CHAPTER II

Numerical Functions

By a *numerical* function on a set E we shall mean a mapping of E into the extended line $\bar{\mathbf{R}} = [-\infty, +\infty]$; the mappings into $\mathbf{R}$ will be called *finite* numerical functions, or simply numerical functions when it is clear from the context that they are finite.

In this chapter we shall study the special properties of numerical functions which result from the fact that $\mathbf{R}$ is a field and an ordered set; in particular we shall study certain classes of mappings of an interval of $\mathbf{R}$ into $\mathbf{R}$.

1. NUMERICAL FUNCTIONS DEFINED ON AN ARBITRARY SET

1. ORDER RELATION ON $\mathscr{F}(\mathrm{E}, \mathbf{R})$ AND ON $\mathscr{F}(\mathrm{E}, \bar{\mathbf{R}})$

We shall denote by $\mathscr{F}(\mathrm{E}, \bar{\mathbf{R}})$ (respectively, $\mathscr{F}(\mathrm{E}, \mathbf{R})$) the set of numerical (respectively, finite numerical) functions on E.

The set $\mathscr{F}(\mathrm{E}, \mathbf{R})$ can evidently be given an algebraic structure. It possesses an order structure as well which is compatible with its algebraic structure:

For every $f \in \mathscr{F}(\mathrm{E}, \mathbf{R})$ we shall say that f is *positive*, and write $f \geqslant 0$, if $f(x) \geqslant 0$ for every $x \in \mathrm{E}$.

It is immediate that if $f \geqslant 0$, $g \geqslant 0$, and if $\lambda \in \mathbf{R}_+$, then

$$f + g \geqslant 0, \quad fg \geqslant 0, \quad \text{and} \quad \lambda f \geqslant 0.$$

More generally, for all $f, g \in \mathscr{F}(\mathrm{E}, \mathbf{R})$ we put

$$f \leqslant g \quad \text{if} \quad g - f \geqslant 0$$

(which is equivalent to $f(x) \leqslant g(x)$ for every $x \in \mathrm{E}$). It is immediate that the relation $\leqslant$ is an order relation on $\mathscr{F}(\mathrm{E}, \mathbf{R})$ (a nontotal order if E contains more than one point); and one can verify that for all $f, g, h \in \mathscr{F}(\mathrm{E}, \mathbf{R})$ and for every $\lambda \in \mathbf{R}_+$, the relation $f \leqslant g$ implies

$$f + h \leqslant g + h; \quad \lambda f \leqslant \lambda g; \quad fh \leqslant gh \quad \text{if} \quad h \geqslant 0.$$

Similarly, one can verify that the relation defined on $\mathscr{F}(E, \bar{\mathbf{R}})$ by

$$f \leqslant g \quad \text{if} \quad f(x) \leqslant g(x) \quad \text{for every} \quad x \in E$$

is an order relation. It is, however, well to note that the operations in $\mathscr{F}(E, \mathbf{R})$ do not extend to $\mathscr{F}(E, \bar{\mathbf{R}})$; in particular, this last set is not a vector space.

For every $f \in \mathscr{F}(E, \bar{\mathbf{R}})$ we denote by $|f|$ the function such that $|f|(x) = |f(x)|$ for every $x \in E$.

We evidently have the relation

$$-|f| \leqslant f \leqslant |f|.$$

REMARK. It should be noted that the relation $f \geqslant 0$ does not imply that either $f(x) = 0$ for every x, or $f(x) > 0$ for every x.

2. BOUNDS OF A NUMERICAL FUNCTION

We recall that $\bar{\mathbf{R}}$ is isomorphic (for the order relations) to $[0, 1]$; it follows that every nonempty subset of $\bar{\mathbf{R}}$ has a supremum and an infimum; we can therefore give the following definition:

2.1. Definition. FOR EVERY $f \in \mathscr{F}(E, \bar{\mathbf{R}})$ AND EVERY NONEMPTY SUBSET X OF E, THE *SUPREMUM* (*INFIMUM*) OF f ON X IS DEFINED AS THE SUPREMUM (INFIMUM) OF $f(X)$ IN $\bar{\mathbf{R}}$.

THESE ARE DENOTED BY $\sup_{x \in X} f(x)$ AND $\inf_{x \in X} f(x)$.

The properties of the supremum and infimum of a subset of $\bar{\mathbf{R}}$ imply, for example, that the number $a = \sup_{x \in X} f(x)$ is characterized by the following properties:

1. For every $x \in X$ we have $f(x) \leqslant a$;
2. For every $b < a$, there exists $x \in X$ such that $b < f(x)$.

When f is everywhere finite on X, the oscillation of f on X (see Chapter I, Section 15) is equal to

$$(\sup_{x \in X} f(x) - \inf_{x \in X} f(x)).$$

When f is not everywhere finite on X, this difference will by definition be the oscillation of f on X, at least when this difference is meaningful, that is, when $f(X)$ does not consist of the single point $+\infty$ or $-\infty$.

2.2. Definition. WE SHALL SAY THAT f IS *BOUNDED FROM ABOVE* (*BELOW*) ON X WHEN ITS SUPREMUM (INFIMUM) ON X IS $< +\infty$ $(> -\infty)$.

WE SHALL SAY THAT f IS *BOUNDED* ON X IF IT IS BOUNDED FROM ABOVE AND FROM BELOW.

It is immediate that $\inf_{x \in X} f(x) = -\sup_{x \in X} (-f(x))$; this relation often makes it possible to restrict oneself to the study of the supremum.

Z To say that f is bounded on X is equivalent to saying that there exist $a, b \in \mathbf{R}$ such that $a \leqslant f(x) \leqslant b$ for every $x \in X$. Thus if f is bounded on X, it is finite on X; on the other hand, f can be finite on X without being bounded:

This is the case for the mapping $x \to x^2$ of $\mathbf{R}$ into $\mathbf{R}$.

2.3. Proposition. *Let* (f_i) *be a finite family of elements of* $\mathscr{F}(E, \mathbf{R})$; *then*

$$\sup_{x \in X} \sum_i f_i(x) \leqslant \sum_i \sup_{x \in X} f_i(x),$$

and if the f_i *are* $\geqslant 0$, *then*

$$\sup_{x \in X} \prod_i f_i(x) \leqslant \prod_i \sup_{x \in X} f_i(x).$$

We shall, for example, prove the first relation; we have

$$f_i(x) \leqslant \sup_{x \in X} f_i(x), \qquad \text{whence} \qquad \sum_i f_i(x) \leqslant \sum_i \sup_{x \in X} f_i(x),$$

which gives the desired relation.

3. UPPER AND LOWER ENVELOPES OF A FAMILY OF FUNCTIONS

Let $(f_i)_{i \in I}$ be a family of numerical functions on a set E. In order that a numerical function g be an upper bound for this family, it is necessary and sufficient that $f_i(x) \leqslant g(x)$ for every $x \in E$ and every $i \in I$. Among these functions g there thus exists one which is smaller than all the others, namely the function f defined by $f(x) = \sup_{i \in I} f_i(x)$; in other words, the ordered set $\mathscr{F}(E, \bar{\mathbf{R}})$ is a complete lattice.

3.1. Definition. THE *UPPER ENVELOPE* OF THE FAMILY OF FUNCTIONS $(f_i)_{i \in I}$, WHICH IS DENOTED BY $\sup_{i \in I} f_i$, IS DEFINED AS THE FUNCTION f SUCH THAT

$$f(x) = \sup_{i \in I} f_i(x) \qquad \text{for every} \quad x \in E.$$

THE LOWER ENVELOPE $\inf_{i \in I} f_i$ IS DEFINED SIMILARLY.

REMARK. One should note the similarity of the notations for the supremum of a function and the upper envelope of a family of functions; this similarity is in no way fortuitous, for the two formulas represent the supremum of a subset of a complete lattice, namely $\bar{\mathbf{R}}$ in the first case, and $\mathscr{F}(\mathrm{E}, \bar{\mathbf{R}})$ in the second.

EXAMPLE 1. Let f_n be the mapping $x \to \sin 2\pi n x$ of $\mathbf{R}$ into $\mathbf{R}$; the upper envelope of the family $(f_n)_{n\in \mathbf{Z}}$ is a function f such that $f(x) = 1$ if x is irrational or equal to an irreducible fraction of the form $p/4q$, and $f(x) \neq 1$ everywhere else.

EXAMPLE 2. Let φ_{A} be the characteristic function of a closed set $\mathrm{A} \subset \mathbf{R}$. This function is the lower envelope of the continuous functions which are everywhere $\geqslant \varphi_{\mathrm{A}}$.

3.2. GEOMETRIC INTERPRETATION. For every $f \in \mathscr{F}(\mathrm{E}, \bar{\mathbf{R}})$, let $\mathrm{A}(f)$ be the set of points of the product space $\mathrm{E} \times \bar{\mathbf{R}}$ which lie on or above the graph of f, in other words, the points (x, y) such that $y \geqslant f(x)$.

Specifying $\mathrm{A}(f)$ is equivalent to specifying f. It is immediate that for every family $(f_i)_{i\in\mathrm{I}}$ we have

$$\mathrm{A}(\sup_{i\in\mathrm{I}} f_i) = \bigcap_i \mathrm{A}(f_i).$$

Similarly, when I is finite,

$$\mathrm{A}(\inf_{i\in\mathrm{I}} f_i) = \bigcup_i \mathrm{A}(f_i).$$

When I is infinite, the left side of the last relation still contains the right side, but is not necessarily identical with it.

UPPER ENVELOPE IN $\mathscr{F}(\mathrm{E}, \mathbf{R})$. UNIFORMLY BOUNDED FAMILIES. (a) If $(f_i)_{i\in\mathrm{I}}$ is a finite family of finite numerical functions, its upper envelope is again finite; but if I is infinite, its upper envelope need not be finite; in order that it be finite, it is necessary and sufficient that for every x the family of numbers $(f_i(x))_{i\in\mathrm{I}}$ be bounded from above.

(b) If $(f_i)_{i\in\mathrm{I}}$ is a family of numerical functions, each of which is bounded from above, this does not imply that $\sup_{i\in\mathrm{I}} f_i$ is bounded from above.

For example, the family (f_n) of constant mappings $x \to n$ of an arbitrary E into $\mathbf{R}$ consists of bounded functions, but its upper envelope is not bounded from above.

When $\sup_{i\in\mathrm{I}} f_i$ is bounded from above, one says that the family $(f_i)_{i\in\mathrm{I}}$

is *uniformly bounded from above*; this amounts to saying that there exists a finite number λ such $f_i(x) \leqslant \lambda$ for every $i \in \mathrm{I}$ and every $x \in \mathrm{E}$.

Families *uniformly bounded from below* are defined similarly. A family which is uniformly bounded from above and from below is said to be *uniformly bounded.*

DEFINITION OF f^+ AND f^-. For every numerical function f, we put $f^+ = \sup(f, 0)$. In other words, f^+ is defined by the relation

$$f^+(x) = (f(x))^+ \qquad \text{for every} \quad x \in \mathrm{E}.$$

Similarly, we put

$$f^- = \sup(-f, 0) = (-f)^+.$$

It should be observed that we always have $f^+ \geqslant 0$ and $f^- \geqslant 0$.

It follows from the relations for real numbers

$$a = a^+ - a^-; \qquad |a| = a^+ + a^-,$$

that

$$f = f^+ - f^-; \qquad |f| = f^+ + f^-.$$

Hence

$$f^+ = \tfrac{1}{2}(|f| + f); \qquad f^- = \tfrac{1}{2}(|f| - f).$$

More generally, we have

$$\sup(f, g) = \tfrac{1}{2}[(f+g) + |f-g|]; \qquad \inf(f, g) = \tfrac{1}{2}[(f+g) - |f-g|].$$

It follows at once that

$$\sup(f, g) + \inf(f, g) = f + g.$$

II. LIMIT NOTIONS ASSOCIATED WITH NUMERICAL FUNCTIONS

With every $f \in \mathscr{F}(\mathrm{E}, \bar{\mathbf{R}})$ and with every $x \in \mathrm{E}$ there is associated the element $f(x)$ of $\bar{\mathbf{R}}$. Since $\bar{\mathbf{R}}$ is a topological space, with every sequence of elements of E or of $\mathscr{F}(\mathrm{E}, \bar{\mathbf{R}})$ or, more generally, with every filter base on E or on $\mathscr{F}(\mathrm{E}, \bar{\mathbf{R}})$ we shall therefore be able to associate limit elements.

We will first examine the notions associated with a filter base on E, and then those associated with a filter base on $\mathscr{F}(\mathrm{E}, \bar{\mathbf{R}})$.

4. LIMITS SUPERIOR AND INFERIOR OF A FUNCTION ALONG A FILTER BASE ON E

Let f be a numerical function on E, and let $\mathscr{B}$ be a filter base on E. We have defined (in Chapter I, Section 8) the adherence of f along $\mathscr{B}$ as the set

$$\mathring{f}(\mathscr{B}) = \bigcap_{B \in \mathscr{B}} \overline{f(B)}.$$

Since the family of sets $\overline{f(B)}$ has the finite intersection property (Chapter I, Section 11), and since $\bar{\mathbf{R}}$ is compact, $\mathring{f}(\mathscr{B})$ is nonempty. Since, besides, every nonempty subset of $\bar{\mathbf{R}}$ has a supremum, we can establish the following definition:

4.1. Definition. THE *LIMIT SUPERIOR* OF f ALONG THE FILTER BASE $\mathscr{B}$ IS DEFINED AS THE SUPREMUM OF THE SET $\mathring{f}(\mathscr{B})$. IT IS DENOTED BY

$$\overline{\lim_{\mathscr{B}}} f \qquad \text{or} \qquad \limsup_{\mathscr{B}} f.$$

THE LIMIT INFERIOR IS DEFINED SIMILARLY.

SPECIAL CASES. 1. Let (x_n) be a sequence of points of E; the sets $B_n = \{x_i : i \geqslant n\}$ form a filter base $\mathscr{B}$ on E. The limit superior of f along $\mathscr{B}$ is also called the limit superior of the sequence $(f(x_n))$.

2. Suppose E has a topology; let A be a nonempty subset of E, and let $a \in \bar{A}$. Let $\mathscr{B}$ denote the collection of subsets of E of the form $A \cap V$, where V is a neighborhood of a; since $a \in \bar{A}$, it is clear that $\mathscr{B}$ is a filter base.

In this case, $\limsup_{\mathscr{B}} f$ is denoted by $\limsup_{x \to a, x \in A} f(x)$. For example, if $E = [\alpha, \beta] \subset \mathbf{R}$ and if $A = (a, \beta]$, then $\limsup_{\mathscr{B}} f$ is called the limit superior of f from the right at the point a.

3. Suppose that E is an ordered set which is directed in the increasing direction, that is, every finite subset of E has an upper bound, and suppose that f is an increasing mapping of E into $\bar{\mathbf{R}}$.

If $\mathscr{B}$ denotes the collection of subsets of E of the form $\{x : x \geqslant a\}_{a \in E}$, one can verify that $\mathscr{B}$ is a filter base and that

$$\lim_{\mathscr{B}} f = \limsup_{\mathscr{B}} f = \sup_{x \in E} f(x).$$

IMMEDIATE PROPERTIES. 1. The limit superior of f along $\mathscr{B}$ belongs to the adherence of f along $\mathscr{B}$.

2. $\liminf_{\mathscr{B}} f = -\limsup_{\mathscr{B}} (-f)$.

4.2. Lemma. *Let f be a mapping of a set* E *into a compact space* X; *let* $\mathscr{B}$ *be a filter base on* E, *and let* $\bar{f}(\mathscr{B})$ *be the adherence of f along* $\mathscr{B}$.

Then for every open set ω *in* X *such that* $\bar{f}(\mathscr{B}) \subset \omega$, *there exists* $B \in \mathscr{B}$ *such that* $\overline{f(B)} \subset \omega$.

Indeed, the traces of the closed sets $\overline{f(B)}$ on the compact set $\complement\omega$ have intersection $\bar{f}(\mathscr{B}) \cap \complement\omega = \emptyset$; therefore there exists a finite family $(\overline{f(B_i)})$ of these closed sets whose intersection does not meet $\complement\omega$. But there exists $B \in \mathscr{B}$ contained in $\bigcap B_i$; this is the desired B.

4.3. Proposition. *If f is a numerical function on* E, *to say that f converges to a along a filter base $\mathscr{B}$ is equivalent to saying that*

$$a = \limsup_{\mathscr{B}} f = \liminf_{\mathscr{B}} f, \qquad \textit{or that} \qquad \{a\} = \bar{f}(\mathscr{B}).$$

Indeed, if $a = \lim_{\mathscr{B}} f$, then every closed neighborhood V of a in $\overline{\mathbf{R}}$ contains a set $f(B)$, therefore also $\overline{f(B)}$; hence $\bar{f}(\mathscr{B}) \subset V$. It follows that $\bar{f}(\mathscr{B}) = \{a\}$.

