIRCUMFERENCE.

EXAMPLE. In $\mathbf{R}^n$ with the metric $d'(x, y) = \sup | x_i - y_i |$, the ball B$(a, \rho)$ is a cube with sides parallel to the axes.

It is immediate that every union of open balls with center a is again an open ball with center a; similarly every intersection of closed balls with center a is a closed ball with the same center.

It should be mentioned here that the balls and spheres of a space E do not in general have any of the geometric properties of the balls and spheres of $\mathbf{R}^n$. One can convince oneself of this by taking for E an arbitrary metric subspace of $\mathbf{R}^n$.

Diameter. Distance between two sets

15.6. Definition. THE *DIAMETER* OF A SUBSET A OF A METRIC SPACE E IS THE SUPREMUM δ(A) OF THE DISTANCES $d(x, y)$, WHERE $x, y \in$ A.

A SET A IS SAID TO BE *BOUNDED* WHEN ITS DIAMETER IS FINITE.

FOR EVERY MAPPING f OF A SET X INTO A METRIC SPACE E AND EVERY SUBSET Y OF X, THE DIAMETER OF $f(Y)$ IS CALLED THE *OSCILLATION* OF f ON Y.

EXAMPLE 1. The diameter of a plane triangle is equal to the length of its longest side.

EXAMPLE 2. The diameter of a ball of $\mathbf{R}^n$ of radius ρ is equal to 2ρ; however, in every metric space of diameter ρ_0, the diameter of every ball of radius $\rho > \rho_0$ is equal to ρ_0.

It is immediate that the bounded subsets of a metric space are simply the subsets of balls with finite radii, that the union of two bounded sets is bounded, and that

$$(A \cap B \neq \emptyset) \Rightarrow (\delta(A \cup B) \leqslant \delta(A) + \delta(B)).$$

15.7. Definition. LET A AND B BE SUBSETS OF A METRIC SPACE E WITH THE METRIC d. THE *DISTANCE* BETWEEN A AND B IS DEFINED AS THE INFIMUM $d(A, B)$ OF THE DISTANCES $d(x, y)$, WHERE $x \in A$ AND $y \in B$.

IN PARTICULAR, FOR EVERY $x \in E$ THE DISTANCE FROM x TO B IS DEFINED AS THE NUMBER $d(x, B) = d(\{x\}, B) = \inf_{y \in B} d(x, y)$.

EXAMPLE 1. In the Euclidean plane the distance from a point to a line D is equal to the distance from the point to its projection on D.

EXAMPLE 2. In $\mathbf{R}$, the distance between $\mathbf{Q}$ and $\complement \mathbf{Q}$ is zero.

EXAMPLE 3. The distance from a branch of a hyperbola to one of its asymptotes is zero.

Z Despite its name, $d(A, B)$ is not a metric, nor even an ecart, on the set of subsets of E, for the triangle inequality is not satisfied. For example, if

$$A = [0, 1], \qquad B = [1, 2], \qquad C = [2, 3],$$

then

$$d(A, B) = d(B, C) = 0,$$

while

$$d(A, C) = 1.$$

16. TOPOLOGY OF A METRIC SPACE

Among all the topologies which can be defined on a set E having a metric space structure, there is one which is directly related to the metric, and which is called the topology of the metric space E.

16.1. Definition. A SUBSET A OF A METRIC SPACE E IS SAID TO BE OPEN IF IT IS EMPTY OR IF FOR EVERY $x \in$ A THERE EXISTS AN OPEN BALL WITH CENTER x AND NONZERO RADIUS CONTAINED IN A.

It is immediate that the collection of open sets in E satisfies axioms O_1, O_2, O_3 of a topological space; the topology on E defined by these open sets is called the topology of the metric space E.

Every open ball $B(a, \rho)$ is an open set. This is evident if $\rho = 0$. If $\rho \neq 0$, let $x \in B(a, \rho)$; the open ball $B(x, \rho - d(a, x))$ is contained in $B(a, \rho)$, as

$$d(x, y) < \rho - d(a, x) \quad \text{implies} \quad d(a, y) \leqslant d(a, x) + d(x, y) < \rho.$$

It follows that every union of open balls is an open set.

Conversely, the definition of an open set implies that every open set is a union of open balls. The open sets in E and the unions of open balls are thus identical.

16.2. Proposition. *The topology of every metric space is separated.*

Indeed, if x and y are distinct points of E, the open balls $B(x, \rho)$ and $B(y, \rho)$, where $\rho \leqslant d(x, y)/2$, are disjoint neighborhoods of x and y.

Corollary. *A sequence of points of a metric space* E (*or, more generally, a filter base on* E) *can have at most one limit point.*

Note that the topology on a set E which can be associated with an ecart d on E by a definition analogous to Definition 16.1 is also separated if the condition

$$(d(x, y) = 0) \Rightarrow (x = y)$$

is satisfied.

More generally, for every ecart d, two distinct points x, y such that $d(x, y) \neq 0$ can be separated; on the other hand, if $d(x, y) = 0$, every neighborhood of x contains y, and conversely.

16.3. Proposition. *Every point of a metric space has a countable neighborhood base.*

More precisely, for every sequence (ρ_n) of numbers > 0 tending to 0, the balls $B(a, \rho_n)$, either open or closed, form a neighborhood base of a. Indeed, every open set containing a contains a ball $B(a, \rho)$ where $\rho > 0$, and therefore also contains some ball $B(a, \rho_n)$; on the other hand, every $B(a, \rho_n)$ is a neighborhood of a.

Here are several consequences of this property:

16.4. Proposition. *Let* E *be a metric space and* A *a subset of* E. *Then*

$(a \in \bar{A}) \Leftrightarrow$ (*there exists a sequence* (x_n) *of points of* A *which converges to* a).

Indeed, if $a \in \bar{A}$, the set $A \cap B(a, 1/n)$ is nonempty. Let x_n be one of its points. The sequence (x_n) clearly converges to a.

The converse is true in every topological space.

16.5. Proposition. *Let* E *be a metric space and let* (x_n) *be a sequence of points of* E. *Then*

(*a is an adherent point of the sequence* (x_n)) $\Leftrightarrow$ (*there exists a subsequence* (x_{n_i}) *which converges to* a).

The proof is completely analogous to the preceding proof.

16.6. Proposition. *Let* E *be a metric space,* A *a subset of* E, *and* f *a mapping of* A *into a topological space* F.

For every $a \in \bar{A}$ *and* $b \in F$, *the following properties are equivalent*:

1. *For every sequence* (x_n) *of points of* A *which tends to* a, $\lim f(x_n) = b$.
2. $\lim_{x \to a} f(x) = b$.

Proof. We shall show that (1) $\Rightarrow$ (2).

Indeed, if $f(x)$ does not converge to b as x tends to a, there exists a neighborhood V of b such that, for every neighborhood v of a in A, we have $f(v) \not\subset V$; in particular, there exists a point $x_n \in A \cap B(a, 1/n)$ such that $f(x_n) \notin V$; the sequence (x_n) converges to a while $f(x_n)$ does not converge to b. This is excluded by the hypothesis.

The converse (2) $\Rightarrow$ (1) is true in every topological space.

Corollary 1. *To say that a mapping* f *of a metric space* E *into a topological space* F *is continuous at the point* a *is equivalent to saying that for every sequence* (x_n) *of points of* E *which converges to* a, *the sequence* $(f(x_n))$ *converges to* $f(a)$.

This is a special case of the preceding proposition.

Corollary 2. *To say that a mapping* f *of a metric space* E *into a topological space* F *is continuous in* E *is equivalent to saying that the restriction of* f *to every compact set in* E *is continuous.*

In one direction, this is evident. Conversely, suppose that the restriction of f to every compact set is continuous. For every $a \in E$ and every sequence (x_n) which tends to a, the set $\{a, x_1, x_2, \ldots\}$ is compact; therefore $f(x_n)$ tends to $f(a)$. By the preceding corollary, f is continuous.

Relation between the metric and the topology

We have associated a topology with every metric on a set. It is sometimes convenient to use this metric in studying the topological properties of E; for example, the fact that for every point x of E the balls with center at x constitute a neighborhood base of x implies the following equivalences:

For every sequence (x_n) of points of E,

$$(\lim x_n = a) \Leftrightarrow (\lim d(a, x_n) = 0).$$

For every mapping f of E into a metric space F, the continuity of f at the point a is equivalent to the following condition:

For every $\epsilon > 0$ there exists $\eta > 0$ such that $(d(a, x) < \eta) \Rightarrow (d(f(a), f(x)) < \epsilon)$.

Nevertheless an overusage of the metric frequently complicates proofs and hides the actual causes of phenomena.

This comes in part from the fact that the same topology on a set E may be associated with many different metrics; these metrics do not therefore constitute an intrinsic tool for the study of this topology.

Another reason which tends to limit the use of metrics is that certain topological spaces which are very useful in the study of the most classical kinds of questions are not *metrizable*, that is, cannot be defined starting with a metric.

We shall make precise the connections between the metric and the topology by several results.

16.7. Proposition. *Let d and d' be metrics on a set* E, *and let φ and φ' be increasing mappings of $\mathbf{R}_+$ into $\bar{\mathbf{R}}_+$, continuous at* 0 *and such that* $\varphi(0) = \varphi'(0) = 0$.

If for every $x, y \in$ E we have

$$d'(x, y) \leqslant \varphi(d(x, y)) \quad \textit{and} \quad d(x, y) \leqslant \varphi'(d'(x, y)),$$

then the topologies associated with d and d' are identical.

Indeed, it follows from the hypothesis that the mapping $x \rightarrow x$ of E, with the topology associated with d, onto E with the topology associated with d', is bicontinuous; hence it is a homeomorphism.

Example 1. For every metric d on E, the topologies on E associated with the metrics d, $d/(1 + d)$, $\inf(1, d)$, are identical.

Example 2. Let E be a product of metric spaces E_i with the metrics d_i. The topologies on E associated with the metrics

$$\left(\sum d_i^2\right)^{1/2}, \qquad \sup d_i, \qquad \sum d_i,$$

are identical; indeed, we know that the ratio of any two of these metrics is bounded.

Z It is false that when the topologies on E associated with two metrics are identical, these metrics satisfy relations of the kind used in Proposition 16.7. For example, the metrics $|x - y|$ and $|1/x - 1/y|$ on $\mathbf{R}_+^*$ give $\mathbf{R}_+^*$ the usual topology, although $|1/x - 1/y|$ does not tend to 0 as $|x - y|$ tends to 0.

16.8. Proposition. 1. *Every metric subspace of a metric space* E *has for its topology the topology induced by that of* E.

2. *Every metric space* E *which is the product of a finite number of metric spaces* E_i *has as its topology the product of the topologies of the* E_i.

PROOF. 1. This is immediate since, for every $A \subset E$, the open balls of A are the traces on A of the open balls of E.

2. We already know that the three metrics $(\Sigma d_i^2)^{1/2}$, $\sup d_i$, Σd_i define the same topology on E. Let us for example use the second of these metrics: For every point $a = (a_i)$ of E, the open ball $B(a, \rho)$ is the product of the open balls $B_i(a_i, \rho) \subset E_i$.

Thus these balls constitute a neighborhood base of a, both for the topology associated with the metric of E, and for the product of the topologies of the E_i.

EXAMPLE. In $\mathbf{R}^n$ the product topology used up to now is identical with the topology associated with the Euclidean metric.

17. UNIFORM CONTINUITY

The possibility of speaking of the smallness of a set will enable us, as in the case of topological groups, to speak of the uniform continuity of a function.

17.1. Definition. A MAPPING f OF A METRIC SPACE E INTO A METRIC SPACE F IS SAID TO BE *UNIFORMLY CONTINUOUS* IF, FOR EVERY $\epsilon > 0$, THESE EXISTS AN $\eta > 0$ SUCH THAT

$$(d_E(x, y) \leqslant \eta) \Rightarrow (d_F(f(x), f(y)) \leqslant \epsilon).$$

An equivalent definition, perhaps more suggestive, is the following:

17.2. Definition. f IS SAID TO BE UNIFORMLY CONTINUOUS IF, FOR EVERY $\epsilon > 0$, THERE EXISTS AN $\eta > 0$ SUCH THAT FOR EVERY $X \subset E$,

$$(\delta(X) \leqslant \eta) \Rightarrow (\delta(f(X)) \leqslant \epsilon),$$

WHERE δ DENOTES THE DIAMETER, IN E AND IN F.

REMARK 1. This definition points up the difference between uniform continuity and continuity at every point.

The continuity of f is expressed by

$$(\forall x \in \mathrm{E})(\forall \epsilon > 0)(\exists \eta > 0) : (d_{\mathrm{E}}(x, y) \leqslant \eta) \Rightarrow (d_{\mathrm{F}}(f(x), f(y)) \leqslant \epsilon).$$

The uniform continuity of f is expressed by

$$(\forall \epsilon > 0)(\exists \eta > 0) : (d_{\mathrm{E}}(x, y) \leqslant \eta) \Rightarrow (d_{\mathrm{F}}(f(x), f(y)) \leqslant \epsilon).$$

In the first case, η depends on the choice of x and ϵ; in the second case, η depends only on the choice of ϵ.

This remark clearly shows that if f is uniformly continuous, it is continuous. But the converse is false. For example, the mapping $x \to x^2$ of $\mathbf{R}$ into $\mathbf{R}$ is not uniformly continuous since, for every $\eta > 0$, the oscillation $\omega(a)$ of f on the interval $[a, a + \eta]$ is $\geqslant |\, 2a\eta + \eta^2 \,|$, and $\omega(a)$ is not bounded independently of a.

Similarly the mapping $f : x \to \sin 1/x$ of $(0, 1)$ into $[-1, 1]$ is not uniformly continuous, although it is bounded; indeed, the oscillation of f on each of the intervals $(0, \eta]$ is equal to 2.

We shall see, on the other hand, that when E is compact, the converse is true.

REMARK 2. One could easily formulate a notion of uniform continuity for a mapping of a topological group into a metric space, or vice versa. This would be another example of a general notion of uniform continuity which can be formulated in terms of a *uniform structure* on a set. Metric spaces and topological groups are two important examples of such uniform structures.

Modulus of continuity. Mappings of Lipschitz class

Let φ be an increasing mapping of $\bar{\mathbf{R}}_+$ into $\bar{\mathbf{R}}_+$, continuous at the point 0, and such that $\varphi(0) = 0$; and let f be a mapping of a metric space E into a metric space F.

Then f is said to admit φ as a *modulus of continuity* if, for all $x, y \in \mathrm{E}$,

$$d(f(x), f(y)) \leqslant \varphi(d(x, y)).$$

Since $\lim_{x \to 0} \varphi(x) = 0$, f is then uniformly continuous.

Conversely, if f is uniformly continuous, let us put for every $u \geqslant 0$

$$\varphi(u) = \sup\,(d(f(x), f(y)))$$

over all $x, y \in \mathrm{E}$ such that $d(x, y) \leqslant u$.

It is immediate that φ is a modulus of continuity for f.

Thus the notion of uniform continuity can be expressed in terms of moduli of continuity.

If f maps E into F, if g maps F into G, and if f and g have φ and γ for moduli of continuity, then the mapping $g \circ f$ has $\gamma \circ \varphi$ for a modulus of continuity.

The most-used moduli of continuity in analysis are the functions φ of the type $u \to ku^\alpha$ ($\alpha > 0$); the case $\alpha = 1$ yields the mappings of Lipschitz class. More explicitly:

17.3. Definition. LET k BE A NUMBER >0. A MAPPING f IS SAID TO BE OF *LIPSCHITZ CLASS* WITH RATIO k IF, FOR ALL $x, y \in$ E,

$$d(f(x), f(y)) \leqslant k\, d(x, y).$$

WHEN $k < 1$, f IS SAID TO BE A *CONTRACTIVE* MAPPING.

EXAMPLE 1. Let f be a numerical function defined and differentiable on an interval of **R**. If f is of Lipschitz class with ratio k, the relation $|\Delta f/\Delta x| \leqslant k$ shows that $|f'| \leqslant k$; conversely, if $|f'| \leqslant k$, the mean value theorem shows that

$$|\Delta| f = |f(x) - f(y)| = |(x - y)f'(z)| \leqslant k|\Delta x|,$$

so that f is of Lipschitz class with ratio k.

EXAMPLE 2. For every product E of metric spaces E_i, the projection f_i of E on the space E_i is of Lipschitz class with ratio 1 (for any one of the three usual metrics on E).

