Homework 2

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Problem 1

a)

$$
\begin{aligned}
|\psi|^2 &= \langle\psi|\psi\rangle \\
&= \begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \\
&= \cos^2\theta + \sin^2\theta = 1
\end{aligned}
$$

$\therefore |\psi\rangle$ is normalized.

b)

Probability of $|0\rangle$ is $\cos^2\theta$. Probability of $|1\rangle$ is $\sin^2\theta$

c)

Guess it's $\cos^2(\theta - \frac{\pi}{4})$ and $\sin^2(\theta - \frac{\pi}{4})$

$$\langle +|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} = \frac{1}{\sqrt{2}}(\sin\theta + \cos\theta) = \cos(\theta - \frac{\pi}{4})$$

$$\langle -|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} = \frac{1}{\sqrt{2}}(\sin\theta - \cos\theta) = \sin(\theta - \frac{\pi}{4})$$

$\therefore$ the probability to get $|+\rangle$ is $\cos^2(\theta - \frac{\pi}{4})$, and the probability to get $|-\rangle$ is $\sin^2(\theta - \frac{\pi}{4})$

d)

$$\mathcal{H}\cdot|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}\cdot\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos\theta + \sin\theta \\ \cos\theta - \sin\theta\end{bmatrix} = \begin{bmatrix}\cos(\theta - \frac{\pi}{4}) \\ \sin(\theta - \frac{\pi}{4})\end{bmatrix}$$

Problem 2

a)

Possible outcomes for each round: $\{|0\rangle, |1\rangle\}$

Define. $|\chi_0\rangle$ is the guessing state when the outcome is $|0\rangle$, and $|\chi_1\rangle$ is the guessing state when the outcome is $|1\rangle$

$$|\chi_0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$|\chi_1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

b)

Define. F_0 is the fidelity when the outcome is $|0\rangle$, and F_1 is the fidelity when the outcome is $|1\rangle$

$$F_0 = |\langle\psi|\chi_0\rangle|^2 = |\begin{bmatrix}\cos\theta & e^{i\Phi}\sin\theta\end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}|^2 = \cos^2\theta$$
$$F_1 = |\langle\psi|\chi_1\rangle|^2 = |\begin{bmatrix}\cos\theta & e^{i\Phi}\sin\theta\end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}|^2 = \sin^2\theta$$

c)

Probability p_0 on χ_0 is $\cos^2\theta$
Probability p_1 on χ_1 is $\sin^2\theta$
Average fidelity is

$$p_0 \cdot F_0 + p_1 \cdot F_1 = \cos^4\theta + \sin^4\theta$$

d)

The average fidelity is $\int(\cos^4\theta + \sin^4\theta)d\theta$

$$
\begin{aligned}
&\int_0^\pi (\cos^4\theta + \sin^4\theta)d\theta \\
=&\int_0^\pi \cos^4\theta d\theta + \int_0^\pi \sin^4\theta d\theta \\
=&\int_0^\pi (\frac{\cos 2\theta + 1}{2})^2 d\theta + \int_0^\pi (\frac{1-\cos 2\theta}{2})^2 d\theta \\
=&\int_0^\pi (\frac{1}{4}\cos^2 2\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4})d\theta + \int_0^\pi (\frac{1}{4}\cos^2 2\theta - \frac{1}{2}\cos 2\theta + \frac{1}{4})d\theta \\
=&\int_0^\pi (\frac{1}{8}\cos 4\theta + \frac{1}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{4})d\theta + \int_0^\pi (\frac{1}{8}\cos 4\theta + \frac{1}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{4})d\theta \\
=&(\frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta)\Big|_0^\pi + (\frac{1}{32}\sin 4\theta - \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta)\Big|_0^\pi \\
=&\frac{3}{4}\pi
\end{aligned}
$$