HW 12

Mixture Models

Two-Component Normal Mixture Models

Data $y_1, y_2, \ldots, y_n$ come from one of two normal distributions. Not all from same. Find which y_is go with each dist. and find μ and σ^2 of each.

Let μ_1 and σ_1^2 represent the mean and variance of one of the two normals.
Let μ_2 and σ_2^2 represent the mean and variance of the other.

How can we write the joint density for the Y_is?

(From Joint Density, likelihood is derived so important)

First introduce latent indicator variables.

Z_i switch to turn on approp. normal

Let $Z_i = \begin{cases} 0 & \text{if } Y_i \text{ is drawn from } N(\mu_1, \sigma_1^2) \\ 1 & \text{if } Y_i \text{ is drawn from } N(\mu_2, \sigma_2^2) \end{cases}$

latent indicator variable

Use the Z_i to write the joint density for the Y_is:

$$f(Y_1, Y_2, \ldots, Y_n \mid \vec{Z}, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2, p)$$

p = prob from $N(\mu_1, \sigma_1^2)$

$$= \left[\frac{p}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}(y_1-\mu_1)^2}\right]^{1-Z_1} \cdot \left[\frac{(1-p)}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\sigma_2^2}(y_2-\mu_2)^2}\right]^{Z_1} \cdots$$

$$\left[\frac{p}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}(y_n-\mu_1)^2}\right]^{1-Z_n} \cdot \left[\frac{(1-p)}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\sigma_2^2}(y_n-\mu_2)^2}\right]^{Z_n} =$$

$\longrightarrow$

$$= p^{n-\sum z_i} \cdot (1-p)^{\sum z_i} \cdot \left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\right)^{n-\sum z_i} \cdot \left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\right)^{\sum z_i} \cdot$$

$$e^{-\frac{1}{2\sigma_1^2}\sum_{i \ni z_i=0}(y_i-\mu_1)^2} \cdot e^{-\frac{1}{2\sigma_2^2}\sum_{i \ni z_i=1}(y_i-\mu_2)^2}$$

joint density as a function of $Y_1, Y_2, \dots Y_n$

likelihood function, as a function of $p, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \vec{Z}$

For simplicity, suppose $\mu_1=0$, and that $\mu_2>0$. Furthermore, we will fix $\sigma_1^2=\sigma_2^2=1$ for this week. Combine with joint prior

$\pi(p, \vec{z}, \mu_2) \propto 1$ to yield the following posterior distribution for p, $\vec{z}$, and μ_2:

$$\pi(p, \vec{z}, \mu \mid \vec{y}) \propto p^{n-\sum z_i} \cdot (1-p)^{\sum z_i} \cdot e^{-\frac{1}{2}\sum_{i \ni z_i=0}(y_i)^2} \cdot e^{-\frac{1}{2}\sum_{i \ni z_i=1}(y_i-\mu_2)^2}$$

find full conditional for each parameter
sample from marginal posterior

$y_i-\mu_2 = (y_i-\bar{y})+(\bar{y}-\mu_2)$

<u>Full Conditionals</u>

μ_2: $N(\bar{y}, 1/\sum z_i)$

p: $\text{Beta}(n-\sum z_i+1, \sum z_i+1)$ $\qquad \alpha = n-\sum z_i \quad \lambda = \sum z_i$

z_i: $\text{Bern}\left(\frac{(1-p)e^{-\frac{1}{2}(y_i-\mu_2)^2}}{pe^{-\frac{1}{2}y_i^2} + e^{-\frac{1}{2}(y_i-\mu_2)^2}(1-p)}\right)$ (probability when $z=1$)

when $z=1$

when $z=0$ + when $z=1$

Bern is Bin w/ $n=1$
rbinom

$f(x) = \frac{\Gamma(\alpha+\lambda)}{\Gamma(\alpha)\Gamma(\lambda)} x^{\alpha-1}(1-x)^{\lambda-1}$ for $x \in (0,1)$
(p plays role of x in beta density minus multiplicative constants)

realizations

Z <- rbinom(N, 1, pvec)

n parameter = 1

pvec is vector of probabilities Z

HW #12

1. Simulate two-component normal mixture data
   - $N = 200$
   - $p = 0.7$
   - $\mu_2 = 3$
   - $\sigma_1^2 = \sigma_2^2 = 1$

K <- rbinom(1, 200, 0.7)

⋮

keep going

give the number of the 200 y_is that will be drawn from $N(\mu_1, \sigma_1^2)$

2. Use Gibbs sampling to sample from the joint posterior for μ_2, $\vec{z}$, and p. (use simulated data)

   Plot histograms and trace plots of the μ_2 and p realizations (after appropriate lagging).

   Also plot the $\bar{z}_i$ vs. y_i for all i
   Explain this plot. What plot communicates.