Laying a Foundation for Problem Solving, Reasoning and Proof, Communication, Concept Connections, and Representation of Concepts and Data

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Introduction

This nine-week curriculum is designed to lay a foundation for higher level mathematical thinking while also maintaining a strong focus on content. Through this curriculum, 7^{th} or 8^{th} grade pre-algebra students will have numerous developmentally-appropriate opportunities to practice problem solving techniques, engage in reasoning and proof, develop communication skills, discover connections among mathematical topics, and explore various representations of concepts and data (NCTM, 2010).

Project Significance and Background

After surveying college mathematics faculty and analyzing educational statistics, Corbishley and Truxaw (2010) found that an overwhelming majority of American high school students are not prepared for the higher level thinking required in college mathematics courses. Based on a 2006 report from the National Center for Educational Statistics (NCES), Corbishley and Truxaw (2010) concluded that “less than one quarter of US 12th-grade students who were tested performed at or above the proficient level in mathematics, and only 2% achieved at the advanced level” (Mathematical Knowledge and Achievement section, para. 2). For students to be proficient in mathematics, they must “transition from a simple to a more complex view” of mathematics, which involves “making use of mathematical properties in order to deductively construct concepts” (Corbishley and Truxaw, 2010, Mathematical Knowledge and Achievement section, para. 1). However, according to Wells (2009), students in K-12 are not exposed to tasks or problems that cultivate a transition from simple to complex thinking. Wells (2009) examined the National Curriculum’s various levels of mathematical competencies and concluded that they lack sufficient emphasis on reasoning and proof. At the highest level, the National Curriculum merely expects students to “comment constructively on the reasoning and logic, the process employed and the results obtained” (Wells, 2009, p. 27). Wells (2009) responded to this requirement of the National Curriculum by stating, “No wonder proof is problematic if the ability to distinguish mere evidence from proof is confirmation of *exceptional performance*” (p. 27). Wells also searched for opportunities for proof in a textbook that claimed to prepare students for the General Certificates of Secondary Education (GCSE). Wells (2009) critiqued the textbook *Edexcel GCSE Mathematics: Higher Course* (Pledger, 2001), which followed the National Curriculum, and found very few opportunities for proof; in fact, “the *term* proof does not appear in the Index, nor does justification” (Wells, 2009, p. 27). In the table of contents, there were only four sections in the whole textbook that said anything at all about proving or justifying. (Wells, 2009, p. 27) This research clearly indicates a need for a new curriculum if students are to be ready for the higher level thinking required in college and life. The curriculum proposed in this document provides many opportunities for students to develop higher level thinking skills in mathematics. It proposes the use of a textbook that has plenty of problems specifically designed to facilitate students' transitions from simple to complex thinking. However, while a textbook is a wonderful and necessary tool for teachers to use, a particular textbook is not at the heart of this curriculum. The core of this curriculum involves the teacher who will design and present lessons that are challenging and developmentally appropriate and that move students seamlessly from simple to complex thinking while not ignoring content.

Corbishley and Truxaw (2010) referred to the content and process standards established by the National Council of Teachers of Mathematics (NCTM) as standards that, if followed, lead students through the transition from simple to complex thinking. The content standards include number and operations, algebra, geometry, measurement, and data analysis and probability (NCTM, 2010). The process standards include problem solving, reasoning and proof, communication, connections, and representation (NCTM, 2010). This nine-week curriculum includes content from four of the five content categories—number and operations, algebra, geometry, and measurement—as outlined in the included Scope and Sequence. This curriculum also involves students in all five process categories by providing opportunities for students to practice problem solving techniques, engage in reasoning and proof, develop communication skills, discover connections among mathematical topics, and explore various representations of concepts and data. Because the importance of content is well established in the concept of basics-in-learning, the significance of this new curriculum is in how it addresses the process standards. Process standards are critical to mathematics education because they "create greater cohesion within the mathematics curriculum by describing specific mathematical strategies that can be applied across content areas" (Corbishley and Truxaw, 2010, Mathematics Education Reform section, para. 1). Like the content standards, process standards have their own set of basics. While content basics act as building blocks for greater understanding, process basics build intersections that enable students to move freely within, around, and between different topics, concepts, and skills of mathematics. These basics of mathematical processes are embedded in the following descriptions.