EXAMPLE 3. Let E be a metric space, and d its metric. The mapping $(x, y) \to d(x, y)$ of $E \times E$ into **R** is of Lipschitz class with ratio 1 (therefore also continuous) when $E \times E$ is taken with the metric d'' defined by

$$d''((x, y), (x', y')) = d(x, x') + d(y, y').$$

Indeed, the triangle inequality gives

$$d(x', y') \leqslant d(x', x) + d(x, y) + d(y, y')$$

and a similar relation for $d(x, y)$, whence

$$|d(x', y') - d(x, y)| \leqslant d(x, x') + d(y, y') = d''((x, y), (x', y')).$$

For the other two usual metrics, the ratio 1 has to be replaced by $\sqrt{2}$ and 2, respectively.

Similarly, for every $a \in E$, the mapping $x \to d(a, x)$ is of Lipschitz class with ratio 1.

It follows from the continuity of the metric that for every $A \subset E$ we have $\delta(A) = \delta(\bar{A})$. From it we also deduce the following proposition:

17.4. Proposition. *For any compact sets* $A, B \subset E$, *there exists* $a \in A$ *and* $b \in B$ *such that* $d(a, b) = d(A, B)$.

PROOF. The function d is continuous on the compact set $A \times B$, and is therefore bounded and attains its infimum at some point (a, b), which is the assertion of the proposition.

Similarly, there exists a point (a', b') at which d attains its supremum; in particular, if $A = B$, $d(a', b')$ is equal to the diameter of A.

Isomorphism of uniform structures. Equivalent metrics

We have defined a notion of isomorphism for topological spaces (homeomorphisms), and then for metric spaces (isometries). We shall see that there exists an intermediate notion which implies a certain preservation of the notion of smallness.

17.5. Definition. LET E AND E′ BE METRIC SPACES, AND LET f BE A BIJECTION OF E TO E′. IF f AND f^{-1} ARE UNIFORMLY CONTINUOUS, f IS SAID TO BE AN ISOMORPHISM OF THE UNIFORM STRUCTURES OF E AND E′.

If d, d' denote the metrics on E, E′, this definition can at once be stated as follows: The bijection $f : x \to x'$ is an isomorphism if there exist two moduli of continuity φ and φ' such that for all $x, y \in E$,

$$d'(x', y') \leqslant \varphi(d(x, y)) \quad \text{and} \quad d(x, y) \leqslant \varphi'(d'(x', y')),$$

or more briefly, if $d(x, y)$ and $d'(x', y')$ tend to 0 simultaneously.

It is evident that the product of two isomorphisms is an isomorphism.

EXAMPLE 1. Let f be a homeomorphism between a *compact metric* space E and a compact metric space E′. Since f and f^{-1} are continuous and E, E′ are compact, these mappings are uniformly continuous (See Section 18). Therefore f is an isomorphism of the uniform structures of E and E′.

EXAMPLE 2. On the other hand, the mapping $x \to x/(1 + |x|)$ of $\mathbf{R}$ onto $(-1, 1)$ is uniformly continuous and is a homeomorphism. But f^{-1} is not uniformly continuous; therefore f is not an isomorphism.

17.6. Definition. TWO METRICS d, d' ON A SET E ARE SAID TO BE *EQUIVALENT* IF THE IDENTITY MAPPING $x \to x$ OF E WITH THE METRIC d ONTO E WITH THE METRIC d', AND ALSO ITS INVERSE, IS UNIFORMLY CONTINUOUS.

This is evidently a special case of isomorphism; the condition can be expressed briefly by the assertion that $d(x, y)$ and $d'(x, y)$ tend to 0 simultaneously.

EXAMPLE. The three usual metrics on a product of metric spaces are equivalent.

It is evident that the notions of continuity and uniform continuity on a metric space do not change when the metric is replaced by an equivalent metric.

18. COMPACT METRIC SPACES

We shall base our study of such spaces on the following fundamental lemma.

18.1. Lemma. *Let* E *be a metric space, and let* K *be a closed set in* E *such that every infinite sequence of points of* K *contains a convergent subsequence.*

For every family $(\omega_i)_{i \in I}$ *of open sets in* E *covering* K, *there exists a number* $\rho > 0$ *such that, for every* $x \in K$, *the open ball* $B(x, \rho)$ *is contained in at least one* ω_i.

PROOF. It is, in effect, a question of showing that not only do the ω_i cover K, they cover it ρ-uniformly, in a sense made clear by the assertion of the lemma.

Let us suppose that such a ρ does not exist; then for every integer n there exists a point x_n of K such that $B(x_n, 1/n)$ is not contained in any ω_i. The sequence (x_n) contains a convergent subsequence (x_{n_i}); let a be the limit of this subsequence.

Since K is closed, a belongs to K. Therefore there exists an open set ω of the family $(\omega_i)_{i \in I}$ which contains a; let $B(a, \lambda)$ be an open ball with center a which is contained in ω.

The ball $B(x_{n_i}, (\lambda - \epsilon_{n_i}))$, where $\epsilon_{n_i} = d(a, x_{n_i})$, is contained in $B(a, \lambda)$ by the triangle inequality, and therefore *a fortiori* in ω. Since ϵ_{n_i} tends to 0 as $n_i \to \infty$, the ball $B(x_{n_i}, 1/n_i)$ is contained in ω for n_i sufficiently large, contrary to hypothesis.

This contradiction proves the lemma.

Z It is false that the property stated in the lemma is true for every subset K of a metric space. For example if $E = \mathbf{R}$ and $K = (0, 1)$, the family (ω_i) consisting of the single open set $(0, 1)$ does not have the stated property. A similar example is obtained with $E = K = (0, 1)$; moreover, here K is closed in E.

This lemma has important consequences, in particular Theorems 18.2 and 18.4 which follow.

18.2. Theorem. *For every metric space* E, *the following four properties are equivalent*:

1. E *is compact.*
2. *Every infinite sequence of points of* E *has at least one adherent point.*
3. *Every infinite sequence of points of* E *has a convergent subsequence.*
4. *Every infinite subset of* E *has at least one accumulation point.*

PROOF.

(1) $\Rightarrow$ (2) by Proposition 11.5.

(2) $\Rightarrow$ (3) by Proposition 16.5.

(3) $\Rightarrow$ (4). Indeed, if A is an infinite subset of E, there exists an infinite sequence (x_n) of *distinct* points of A; this sequence contains a convergent subsequence; if x is its limit, clearly x is an accumulation point of A.

(4) $\Rightarrow$ (1). Indeed, suppose that every infinite subset of E has at least one accumulation point.

Let $(\omega_i)_{i \in \mathrm{I}}$ be an open covering of E; we shall find a finite subcovering of E, which will prove the compactness of E.

By the preceding lemma, there exists a number $\rho > 0$ such that every open ball $\mathrm{B}(x, \rho)$ is contained in some ω_i.

Let x_1 be a point of E; if $\mathrm{B}(x_1, \rho)$ does not cover E, there exists a point x_2 such that $d(x_1, x_2) \geqslant \rho$. Suppose, proceeding inductively, that the points $x_1, x_2, \ldots, x_p$ have mutual distances $\geqslant \rho$. If the union of the $\mathrm{B}(x_n, \rho)$ $(n = 1, 2, \ldots, p)$ does not cover E, there exists a point x_{p+1} such that the distances from x_{p+1} to the x_n $(n \leqslant p)$ are $\geqslant \rho$.

The sequence of x_n cannot be infinite, for otherwise the x_n would form an infinite set of points whose mutual distances are $\geqslant \rho$, which excludes the possibility of having an accumulation point, contrary to hypothesis.

Thus there exists an integer p such that the family of balls $\mathrm{B}(x_n, \rho)$ $(n \leqslant p)$ covers E; each of them is contained in some ω_i; these ω_i form the desired finite subcovering.

Corollary. *Let* X *be a subset of a metric space* E. *To say that* X *is relatively compact in* E *is equivalent to saying that every infinite sequence of points of* X *has a subsequence which converges to a point of* E.

PROOF. 1. Indeed, if $\bar{\mathrm{X}}$ is compact, every infinite sequence of points of X is an infinite sequence of points of $\bar{\mathrm{X}}$, and therefore contains a subsequence which converges in $\bar{\mathrm{X}}$.

2. Conversely, we shall show that if this holds, then $\bar{\mathrm{X}}$ is compact.

Let (x_n) be a sequence of points of $\bar{X}$; for every n there exists an $x'_n \in X$ such that $d(x_n, x'_n) < 1/n$. The sequence (x'_n) contains a subsequence (x'_{n_i}) which converges to a point a of E; evidently the sequence (x_{n_i}) also converges to a, which is thus adherent to $\bar{X}$, and therefore in $\bar{X}$.

Thus $\bar{X}$ is compact by Theorem 18.2.

One can easily state equivalent criteria in terms of accumulation points, or adherent points of a sequence.

18.3. Proposition. *For every compact metric space* E, *there exists a countable family of open balls of* E *such that every open set in* E *is the union of a subfamily of these balls.*

Proof. Indeed, for every n the open balls $B(x, 1/n)$ cover E; therefore there exists a finite family $\mathscr{F}_n$ of these balls which covers E. The union of these finite families is the desired family. Indeed, for every open set ω in E, and for every $x \in \omega$, there exist elements B of this family which contain x and have arbitrarily small diameter; one of them, B_x, is therefore contained in ω, and ω is evidently the union of these B_x.

Corollary. *Every compact metric space contains a countable everywhere dense subset.*

Indeed, if (B_i) is the countable family of balls just constructed, and if x_i denotes an arbitrary point of B_i, the set consisting of these x_i has the required property.

18.4. Theorem. *Every continuous mapping f of a compact metric space* E *into another metric space* F *is uniformly continuous.*

Because of the importance of this theorem, we shall give two proofs of it.

Proof 1. Let $\epsilon > 0$. Since f is continuous, with every $x \in E$ we can associate an open neighborhood ω_x of x such that the oscillation of f on ω_x is $\leqslant \epsilon$.

Let ρ be the number associated with the family of ω_x by Lemma 18.1. Every ball $B(y, \rho)$ of E is contained in at least one ω_x; therefore the oscillation of f on this ball is $\leqslant \epsilon$. This proves the uniform continuity of f.

Proof 2. If f is not uniformly continuous, there exists a number $\epsilon > 0$ such that, for every integer $n > 0$, there exist two points $x_n, y_n \in E$ such that

$$d(x_n, y_n) \leqslant 1/n \qquad \text{and} \qquad d(f(x_n), f(y_n)) \geqslant \epsilon.$$

The sequence (x_n) has a subsequence (x_{n_i}) which converges to some point a of E; since $\lim d(x_n, y_n) = 0$, the sequence (y_{n_i}) also converges to a. Therefore every neighborhood V of a contains pairs (x_n, y_n); the oscillation of f on every V is thus at least ϵ, hence f is not continuous at a. But this contradicts the hypothesis.

One can extend Theorem 18.4 and obtain a result which is slightly more general and very convenient:

18.5. Theorem. *Let* E *be a metric space,* K *a compact set in* E, *and let f be a mapping of* E *into a metric space* F. *If f is continuous at every point of* K, *then f is uniformly continuous about* K *in the sense that for every* $\epsilon > 0$ *there exists a number* $\rho > 0$ *such that for every* $x \in$ K *the oscillation of f on the ball* $B(x, \rho)$ *is* $\leqslant \epsilon$.

The proof can be carried out by adapting one or the other of the proofs of Theorem 18.4, for example by applying Lemma 18.1 to the family of open sets ω_x defined as follows: For every $x \in$ K, ω_x is an open neighborhood of x in E on which the oscillation of f is $\leqslant \epsilon$.

Note that the statement of the theorem assumes nothing about the continuity of f outside K.

19. CONNECTED METRIC SPACES

We shall see that for metric spaces one can make the intuitive notion of connectedness more precise.

19.1. Definition. A METRIC SPACE E IS SAID TO BE *WELL-LINKED* IF FOR EVERY PAIR (a, b) OF POINTS OF E AND FOR EVERY $\epsilon > 0$, THERE EXISTS A FINITE SEQUENCE $a_1, \ldots, a_n$ OF POINTS OF E, WITH $a_1 = a$ AND $a_n = b$, SUCH THAT $d(a_i, a_{i+1}) \leqslant \epsilon$ FOR EVERY $i < n$; IN OTHER WORDS, a AND b CAN BE JOINED BY A CHAIN OF STEPS AT MOST EQUAL TO ϵ.

19.2. Proposition. *Every connected metric space* E *is well-linked.*

Proof. Let $a \in$ E and let $E(a, \epsilon)$ be the set of points x of E which can be joined to a by a chain of steps at most equal to ϵ. This set is not empty, as it contains a; it is open, since if $x \in E(a, \epsilon)$, the same is true for every y such that $d(x, y) < \epsilon$; it is closed since if x is an accumulation point of $E(a, \epsilon)$, there exist points y of $E(a, \epsilon)$ such that $d(x, y) < \epsilon$.

Since E is connected, we have $E(a, \epsilon) = E$; in other words, every point b of E can be joined to a by a chain of steps at most equal to ϵ. Thus E is well-linked.

Z It is false that, conversely, every well-linked metric space is connected. For example, the set **Q** of rationals is well-linked but not connected. However, this converse holds if E is compact:

19.3. Proposition. *For a compact metric space, the properties of being connected and being well-linked are equivalent.*

PROOF. We have only to show one half of this equivalence. Thus, let E be a compact metric space. If it is not connected, there exists a partition of E into two nonempty closed sets E_1, E_2. Since E_1 and E_2 are compact, the distance δ between them is not zero. A point of E_1 cannot be joined to a point of E_2 by a chain of steps less than $\delta/2$, for if $(a_1, a_2, ..., a_n)$ is such a chain, let i be the smallest index such that $a_i \in E_2$; then $a_{i-1} \in E_1$ and $d(a_{i-1}, a_i) < \delta/2$, in contradiction with

$$d(E_1, E_2) = \delta.$$

In other words, if E is well-linked, it is also connected.

Corollary. *Every compact interval of* **R** *is connected.*

Indeed, every interval $[a, b]$ is compact and is clearly well-linked (use the points $a + n\epsilon$).

More generally, let E be an arbitrary interval of **R**, and $x_0 \in E$; for every $x \in E$, we have $[x_0, x] \subset E$; therefore E is the union of the compact intervals $[x, x_0]$, hence connected.

Conversely, we have seen during the study of connected topological spaces that a subset of **R** which is not an interval is not connected. To sum up:

19.4. Proposition. *The only connected subsets of* **R** *are the intervals (open, semi-open, or closed).*

Corollary. *For every continuous numerical function f on a connected topological space* E, *the set f(E) is an interval of* **R**.

Therefore if f takes on positive and negative values on E, it vanishes at one point at least of E.

19.5. APPLICATION. Let f be a continuous and strictly increasing numerical function defined on an interval E of **R**.

Since f is an isomorphism for the order relations of E and f(E), it is also a homeomorphism for the topologies associated with these orders. But f being continuous, f(E) is an interval. Thus the order topology on

$f(\mathrm{E})$ is identical with the topology induced by that of $\mathbf{R}$; the same is true for E. Therefore f is a homeomorphism of the intervals E, $f(\mathrm{E})$; hence the inverse function f^{-1} is also continuous and strictly increasing.

Evidently an analogous result holds if f is strictly decreasing.

20. CAUCHY SEQUENCES AND COMPLETE SPACES

During the study of $\mathbf{R}$, we defined the notion of a Cauchy sequence, and showed that every Cauchy sequence in $\mathbf{R}$ is convergent.

The notion of a Cauchy sequence is not a topological notion since, for example, the sequence $(1/n)$, which is a Cauchy sequence in $(0, \infty)$, is transformed by the homeomorphism $x \to x^{-1}$ of $(0, \infty)$ with itself into the sequence (n), which is not a Cauchy sequence.

Therefore we cannot hope to define the notion of a Cauchy sequence in general topological spaces.

However, we shall see that this can easily be done in metric spaces.

20.1. Definition. Let E be a metric space, and let (x_n) be an infinite sequence of points of E. We say that (x_n) is a *Cauchy sequence* if $d(x_p, x_q)$ tends to 0 as p and q tend to $+\infty$; in other words, if for every $\epsilon > 0$ there exists an integer n such that $d(x_p, x_q) \leqslant \epsilon$ for all $p, q \geqslant n$.

Briefly:

$$(\forall \epsilon > 0)(\exists n, n \in \mathbf{N})(\forall p, q \geqslant n) : (d(x_p, x_q) \leqslant \epsilon).$$

An equivalent and perhaps more suggestive definition is obtained by using the set A_n of points x_p such that $p \geqslant n$:

20.2. Definition. (x_n) is a Cauchy sequence if $\lim \delta(\mathrm{A}_n) = 0$ (where $\delta(\mathrm{A}_n)$ denotes the diameter of A_n).

Every subsequence of a Cauchy sequence is a Cauchy sequence. If a subsequence of a Cauchy sequence converges to a point x, the given sequence also converges to x.

Example 1. For every metric space E, every sequence (x_n) which converges to a point of E is a Cauchy sequence.

