PHY407 Lab 10: Random Numbers, Monte Carlo Integration

Emma Jarvis: Q2, Q3
Lisa Nasu-Yu: Q1

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1 Random Points on the Surface of the Earth

1.a

The probability of a point falling in a particular element is

$$p(\theta,\phi)d\theta d\phi = \frac{sin\theta d\theta d\phi}{4\pi} \tag{1}$$

The ranges of the variables are $\theta\epsilon[0,\pi)$ and $\phi\epsilon[0,2\pi)$. We show below that $p_1(\theta)$ and $p_2(\phi)$ integrates to 1 over this range.

$$\int_0^{2\pi}\int_0^{\pi} p(\theta,\phi)d\theta d\phi = \int_0^{\pi}\frac{sin\theta d\theta}{2} \times \int_0^{2\pi}\frac{d\phi}{2\pi} \tag{2}$$

$$= \frac{2\pi}{2\pi} \times \frac{[] - cos(\pi) + cos(0)]}{2} \tag{3}$$

$$= 1 \times 1 = 1 \tag{4}$$

Eq. 5 and Eq. 6 give the variables from their respective probability distributions. For Eq. 6, we require $\int_0^{\theta} p_1(\theta')d\theta') < 1$.

$$\phi = 2\pi\int_0^{\phi} p_2(\phi')d\phi' = 2\pi z_\phi \tag{5}$$

$$\theta = arccos(1 - 2\int_0^{\theta} p_1(\theta')d\theta') = arccos(1 - 2z_\theta) \tag{6}$$

1.b

We plotted the latitude and longitude for N=5000 random points generated using Eq. 5 and Eq. 6, where z was a random number in the half closed interval $[0, 1)$.

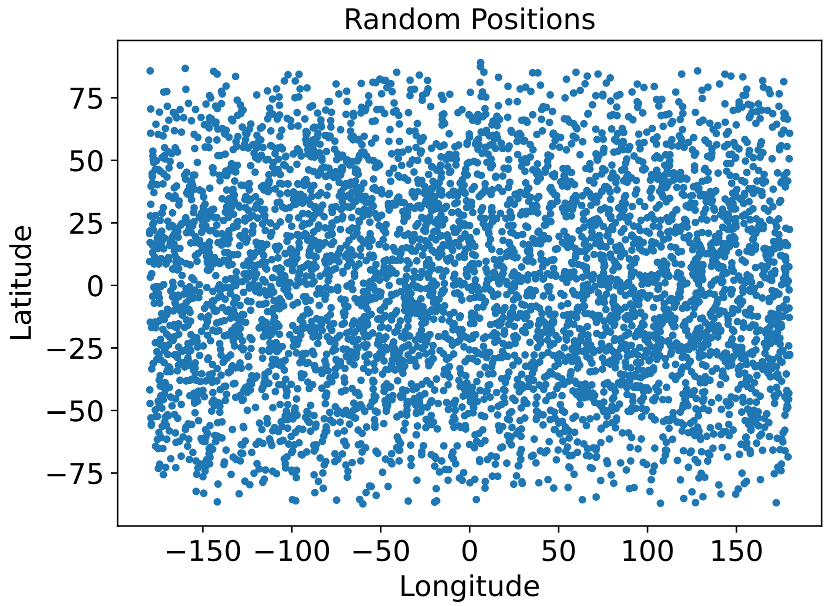


Figure 1: 500 randomly generated points in spherical coordinates, (θ, ϕ), plotted as longitude and latitude. We defined $\phi = 0$ as longitude 180° and $\theta = 0$ as latitude +90°.

1.c

From the file Earth.npy, we calculated the land fraction of the Earth to be 0.33185999657064474.

1.d

N	Land Fraction
50	0.18
500	0.284
5000	0.2784
50000	0.2864

Table 1: Land fraction of Earth, calculated as the land to water ratio of N randomly sampled points from the file Earth.npy.