Problem solving is a process that helps students to move beyond simple rote memory and formula calculations to more complex thinking (Corbishley and Truxaw, 2010). To improve students' problem solving abilities, Ferrucci, Yeap, and Carter (2003) described a modeling approach that involves drawing pictures and using symbols to represent data or ideas. Used in Singapore schools, Ferrucci, Yeap, and Carter (2003) suggested that "this approach may be a contributing factor to the country's success on international comparisons in mathematics" (p. 471). Ferrucci, Yeap, and Carter (2003) gave examples of how one teacher used modeling to help students solve a multistep word problem that involved multiplying fractions. The teacher, referred to as Ms. Y, talked the students through the problem solving process by using sectioned rectangles to represent fractions. Through modeling, students "develop the ability to use representations to solve problems that involve algebraic thinking before they study algebra" (Ferrucci, Yeap, and Carter, 2003, p. 472). Ferrucci, Yeap, and Carter (2003) further concluded that representing a problem through illustrations and symbols helps students see the underlying mathematical concepts of the problem and its solution. This better prepares students for the more abstract algebraic methods of solving problems that they will encounter later. Wallace (2007) presented another method to teach problem solving. Wallace (2007) recommended problem solving lessons but points out that a problem solving lesson involves more than just "beginning with a problem or task" (p. 504). Wallace (2007) suggested creating a list of possible probing questions along with the reasoning behind them. For example, a particular question may redirect a student who has a misconception. By anticipating student responses, the teacher can "orchestrate the work of the students without taking over the process of thinking for them" (Wallace, 2007, p. 511).

Reasoning and proof is another process area that is critical to mathematics. Wells (2009) stated that "pure mathematics is concerned with establishing the truth of statements" and that school students need "to develop these skills of logical reasoning to support their further studies, their employment and their critical view of life" (p. 27). Wells (2009) also asserted that "children are very good at arguing, *if given the chance*" (p. 29). Therefore, developmentally appropriate opportunities to engage in logic and proof exercises are a central part of this curriculum proposal. One way to implement reasoning and proof into a middle school curriculum is through deductive puzzling. Wanko (2010) explored how visual-based logic puzzles can help students improve their deductive reasoning skills. In a ten-week curriculum, Wanko (2010) created a learning environment that enabled students to discover and explore strategies and solutions to various Japanese logic puzzles such as Shikaku and Hashiwokakero. At the end of the ten-weeks, Wanko (2010) found that most students' deductive reasoning skills had improved; "it became clear that [students] were becoming comfortable applying the discourse of deductive reasoning to the statements of their solutions" (p. 528). Perhaps even more significantly, Wanko (2010) found that students showed "great interest in solving puzzles, discussing their problem-solving strategies, and applying what they have learned to other logic problems" (p. 525).

Communication is a foundational process that is essential not only for understanding and conveying mathematical content but for developing proficiency in all other processes. Words give shape and structure to our understanding. Students use words to construct and convey meaning as they solve problems, reason, and connect ideas. To communicate effectively about mathematics, students need to be proficient in the language of mathematics. However, according to Herbel-Eisenmann (2002), proficiency in communication comes gradually with practice and guidance. Herbel-Eisenmann (2002) provided "suggestions for how to move students from using their own language to using words that are more mathematically appropriate" (p. 100). She describes four different levels, or transitions, of mathematical language through which students must progress if they are to internalize the meaning of the technical mathematical terms. These include “Contextual Language (CL),” which is specific to particular problems or experiences and their context, e.g. “miles per hour;” “Classroom Generated Language (CGL),” which is colloquial, e.g. “slantiness;” “Transitional Mathematical Language (TML),” which is more specific than CGL but does not express the context of the problem, e.g. “As x goes up by…y goes up by….;” and “Official Mathematical Language (OML)” which is the language used by the mathematical community, e.g. “slope” (Herbel-Eisenmann, 2002, p. 102). All of the previous examples are ways of expressing slope. Through her observation and research, Herbel-Eisenmann (2002) concluded that these different levels “interacted and supported each other” as teachers used them interchangeably to “introduce mathematical language that was more natural to the flow of conversation than simply writing a list of vocabulary words for students to memorize” (p. 103-104). In addition to modeling these transitions, teachers also must give students opportunities to practice using them. Herbel-Eisenmann and Breyfogle (2005) examined “what happens in the exchanges *after* an initial question is posed” by a teacher, and they refer to this as examining the “interaction patterns” that occur between teachers and students (p. 484). Herbel-Eisenmann and Breyfogle started with the Initiation-Response-Feedback pattern, which was first described by Mehan in 1979 and is widely used among educators. In this pattern, teachers ask a question, students respond, and teachers provide feedback. The disadvantage of this pattern is that it “does little to encourage students to express their thinking” (Herbel-Eisenmann and Breyfogle, 2005, p. 484). Herbel-Eisenmann and Breyfogle (2005) then described two alternative patterns that allow students to practice communicating concepts and ideas. One pattern, which Herbel-Eisenmann and Breyfogle (2005) refered to as funneling, “occurs when the teacher asks a series of questions that guide the students through a procedure or to a desired end” (Herbel-Eisenmann and Breyfogle, 2005, p. 485). Herbel-Eisenmann and Breyfogle (2005) described funneling as a form of scaffolding where teachers model the questions that lead up to a solution or an understanding. Herbel-Eisenmann and Breyfogle (2005) described another pattern that allows the teacher “to see more clearly what the students [are] thinking” or that requires the students “to make their thinking clear and articulate so that others can understand what they are saying." In this pattern, which Herbel-Eisenmann and Breyfogle (2005) refered to as focusing, teachers "listen to students' responses and guide them based on what the students are thinking rather than how the teacher would solve the problem" (Herbel-Eisenmann and Breyfogle, 2005, p. 486). Focusing student thinking allows them to practice using transitional and official mathematical terminology, which is essential to becoming proficient in communicating mathematical ideas and concepts.