Example 2. In the metric subspace $\mathrm{E} = (0, \infty)$ of $\mathbf{R}$, the sequence $(1/n)$ is a nonconvergent Cauchy sequence.

EXAMPLE 3. More generally, let E be a metric subspace of a metric space F; for every $x \in \bar{E}$ there exists a sequence (x_n) of points of E which converges to x; this sequence is a Cauchy sequence in E; it converges in E only if $x \in E$.

EXAMPLE 4. Let E be the collection of continuous numerical functions on [0, 1]; for every $f, g \in E$ we put

$$d(f, g) = \int_0^1 |f(x) - g(x)| \, dx.$$

One can verify that d is a metric on E. If we put

$$f_n(x) = \inf(n, x^{-1/2}),$$

then (f_n) is a Cauchy sequence in E.

20.3. Proposition. *Let* E *and* F *be metric spaces, and let f be a uniformly continuous mapping of* E *into* F; *then the image under f of every Cauchy sequence (x_n) in* E *is a Cauchy sequence in* F.

Indeed, if A_n and B_n denote, respectively, the set of x_p and of $f(x_p)$ such that $p \geqslant n$, then $f(A_n) = B_n$; but $\lim \delta(A_n) = 0$; thus since f is uniformly continuous, $\lim \delta(B_n) = 0$, that is, $(f(x_n))$ is a Cauchy sequence.

Z If f were only continuous, it would still transform every *convergent* Cauchy sequence into a convergent sequence, and thus into a Cauchy sequence; but Example 2 above showed that f could transform certain nonconvergent Cauchy sequences into sequences which are not Cauchy.

Corollary. *If f is an isomorphism of the uniform structures of* E *and* F, *then f interchanges the Cauchy sequences of* E *and* F.

In particular, if d and d' are equivalent metrics on a set E, the same sequences are Cauchy sequences for d and d'. The notion of a Cauchy sequence in E is thus related not to the metric structure of E, but to its uniform structure.

20.4. Proposition. *Let* E *be a finite product of metric spaces* E_i . *Let* (x_n) *be a sequence of points of* E *and let* $(x_{n,i})$ *be its projection on* E_i . *Then*

$((x_n)$ *is a Cauchy sequence in* E$) \Leftrightarrow (\forall i,\ (x_{n,i})$ *is a Cauchy sequence in* $E_i)$.

PROOF. Let d_i be the metric on E_i ; since the three usual metrics on E are equivalent, the corollary above shows that we can take any one of these metrics on E; we shall take

$$d''(x, y) = \sum d_i(x_i, y_i).$$

The projection $x \to x_i$ of E on E_i is of Lipschitz class with ratio 1, therefore uniformly continuous; hence if (x_n) is Cauchy, so is $(x_{n,i})$.

Conversely, if each $(x_{n,i})$ is a Cauchy sequence, the relation

$$d''(x_p, x_q) = \sum_i d_i(x_{p,i}, x_{q,i})$$

shows that (x_n) is a Cauchy sequence.

Complete spaces

Every Cauchy sequence in **R** converges; on the other hand, several of the examples above show that there exist metric spaces in which certain Cauchy sequences do not converge. We are thus led to study the spaces in which every Cauchy sequence converges.

20.5. Definition. A METRIC SPACE E IS SAID TO BE A *COMPLETE SPACE* IF EVERY CAUCHY SEQUENCE OF POINTS OF E IS CONVERGENT IN E.

EXAMPLES. **R**, with the usual metric, is complete. On the other hand, (0, 1) and **Q** are not complete.

20.6. Proposition. *Let* E *be a complete metric space. For every decreasing sequence* (X_n) *of nonempty closed sets in* E *such that* $\lim \delta(X_n) = 0$, *the intersection* X *of the* X_n *contains exactly one point.*

PROOF. We choose an arbitrary point x_n in each X_n. If $p \geqslant n$, then $X_p \subset X_n$, hence $x_p \in X_n$.

Thus the set A_n of x_p such that $p \geqslant n$ is contained in X_n; it follows that $\lim \delta(A_n) = 0$, and so (x_n) is a Cauchy sequence. Since E is complete, (x_n) converges to a point x.

But for every fixed n, x is the limit of the points x_{n+p}, which belong to X_n; since X_n is closed, we have $x \in X_n$ for every n, hence $x \in X$.

Finally, since $\delta(X) \leqslant \delta(X_n)$ for every n, $\delta(X) = 0$; thus X can contain only one point.

Z One might expect that if $\lim \delta(X_n) > 0$, then not only is the intersection of the X_n not empty, but moreover it contains more than one point. This is not so, as the following example shows:

In the complete space **R**, the intervals $[n, \infty)$ form a decreasing sequence of closed sets with infinite diameter, and yet their intersection is empty.

In other words, the fact that a space is complete only manifests itself on the small sets. We shall confirm this fact by a result which completes Proposition 20.6.

20.7. Definition. LET E BE A METRIC SPACE, AND LET $\mathscr{B}$ BE A FILTER BASE ON E. IF FOR EVERY $\epsilon > 0$ THERE EXISTS AN $X \in \mathscr{B}$ SUCH THAT $\delta(X) < \epsilon$, THEN $\mathscr{B}$ IS CALLED A *CAUCHY FILTER BASE*.

For example, for every decreasing sequence (A_n) of nonempty subsets of E, the A_n form a Cauchy filter base if $\lim \delta(A_n) = 0$.

20.8. Proposition. *Every Cauchy filter base on a complete metric space* E *is convergent.*

PROOF. Let $\mathscr{B}$ be a Cauchy filter base on E. For every integer n, there exists an $X_n \in \mathscr{B}$ such that $\delta(X_n) < 1/n$. Let x_n be an arbitrary point of X_n.

For all p, q we have

$$X_p \cap X_q \neq \emptyset; \qquad \text{therefore} \qquad \delta(X_p \cup X_q) < 1/p + 1/q;$$

hence

$$d(x_p, x_q) < 1/p + 1/q.$$

Thus (x_n) is a Cauchy sequence; let x be its limit. For every $\epsilon > 0$, there exists an n such that

$$d(x, x_n) < \epsilon/2 \qquad \text{and} \qquad \delta(X_n) < \epsilon/2.$$

Since $x_n \in X_n$, it follows that $X_n \subset B(x, \epsilon)$. This says that $\mathscr{B}$ converges to x.

EXAMPLE. Let f be a numerical function on a locally compact but noncompact space E. We assume that for each $\epsilon > 0$ there exists a compact set $K \subset E$ such that the oscillation of f on $\complement K$ is $< \epsilon$. Then the sets $f(\complement K)$ form a Cauchy filter base on $\mathbf{R}$; this filter base converges to a number l which is called the limit of $f(x)$ as x tends to infinity.

An important class of complete spaces is given by the following theorem:

20.9. Theorem. *Every compact metric space is complete.*

PROOF. With the notation already used, for every sequence (x_n) of points of E the adherence of this sequence is

$$A = \bigcap \bar{A}_n .$$

If the sequence is a Cauchy sequence, $\lim \delta(A_n) = 0$, hence $\delta(A) = 0$. Moreover, by Proposition 11.5, A is nonempty, and therefore consists of a single point; by the same proposition, the sequence converges to this point.

REMARK. There are other complete spaces besides the compact metric spaces; the real line $\mathbf{R}$ is the classical example. The fact that $\mathbf{R}$ is locally compact might lead one to believe that every locally compact metric space is complete or that every complete space is locally compact; neither of these statements is true. For example, the metric subspace $(0, 1)$ of $\mathbf{R}$ is locally compact but not complete; while the space $\mathscr{C}([0, 1], \mathbf{R})$ which we shall define in Section 22 is a complete but not locally compact space.

Analogy between complete spaces and compact spaces

One could note, in the preceding proofs, a certain analogy between complete spaces and compact spaces. We shall clarify this by a series of theorems analogous to those previously established for compact spaces. On the other hand, we shall also point out several important differences.

20.10. Theorem. *Every closed subset* A *of a complete metric space* E *is a complete metric subspace.*

Indeed, let (x_n) be a Cauchy sequence of points in A. This sequence is also a Cauchy sequence of points in E, and therefore converges to a point x of E; but since the points of the sequence belong to A, which is closed, we also have $x \in \mathrm{A}$. Therefore the sequence converges to a point of A.

20.11. Theorem. *Every complete metric subspace* A *of a metric space* E *is a closed subset of* E.

We shall show that A contains its accumulation points. Let x be such a point; then x is the limit of a sequence (x_n) in A. This sequence being convergent, it is a Cauchy sequence; by hypothesis it converges to a point of A. Therefore $x \in \mathrm{A}$.

Corollary. *In every complete metric space* E, *the closed subsets of* E *and the complete subspaces of* E *are identical.*

20.12. Theorem. *In every metric space* E, *the union of two complete subspaces is complete*; *every intersection of complete subspaces is complete.*

1. Let A and B be complete subspaces of E, and let (x_n) be a Cauchy sequence in $\mathrm{A} \cup \mathrm{B}$; every subsequence of this sequence is a Cauchy sequence. But one or the other at least of A and B contains a subse-

quence of (x_n); since both A and B are complete, this subsequence converges to a point of A or of B; the sequence (x_n) converges to the same point.

2. If the $(A_i)_{i \in I}$ are complete, each of them is closed in E; therefore their intersection is closed in E and *a fortiori* in any one of the A_i, all of which are complete. By Theorem 20.10, this intersection is a complete space.

Z Let E and F be metric spaces, and let f be a continuous surjection of E to F. It is false that if E is complete, then F is necessarily complete, even in the case where f is uniformly continuous and is a homeomorphism.

Indeed, the mapping $x \to x/(1 + |x|)$ of **R** onto $(-1, 1)$ has these properties, and $(-1, 1)$ is not complete although **R** is.

However, if f is an isomorphism of the uniform structures of E and F, these two spaces are simultaneously complete or incomplete; in particular, if d and d' are equivalent metrics on a set E, the associated metric spaces are simultaneously complete or incomplete.

More generally, let f be a homeomorphism between E and F; if f is uniformly continuous and if F is complete, then E is complete.

20.13. Theorem. *Every finite product of complete metric spaces is complete.*

Proof. Let E_i be a finite family of complete metric spaces. For every Cauchy sequence (x_n) in $E = \Pi E_i$, the projection of (x_n) on E_i is a Cauchy sequence (Proposition 20.4); let a_i be its limit. The sequence (x_n) converges in E to $a = (a_i)$ (Proposition 10.3); therefore E is complete.

Example. Since **R** is complete, so is $\mathbf{R}^n$. Every closed set in $\mathbf{R}^n$ is therefore complete.

Extension of uniformly continuous functions

If f denotes a continuous mapping of a metric space E into another metric space F, the restriction of f to every $X \subset E$ is continuous; conversely, given a continuous mapping of X into F, the question arises whether f can be extended to E, that is, whether f is the restriction to X of a continuous mapping of E into F.

We shall examine here only a highly useful special case of this question.

20.14. Theorem. *Let* X *be an everywhere dense subset of a metric space* E, *and let f be a uniformly continuous mapping of* X *into a complete metric* *space* F. *Then there exists a unique continuous mapping* g *of* E *into* F *whose restriction to* X *is* f; *this mapping* g *is uniformly continuous.*

Proof. If g is such an extension, then for every $a \in \mathrm{E}$ we have

$$g(a) = \lim_{x \in \mathrm{X}, x \to a} f(x).$$

This relation shows, on the one hand, that if g exists it is unique, and, on the other hand, gives the possible value of $g(a)$ for every $a \in \mathrm{E}$.

Since f is uniformly continuous, the diameter of $f(\mathrm{B}(a, \rho))$ tends to 0 with ρ; therefore for every $a \in \mathrm{E}$ the sets $f(\mathrm{B}(a, \rho))$ form a Cauchy filter base on F, and since F is complete, this filter base converges (Proposition 20.8) to a point of F which we denote by $g(a)$.

If $a \in \mathrm{X}$, then $f(a) \in f(\mathrm{B}(a, \rho))$, hence $f(a) = g(a)$; thus if we show that g is uniformly continuous, g will be the desired extension of f.

By hypothesis, for every $\epsilon > 0$ there exists an $\eta = \varphi(\epsilon) > 0$ such that $d(f(x), f(y)) \leqslant \epsilon$ for all $x, y \in \mathrm{X}$ satisfying $d(x, y) < \eta$.

Let then $a, b \in \mathrm{E}$ be such that $d(a, b) < \eta$; there exist two sequences (a_n), (b_n) of points of X which converge to a and b, respectively, and such that $d(a_n, b_n) < \eta$ for all n. The relations

$$g(a) = \lim f(a_n); \qquad g(b) = \lim f(b_n); \qquad d(f(a_n), f(b_n)) \leqslant \epsilon$$

imply $d(g(a), g(b)) \leqslant \epsilon$ since the metric d is a continuous function.

Thus g is uniformly continuous; and the proof even shows that if a modulus of continuity φ for f is continuous, it is also a modulus of continuity for its extension g; in particular, if f is of Lipschitz class with ratio k, the same is true of g.

21. IDEA OF THE METHOD OF SUCCESSIVE APPROXIMATIONS

One of the most powerful methods for proving the existence of solutions of an equation, be it numerical, differential, partial differential, or integral, and often also for effectively calculating this solution, is the so-called method of successive approximations, which was systematically used for the first time by Emile Picard. In many cases this method can be reduced to a scheme which we shall make specific by the following theorem:

21.1. Theorem. *Let* E *be a* **complete** *metric space and let* f *be a contractive mapping of* E *into itself.*

For every $x_0 \in \mathrm{E}$, *the sequence of successive transforms* $x_n = f^n(x_0)$ *of the point* x_0 *converges to a point* a *which is a solution of the equation* $x = f(x)$. *This solution* a *is the only solution of this equation.*

PROOF. By hypothesis f is of Lipschitz class with ratio $k < 1$. Put $x_1 = f(x_0)$, and more generally $x_{n+1} = f(x_n)$. For every n, the transform of the pair (x_{n-1}, x_n) is the pair (x_n, x_{n+1}),

We therefore have

$$d(x_n, x_{n+1}) \leqslant k\, d(x_{n-1}, x_n).$$

We conclude from the first n relations of this form that

$$d(x_n, x_{n+1}) \leqslant k^n\, d(x_0, x_1),$$

from which

$$d(x_n, x_{n+p}) \leqslant \sum_{n}^{n+p-1} d(x_i, x_{i+1}) \leqslant d(x_0, x_1) \sum_{n}^{\infty} k^p$$
$$= d(x_0, x_1)k^n/(1-k).$$

Since $0 \leqslant k < 1$, the sequence (x_n) is thus a Cauchy sequence. Let a be its limit; since f is continuous, $\lim f(x_n) = f(a)$. The relation $x_{n+1} = f(x_n)$ therefore has the limit $a = f(a)$.

This solution a is the only solution of the equation $x = f(x)$, since for every solution x of this equation we have

$$d(x, a) = d(f(x), f(a)) \leqslant k\, d(x, a),$$

which implies $d(x, a) = 0$, hence $x = a$.

Let us remark that the method used constitutes an effective calculational procedure, since the series with general term $d(x_n, x_{n+1})$ converges at least as fast as a geometric series with ratio k. More precisely,

$$d(x_n, a) = \lim_{p\to\infty} d(x_n, x_{n+p}) \leqslant d(x_0, x_1)k^n/(1-k).$$

It can happen that the convergence is still more rapid, for example when the restriction of f to a neighborhood V of a is contractive with ratio $k(\mathrm{V})$ and $k(\mathrm{V})$ tends to 0 with the diameter of V.

Z 1. Every contractive mapping f of E into itself is such that for all $x, y \in \mathrm{E}$ $(x \neq y)$, $d(f(x), f(y)) < d(x, y)$; but the converse is false. It can even happen that f is strictly distance decreasing in this sense, and that E is complete, without f having a fixed point; this is the case for the mapping $x \to (x^2 + 1)^{1/2}$ of $\mathbf{R}_+$ into itself.

2. When E is not complete, a contractive mapping of E into E may have no fixed point; for example, the mapping $x \to x/2$ of $(0, 1]$ into itself has no fixed point.

REMARK 1. Let f be a contractive mapping with ratio k of E (complete) into itself; let a be the fixed point of E.

For every number $\rho > 0$,

$$f^n(\mathrm{B}(a, \rho)) \subset \mathrm{B}(a, k^n\rho).$$

In particular, if E is bounded the decreasing sequence $\delta(f^n(\mathrm{E}))$ has limit 0; the sequences $(f^n(x))$ therefore converge to a uniformly.

REMARK 2. It can happen that f is not contractive, but that a suitable power f^p of f is; we set $f^p = g$.

We can apply Theorem 21.1 to g. If a is the fixed point of g, the relation $g(a) = a$ gives

$$f(g(a)) = f(a); \qquad \text{but} \qquad f^{p+1} = f(g) = g(f);$$

therefore

$$g(f(a)) = f(a).$$

Hence $f(a)$ is a fixed point of g; the uniqueness of this point shows that $f(a) = a$; in other words, a is also a fixed point of f.