Conversely, if $\{a\} = \bar{f}(\mathscr{B})$, for every neighborhood V of a there exists by Lemma 4.2 a $B \in \mathscr{B}$ such that $\overline{f(B)} \subset V$; thus f converges to a along $\mathscr{B}$.

4.4. Definition. LET f BE A MAPPING OF A TOPOLOGICAL SPACE E INTO $\mathbf{R}$. FOR EVERY $a \in E$, THE *OSCILLATION* OF f AT THE POINT a IS DEFINED AS THE NUMBER

$$\omega(f, a) = \limsup_{x \to a} f(x) - \liminf_{x \to a} f(x).$$

This difference is always meaningful, since the set of adherent values of f at a always contains $f(a)$, which is finite.

4.5. Proposition. *Let $f \in \mathscr{F}(E, \mathbf{R})$, where* E *is a topological space. To say that f is continuous at a is equivalent to saying that the oscillation of f at a is zero.*

This is an immediate consequence of Proposition 4.3.

4.6. Proposition. *Let $f, g \in \mathscr{F}(E, \overline{\mathbf{R}})$ with $f \leqslant g$, and let $\mathscr{B}$ be a filter base on* E; *then*

$$\limsup_{\mathscr{B}} f \leqslant \limsup_{\mathscr{B}} g; \qquad \liminf_{\mathscr{B}} f \leqslant \liminf_{\mathscr{B}} g.$$

Indeed, denote the members of the first inequality by α and β, respectively.

If $\beta = +\infty$, the first relation is satisfied. If $\beta < +\infty$, Lemma 4.2 shows that for every k such that $\beta < k$, there exists $B \in \mathscr{B}$ such that $g(B) < k$, hence $f(B) < k$ and so $\overline{f(B)} \leqslant k$; therefore $\alpha \leqslant k$. Since this relation holds for all $k > \beta$, we have $\alpha \leqslant \beta$.

The second inequality is deduced from the first by replacing f by $-g$ and g by $-f$.

4.7. Corollary. *If $f \leqslant g$ and if f and g have the respective limits α and β along $\mathscr{B}$, then $\alpha \leqslant \beta$.*

Z The compactness of X is essential for the validity of Lemma 4.2; also, the consequences of this lemma which we have obtained for $\overline{\mathbf{R}}$ do not extend to $\mathbf{R}$.

For example, the finite numerical function g defined on $\mathbf{R}$ by

$$g(x) = 1/x \quad \text{if} \quad x > 0; \qquad g(x) = 0 \quad \text{if} \quad x \leqslant 0$$

has only the point 0 as an adherent value in $\mathbf{R}$ at $x = 0$, although it is not continuous at 0.

Similarly, if we denote by f the lower envelope of g and 1, then $f \leqslant g$, and yet the adherent values in $\mathbf{R}$ of f and g at the point $x = 0$ are, respectively, $\{0, 1\}$ and $\{0\}$.

5. LIMITS SUPERIOR AND INFERIOR OF A FAMILY OF FUNCTIONS

Let $(f_i))_{i \in I}$ be a family of elements of $\mathscr{F}(E, \overline{\mathbf{R}})$ and let $\mathscr{B}$ be a filter base on I.

For every $x \in E$, the mapping $\varphi_x : i \to f_i(x)$ of I into $\overline{\mathbf{R}}$ has an adherence $\overline{\varphi_x}(\mathscr{B})$ along $\mathscr{B}$.

Every mapping f of E into $\overline{\mathbf{R}}$ such that $f(x) \in \overline{\varphi_x}(\mathscr{B})$ for every $x \in E$ is called an adherent point of the family (f_i) along $\mathscr{B}$.

The set of these adherent points contains its infimum and supremum, which are simply the mappings

$$x \to \liminf_{\mathscr{B}} f_i(x) \qquad \text{and} \qquad x \to \limsup_{\mathscr{B}} f_i(x).$$

We will denote them by

$$\liminf_{\mathscr{B}}(f_i) \qquad \text{and} \qquad \limsup_{\mathscr{B}}(f_i).$$

5.1. Proposition. *Let $(f_i)_{i \in I}$ and $(g_i)_{i \in I}$ be two families of elements of $\mathscr{F}(E, \overline{\mathbf{R}})$, and let $\mathscr{B}$ be a filter base on* I.

If $f_i \leqslant g_i$ *for every* $i \in \mathrm{I}$, *then we also have*

$$\lim_{\mathscr{B}} \sup(f_i) \leqslant \lim_{\mathscr{B}} \sup(g_i); \qquad \lim_{\mathscr{B}} \inf(f_i) \leqslant \lim_{\mathscr{B}} \inf(g_i).$$

Indeed, for every $x \in \mathrm{E}$, Proposition 4.6 applied to φ_x shows, for example, that

$$\lim_{\mathscr{B}} \sup f_i(x) \leqslant \lim_{\mathscr{B}} \sup g_i(x).$$

Z If $f_i < g_i$ for every i, that is, $f_i(x) < g_i(x)$ for every i and every x, the strict inequality does not pass to the limit, that is, one does not in general have

$$\lim_{\mathscr{B}} \sup(f_i) < \lim_{\mathscr{B}} \sup(g_i).$$

6. OPERATIONS ON CONTINUOUS FUNCTIONS

6.1. Proposition. *Let* E *be a topological space and let* $a \in \mathrm{E}$. *The subset* A *of* $\mathscr{F}(\mathrm{E}, \mathbf{R})$ *consisting of the functions which are continuous at* a *is a subalgebra of* $\mathscr{F}(\mathrm{E}, \mathbf{R})$ *which is a lattice.*

Indeed, the continuity of addition and multiplication on $\mathbf{R}$ implies that if f and g are continuous at a, the same is true of $f + g$, fg, and of λf for every scalar λ. On the other hand, the mapping $\varphi : u \to |u|$ of $\mathbf{R}$ into $\mathbf{R}$ is continuous; thus $|f| = \varphi \circ f$ is in A for every $f \in \mathrm{A}$. More generally, then, the relation

$$\sup(f, g) = \tfrac{1}{2}[(f+g) + |f-g|]$$

shows that if $f, g \in \mathrm{A}$, then $\sup(f, g) \in \mathrm{A}$. This conclusion evidently extends to the upper and lower envelopes of every finite family of elements of A.

6.2. Proposition. *Let* E *be a metric space, and let* U (*respectively,* L) *be the subset of* $\mathscr{F}(\mathrm{E}, \mathbf{R})$ *consisting of the functions which are uniformly continuous* (respectively, of Lipschitz class) *on* E.

Then U *and* L *are lattices and vector subspaces of* $\mathscr{F}(\mathrm{E}, \mathbf{R})$.

The proof goes through like that of the preceding proposition, upon observing that the mappings $(u, v) \to u + v$ of $\mathbf{R}^2$ into $\mathbf{R}$ and $u \to |u|$ of $\mathbf{R}$ into $\mathbf{R}$ are of Lipschitz class and therefore uniformly continuous.

Z It is not true that U and L are closed under multiplication. For example, the numerical function $x \to x$ belongs to U and L, but its square $x \to x^2$ belongs to neither one.

6.3. Proposition. *Let $(f_i)_{i \in I}$ be an arbitrary family of elements of $\mathscr{F}(E, \mathbf{R})$ and let f be its upper envelope.*

If each f_i is of Lipschitz class with ratio k, and if f is finite at one point at least, then f is everywhere finite, and is of Lipschitz class with ratio k.

Indeed, for all $x, y \in E$ we have by hypothesis

$$f_i(y) \leqslant k\, d(x, y) + f_i(x),$$

hence

$$f(y) \leqslant k\, d(x, y) + f(x).$$

Therefore if $f(x) < \infty$, then also $f(y) < \infty$, hence $f(y)$ is finite. We can therefore write, for all $x, y \in E$,

$$f(y) - f(x) \leqslant k\, d(x, y) \qquad \text{and similarly} \qquad f(x) - f(y) \leqslant k\, d(x) y).$$

Therefore f is of Lipschitz class with ratio k.

Clearly a similar assertion holds for the lower envelope.

Example. Let E be a metric space, and let $A \subset E$.

For every $x \in E$ we put

$$f(x) = \sup_{a \in A} d(x, a); \qquad g(x) = \inf_{a \in A} d(x, a).$$

Each of the functions $x \rightarrow d(x, a)$ is of Lipschitz class with ratio 1; therefore g is of Lipschitz class with ratio 1, and if f is finite at one point (which is equivalent to saying that A is bounded), then f is finite everywhere, and is of Lipschitz class with ratio 1.

It is false that an upper or lower envelope of uniformly continuous functions is uniformly continuous. It need not be even continuous; for example, the family of functions $1/(1 + x^2)^n$ on $[0, 1]$ has a discontinuous lower envelope. To obtain an assertion of this kind, it would be necessary to assume that the family (f_i) have an additional property, for example, that it be equicontinuous.

We shall see in the following section that the upper or lower envelopes of families of continuous functions, while not being necessarily continuous, still have striking properties.

III. SEMICONTINUOUS NUMERICAL FUNCTIONS

Let f be a numerical function defined on a topological space E. To say that f is continuous at $x_0 \in E$ is equivalent to saying that each of the following conditions holds:

1. For every $\lambda < f(x_0)$ there exists a neighborhood V of x_0 such that $\lambda < f(\mathrm{V})$.

2. For every $\lambda > f(x_0)$ there exists a neighborhood V of x_0 such that $\lambda > f(\mathrm{V})$.

When only one of these conditions is retained, one is led to the notion of semicontinuity.

7. SEMICONTINUITY AT A POINT

7.1. Definition. LET E BE A TOPOLOGICAL SPACE, AND LET $f \in \mathscr{F}(\mathrm{E}, \bar{\mathbf{R}})$. WE SHALL SAY THAT f IS *LOWER* SEMICONTINUOUS AT THE POINT $a \in \mathrm{E}$ IF, FOR EVERY $\lambda < f(a)$, THERE EXISTS A NEIGHBORHOOD V OF a SUCH THAT $\lambda < f(\mathrm{V})$.

WHEN THIS CONDITION HOLDS AT EVERY POINT $a \in \mathrm{E}$, THEN f IS SAID TO BE LOWER SEMICONTINUOUS ON E.

THE DEFINITION OF UPPER SEMICONTINUITY IS OBTAINED BY REVERSING THE DIRECTION OF THE INEQUALITIES.

It is immediate that the lower semicontinuity of f at a is equivalent to the upper semicontinuity of $-f$ at the same point. This remark will permit us to formulate most of our results for lower semicontinuous functions only.

EXAMPLE 1. Let f_n be the mapping of **R** into **R** such that $f(0) = 0$ and $f(x) = x^{-n}$ for $x \neq 0$.

For every even integer n, f_n is lower semicontinuous at the point 0.

For every odd integer n, f_n is neither lower nor upper semicontinuous at the point 0.

EXAMPLE 2. Let f be the mapping of **R** into **R** such that $f(x) = 0$ if x is rational, and $f(x) = 1$ if x is irrational.

The function f is lower semicontinuous at every rational point, and upper semicontinuous at every irrational point.

7.2. Proposition. *To say that f is lower semicontinuous at a is equivalent to saying that*

$$f(a) = \liminf_{x \to a} f(x).$$

PROOF. (a) Suppose that f is lower semicontinuous at a, and let $\lambda < f(a)$. Then there exists a neighborhood V of a such that $\lambda < f(\mathrm{V})$; therefore

$$\lambda \leqslant \overline{f(\mathrm{V})}, \qquad \text{from which} \qquad \lambda \leqslant \liminf_{x \to a} f(x).$$

Since this relation holds for all $\lambda < f(a)$, we also have

$$f(a) \leqslant \liminf_{x \to a} f(x).$$

But the reverse inequality is true, since $f(a) \in f(\mathrm{V})$ for every V, which implies the desired equality.

(b) Conversely, suppose $f(a) = \liminf_{x \to a} f(x)$. For every $\lambda < f(a)$ there exists, by Lemma 4.2, a neighborhood V of a such that $\lambda < f(\mathrm{V})$; that is, f is lower semicontinuous at a.

7.3. Proposition. *The set $\mathscr{I}(a)$ of functions which are lower semicontinuous at a and $> -\infty$ is closed under addition.*

Indeed, let $f, g \in \mathscr{I}(a)$, and let $\lambda < f(a) + g(a)$. We can write $\lambda = \alpha + \beta$, where $\alpha < f(a)$ and $\beta < g(a)$. Then there exists a neighborhood U of a such that $\alpha < f(\mathrm{U})$, and a neighborhood V of a such that $\beta < g(\mathrm{V})$. Therefore

$$\lambda = \alpha + \beta < (f+g)(x) \qquad \text{for all} \qquad x \in \mathrm{U} \cap \mathrm{V}.$$

This proves that $f + g \in \mathscr{I}(a)$.

7.4. Corollary. *The subset of $\mathscr{F}(\mathrm{E}, \mathbf{R})$ consisting of the functions which are lower semicontinuous at a is a convex cone.*

We are going to break off the study of semicontinuity at a point in order to study the much more interesting case of semicontinuity on the entire space.

8. FUNCTIONS, LOWER SEMICONTINUOUS ON THE ENTIRE SPACE

In order that a mapping f of a space E into a space F be continuous in E, it is necessary and sufficient that the inverse image of every closed set in F be a closed set in E. We shall see that there exists a similar characterization of semicontinuous functions.

8.1. Proposition. *To say that a mapping f of a space* E *into $\overline{\mathbf{R}}$ is lower semicontinuous on* E *is equivalent to saying that for every $\lambda \in \overline{\mathbf{R}}$, the set of x such that $f(x) \leqslant \lambda$ is closed (or, what amounts to the same, that the set of x for which $\lambda < f(x)$ is open).*

Indeed, the lower semicontinuity of f at a point a is equivalent to the statement that, for every $\lambda < f(a)$, the set of x such that $\lambda < f(x)$ is a neighborhood of a; in other words, that for every λ the set of x such that $\lambda < f(x)$ is a neighborhood of each of its points, hence an open set.

8.2. Corollary. *Let φ_{A} be the characteristic function of a subset* A *of* E. *Then the lower semicontinuity of φ_{A} is equivalent to the assertion that* A *is open.*

Indeed, the set of x such that $\lambda < \varphi_A(x)$ is either $\emptyset$, if $\lambda \geqslant 1$, or A if $\lambda < 1$.

Similarly, the upper semicontinuity of φ_A *is equivalent to the assertion that* A *is closed.*

Geometric interpretation of lower semicontinuity

In Section 3.2 we associated with every mapping f of a set E into $\bar{\mathbf{R}}$ a subset $A(f)$ of the product space $E \times \bar{\mathbf{R}}$.

We shall give a simple interpretation of lower semicontinuity by means of $A(f)$ (see Fig. 6).

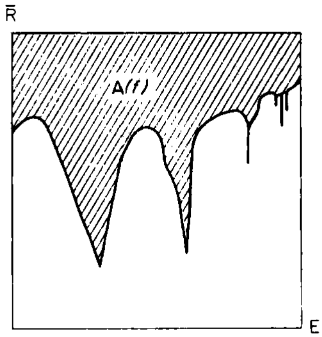

FIG. 6.

8.3. Proposition. *To say that a mapping f of a topological space* E *into* $\bar{\mathbf{R}}$ *is lower semicontinuous is equivalent to saying that* $A(f)$ *is closed in* $E \times \bar{\mathbf{R}}$.

PROOF. The assertion that $A(f)$ is closed is equivalent to the assertion that its complement is open, or again that the complement $\complement A(f)$ is a neighborhood of each of its points.

To say that f is lower semicontinuous in E is equivalent to saying that for every pair (a, λ) such that $\lambda < f(a)$ (that is, for every $(a, \lambda) \in \complement A(f)$) and for every μ satisfying $\lambda < \mu < f(a)$, we have $\mu < f(x)$ for every x in some neighborhood V of a. In other words, this is equivalent to saying that there exists a neighborhood of (a, λ) (namely $V \times [-\infty, \mu)$) contained in $\complement A(f)$.

This proves the stated equivalence.

Operations on lower semicontinuous functions

Proposition 7.3 shows us that the class of lower semicontinuous functions is closed under addition; we are going to see that it is closed under other operations as well.

8.4. Proposition. *Let f be a lower semicontinuous mapping of a space* E *into an interval* $[\alpha, \beta]$ *of* $\overline{\mathbf{R}}$, *and let* φ *be an increasing and continuous mapping of* $[\alpha, \beta]$ *into* $\overline{\mathbf{R}}$. *Then the composition* $g = \varphi \circ f$ *is lower semicontinuous.*

This is a direct consequence of Proposition 8.1; indeed, for every λ, $\varphi^{-1}([-\infty, \lambda])$ is of the form $[\alpha, \mu]$, and $f^{-1}([\alpha, \mu])$ is closed since it is identical with $f^{-1}([-\infty, \mu])$. But this last set is simply $g^{-1}([-\infty, \lambda])$, which proves the proposition.