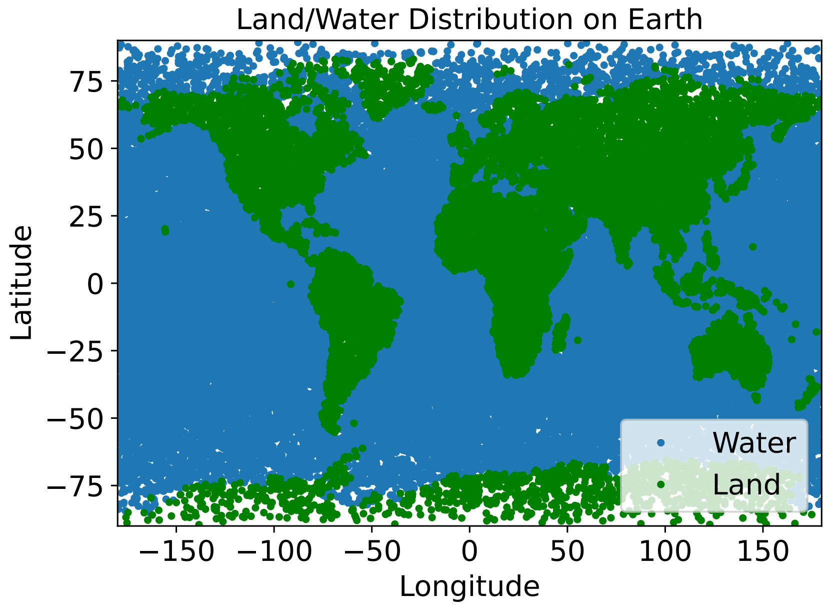


Figure 2: Land and water distribution of Earth generated with 50000 randomly sampled points from the provided file Earth.npy.

2 Limb Darkening of a Star

2.a

We wrote a function to emit a function from the stellar core and follow its path until it leaves the atmosphere.

2.b

Using the function from part (a), we followed the path for 10^5 photons starting from $\tau_{max} = 10$ and made a histogram of the final μ values where $\mu = \cos\theta$ and θ is the scattering angle. Figure 3 depicts this histogram where the y-axis is $N/N(\mu = 1)$ where N_1 is the number of of photons with an angle in the last bar of the histogram. Here we see that the photon is more likely to leave the atmosphere if μ is closer to 1.

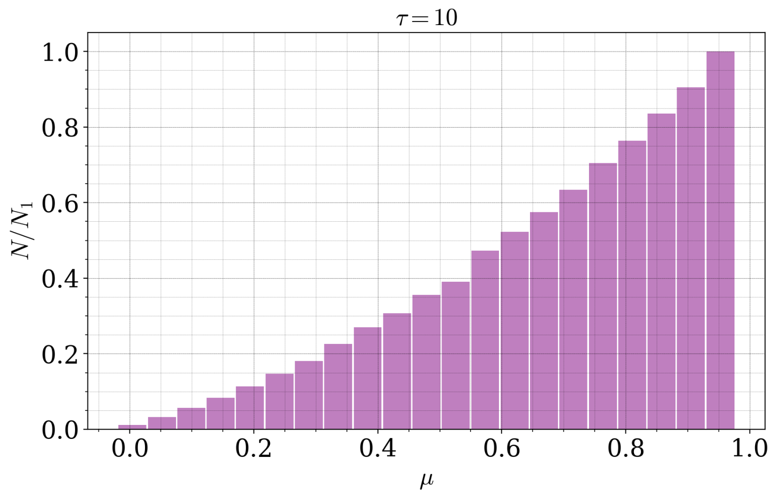


Figure 3: Histogram depicting the final μ values for a photon emitted from the stellar core with $\tau = 10$.