For every $x_0 \in \mathrm{E}$, the $g^n(x_0)$ $(\equiv f^{np}(x_0))$ converge to a; since f is continuous, the same is true of the sequence $(f^{np+1}(x_0))_{n\in\mathbf{N}}$, and more generally of $(f^{np+i}(x_0))_{n\in\mathbf{N}}$ (where $i \leqslant p$); therefore the sequence $(f^n(x_0))$ also converges to a.

EXAMPLE 1. E is a Euclidean space $\mathbf{R}^n$ and f is a similarity transformation (a transformation which multiplies distances by a constant factor) with ratio $k < 1$. The fixed point a is called the center of the similarity transformation.

EXAMPLE 2. Let f be a differentiable numerical function on a bounded closed interval $\mathrm{E} = [a, b]$, such that the set of its values is also contained in E, and such that $|f'| \leqslant k$ where $k < 1$.

The mapping f of $[a, b]$ into itself is thus contractive with ratio k and the theorem applies.

This is one of the well-known methods for solving numerical equations.

For example, let f be the mapping $x \to x/2 + 1/x$ of $[1, \infty)$ into itself; f is contractive with ratio $1/2$ and its fixed point is $\sqrt{2}$. The fact that $f'(\sqrt{2}) = 0$ renders the convergence of the procedure particularly rapid.

We shall see other applications of this theorem to the theory of implicit functions, and to the solution of differential equations; nevertheless it is convenient, for these applications, to strengthen Theorem 21.1 by introducing a parameter:

21.2. Theorem. *Let* L *be a topological space,* E *a complete metric space, and f a continuous mapping of* $L \times E$ *into* E *such that, for every* $\lambda \in L$, *the mapping* $x \to f(\lambda, x)$ *of* E *into* E *is contractive with ratio k (where $k < 1$ does not depend on λ).*

If for every λ we denote by a_λ the point x of E *such that $x = f(\lambda, x)$, then the mapping $\lambda \to a_\lambda$ of* L *into* E *is continuous.*

Proof. Let $\lambda_0 \in L$; we shall show that the mapping $\lambda \to a_\lambda$ is continuous at the point λ_0.

Let $\epsilon > 0$; since f is continuous, there exists a neighborhood V of λ_0 such that, for every $\lambda \in V$,

$$d(f(\lambda, a_{\lambda_0}), f(\lambda_0, a_{\lambda_0})) \leqslant \epsilon.$$

But the triangle inequality gives

$$\begin{aligned} d(a_\lambda, a_{\lambda_0}) &= d(f(\lambda, a_\lambda), f(\lambda_0, a_{\lambda_0})) \\ &\leqslant d(f(\lambda, a_\lambda), f(\lambda, a_{\lambda_0})) + d(f(\lambda, a_{\lambda_0}), f(\lambda_0, a_{\lambda_0})) \\ &\leqslant k\, d(a_\lambda, a_{\lambda_0}) + \epsilon, \end{aligned}$$

from which

$$d(a_\lambda, a_{\lambda_0}) \leqslant \epsilon/(1-k) \qquad \text{for all} \quad \lambda \in V.$$

Remark. We have, in fact, used only the partial continuity of f with respect to λ. But this remark does not allow us to weaken the hypotheses, as every mapping f of $L \times E$ into E which is partially continuous with respect to λ and of Lipschitz class with ratio k with respect to x is continuous on $L \times E$; this is immediate from the inequality

$$\begin{aligned} d(f(\lambda, x), f(\lambda_0, x_0)) &\leqslant d(f(\lambda, x), f(\lambda, x_0)) + d(f(\lambda, x_0), f(\lambda_0, x_0)) \\ &\leqslant k\, d(x, x_0) + d(f(\lambda, x_0), f(\lambda_0, x_0)). \end{aligned}$$

The last expression evidently tends to 0 as $x \to x_0$ and $\lambda \to \lambda_0$.

22. POINTWISE CONVERGENCE AND UNIFORM CONVERGENCE

The most useful topological spaces in analysis are the function spaces, that is, spaces whose elements are functions. One can, in such spaces, speak of the convergence of functions to another function, in a sense defined by the topology of the space. Conversely, when one wishes to study certain given aspects of given functions, it is often convenient to put on this set of functions a topology adapted to the study of these aspects.

Here are some examples of such situations:

Let E be the collection of numerical functions on [0, 1] which have derivatives of all orders. This is a vector space; if one wishes to study properties related to this vectorial structure, one should require of the topology of E that it be compatible with the vector space structure. In particular, if one seeks a topology defined by a metric, this metric ought to be invariant under the translations of E; it suffices therefore to define the distance between the element O of E and an arbitrary element f; we will denote this distance by $p(f)$.

1. If we now wish to say that the function $x \to f(x)$ is close to O when it is small for every x, we can put

$$p(f) = \sup_{x \in E} |f(x)|.$$

2. If, on the other hand, we only wish to say that f is small "on the average," we can put

$$p(f) = \int_0^1 |f(x)|\, dx$$

or, if we wish to avoid large values of $f(x)$,

$$p(f) = \left(\int_0^1 f^2(x)\, dx\right)^{1/2}.$$

3. We might want, on the other hand, to express the fact that f is not only small, but does not oscillate too much.

Depending on the circumstance, we can put

$$p(f) = |f(0)| + \text{Total variation of } f \text{ on } [0, 1]$$

or

$$p(f) = |f(0)| + \sup |f'(x)|.$$

4. If we wish to express the fact that the derivatives up to order n are small, we can put

$$p(f) = |f(0)| + |f'(0)| + \cdots + |f^{(n-1)}(0)| + \sup |f^{(n)}(x)|.$$

We shall discuss these procedures in detail in the study of normed spaces. For the moment we shall only briefly study two of the most used modes of convergence.

Pointwise convergence

Let X be a set, with or without a topology, and let Y be a topological space. Let f be a mapping of X into Y, and let (f_n) be a sequence of mappings of X into Y.

22.1. Definition. THE SEQUENCE (f_n) IS SAID TO *CONVERGE POINTWISE* TO f IF, FOR EVERY $x \in X$, THE SEQUENCE $(f_n(x))$ CONVERGES TO $f(x)$.

EXAMPLE 1. Let f_n be the mapping $x \to x^n$ of $[0, 1]$ into **R**. This sequence converges pointwise to the function f defined by

$$f(x) = 0 \quad \text{if} \quad x \neq 1; \qquad f(1) = 1.$$

EXAMPLE 2. Let f_n be the mapping $x \to nx/(1 + |nx|)$ of **R** into **R**. This sequence converges pointwise to the function f defined by

$$f(x) = -1 \quad \text{if} \quad x < 0; \qquad f(x) = 1 \quad \text{if} \quad x > 0; \qquad f(0) = 0.$$

EXAMPLE 3. Let f_n be the mapping $x \to 1/[1 + (x - n)^2]$ of **R** into **R**. This sequence converges pointwise to $f = 0$.

More generally, it is convenient to be able to speak of the pointwise convergence of a family of functions:

22.2. Definition. LET $(f_i)_{i \in I}$ BE A FAMILY OF MAPPINGS OF X INTO Y, AND LET $\mathscr{B}$ BE A FILTER BASE ON I. THE FAMILY $(f_i)_{i \in I}$ IS SAID TO CONVERGE POINTWISE TO f ALONG $\mathscr{B}$ IF, FOR EVERY $x \in X$, $f_i(x)$ CONVERGES TO $f(x)$ ALONG $\mathscr{B}$.

EXAMPLE 1. One can take the preceding examples after replacing the integer n by an arbitrary number, and taking for $\mathscr{B}$ the filter base on **R** consisting of the intervals $[\alpha, \infty)$.

EXAMPLE 2. Let f be a differentiable numerical function defined on **R**. The family (g_α) (where $\alpha \in \mathbf{R}^*$) of functions defined by

$$g_\alpha(x) = (f(x + \alpha) - f(x))/\alpha$$

converges pointwise to the derivative f' as α tends to 0 (in other words, along the filter base $\mathscr{B}$ consisting of the traces on $\mathbf{R}^*$ of the neighborhoods of 0 in **R**).

EXAMPLE 3. Let $f_{p,q}$ be the mapping of $\mathbf{R}^2$ into **R** defined by

$$f_{p,q}(x, y) = \exp(-px^2 - qy^2) \qquad (\text{where } p, q \in \mathbf{N}).$$

The doubly indexed sequence $(f_{p,q})$ converges pointwise to the function f (where $f = 1$ at the origin and 0 everywhere else) as $p, q \to \infty$.

We have defined pointwise convergence without using a topology on the family of mappings of X into Y; in fact there is an underlying topology on this family (see Problem 107) called the topology of pointwise convergence, but we shall not have occasion to use it explicitly.

It is useful to note here that this topology cannot in general be defined by a metric: Let (f_n) be a sequence of numerical functions on $[0, 1]$ which converges pointwise to f; when each f_n is itself the pointwise limit of a sequence $(f_{n,p})$, one might think that f is the pointwise limit of a suitable chosen sequence of functions among the $f_{n,p}$. It is immediate that this would be the case if the topology of pointwise convergence were metrizable (see Problem 67); but simple examples show that this is not the case; therefore this topology is not metrizable (see also Problem 107).

Uniform convergence

Again let X be an arbitrary set, with or without a topology, and let Y be a space which we shall now suppose *metric*; f and f_n again denote mappings of X into Y.

22.3. Definition. THE SEQUENCE (f_n) IS SAID TO *CONVERGE UNIFORMLY* TO f (OR, f IS THE UNIFORM LIMIT OF THE f_n) IF FOR EVERY $\epsilon > 0$ THERE EXISTS AN INTEGER n_0 SUCH THAT $d(f_n(x), f(x)) \leqslant \epsilon$ FOR EVERY $n \geqslant n_0$ AND FOR EVERY $x \in X$.

We can similarly define the uniform convergence of a family (f_i) to f along a filter base $\mathscr{B}$; however, this definition will become superfluous once we have defined the topology of uniform convergence.

Every sequence (f_n) which converges uniformly to f converges pointwise to f, but it is important to note that the converse is incorrect. Here are several examples:

1. Let f_n be the continuous numerical function (see Fig. 3) defined on $X = [0, 1]$ by

$$f_n(x) = n^2x(1 - nx) \quad \text{on} \quad [0, 1/n],$$
$$f_n(x) = 0 \quad \text{on} \quad [1/n, 1].$$

It is easily verified that the sequence (f_n) converges pointwise to the function $f = 0$, but this convergence is not uniform, as

$$\sup | f_n(x) - f(x) | = n/4$$

which, far from tending to 0, tends to $+\infty$.

2. In each of the three examples which illustrate pointwise convergence, the convergence is nonuniform; this is particularly clear in the first two, from the fact that the limit f is not continuous; in the third example, there is uniform convergence on every bounded interval, but not on all of **R**.

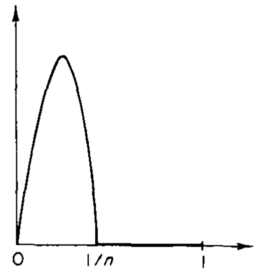

FIG. 3.

Metric and topology of uniform convergence

The study of uniform convergence can be simplified by introducing an ecart on the set $\mathscr{F}(X, Y)$ of mappings of X into Y.

Let f and g be mappings of a set X into a metric space Y, and put

$$d(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

This is either a finite number or $+\infty$; by the previous study of ecarts on a set, for every $x \in X$

$$d_x(f, g) = d(f(x), g(x))$$

is an ecart on $\mathscr{F}(X, Y)$, and

$$d(f, g) = \sup d_x(f, g)$$

is also an ecart. Moreover

$$(f \neq g) \Rightarrow (d(f, g) \neq 0).$$

When the diameter of Y is finite, $d(f, g)$ is always finite, therefore d is a metric on $\mathscr{F}(X, Y)$; when Y is arbitrary, it is often convenient to replace the ecart d by $d/(1 + d)$, which is a metric and which defines the same topology and the same uniform structure on $\mathscr{F}(X, Y)$ as does d.

The topology on $\mathscr{F}(X, Y)$ associated with the ecart d is called the *topology of uniform convergence* (sometimes, *uniform topology*); this terminology is justified by the following result:

22.4. Proposition. *To say that a sequence* (f_n) *of mappings of* X *into* Y *converges uniformly to a mapping* f *is equivalent to saying that in the space* $\mathscr{F}(X, Y)$ *with the topology of uniform convergence, the sequence of points* f_n *converges to the point* f.

Indeed, to say that the sequence of functions f_n converges uniformly to f is equivalent to saying that for every $\epsilon > 0$ we have, for n sufficiently large,

$$d(f(x), f_n(x)) \leqslant \epsilon \qquad \text{for all} \qquad x \in X,$$

which is equivalent to saying that $d(f, f_n) \leqslant \epsilon$.

Interpretation of uniform convergence by means of the graphs

The use of the graphs of mappings f of X into Y gives a convenient and intuitive interpretation of uniform convergence (Fig. 4):

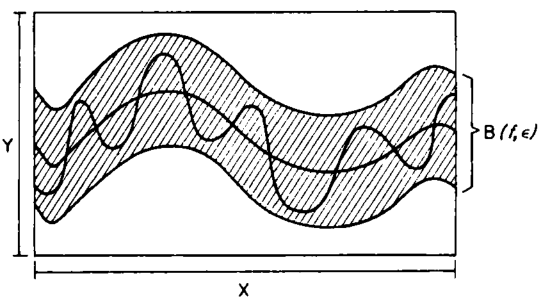

FIG. 4.

Let $B(f, \epsilon)$ be the set of points (x, y) of $X \times Y$ such that

$$d(y, f(x)) \leqslant \epsilon;$$

this set forms a sort of *tube* of radius ϵ about the graph of f.

To say that $d(f, g) \leqslant \epsilon$ is equivalent to saying that

$$d(f(x), g(x)) \leqslant \epsilon$$

for every $x \in X$, or that the graph of g is contained in $B(f, \epsilon)$.

To say that the sequence (f_n) converges uniformly to f amounts to saying that for every $\epsilon > 0$, all the f_n from some n_0 on have their graphs in $B(f, \epsilon)$.

One can now better see the reason why, in Example 1 above, the f_n do not converge uniformly to 0: For every $\epsilon < 1/4$, the graph of no one of the f_n is contained in $B(0, \epsilon)$.

Case of complete spaces

The introduction of the ecart d on $\mathscr{F}(X, Y)$ allows us to speak of Cauchy sequences of mappings, and to state the following theorem:

22.5. Theorem. *When the metric space* Y *is complete, the space* $\mathscr{F}(X, Y)$ *is also complete.*

Proof. Let (f_n) be a Cauchy sequence in $\mathscr{F}(X, Y)$. For every $x \in X$, the inequality $d(f_p(x), f_q(x)) \leqslant d(f_p, f_q)$ shows that the sequence $(f_n(x))$ of points of Y is a Cauchy sequence. Since Y is complete, this sequence has a limit which we denote by $f(x)$.

But since (f_n) is a Cauchy sequence, for every $\epsilon > 0$ there exists an integer $n(\epsilon)$ such that for all $p, q \geqslant n(\epsilon)$ and every $x \in X$,

$$d(f_p(x), f_q(x)) \leqslant \epsilon.$$

If in this inequality we fix x and p and let $q \to \infty$, then $f_q(x)$ tends to $f(x)$, and we obtain the inequality

$$d(f_p(x), f(x)) \leqslant \epsilon \qquad \text{for all} \quad p \geqslant n(\epsilon).$$

It follows from this inequality that $d(f_p, f) \leqslant \epsilon$, and therefore the sequence (f_n) converges to f in the space $\mathscr{F}(X, Y)$ taken with the ecart d. Hence this space is complete.

Preservation of continuity by uniform convergence

We now assume that X is a topological space, with Y still a *metric* space. One can then speak of the continuity of a mapping of X into Y. We shall see that this property is preserved by uniform convergence.

22.6. Theorem. *Let* (f_n) *be a sequence of mappings of* X *into* Y *which converges uniformly to* f, *and let* a *be a point of* X.

If all the f_n *are continuous at* a, *then* f *is continuous at* a.

Proof. Let $\epsilon > 0$. Since the convergence of the f_n is uniform, there exists an integer n_0 such that

$$d(f(x), f_{n_0}(x)) \leqslant \epsilon \qquad \text{for every} \quad x \in X.$$

Since f_{n_0} is continuous at a, there exists a neighborhood V of a such that

$$d(f_{n_0}(x), f_{n_0}(a)) \leqslant \epsilon \qquad \text{for every} \quad x \in \mathrm{V}.$$

It follows that for every $x \in \mathrm{V}$ we have

$$d(f(x), f(a)) \leqslant d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(a)) + d(f_{n_0}(a), f(a)) \leqslant 3\epsilon.$$

Since ϵ is arbitrary, this inequality proves the continuity of f at a.