8.5. Corollary. *If f and g are > 0 and lower semicontinuous on* E, *so is* fg.

We apply Proposition 8.4, taking for φ successively a logarithm and an exponential:

Indeed, $\log f$ and $\log g$ are $> -\infty$ and lower semicontinuous; therefore $\log fg = \log f + \log g$ is lower semicontinuous, hence fg is also.

8.6. Theorem. *The upper envelope of every family $(f_i)_{i \in \mathrm{I}}$ of numerical lower semicontinuous functions is lower semicontinuous.*

The lower envelope of every **finite** *family of numerical lower semicontinuous functions is lower semicontinuous.*

This is an immediate consequence of Proposition 8.3 and the formulas of Section 3.2.

Indeed, if each $\mathrm{A}(f_i)$ is closed, then $\bigcap \mathrm{A}(f_i)$ is closed, and therefore $\sup(f_i)$ is lower semicontinuous. Similarly if I is finite, $\bigcup \mathrm{A}(f_i)$ is closed; therefore $\inf_i (f_i)$ is lower semicontinuous.

Special case. The upper envelope of every family of *continuous* functions is lower semicontinuous.

More particularly, if (f_n) is an increasing sequence of continuous numerical functions, its limit is identical with $\sup(f_n)$; therefore f is lower semicontinuous. The converse of this proposition is studied in Problems 9 and 10.

Similar results are obtained for upper semicontinuity by replacing the upper envelope by the lower envelope and vice versa.

Theorem 8.6 does not extend to the lower envelope of every infinite family of lower semicontinuous functions.

Indeed, let f be an *arbitrary* numerical function on E. Then f is the lower envelope of the functions f_a defined by

$$f_a(x) = +\infty \quad \text{for} \quad x \neq a; \qquad f_a(a) = f(a),$$

and it is immediate that each of the functions f_a is lower semicontinuous.

9. CONSTRUCTION OF LOWER SEMICONTINUOUS FUNCTIONS

Semicontinuous functions are at least as close to ordinary experience as continuous functions. An example will explain this. When we look at an opaque object, we see only one point of this object along every halfline issuing from the eye; the distance from this point to the eye is a function of the direction of the halfline; this function is not continuous, but lower semicontinuous, if we take the object under consideration to be a closed set.

This example, suitably generalized, is incidentally capable of giving the most general semicontinuous functions.

Indeed, with E a topological space, let A be a closed subset of the product space $E \times \bar{\mathbf{R}}$. For every $x \in E$ let $f_A(x)$ be the infimum of the ordinates of the points of A having abscissa x (or $+\infty$ if there is no such point).

It is easily shown that the function f_A thus defined is lower semicontinuous. Conversely, by Proposition 8.3 every lower semicontinuous function on E can be obtained in this way.

Example. Let g be an arbitrary mapping of a topological space E into $\bar{\mathbf{R}}$, and let Γ be the graph of g. The set $\bar{\Gamma}$ is a closed subset of $E \times \bar{\mathbf{R}}$. The function $f_{\bar{\Gamma}}$ associated with $\bar{\Gamma}$ by the above procedure is lower semicontinuous; one can easily verify that

$$f_{\bar{\Gamma}}(x) = \liminf_{t \to x} g(t)$$

for every $x \in E$. Similarly, the function $\limsup_{t \to x} g(t)$ is upper semicontinuous. Their difference $\omega(f; x)$ is therefore upper semicontinuous.

10. SEMICONTINUOUS FUNCTIONS ON A COMPACT SPACE

10.1. Theorem. *For every lower semicontinuous mapping f of a compact space* E *into* $\bar{\mathbf{R}}$, *there exists at least one point a of* E *such that*

$$f(a) = \inf_{x \in E} f(x).$$

Indeed, let us put $m = \inf_{x \in E} f(x)$.

For every $\lambda > m$ the set E_λ of x such that $f(x) \leqslant \lambda$ is closed and nonempty. On the other hand, the family of sets E_λ is totally ordered by inclusion, as E_λ is an increasing function of λ; therefore (Chapter I, Proposition 11.4) the intersection of the E_λ is nonempty. At every point a of this intersection we have $f(a) \leqslant \lambda$ for every $\lambda > m$; hence $f(a) \leqslant m$.

Since on the other hand $f \geqslant m$ by the definition of m, we have $f(a) = m$.

10.2. Corollary. *Every lower semicontinuous mapping f of a compact space* E *into* $(-\infty, +\infty]$ *is bounded from below on* E.

Indeed, we then have $m = f(a) > -\infty$, and therefore $f \geqslant f(a) > -\infty$.

Analogous results hold for upper semicontinuous functions.

If we apply these results to a continuous function, we obtain an earlier result stating that a continuous function on a compact space attains its infimum and supremum on the space.

We are now going to study an important application of these results to the calculus of variations.

11. SEMICONTINUITY OF LENGTH

The length of a curve is a function of the curve; when the curve varies continuously, in a sense which we shall make precise, one might expect that its length also varies continuously. This is not so, as the following elementary example shows:

Let C_n be the plane curve with equation $y = n^{-1} \sin nx$, where $0 \leqslant x \leqslant \pi$ and $n \in \mathbf{N}^*$ (in Cartesian coordinates).

It is immediate that all these curves have the same length and that this length is a number $l > \pi$. But as $n \to +\infty$, these curves converge uniformly to the segment $[0, \pi]$. Therefore uniform convergence does not imply the convergence of the lengths.

This example can be modified to go through with any number $l \geqslant \pi$. But it is noteworthy that one cannot take $l < \pi$.

In other words, the limit inferior of the lengths of the curves which converge to the segment $[0, \pi]$ is equal to π. This is nothing more than lower semicontinuity, which we shall now make precise.

The space of parametrized curves

Let T be a compact interval of $\mathbf{R}$ and let E be a metric space. By an earlier definition (Chapter I, Section 24) every continuous mapping f of T into E defines a parametrized curve. Therefore the set $\mathscr{C}(\mathrm{T}, \mathrm{E})$ of continuous mappings of T into E can be regarded as the set of parametrized curves in E defined on T.

We shall take as the topology on $\mathscr{C}(\mathrm{T}, \mathrm{E})$ the topology associated with the metric of uniform convergence, defined by

$$d(f, g) = \sup_{t \in \mathrm{T}} d(f(t), g(t)).$$

For every $f \in \mathscr{C}(\mathrm{T}, \mathrm{E})$ we denote by $\mathrm{L}(f)$ the length of the curve defined by f. We thus have a numerical function defined on the topological space $\mathscr{C}(\mathrm{T}, \mathrm{E})$.

11.1. Theorem. *The length* $\mathrm{L}(f)$ *is a lower semicontinuous function of* f *in* $\mathscr{C}(\mathrm{T}, \mathrm{E})$.

Proof. For every finite subset $\sigma = \{t_1, t_2, \ldots, t_n\}$ of T, where $t_1 < t_2 < \cdots < t_n$, and for every $f \in \mathscr{C}(\mathrm{T}, \mathrm{E})$, we put

$$\mathrm{V}_\sigma(f) = \sum_i d(f(t_i), f(t_{i+1})).$$

For every $a \in \mathrm{T}$, the mapping $f \to f(a)$ of $\mathscr{C}(\mathrm{T}, \mathrm{E})$ into E is continuous; therefore the mapping $f \to \mathrm{V}_\sigma(f)$ is continuous for every σ.

But $\mathrm{L}(f) = \sup_\sigma \mathrm{V}_\sigma(f)$ (see Chapter I, Section 24); hence L is the upper envelope of the continuous functions $\mathrm{V}_\sigma(f)$. Therefore it is lower semicontinuous by Theorem 8.6.

11.2. Corollary. *The mapping* $f \to$ *(total variation of* f*) of* $\mathscr{C}(\mathrm{T}, \mathbf{R})$ *into* $\bar{\mathbf{R}}$ *is lower semicontinuous.*

Application to the calculus of variations

One of the problems of the calculus of variations in one variable consists in finding, among a given set of curves, one whose length is the smallest possible.

The solution of this problem is furnished by the following lemma, which is a consequence of Theorems 10.1 and 11.1 taken together.

11.3. Lemma. *For every compact subset* K *of* $\mathscr{C}(\mathrm{T}, \mathrm{E})$, *there exists an element* f_0 *of* K *such that*

$$\mathrm{L}(f_0) = \inf_{f \in \mathrm{K}} \mathrm{L}(f).$$

We are therefore led to investigate the compact subsets of $\mathscr{C}(\mathrm{T}, \mathrm{E})$.

11.4. Lemma. *Let* T *be an interval* $[a, b]$, *where* $a < b$. *For every* $f \in \mathscr{C}(\mathrm{T}, \mathrm{E})$ *such that* $\mathrm{L}(f) < l$, *there exists an increasing homeomorphism* α *of* $[0, 1]$ *with* $[a, b]$ *such that* $f \circ \alpha$ *is of Lipschitz class with ratio* l.

Proof. For every $x \in [a, b]$, let $\mathrm{V}(x)$ be the total variation of f on $[a, x]$; we then put

$$\beta(x) = k_1 \mathrm{V}(x) + k_2(x - a).$$

One can verify that if $k_1 = l^{-1}$ and $k_2 = (l - \mathrm{L}(f))/(b - a)l$, the mapping β is an increasing homeomorphism of $[a, b]$ with $[0, 1]$. We denote the inverse mapping β^{-1} by α.

Since an increasing change of variable does not change the total variation (Chapter I, Proposition 24.5), for every $x, y \in [0, 1]$ such that $x < y$ we have

$$\begin{aligned}
|f \circ \alpha(y) - f \circ \alpha(x)| &\leqslant \text{total variation of } f \circ \alpha \text{ on } [x, y] \\
&= \text{total variation of } f \text{ on } [\alpha(x), \alpha(y)] \\
&= V(\alpha(y)) - V(\alpha(x)) \leqslant l(\beta(\alpha(y)) - \beta(\alpha(x))) = l(y - x).
\end{aligned}$$

The inequality $|f \circ \alpha(y) - f \circ \alpha(x)| \leqslant l(y - x)$ establishes the stated property.

11.5. Corollary. *For every family $(C_i)_{i \in I}$ of parametrized curves in E of length $< l$, there exists a family $(C_i')_{i \in I}$ of parametrized curves belonging to $\mathscr{C}([0, 1], E)$ such that, for every $i \in I$, C_i and C_i' are equivalent by a change of variable, and C_i' is of Lipschitz class with ratio l.*

We can now apply these results to the study of geodesics of a metric space (giving the name "geodesic of E" to every simple rectifiable arc whose length is $\leqslant$ the length of every other arc having the same end points).

11.6. Theorem. *Let E be a compact metric space, and let A and B be disjoint closed subsets of E.*

If there exist rectifiable curves in E with endpoints in A and B, respectively, and if k denotes the infimum of their lengths, then there also exists a simple arc with length k and endpoints in A and B, respectively.

Indeed, let l denote any finite number such that $k < l$. Let K be the subset of $\mathscr{C}([0, 1], E)$ consisting of the mappings f of Lipschitz class with ratio l such that $f(0) \in A$, $f(1) \in B$; K is equicontinuous and closed in $\mathscr{C}([0, 1], E)$, hence compact by Ascoli's theorem (Chapter I, Section 23).

Corollary 11.5 shows that $k = \inf_{f \in K} L(f)$; therefore by Lemma 11.3 there exists $f_0 \in K$ such that

$$L(f_0) = \inf_{f \in K} L(f) = k.$$

This mapping f_0 is not necessarily one-to-one, but the use of an intrinsic parametrization by $[0, k]$ will eliminate this deficiency:

For every $t \in [0, 1]$, let $\varphi(t)$ be the total variation of f_0 on $[0, t]$. For all $t_1, t_2 \in [0, 1]$ we have

$$d(f_0(t_1), f_0(t_2)) \leqslant |\varphi(t_1) - \varphi(t_2)|. \tag{1}$$

Therefore the relation $\varphi(t_1) = \varphi(t_2)$ implies $f_0(t_1) = f_0(t_2)$, which shows that f_0 is a function of φ, that is, of the form $f_0 = g \circ \varphi$. Then inequality (1) can be written as

$$d(g(u_1), g(u_2)) \leqslant | u_1 - u_2 |,$$

where

$$u_1 = \varphi(t_1) \quad \text{and} \quad u_2 = \varphi(t_2).$$

Thus g is of Lipschitz class with ratio 1 on $[0, k]$; g is called the *intrinsic parametrization* associated with f_0.

The total variation of g is $\leqslant k$, but since

$$g(0) = f_0(0) \in \mathrm{A} \qquad \text{and} \qquad g(k) = f_0(1) \in \mathrm{B},$$

the total variation of g is $\geqslant k$; thus it is exactly equal to k, and the total variation of g on every interval $[u_1, u_2]$ is $(u_2 - u_1)$.

We assert that g is one-to-one; indeed, if there existed u_1, u_2 with $u_1 < u_2$ such that $g(u_1) = g(u_2)$, then by removing from the curve the arc corresponding to the interval (u_1, u_2) we would obtain a curve with end points in A and B, respectively, and whose length would be

$$k - (u_2 - u_1) < k.$$

Example. Let Δ be the closed disc $x^2 + y^2 \leqslant r^2$ of $\mathbf{R}^2$; let f be a continuous mapping of Δ into $\mathbf{R}$, and let E be the graph of f. The metric subspace E of $\mathbf{R}^3$ is homeomorphic to Δ, hence compact.

If f is of Lipschitz class, then through any two points p and q of E there passes a rectifiable arc in E, namely the image of the line segment joining the projections of p and q on Δ.

Therefore by Theorem 11.6 a geodesic of E passes through p and q.

Let us call the image of the interior of Δ the *interior* of E, and the image of the circle $x^2 + y^2 = r^2$ the *frontier* of E, and let O denote the image of the center of Δ.

The lengths of the arcs of E joining O to a point of the frontier have an infimum $\geqslant r$; on the other hand, the length of the geodesics joining O to q tends to 0 as q tends to O. It follows that if q is taken in a suitable neighborhood of O, every geodesic joining O and q is interior to E. We can therefore state:

11.7. Proposition. *If p is a point of an open Lipschitz surface* S *of* $\mathbf{R}^3$ *of the form $z = f(x, y)$, then for every point q of* S *close enough to p there exists a geodesic of* S *with endpoints p and q.*

IV. THE STONE-WEIERSTRASS THEOREM (Section 12)

We are here going to establish several theorems which will show us that every family of numerical functions, on a compact space, which is sufficiently rich and which is closed under certain operations, can be used to uniformly approximate every continuous function on this space.

Let X be a compact space, and $\mathscr{C}(X, \mathbf{R})$ the algebra of continuous mappings of X into $\mathbf{R}$, taken with the topology of uniform convergence.

We will say that a subset A of $\mathscr{C}(X, \mathbf{R})$ is a *lattice* if for all $f, g \in A$, the envelopes $\sup(f, g)$ and $\inf(f, g)$ also belong to A.

12.1. Proposition. *If* A *is a lattice subset of* $\mathscr{C}(X, \mathbf{R})$, *then for every* $f \in \mathscr{C}(X, \mathbf{R})$, *the statement that* $f \in \bar{A}$ *is equivalent to the statement that for every* $x, y \in X$, f *is the limit on* $\{x, y\}$ *of elements of* A.

PROOF. Indeed, let f be a continuous function which can be thus approximated on every set $\{x, y\}$, and let ϵ be a number > 0.

Then for any $x, y \in X$ there exists $g_{x,y} \in A$ such that

$$|f(x) - g_{x,y}(x)| < \epsilon; \tag{1}$$

$$|f(y) - g_{x,y}(y)| < \epsilon. \tag{2}$$

We put

$$\omega_{x,y} = \{z : g_{x,y}(z) < f(z) + \epsilon\}.$$

Since the function $g_{x,y} - f$ is continuous, the set $\omega_{x,y}$ is open, and it contains y by relation (2); therefore for every fixed x, the $\omega_{x,y}$ constitute an open covering of X. We can therefore find a finite subcovering (ω_{x,y_i}).

Put

$$g_x = \inf_i(g_{x,y_i});$$

then $g_x < f + \epsilon$ on X; and $g_x(x) > f(x) - \epsilon$ from (1).

Put

$$\omega_x = \{z : g_x(z) > f(z) - \epsilon\}.$$

Since the function $g_x - f$ is continuous, the set ω_x is open, and it contains x. Therefore the ω_x constitute an open covering of X, and we can find a finite subcovering (ω_{x_i}).

We now put

$$g = \sup_j(g_{x_j});$$

then $g \in A$; $g < f + \epsilon$, and $g > f - \epsilon$ on X.

We have therefore found a $g \in \mathrm{A}$ which uniformly approximates f to within ϵ.

EXAMPLE. Let X be a compact set in $\mathbf{R}^n$ and let A be the set of traces on X of continuous functions f on $\mathbf{R}^n$ which are piecewise affine (that is, there exists a finite covering of $\mathbf{R}^n$ by convex polyhedra on each of which f is affine). This set A is clearly a lattice, and for all x and y there exist functions $f \in \mathrm{A}$ which take any assigned values at x and y.

