① HW 7

2-24-15

Metropolis Sampling

Let $f(x)$ denote the density from which we wish to sample. The Metropolis sampling algorithm is as follows:

Single variable (parameter) model:

① Choose an initial value x_0.
② Propose a new value x^* from a proposal density $g(x|x_0)$
③ Accept x^* as a draw from the target density f with probability α^*, where

$$\alpha^* = \min\left\{\frac{f(x^*)}{f(x_0)}, 1\right\}$$

④ If x^* is accepted, set $x_0 = x^*$ (save x_0) and go to ② else, set $x_0 = x_0$ (save x_0) and go to ②

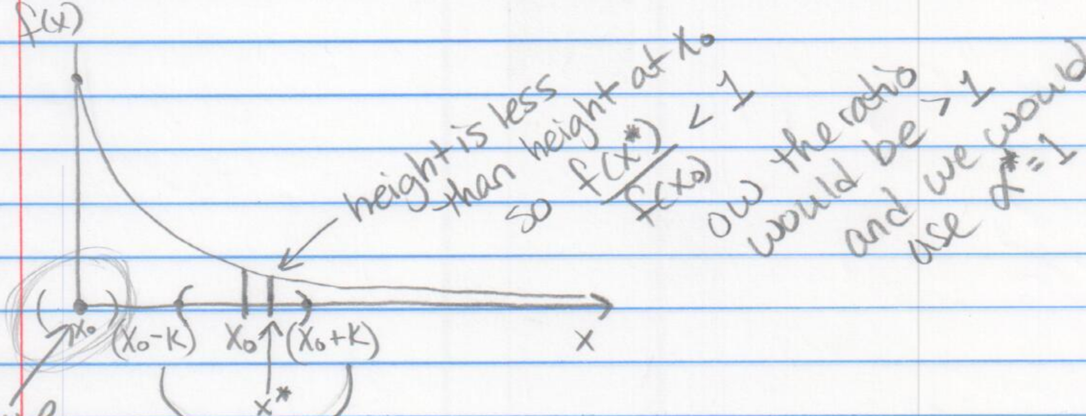
Notes: The proposal density must be chosen to satisfy $g(x^*|x_0) = g(x_0|x^*)$

→ It is possible to "sample" the same value as the previous iteration. This happens whenever x^* is rejected.

Example: Sample from an exponetial density.

Target density: $f(x) = \lambda e^{-\lambda x}$ for $x > 0$

Proposal density: $g(x \mid x_0) \Rightarrow$ uniform $(x_0 - K, x_0 + K)$



issue w/ possible negative x^* which is not valid in target

Select x^* within this interval

call x^* a draw from exponential

- drawn from proposal but call from target w/ certain prob. α^*
- prob. = height of target at x^* ÷ height of target at x_0

$$= \alpha^* = \min\left(\frac{f(x^*)}{f(x_0)}, 1\right)$$

$x \mid x_0 \sim$ uniform$(x_0 - k, x_0 + k)$

$g(x \mid x_0) = \frac{1}{2k}$ for $x \in (x_0 - k, x_0 + k)$

$$g(x^* \mid x_0) = g(x_0 \mid x^*)$$

$$g(x^* \mid x_0) = \frac{1}{2k} \text{ for } x^* \in (x_0 - k, x_0 + k)$$

$$g(x_0 \mid x^*) = \frac{1}{2k} \text{ for } x_0 \in (x^* - k, x^* + k)$$

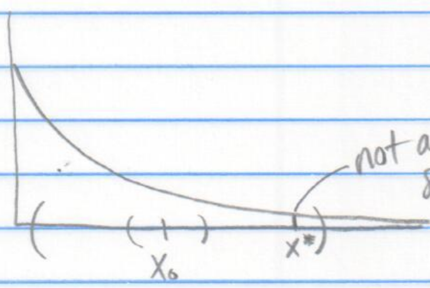
	Acceptance probability	realizations
small k	high b/c proposing close to current value	highly correlated
large k	low(er)	less highly correlated, then more highly correlated as k grows -accept a lot so values close together

Keep track of acceptance probabilities - proportion of times accept
- if did not change x_0, don't increment

There is a sweet spot for k to get best non-correlation

smaller the k, the higher the lag

pick k to get acceptance in the 50-60% range



not a lot of support — very small prob.
so fewer acceptances even though still have 50% chance of guaranteed acceptance

50% of guaranteed acceptance

If propsal is normal

$$q(x|x_0) \Rightarrow N(x_0, c^2)$$

Metropolis Sampling for a multi-parameter Bayesian model:

Let $(\theta_1, \theta_2, \ldots, \theta_m)$ denote the parameters of interest.

Let $\pi_1(\theta_1, \theta_2, \ldots, \theta_m \mid \vec{y})$ denote the joint posterior distribution of $\vec{\theta}$

Use Metropolis sampling to draw from the m full conditional posterior distributions, using m potentially distinct proposal densities.