Corollary 1. *If the f_n are continuous in* X *and converge uniformly to f, then f is continuous in* X.

Corollary 2. *Let (f_n) be a sequence of continuous mappings of a metric space* X *into a metric space* Y. *If for every compact* K *in* X *the restrictions of the f_n to* K *converge uniformly, then the sequence (f_n) converges on* X *and its limit f is continuous.*

This is an immediate consequence of the preceding theorem and of Corollary 2 of Proposition 16.6.

We shall often have use of this corollary.

Z Theorem 22.6 and its corollaries extend immediately to the uniform limit of a family of continuous functions along a filter base.

However, it should be observed here that Theorem 22.6 does not extend to pointwise convergence:

When a sequence (f_n) of continuous functions converges pointwise to a function f, then f has a certain degree of regularity but is not always continuous. For example, the sequence of monomials $f_n(x) = x^n$ converges pointwise on $[0, 1]$ to the function f equal to 0 on $[0, 1)$ and to 1 for $x = 1$.

The space $\mathscr{C}$(X, Y)

We shall denote by $\mathscr{C}$(X, Y) the subspace of $\mathscr{F}$(X, Y) consisting of the continuous mappings of the topological space X into the metric space Y. It follows from the preceding theorem that $\mathscr{C}$(X, Y) is a *closed* subset of $\mathscr{F}$(X, Y) taken with the topology of uniform convergence; when $\mathscr{F}$(X, Y) is complete, then $\mathscr{C}$(X, Y) is also complete. We can therefore state:

22.7. Theorem. *When* Y *is complete, the subspace $\mathscr{C}$(X, Y) of $\mathscr{F}$(X, Y) consisting of the continuous mappings of* X *into* Y *is complete with respect to the ecart of uniform convergence.*

REMARK. Even if very restrictive regularity hypotheses are imposed on X and Y, the space $\mathscr{C}(X, Y)$ is in general noncompact.

For example, suppose that X and Y are identical with the compact interval [0, 1]. The set $\mathscr{C}(X, Y)$ is then simply the set of continuous numerical functions defined on [0, 1] with values in [0, 1].

With the metric chosen on $\mathscr{C}(X, Y)$, this space is complete, but not compact or even locally compact. For example, the element O of this space does not have any compact neighborhood, since for every $k > 0$ the sequence (f_n) defined by $f_n(x) = k \sin^2 nx$ has no convergent subsequence.

Uniform convergence on a collection of subsets

The sequence of numerical functions $x \to 1/[1 + (x - n)^2]$ converges pointwise to the function O; but this convergence is not uniform. However, for every interval $[a, b]$ of $\mathbf{R}$, the restrictions of these functions to $[a, b]$ converge uniformly to 0.

More generally, Corollary 2 of Theorem 22.6 shows the interest of uniform convergence on every compact set.

The latter is a special case of an important general notion.

22.8. Definition. LET X BE AN ARBITRARY SET; LET Y BE A METRIC SPACE, AND LET $\mathscr{A}$ BE A COLLECTION OF SUBSETS OF X. WE DENOTE BY f, f_n MAPPINGS OF X INTO Y.

THE SEQUENCE (f_n) IS SAID TO CONVERGE UNIFORMLY TO f ON EVERY $A \in \mathscr{A}$ IF FOR EVERY $A \in \mathscr{A}$ THE RESTRICTIONS OF THE f_n TO A CONVERGE UNIFORMLY TO THE RESTRICTION OF f TO A.

EXAMPLE 1. If X is the only element of $\mathscr{A}$, we are back to uniform convergence.

EXAMPLE 2. If $\mathscr{A}$ is the collection of one-point subsets of X, we are back to pointwise convergence.

EXAMPLE 3. If $\mathscr{A}$ is the collection of compact sets in the topological space X, we obtain uniform convergence on every compact set.

Relation between uniform convergence and pointwise convergence

Several examples have shown us that the pointwise convergence of a sequence of functions does not necessarily imply its uniform convergence. Here, however, is an important case in which this implication is true:

22.9. Theorem (of Dini). *Let* X *be a compact space, and let* (f_n) *be a sequence of continuous numerical functions on* X *which converges pointwise to a numerical function* f.

If **the sequence** (f_n) **is increasing,** *that is if* $(p \leqslant q) \Rightarrow (f_p \leqslant f_q)$, *and if* f *is* **continuous,** *the convergence of the* f_n *to* f *is uniform.*

PROOF. We take an $\epsilon > 0$. For every $x \in X$, there exists an integer p_x such that

$$f_{p_x}(x) > f(x) - \epsilon.$$

Let ω_x denote the set of y such that

$$f_{p_x}(y) > f(y) - \epsilon.$$

Since f and f_{p_x} are continuous, ω_x is open; by hypothesis ω_x contains x, and therefore the ω_x form an open covering of X, which is compact. Therefore there exists a finite set of points x_i of X such that the ω_{x_i} cover X. We put

$$p = \sup_i p_{x_i}.$$

Since the sequence (f_n) is increasing, we have

$$f_p(x) > f(x) - \epsilon$$

for all $x \in \bigcup \omega_{x_i}$, hence everywhere. We thus have, for all $q \geqslant p$,

$$f - \epsilon \leqslant f_q \leqslant f;$$

therefore the convergence is uniform.

Of course, an analogous result holds for *decreasing* sequences.

Z It is the sequence (f_n) which is increasing, and not each of the f_n; however, this last interpretation could be meaningful only if X had an order relation, for example if $X = [0, 1]$.

All the same this confusion would not lead to error; it can in fact be verified that if an arbitrary sequence of monotone numerical functions on an interval $[a, b]$ converges pointwise to a continuous function, the convergence is uniform (see Problem 85).

23. EQUICONTINUOUS SPACES OF FUNCTIONS

Despite the interest presented by complete spaces, as evidenced by the preceding theorems, it is often valuable to be able to deal with compact spaces of functions. The notion of *equicontinuity* will give us an important class of such spaces.

23.1. Definition. LET E BE A FAMILY OF MAPPINGS OF A METRIC SPACE X INTO A METRIC SPACE Y. WE SAY THAT E IS *EQUICONTINUOUS* IF, FOR

EVERY $\epsilon > 0$, THERE EXISTS AN $\eta > 0$ SUCH THAT $d_Y(f(x_1), f(x_2)) \leqslant \epsilon$ FOR EVERY $f \in E$ AND FOR ALL $x_1, x_2 \in X$ SUCH THAT $d_X(x_1, x_2) \leqslant \eta$.

It follows from this definition that every $f \in E$ is not only continuous, but also uniformly continuous. The equicontinuity of E says that in addition this uniform continuity is of the same type for all the $f \in E$, or more precisely, that $\eta(\epsilon)$ does not depend on f.

EXAMPLE 1. The set E of numerical functions x^n defined on $[0, 1]$ *is not equicontinuous*, although each of these functions is uniformly continuous on X.

EXAMPLE 2. For any metric spaces X and Y, the collection E_k (where $k > 0$) of mappings of Lipschitz class with ratio k of X into Y is equicontinuous.

Interpretation of equicontinuity

It is convenient to express Definition 23.1 by again using the notion of modulus of continuity which was already used in studying uniform continuity:

To say that E is an equicontinuous set of functions is equivalent to saying that there exists a common modulus of continuity φ for all the $f \in E$.

Let $\alpha > 0$ be arbitrary, and let A be a subset of X which is α-dense in the sense that $d(x, A) < \alpha$ for every $x \in X$. For every $f \in E$, the relation $d(x_1, x_2) \leqslant \alpha$ implies

$$d(f(x_1), f(x_2)) \leqslant \varphi(\alpha);$$

therefore knowing the restriction of f to A entails a knowledge of f on X up to an accuracy $\varphi(\alpha)$.

In other words, the functions f of E have a certain "rigidity" which is measured by their modulus of continuity φ.

For example, a numerical function which is of Lipschitz class with ratio k on $[0, 1]$ is known everywhere with an accuracy k/n whenever $f(1/n), f(2/n), \ldots, f(p/n), \ldots$ are known.

All the consequences of equicontinuity follow from this "rigidity."

23.2. Theorem (of Ascoli). *Let* X *and* Y *be compact metric spaces and let* E *be a subset of the space* $\mathscr{C}(X, Y)$ *taken with the topology of uniform convergence.*

To say that E *is equicontinuous is equivalent to saying that* E *is relatively compact in* $\mathscr{C}(X, Y)$.

PROOF. 1. Suppose that E is equicontinuous; to show that $\bar{E}$ is compact it suffices, by the corollary of Theorem 18.2, to prove that every infinite sequence S of elements of E contains a subsequence which converges in $\mathscr{C}(X, Y)$; as $\mathscr{C}(X, Y)$ is complete (since Y is), it suffices to prove that S contains a Cauchy subsequence.

Let φ be a common modulus of continuity for all the $f \in E$. For every $\epsilon > 0$ there exists a finite covering of X by open balls of radius ϵ with centers x_i $(i = 1, 2, ..., n)$.

Since Y is compact, we can find a subsequence $S_1(\epsilon)$ of S such that, for all $f, g \in S_1(\epsilon)$,

$$d_Y(f(x), g(x)) \leqslant \varphi(\epsilon) \qquad \text{at the point} \quad x = x_1 . \tag{1}$$

We can find a subsequence $S_2(\epsilon)$ of the sequence $S_1(\epsilon)$ having the same property at the point x_2, and so on. After n operations we will have constructed a subsequence $S_n(\epsilon)$ satisfying inequality (1) at each of the points x_i $(i = 1, 2, ..., n)$; we denote $S_n(\epsilon)$ by $S(\epsilon)$.

But by construction, for every $x \in X$ there exists a point x_i such that

$$d_X(x, x_i) \leqslant \epsilon,$$

which implies that

$$d_Y(f(x), f(x_i)) \leqslant \varphi(\epsilon) \qquad \text{for every} \quad f \in S.$$

For all $f, g \in S(\epsilon)$ we therefore have

$$d_Y(f(x), g(x)) \leqslant d_Y(f(x), f(x_i)) + d_Y(f(x_i), g(x_i)) + d_Y(g(x_i), g(x)) \leqslant 3\varphi(\epsilon),$$

in other words

$$d(f, g) \leqslant 3\varphi(\epsilon). \tag{2}$$

We have thus exhibited a procedure which associates with every sequence S a subsequence $S(\epsilon)$ every two elements f, g of which satisfy the relation (2).

It suffices to iterate this procedure, giving ϵ the successive values $(1, 1/2, ..., 1/n, ...)$, to obtain successive sequences

$$S, \quad S(1), \quad S(1, 1/2), ..., S(1, 1/2, ..., 1/n), ...$$

each of which is a subsequence of the preceding.

Since the sequence $(\varphi(1/n))$ has limit 0, the diagonal of this sequence of sequences is a Cauchy sequence. This is the desired sequence.

2. Let E be a compact subset of $\mathscr{C}(X, Y)$.

For every $\epsilon > 0$ there exists a finite sequence $(f_1, f_2, \ldots, f_n)$ of points of E such that the open balls $B(f_i, \epsilon)$ form an open covering of E. For each of these f_i there exists an $\eta_i > 0$ such that the inequality

$$d_X(x_1, x_2) < \eta_i$$

implies

$$d_Y(f_i(x_1), f_i(x_2)) < \epsilon.$$

Let η be the smallest of the η_i; for every $f \in E$ we therefore have, noting that f belongs to some ball $B(f_i, \epsilon)$,

$$d_Y(f(x_1), f(x_2)) \leqslant 3\epsilon$$

whenever

$$d_X(x_1, x_2) \leqslant \eta.$$

In other words, E is an equicontinuous family.

Z Theorem 23.2 does not extend to the case where either X or Y is locally compact but not compact.

For example, the collection E of constant mappings of [0, 1] into **R** is clearly equicontinuous; nevertheless it is not compact.

Similarly, the family of mappings $x \to 1/[1 + (x - n)^2]$ of **R** into [0, 1] is equicontinuous but not compact.

Examples. The family of mappings of Lipschitz class with ratio k of the interval [0, 1] into an interval $[a, b]$ is compact in the topology of uniform convergence.

An important example of equicontinuity is given by the following proposition, which will be useful in the theory of integration.

23.3. Proposition. *Let* X *and* Y *be compact metric spaces; let f be a mapping of* $X \times Y$ *into another metric space* Z, *and for every* $x \in X$ *let* f_x *denote the partial mapping* $y \to f(x, y)$.

If f is continuous, the mapping $x \to f_x$ *of* X *into* $\mathscr{C}(Y, Z)$ *is continuous, and the* f_x *form an equicontinuous family.*

Proof. Since $X \times Y$ is compact, f is uniformly continuous; thus for every $\epsilon > 0$ there exists an $\eta > 0$ such that

$$(d(x_1, x_2) < \eta \quad \text{and} \quad d(y_1, y_2) < \eta) \Rightarrow (d(f(x_1, y_1), f(x_2, y_2)) < \epsilon).$$

In particular, $d(x_1, x_2) < \eta$ implies for every y the inequality

$$d(f(x_1, y), f(x_2, y)) < \epsilon,$$

in other words,

$$d(f_{x_1}, f_{x_2}) < \epsilon.$$

Therefore the mapping $x \to f_x$ is continuous.

It follows from this, since X is compact, that the family of the f_x is a compact set in $\mathscr{C}(Y, Z)$; hence by Theorem 23.2 it is an equicontinuous family. This can also be obtained directly. By the foregoing, $d(y_1, y_2) < \eta$ implies, for every x, the inequality

$$d(f_x(y_1), f_x(y_2)) < \epsilon,$$

from which the desired equicontinuity follows.

24. TOTAL VARIATION AND LENGTH

The notions which we have studied up to now, continuity, uniform continuity, uniform convergence, do not have a metric character, even though it is convenient to make use of a metric in studying them.

On the other hand, the total variation which we are going to define has an essentially metric character.

Let T be a totally ordered set (which will play the role of a parameter set) and let f be a mapping of T into a metric space E; to simplify the notation, we shall denote the distance between two points $x, y \in E$ by $|xy|$.

24.1. Definition. FOR EVERY FINITE SUBSET σ OF T, THE NUMBER

$$V_\sigma = \sum_i |f(t_i)f(t_{i+1})|,$$

WHERE THE t_i $(t_1 < t_2 \cdots < t_n)$ DENOTE THE POINTS OF σ, IS CALLED THE TOTAL VARIATION OF f ON σ.

The triangle inequality shows at once that $(\sigma \subset \sigma') \Rightarrow (V_\sigma \leqslant V_{\sigma'})$; in other words, V is an increasing function of σ. It is therefore natural to introduce the following definition:

24.2. Definition. FOR EVERY $A \subset T$, THE POSITIVE, FINITE OR INFINITE NUMBER V_A DEFINED BY

$$V_A = \sup V_\sigma \qquad \text{(over all the finite subsets } \sigma \text{ of A)}$$

IS CALLED THE *TOTAL VARIATION* OF f ON A. WHEN $V_A < \infty$, f IS SAID TO HAVE *BOUNDED VARIATION* (OR TO BE *OF BOUNDED VARIATION*) ON A.

It is evident that V_A is an increasing function of A.

In particular, if T has a first point α and a last point β, then

$$V_T \geqslant V_{\{\alpha,\beta\}}, \qquad \text{that is,} \qquad V_T \geqslant |f(\alpha)f(\beta)|.$$

EXAMPLE 1. Let f be an *increasing* mapping of T into **R**.

The relation

$$V_\sigma = \sum (f(t_{i+1}) - f(t_i)) = f(t_n) - f(t_1)$$

shows that if T has a first point α and a last point β, then

$$V_T = f(\beta) - f(\alpha).$$

If f is decreasing, then $V_T = |f(\beta) - f(\alpha)|$.

EXAMPLE 2. Let f be a mapping of Lipschitz class with ratio k of the interval $[a, b]$ of **R** into a metric space E.