Therefore

$$\bar{\mathrm{A}} = \mathscr{C}(\mathrm{X}, \mathbf{R}).$$

12.2. Lemma. *Every closed subalgebra* A *of* $\mathscr{C}(\mathrm{X}, \mathbf{R})$ *is a lattice.*

PROOF. By virtue of the relations

$$\sup(f, g) = \tfrac{1}{2}[(f+g) + |f-g|]; \quad \inf(f, g) = \tfrac{1}{2}[(f+g) - |f-g|],$$

it suffices to prove that if $f \in \mathrm{A}$, then also $|f| \in \mathrm{A}$.

To do this, we shall show that $|f|$ is the uniform limit of polynomials in f of the form $\sum_1^n a_p f^p$; we can clearly restrict ourselves to the case where $\|f\| \leqslant 1$.

But for every $\epsilon > 0$ we have

$$0 \leqslant (x^2 + \epsilon^2)^{1/2} - |x| \leqslant \epsilon.$$

On the other hand,

$$x^2 + \epsilon^2 = 1 + \epsilon^2 + (x^2 - 1) = (1 + \epsilon^2)(1 + u),$$

where $u = (x^2 - 1)/(1 + \epsilon^2)$.

For every $x \in [-1, 1]$ we have

$$|u| \leqslant (1 + \epsilon^2)^{-1} < 1;$$

therefore the Taylor series* of $(1 + u)^{1/2}$ converges uniformly to $(1 + u)^{1/2}$ when $x \in [-1, 1]$; there thus exists a polynomial $\mathrm{P}(x)$ such that

$$|(x^2 + \epsilon^2)^{1/2} - \mathrm{P}(x)| \leqslant \epsilon \qquad \text{for all} \quad x \in [-1, 1].$$

In particular, we therefore have $|\mathrm{P}(0)| \leqslant 2\epsilon$, so that if finally we put $\mathrm{Q} = \mathrm{P} - \mathrm{P}(0)$, then

$$\big|\mathrm{Q}(x) - |x|\big| \leqslant \epsilon + 2\epsilon + \epsilon = 4\epsilon \qquad \text{for all} \quad x \in [-1, 1].$$

* There exist proofs of Lemma 12.2 which avoid the use of Taylor's formula (see Problem 16).

Since Q does not have a constant term, $Q(f) \in A$; and since we have assumed $\|f\| \leqslant 1$, then

$$\| Q(f) - |f| \| \leqslant 4\epsilon.$$

12.3. Definition. LET A BE A FAMILY OF MAPPINGS OF A SET X INTO A SET Y. THEN A IS SAID TO *SEPARATE* THE POINTS OF X IF, FOR ANY $x, y \in X$ WITH $x \neq y$, THERE EXISTS AN $f \in A$ SUCH THAT $f(x) \neq f(y)$.

We can now state the desired fundamental theorem:

12.4. Theorem (of Stone-Weierstrass). *Let* A *be a subalgebra of* $\mathscr{C}(X, \mathbf{R})$ *such that*:

1. A *separates the points of* X;
2. *For every* $x \in X$ *there exists an* $f \in A$ *such that* $f(x) \neq 0$.

Then

$$\bar{A} = \mathscr{C}(X, \mathbf{R}).$$

PROOF. By virtue of Lemmas 12.1 and 12.2, it suffices to show that for all $x, y \in X$ with $x \neq y$, and for all scalars α, β, there exists an $f \in A$ with $f(x) = \alpha$ and $f(y) = \beta$.

If there exists a $g \in A$ with $g(x) \neq g(y)$ and $g(x), g(y) \neq 0$, we put

$$f = a_1 g + a_2 g^2.$$

The existence of scalars a_1, a_2 such that $f(x) = \alpha$, $f(y) = \beta$ is guaranteed by the condition

$$g(x)g^2(y) - g(y)g^2(x) = g(x)g(y)(g(y) - g(x)) \neq 0,$$

which is satisfied by hypothesis.

But there exists such a g; for there exists g_1 such that $g_1(x) \neq g_1(y)$; if $g_1(x)$ and $g_1(y)$ are $\neq 0$, we take $g = g_1$; if, however, $g_1(x) = 0$ for example, there exists g_2 such that $g_2(x) \neq 0$, and we take $g = g_1 + \epsilon g_2$, where $\epsilon \neq 0$ and is so small that $g(x) \neq g(y)$ and $g(y) \neq 0$.

REMARK 1. It is convenient in applications to formulate this theorem as follows:

If a family (f_i) of elements of $\mathscr{C}(X, \mathbf{R})$ separates the points of X, and if the f_i do not all vanish at any one point of X, then every $f \in \mathscr{C}(X, \mathbf{R})$ is the uniform limit of polynomials (without constant terms) with respect to the f_i.

REMARK 2. If A contains the constants, condition 2 of Theorem 12.4 is satisfied.

12.5. Corollary. *Let* A *be a subalgebra* (*over the field* **C**) *of* $\mathscr{C}(\mathrm{X}, \mathbf{C})$ *such that*:

1. A *separates the points of* X;
2. *For every* $x \in \mathrm{X}$, *there exists an* $f \in \mathrm{A}$ *such that* $f(x) \neq 0$;
3. *For every* $f \in \mathrm{A}$, *we also have* $\bar{f} \in \mathrm{A}$ (*where* $\bar{f}$ *denotes the conjugate of* f).

Then

$$\bar{\mathrm{A}} = \mathscr{C}(\mathrm{X}, \mathbf{C}).$$

Indeed, the subalgebra A_r (over **R**) of the real-valued functions in A satisfies conditions 1 and 2 of Theorem 12.4, for if f separates x and y, the same is true of $\mathscr{R}(f)$ or $\mathscr{R}(if)$, and if $f(x) \neq 0$, the same is true of $\mathscr{R}f(x)$ or $\mathscr{R}(if)(x)$.

Therefore

$$\bar{\mathrm{A}}_r = \mathscr{C}(\mathrm{X}, \mathbf{R}); \qquad \text{hence} \qquad \overline{\mathrm{A}_r + i\mathrm{A}_r} = \mathscr{C}(\mathrm{X}, \mathbf{C}).$$

Z Condition 3 of the corollary is essential. Indeed, let us take for X the unit disk of **C**, and let A be the set of traces on X of polynomials in the complex variable z. The algebra A satisfies conditions 1 and 2, but not condition 3. One can verify that $\bar{\mathrm{A}} \neq \mathscr{C}(\mathrm{X}, \mathbf{C})$, for example by noting that for every $f \in \mathrm{A}$, hence also for every $f \in \bar{\mathrm{A}}$, $f(0)$ is the mean of f over the unit circle, which is not true for every $f \in \mathscr{C}(\mathrm{X}, \mathbf{C})$.

APPLICATION 1. Let X be a compact subset of $\mathbf{R}^n$; the family (f_i) of the n coordinate functions $x \to x_i$ separates the points of X. Therefore every function $f \in \mathscr{C}(\mathrm{X}, \mathbf{C})$ is the uniform limit of polynomials in n variables with complex coefficients (with a constant term if $\mathrm{O} \in \mathrm{X}$; without constant term, if one wishes, if $\mathrm{O} \notin \mathrm{X}$).

APPLICATION 2. Let X be the unit circle $|z| = 1$ of **C**; the function $z \to z$ separates the points of X and does not vanish on X. Therefore the algebra generated by z and $\bar{z}$ is everywhere dense on $\mathscr{C}(\mathrm{X}, \mathbf{C})$. Let φ denote the mapping $t \to e^{it}$ of **R** into X. For every $f \in \mathscr{C}(\mathrm{X}, \mathbf{C})$, $f \circ \varphi$ is a continuous function on **R**, periodic with period 2π, and we know (see Volume 1, Chapter III) that every continuous periodic function with period 2π on **R** is of this form.

Since f is the uniform limit of polynomials in z and $\bar{z}$, $f \circ \varphi$ is the uniform limit of polynomials in e^{it} and e^{-it}; in other words, every continuous complex-valued function on **R** which is periodic with period 2π is the uniform limit of trigonometric polynomials $\sum_{-n}^{n} a_p e^{ipt}$.

Extension of continuous functions

Let X be a topological space, Y a closed subset of X, and f a continuous mapping of Y into $\mathbf{R}$. The question presents itself of determining whether f can be extended to a continuous mapping of X into $\mathbf{R}$. We shall see that the Stone-Weierstrass theorem furnishes an easy answer when X is compact and metrizable.

12.6. Proposition. *If* Y *denotes a closed subset of a compact metric space* X, *every* $f \in \mathscr{C}(\mathrm{Y}, \mathbf{R})$ *is the restriction to* Y *of an element of* $\mathscr{C}(\mathrm{X}, \mathbf{R})$.

PROOF. 1. $\mathscr{C}(\mathrm{X}, \mathbf{R})$ separates the points of X since, if $a, b \in \mathrm{X}$ with $a \neq b$, the continuous function $x \to d(a, x)$ separates a and b. Therefore Theorem 12.4 shows that for every $g \in \mathscr{C}(\mathrm{Y}, \mathbf{R})$ and for every $\epsilon > 0$, there exists $g_\epsilon \in \mathscr{C}(\mathrm{X}, \mathbf{R})$ such that $|g_\epsilon(y) - g(y)| < \epsilon$ for every $y \in \mathrm{Y}$. We shall assume in addition that g and g_ϵ have the same infimum and supremum (if this is not the case, we replace g_ϵ by $\sup(\alpha, \inf(\beta, g_\epsilon))$, where α and β are the infimum and supremum, respectively, of g).

2. Let us define a sequence $(g^{(n)})$ of elements of $\mathscr{C}(\mathrm{X}, \mathbf{R})$ recursively by the following conditions:

$$g^{(1)} = f_{1/2}; \qquad g^{(n)} = \Big(f - \sum_1^{n-1} g^{(i)}\Big)_{1/2^n} \qquad \text{for} \quad n > 1.$$

We deduce that

$$\Big|f(y) - \sum_1^n g^{(i)}(y)\Big| \leqslant (\tfrac{1}{2})^n \qquad \text{on} \quad \mathrm{Y};$$

$$|g^{(n)}(x)| \leqslant (\tfrac{1}{2})^{n-1} \qquad \text{on} \quad \mathrm{X}.$$

It follows from this that on the one hand the series with general term $g^{(n)}$ converges uniformly on X, and on the other hand that its sum g is equal to f on Y.

V. FUNCTIONS DEFINED ON AN INTERVAL OF R

The existence of an order structure and an affine structure on the intervals of $\mathbf{R}$ makes it possible to define, for functions on such intervals, various notions related to these structures, such as that of the limit from the left or right, monotonicity, differentiability, and convexity.

We are here going to examine these various notions.

13. LEFT AND RIGHT LIMITS

Let us recall (Chapter I, Section 8) that if I is an interval of $\mathbf{R}$, E is a separated topological space, and f is a mapping of I into E, then f is said to have a right limit at the point a of I if $\lim_{x>a, x\to a} f(x)$ exists.

This limit is denoted by $f(a_+)$; $f(a_-)$ is defined in a similar way.

13.1. Definition. THE POINT a IS SAID TO BE A DISCONTINUITY OF THE *FIRST KIND* OF f IF, ON THE ONE HAND, $f(a_-)$ AND $f(a_+)$ EXIST, AND ON THE OTHER HAND, WE DO NOT HAVE $f(a_-) = f(a) = f(a_+)$.

EXAMPLE 1. The mapping $x \to$ (integer part of x) of $\mathbf{R}_+$ into $\mathbf{R}_+$ has the integers 1, 2, ... as its discontinuities of the first kind.

EXAMPLE 2. The mapping f of $\mathbf{R}^+$ into $\mathbf{R}^+$ defined by:

$$f(x) = 0 \qquad \text{for } x = 0 \text{ and for } x \text{ irrational,}$$

$$f(x) = q^{-1} \qquad \text{if } x \text{ is the irreducible fraction } p/q,$$

has as its discontinuities of the first kind all the rational points $\neq 0$.

EXAMPLE 3. The mapping f of $\mathbf{R}$ into $\mathbf{R}$ defined by $f(x) = 0$ if x is rational, and $f(x) = 1$ if x is irrational, has no discontinuities of the first kind.

An example will show us that if no restrictions are placed on E, then every point of I can be a discontinuity of the first kind:

Let E denote the product $\mathbf{R} \times \{1, 2, 3\}$, with the lexicographic order defined as follows:

For every $x \in \mathbf{R}$ we take $(x, 1) < (x, 2) < (x, 3)$, and for every $x, y \in \mathbf{R}$ such that $x < y$, we take $(x, i) < (y, j)$ for any $i, j \in \{1, 2, 3\}$.

Let E be taken with the order topology, and let f denote the mapping $x \to (x, 2)$ of $\mathbf{R}$ into E.

One can verify that E is locally compact and that for every $x \in \mathbf{R}$,

$$f(x_-) = (x, 1) \qquad \text{and} \qquad f(x_+) = (x, 3).$$

However, this kind of singular occurrence cannot take place if E is metrizable:

13.2. Proposition. *If f is a mapping of an interval* I *of* $\mathbf{R}$ *into a metric space* E, *the set of points of discontinuity of the first kind of f is at most countable.*

Indeed, let D_n be the set of points of discontinuity of the first kind of f at which the oscillation of f is $\geqslant n^{-1}$. Then D_n has only isolated points, for if a denotes an accumulation point of D_n, a is the limit of a monotone sequence of distinct points of D_n; if for example this sequence is decreasing, then $f(a_+)$ cannot exist, therefore $a \notin D_n$.

Since D_n has only isolated points, it is at most countable; and since $D = \bigcup D_n$, D is at most countable.

There exist functions which are discontinuous at every point and have no discontinuities of the first kind; this is the case for Example 3 above.

13.3. Definition. LET f BE A MAPPING OF AN INTERVAL I OF $\mathbf{R}$ INTO A METRIC SPACE E.

1. f IS SAID TO BE *REGULATED* IF ITS POINTS OF DISCONTINUITY ARE ALL OF THE FIRST KIND.

2. f IS SAID TO BE A *STEP FUNCTION* IF THERE EXISTS A FINITE PARTITION OF I INTO SUBINTERVALS (SOME OF WHICH MAY BE ONE POINT) ON EACH OF WHICH f IS CONSTANT.

Proposition 13.2 shows that the set of points of discontinuity of a regulated function is at most countable. It is evident, incidentally, that every step function is regulated, but the converse is false, as every continuous function is regulated.

13.4. Proposition. *The collection of regulated functions is closed under uniform limit.*

Indeed, suppose f is the uniform limit of regulated functions f_n; for every $a \in I$, $f_n(a_+)$ and $f_n(a_-)$ exist, hence the same is true of $f(a_+)$ and $f(a_-)$.

Let us note here that if the metric space E is a normed vector space, the set of regulated functions in $\mathscr{F}(I, E)$ is a vector subspace of $\mathscr{F}(I, E)$.

13.5. Proposition. *Suppose the interval* I *is compact, while the metric space* E *is arbitrary.*

Then the class of regulated functions and the class of uniform limits of step functions are identical.

PROOF. Since every step function is regulated, Proposition 13.4 shows that the same is true of every uniform limit of step functions.

Conversely, let f be a regulated function, and let ϵ be a number >0. For every $x \in I$ there exist two nonempty intervals (α_x, x) and (x, β_x) on each of which the oscillation of f is $<\epsilon$. Since I is compact, there exists a finite covering of I by intervals of the form (α_x, β_x); hence there exists a finite partition of I into intervals I_n, some of them possibly consisting of one point, on each of which the oscillation of f is $<\epsilon$. The function f_ϵ which is constant on each interval I_n and which coincides with f at the midpoint of each I_n clearly approximates f to within ϵ.

Corollary. *Every regulated mapping of a compact* I *into a metric space* E *is bounded.*

14. MONOTONE FUNCTIONS

If f is an increasing mapping of an interval I into **R**, then for every $a \in \mathrm{I}$ we have

$$f(a_-) = \sup_{x<a} f(x); \qquad f(a_+) = \inf_{x>a} f(x).$$

Thus *every increasing function* (and similarly every decreasing function) *is a regulated function*; in particular, the set of its points of discontinuity is at most countable.

Therefore every difference of two increasing functions, in other words every function of bounded variation (Chapter I, Proposition 24.8) is regulated. This result might lead us to expect that, conversely, every regulated numerical function is of bounded variation; this is not so, as there exist even continuous functions on [0, 1] with unbounded variation.

Examples. It is well to be familar with several classical examples of increasing functions of a paradoxical nature, which serve as guide lines in trying to discover mathematical propositions.

Example 1. Let $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ be a countable set of points of the open interval (0, 1) such that $\bar{A} = [0, 1]$, and let (α_n) be an infinite sequence of numbers >0, with sum 1.

For every $x \in [0, 1]$ we put

$$f(x) = \sum_{a_n < x} \alpha_n .$$

In other words, the sum is taken over all n such that $a_n < x$. It is immediate that f is strictly increasing on [0, 1], with $f(0) = 0$ and $f(1) = 1$, and that the set of its points of discontinuity is A.