Understanding the relationships, or connections, among the different topics of mathematics is another process area that is critical to mathematics. Woodbury (2000) said that students "build relational understanding by connecting what they are learning…with what they already know" (p. 227). Woodbury (2000) suggested that teachers change their instructional focus "from the specific topics at hand to the conceptual arenas in which those topics reside" (p. 230). However, Woodbury (2000) did not mean that teachers should stop teaching procedures and rules. Instead, Woodbury (2000) said that teachers must find "ways to help students connect the currently studied rules, procedures, and individual topics with larger conceptual ideas" (p. 230). Woodbury (2000) provided an example from a seventh grade classroom where students developed relational ideas about real numbers, integers, and rational numbers. Rather than just memorizing the rule that prime numbers only have two factors, students explored why that rule made sense. These seventh graders developed a deep understanding of what it means for a number to be prime because they were asked to think about whether or not the number 5 was a prime number when considered within the real number system, which includes rational numbers, or fractions. The lesson on prime numbers was also framed around a problem that helped students to understand why the concepts are important and how the concepts can be used in real life. According to Woodbury (2000), "the students showed that they had become very interested in, and conversant about, a concept that they had previously understood only as a rule to follow" (p. 229). Barger and McCoy (2010) also promote connections within mathematics and suggest that teachers facilitate student connections with the conceptual ideas of another mathematical content area—calculus. Barger and McCoy (2010) described ways that teachers can introduce the concept of change by having students interpret, compare, and write stories about what is changing in a set of graphs. For example, one graph represented the distance and time of two female middle school students in a swimming race. The graph indicated that the winner started out behind her opponent but then picked up speed near the end to win the race. Barger and McCoy (2010) suggested that exploring these concepts in middle school would encourage students to take calculus in high school or college; "by showing middle school students the connections between what they are studying and actual problems from calculus textbooks, we can help to demystify calculus" (Barger and McCoy, 2010, p. 352). They also suggested that students who have a basic understanding of calculus concepts from an early age "will eventually enter calculus with a stronger basis for understanding the ideas at that [higher] level" (Barger and McCoy, 2010, p. 350).

Representation is another foundational process in that it is necessary for understanding mathematical content as well as developing skills in other process areas such as problem solving and communication. To understand and communicate mathematical content and concepts, students need to be able to use, evaluate, and interpret expressions, graphs, drawings, and tables. According to Preston and Garner (2003), representations are "tools that are vital for recording, analyzing, solving, and communicating mathematical data, problems, and ideas" (p. 39). Preston and Garner (2003) describe how one class of middle school students used representations to solve a word problem. These students were asked to work in groups to compare three different party options and to select one based on cost, i.e. they were to select the least expensive party. Students used tables to organize their information to determine the cost of each depending on the number students in attendance. Some party options charged a base price in addition to a "per student" price while others charged only "per student." Some students found an expression for their data, e.g. cost = 2.5s + 100, meaning that there was a $100 reservation rate plus $2.50 per student. Most students drew a line graph to visualize the data in their tables. Preston and

Garner (2003) found that “students used numerical approaches for their work and then when they were satisfied with the solution, turned to graphs to convince others” (p. 43). The advantage of knowing how to use and interpret various kinds of representations is to give students “choices when solving and communicating” (Preston and Garner, 2003, p. 43) so they will learn “which style of graph worked best” for a particular situation and why “particular graphs are used for certain problem situations” (Preston and Garner, 2003, p. 41).

