① HW 8

3-10-15

Metropolis Sampling → uses a symmetric proposal distribution

when sampling a proposal parameter value θ^* from proposal density $q(\theta^* | \theta_c)$, this means

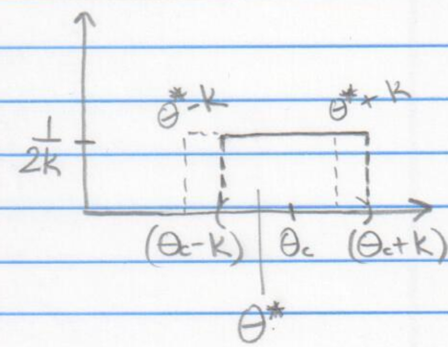
$$q(\theta^* | \theta_c) = q(\theta_c | \theta^*)$$

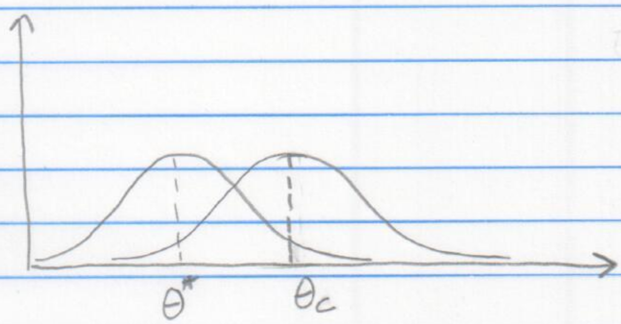
proposal density   centered at   evaluated at

θ_c = current value of θ

Example

let $q(\theta^* | \theta_c) = \frac{1}{2k}$ for $\theta^* \in (\theta_c - k, \theta_c + k)$





Example:

$$q(\theta^* | \theta_c) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(\theta^* - \theta_c)^2} \quad \text{for } \theta^* \in \mathbb{R}$$

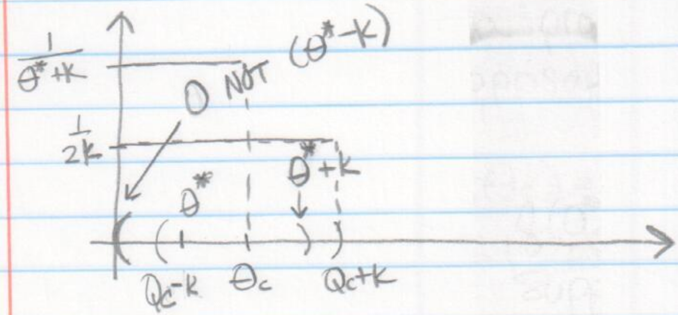
- Equality $q(\theta^* | \theta_c) = q(\theta_c | \theta^*)$

Always satisfied for symmetric distributions

except →

Ex. $q(\theta^* | \theta_c) = \frac{1}{2k}$ for $\theta^* \in (\theta^*+k, \theta^*-k)$

Suppose θ must be > 0.



☆ Need to adjust acceptance probability when near parameter bound - Metropolis Hastings

Metropolis-Hastings Algorithm

Let $\pi(\theta)$ be the distribution from which we wish to sample.

Use the Metropolis algorithm to sample values of θ from $\pi(\theta)$, with the following adjustment to the acceptance probability:

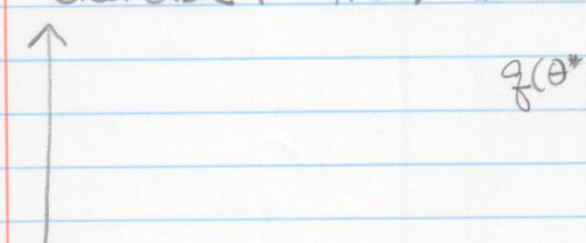
$$\alpha^* = \min\left\{ 1, \underbrace{\frac{\pi(\theta^*)}{\pi(\theta_c)}}_{\text{target density}} \cdot \underbrace{\frac{q(\theta_c | \theta^*)}{q(\theta^* | \theta_c)}}_{\text{proposal density}} \right\}$$

Hastings Correction

HW 8

Exercise 1 $\pi(\theta) = 1$ for $\theta \in (0,1)$.

