HW 6

1

2-17-15

quantified our uncertainty from a prior distribution

- combined, before obs. the data, with the likelihood function to get posterior distribution
- posterior incorporates the data w/ non-informative prior for (μ, σ^2)
- credible interval - what is the prob. our parameter falls in the interval? 95%

Data Model (from last week):

$Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Likelihood for (μ, σ^2)

$$L(\mu, \sigma^2) \propto \left(\frac{1}{\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}[(n-1)s^2 + n(\bar{y}-\mu)^2]}$$

$$s^2 = \frac{\sum (y_i - \bar{y})^2}{n-1}$$

NEW Informative Prior for (μ, σ^2)

$$\pi_0(\mu, \sigma^2) = \pi_0(\mu \mid \sigma^2) \cdot \pi_0(\sigma^2)$$

where $\mu \mid \sigma^2 \sim N\left(\mu_0, \frac{\sigma^2}{k_0}\right)$ — hyperparameters

$\sigma^2 \sim IG\left(\frac{\nu_0}{2}, \frac{\nu_0 \sigma_0^2}{2}\right)$ IG inverse gamma

Goal sample from the marginal posteriors for μ and σ^2 using GIBBS sampling

marginal not conditional

(each μ could be the μ that generated the data.)

- drew from full conditional but because of lag / burnin, as a whole are considered marginal

two priors to tinker with: μ_0, K_0
ν_0, σ_0^2

parameters of prior — hyperparameters

Joint Posterior for (μ, σ^2):

$$\pi_1(\mu, \sigma^2 | y_1, y_2, \ldots, y_n) \propto$$

likelihood · prior · prior

$$\propto \left(\frac{1}{\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}[(n-1)s^2 + n(\bar{y}-\mu)^2]} \cdot \frac{1}{\sqrt{2\pi\frac{\sigma^2}{K_0}}} e^{-(\mu-\mu_0)^2 / \left(\frac{2\sigma^2}{K_0}\right)} \cdot \frac{\left(\frac{\nu_0\sigma_0^2}{2}\right)^{\frac{\nu_0}{2}}}{\Gamma\left(\frac{\nu_0}{2}\right)} \cdot \left(\frac{1}{\sigma^2}\right)^{\nu_0/2+1} \cdot e^{-\left(\frac{\nu_0\sigma_0^2}{2}\right)/\sigma^2}$$

Full conditional for μ:

$$\pi_1(\mu | \sigma^2, y_1, \ldots, y_n) \propto e^{\frac{-n(\bar{y}-\mu)^2}{2\sigma^2}} \cdot e^{-(\mu-\mu_0)^2/\left(\frac{2\sigma^2}{K_0}\right)}$$

$$= e^{\frac{-n}{2\sigma^2}[\bar{y}^2 - 2\bar{y}\mu + \mu^2] - \frac{K_0}{2\sigma^2}[\mu^2 - 2\mu_0\mu + \mu_0^2]}$$

$$= e^{(n+K_0)\frac{-1}{2\sigma^2}\left[\frac{(n+K_0)\mu^2}{n+K_0} - \frac{2(\bar{y}n + \mu_0 K_0)}{n+K_0}\mu + \frac{(n\bar{y}^2 + K_0\mu_0^2)}{n+K_0}\right]}$$

$$= e^{-\frac{n+K_0}{2\sigma^2}\left[\mu^2 - \frac{2(\bar{y}n+\mu_0K_0)\mu}{n+K_0} + \left(\frac{n\bar{y}^2+K_0\mu_0^2}{n+K_0}\right)\right]}$$

complete the square

All terms w/ no μ are absorbed

$$\propto e^{-\frac{(n+K_0)}{2\sigma^2}\left(\mu - \frac{\bar{y}n + \mu_0K_0}{n+K_0}\right)^2}$$

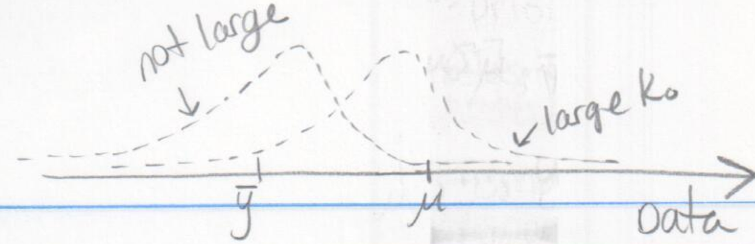
full conditional for μ

$$\boxed{\mu | \sigma^2, y_1, \ldots, y_n \sim N\left(\frac{\bar{y}n + \mu_0K_0}{n+K_0}, \frac{\sigma^2}{n+K_0}\right)}$$

weighted average $\frac{n}{n+K_0}$ and $\frac{K_0}{n+K_0}$

μ_0 mean of prior

$\bar{y}$ mean of data



$$\frac{1}{\sqrt{2\pi\frac{\sigma^2}{K_0}}} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\sigma^2}}\sqrt{K_0} \propto \frac{1}{2}$$

full conditional for σ^2

(From likelihood $\cdot$ priors – lots of σ^2 terms)

$$\pi(\sigma^2 | \mu, y_1, \ldots, y_n) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+\frac{1}{2}+\frac{\nu_0}{2}+1} e^{\frac{-1}{2\sigma^2}\left[(n-1)s^2+n(\bar{y}-\mu)^2 + (\mu-\mu_0)^2 K_0 + \nu_0\sigma_0^2\right]}$$

full conditional for σ^2

$$\boxed{\sigma | \mu, y_1 \ldots y_n \sim IG\left(\frac{n+1+\nu_0}{2}, \frac{[(n-1)s^2+n(\bar{y}-\mu)^2+(\mu-\mu_0)^2K_0+\nu_0\sigma_0^2]}{2}\right)}$$

HW 6 Complete HW #5 again but replace the prior $\pi_0(\mu, \sigma^2)$ therein with the prior $\pi_0(\mu|\sigma^2)\cdot\pi_0(\sigma^2)$ introduced this week. Use only Gibbs sampling method and compute the posterior predictive distribution for a new y-value (i.e. complete problem #2)

Address the following: 3 options each

How do different choices of K_0, ν_0, and σ_0^2 affect the marginal posteriors for μ and σ^2.

Set $\mu_0 = 20$

2x2 table small K_0 large K_0

small σ^2

small ν_0

large ν_0

replicate w/ large σ^2