Example 2. We now put $f([0, 1]) = B$, where f is the function defined in the preceding example. The mapping f^{-1} of B onto [0, 1] is strictly increasing; one can verify:

(a) That f^{-1} can be extended in a unique way to an increasing mapping g of [0, 1] onto [0, 1];

(b) That g is continuous;

(c) That the sum of the lengths of the intervals of $[0, 1]$ on which g is constant is equal to 1, which implies that the change of g takes place on a set of "measure zero" in a sense which will be made precise in the theory of integration.

15. THEOREMS OF FINITE INCREASE

We shall not review here the elementary classical properties of the derivatives of a function of a real variable. On the other hand, we shall prove several theorems of "finite increase" which enable one to pass from a local to a global situation and which will advantageously replace the classical theorem of finite increase (the mean value theorem), which is valid only for real-valued differentiable functions.

In all of what follows, I denotes an arbitrary interval of $\mathbf{R}$, and D a finite or countable subset of I; in the more classical results, this set D is either empty, or consists of the end points of I.

15.1. Lemma. *Let f be a continuous mapping of* I *into* $\mathbf{R}$ *such that, for every x interior to* I *with $x \notin$ D and for every $\delta > 0$, there exists $y \in (x, x + \delta)$ such that $f(x) \leqslant f(y)$.*

Then f is increasing on I.

Proof. Let $u, v \in$ I, with $u < v$, and let k be any number such that

$$k < f(u); \qquad k \notin f(\mathrm{D}).$$

Let A be the set of $x \in [u, v]$ such that $k \leqslant f(x)$; this set is nonempty since it contains u, and is closed since f is continuous; therefore it contains the number $\alpha = \sup$ A.

Suppose $\alpha < v$; we cannot have $k < f(\alpha)$, for then α would be in the interior of A; therefore $k = f(\alpha)$. But $\alpha \notin$ D since $f(\alpha) = k \notin f(\mathrm{D})$; therefore by the hypothesis of the lemma there exists $\beta \in (\alpha, v)$ such that $f(\alpha) \leqslant f(\beta)$; hence $\beta \in$ A, which contradicts $\alpha = \sup$ A.

We therefore have $\alpha = v$, in other words $k \leqslant f(v)$; since $f(\mathrm{D})$ is at most countable, there exist numbers k arbitrarily close to $f(u)$; therefore $f(u) \leqslant f(v)$.

In other words, f is increasing.

15.2. Lemma. *Let φ be a continuous mapping of* I *into a metric space* E, *and let g be a continuous increasing mapping of* I *into* $\mathbf{R}$.

Suppose that for every x in the interior of I, *with $x \notin$ D, and for every $\delta > 0$, there exists $y \in (x, x + \delta)$ such that*

$$d(\varphi(y), \varphi(x)) \leqslant g(y) - g(x). \tag{1}$$

Then for all $u, v \in \mathrm{I}$, *with* $u < v$, *we have*

$$d(\varphi(v), \varphi(u)) \leqslant g(v) - g(u).$$

Indeed, given u and v, we put

$$f(x) = g(x) - d(\varphi(u), \varphi(x)) \qquad \text{for every} \quad x \in [u, v].$$

Then for all $x, y \in [u, v]$ such that $x < y$ and such that (1) is satisfied, we have

$$\begin{aligned} f(x) - f(y) &= g(x) - g(y) + d(\varphi(u), \varphi(y)) - d(\varphi(u), \varphi(x)) \\ &\leqslant g(x) - g(y) + d(\varphi(y), \varphi(x)) \leqslant 0. \end{aligned}$$

Therefore, by Lemma 15.1, f is increasing on $[u, v]$; hence

$$0 \leqslant f(v) - f(u), \qquad \text{that is} \qquad d(\varphi(u), \varphi(v)) \leqslant g(v) - g(u).$$

To now obtain statements of a more classical nature, we give a definition:

15.3. Definition. LET φ BE A MAPPING OF I INTO A NORMED SPACE E. THEN φ IS SAID TO BE *DIFFERENTIABLE FROM THE RIGHT* (OR *RIGHT DIFFERENTIABLE*) AT THE POINT a OF I (WHERE a IS NOT THE RIGHT END POINT OF I) IF THE ELEMENT $h^{-1}(\varphi(a+h) - \varphi(a))$ OF E TENDS TO A LIMIT AS $h \to 0$ ALONG VALUES >0. THIS LIMIT IS CALLED THE RIGHT DERIVATIVE OF φ AT a, AND IS DENOTED BY $\varphi_r'(a)$.

We can then write

$$\Delta\varphi = \varphi(a+h) - \varphi(a) = h(\varphi_r'(a) + \delta(h)),$$

where $\lim_{h \to 0} \delta(h) = 0$. Thus for every $\epsilon > 0$ we have

$$h(\|\varphi_r'(a)\| - \epsilon) \leqslant \|\Delta\varphi\| \leqslant h(\|\varphi_r'(a)\| + \epsilon)$$

for all $h > 0$ sufficiently small.

In particular, φ is right continuous at the point a.

Differentiability from the left is defined similarly; if φ is differentiable from the right and left at the point a, with the derivatives equal, then φ is said to be differentiable at a.

15.4. Theorem. *If the continuous numerical function* φ *is right differentiable at every point of* $\mathrm{I} \dot{-} \mathrm{D}$, *and if* $m \leqslant \varphi_r'(x) \leqslant \mathrm{M}$ *for every* $x \in \mathrm{I} \dot{-} \mathrm{D}$, *then*

$$m(v - u) \leqslant \varphi(v) - \varphi(u) \leqslant \mathrm{M}(v - u) \qquad \text{for} \quad u < v.$$

The inequalities are strict when f *is not affine on* $[u, v]$.

PROOF. Let us first show that if $\varphi_r'(x) \geqslant 0$ for every $x \notin D$, then φ is increasing. Indeed, given $\epsilon > 0$, for every $x \notin D$ we have, when h is sufficiently small,

$$\varphi(x + h) - \varphi(x) \geqslant -\epsilon h.$$

Therefore the function $f : x \to \varphi(x) + \epsilon x$ satisfies the conditions of Lemma 15.1; hence it is increasing. Since this is true for all $\epsilon > 0$, φ itself is increasing.

But the functions $\varphi_1 : x \to Mx - \varphi(x)$ and $\varphi_2 : x \to \varphi(x) - mx$ have right derivatives $\geqslant 0$ at the points of $I \dot{-} D$; hence they are increasing, which implies the desired inequalities.

Finally, if f is not affine with derivative M on $[u, v]$, the increasing function $Mx - \varphi(x)$ is not constant on $[u, v]$, whence

$$Mu - \varphi(u) < Mv - \varphi(v).$$

A similar argument holds for $\varphi(x) - mx$.

15.5. Theorem. *Let φ be a continuous mapping of* I *into a normed space* E, *and let g be a continuous increasing mapping of* I *into* **R**.

Suppose that φ and g are right differentiable at every point of $I \dot{-} D$, *and that*

$$\| \varphi_r'(x)\| \leqslant g_r'(x).$$

Then for all $u, v \in I$ with $u < v$ we have

$$\| \varphi(v) - \varphi(u)\| \leqslant g(v) - g(u).$$

PROOF. Given a number $\epsilon > 0$, for every x in the interior of I, with $x \notin D$, there exist $h > 0$ arbitrarily small such that

$$\begin{aligned} \| \varphi(x + h) - \varphi(x)\| &\leqslant h(\| \varphi_r'(x)\| + \epsilon) \leqslant h(g_r'(x) + \epsilon) \\ &\leqslant g(x + h) - g(x) + 2\epsilon h. \end{aligned}$$

By virtue of Lemma 15.2 applied to the functions φ and $x \to g(x) + 2\epsilon x$, we therefore have

$$\| \varphi(v) - \varphi(u)\| \leqslant g(v) - g(u) + 2\epsilon(v - u).$$

This relation holds for all $\epsilon > 0$, which implies the desired relation.

15.6. Corollary. *If $\| \varphi_r'(x)\| \leqslant M$ for every $x \in I \dot{-} D$, then φ is of Lipschitz class with ratio* M *on* I.

Z All the results which we have just established concern functions defined on an interval I of **R**; we shall see later that certain of them can be extended to the case where I is replaced by a convex subset of $\mathbf{R}^n$, by making use of the fact that any two points of a convex set are the endpoints of a line segment contained in the convex set.

But we shall show by an example that it is not possible to replace I by a general metric space:

Let J be the subset of the Euclidean plane $\mathbf{R}^2$ defined as follows: J is the union of the two line segments $[(1,0),(0,k)]$ and $[(-1,0),(0,k)]$.

We define the mapping φ of J into **R** as the restriction to J of the linear form $(x_1, x_2) \to kx_1$.

It is easily verified that if J is taken with the Euclidean metric, then φ is of Lipschitz class with ratio 1 in the neighborhood of each point of J. But the smallest number k' such that φ is of Lipschitz class with ratio k' is k, which can be arbitrarily large.

2. The proof of Lemma 15.1 only uses the fact that $f(\mathrm{D})$ does not contain any interval; one could therefore hope to enlarge the class of sets D entering into the assertions obtained. But careful examples show that such an extension would be of no convenience and little interest; thus, for example, the continuous increasing function g constructed in 13.5 has derivative zero, hence $\leqslant 0$, except at the points of a closed set of "measure zero," and yet is increasing and nonconstant.

3. In all of our results, the right derivatives have played a priviledged role; one can evidently replace right by left in these results. Besides, in general the functions studied will have a two-sided derivative at every point of $\mathrm{I} \dot{-} \mathrm{D}$.

16. DEFINITION OF CONVEX FUNCTIONS. IMMEDIATE PROPERTIES

In Section 8 we have seen the importance of the operation "upper envelope" in creating new classes of functions; this operation will arise once again in connection with convex functions, which will turn out to be identical with the upper envelopes of affine functions.

16.1. Definition. LET f BE A *FINITE* NUMERICAL FUNCTION, DEFINED ON AN INTERVAL I OF **R**.

THEN f IS SAID TO BE *CONVEX* IF, FOR ALL $x_1, x_2 \in \mathrm{I}$, EVERY POINT $\mathrm{M}(x)$ OF THE GRAPH Γ OF f SUCH THAT $x \in [x_1, x_2]$ LIES BELOW* THE

* The terms "below" and "above" are used, here and in the sequel, in the weak sense; for example, the graph of f is included in the set of points lying above the graph of f as well as in the set of points lying below the graph of f.

LINE SEGMENT JOINING $M(x_1)$ AND $M(x_2)$ (WHERE $M(x)$ DENOTES THE POINT $(x, f(x))$); IN OTHER WORDS:

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leqslant \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

FOR ALL $\alpha_1, \alpha_2 \geqslant 0$ SUCH THAT $\alpha_1 + \alpha_2 = 1$.

For example, every affine function $x \to ax + b$ is convex.

16.2. Proposition. *To say that f is convex is equivalent to saying that the set $A(f)$ of points of the plane $\mathbf{R}^2$ situated above the graph of f is convex.*

Indeed, if f is convex and if P_1 and P_2 are points of $A(f)$ with abscissas x_1 and x_2, the segment $M(x_1)M(x_2)$ lies above Γ, hence so does the segment P_1P_2 since P_1 and P_2 lie above $M(x_1)$ and $M(x_2)$, respectively.

Conversely, if $A(f)$ is convex and if $M(x_1)$ and $M(x_2)$ are two points of Γ, these points belong to $A(f)$; the segment $M(x_1)M(x_2)$ belongs to $A(f)$, and therefore lies above Γ.

16.3. Definition. A FINITE NUMERICAL FUNCTION f DEFINED ON AN INTERVAL I OF $\mathbf{R}$ IS SAID TO BE *STRICTLY CONVEX* IF FOR ALL $x_1, x_2 \in I$, EVERY POINT $M(x)$ OF THE GRAPH Γ OF f SUCH THAT $x \in (x_1, x_2)$ LIES *STRICTLY* BELOW THE SEGMENT $M(x_1)\,M(x_2)$.

This is equivalent to saying that f is convex and that its graph Γ does not contain three colinear points. For if M_1, M_2, M_3 are three colinear points of Γ with abscissas x_1, x_2, x_3 and $x_1 < x_2 < x_3$, then for every $x \in [x_1, x_3]$ the point $M(x)$ lies on the segment M_1M_3, for otherwise one would have, for example, $x_1 < x < x_2$ with $M(x)$ lying strictly below M_1M_2, which would imply that M_2 lies strictly above $M(x)M_3$, which is impossible.

Therefore to say that f is strictly convex in I is equivalent to saying that f is convex and that there does not exist any open subinterval of I on which f is affine.

16.4. Definition. A FINITE NUMERICAL FUNCTION f DEFINED ON AN INTERVAL I OF $\mathbf{R}$ IS SAID TO BE *CONCAVE* IF $(-f)$ IS CONVEX.

This amounts to saying that the set of points of the plane lying below the graph of f is convex.

Operations on convex functions

16.5. Proposition. (a) *Every linear combination with positive coefficients of convex functions is convex.*

(b) *Every pointwise limit of convex functions is convex.*

(c) *Every finite upper envelope of convex functions is convex.*

PROOF. Assertions (a) and (b) follow from the inequalities

$$f_i(\alpha_1 x_1 + \alpha_2 x_2) \leqslant \alpha_1 f_i(x_1) + \alpha_2 f_i(x_2),$$

which are preserved by positive linear combination and by passage to the limit.

To prove (c), let (f_i) be a family of convex functions, with upper envelope f (assumed everywhere finite). Each of the sets $A(f_i)$ is convex; we know that $A(f) = \bigcap_i A(f_i)$; therefore $A(f)$ is convex. In other words (Proposition 16.2) f is convex.

16.6. Corollary. *Every finite numerical function on* I *which is the upper envelope of affine functions is convex.*

17. CONTINUITY AND DIFFERENTIABILITY OF CONVEX FUNCTIONS

17.1. Proposition. *To say that f is convex is equivalent to saying that the function p,*

$$p(x, y) = \frac{f(x) - f(y)}{x - y} \qquad \text{(where } x \neq y\text{)},$$

is increasing with respect to each of the variables.

Indeed, $p(x, y)$ is the slope of the line $M(x)M(y)$, and the condition of the proposition is equivalent to the assertion that if a, b, c are three points of I with $a < c < b$, then

$$\text{slope } M(a)M(c) \leqslant \text{slope } M(a)M(b) \leqslant \text{slope } M(c)M(b),$$

which amounts to saying that $M(c)$ lies below the segment $M(a)M(b)$ (Fig. 7).

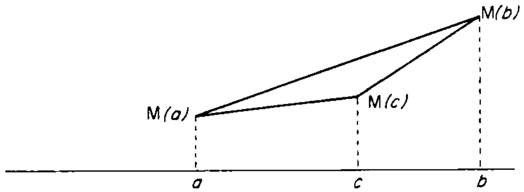

FIG. 7

17.2. Corollary. *To say that f is both convex and concave is equivalent to saying that f is affine.*

We note, in fact, that the concavity of f is characterized by the decreasing nature of p in each variable.

Thus if f is convex and concave, p is a constant; we therefore have

$$f(x) - f(y) = k(x - y).$$

Hence f is affine. The converse is obvious.

17.3. Proposition. *Every function f which is convex in an open interval* I *has a right derivative and a left derivative at every point of* I *(hence is continuous), and if $a < b$, then*

$$f_l'(a) \leqslant f_r'(a) \leqslant \frac{f(b) - f(a)}{a - b} \leqslant f_l'(b) \leqslant f_r'(b).$$

PROOF. Indeed, let $x < a < y$ and put

$$p(t) = \frac{f(t) - f(a)}{t - a}.$$

Proposition 17.1 shows that $p(x) \leqslant p(y)$.

Therefore the infimum of the $p(y)$ (where $a < y$) is finite and equal to $\lim_{y \to a} p(y)$; similarly the supremum of the $p(x)$ (where $x < a$) is finite and equal to $\lim_{x \to a} p(x)$. Therefore $f_r'(a)$ and $f_l'(a)$ exist, and

$$p(x) \leqslant f_l'(a) \leqslant f_r'(a) \leqslant p(y).$$

Letting $y = b$ in this relation, we obtain several of the desired inequalities; the rest are obtained by interchanging the roles of a and b.

Z 1. A function which is convex in a *closed* interval $[a, b]$ need not be continuous at the endpoints; for example, the function f which equals 0 in the interior of $[a, b]$ and 1 at the end points is convex in $[a, b]$ but not continuous.

2. A function which is convex in a bounded open interval may be unbounded; this is the case for the function $(1 - x^2)^{-1}$ on the interval $(-1, 1)$.

3. A function which is continuous and convex on a closed interval may be nondifferentiable at the end points; this is the case for the function $-(1 - x^2)^{1/2}$ on $[-1, 1]$. Nevertheless there is then a generalized derivative which equals $-\infty$ at a and $+\infty$ at b.

