Homework 1

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Problem 1

Define.

$Cov =$ Getting COVID

$DF =$ Disease Free

$+ =$ Test positive

$- =$ Test negative

From the question, we learned

$Sensitivity = P(+|Cov)$

$Specificity = P(-|DF)$

a) $P(Cov) = 0.5\%$

b) $P(Cov) = 5\%$

We want $P(Cov|+)$ for bose cases.

$$\begin{aligned}
P(Cov|+) &= \frac{P(+|Cov) \cdot P(Cov)}{P(+)} \\
&= \frac{P(+|Cov) \cdot P(Cov)}{P(+|Cov) \cdot P(Cov) + P(+|DF) \cdot P(DF)} \\
&= \frac{Sensitivity \cdot P(Cov)}{Sensitivity \cdot P(Cov) + (1 - Specificity) \cdot (1 - P(Cov))}
\end{aligned}$$

For *a)*

$$P(Cov|+) = \frac{0.65 \times 0.005}{0.65 \times 0.005 + 0.01 \times 0.995} = \frac{65}{264} \approx 0.246 = 24.6\%$$

For *b)*

$$P(Cov|+) = \frac{0.65 \times 0.05}{0.65 \times 0.05 + 0.01 \times 0.95} = \frac{65}{84} \approx 0.774 = 77.4\%$$

Problem 2

a). number of bits needed for keyboard symbols: 8bits
2615 characters total size = 2615 bytes
ls -alF shows 2659 bytes
not all characters on the keyboard, unicode resulting in this

b).

$1byte = 2^3 bits$
$1KiB = 2^1 3bits$
$1MiB = 2^2 3bits$
$1GiB = 2^3 3bits$

$1byte = 8bits$
$1KB = 8 \times 10^3 bits$
$1MB = 8 \times 10^6 bits$
$1GB = 8 \times 10^9 bits$

c).

$80GB = 8 \times 10^9 bytes$
number of pages $= 8 \times 10^9/2659 \approx 3008649.868 \approx 3008650$

d). it means our information reduces 1GB $= 8 \times 10^9 bits$
$\Delta S = (k_B ln2) \times 8 \times 10^9$
$\therefore Q = T \cdot \Delta S = 300 \times k_B \times ln2 \times 8 \times 10^9 = 300 \times 1.38 \times 10^{-23} \times ln2 \times 8 \times 10^9 J \approx 2.87 \times 10^{-12}$

e). We need $80 \times 2.87 \times 10^{-12} \approx 2.30 \times 10^{-10} J$

Problem 3

a).

$$
\begin{aligned}
H(X,Y) &= -\sum_{i,j} P(X_i, Y_j) log P(X_i, Y_j) \\
&= -\sum_{i,j} P(Y_j, X_i) log P(Y_j, X_i) \\
&= H(Y, X) \\
H(X:Y) &= H(X) + H(Y) - H(X,Y) \\
&= H(Y) + H(X) - H(Y,X) \\
&= H(Y:X)
\end{aligned}
$$

b). It's obvious the information of any event ≥ 0, $H(Y|X)$ is always true

$$H(X:Y) = H(Y) + H(X) - H(X,Y)$$

$$
\begin{aligned}
\because H(X) - H(X,Y) &= -\sum_{i} P(X_i) log P(X_i) + \sum_{i,j} P(Y_j|X_i)P(X_i) log(P(Y_j|X_i)P(X_i)) \\
&= \sum_{i} P(X_i) \sum_{j} (P(Y_j|X_i) log(P(Y_j|X_i)P(X_i)) - log(P(X_i)))
\end{aligned}
$$

$$
\begin{aligned}
&\because P(Y_j|X_i) \leq 1 \\
&\therefore P(Y_j|X_i) log(P(Y_j|X_i)P(X_i)) \leq log(P(Y_j|X_i)P(X_i)) \leq log(P(X_i)) \\
&\therefore \sum_{i} P(X_i) \sum_{j} (P(Y_j|X_i) log(P(Y_j|X_i)P(X_i)) - log(P(X_i))) \leq 0 \\
&\therefore H(X) \leq H(X,Y)
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
&P(Y_j|X_i) log(P(Y_j|X_i)P(X_i)) = log(P(Y_j|X_i)P(X_i)) = log(P(X_i)) \\
&\iff P(Y_j|X_i) = 1 \\
&\iff Y = f(X)
\end{aligned}
$$

c).

$$
\begin{aligned}
&H(X,Y) = H(X) + H(Y) - H(X:Y) \\
&\because H(X:Y) \geq 0 \\
&\therefore H(X,Y) \leq H(X) + H(Y)
\end{aligned}
$$

When equality,

$$
\begin{aligned}
& H(X:Y) = 0 \\
\iff & log(\frac{P(X,Y)}{P(X)P(Y)}) = 1 \\
\iff & P(X,Y) = P(X)P(Y) \\
\iff & \text{X and Y are independent}
\end{aligned}
$$