$$q(\theta^* | \theta_c) = \begin{cases} \frac{1}{2k} & \text{if } \theta_c+k \le 1 \text{ and } \theta_c - k \ge 0. \\ \frac{1}{\theta_c+k} & \text{if } \theta_c < k \\ \frac{1}{(1-\theta_c)+k} & \text{if } \theta_c > (1-k) \end{cases}$$



when $\theta_c > 1-K$ $\qquad \frac{1}{\theta_c+K} < 1$, $\qquad \frac{1}{2K} < 1$,

$$\frac{1}{1-\theta_c+K}$$

when $\theta_c < K$ $\qquad \frac{1}{1-\theta_c+K} > \frac{1}{\theta_c+K}$

max $(\sim\theta^* \sim)$ in denom.
max $(\sim\theta_c\sim)$ in num.

CPMA 573 — Homework #8

Exercise 1: The Hastings Ratio. Sample 5000 independent realizations from the uniform $(0, 1)$ density for the following methods:

target density

Method 1: Metropolis sampling with uniform proposal density $q(x^*|x^c)$ on $(x^c - 0.2, x^c + 0.2)$. If a value x^* is proposed outside of the interval $(0, 1)$, then continue to propose from q until an $x^* \in (0, 1)$ is obtained.

Method 2: Metropolis-Hastings sampling with uniform proposal density $q(x^*|x^c)$ on $(x^c - 0.2, x^c + 0.2)$. If a value x^* is proposed outside of the interval $(0, 1)$, then continue to propose from q until an $x^* \in (0, 1)$ is obtained.

Complete the following for both methods:

- Report the lag used to obtain independent draws.
- Report the overall acceptance probability.
- Plot a histogram of the 5000 independent realizations.

Describe, in your own words, the differences between the two histograms, and, *why* the histogram based on Method 1 is non-uniform.

Exercise 2: Metropolis-Hastings sampling and the normal model. Let $Y_1, \ldots, Y_{155}$ be normal random variables from the density

$$f(y|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\} .$$

Realizations from this data model are located at

`www.mathcs.duq.edu/~kern/hw8.dat`

It is your goal to make inference on the values of μ and σ^2 used to generate these data.

likelihood $\cdot\ \pi_0(\mu) \cdot \pi_0(\sigma^2)$ = joint posterior

Assuming μ and σ^2 have prior densities

$$\mu|\sigma^2 \sim \mathrm{N}(0, 10 \cdot \sigma^2) \quad \text{and} \quad \sigma^2 \sim \mathrm{IG}(2.5, 1),$$

obtain 10000 independent draws from the marginal posteriors $\pi(\mu|\vec{y})$ and $\pi(\sigma^2|\vec{y})$ using Metropolis-Hastings sampling in conjunction with proposal densities

$$\mu^*_{i+1} \sim N(\mu_i, b^2) \quad \text{and} \quad \sigma^{2*}_{i+1} \sim N(\sigma^2_i, c^2)$$

for your choice of b and c. Notice that these prior densities are special cases of those used in HW #6. Using starting values $\mu_0 = \bar{y}$ and $\sigma^2_0 = s^2$, report the acceptance probability and the lag necessary to obtain independent realizations for your choice of b and c. Then, provide trace plots and histograms of your μ and σ^2 realizations. As with HW #6, superpose the theoretical marginal density of σ^2 on the histogram of the σ^2 realizations. For this analysis, also answer the following:

i. Write down the joint posterior density $\pi(\mu, \sigma^2|\bar{y})$. likelihood × prior ($\mu$) × prior ($\sigma^2$)

ii. Write down the density for the marginal posterior of σ^2. Hold μ constant and pull out constants

iii. Which (if any) of the parameters (μ, σ^2) require use of the Hastings ratio, and why?

iv. State explicitly the necessary Hastings ratio(s) from **iii**.

v. Explain carefully why it was acceptable in HW #7 to sample σ^2 using the Metropolis algorithm *without* a Hastings ratio.