This number of photons is proportional to the specific intensity, I. To obtain this intensity, we divided N (the y axis of the histogram) by μ where the μ values were taken to be the centre of the bars in the histogram 3. With limb darkening, the specific intensity is approximated by 7 by assuming that the stellar photosphere is very thin compared to the stellar radius.

$$I(\mu) = (0.4 + 0.6\mu)I_1 \tag{7}$$

We fitted the solution for I to a model given by 8 and used `scipy.optimize.curvefit` to solve for the constants a and b. We found that $a = 0.64889587 \pm 0.00006328$ and $b = 0.38574468 \pm 0.00003179$ which are close to the limb darkening values of 0.6 and 0.4. Figure 4 depicts $I(\mu)$ for the analytic solution and the line of best fit.

$$I(\mu) = a\mu + b \tag{8}$$

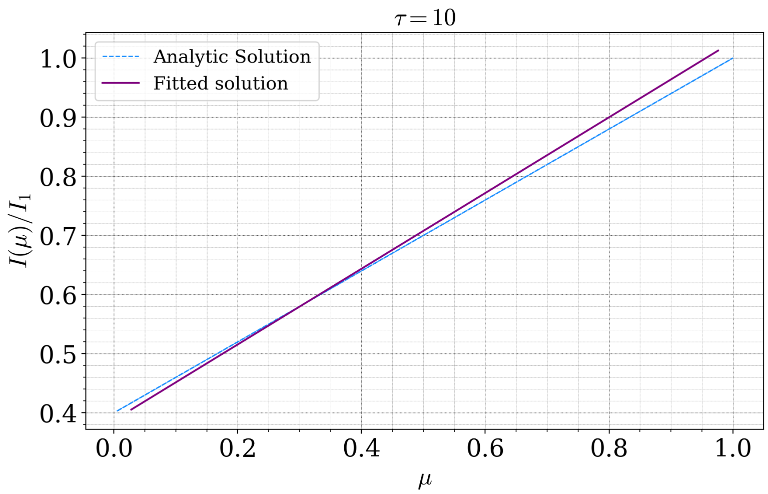


Figure 4: $I(\mu)$ for the analytic solution (blue dashed line) obtained from 7 and the line of best fit (purple line) obtained using `curvefit` with $\tau = 10$.

2.c

We then repeated this process, but changed τ_{max} to 10^{-4}. Figures 5 and 6 depict the histogram and intensities respectively. Here we see that the there is no preference for the value of μ as the histogram is evenly distributed and the photon will always leave the atmosphere as there is no scattering with a small τ_{max}. This also shows that limb darkening is a consequence of scattering and in this case because there is no scattering, there is also no limb darkening.

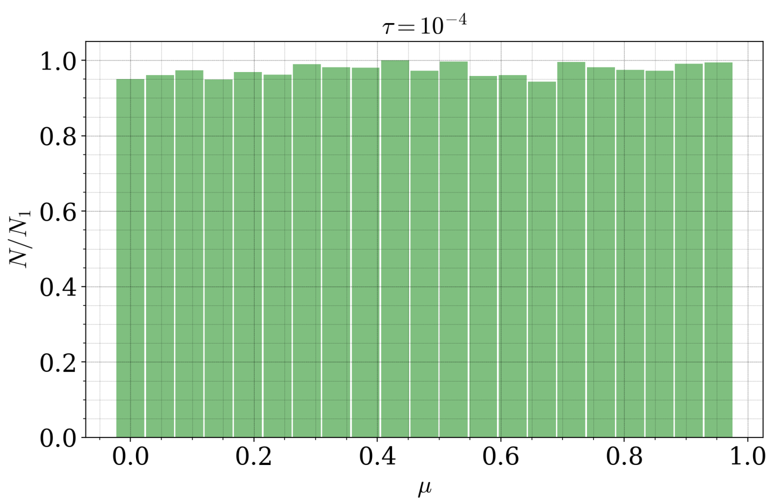


Figure 5: Histogram depicting the final μ values for a photon emitted from the stellar core with $\tau = 10^{-4}$.