For every $\sigma = \{t_i\} \subset [a, b]$ we have

$$V_\sigma \leqslant k \sum (t_{i+1} - t_i) \leqslant k(b - a).$$

Therefore

$$V_{[a,b]} \leqslant k(b - a).$$

24.3. Proposition. *The total variation is additive in the sense that, for for every $x \in$ T, if we set*

$$T_1 = (-\infty, x] \cap T \qquad \textit{and} \qquad T_2 = [x, \infty) \cap T,$$

then

$$V_T = V_{T_1} + V_{T_2}.$$

Indeed, since V_σ is an increasing function of σ, we also have

$$V_T = \sup V_\sigma \qquad \text{over all finite } \sigma \text{ containing } x.$$

But such a σ is simply the union of an arbitrary σ_1 of T_1 containing x and an arbitrary σ_2 of T_2 containing x. But

$$V_\sigma = V_{\sigma_1} + V_{\sigma_2};$$

therefore

$$V_T = \sup V_\sigma = \sup (V_{\sigma_1} + V_{\sigma_2}) = \sup V_{\sigma_1} + \sup V_{\sigma_2} = V_{T_1} + V_{T_2}.$$

24.4. Definition. LET T AND T′ BE TOTALLY ORDERED SETS, AND LET f AND f' BE MAPPINGS OF T AND T′, RESPECTIVELY, INTO E. THEN f AND f' ARE SAID TO BE EQUIVALENT IF THERE EXISTS AN INCREASING BIJECTION φ OF T′ TO T SUCH THAT $f' = f \circ \varphi$.

Such a relation is clearly reflexive, symmetric, and transitive; hence it is an equivalence relation.

24.5. Proposition. *If f and f' are equivalent, the total variations* V, V′ *of f and f' on* T *and* T′, *respectively, are equal.*

Indeed, for all $x', y' \in \mathrm{T}'$ we have

$$|f'(x')f'(y')| = |f(\varphi(x'))f(\varphi(y'))|;$$

therefore for every $\sigma' \subset \mathrm{T}'$, the total variation of f' on σ' is equal to the total variation of f on $\sigma = \varphi(\sigma')$.

It follows that $\mathrm{V}' \leqslant \mathrm{V}$; in the same way $\mathrm{V} \leqslant \mathrm{V}'$, from which the equality follows.

Parametrized curves

From now on we shall consider only the case where T is an interval $[a, b]$ of **R** and where f is continuous; in this case the pair (T, f) is called a *parametrized curve* or a *path, and* $\mathrm{V_T}$ is called the *length* of this curve; when this length is finite, we say that the curve is *rectifiable.*

For every finite subset σ of T, with

$$\sigma = \{t_1, t_2, \ldots, t_n\} \qquad \text{where} \qquad a = t_1 < t_2 < \cdots < t_n = b,$$

the number

$$\mu(\sigma) = \sup_i |t_{i+1} - t_i|$$

is called the *modulus of* σ.

24.6. Theorem. *For every parameterized curve* (T, f), $\mathrm{V_T}$ *is the limit of the* V_σ *as the modulus* $\mu(\sigma)$ *tends to* 0.

In other words (assuming $\mathrm{V_T} < \infty$ for definiteness), for every $\epsilon > 0$ there exists an $\eta > 0$ such that

$$(\mu(\sigma) < \eta) \Rightarrow (\mathrm{V_T} - \mathrm{V}_\sigma < \epsilon).$$

This theorem is sometimes stated in the following form: The length of the curve is the limit of the lengths of *polygons inscribed* in the curve, where by an inscribed polygon is meant a pair (σ, f), with σ a finite subset of T containing a and b.

PROOF. We shall assume $V_T < \infty$; the proof is completely analogous if $V_T = \infty$.

By the definition of V_T, for every $\epsilon > 0$ there exists a finite σ_0, $\sigma_0 = \{t_1, t_2, \cdots, t_{n_0}\}$, such that

$$V_T - V_{\sigma_0} \leqslant \epsilon.$$

Since f is uniformly continuous on T, there exists a number $\eta > 0$ such that $|u - v| < \eta$ implies

$$|f(u)f(v)| < \epsilon/n_0\ ;$$

we shall assume in addition that η is smaller than the length of the smallest of the intervals $[t_i, t_{i+1}]$ associated with σ_0.

Now let us take any σ with modulus $\mu(\sigma) < \eta$. Each of the intervals $[t_i, t_{i+1}]$ of σ_0 contains at least one point of σ distinct from t_i and t_{i+1}; we denote by t_i' (t_i'') the largest (smallest) of the points t of σ such that $t < t_i$ $(t > t_i)$ (see Fig. 5), and put

$$f(t_i) = m_i\ ; \qquad f(t_i') = m_i'; \qquad f(t_i'') = m_i''\ .$$

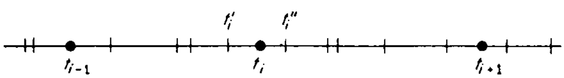

FIG. 5.

We have

$$V_{\sigma\cup\sigma_0} - V_\sigma = \sum_i (|m_i'm_i| + |m_im_i''| - |m_i'm_i''|) \leqslant \sum_i (|m_i'm_i| + |m_im_i''|)$$
$$\leqslant 2(\epsilon/n_0)n_0 = 2\epsilon,$$

where the summation is over only those i for which $t_i \notin \sigma$.

Adding the relations

$$V_T - V_{\sigma_0} \leqslant \epsilon$$
$$V_{\sigma_0} \leqslant V_{\sigma\cup\sigma_0}$$
$$V_{\sigma\cup\sigma_0} - V_\sigma \leqslant 2\epsilon$$

yields

$$V_T - V_\sigma \leqslant 3\epsilon;$$

this is the desired inequality, up to replacement of ϵ by 3ϵ.

Corollary 1. *If* $V_T < \infty$, *the mapping* $t \to V_{[a,t]}$ *of* T *into* **R** is *increasing and continuous.*

PROOF. Given $\epsilon > 0$, there exists an $\eta > 0$ such that

$$(\mu(\sigma) < \eta) \Rightarrow (V_T - V_\sigma < \epsilon)$$

and

$$(|y - x| < \eta) \Rightarrow (|f(x)f(y)| < \epsilon).$$

Now let x, y be arbitrary points of $[a, b]$ such that $0 \leqslant y - x \leqslant \eta$.

There exists a σ containing x, y but no other points of $[x, y]$, and with modulus $\mu(\sigma) < \eta$; let σ_1 and σ_2 denote the traces of σ on $[a, x]$ and $[y, b]$.

The relation $V_T < V_\sigma + \epsilon$ can be written as

$$\begin{aligned} V_{[a,x]} + V_{[x,y]} + V_{[y,b]} &= V_{[a,b]} \leqslant V_\sigma + \epsilon \\ &= V_{\sigma_1} + |f(x)f(y)| + V_{\sigma_2} + \epsilon \leqslant V_{[a,x]} + \epsilon + V_{[y,b]} + \epsilon, \end{aligned}$$

from which $V_{[x,y]} < 2\epsilon$ for all x, y such that $|y - x| < \eta$.

Corollary 2. *If* E *is the Euclidean space* $\mathbf{R}^n$ *and if the mapping* $f = (f_p)$ *of* $[a, b]$ *into* E *has a continuous derivative, the length of the curve defined by* f *is equal to*

$$\int_a^b \left(\sum f_p'^2(t)\right)^{1/2} dt.$$

PROOF. To simplify the notation we shall carry out the proof only for $n = 2$; we denote f_1 and f_2 by g and h. Let σ_n be the finite subset of $[a, b]$ consisting of the points

$$t_i = a + i(b - a)/n \qquad (\text{where} \quad i \leqslant n).$$

We set

$$\Delta x_i = g(t_{i+1}) - g(t_i); \qquad \Delta y_i = h(t_{i+1}) - h(t_i); \qquad \Delta t_i = t_{i+1} - t_i\,.$$

Then

$$\Delta x_i = g'(\tau_i^1)\,\Delta t_i \qquad \text{and} \qquad \Delta y_i = h'(\tau_i^2)\,\Delta t_i$$

where

$$\tau_i^1, \tau_i^2 \in [t_i\,, t_{i+1}];$$

therefore

$$V_{\sigma n} = \sum_i (\Delta^2 x_i + \Delta^2 y_i)^{1/2} = \sum_i (g'^2(\tau_i^1) + h'^2(\tau_i^2))^{1/2}\,\Delta t_i\,.$$

This expression is the integral of the step function φ_n, whose value on each $[t_i, t_{i+1})$ is

$$(g'^2(\tau_i^1) + h'^2(\tau_i^2))^{1/2}.$$

But the sequence (φ_n) converges uniformly to the function φ:

$$x \to (g'^2(x) + h'^2(x))^{1/2}.$$

Indeed, for every $t \in [t_i, t_{i+1})$ we have, by the triangle inequality in $\mathbf{R}^2$,

$$|\varphi_n(t) - \varphi(t)| \leqslant [(g'(\tau_i^1) - g'(t))^2 + (h'(\tau_i^2) - h'(t))^2]^{1/2}.$$

Since

$$|\tau_i^1 - t|, \quad |\tau_i^2 - t| \leqslant (b-a)/n$$

and since g' and h' are uniformly continuous, the right side of the above inequality tends to 0 as $n \to \infty$.

Therefore the integral V_{σ_n} of φ_n converges to that of φ; but the length of the curve is equal to the limit of the V_{σ_n}, from which the assertion follows.

Parametrized curves in a product space

Let E_i be a finite family of metric spaces; let d_i be the metric on E_i, and let d be any one of the usual metrics on the product E of the E_i.

Let $f = (f_i)$ be a mapping of $[a, b]$ into E, and let V (V_i) be the total variation of f (f_i).

The inequality

$$d_i(x_i, y_i) \leqslant d(x, y) \leqslant \sum d_i(x_i, y_i)$$

implies a similar inequality for the variations on every finite subset σ of $[a, b]$, from which we readily obtain

$$V_i \leqslant V \leqslant \sum V_i.$$

In particular we can state:

24.7. Proposition. *To say that a parametrized curve in a finite product of metric spaces is rectifiable is equivalent to saying that each of its projections on the coordinate spaces is rectifiable.*

Example 1. Let Γ be a curve in the Euclidean space $\mathbf{R}^n$ defined by a continuous mapping $f = (f_i)$ of $[a, b]$ into $\mathbf{R}^n$; to say that Γ is rectifiable is equivalent to saying that each of the f_i is of bounded variation.

EXAMPLE 2. Let Γ be the graph in $\mathbf{R}^2$ of a continuous mapping f of $[a, b]$ into $\mathbf{R}$; here the graph Γ can be identified with the curve $x \to (x, f(x))$. Therefore this graph has finite length when f has bounded variation.

Z It is essential to note that the total variation V of f (or the length of the path in $\mathbf{R}$ which f traverses) is never equal to the length of the graph of f; the latter belongs to the interval $(\mathrm{V}, \mathrm{V} + b - a]$.

Numerical functions of bounded variation

The preceding examples show the importance of numerical functions of bounded variation; thus, we shall undertake a short study of such functions. More precisely, we shall study the structure of the family $\mathscr{V}$ of finite numerical functions of bounded variation on $[a, b]$.

For every $f \in \mathscr{V}$, we denote by $\mathrm{V}(f)$ the total variation of f on $[a, b]$.

24.8. Proposition. 1. *$\mathscr{V}$ is a vector space, and* V *is a semi-norm on $\mathscr{V}$ in the sense that*:

$$\mathrm{V} \geqslant 0; \qquad \mathrm{V}(\lambda f) = |\lambda| (\mathrm{V}(f); \qquad \mathrm{V}(f_1 + f_2) \leqslant \mathrm{V}(f_1) + \mathrm{V}(f_2).$$

2. *$\mathscr{V}$ contains the convex cone $\mathscr{V}_0$ of increasing functions, and every element of $\mathscr{V}$ is the difference of two elements of $\mathscr{V}_0$.*

PROOF. 1. Let f and g be arbitrary numerical functions on $[a, b]$, and put $h = f + g$. For all $u, v \in [a, b]$ we have

$$|h(u) - h(v)| = |(f(v) - f(u)) + (g(v) - g(u))| \leqslant |f(v) - f(u)| + |g(v) - g(u)|.$$

Therefore for every finite subset σ of $[a, b]$ we have

$$\mathrm{V}_\sigma(h) \leqslant \mathrm{V}_\sigma(f) + \mathrm{V}_\sigma(g) \leqslant \mathrm{V}_f + \mathrm{V}_g,$$

hence

$$\mathrm{V}(f + g) = \mathrm{V}(h) \leqslant \mathrm{V}(f) + \mathrm{V}(g).$$

Therefore if f and $g \in \mathscr{V}$, then also $f + g \in \mathscr{V}$.

On the other hand, it is evident that $\mathrm{V}(\lambda f) = |\lambda| \mathrm{V}(f)$ for every $f \in \mathscr{V}$. Therefore $\mathscr{V}$ is indeed a vector space and V is a semi-norm on $\mathscr{V}$.

2. We already know that if f is increasing, $\mathrm{V}(f) = f(b) - f(a)$; therefore $\mathscr{V}_0 \subset \mathscr{V}$. Since $\mathscr{V}$ is a vector space, $f, g \in \mathscr{V}_0$ implies $f - g \in \mathscr{V}$

To prove the second part of 2, suppose $f \in \mathscr{V}$. For every $t \in [a, b]$ let $\varphi(t)$ denote the total variation of f on $[a, t]$, and put

$$\psi(t) = \varphi(t) + f(t).$$

For all $u, v \in [a, b]$ such that $u < v$ we have

$$\psi(v) - \psi(u) = (\varphi(v) - \varphi(u)) + (f(v) - f(u)).$$

But it follows from the additivity of the total variation that $\varphi(v) - \varphi(u)$ is the total variation of f on $[u, v]$; therefore

$$\varphi(v) - \varphi(u) \geqslant |f(v) - f(u)|, \qquad \text{whence} \qquad \psi(v) - \psi(u) \geqslant 0.$$

Thus $\varphi, \psi \in \mathscr{V}_0$; since $f = \psi - \varphi$, the desired property is established.

24.9. Corollary. *Every numerical function of bounded variation on $[a, b]$ has a limit from the right and from the left at every point (with the obvious exceptions at a and b).*

Indeed, the increasing functions have this property, and this property is preserved by the subtraction of functions.

Case of continuous functions

With the above notation, if f is continuous and of bounded variation, the function φ is continuous (Corollary 1 of Theorem 24.6); therefore ψ is also continuous.

Thus, *every continuous numerical function of bounded variation is equal to the difference of two continuous increasing functions.*

Examples of numerical functions of bounded variation

1. Every function f on a finite interval I of **R** which is bounded and piecewise monotone, that is, for which there exists a finite partition of I into intervals, some of which may consist of one point, in each of which f is monotone (ignoring the one-point intervals). When f is moreover continuous, the total variation of f on I is the sum of its variations on each of these subintervals.

All the elementary functions are of this nature.

2. If f is of Lipschitz class with ratio k on $[a, b]$, then

$$\mathrm{V}(f) \leqslant k(b - a).$$

This is the case, in particular, for functions f which are differentiable and whose derivative f' is bounded; in fact, if $|f'| \leqslant k$, then f is of Lipschitz class with ratio k.

More precisely if f' is continuous, then

$$V(f) = \int_a^b |f'(t)|\, dt.$$

This is a particular case of the relation established in Corollary 2 of Theorem 24.6 (for $n = 1$).

3. On the other hand, here is a simple example of a continuous function which is differentiable everywhere on $[a, b]$ and has infinite total variation on $[a, b]$:

The function f in question is defined on $[-1, 1]$ by

$$f(0) = 0; \qquad f(x) = x^2 \cos^2(\pi/x^2) \qquad \text{for all} \quad x \neq 0.$$

Indeed, for every integer $n > 1$, the total variation of f on $[n^{-1/2}, 1]$ is equal to $(1 + 1/2 + \cdots + 1/n)$, and this sum tends to $+\infty$ with n.

PROBLEMS

Note: The rather difficult problems are marked with an asterisk.

THE LINE R AND THE SPACE $\mathbf{R}^n$

1. Let f be a strictly increasing surjection of an interval $[\alpha, \beta]$ to an interval $[a, b]$. Show that f is continuous and that the inverse function f^{-1} is defined and continuous on $[a, b]$ (in other words, that f is a homeomorphism).

***2.** Let A and B be countable everywhere dense subsets of $(0, 1)$.

(a) Show that there exists an infinity of increasing bijections of A to B.

(b) Show that such a bijection extends in a unique way to a homeomorphism of $[0, 1]$ with itself.

3. Show that there does not exist any countable partition of the interval $[0, 1]$ into nonempty closed subsets.

4. Let $0.a_1a_2 \cdots a_n \cdots$ denote the reduced decimal expansion of an $x \in [0, 1)$; put $S_n(x) = a_1 + a_2 + \cdots + a_n$, and let $\lambda \in [0, 9)$. Show that the set of x such that $S_n \leqslant \lambda n$ for every n is compact, has no isolated points, and does not contain any interval.

5. Show that every isometry of $\mathbf{R}$ into $\mathbf{R}$ is of the form $x \to a + x$ or $x \to a - x$.

6. Show that in $\mathbf{R}^n$ $(n \geqslant 2)$ the complement of every compact interval is connected.

7. Let D be a domain of $\mathbf{R}^n$ $(n \geqslant 2)$.

(a) Show that for any distinct points a, b, c of D, there exists a polygonal line in D containing a and b but not c.