$$\mu^2 - 2\cdot\beta\cdot\mu + K$$

$$(\mu^2 - \beta)^2 + K_2$$

$$\mu^2 - 2\beta\mu \boxed{+ \beta^2 + K^2}$$

$$\mu \sim N\left(\beta, \frac{\sigma^2}{(n + \frac{1}{10})}\right)$$

Lisa Over
HW 8 prep/notes

$\mu|\sigma^2 \sim N(\mu_0, \frac{\sigma^2}{k_0})$ $\sigma^2 \sim IG(\frac{\nu_0}{2}, \frac{\nu_0 \sigma_0^2}{2})$

prior densities $\mu|\sigma^2 \sim N(0, 10\cdot\sigma^2)$ $\sigma^2 \sim IG(2.5, 1)$

$\therefore \mu_0 = 0$

$\frac{\sigma^2}{k_0} = 10\cdot\sigma^2$

$\frac{\nu_0}{2} = 2.5$

$\frac{\nu_0 \cdot \sigma_0^2}{2} = 1$

joint posterior = likelihood · prior $\pi_0(\mu)$ · prior $\pi_0(\sigma^2)$

likelihood: $\left(\frac{1}{\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}[(n-1)s^2 + n(\bar{y}-\mu)^2]}$

$\pi_0(\mu)$: $\frac{1}{\sqrt{2\pi\frac{\sigma^2}{k_0}}} e^{-(\mu-\mu_0)^2/(\frac{2\sigma^2}{k_0})}$

$\pi_0(\sigma^2)$: $\frac{\left(\frac{\nu_0\sigma_0^2}{2}\right)^{\nu_0/2}}{\Gamma(\frac{\nu_0}{2})} \cdot \left(\frac{1}{\sigma^2}\right)^{\nu_0/2+1} \cdot e^{-(\frac{\nu_0\sigma_0^2}{2})/\sigma^2}$

with substitutions...

$\pi_0(\mu)$: $\frac{1}{\sqrt{2\pi 10\sigma^2}} e^{-(\mu)^2/(2\cdot 10\sigma^2)} \rightarrow \frac{1}{\sqrt{20\pi\sigma^2}} e^{-\mu^2/(20\sigma^2)}$

$\pi_0(\sigma^2)$: $\frac{1}{\Gamma(2.5)} \cdot \left(\frac{1}{\sigma^2}\right)^{3.5} e^{-1/\sigma^2}$

Hold σ^2 as constant to obtain full conditional for μ.
constants are ignored b/c proportionality

likelihood: $e^{\frac{-1}{2\sigma^2}(n(\bar{y}-\mu)^2)} \rightarrow e^{\frac{-n(\bar{y}-\mu)^2}{2\sigma^2}}$

All of $\pi_0(\sigma^2)$ is a constant - no μ and σ^2 is constant

$\pi_0(\mu)$ has constant $\frac{1}{\sqrt{20\pi\sigma^2}}$ so use $e^{-\mu^2/20\sigma^2}$

Full conditional for μ:

$$\pi_1(\mu|\sigma^2, y_1, \ldots, y_n) \propto e^{\frac{-n(\bar{y}-\mu)^2}{2\sigma^2}} \cdot e^{-\mu^2/20\sigma^2}$$

$$= e^{\frac{-n(\bar{y}-\mu)^2}{2\sigma^2} - \frac{\mu^2}{10 \cdot 2\sigma^2}}$$

$$= e^{-\frac{1}{2\sigma^2}\left[n(\bar{y}-\mu)^2 + \frac{\mu^2}{10}\right]}$$

alpha

target ratio $e^{-\frac{1}{2\sigma^2}\left[n(\bar{y}-\mu^*)^2 + \frac{(\mu^*)^2}{10}\right] - \left(\frac{-1}{2\sigma^2}\right)\left[n(\bar{y}-\mu_0)^2 - \frac{(\mu_0)^2}{10}\right]}$

$$= e^{\frac{-1}{2\sigma^2}\left[\left[n(\bar{y}-\mu^*)^2 + \frac{(\mu^*)^2}{10}\right] - \left[n(\bar{y}-\mu_0)^2 - \frac{(\mu_0)^2}{10}\right]\right]}$$

$$\frac{e^{-\frac{1}{2\sigma^2}\left[n(\bar{y}-\mu^*)^2 + \frac{\mu^{*2}}{10}\right]}}{e^{-\frac{1}{2\sigma^2}\left[n(\bar{y}-\mu_c)^2 + \frac{\mu_c^2}{10}\right]}}$$

