PHY407 Lab 8: Partial Differential Equations Pt. I

Work Distribution:
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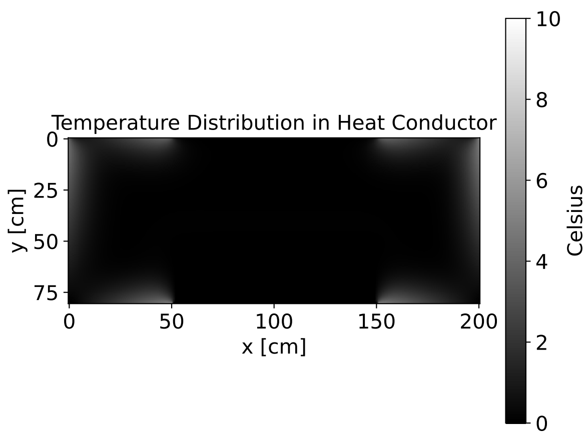
1 Temperature distribution in a heat conductor

1.a

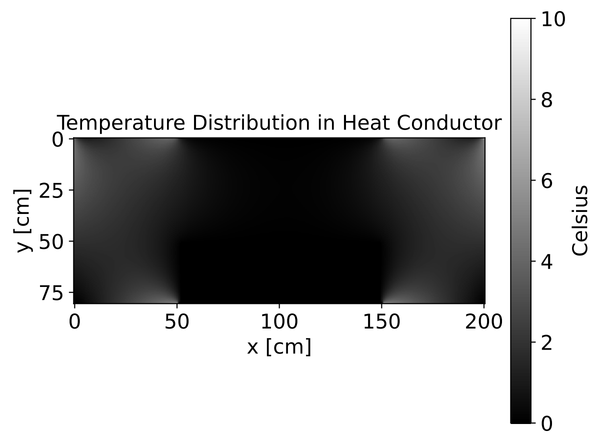
We wrote a program to simulate the temperature distribution in a heat conductor using Gauss-Seidel relaxation and overrelaxation. We used a grid spacing of 0.1cm.

1.b

We ran the simulation for overrelaxation parameter, $w = 0$ and $w = 0.9$ for 100 iterations each. In the same number of iterations, we see that the heat has diffused more in Fig. 1b with $w = 0.9$, than in Fig. 1a with $w = 0$. This shows that overrelaxation increases the rate of the simulation, as expected.



(a) w=0



(b) w=0.9

Figure 1: Temperature distribution in a heat conductor, with overrelaxation parameters $w = 0$ (Fig. a), and $w = 0.9$ (Fig. b). Note that the shape of the conductor was a rectangle with a 10cm x 3cm cutout at the bottom-centre.

1.c

We ran the simulation again with $w = 0.9$, but this time up to a target accuracy of 10^{-6}°C at each point. By visual inspection, Fig. 2 is symmetric about the vertical bisector, x=10cm. This is as expected, because the shape and the initial temperatures on the boundary of the conductor were symmetric about this axis. After the solution converged to the target accuracy, the temperature at x=2.5cm, y=1cm was 1.6355393542905465 °C.

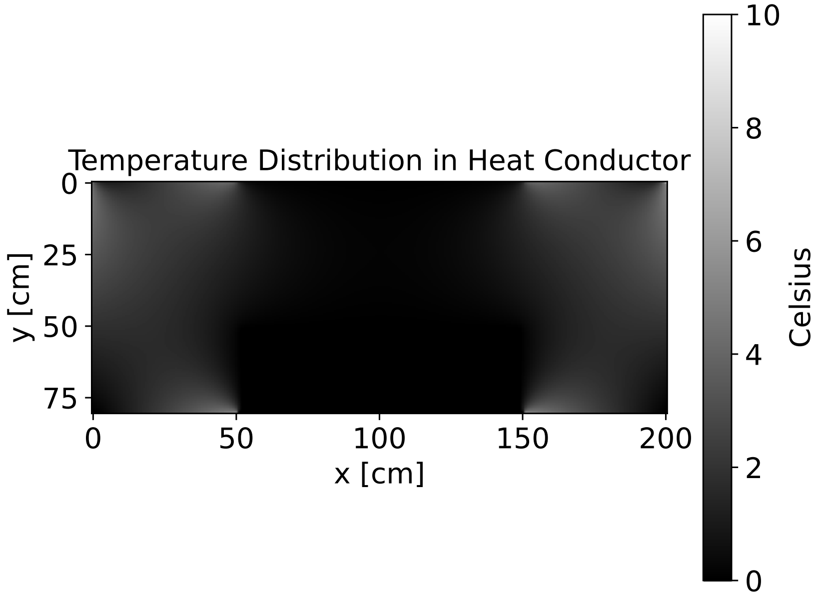


Figure 2: Temperature distribution in a heat conductor, with overrelaxation parameter $w = 0.9$. The simulation was run until each grid point had an accuracy of 10^{-6}°C. Note that the shape of the conductor was a rectangle with a 10cm x 3cm cutout at the bottom-centre.