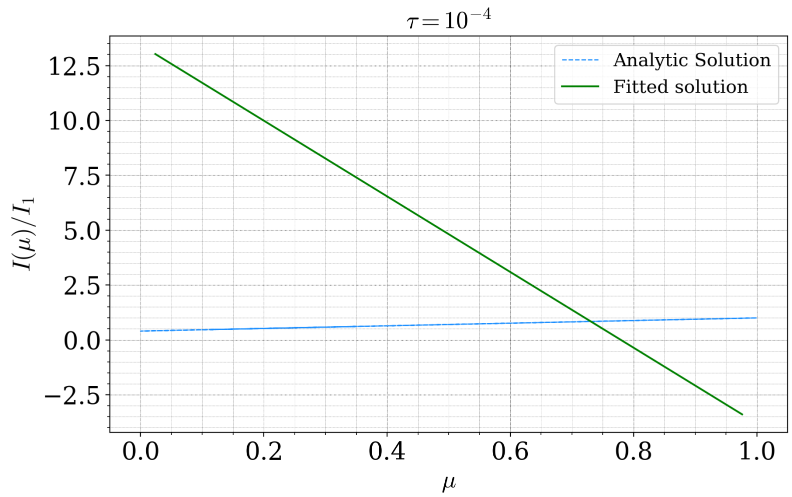


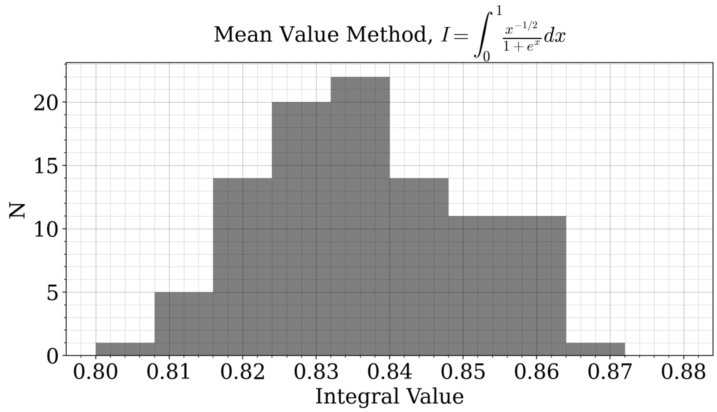
Figure 6: $I(\mu)$ for the analytic solution (blue dashed line) obtained from 7 and the line of best fit (purple line) obtained using `curvefit` with $\tau = 10^{-4}$.

3 Importance Sampling

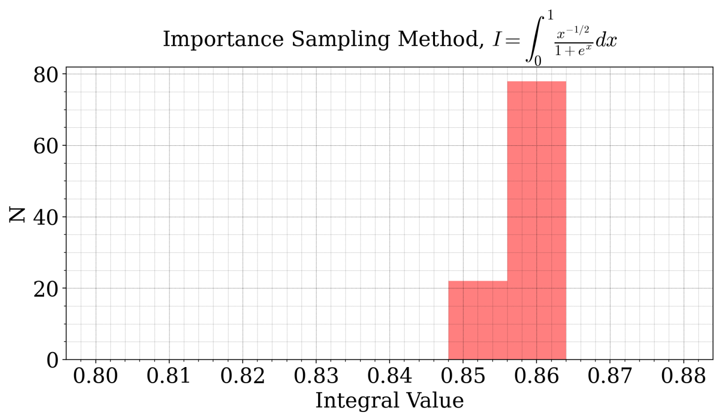
3.a

We evaluated the integral 9 using the mean value method and importance sampling method. For both of these methods, we used 10,000 sample points and repeated the calculation for 100 times for each method. Figures 7a and 7b depict the histograms of the evaluated integral for the mean value and importance sampling methods respectively. For the mean value method, the x values were taken from a random distribution using `numpy.random.random`. For the importance sampling method, the probability distribution is $p(x) = \frac{1}{2\sqrt{x}}$ so the x values were taken to be $\int_0^z \frac{1}{2\sqrt{z'}} = \sqrt{z}$ where z is the values obtained using `numpy.random.random`.