$$= e^{-\frac{1}{2\sigma^2}\left[n(\bar{y}-\mu^*)^2 + \frac{\mu^{*2}}{10} - n(\bar{y}-\mu_c^2) - \frac{\mu_c^2}{10}\right]}$$

$$\frac{A^* e^{-B^*}}{A_0 e^{-\beta_0}} = \ln(A^*) - B^* - \ln(A_0) + \beta_0$$

Full conditional for σ^2: $3\left(\frac{1}{\sigma^2}\right)$ terms

$$\hookrightarrow \left(\frac{1}{\sigma^2}\right)^{n/2}, \left(\frac{1}{\sigma^2}\right)^{3.5}, \frac{1}{\sqrt{2\pi 10\sigma^2}}$$

likelihood $\cdot \pi_0(\mu) \cdot \pi_0(\sigma^2)$

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{10}} \cdot \frac{1}{\sqrt{\sigma^2}} \rightarrow \left(\frac{1}{\sigma^2}\right)^{1/2}$$

$$\left(\frac{1}{\sigma^2}\right)^{n/2+3.5+0.5} e^{-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right] - \frac{\mu^2}{20\sigma^2} - \frac{1}{\sigma^2}}$$

$$\left(\frac{1}{\sigma^2}\right)^{n/2+4} e^{\underbrace{-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2 + \frac{\mu^2}{10} + 2\right]}}$$

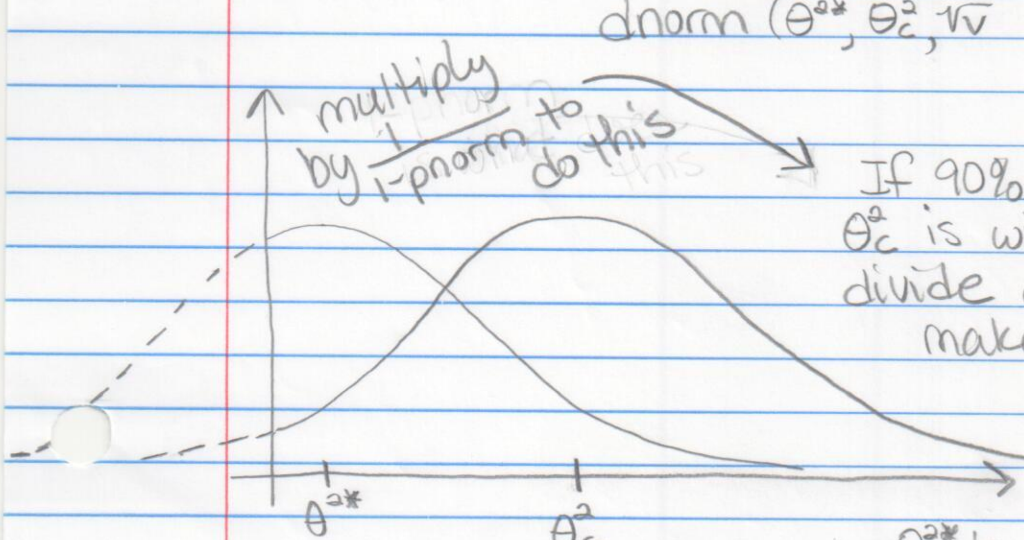
A.star　　　　B.star

A.zero　　　　B.zero

Hastings Correction

$$\frac{q(\theta_c^2 \mid \theta^{2*})}{q(\theta^{2*} \mid \theta_c^2)} = \frac{\frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(\theta_c^2 - \theta^{2*})^2} \cdot \left(\frac{1}{\int_0^\infty \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(\theta^{2*} - \theta_c^2)^2} d\theta_c^2}\right)}{\frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(\theta^{2*} - \theta_c^2)^2} \cdot \left(\frac{1}{\int_0^\infty \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(\theta_c^2 - \theta^{2*})^2} d\theta^{2*}}\right)}$$

$$\frac{\text{dnorm}(\theta_c^2, \theta^{2*}, \sqrt{v})}{\text{dnorm}(\theta^{2*}, \theta_c^2, \sqrt{v})} \cdot \frac{1 - \text{pnorm}(0, \theta_c^2, \sqrt{v})}{1 - \text{pnorm}(0, \theta^{2*}, \sqrt{v})}$$



