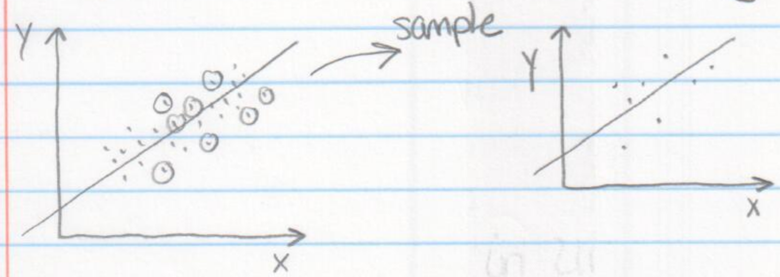
Bayesian Inference in Linear regression



$\hat{y} = \hat{\alpha} + \hat{\beta} x$

maximum likehood or least squares method

population of (x, y) coordinates

$Y_i \sim N(\alpha + \beta x_i, \sigma^2)$

Y_i is normal w/ mean $\alpha + \beta x_i$ (pt on line at x_i)

constant var not dependent on i

Maximum likelihood estimates for α and β are:

$$\begin{cases} \hat{\beta} = \frac{s_y}{s_x} r \\ \hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \end{cases}$$

Assignment

Use Gibbs sampling to obtain realizations from the marginal posteriors of α, β, and σ^2 in the simple linear regression model, as applied to the old faithful eruption data built-in R (faithful)

Dependent variable: waiting time until next eruption

Independent variable: Duration of current eruption likelihood $N(\alpha+\beta x_i, \sigma^2)$

priors

Joint posterior $\underbrace{\left[\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i-(\alpha+\beta x_i))^2}\right]}_{\text{likelihood}} \cdot \underbrace{\frac{1}{\sigma^2} \cdot \frac{1}{10000} \cdot \frac{1}{10000}}_{\text{uniformed priors}}$

- Jeffrey's uniformed prior for σ^2 and
- uniform uninformed priors for α and β

Inverse Gamma (α, λ)

$$\frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \left(\frac{1}{x}\right)^{\alpha+1} e^{-\lambda/x}$$

$$\left[\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i-(\alpha+\beta x_i))^2}\right] \cdot \frac{1}{\sigma^2} \cdot \frac{1}{10000} \cdot \frac{1}{10000}$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+1} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-(\alpha+\beta x_i))^2}$$

$$\alpha = \frac{n}{2} \qquad \lambda = \frac{\sum(y_i-(\alpha+\beta x_i))^2}{2}$$

Full Conditional for σ^2: $(\sigma^2 | \alpha, \beta, \vec{y}, \vec{x}) \sim IG\left(\frac{n}{2}, \frac{\sum(y_i-(\alpha+\beta x_i))^2}{2}\right)$

Full Conditional for α: $(\alpha | \sigma^2, \beta, \vec{y}, \vec{x}) \sim N\left(\sum_{i=1}^{n}\frac{(y_i-\beta x_i)}{n}, \frac{\sigma^2}{n}\right)$

work for

$$\prod_{i=1}^{n} e^{-\frac{1}{2\sigma^2}(\underbrace{y_i-\beta x_i}_{\text{residual}}-\alpha)^2} = \prod_{i=1}^{n} e^{-\frac{1}{2\sigma^2}((y_i-\beta x_i)^2-2\alpha(y_i-\beta x_i)+\alpha^2)}$$

$$= e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(y_i-\beta x_i)^2-2\alpha\sum_{i=1}^{n}(y_i-\beta x_i)+n\alpha^2\right)}$$

$$= e^{-\frac{n}{2\sigma^2}\left(\sum_{i=1}^{n}\frac{(y_i-\beta x_i)^2}{n}-2\alpha\sum_{i=1}^{n}\frac{(y_i-\beta x_i)}{n}+\alpha^2\right)}$$

$$\propto e^{-\frac{n}{2\sigma^2}\left(\alpha - \sum_{i=1}^{n}\frac{(y_i - \beta x_i)}{n}\right)^2}$$

$$\text{mean} = \sum_{i=1}^{n}\frac{(y_i - \beta x_i)^2}{n} \qquad \text{var} = \frac{\sigma^2}{n}$$

Full conditional for β:

$$(\beta \mid \sigma^2, \alpha, \vec{x}, \vec{y}) \sim N\left(\frac{\sum x_i(y_i - \alpha)}{\sum x_i^2}, \frac{\sigma^2}{\sum x_i^2}\right)$$

work for above

$$\prod_{i=1}^{n} e^{-\frac{1}{2\sigma^2}(y_i - \alpha - \beta x_i)^2} = \prod_{i=1}^{n} e^{-\frac{1}{2\sigma^2}\left((y_i-\alpha)^2 - 2\beta x_i(y_i - \alpha) + (\beta x_i)^2\right)}$$

$$= e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(y_i-\alpha)^2 - 2\beta\sum_{i=1}^{n}x_i(y_i-\alpha) + \beta^2\sum_{i=1}^{n}x_i^2\right)}$$

$$= e^{-\frac{\sum_{i=1}^{n}x_i^2}{2\sigma^2}\left(\frac{1}{\sum x_i^2}\sum(y_i-\alpha)^2 - \frac{2\beta\sum x_i(y_i-\alpha)}{\sum x_i^2} + \beta^2\right)}$$

$$\propto e^{-\frac{\sum x_i^2}{2\sigma^2}\left(\beta - \frac{\sum x_i(y_i-\alpha)}{\sum x_i^2}\right)^2}$$

Starting values for β and α to draw σ^2 first

each set of draws is a regression line...

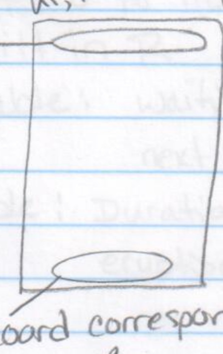
$\alpha_1, \beta_1, \sigma_1^2$ - is a line

$\alpha_2, \beta_2, \sigma_2^2$ - another line

HW 9.

y coord correspond to α_1, β_1

matrix



y coord correspond to α_N, β_N

☆ average "posterior" regression line along w/ 95% credible bounds for the line

☆ Trace plots, Hist, Acf for σ^2, α, β

average columns

$$\begin{array}{cc} \begin{bmatrix} \hat{y}_1 & \hat{y}_2 \dots & \hat{y}_{272} \\ \hat{y}_1 & \hat{y}_2 \dots & \hat{y}_{272} \\ & & \\ & & \end{bmatrix} & \begin{array}{l} \alpha_1, \beta_1 \\ \alpha_2, \beta_2 \\ \\ \\ \end{array} \end{array}$$

avg $\hat{y}$ to plot line