17.4. Corollary. *The functions f_l' and f_r' are increasing, and the set of points of* I *at which f is not differentiable is at most countable.*

PROOF. That f_l' is increasing, for example, is contained in the inequalities of Proposition 17.3. These inequalities also show that if $a < b$, the intervals $(f_l'(a), f_r'(a))$ and $(f_l'(b), f_r'(b))$ are disjoint; therefore there cannot be more than countably many such intervals which are nonempty; in other words, the sets of points x at which $f_l'(x) \neq f_r'(x)$ is at most countable.

17.5. Definition. LET f BE A CONVEX FUNCTION IN AN OPEN INTERVAL I, WITH GRAPH Γ. EVERY LINE Δ PASSING THROUGH A POINT M(a) OF Γ AND LYING BELOW Γ IS CALLED A *SUPPORTING LINE* AT THE POINT M(a).

17.6. Proposition. *There exists at least one supporting line at every point of the graph of a function which is convex in an open interval.*

Indeed, the inequalities of Proposition 17.3 show that in order for Δ to be a supporting line at M(a), it is necessary and sufficient that its slope p satisfy the relation

$$f_l'(a) \leqslant p \leqslant f_r'(a).$$

In particular, if f is differentiable at a, the only supporting line at M(a) is the tangent.

17.7. Corollary. *Every function f which is convex in an open interval is the upper envelope of a family of affine functions.*

Indeed, it suffices to take those affine functions whose graphs are supporting lines of the graph of f.
This corollary is the converse of Corollary 16.6.

18. CRITERIA FOR CONVEXITY

Proposition 17.1 already furnishes a convenient criterion for convexity; here is another:

18.1. Proposition. *Let f be a finite numerical function on an open interval* I, *and let* D *be an at most countable subset of* I. *In order that f be convex, it is necessary and sufficient that f be continuous, have a right derivative f_r' at every point of* $\mathrm{I} \dot{-} \mathrm{D}$, *and that f_r' be increasing on* $\mathrm{I} \dot{-} \mathrm{D}$.

PROOF. By Corollary 17.4, we already know that the condition is necessary. Conversely, suppose it is satisfied. Let a, b, c be points of I with $a < b < c$; we put

$$k_1 = \sup_{x \leqslant b} f_r'(x) \qquad \text{and} \qquad k_2 = \inf_{x \geqslant b} f_r'(x).$$

Theorem 15.4 shows that

$$f(b) - f(a) \leqslant k_1(b - a) \qquad \text{and} \qquad k_2(c - b) \leqslant f(c) - f(b).$$

Since $k_1 \leqslant k_2$, we therefore have

$$\frac{f(b) - f(a)}{b - a} \leqslant \frac{f(c) - f(b)}{c - b}.$$

This relation shows that $\mathrm{M}(c)$ lies below the segment $\mathrm{M}(a)\mathrm{M}(b)$. Therefore f is convex.

18.2. Corollary. *If a function f has a second derivative f'' at every point of the open interval* I, *the convexity of f is equivalent to the relation $f'' \geqslant 0$.*

Here is a criterion which enables one to pass from a local statement to a global one.

18.3. Proposition. *Let f be a finite numerical function on an open interval* I. *If every point of* I *is interior to an interval in which f is convex, then f is convex in* I.

Indeed, the local convexity implies that f_r' exists at every point of I, and is locally increasing. Every interval $[u, v]$ of I can be covered by a finite number of open intervals, on each of which f_r' is increasing; it follows at once that $f_r'(u) \leqslant f_r'(v)$. Therefore f_r' is increasing on I. By virtue of Proposition 18.1, f is therefore convex.

Finally, here is a criterion which is of little more than historical interest:

18.4. Proposition. *Let f be a finite numerical function which is continuous (or even only lower semicontinuous) on an open interval* I.

If for all $a, b \in$ I we have

$$f\left(\frac{a + b}{2}\right) \leqslant \frac{1}{2}(f(a) + f(b)),$$

then f is convex.

PROOF. Suppose $a, b \in \mathrm{I}$ with $a < b$; let $y = d(x)$ be the equation of the line $\mathrm{M}(a)\mathrm{M}(b)$. If we did not have $f(x) \leqslant d(x)$ on $[a, b]$, the set ω of points x of $[a, b]$ at which $f(x) > d(x)$ would be nonempty and open, since $f - d$ is lower semicontinuous. Let (α, β) be a connected component of ω; then $f(x) > d(x)$ on (α, β) and $f(x) - d(x) \leqslant 0$ for $x = \alpha$ or β. Therefore

$$f\left(\frac{\alpha+\beta}{2}\right) > d\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}(d(\alpha) + d(\beta)) \geqslant \frac{1}{2}(f(\alpha) + f(\beta))$$

or

$$f\left(\frac{\alpha+\beta}{2}\right) > \frac{1}{2}(f(\alpha) + f(\beta)),$$

contrary to hypothesis.

Z There exist everywhere discontinuous functions on **R** for which $f(x+y) = f(x) + f(y)$ identically; such a function satisfies the inequality of Proposition 18.4 and yet is not convex, for otherwise it would be continuous.

19. CONVEX FUNCTIONS ON A SUBSET OF A VECTOR SPACE

The criterion for convexity of Proposition 16.2 suggests a convenient procedure for defining the convexity of a numerical function defined on a subset X of a vector space E over the field **R**.

19.1. Definition. LET f BE A FINITE NUMERICAL FUNCTION, DEFINED ON A SUBSET X OF A VECTOR SPACE E (OVER **R**).

WE SAY THAT f IS CONVEX IF THE SET $A_X(f)$ OF POINTS OF THE VECTOR SPACE $E \times$ **R** LYING ABOVE THE GRAPH OF f IS CONVEX.

19.2. Proposition. *In order that a finite numerical function f defined on a subset X of a vector space E be convex, it is necessary and sufficient that X be a convex set and that the restriction of f to every segment I of X be convex.*

PROOF. The condition is necessary, for if f is convex, the set $A_X(f)$ is convex; hence X, which is its projection on E, is also convex. On the other hand, for every segment $I \subset X$ the set $A_I(f)$ is the intersection of $A_X(f)$ with the convex set $I \times$ **R**; hence $A_I(f)$ is convex.

The condition is sufficient, for, let P_1 and P_2 be points of $A_X(f)$ and let p_1 and p_2 be their projections on X. If the restriction of f to the segment $I = [p_1p_2]$ is convex, the segment $[P_1P_2]$ belongs to $A_I(f)$, hence also to $A_X(f)$.

Several of the propositions established for convex functions of a real variable extend at once to these generalized convex functions. In particular, linear combinations with coefficients $\geqslant 0$ of convex functions, the limits of convex functions, and the upper envelopes of convex functions are convex functions.

If we restrict ourselves to convex functions defined on an open convex subset X of $\mathbf{R}^n$, other properties can be generalized. For example, such an f is continuous and there exists at least one supporting hyperplane at every point of the graph of f. We shall not prove these last two properties.

19.3. Proposition. *If f is convex, the set $\{x : f(x) \leqslant 0\}$ is convex.*

More generally, if f is convex and g is concave, the set of points x such that $f(x) \leqslant g(x)$ is convex.

Indeed, if f is convex and if $f(a) \leqslant 0$, $f(b) \leqslant 0$, we also have $f(x) \leqslant 0$ for every $x \in [a, b]$; therefore the set in question is convex.

The second statement follows from the first on noting that $f - g$ is convex.

A similar result holds for the set $\{x : f(x) < g(x)\}$.

EXAMPLES. The solid ellipsoid in $\mathbf{R}^n$ defined by $\Sigma\, x_p{}^2/a_p{}^2 \leqslant 1$ is convex.

Similarly, the set of $x \in \mathbf{R}^n$ such that $\Sigma\, \alpha_i \| x - a_i \| \leqslant 1$ is convex when the α_i are $\geqslant 0$.

19.4. Proposition. *Let* X (Y) *be a convex subset of a vector space* E (F); *let f be a convex function on* Y, *and let φ be an affine mapping of* X *into* Y.

Then $f \circ \varphi$ is a convex function on X.

This is an immediate consequence of the relation

$$\varphi(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \varphi(x_1) + \alpha_2 \varphi(x_2) \qquad \text{when} \qquad \alpha_1 + \alpha_2 = 1.$$

EXAMPLE. If f is a convex function on Y, the function on Y^2 defined by $(x, y) \to f(x - y)$ is convex since the mapping $(x, y) \to x - y$ is linear.

Positive-homogeneous convex functions

19.5. Definition. LET f BE A NUMERICAL FUNCTION DEFINED ON A SUBSET X OF A VECTOR SPACE E (OVER $\mathbf{R}$). THEN f IS SAID TO BE *POSITIVE--HOMOGENEOUS* IF X IS A CONE WITH VERTEX O AND IF $f(\lambda x) = \lambda f(x)$ FOR EVERY $x \in$ X AND $\lambda > 0$.

19.6. Proposition. *Let f be a finite positive-homogeneous numerical function defined on a convex cone* X *of a vector space* E.

1. *In order that f be a convex function, it is necessary and sufficient that $f(x_1 + x_2) \leqslant f(x_1) + f(x_2)$ for all $x_1, x_2 \in X$.*

2. *When in addition $f \geqslant 0$, then for f to be convex it is necessary and sufficient that the set B of points x of X at which $f(x) \leqslant 1$ be convex.*

PROOF. 1. If f is convex, then for all $x_1, x_2 \in X$ we have

$$f(x_1 + x_2) = 2f\left(\frac{x_1 + x_2}{2}\right) \leqslant 2\left[f\left(\frac{x_1}{2}\right) + f\left(\frac{x_2}{2}\right)\right] = f(x_1) + f(x_2).$$

Conversely, if $f(u_1 + u_2) \leqslant f(u_1) + f(u_2)$ for all $u_1, u_2 \in X$, then for all $\alpha_1, \alpha_2 \geqslant 0$ we have

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leqslant f(\alpha_1 x_1) + f(\alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2).$$

Therefore f is convex.

2. Let B be the set of $x \in X$ such that $f(x) \leqslant 1$.

If f is convex, Proposition 19.3 shows that B is convex. Conversely, suppose that $f \geqslant 0$ and B is convex.

If $x_1, x_2 \in X$, and if $k_1 > f(x_1)$, $k_2 > f(x_2)$, then

$$k_1^{-1}x_1 \in B \quad \text{and} \quad k_2^{-1}x_2 \in B.$$

Thus, since B is convex, the barycenter $(k_1 + k_2)^{-1}(x_1 + x_2)$ of these points also belongs to B; therefore

$$f(x_1 + x_2) \leqslant k_1 + k_2 \quad \text{when} \quad k_1 > f(x_1) \quad \text{and} \quad k_2 > f(x_2).$$

It follows that

$$f(x_1 + x_2) \leqslant f(x_1) + f(x_2).$$

Thus f is convex.

EXAMPLE. For every scalar $\alpha > 0$, the function f

$$(x_p) \to \left(\sum |x_p|^\alpha\right)^{1/\alpha}$$

is positive-homogeneous on $\mathbf{R}^n$. But if $\alpha \geqslant 1$, each of the functions $|x_p|^\alpha$ is convex; hence f^α is convex and the set $\{x : f(x) \leqslant 1\}$ is convex; therefore f is convex.

Twice-differentiable convex functions

Let f be a numerical function defined on a convex open subset X of $\mathbf{R}^n$. We assume that f has continuous second partial derivatives.

For every $a \in X$ and every $\alpha \in \mathbf{R}^n$, the mapping $\varphi : t \to f(a + t\alpha)$, which is defined on an open interval of $\mathbf{R}$ containing 0, has a second derivative at 0:

$$\varphi''(0) = \sum \alpha_i \alpha_j \frac{\partial^2 f}{\partial x_i \, \partial x_j}(a).$$

But the convexity of f is equivalent to that of its restrictions to the segments of X, that is, to that of the functions φ. This convexity is also expressed (Corollary 18.2) by the relation $\varphi''(0) \geqslant 0$ for all a, α; we can therefore state:

19.7. Proposition. *In order that a numerical function f with continuous second partial derivatives defined on an open convex set X in $\mathbf{R}^n$ be convex, it is necessary and sufficient that for every $a \in X$ the quadratic form*

$$\sum \alpha_i \alpha_j \frac{\partial^2 f}{\partial x_i \, \partial x_j}(a) \qquad \textit{be} \quad \geqslant 0.$$

Z 1. It should be noted that the relations $\partial^2 f / \partial x_i^2 \geqslant 0$ by no means suffice for the convexity of f. For example, the function $(x^2 + 3xy + 2y^2)$ is not convex in any open convex subset of $\mathbf{R}^2$.

2. If f is defined on an open nonconvex subset X of $\mathbf{R}^n$ and satisfies the condition of Proposition 19.7, then it is convex in every open convex subset of X, but cannot in general be extended to a convex function on a convex set containing X. An example is the function $f(x, y) = (y - x^2)^2 + x^4$ on the subset of $\mathbf{R}^2$ defined by $y < 4x^2$.

20. THE MEAN RELATIVE TO A MONOTONE FUNCTION

Even elementary analysis uses various averaging operations such as the arithmetic, geometric, quadratic, or harmonic mean. We are going to generalize these elementary notions and indicate some inequalities based on convexity.

To simplify the language, we shall here call every finite family $\mu = (\alpha_i, x_i)_{i \in I}$, where $x_i \in E$ and the α_i are numbers >0, a *discrete measure* on the set E. We say that μ is carried by a subset X of E if X contains all the points x_i.

The total mass of $\mu = (\alpha_i, x_i)_{i \in I}$ is the number $\| \mu \| = \sum \alpha_i$. If φ denotes a mapping of E into another set F, and if $\mu = (\alpha_i, x_i)_{i \in I}$ is carried by E, we call the discrete measure $\varphi(\mu) = (\alpha_i, \varphi(x_i))_{i \in I}$ on F the image of μ under φ; clearly $\| \varphi(\mu) \| = \| \mu \|$.

If E is a vector space, the barycenter of μ is the point $\| \mu \|^{-1} \sum \alpha_i x_i$ of E, which we shall denote by $\mathscr{M}_1(\mu)$; for every scalar $k > 0$, the measure $k\mu = (k\alpha_i, x_i)_{i\in I}$ has the same barycenter as μ; in particular, for $k = \|\mu\|^{-1}$ the measure $k\mu$ has total mass 1, which is sometimes convenient.

The barycenter $\mathscr{M}_1(\mu)$ belongs to every convex set which carries μ; in particular, if $E = \mathbf{R}$, then

$$\inf(x_i) \leqslant \mathscr{M}_1(\mu) \leqslant \sup(x_i).$$

20.1. Definition. LET f BE A STRICTLY MONOTONE CONTINUOUS NUMERICAL FUNCTION ON AN INTERVAL A OF $\mathbf{R}$; LET $\mu = (\alpha_i, x_i)_{i\in I}$ BE A DISCRETE MEASURE ON A, AND LET f^{-1} DENOTE THE FUNCTION INVERSE TO f.

THE *MEAN OF μ RELATIVE TO f* IS DEFINED AS THE NUMBER $f^{-1}(\mathscr{M}_1(f(\mu)))$, THAT IS, THE NUMBER a SUCH THAT

$$\left(\sum_i \alpha_i\right) f(a) = \sum_i \alpha_i f(x_i).$$

WE SHALL DENOTE THIS NUMBER a BY $\mathscr{M}_f(\mu)$.

The existence of the number a follows from the fact that $\mathscr{M}_1(f(\mu)) \in f(A)$; its uniqueness follows from the fact that f is injective.

It is clear that $\mathscr{M}_f(\mu) \in A$, and more precisely that

$$\inf(x_i) \leqslant \mathscr{M}_f(\mu) \leqslant \sup(x_i).$$

We also note for future use that $\mathscr{M}_f = \mathscr{M}_{-f}$.

EXAMPLE 1. If we take for f the identity mapping $x \to x$ of $\mathbf{R}$ into $\mathbf{R}$, then

$$\mathscr{M}_f(\mu) = \mathscr{M}_1(\mu);$$

this mean will be called the *arithmetic* mean because, in the case where the α_i are equal, we have

$$\mathscr{M}_1(\mu) = n^{-1}\left(\sum x_i\right),$$

where n is the number of elements of I.

More generally, we will denote by $\mathscr{M}_r$ the mean relative to the numerical function $x \to x^r$, defined on $\mathbf{R}_+$ if $r > 0$, and on $\mathbf{R}_+^*$ if $r < 0$. For example, $\mathscr{M}_{-1}$ will be the *harmonic* mean, $\mathscr{M}_2$ the *quadratic* mean.

EXAMPLE 2. For reasons which will be better understood shortly,

$\mathscr{M}_0$ denotes the mean relative to the function $x \to \log x$ defined on $\mathbf{R}_+^*$. When all the α_i are equal to 1, the relation

$$n \log a = \sum \log x_i = \log \left(\prod x_i\right)$$

shows that $\mathscr{M}_0(\mu) = (\prod x_i)^{1/n}$, that is, the geometric mean of the x_i. This is why $\mathscr{M}_0$ will be called the *geometric* mean.