If 90% of curve centered at θ_c^2 is within bounds, must divide density by 0.9 to make it equal to 1 when integrated. Likewise, if 60% of curve centered at θ^{2*} is within bounds must divide by 0.6

Lisa Over

i. $\pi(\mu, \sigma | \vec{y}) =$

$$\frac{1}{\Gamma(2.5)\sqrt{20\pi}} \left(\frac{1}{\sigma^2}\right)^{n/2+4} e^{-1/2\sigma^2 [(n-1)s^2 + n(\bar{y}-\mu)^2] - \mu^2/20\sigma^2 - \frac{1}{\sigma^2}}$$

$$\frac{1}{\Gamma(2.5)\sqrt{20\pi}} \left(\frac{1}{\sigma^2}\right)^{n/2+4} e^{-1/2\sigma^2 [(n-1)s^2 + n(\bar{y}-\mu)^2 + \mu^2/10 + 2]}$$

ii. $$\frac{1}{\Gamma(2.5)\sqrt{20\pi}} \left(\frac{1}{\sigma^2}\right)^{n/2+4} e^{-\frac{(n-1)s^2+2}{2\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2\sigma^2}[n(\bar{y}-\mu)^2 + \mu^2/10]} d\mu$$

$$\Rightarrow \frac{-1}{2\sigma^2}[n(\bar{y}-\mu)^2 + \mu^2/10]$$

$$= \frac{-1}{2\sigma^2}[n(\bar{y}^2 - 2\bar{y}\mu + \mu^2) + \mu^2/10]$$

$$= \frac{-1}{2\sigma^2}[n\bar{y}^2 - 2n\bar{y}\mu + n\mu^2 + \mu^2/10]$$

divide by $n + \frac{1}{10}$

$$= -\frac{n+\frac{1}{10}}{2\sigma^2}\left[\frac{n\bar{y}^2}{n+1/10} - \frac{2n\bar{y}\mu}{n+1/10} + \mu^2\right]$$

$$= -\frac{n+1/10}{2\sigma^2}\left[\mu^2 - \frac{2n\bar{y}\mu}{n+1/10} + \frac{n\bar{y}^2}{n+1/10}\right]$$

$$= -\frac{n+1/10}{2\sigma^2}\left[\mu^2 - \frac{n\bar{y}}{n+1/10}\right]^2$$

$$\frac{1}{\Gamma(2.5)\sqrt{20\pi}} \left(\frac{1}{\sigma^2}\right)^{n/2+4} e^{\frac{-(n-1)s^2+2}{2\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{n+1/10}{2\sigma^2}\left[\mu - \frac{n\bar{y}}{n+1/10}\right]^2} d\mu$$

multiply by $\frac{1}{\sqrt{2\pi\sigma^2/(n+1/10)}}$ so integrates to 1

$\left(\frac{1}{\sigma^2}\right)^{-\frac{1}{2}} = \sqrt{\sigma^2} = \sigma$

marginal posterior for σ^2:

$$\underbrace{\frac{\sqrt{2\pi\sigma^2/(n+1/10)}}{\Gamma(2.5)\sqrt{20\pi}} \left(\frac{1}{\sigma^2}\right)^{n/2+4} \cdot e^{\frac{-(n-1)s^2+2}{2\sigma^2}}}_{IG}$$

$$\boxed{\left(\frac{1}{\sigma^2}\right)^{-1/2} \cdot \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+4-\frac{1}{2}} = \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+3.5}}$$

$\alpha = \frac{n}{2} + 3.5$

$\text{dnorm}(\sigma_c^2, \sigma^{2*}, \sqrt{v})$

$1-\text{pnorm}(0, \sigma_c^{2*}, \sqrt{v})$

$$\frac{q(\sigma_c^2 \mid \sigma^{2*})}{q(\sigma_*^2 \mid \sigma_c^2)} = \frac{\frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(\sigma_c^2-\sigma^{2*})^2} \cdot \left(\frac{1}{\int_0^\infty \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(\sigma^{2*}-\sigma_c^2)^2}\, d\sigma_c^2}\right)}{\frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(\sigma^{2*}-\sigma_c^2)^2} \left(\frac{1}{\int_0^\infty \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2v}(\sigma_c^2-\sigma^{2*})^2}\, d\sigma^{2*}}\right)}$$