* initialize $\theta_{1(0)}, \theta_{2(0)}, \ldots, \theta_{m(0)}$

① Proposal θ_1^* from $q_1(\theta^* \mid \theta_{1(0)})$

② Accept θ_1^* with probability $\alpha^* =$

$$\min\left\{\frac{\pi_1(\theta_1^* \mid \theta_{2(0)}, \theta_{3(0)}, \ldots, \theta_{m(0)}, \vec{y})}{\pi_1(\theta_{1(0)}^* \mid \theta_{2(0)}, \theta_{3(0)}, \ldots, \theta_{m(0)}, \vec{y})}, 1\right\}$$

③ Propose θ_2^* from $q_2(\theta_2^* \mid \theta_{2(0)})$.

④ Accept θ_2^* with probability

$$\alpha^* = \min\left\{\frac{\pi_1(\theta_2^* \mid \theta_1^*, \theta_{3(0)}, \ldots, \theta_{m(0)}, \vec{y})}{\pi_1(\theta_{2(0)}^* \mid \theta_1^*, \theta_{3(0)}, \ldots, \theta_{m(0)}, \vec{y})}, 1\right\}$$

θ_1^*

⑤ Repeat in similar fasion for remaining parameters

⑥ Repeat steps 1-5 lots of times

HW7

(5)

for μ

$$\alpha^* = \min\left\{ \frac{\cancel{\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+1}} e^{-\frac{1}{2\sigma^2}[(n-1)s^2+n(\bar{y}-\mu^*)^2]}}{\cancel{\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+1}} e^{-\frac{1}{2\sigma^2}[(n-1)s^2+n(\bar{y}-\mu_0)^2]}}, 1 \right\}$$

$$= \min\left\{ e^{\frac{1}{2\sigma^2}(n(\bar{y}-\mu^*)^2 - n(\bar{y}-\mu_0)^2)}, 1 \right\}$$

$$= \min\left\{ e^{-\frac{n}{2\sigma^2}[(\bar{y}-\mu^*)^2 - (\bar{y}-\mu_0)^2]}, 1 \right\}$$

for σ^2

$$\alpha^* = \min\left\{ \frac{\overbrace{\left(\frac{1}{\sigma^{2*}}\right)^{\frac{n}{2}+1}}^{A^*} e^{\overbrace{-\frac{1}{2\sigma^{2*}}[(n-1)s^2+n(\bar{y}-\mu^*)^2]}^{B^*}}}{\left(\frac{1}{\sigma_0^2}\right)^{\frac{n}{2}+1} e^{-\frac{1}{2\sigma_0^2}[(n-1)s^2+n(\bar{y}-\mu_0)^2]}}, 1 \right\}$$

$\mu^* = \mu_0$ use either one

$$\frac{A^* e^{-B^*}}{B^* e^{-B_0}} = e^{\ln\left(\frac{A^* e^{-B^*}}{A_0 e^{-B_0}}\right)} = e^{\ln(A^* e^{-B^*}) - \ln(A_0 e^{-B_0})}$$

$$= e^{\ln A^* - B^* - \ln(A_0) + B_0}$$

• posterior + full conditionals from HW5

CPMA 573 — Homework #7

Exercise 1: Metropolis sampling: Uniform proposal. Let $Y_1, \ldots, Y_{43}$ be iid normal random variables from the density

$$f(y|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\}.$$

Realizations from this data model are located at

`www.mathcs.duq.edu/~kern/hw5.dat`

It is your goal to make inference on the values of μ and σ^2 used to generate these data.

Assuming μ and σ^2 are a priori independent, with prior densities $\pi(\mu) \propto 1$ and $\pi(\sigma^2) \propto \sigma^{-2}$, obtain 25000 draws (NOT necessarily independent!) from the marginal posteriors $\pi(\mu|\vec{y})$ and $\pi(\sigma^2|\vec{y})$ using Metropolis sampling in conjunction with proposal densities

$$\mu^*_{i+1} \sim U(\mu_i - b, \mu_i + b) \quad \text{and} \quad \sigma^{2*}_{i+1} \sim U(\sigma^2_i - c, \sigma^2_i + c)$$

for

Case 1: $b = 0.5$, $c = 1$.

Case 2: $b = 2$, $c = 4$.

Using starting values $\mu_0 = \bar{y}$ and $\sigma_0^2 = s^2$, report the acceptance probability and the lag necessary to obtain independent realizations for both cases. Then, also for both cases, provide trace plots and histograms of your μ and σ^2 realizations. As with HW #5, superpose the theoretical marginal density of σ^2 on both histograms of σ^2 realizations. The four plots for Case 1 can be produced in **R** in a fashion analogous to those required for HW #5.

Exercise 2: Metropolis sampling: Normal proposal. Draw realizations from the marginal posteriors of μ and σ^2 from the Exercise 1 model, but replace the uniform proposals with normal proposals, centered at the current parameter values. If b is the standard deviation of the normal proposal for μ, and c is the standard deviation of the normal proposal for σ^2, then use the values specified in Case 1 and Case 2 above to produce two separate sets of independent realizations. For this exercise, you need only compare the acceptance probabilities and the lags necessary to obtain independent realizations with those you found based on the uniform proposal densities in Exercise 1. Comment on any similarities/differences, and argue for which of the four proposal densities you feel performs "best."

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