COMPARISON OF MEANS. For every continuous strictly monotone function f on A, $\mathscr{M}_f$ is a numerical function on the set of discrete measures on A. We propose to investigate under what conditions $\mathscr{M}_f = \mathscr{M}_g$ or, more generally, $\mathscr{M}_f \leqslant \mathscr{M}_g$.

20.2. Lemma. *Let f be a convex numerical function on an interval* A *of* $\mathbf{R}$. *Then for every discrete measure μ on* A

$$f(\mathscr{M}_1(\mu)) \leqslant \mathscr{M}_1(f(\mu)).$$

The equality holds only if f is affine on the smallest interval which carries μ.

PROOF. If $\mu = (\alpha_i, x_i)_{i\in I}$, let G be the barycenter of the measure $(\alpha_i, M(x_i))_{i\in I}$, where $M(x_i)$ denotes the point of the graph Γ of f with abscissa x_i.

The coordinates x, y of G are

$$x = \mathscr{M}_1(\mu) \qquad \text{and} \qquad y = \mathscr{M}_1(f(\mu)).$$

Since f is convex, the set $A(f)$ is convex, and therefore contains the barycenter G; in other words, G lies above the graph of f, which is expressed by the relation $f(x) \leqslant y$, and which is just the desired inequality.

If f is affine on the interval $[\inf(x_i), \sup(x_i)]$, G evidently lies on the graph of f, hence $f(x) = y$. If f is not affine on the above interval, denote $\inf(x_i)$ by x_1, and $\sup(x_i)$ by x_2; for every $x \in (x_1, x_2)$, $M(x)$ lies strictly above Γ; this is in particular the case for the barycenter of the discrete measure $\nu' = (\alpha_i, M(x_i))_{i=1,2}$; we also put $\nu'' = (\alpha_i, M(x_i))_{i\neq 1,2}$.

A well-known property of barycenters shows that G belongs to the segment joining the barycenters of ν' and ν''; since both of these lie above Γ, and one of them strictly above Γ, G lies strictly above Γ; in other words, $f(x) < y$.

EXAMPLE. If f is strictly convex, then for every μ which is not carried by a single point we have

$$f(\mathscr{M}_1(\mu)) < \mathscr{M}_1(f(\mu)).$$

20.3. Corollary. *Let f be a continuous numerical function which is strictly increasing on an interval* A *of* **R**; *then*

$$(\mathscr{M}_1 \leqslant \mathscr{M}_f) \Leftrightarrow (f \text{ is convex}); \qquad (\mathscr{M}_f \leqslant \mathscr{M}_1) \Leftrightarrow (f \text{ is concave}).$$

When f is decreasing, the analogous criteria are obtained by interchanging the words "convex" and "concave."

In these various cases (f convex or concave) $\mathscr{M}_1(\mu) = \mathscr{M}_f(\mu)$ only if f is affine on an interval which carries μ.

Proof. Suppose f is increasing. If f is convex, Lemma 20.2 shows that

$$f(\mathscr{M}_1(\mu)) \leqslant \mathscr{M}_1(f(\mu)), \qquad \text{whence} \qquad \mathscr{M}_1(\mu) \leqslant f^{-1}(\mathscr{M}_1(f(\mu))) = \mathscr{M}_f(\mu).$$

Conversely, if $\mathscr{M}_1 \leqslant \mathscr{M}_f$, which can be expressed by

$$f(\mathscr{M}_1(\mu)) \leqslant \mathscr{M}_1(f(\mu)) \qquad \text{for every} \quad \mu,$$

then in particular we have, for all μ of the form $(\alpha_i, x_i)_{i=1,2}$ with $\alpha_1 + \alpha_2 = 1$,

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leqslant \alpha_1 f(x_1) + \alpha_2 f(x_2),$$

that is, f is convex.

It follows from Lemma 20.2 that if f is convex, $\mathscr{M}_1(\mu) = \mathscr{M}_f(\mu)$ only if f is affine on an interval which carries μ.

The case where f is concave is treated similarly. If f is decreasing, then $-f$ is increasing, and since $\mathscr{M}_f = \mathscr{M}_{-f}$, the stated equivalences follow from those which have just been established.

20.4. Theorem. *Let f and g be continuous strictly monotone functions on an interval* A *of* **R**.

1. *$\mathscr{M}_f = \mathscr{M}_g$ is equivalent to the relation*

$$g = \alpha f + \beta, \qquad \textit{where} \qquad \alpha, \beta \in \mathbf{R}, \quad \textit{with} \quad \alpha \neq 0.$$

2. *The relation $\mathscr{M}_f \leqslant \mathscr{M}_g$ is equivalent to the assertion that either g is increasing and $g \circ f^{-1}$ is convex, or g is decreasing and $g \circ f^{-1}$ is concave.*

In these two cases, when $g \circ f^{-1}$ is strictly convex or concave, $\mathscr{M}_f(\mu) = \mathscr{M}_g(\mu)$ holds only when μ is carried by a single point.

Proof. 1. The relation $\mathscr{M}_f = \mathscr{M}_g$ means that for every measure $(\alpha_i, x_i)_{i \in I}$ such that $\sum \alpha_i = 1$, we have

$$f^{-1}\left(\sum \alpha_i f(x_i)\right) = g^{-1}\left(\sum \alpha_i g(x_i)\right).$$

If we put

$$h = g \circ f^{-1} \qquad \text{and} \qquad y_i = f(x_i),$$

this relation becomes

$$h\left(\sum \alpha_i y_i\right) = \sum \alpha_i (h \circ f)(x_i) = \sum \alpha_i h(y_i)$$

or

$$\mathscr{M}_1(f(\mu)) = \mathscr{M}_h(f(\mu)).$$

Since the discrete measure $f(\mu)$ can be an arbitrary discrete measure on $f(\mathrm{A})$, this relation is written simply as $\mathscr{M}_1 = \mathscr{M}_h$. Corollary 20.3 shows that this implies that h is both convex and concave, hence affine; in other words, we have

$$(g \circ f^{-1})(y) = \alpha y + \beta \qquad \text{with} \quad \alpha \neq 0,$$

or, putting $f^{-1}(y) = x$,

$$g(x) = \alpha f(x) + \beta.$$

2. We now study the inequality $\mathscr{M}_f \leqslant \mathscr{M}_g$, assuming to begin with f and g are increasing. A calculation parallel to the preceding shows that the relation $\mathscr{M}_f \leqslant \mathscr{M}_g$ is equivalent to $\mathscr{M}_1 \leqslant \mathscr{M}_h$ which, by virtue of Corollary 20.3, is equivalent to saying that $h = g \circ f^{-1}$ is convex.

If f is increasing and g decreasing, since $\mathscr{M}_g = \mathscr{M}_{-g}$, the inequality $\mathscr{M}_f \leqslant \mathscr{M}_g$ expresses the convexity of the function $-g \circ f^{-1}$, hence the concavity of $g \circ f^{-1}$.

The other two cases are deduced from this by observing that the two functions $u \to g(u)$ and $u \to g(-u)$ are simultaneously either convex or concave.

When h is strictly convex or concave, Corollary 20.3 shows that $\mathscr{M}_1(f(\mu)) = \mathscr{M}_h(f(\mu))$ only if $f(\mu)$, hence also μ, is carried by a single point, which proves the last assertion.

20.5. A practical criterion. *Suppose that f and g have second derivatives with f' and g' everywhere $\neq 0$; then*

$$(\mathscr{M}_f \leqslant \mathscr{M}_g) \Leftrightarrow (f''/f' \leqslant g''/g').$$

Proof. Since replacing f by $-f$ or g by $-g$ leaves invariant all the quantities entering into the relation to be established, we can consider only the case of f, g increasing.

The relation $\mathscr{M}_f \leqslant \mathscr{M}_g$ is then equivalent to the convexity of $g \circ f^{-1}$; let us put

$$F(y) = (g \circ f^{-1})(y) \qquad \text{and} \qquad y = f(x).$$

Then

$$\frac{dF}{dy} = \frac{dF}{dx}\frac{dx}{dy} = \frac{g'(x)}{f'(x)}.$$

The convexity of F is expressed by dF/dy being an increasing function of y or, since f is increasing with respect to x, by g'/f' being an increasing function of x, that is $(g''f' - g'f'') \geqslant 0$, or $f''/f' \leqslant g''/g'$.

Example. Put

$$f_r(x) = x^r \qquad \text{and} \qquad f_0(x) = \log x.$$

Then for all r

$$f_r''/f_r' = (r-1)x^{-1} \qquad \text{on} \quad (0, \infty).$$

But $(r-1)x^{-1}$ is an increasing function of r; therefore *$\mathscr{M}_r$ is also an increasing function of r.* More precisely, since $(r-1)x^{-1}$ is a strictly increasing function of r, then if $r < r'$ and if μ is not carried by a point,

$$\mathscr{M}_r(\mu) < \mathscr{M}_{r'}(\mu).$$

We note here that the case $r = 0$ is not exceptional, which justifies the notation adopted.

In particular we obtain the classical inequalities:

harmonic mean $\leqslant$ geometric mean $\leqslant$ arithmetic mean $\leqslant$ quadratic mean.

Remark. Criterion 20.5 is particularly interesting for, in the family of functions being considered, it shows that each of the classes of functions f which give the same mean is characterized by the quotient f''/f'; these quotients constitute a vector space of functions on A, whose natural order goes over into the order on the family of means.

20.6. Proposition. Hölder's inequality. *Let $\alpha, \beta, \ldots, \lambda$ be a finite sequence of numbers >0 such that $\alpha + \beta + \cdots + \lambda = 1$, and let $(a_i)_{i\in I}, (b_i)_{i\in I}, \ldots, (l_i)_{i\in I}$ be a sequence of the same length of finite families of numbers $\geqslant 0$.*

Then

$$\sum_i a_i^\alpha b_i^\beta \cdots l_i^\lambda \leqslant \left(\sum a_i\right)^\alpha \left(\sum b_i\right)^\beta \cdots \left(\sum l_i\right)^\lambda. \tag{1}$$

The equality holds only when either all the elements of one family are zero or all of the families are proportional.

PROOF. If all the elements of one family are zero, the relation becomes $0 = 0$. In the contrary case, we can write

$$\frac{\sum a_i^\alpha b_i^\beta \cdots l_i^\lambda}{(\sum a_i)^\alpha (\sum b_i)^\beta \cdots (\sum l_i)^\lambda} = \sum \left(\frac{a_i}{\sum a_i}\right)^\alpha \left(\frac{b_i}{\sum b_i}\right)^\beta \cdots \left(\frac{l_i}{\sum l_i}\right)^\lambda. \qquad (2)$$

But the relation $\mathscr{M}_0 \leqslant \mathscr{M}_1$ shows that

$$\mathrm{A}^\alpha \mathrm{B}^\beta \cdots \mathrm{L}^\lambda \leqslant \alpha \mathrm{A} + \beta \mathrm{B} + \cdots + \lambda \mathrm{L},$$

with equality only if $\mathrm{A} = \mathrm{B} = \cdots = \mathrm{L}$; the right side of (2) is therefore bounded from above by

$$\sum \left(\alpha \frac{a_i}{\sum a_i} + \beta \frac{b_i}{\sum b_i} + \cdots + \lambda \frac{l_i}{\sum l_i}\right) = \alpha + \beta + \cdots + \lambda = 1,$$

with equality only if, for all i,

$$\frac{a_i}{\sum a_i} = \frac{b_i}{\sum b_i} = \cdots = \frac{l_i}{\sum l_i},$$

which asserts the proportionality of the families (a_i), (b_i), ...

Another form of Hölder's inequality

Let r be a number >0; we replace, in (1), a_i by $\omega_i a_i^{r/\alpha}$, b_i by $\omega_i b_i^{r/\beta}$, ..., where the ω_i are >0, obtaining

$$\sum \omega_i (a_i b_i \cdots l_i)^r \leqslant \left(\sum \omega_i a_i^{r/\alpha}\right)^\alpha \left(\sum \omega_i b_i^{r/\beta}\right)^\beta \cdots .$$

With a change of notation for the exponents, this relation can be written as

$$\left(\sum \omega_i (a_i b_i \cdots l_i)^r\right)^{1/r} \leqslant \left(\sum \omega_i a_i^{p_1}\right)^{1/p_1} \left(\sum \omega_i b_i^{p_2}\right)^{1/p_2} \cdots,$$

where $p_1, p_2, \ldots > 0$ with

$$1/p_1 + 1/p_2 + \cdots = 1/r.$$

This relation can be written more concisely as

$$\mathscr{M}_r(ab \cdots l) \leqslant \mathscr{M}_{p_1}(a) \mathscr{M}_{p_2}(b) \cdots .$$

A particularly important case is that in which $r = 1$ and where there are only two families.

20.7. Minkowski's inequality. This inequality says simply that a certain positive-homogeneous function is convex in $\mathbf{R}^n$:

Given numbers $\alpha_1, \alpha_2, \ldots, \alpha_n \geqslant 0$ and a number $p \geqslant 1$, we consider the function f defined on $\mathbf{R}^n$ by

$$x = (x_i) \to \left(\sum \alpha_i \mid x_i \mid^p\right)^{1/p}.$$

The function f is clearly positive-homogeneous and, as in the example following Proposition 19.6, is convex. We therefore have the inequality (of Minkowski)

$$\left(\sum \alpha_i \mid x_i + y_i \mid^p\right)^{1/p} \leqslant \left(\sum \alpha_i \mid x_i \mid^p\right)^{1/p} + \left(\sum \alpha_i \mid y_i \mid^p\right)^{1/p}.$$

When the α_i are >0, with $p > 1$, f is strictly convex (except on the rays issuing from the origin O), and therefore the inequality is strict when the families (x_i) and (y_i) are not proportional.

This inequality extends at once to x_i, y_i complex.

PROBLEMS

Note: The rather difficult problems are marked with an asterisk.

NUMERICAL FUNCTIONS DEFINED ON AN ARBITRARY SET

*1. Let $(x_i)_{i\in \mathrm{I}}$ and $(y_j)_{j\in \mathrm{J}}$ be finite families of real numbers $\geqslant 0$ such that

$$\sum_{\mathrm{I}} x_i = \sum_{\mathrm{J}} y_j .$$

Show that there exists a finite family $(z_{ij})_{(i,j)\in \mathrm{I}\times\mathrm{J}}$ of real numbers $\geqslant 0$ such that

$$x_i = \sum_j z_{ij} \quad \text{for every} \quad i \in \mathrm{I}$$

and

$$y_j = \sum_i z_{ij} \quad \text{for every} \quad j \in \mathrm{J}.$$

*2. Extend the preceding result to families of elements $\geqslant 0$ of the ordered space $\mathscr{F}(\mathrm{E}, \mathbf{R})$, where E is an arbitrary set.

NUMERICAL FUNCTIONS DEFINED ON A TOPOLOGICAL SPACE

3. Let f be a numerical function defined on $\mathbf{R}^2$, such that for every x the mapping $y \to f(x, y)$ is increasing, and for every y the mapping $x \to f(x, y)$ is increasing. Show that f tends to a limit when x and $y \to +\infty$, in a sense which it is required to make precise.

4. Let f be a numerical function defined on $\mathbf{R}$, and define A as the set of points a of $\mathbf{R}$ such that

$$\limsup_{x \to a, x > a} f(x) \neq \limsup_{x \to a, x < a} f(x).$$

Show that A is at most countable.

***5.** Let f be a numerical function defined on $\mathbf{R}$. Show that the set of points a of $\mathbf{R}$ such that $\lim_{x \to a, x \neq a} f(x)$ exists and is different from $f(a)$ is at most countable.

***6.** Let f be a numerical function defined on $\mathbf{R}$. One says that f has a *relative maximum* at a point $a \in \mathbf{R}$ if there exists a neighborhood V of a such that $f(x) \leqslant f(a)$ for every $x \in \mathrm{V}$. Let A be the set of all such points a. Show that $f(\mathrm{A})$ is at most countable.

7. Let $(f_i)_{i \in \mathrm{I}}$ be an equicontinuous family of finite numerical functions defined on a metric space E.

(a) Show that the functions $\sup_{i \in \mathrm{I}} f_i$ and $\inf_{i \in \mathrm{I}} f_i$ are uniformly continuous.

(b) Show that the family (f_J) of finite functions of the form $\sup_{i \in \mathrm{J}} f_i$ or $\inf_{i \in \mathrm{J}} f_i$, where J is an arbitrary subset of I, is equicontinuous.

SEMICONTINUOUS NUMERICAL FUNCTIONS

8. Let f and g be lower semicontinuous mappings of a topological space E into $\mathbf{R}$. Show that if $f + g$ is continuous, then f and g are also.

***9.** Let f be a finite numerical lower semicontinuous function on the interval $[0, 1]$.

(a) Show that f is the upper envelope of the continuous functions $g \leqslant f$.

(b) Show that f is the limit of an increasing sequence of continuous functions.

***10.** Extend the preceding result to every finite and lower semicontinuous function f on a metric space with a countable base.