(b) Deduce from this that D is not homeomorphic to any subset of $\mathbf{R}$.

8. Show that every simple arc of $\mathbf{R}^n$ is nondense on $\mathbf{R}^n$ (use Problem 7).

9. Show that in $\mathbf{R}^n$ the set A obtained from the sphere $\Sigma\, x_i^2 = 1$ by removing the point $(0, ..., 0, 1)$ is homeomorphic to $\mathbf{R}^{n-1}$. (Start with $n = 2$ or 3 by using a stereographic projection and generalize this procedure.)

10. Show that every open ball of $\mathbf{R}^n$ is homeomorphic to $\mathbf{R}^n$.

11. Show that every compact convex subset of $\mathbf{R}^2$ is either a segment, or is homeomorphic to a closed disk. State and prove a similar result for $\mathbf{R}^n$.

12. Let ω be an open set in $\mathbf{R}^n$ containing O. Let E be the set of $x \in \omega$ such that $[\mathrm{O}, x] \subset \omega$ (the *star* of O in ω); show that E is open and intersects every halfline δ from the origin in an interval.

For every δ, put $\varphi(\delta) =$ length of $\delta \cap \mathrm{E}$. Show that φ is a lower semicontinuous function of δ (see Chapter II).

13. Let f be an isometric surjection of $\mathbf{R}^n$ to a subset A of $\mathbf{R}^n$.

(a) Show that the image $f(\Delta)$ of every line Δ in $\mathbf{R}^n$ is a line.

(b) Deduce from this that f is an affine transformation, and that $\mathrm{A} = \mathbf{R}^n$.

TOPOLOGICAL SPACES

14. Let E be a topological space containing a countable everywhere dense subset. Show that every family of nonempty open and disjoint subsets of E is finite or countable.

***15.** Consider the family $\mathcal{O}$ of subsets of $\mathbf{R}$ of the form:

$$\mathrm{X} = (\text{open set of } \mathbf{R}) - (\text{finite or countable set of points}).$$

Show that $\mathcal{O}$ is the family of open sets for a topology $\mathcal{T}$ on $\mathbf{R}$, and study this topology. In particular, determine the compact sets of $\mathcal{T}$.

16. Let A be a closed subset of $\mathbf{R}^2$; let S be the set of points x of A which have in A a neighborhood contained in an angle $< \pi$ and with vertex x. Show that S is finite or countable.

17. Let X^* denote the boundary of a subset X of a topological space E; show that

$$X^* = (X \cap (\overline{\complement X})) \cup (\overline{X} \dot{-} X).$$

***18.** Let E be a topological space, and let X be a subset of E. The *sequential adherence* of X is defined as the set of points of E which are limits of a sequence of points of X; X is said to be *sequentially closed* if it is identical with its sequential adherence. Show by examples that the sequential adherence of A is not always sequentially closed (use Problem 107).

What might one call the *sequential closure* of X? Does it always exist?

19. Let E be a topological space, and X a closed subset of E. The set $X^{(1)}$ of accumulation points of X is called the *derived set* of X; the successive derived sets $X^{(n)}$ are defined by the condition that $X^{(n+1)}$ is the derived set of $X^{(n)}$. We then set

$$X^{\omega} = \bigcap X^{(n)}.$$

Construct a closed set A in $\mathbf{R}$ such that $A^{(n)}$ (respectively, A^{ω}) consists of a single point.

20. Let f be a mapping of a topological space E into $\mathbf{R}$; show that if for every $\lambda \in \mathbf{R}$, $\{x : f(x) < \lambda\}$ and $\{x : f(x) > \lambda\}$ are open, then f is continuous. More generally, let f be a mapping of E into a topological space F, and let (ω_i) be a family of open sets in F such that every open set in F is the union of finite intersections of the open sets ω_i (the family (ω_i) is said to *generate* the topology of F). How can one characterize the continuous mappings f in terms of the open sets ω_i?

21. A mapping f of a topological space E into a topological space F is said to be *open* if $f(\omega)$ is an open set in F for every open set ω in E.

(a) Show that the projection of a product space on each of the factor spaces is an open mapping.

(b) Characterize the continuous mappings of $\mathbf{R}$ (respectively, $\mathbf{R}^n$) into $\mathbf{R}$ which are open.

(c) Show that every holomorphic mapping of $\mathbf{C}$ into $\mathbf{C}$ which is not constant, is open.

22. Let f be a continuous mapping of a topological space E into $\mathbf{R}$. Show that for every open set ω in $\mathbf{R}$, $f^{-1}(\omega)$ is an open F_σ set (see Problem 73).

23. Let E be a topological space with a *countable base* (see Problem 88), and let f be a numerical function on E. Show that if M denotes the set of $x \in$ E at which f has a local maximum, then f(M) is finite or countable.

24. Let E be a separated topological space with a countable base; show that the set P of points of E which do not have any countable neighborhood is perfect (i.e., closed and without isolated points). Show that E $\dot{-}$ P is at most countable.

25. Show that the family of open (and of closed) sets in **R** has the cardinality of the continuum; same problem for $\mathbf{R}^n$. Deduce from this the cardinality of the family of continuous mappings of $\mathbf{R}^n$ into $\mathbf{R}^p$.

26. We say that a quadratic form $\sum a_{ij}x_i x_j$ with real coefficients and in n variables is positive definite if it is > 0 for every $(x_i) \neq 0$. Show that the set of points $a = (a_{ij})$ in $\mathbf{R}^{\mathrm{N}}$, where $\mathrm{N} = n(n - 1)/2$, representing these forms is open.

27. Does there exist a continuous numerical function on [0, 1] such that $f(x)$ is rational for x irrational, and irrational for x rational?

28. Find two subsets X, Y of **R**, each of which is the image of the other under a continuous bijection, without being homeomorphic.

29. Let X be a subset of a topological space E. Under what conditions is the characteristic function of X continuous?

30. Let X, Y be subsets of a separated topological space E, such that E = X $\cup$ Y. When the restrictions of f to X and Y are continuous, is f continuous? Examine the case where X and Y are closed (or open).

31. Show that for every topological space E the diagonal Δ of E $\times$ E is homeomorphic to E.

32. Let f be a continuous mapping of a topological space E into a topological space F, and let Γ be the graph of f in E $\times$ F. Show that E and Γ are homeomorphic.

33. Let E be a separated topological space, and let f be a continuous mapping of E into itself. Show that the set I of points $x \in$ E such that $f(x) = x$ is closed.

Give various examples for which I is empty and E is compact. Show that then $d(x, f(x))$ is always greater than some constant $k > 0$.

Show that I is nonempty if E is a finite union of segments of $\mathbf{R}^2$ containing O.

34. Show that in order for a space E to be separated, it is necessary and sufficient that the diagonal of E^2 be closed in E^2.

35. Let f and g be continuous mappings of a space X into a *separated* space Y. Show that the set of points $x \in E$ such that $f(x) = g(x)$ is closed.

36. Let f and g be continuous mappings of a topological space E into a separated topological space F. If the restrictions of f and g to an everywhere dense subset of E coincide, show that $f = g$.

37. Let $(a_{m,n})$ be a mapping of $\mathbf{N}^2$ into a separated topological space. Show that if

$$\lim_{m\to\infty, n\to\infty} a_{m,n} = a \quad \text{and} \quad \lim_{m\to\infty} a_{m,n} = b_n\,, \quad \text{then} \quad \lim_{n\to\infty} b_n = a.$$

38. Let E be a separated topological space, and let (a_n) be a sequence of points of E which converges to a. Show that the set $\{a, a_1, a_2, ...\}$ is compact.

39. Construct a mapping $(x, y) \to f(x, y)$ of $\mathbf{R}^2$ into $\mathbf{R}$ such that:

(1) The partial mappings $x \to f(x, y)$ and $y \to f(x, y)$ are continuous.

(2) The set of points of discontinuity of f is everywhere dense on $\mathbf{R}^2$.

40. Let E be a topological space, and (ω_i) a base of open sets for E (see Problem 88). Show that if every covering of E by sets ω_i contains a finite subcovering, then E is compact.

41. Let (K_n) be a decreasing sequence of nonempty compact sets in a separated space E. Show that $K = \bigcap K_n$ is nonempty, and that for every open set ω containing K, there exists a K_n contained in ω.

42. Let E be an infinite compact space; we denote by Δ the diagonal of $E \times E$. Show that there exists a countable subset A of $E \times E$ such that

$$A \cap \Delta = \emptyset \quad \text{and} \quad \bar{A} \cap \Delta \neq \emptyset.$$

43. We denote by $\prec$ the order relation on the set $E = [0, 1]^2$ defined by

$$(x, y) \prec (x', y') \quad \text{when either} \quad x < x', \quad \text{or} \quad x = x' \text{ and } y \leqslant y'.$$

(a) Show that this is a total ordering (the lexicographic order).

(b) Show that E, taken with the topology associated with this order, is compact.

44. Let (A_n) be a decreasing sequence of subsets of $\mathbf{R}$, each of which is a finite union of pairwise disjoint closed intervals. We assume that each of the intervals making up A_n contains exactly two of the intervals which make up A_{n+1}, and that the diameter of these intervals tends to 0 with $1/n$. Show that the set $A = \bigcap A_n$ is compact and without isolated points; show also that any two such sets A are homeomorphic.

45. Show that in every separated space X, if two compact subsets A and B are disjoint, then they have disjoint neighborhoods. (Start by treating the case where B consists of a single point.)

46. Let X be a compact space, and Y an arbitrary topological space. Show that if A is closed in $X \times Y$, its projection on Y is closed.

47. Let f be a mapping of a space X into a *separated* space Y.

(a) Show that if f is continuous, the graph of f is closed in $X \times Y$. Show by an example (where $X = Y = \mathbf{R}$) that the converse is false.

(b) Show, on the other hand, that if Y is compact, these two assertions are equivalent.

48. Let L and X be compact spaces, and let f be a continuous mapping of $L \times X$ into a separated space Y such that, for every $\lambda \in L$, the mapping $x \to f(\lambda, x)$ of X into Y is injective. Let $y_0 \in Y$.

(1) Show that the set L_0 of $\lambda \in L$ such that the equation $y_0 = f(\lambda, x)$ has a solution is closed in L.

(2) Show that the solution $x = \varphi(\lambda)$ of this equation is a continuous function of λ on L_0.

49. Let C_1 and C_2 be circles in $\mathbf{R}^3$. Show, making use of the properties of the maximum of the distance between a point of C_1 and a point of C_2, that there exists at least one line which meets C_1, C_2 and their axes.

50. In the plane $\mathbf{R}^2$ referred to two arbitrary axes, a line is defined by its equation $ax + by + c = 0$, where a and b are not both zero. Using this form, establish a correspondence between the set D of these lines and a set of lines in $\mathbf{R}^3$ passing through the origin. What natural topology on D can one deduce from this; does this topology depend on the axes chosen?

What element can be added to D to make it compact? Indicate other spaces homeomorphic to this compact set.

Will this procedure work for the lines in $\mathbf{R}^3$?

51. Show that if the product $X \times Y$ is compact, then X and Y are compact. What is the analogous statement when $X \times Y$ is locally compact?

52. Let $\Delta_1, \Delta_2, \Delta_3$ be lines in $\mathbf{R}^3$, no two of which are parallel or perpendicular to each other. Show that among the triangles (x_1, x_2, x_3) where $x_i \in \Delta_i$ $(i = 1, 2, 3)$, there is one whose area (respectively, perimeter) is minimum.

53. Show that the set of points in $\mathbf{R}^2$ which have at least one irrational coordinate is connected.

54. Let (C_n) be a decreasing sequence of continua in a topological space E. Show that their intersection $\mathrm{C} = \cap\, \mathrm{C}_n$ is also a continuum.

55. Show that the graph of the function $y = \sin 1/x$ $(x \in (0, 1])$ is connected. Determine its closure and show that it is a continuum which is not a simple arc.

56. Let G be an open set in $\mathbf{R}^n$.

(a) Show that each of the connected components of G is open (therefore also a domain).

(b) Show that the family of these connected components is at most countable.

57. Let A be a totally ordered set, taken with the topology associated with this order. Show that if A is connected, every subset of A which is bounded from above has a supremum, and that no point of A has a successor (y is the successor of x if $x < y$ and $(x, y) = \emptyset$). Is the converse true?

***58.** Let f be a homeomorphism of [0, 1] with itself. (1) Show that f either leaves the endpoints fixed, or interchanges them. (2) If $f^2 =$ the identity, show that f is either the identity, or, in a sense to be made precise, equivalent to a central symmetry.

59. Let f be a continuous mapping of [0, 1] into itself; show that f has at least one fixed point.

60. Let A be a closed subset of $[0, 1] \times [0, 1]$ such that, for every $x \in [0, 1]$, the set of y for which $(x, y) \in \mathrm{A}$ is a closed interval l_x. Show that there exists an x such that $x \in l_x$.

61. Let E be a topological space and let A be a subset of E. Show that every connected subset of E which meets $\mathring{\mathrm{A}}$ and $\mathring{\complement}\mathrm{A}$ also meets the boundary of A.

METRIC SPACES

62. Let E be the metric space obtained by taking the sphere $x^2 + y^2 + z^2 = 1$ of $\mathbf{R}^3$ with the geodesic distance. For every $m \in \mathrm{E}$

and every number $\rho > 0$, calculate the diameter $\delta(\rho)$ in E of the "circle" with center m and radius ρ. Show that $\delta(\rho)$ is not an increasing function of ρ.

***63**. We use the notation of the problem of Chapter I (Volume 1) concerning the Cantor-Bernstein theorem.

We now assume that A and E are topological spaces (respectively, metric spaces) and that φ, ψ are homeomorphisms (respectively, isometries). To what assertions does the proof suggested in that problem lead?

64. Let (a_n) be a sequence of points of a metric space. Show that the set of adherent points of this sequence is identical with the set of limits of convergent subsequences of (a_n).

65. Let f be a mapping of a metric space E into a topological space F. Show the equivalence of the continuity of f at a point $a \in \mathrm{E}$ with the following property:

For every sequence (a_n) in E such that $\lim a_n = a$, we have $\lim f(a_n) = f(a)$.

66. Let f be a mapping of a topological space X into a metric space Y. For every $x_0 \in \mathrm{X}$ the *oscillation* of f at x_0 is the positive number $\omega(f, x_0) =$ infimum of diameter of $f(\mathrm{V})$, where V runs through the family of neighborhoods of x_0.

(a) Show that the continuity of f at x_0 is equivalent to $\omega(f, x_0) = 0$.

(b) Show that for every $\epsilon > 0$, the set of points x of E at which $\omega(f, x) \geqslant \epsilon$ is closed.

67. Let E be a metric space, and let $(m, n) \to a_{mn}$ be a mapping of $\mathbf{N}^2$ into E. Assume that

$$\lim_{n\to\infty} a_{mn} = a_m \qquad \text{and} \qquad \lim_{m\to\infty} a_m = a.$$

Show that there exists a subsequence $n \to p_n$ of $\mathbf{N}$ such that

$$\lim_{n\to\infty} a_{n,p_n} = a.$$

Does this result extend to the case of any topological space E?

68. Let X be a metric space and A a subset of X. For every number $\rho > 0$, we denote by $\mathrm{B}(\mathrm{A}, \rho)$ the set of $x \in \mathrm{X}$ such that $d(x, \mathrm{A}) < \rho$.

(a) Show that $\mathrm{B}(\mathrm{A}, \rho) = \bigcup_{x\in \mathrm{A}} \mathrm{B}(x, \rho)$.

(b) Show that $\bar{\mathrm{A}} = \bigcap_{\rho} \mathrm{B}(\mathrm{A}, \rho)$.

69. Let X be a metric space and A, B *closed* subsets of X. We denote by D_A, D_B, I, respectively, the set of points x such that

$$d(x, A) < d(x, B); \qquad d(x, A) > d(x, B); \qquad d(x, A) = d(x, B).$$

(a) Show that D_A and D_B are open. Deduce from this that A and B have disjoint open neighborhoods.
(b) Show that I is closed, and determine this set when X is a plane and when A, B denote lines, circles, or closed disks in X.

70. Show that every metric space is homeomorphic to a bounded metric space. Is there an analogous statement for the uniform structure?

71. Construct examples of continuous (and even uniformly continuous) bijections of one metric space E to another one F, which are not homeomorphisms.

***72.** Let f be a mapping of an everywhere dense subset A of a topological space E into a metric space F such that $\lim_{x \in A, x \to a} f(x)$ exists for every $a \in E$. Show that there exists a unique extension of f to all of E which is a continuous mapping of E into F.

73. We shall say that a subset X of a topological space is an F_σ (respectively, G_δ) set if X is a countable union of closed sets (respectively, a countable intersection of open sets).