$$\int_0^1 \frac{x^{-1/2}}{1+e^x} dx \tag{9}$$



(a) Mean Value Method



(b) Importance Sampling Method

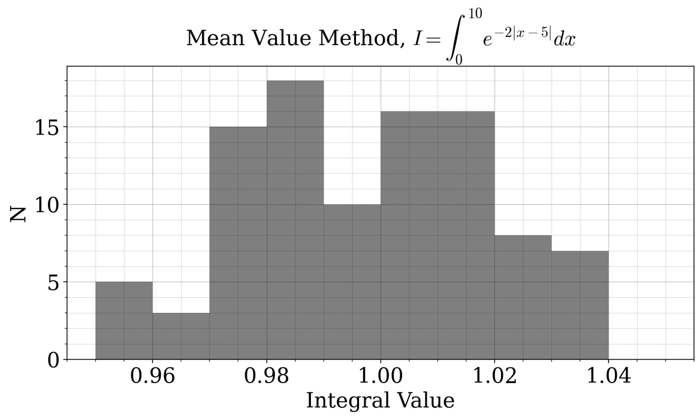
Figure 7: Histograms of 100 solutions to 9 using the mean value method and importance sampling methods using $N = 10000$ sample points.

Both histograms are plotted with the same range of 0.8 to 0.88. The histograms show that the mean value method has a wider spread across this range while the solutions obtained using the importance sampling method have a much narrower spread around 0.85. Because the expected value of 9 is around 0.84, this indicates that the importance sampling method gives consistent results closer to this expected result.

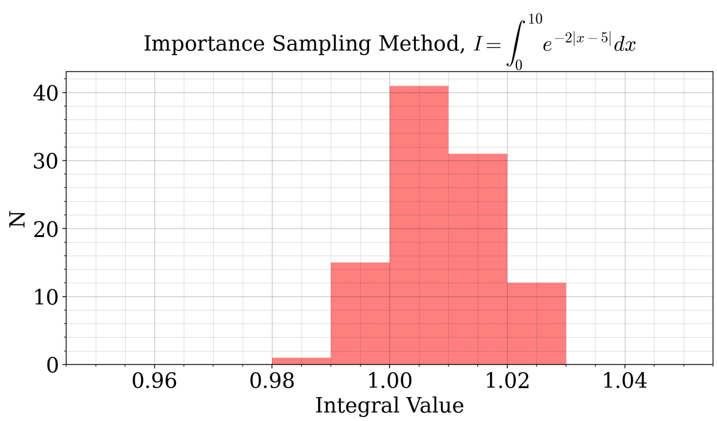
3.b

We then repeated this process for the integral 10. Again for both methods, 10, 000 sample points were used and the calculation was repeated 100 times for each method. Figures 7a and 7b depict the histograms of the evaluated integral for the mean value and importance sampling methods respectively. For the mean value method, the x values were taken from a random distribution using `numpy.random.random`. For the importance sampling method, the probability distribution is $p(x) = \frac{1}{2\sqrt{pi}} e^{-(x-5)^2/2}$ so the x values were taken using `numpy.random.normal` with a mean of 5 and standard deviation of 5.

$$\int_0^{10} e^{-2|x-5|} dx \tag{10}$$



(a) Mean Value Method



(b) Importance Sampling Method

Figure 8: Histograms of 100 solutions to 10 using the mean value method and importance sampling methods using $N = 10000$ sample points.

Both histograms are plotted with the same range of 0.95 to 1.05. The histograms show that the mean value method has a wider spread across this range while the solutions obtained using the importance sampling method have a much narrower spread around 1. Because the expected value of 10 is around 1, this indicates that the importance sampling method gives consistent results closer to this expected result.