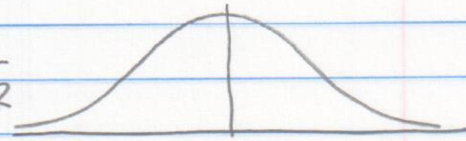
① HW3

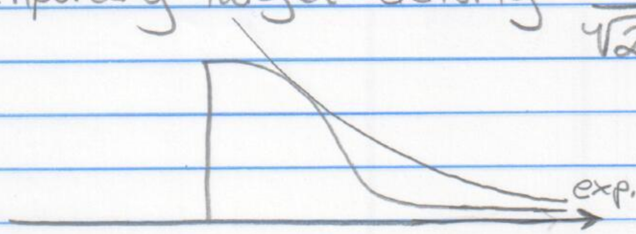
1-27-15

Alternative Rejection Sampling Approach to simulate a standard normal realization.

Target density: $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ for $x \in \mathbb{R}$



Temporary target density $\frac{2}{\sqrt{2\pi}} e^{-x^2/2}$ for $x > 0$



Exponential proposal density

$$g(x) = e^{-x} \quad \text{for } x > 0$$

$$\text{find } A \overset{\text{s.t.}}{\ni} A\,g(x) \geq f(x) \;\; \forall x$$

Using random #s to calculate Probabilities

Let X have a density $f(x)$. We wish to find, for some interval A, $P(X \in A) = \int_A f(x)\,dx$.

(A not same as multiplier above)

- See how many #s from realization are in A
- what proportion are in interval

To approximate this probability, (integral), we can do the following:

① Use Monte Carlo integration

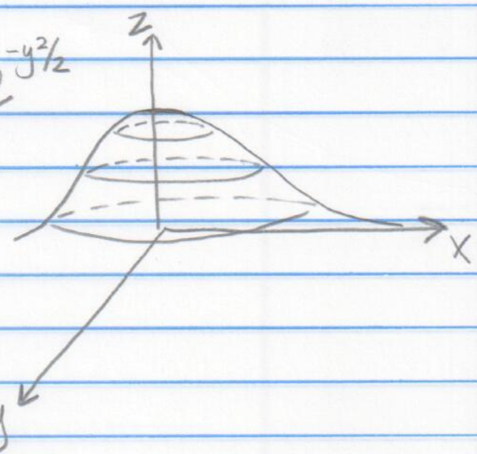
② Generate a large # of r.v. values; observe the proportion that fall in the interval A. This proportion is estimate of $P(X \in A)$

The Polar Method for generating Normal random variables.

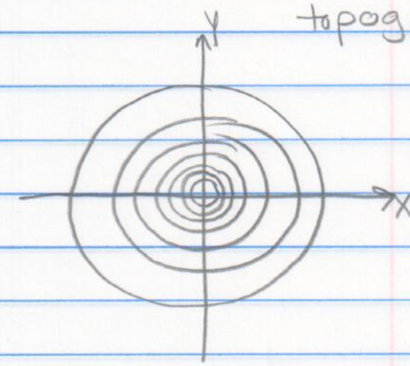
Let X and Y be iid standard normal r.v.s. The joint density $f(x,y)$ of X and Y is given by

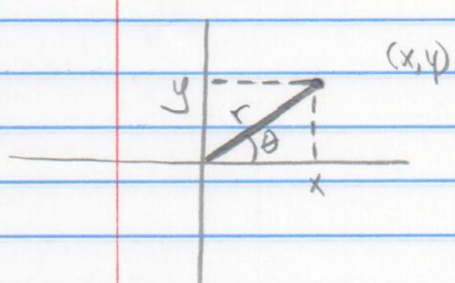
$$f(x,y) = f_X(x)\, f_Y(y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2} \quad \text{for } x \in \mathbb{R},\ y \in \mathbb{R}$$

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$



topographical





$$r = (x^2+y^2)^{1/2}$$

$$\theta = \arctan \frac{y}{x}$$

Rewrite joint density in polar coordinates...

$$f(x,y) \rightarrow f(r,\theta) = \frac{1}{2\pi} \cdot \frac{1}{2} e^{-r^2/2} \quad \text{for } \theta \in (0, 2\pi),\ r^2 \in (0, \infty)$$

θ uniform on interval $(0, 2\pi)$, $(k\pi, 2k\pi)$, $(-2k\pi, 0)$

exponential density with parameter $1/2$

r translate to x, y

To simulate realizations of X and Y simultaneously:

Step 1: Generate $u_1 \sim unif(0,1)$
$u_2 \sim unif(0,1)$

Step 2: $r^2 = -2\log(u_1)$ $\qquad \lambda = 1/2$
$\theta = 2\pi(u_2)$

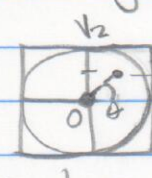
Step 3: $X = r\cos\theta = \sqrt{-2\log(u_1)} \cdot \cos(2\pi(u_2))$
$Y = r\sin\theta = \sqrt{-2\log(u_1)} \cdot \sin(2\pi(u_2))$

To improve computational efficiency, take the following steps to avoid sin and cos calls:

(i) simulate a coordinate (v_1, v_2) in the unit circle as follows:

Let $v_1 = \underbrace{2u_1}_{unif(0,2)} - 1 \qquad unif(-1,1)$ $\quad$ unif on coord.
$v_2 = 2u_2 - 1 \qquad unif(-1,1)$ $\quad$ unit square

If $(v_1^2 + v_2^2)^{1/2} > 1$, sample u_1, u_2 again



- random pt in unit circle
- $\sin\theta = \frac{v_2}{\sqrt{v_1^2+v_2^2}}$
- $\cos\theta = \frac{v_1}{\sqrt{v_1^2+v_2^2}}$

$v_1^2 + v_2^2 \sim unif(0,1)$
Let $S^2 = v_1^2 + v_2^2$

Let $S^2 = v_1^2 + v_2^2$

Faster Step 3

$X = \sqrt{-2\log(S^2)} \cdot \frac{v_2}{S}$

$Y = \sqrt{-2\log(S^2)} \cdot \frac{v_1}{S}$

How often do we accept $(v_1^2+v_2^2)^{1/2}$?

$\frac{\text{Area of circle}}{\text{Area of Square}} = 0.78$

1/5 of time reject

CPMA 573 — Homework #3

Exercise 1: Beta probabilities. Recall the beta density function $f(x|\alpha, \beta)$ given by

$$f(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},$$

for $x \in (0,1)$ and positive α and β. Let $X \sim \text{beta}(2,5)$. Find $P(X > \frac{1}{2})$ by:

a. Monte Carlo (MC) integration.

b. Simulating 25,000 beta(2,5) realizations. Use rejection sampling and uniform proposal density $q(x)$.

c. Using the `pbeta` function in **R**.

Exercise 2: Standard normal probabilities. Let $X \sim \text{N}(0,1)$. Find $P(-1.96 < X < 1.96)$ by:

a. MC integration.

b. Simulating 25,000 N(0,1) realizations. Use rejection sampling and exponential reference density $q(x)$.

c. Using the `pnorm` function in **R**.

Exercise 3: Bivariate normal probabilities (polar method). Let X and Y be independent standard normal random variables. Use the polar method to simulate 25,000 pairs (x, y) of independent, standard normal realizations. Then, use these realizations to find the value of k that satisfies $P(\sqrt{X^2+Y^2} < k) = \frac{1}{2}$.