(a) Show that the class of F_σ (respectively, G_δ) sets is closed under countable union (respectively, countable intersection).
(b) Show that the complement of an F_σ set is a G_δ set, and vice versa.
(c) Give examples of subsets of **R** which are F_σ sets and which are neither open nor closed.
(d) Show that in a metric space, every open set is an F_σ, and every closed set a G_δ.

74. Let E be a set with an ecart d and with the topology associated with this ecart. Show that for every $X \subset E$, the mapping $x \to d(x, X)$ is continuous.

Show that

$$\bar{X} = \{x : d(x, X) = 0\}.$$

75. If a numerical function f is uniformly continuous in $\mathbf{R}^n$, show that $|f(x)| \leqslant a|x| + b$, where a and b are positive constants and $|x|$ is the distance from x to the origin.

76. For every metric space E, show that the metric $d(x, y)$ is uniformly continuous on E^2.

77. Construct an example of a numerical function on [0, 1] which has a finite derivative everywhere, and yet which is not of Lipschitz class.

78. Let E, F be metric spaces, and Φ a family of mappings of E into F. We assume that for every $x \in E$, every $f \in \Phi$, and every $\epsilon > 0$ there exists $\eta(x, f, \epsilon) > 0$ such that

$$(d(x, y) < \eta(x, f, \epsilon)) \Rightarrow (d(f(x), f(y)) < \epsilon).$$

Interpret the meaning of the assertion that η does not depend on ϵ, or does not depend on x, or does not depend on f, or does not depend on either f or x.

79. Let E be a metric space, and $A \subset E$. Show that the compactness of $\bar{A}$ is equivalent to each of the following conditions:

(a) Every sequence of points of A contains a subsequence which converges in E.

(b) Every infinite subset of A has at least one accumulation point in E.

80. Let (F_i) be a finite family of closed sets in a compact metric space whose intersection is empty. Show that there exists an $\epsilon > 0$ such that every set which meets all the F_i has diameter at least equal to ϵ.

***81.** Let K be a compact set in a metric space E; we denote by $B(\epsilon)$ the set of $x \in E$ such that $d(x, K) < \epsilon$. Show that the $B(\epsilon)$ form a base of neighborhoods of K. Does this conclusion extend to noncompact sets K?

82. Let E be a compact metric space. Show that every metric subspace A of E which is isometric to E is identical with E. Show by a simple example that this is not always true when E is not compact.

83. Let X and Y be compact metric spaces, f a mapping of X into Y, and g a mapping of Y into X. Show that if f and g are isometries, then $f(X) = Y$ and $g(Y) = X$.

84. Let E be an incomplete metric space. Show that one can define continuous functions f of each of the following types on E:

(a) f is continuous and unbounded.

(b) f is continuous and bounded, but not uniformly continuous.

Then study the same questions when E is only noncompact.

85. We have emphasized in this chapter that the pointwise convergence of a sequence of functions does not in general imply its uniform convergence. Prove, however, that this implication is true in each of the following circumstances:

(a) X is a compact space and (f_n) is an *increasing* sequence of lower semicontinuous mappings of X into $\mathbf{R}$ (that is, $f_p(x) \leqslant f_q(x)$ for every $x \in X$ if $p \leqslant q$) which converges pointwise to a *continuous* function f (this result extends Dini's theorem).

(b) $X = [0, 1]$ and (f_n) is a sequence of increasing mappings of X into $\mathbf{R}$ (not necessarily continuous) which converges pointwise to a continuous function f.

86. Let X and Y be metric spaces, with X compact. Let f and f_n $(n \in \mathbf{N})$ be continuous mappings of X into Y. Show that if, for every $x \in X$, $d(f(x), f_n(x))$ *decreases* to 0, then (f_n) converges uniformly to f.

More generally, show that if there exists a constant $k > 0$ such that for all integers p, q and for every x,

$$d(f(x), f_{p+q}(x)) \leqslant k\, d(f(x), f_p(x)),$$

then the pointwise convergence of (f_n) to f implies uniform convergence. This result extends Dini's theorem in two directions.

87. Show that every compact metric space is finite or countable, or else has the cardinality of the continuum.

88. A *base* of a topological space E is a family of open sets in E such that every open set in E is the union of a subfamily of the base. Show that:

(a) If E has a countable base, then every open covering of E has a countable subcovering.

(b) If E has a countable base, so does every subspace of E.

(c) If the spaces E_1 and E_2 have countable bases, so does their product $E_1 \times E_2$.

(d) If E has a countable base, there exists a countable everywhere dense subset D of E.

(e) Every compact metric space has a countable base.

(f) $\mathbf{R}^n$ has a countable base.

89. Put a natural topology on the collection of halflines from the origin O in $\mathbf{R}^n$; same problem for the lines containing O. Show that the spaces obtained are compact and metrizable.

90. We denote by E (respectively E′) the collection of circles in the Euclidean plane $\mathbf{R}^2$ of radius $\geqslant 0$ (respectively, > 0). (Every circle in the plane is thus represented by a point of E).

(a) One can show, for a natural topology which it is required to specify, that E is homeomorphic to a closed halfspace of $\mathbf{R}^3$; what does E′ become under this homeomorphism?

(b) Let f be the mapping of E into $\mathbf{R}^2$ which associates, with every circle C, its center $f(C)$. Show that f is continuous. Let A be a closed subset of $\mathbf{R}^2$. What can be said of the set of elements of E whose centers belong to A, and of the set of elements of E such that the corresponding circumference meets A?

(c) Show that the set of elements of E for which the circumference is contained in a compact set K in $\mathbf{R}^2$ is compact; show that among these elements there exists one whose radius is maximum.

91. Show that for every Cauchy sequence (a_n) in a metric space, the set $\{a_1, a_2, \ldots\}$ is bounded. Show by an example that the converse is false, and that there can even exist bounded sequences none of whose subsequences is a Cauchy sequence.

92. Let E be a metric space, (a_n) a Cauchy sequence in E, and (b_n) a sequence of numbers > 0. Show that there exists a subsequence (a_{n_p}) of the given sequence such that $d(a_{n_p}, a_{n_{p+1}}) < b_p$ for all p.

93. Let E and F be metric spaces, and let f be a bijection of E to F. Show that if F is complete, if f is uniformly continuous, and if f^{-1} is continuous, then E is also complete.

94. Show that every metric space in which every closed ball is compact, is complete; show that its compact subsets are closed bounded sets.

95. Let E be a complete metric space; show that one criterion for the compactness of E is that for every $\epsilon > 0$ there exists a finite covering of E by sets of diameter $< \epsilon$.

96. Let f be a continuous mapping of a metric space E into a metric space F which is uniformly continuous on every bounded subset of E.

(a) Show that for every Cauchy sequence (a_n) in E, $(f(a_n))$ is a Cauchy sequence in F.

(b) If $E \subset E'$ with E everywhere dense on E′, and if F is complete, show that f can be extended in a unique way to a continuous mapping of E′ into F.

97. Show that if the product $X \times Y$ of two metric spaces is complete, then X and Y are complete.

98. If E is an arbitrary set and F is a metric space, show that the subset $\mathscr{B}(E, F)$ of $\mathscr{F}(E, F)$ consisting of the bounded mappings of E into F is both open and closed in $\mathscr{F}(E, F)$.

99. Show that the subspace F′ of $\mathscr{F}$(E, F) consisting of the constant mappings of E into F is closed in $\mathscr{F}$(E, F) and isometric to F.

100. Show that for every set E, the mapping $u \to \sup u(x)$ of $\mathscr{B}(\mathrm{E}, \mathbf{R})$ into $\bar{\mathbf{R}}$ is continuous.

101. Let E be a compact metric space, and F a metric space with a countable base. Show that $\mathscr{C}$(E, F) is a space with a countable base.

The following three problems lead to the basic results of the theory of *category*.

***102.** Let E be a complete metric space, and let (G_n) be a sequence of open sets in E, each of which is everywhere dense on E ($\overline{\mathrm{G}_n} = \mathrm{E}$).

(a) Show that the set $\mathrm{G} = \bigcap \mathrm{G}_n$ is also the intersection of a *decreasing* sequence of open everywhere dense subsets of E.

(b) Deduce from this that G is nonempty, and more precisely that G is everywhere dense on E.

103. Let E be a complete metric space. Deduce from Problem 102 that it is not possible to have

$$\mathrm{E} = \bigcup_n \mathrm{A}_n, \qquad \text{where each } \mathrm{A}_n \text{ is nondense on E.}$$

104. Let E be a complete metric space. Deduce from Problem 103 that if

$$\mathrm{E} = \bigcup_n \mathrm{A}_n,$$

where each A_n is closed, then there exists at least one A_n whose interior $\mathring{\mathrm{A}}_n$ is nonempty, and more precisely that the open set $\mathrm{G} = \bigcup \mathring{\mathrm{A}}_n$ is everywhere dense on E.

105. For every rational number $x \in [0, 1)$, we put $x = p/q$, where p and q are mutually prime integers. Let n be an integer $\geqslant 3$ and let $i(x)$ be the open interval of $\mathbf{R}$ with center x and halflength $1/q^n$. Set

$$\mathrm{G}_n = \bigcup_x i(x).$$

(a) Show that for every n, the sum l_n of the lengths of the $i(x)$ is finite and $l_n \to 0$.

(b) Show that $\mathrm{G} = \bigcap \mathrm{G}_n$ contains other points beside the rationals. Give an example of such a point.

106. Let E be a countable set whose points are $a_1, a_2, \ldots$ Put

$$d(a_p, a_p) = 0; \qquad d(a_p, a_q) = 10 + 1/p + 1/q \qquad \text{if} \quad p \neq q.$$

(a) Show that d is a metric, and that E is complete in this metric.

(b) Let f be the mapping of E into E such that $f(a_p) = a_{p+1}$. Show that f is strictly distance decreasing and that yet f has no fixed point.

(c) By slightly modifying this example, construct a complete metric space F and a mapping f of F into itself which is strictly distance decreasing, and which has a fixed point a, and such that nevertheless $(f^{(n)}(x))$ does not tend to a for any $x \neq a$.

***107.** Let E be the collection of arbitrary mappings of [0, 1] into itself. For every $f_0 \in$ E, integer $n > 0$, points $x_1, \ldots, x_n$ of [0, 1] and $\epsilon > 0$, the set $V(f_0, \epsilon; x_1, \ldots, x_n)$ consisting of all f such that $|f(x_i) - f_0(x_i)| < \epsilon$ $(i = 1, \ldots, n)$ is called an *elementary set* with center f_0. Every union of elementary sets is called an "open" set in E.

(a) Verify that these "open" sets satisfy axioms O_1, O_2, O_3, O_4 and therefore define a topology on E.

(b) If we call every function f which is everywhere zero on [0, 1] except at a finite number of points a *simple* function, show that the set of simple functions is everywhere dense on E.

(c) Show that a function which is nonzero for an uncountable infinity of values of x cannot be the limit in E of a *sequence* of simple functions.

(d) Deduce from this that one cannot define a metric on E for which the associated topology is that of E; in other words, E is not metrizable.

(e) Show that every simple function is the limit of a sequence of continuous functions; that the function g which equals 1 on the rationals and 0 everywhere else is the limit of a sequence of simple functions; that g, however, is not the limit of a sequence of continuous functions.

108. Let $\mathscr{F}$ be an equicontinuous family of (continuous) numerical functions on a metric space E, and let f be the upper envelope of the element of $\mathscr{F}$.

Show that if E is connected, then f is everywhere finite or everywhere $+\infty$; show that if f is finite, it is uniformly continuous.

109. Let f be a uniformly continuous numerical function on $\mathbf{R}$. Put

$$f_a(x) = f(x - a).$$

Show that the set of functions f_a is equicontinuous on $\mathbf{R}$. Deduce from this that on every compact set K in $\mathbf{R}$, the set of oscillations of the f_a is bounded from above.

110. Let A be a nonempty subset of a metric space E and let f be a mapping of A into $\mathbf{R}$ which is of Lipschitz class with ratio k. For every $x \in \mathrm{E}$ and every $y \in \mathrm{A}$ we put

$$f_y(x) = f(y) + k\, d(x, y).$$

Show that g, defined by

$$g(x) = \inf_{y \in \mathrm{A}} f_y(x),$$

is finite for every x, and of Lipschitz class with ratio k, and that its restriction to A is identical with f.

111. Let E be a compact metric space, and let J be a compact subset of the space $\mathscr{C}$ of continuous numerical functions on E, taken with the metric of uniform convergence. Show that for every $\epsilon > 0$ there exists a constant $k > 0$ with the following properties:

For every $\alpha \in \mathrm{J}$, there exists $\beta \in \mathscr{C}$ such that $\| \alpha - \beta \| \leqslant \epsilon$, with β of Lipschitz class with ratio k. (Apply the preceding problem, taking for A a suitable finite subset of E.)

***112.** Let f be a continuous numerical function of bounded variation on [0, 1] and let $g(x)$ be the total variation of f on $[0, x]$.

Show that the graphs of f and g (in Cartesian coordinates) have the same length.

Show that the surfaces of revolution about the y axis generated by these graphs have the same area, and more precisely are "isometric" (by a homeomorphism preserving the lengths of curves).

113. Let

$$f(x) = \sum_n 2^{-n} \sin(10^n x);$$

show that the total variation of f on every interval of $\mathbf{R}$ is infinite.

DEFINITIONS AND AXIOMS

AXIOMS OF TOPOLOGICAL SPACES

O_1 : The empty set and the entire space are open sets.
O_2 : Every *finite* intersection of open sets is an open set.
O_3 : Every union of open sets is an open set.

The complement of an open set is called a *closed* set.

PROPERTIES OF CLOSED SETS

F_1 : The entire space and the empty set are closed sets.
F_2 : Every *finite* union of closed sets is a closed set.
F_3 : Every intersection of closed sets is a closed set.

Every subset of a space E which contains an open set containing a given point x is called a *neighborhood* of x.

SEPARATED SPACE. A space E is said to be *separated* if it satisfies the following axiom:

O_4 : Any two distinct points of E have disjoint neighborhoods.

PRODUCT OF SPACES. If E_1, E_2 are topological spaces, the topological product space $E = E_1 \times E_2$ is the space obtained by taking, as the open subsets of the set $E_1 \times E_2$, all unions of elementary sets $\omega_1 \times \omega_2$, where ω_i is an open set in E_i ($i = 1, 2$).

COMPACT SPACE. A space E is said to be *compact* if it is separated and moreover satisfies the axiom for open coverings:

O_5 : Every open covering of E has a finite subcovering.

CONNECTED SPACE. A space E is said to be connected if E and $\emptyset$ are the only subsets which are both open and closed sets in E.

TOPOLOGICAL GROUP, RING AND FIELD. A *topological group* is a group with a topology for which the functions (x^{-1}) and (xy) are continuous.

A *topological ring* is a ring with a topology for which the functions $(-x)$, $(x + y)$, (xy) are continuous.

A *topological field*: Same conditions as for rings, with in addition the continuity of x^{-1} for all $x \neq 0$.

METRIC SPACES. A *metric* space is a set E with which there is associated a *metric* ($d(x, y)$), that is, a numerical function defined on E^2 such that:

$M_1 : d(x, y) > 0$ if $x \neq y$; $d(x, x) = 0$.
$M_2 : d(x, y) = d(y, x)$ (symmetry).
$M_3 : d(x, y) \leqslant d(x, z) + d(z, y)$ (triangle inequality).

A metric space E is said to be *complete* if every Cauchy sequence in E is convergent.

TRACE, INJECTION, SURJECTION, BIJECTION. The *trace* of a subset A (a family $\mathscr{F}$ of subsets) of a space E on a subset B of E is the set $A \cap B$ (the family of sets $F \cap B$, $F \in \mathscr{F}$).

A mapping $f : X \rightarrow Y$ is called an *injection* if $x_1, x_2 \in X$, $x_1 \neq x_2$, implies $f(x_1) \neq f(x_2)$, a *surjection* if $f(X) = Y$, and a *bijection* if it is both an injection and a surjection.

REVIEW OF SOME CLASSICAL NOTATION

The symbols **N**, **Z**, **Q**, **R**, **C** denote,respectively, the set $\{0, 1, 2, ...\}$ of natural integers, the set of integers of arbitrary sign, the set of rational numbers, the set of real numbers, the set of complex numbers.

The symbols $\mathbf{Q}_+$, $\mathbf{R}_+$ denote the set of element $\geqslant 0$ of **Q** and **R**, respectively.

The symbols $\mathbf{N}^*$, $\mathbf{Z}^*$, $\mathbf{Q}^*$, $\mathbf{R}^*$, $\mathbf{C}^*$ denote, respectively, **N**, **Z**, **Q**, **R**, **C** with their element 0 deleted.

Finally, $\bar{\mathbf{R}}$ denotes the set $[-\infty, +\infty]$ obtained from **R** by adjoining two points at infinity.

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