2 FTCS solution of the wave equation

2.a

We wrote a program that uses the FTCS method to solve the wave equation 1 for a piano strong of length L initially at rest that is struck by a hammer a distance d from the end of the string.

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2} \tag{1}$$

To do this with the FTCS method, we used the technique to seperate this into two first order equations 2 and 3.

$$\frac{d\phi}{dt} = \psi(x, t) \tag{2}$$

$$\frac{d\psi}{dt} = \frac{v^2}{a^2} \left[\phi(x + a, t) + \phi(x - a, t) - 2\phi(x, t) \right] \tag{3}$$

In these equations $v = 100$m/s as specified in the question and a is the grid spacing. The initial conditions that were used to solve this were 4 and 5 where $L = 1$m, $d = 0.1$m, $C = 1$m/s and $\sigma = 0.3$m as specified in the question. Since the string is fixed at the ends, we set the boundary conditions of ϕ to be zero (so $\phi(x = 0) = 0$ and $\phi(x = L) = 0$ for all t).

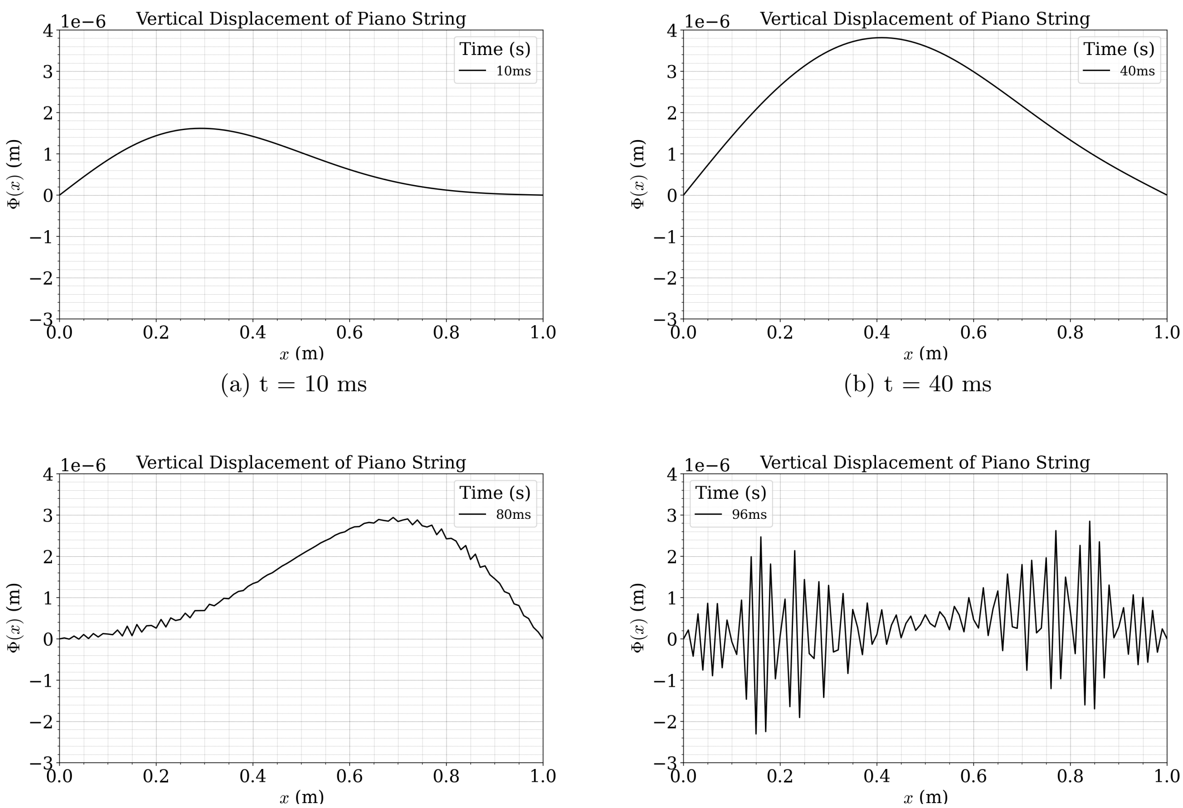
$$\psi(x, t = 0) = 0 \tag{4}$$

$$\phi(x, t = 0) = C\frac{x(L - x)}{L^2}e^{-\frac{(x-d)^2}{2\sigma^2}} \tag{5}$$

We also set the number of divisions in the grid, N, to be 100 and the time-step h to be 10^{-7}s.