11. Let f be a numerical lower semicontinuous function defined on a topological space E. Show that for every nonempty subset A of E we have

$$\sup_{x\in\bar{A}} f(x) = \sup_{x\in A} f(x).$$

12. For every rational number $r = p/q$ in irreducible form ($q > 0$) put $f(r) = q$; show that f is lower semicontinuous on $\mathbf{Q}$ and that at every point $r \in \mathbf{Q}$, the oscillation of f is $+\infty$.

***13.** Let E be a metric space and let K be a compact subset of $\mathbf{R} \times E$. For every $x \in \mathbf{R}$ we put $q(x)$ = diameter of the set of points of K with abscissa x. Show that $q(x)$ is an upper semicontinuous function.

14. Let f be a numerical function defined on a topological space E. Show that the set of points of E at which the oscillation of f is $\geqslant \lambda$ (where $\lambda > 0$) is closed.

Deduce from this that the set of points of continuity of f is a countable intersection of open sets.

***15.** Let f be a continuous numerical function defined on the square $C = [0, 1]^2$; f is said to be piecewise affine if there exists a finite covering of C by triangles, on each of which f is affine. Let $\mathscr{P}$ be the vector space of these functions, taken with the topology of uniform convergence. Let $\alpha(f)$ be the ordinary area of the graph of f in the Euclidean space $\mathbf{R}^3$. Show that $\alpha(f)$ is lower semicontinuous on $\mathscr{P}$.

STONE-WEIERSTRASS THEOREM

16. It is required to show, using an induction argument, that there exists a sequence (p_n) of real polynomials which is increasing on the interval $[0, 1]$ and which converges uniformly to $\sqrt{t}$ on this interval. To show this, put

$$p_{n+1}(t) = p_n(t) + \tfrac{1}{2}(t - p_n^2(t)) \qquad \text{and} \qquad p_1 = 0,$$

and show that the sequence (p_n) thus defined recursively has the required properties.

17. Let E be a compact space and let (f_i), where $i = 1, 2, \ldots, n$, be a family of n elements of $\mathscr{C}(E, \mathbf{R})$ which separates the points of E. Show that E is homeomorphic with a subset of $\mathbf{R}^n$.

18. Let a_1, a_2, a_3 be three noncolinear points of $\mathbf{R}^2$. We denote by f_i the function $x \to \| x - a_i \|$. Show that every continuous numerical function on $\mathbf{R}^2$ is the limit of a sequence of polynomials (without constant terms) in f_1, f_2, f_3 or in the squares of these functions, which converges uniformly on every compact set in $\mathbf{R}^2$.

THEOREMS OF FINITE INCREASE

19. Let f be a finite and continuous numerical function on an interval $\mathrm{I} = [a, b]$ of $\mathbf{R}$.

One says that f is *increasing to the right* at a point $x_0 \in \mathring{\mathrm{I}}$ if there exists a number $x_1 > x_0$ such that $f(x_0) \leqslant f(x)$ for all $x \in [x_0, x_1]$. Decrease to the right is similarly defined.

Show that if $f(a) = f(b) = 0$, there exists a point c_1 of $\mathring{\mathrm{I}}$ at which f is increasing to the right, and a point c_2 of $\mathring{\mathrm{I}}$ at which f is decreasing to the right (same for left).

20. Let f be a continuous and differentiable numerical function on an interval $[a, b]$. Show, using the classical theorem of Rolle, that even if f' is not continuous, f' assumes all the values lying between $f'(a)$ and $f'(b)$ (show that for every λ such that $f'(a) < \lambda < f'(b)$ there exists an $x \in [a, b]$ such that one of the chords $\mathrm{M}(a)\mathrm{M}(x)$ and $\mathrm{M}(x)\mathrm{M}(b)$ has slope λ). Deduce from this that $f'([a, b])$ is connected.

21. Let f be a continuous and differentiable numerical function on an *open* interval (a, b). If $\alpha = \lim_{x \to a} f'(x)$ exists and is finite, show that $\lim_{x \to a} f(x)$ exists and that f has a continuous extension to $[a, b)$ which is right differentiable at a, with derivative equal to α.

CONVEX FUNCTIONS

22. Let $(f_i)_{i \in \mathrm{I}}$ be a family of convex functions on an open interval I of $\mathbf{R}$. Show that if the family (f_i) is uniformly bounded from above and if there exists a $c \in \mathrm{I}$ such that the set $(f_i(c))$ is bounded from below, the family (f_i) is equicontinuous on every compact interval of I.

23. Let (f_n) be a sequence of convex functions on an open interval I of $\mathbf{R}$ which converges pointwise to a finite function f. Show by using the preceding problem that the convergence is uniform on every compact interval of I.

***24.** Extend the preceding problems to convex functions on a convex subset X of $\mathbf{R}^2$ (or more generally of $\mathbf{R}^n$).

25. Let f and g be convex functions on a convex subset X of a vector space; show that if $(f + g)$ is affine on X, then f and g are also.

26. Let f be a convex function on a compact interval $[a, b]$ and such that $f_r'(a)$ and $f_l'(b)$ are finite. Show that f is the restriction to $[a, b]$ of a convex function on $\mathbf{R}$.

27. Let g be a continuous numerical function of the pair (x, t), where $a \leqslant x \leqslant b$ and $\alpha \leqslant t \leqslant \beta$. If g is a convex function of x for every t, prove the convexity of the function

$$f(x) = \int_\alpha^\beta g(x, t)\, dt.$$

28. Let f be a convex function on $\mathbf{R}$ and let φ be a continuous function $\geqslant 0$ on $\mathbf{R}$, which vanishes off a compact set.

(a) Show that the function $F(x) = \int_{\mathbf{R}} f(x - t)\varphi(t)\, dt$ is convex on $\mathbf{R}$.

(b) Let ψ be the function defined by

$$\psi(t) = 0 \qquad \text{for} \quad |t| \geqslant 1;$$
$$\psi(t) = k \exp[(t^2 - 1)^{-1}] \qquad \text{for} \quad |t| < 1,$$

where k is such that the integral of ψ over $\mathbf{R}$ equals 1. Show that ψ has derivatives of all orders on $\mathbf{R}$.

(c) Show that the convex function

$$f_a(x) = a\int_{-\infty}^{+\infty} f(x - t)\psi(at)\, dt = a\int_{-\infty}^{+\infty} f(t)\psi(a(x - t))\, dt$$

is infinitely differentiable and that f_a converges to f uniformly on every compact interval of $\mathbf{R}$ as $a \to +\infty$.

29. Let f be an increasing convex function in $(0, +\infty)$. Show that either $f = \text{constant}$ or $\lim_{x\to+\infty} f(x) = +\infty$.

30. Let f be a convex function in an interval $[a, +\infty)$. Show that $f(x)/x$ tends to a finite limit or to $+\infty$ as $x \to +\infty$. Show that when this limit is $\leqslant 0$, the function f is decreasing on $[a, +\infty)$.

***31.** Let f be a convex function in the interval $[0, +\infty)$.

(a) Show that the function φ defined by

$$\varphi(x) = f(x) - xf_r'(x) \qquad \text{(the } y \text{ intercept of the right tangent)}$$

is decreasing.

(b) Show that if $\lim_{x\to+\infty} \varphi(x)$ is a finite number β, the same is true of $\alpha = \lim_{x\to+\infty} f(x)/x$ and that $f(x) - (\alpha x + \beta)$ is $\geqslant 0$ and tends to 0 as $x \to +\infty$.

32. Extend Proposition 18.4 to semicontinuous functions f such that for all $a, b \in \mathrm{I}$ there exists an $x \in (a, b)$ for which $\mathrm{M}(x)$ lies below the segment $\mathrm{M}(a)\mathrm{M}(b)$.

33. Let f be a strictly increasing (decreasing) function on an open interval I of $\mathbf{R}$. Show that for f to be convex it is necessary and sufficient that its inverse f^{-1} be concave (convex).

34. Let f be a continuous finite numerical function on an interval I of $\mathbf{R}$. Show that if for every $a \in \mathrm{I}$ there exists an open interval (of a line in $\mathbf{R}^2$) containing $\mathrm{M}(a)$ and lying below the graph of f, then f is convex.

35. Using the properties of the projective transformation $(x, y) \to (x^{-1}, yx^{-1})$ of the halfplane $x > 0$ onto itself, show that if f is convex for $x > 0$ the same is true of the function $x \to xf(x^{-1})$ and conversely.
Then give another proof using Problem 28.

36. (a) Using Problem 28, show that if f and g are positive, convex, and increasing (or decreasing) on the same interval I, the product fg is convex.

(b) Using the same method, state a criterion for $f \circ g$ to be convex.

37. Let f be a positive function on an interval $\mathrm{I} \subset \mathbf{R}$. We shall say that f is *logarithmically convex* if $\log f$ is convex in I.
Show that if f is logarithmically convex, f is convex.
Show that if f and g are logarithmically convex, so is fg.
Use Problem 28 here, as well as in the following problems.

***38.** Show that the sum of two logarithmically convex functions is logarithmically convex.

39. Let $f(x, t)$ be a finite numerical function >0, defined and continuous on the product $\mathrm{I} \times \mathrm{J}$ of two open intervals I and J of $\mathbf{R}$ and such that, for every $t \in \mathrm{J}$, $f(x, t)$ is logarithmically convex with respect to x.
Show that if, for every $x \in \mathrm{I}$, the integral

$$g(x) = \int_{\mathrm{J}} f(x, t)\, dt$$

is convergent, then g is logarithmically convex. Use Problem 38.

40. Let

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\, dt.$$

Show, by using Problem 39, that for every $x > 0$ this integral is finite and that the function Γ is logarithmically convex on $(0, \infty)$.
Show that $\Gamma(n + 1) = n!$ for every integer n.

41. Let $\mathscr{K}$ be the set of functions which are concave and positive in an open interval (a, b).

(a) Show that $\mathscr{K}$ is ordered by the relation

$$f \prec g \quad \text{if} \quad g - f \in \mathscr{K}.$$

(b) We will say that a function $f \in \mathscr{K}$ is *extremal* if every relation $f \prec g$, where $g \in \mathscr{K}$, implies that $g = \lambda f$ with $\lambda \in [0, 1]$.

Show that the only extremal elements of $\mathscr{K}$ are the traces of functions f which vanish at a and b and are affine on two intervals $[a, c]$, $[c, b]$ (where $c \in (a, b)$), or of functions which are affine on (a, b) and vanish at a or b.

(c) Show, by using Problem 28, that $\mathscr{K}$ is a lattice for the order $\prec$.

MEANS AND INEQUALITIES

42. Show that the area of a triangle of perimeter $2p$ is maximum when its sides a, b, c are equal (use the relation $\mathscr{M}_0 \prec \mathscr{M}_1$ and the expression for the area as a function of p, $(p - a)$, $(p - b)$, $(p - c)$).

43. Show that the volume of a rectangular parallelepiped of given area is a maximum when it is a cube.

44. Let $a_1, a_2, \ldots, a_n$ be numbers > 0, and let a be their geometric mean. Show that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geqslant (1 + a)^n,$$

with the equality holding only if the a_i are equal.

45. Let μ be a discrete measure on $(0, \infty)$. Show that the function $r \to (\mathscr{M}_r(\mu))^r$ is logarithmically convex.

46. Let φ be a continuous numerical function on an open interval I of $\mathbf{R}$, and let $a \in \mathrm{I}$.

(a) Show that there exists a strictly increasing function Φ (and only one) on $(0, \infty)$ such that

$$\Phi'' = \varphi\Phi'; \qquad \Phi(a) = 0; \qquad \Phi'(a) = 1.$$

(b) If μ denotes an arbitrary discrete measure on I, we put

$$\mathrm{I}(\varphi) = \Phi(\mathscr{M}_\Phi(\mu)).$$

Show that the function I is logarithmically convex in the vector space $\mathscr{C}(\mathrm{I}, \mathbf{R})$.

47. For every $f \in \mathscr{C}([0, 1], \mathbf{R})$ and every $p > 0$ we put

$$\mathrm{N}_p(f) = \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}.$$

(a) Show that if $p > 1$, N_p is a convex function on $\mathscr{C}([0, 1], \mathbf{R})$. From now on we assume that f is fixed and > 0.

(b) Show that the function $p \to \mathrm{N}_p(f)$ is increasing and infinitely differentiable.

(c) Show that $(\mathrm{N}_p(f))^p$ is a logarithmically convex function of p.

(d) Show that $\mathrm{N}_p(f)$ is a logarithmically convex function of p^{-1} (use Problem 35).

48. With the notation of Problem 47, show that if $p^{-1} + q^{-1} = 1$, then

$$\mathrm{N}_1(fg) \leqslant \mathrm{N}_p(f)\mathrm{N}_q(g).$$

***49.** Let $\varphi, \psi, \ldots$ denote strictly increasing continuous functions on $[0, \infty)$, and for every $f \in \mathscr{C}([0, 1], \mathbf{R})$ with $f \geqslant 0$ define $\mathscr{M}_\varphi(f)$ by the relation

$$\varphi(\mathscr{M}_\varphi(f)) = \int_0^1 \varphi(f(x)) \, dx.$$

Show that $\mathscr{M}_\varphi \leqslant \mathscr{M}_\psi$ is equivalent to the convexity of $\psi \circ \varphi^{-1}$.

50. Let φ be a continuous strictly increasing function on an interval $[0, a]$, with $\varphi(0) = 0$, and let ψ be the inverse of φ.

Show that for all x, y such that $x \in [0, a]$, $y \in [0, \varphi(a)]$,

$$xy \leqslant \int_0^x \varphi(t) \, dt + \int_0^y \psi(t) \, dt,$$

with the equality holding only if $y = \varphi(x)$.

51. With the same notation, we denote by Φ and Ψ the primitives of φ and ψ which vanish at 0 (i.e., $\Phi(0) = \Psi(0) = 0$ and $\Phi' = \varphi$, $\Psi' = \psi$).

Now let f and g be functions which are continuous on $[0, 1]$ and have values in $[0, a]$ and $[0, \varphi(a)]$, respectively. Show that

$$\int_0^1 f(u)g(u) \, du \leqslant \int_0^1 \Phi(f(u)) \, du + \int_0^1 \Psi(g(u)) \, du.$$

DEFINITIONS AND AXIOMS

UPPER ENVELOPE. The *upper envelope* of a family $(f_i)_{i \in \mathrm{I}}$ of numerical functions on a set E is the function

$$f = \sup_{i \in \mathrm{I}} f_i$$

defined by

$$f(x) = \sup_{i \in \mathrm{I}} f_i(x) \qquad \text{for every} \quad x \in \mathrm{E}.$$

LIMIT SUPERIOR. Let f be a numerical function on a set E, and let $\mathscr{B}$ be a filter base on E.

The *limit superior* of f along $\mathscr{B}$ is defined as the supremum of $\bar{f}(\mathscr{B})$ (where $\bar{f}(\mathscr{B}) = \bigcap_{\mathrm{B} \in \mathscr{B}} \overline{f(\mathrm{B})}$), and is denoted by $\lim_{\mathscr{B}} \sup f$.

LOWER SEMICONTINUITY. A numerical function f defined on a topological space E is said to be *lower semicontinuous* at the point a if for every $\lambda < f(a)$ there exists a neighborhood V of a such that $\lambda < f(\mathrm{V})$.

CONVEX FUNCTIONS OF ONE VARIABLE. Let f be a finite numerical function defined on an interval I of $\mathbf{R}$. Then f is said to be *convex* if, for all $x_1, x_2 \in \mathrm{T}$, every point $\mathrm{M}(x)$ of the graph of f such that $x \in [x_1, x_2]$ lies below the segment $\mathrm{M}(x_1)\mathrm{M}(x_2)$.

CONVEX FUNCTIONS ON A SUBSET OF A VECTOR SPACE. Let f be a finite numerical function defined on a subset of a vector space E (over $\mathbf{R}$).

Then f is said to be *convex* if the set of points of the vector space $\mathrm{E} \times \mathbf{R}$ lying above the graph of f is convex.

POSITIVE-HOMOGENEOUS FUNCTIONS. Let f be a numerical function defined on a subset X of a vector space E (over $\mathbf{R}$). Then f is said to be *positive-homogeneous* if X is a cone with vertex O and if

$$f(\lambda x) = \lambda f(x) \qquad \text{for every} \quad x \in \mathrm{X} \quad \text{and every} \quad \lambda > 0.$$

THE MEAN RELATIVE TO A MONOTONE FUNCTION. Let f be a continuous strictly monotone numerical function on an interval A of $\mathbf{R}$, and let $\mu = (\alpha_i, x_i)_{i \in \mathrm{I}}$ be a discrete measure on A.

The *mean of μ relative to f* is defined as the number a such that

$$\Big(\sum_i \alpha_i\Big) f(a) = \sum_i \alpha_i f(x_i);$$

the number a is denoted by $\mathscr{M}_f(\mu)$.

TRACE OF A MAPPING. If f is a mapping of a space E into a space F, and X is a subset of E, then the *trace of f on* X (or *restriction of f to* X) is the mapping g of X into F defined by $g(x) = f(x)$, $x \in X$.

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