2.b

We then plotted this solution up to a time of 100ms to show the instability of the solution. Figures 3a, 3b, 3c and 3d depict the solution at times 10ms, 40ms, 80ms and 96ms respectively. At 10ms and 40ms the solution is stable but after 80ms some instability arises and then at 96ms, the instability in the solution is so large that the solution is no longer workable.



(a) t = 10 ms

(b) t = 40 ms

(c) t = 80 ms

(d) t = 96 ms

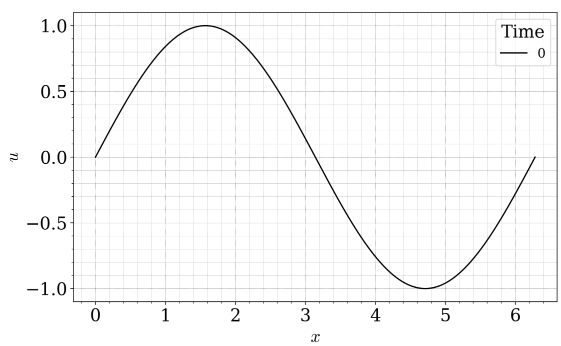
Figure 3: The solution to 1 at times 10ms, 40ms, 80ms and 96ms, demonstrating the instability of the FTCS method as time progresses.

3 Solving Burger's equation

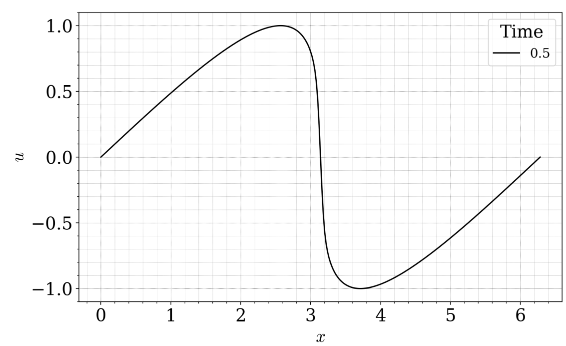
We then wrote a solution to the Burger's equation given by 6 where u is the speed of a fluid, x is a spatial component, t is a temporal component and $\epsilon = 1$. As given in the question, the timestep, Δt, was 0.005, the grid spacing Δx, was 0.02, the end time, T_f, was 2 and the length of the x interval, L_x was 2π. Using this, we computed the number of divisions in the grid $N_x = \frac{L_x}{\Delta t}$ and $N_t = \frac{T_f}{\Delta t}$. To solve the equation, we used the method 7 where $\beta = \epsilon\frac{\Delta t}{\Delta x}$. We also applied 1 forward timestep to the initial condition before applying 7. We then plotted the solution for $t = 0, 0.5, 1.0$ and 1.5 and these plots are depicted in figures 4a, 4b, 4c and 4d respectively.

$$\frac{\partial u}{\partial t} + \epsilon\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \tag{6}$$

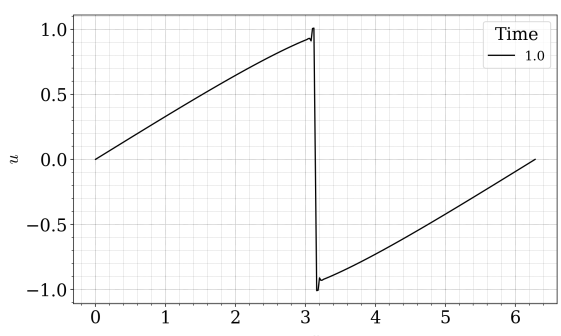
$$u_i^{j+1} = u_i^{j-1} - \frac{\beta}{2}\left[\left(u_{i+1}^{j}\right)^2 - \left(u_{i-1}^{j}\right)^2\right] \tag{7}$$



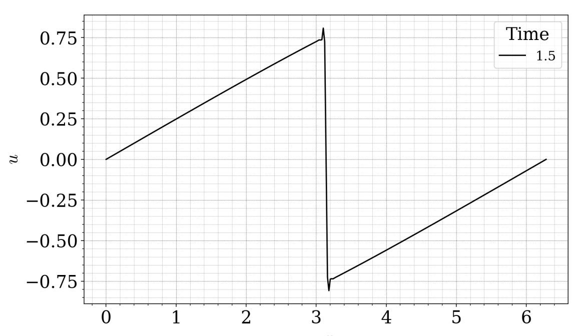
(a) t = 0.0



(b) t = 0.5



(c) t = 1.0



(d) t = 1.5

Figure 4: Solutions to the Burger's equation, 4, for times 0, 0.5, 1.0 and 1.5.

As time progresses, the disturbance becomes less sinusoidal and the disturbance moves forward.