Project Description and Methodology

The primary difference between this curriculum and other middle school curricula is that this curriculum emphasizes the process standards as outlined by the National Council of Teachers of Mathematics (NCTM, 2010), which are problem solving, reasoning and proof, communication, connections, and representation. Furthermore, it reflects the most recent pedagogical research for teaching those process standards while still maintaining a strong focus on content. The lessons in this nine-week Scope and Sequence incorporate four of the five content standards: number and operations, algebra, geometry, and measurement.

Reasoning and proof is incorporated in every lesson either through specific teacher-guided deductive reasoning sessions, such as the lessons outlined below, or through homework. The teacher-guided lessons enable students to practice deductive reasoning by systematically solving logic puzzles and by discovering a geometric proof of the Pythagorean Theorem. The lesson for Day 1 is included in this proposal.

- Day 1: Number Theory: Identifying Factors, Primes, and Composites; Shikaku Puzzles and Deductive Reasoning[i]
- Day 15: Irrational Numbers: The Pythagorean Theorem; Geometrical, Discovery Proof of the Pythagorean Theorem
- Day 23: Geometry: Line and Angles; Hashiwokakero Puzzles and Deductive Reasoning[ii]
- Day 25: Geometry: Polygons; Nurikabe Puzzles and Deductive Reasoning[iii]
- Day 31: Slitherlink Puzzles and Deductive Reasoning[iv]

Problem solving is also incorporated in every lesson either through specific teacher-guided problem solving sessions, such as the lesson outlined below, or through homework. The lesson for Day 2 is included in this proposal.

- Day 2: Number Theory: The Fundamental Theorem of Arithmetic; Divisibility and Prime Factorization

Connections between mathematical ideas and concepts are incorporated in every lesson either through specific teacher-guided concept sessions, such as the lessons outlined below, or through homework. The Calculus Concept Connection lesson on day 45 probably stands out as being the most unique and is included in this proposal.

- Day 11: Number Systems: Algebra Concept Connection
- Day 28: Symbolic Representation of Geometry: Algebra Concept Connection
- Day 45: Exploring Change and Representations of Change: Calculus Concept Connection[v]

The foundational processes, representation and communication, are embedded in homework and teacher-guided lessons throughout the entire curriculum. All three lesson plans included in this proposal incorporate representation concepts, communication concepts, or both.

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[i] Wanko, 2010
[ii] Wanko, 2010
[iii] Wanko, 2009
[iv] Wanko, 2009
[v] Barger and McCoy, 2010

Outcomes

The primary outcomes of this curriculum are for middle school students to master the required mathematical content and to begin transitioning to higher order thinking so they are better prepared for more rigorous mathematics in high school and college. If these outcomes are realized, this curriculum could be expanded to include the entire pre-algebra course. The following sample assessment and rubric show how these outcomes will be measured:

- Day 7: Number Theory Assessment
- Day 1: Shikaku Puzzle Rubric

This curriculum is based on the most recent research in the teaching of mathematics. The Scope and Sequence, sample lesson plans, assessment, and rubric will serve as a guide for teachers as they lay a foundation for higher level mathematical thinking while teaching content and skills.

Budget

The following budget was derived from the Resources column of the Scope and Sequence. The technology items are necessary for presenting the lessons. The SMART Board with projector will enable the students and the teacher to interact with the ideas and data and to explore concepts and solutions together. The student computers and software are necessary for the students to develop computer skills and to explore concepts visually and in a hands-on way as they enter data or draw. The graph paper will be used to teach graphing, area, and volume concepts in a hands-on way. The photocopies are for additional practice worksheets, assessments, and project rubrics that will extend the curriculum beyond what is provided in the textbook. The tools and manipulatives will allow students to engage in hands-on activities designed to help them understand the concepts and give them more practice.

	Quantity	Cost/Item	Total
Technology			
660i with SMART™ UF65 projector*	1	$3,079.00	$3,079.00
SMART™ Board cables (16' USB and VGA)	2	$20.00	$40.00
Computer for SMART™ Board	1	$800.00	$800.00
Student Computers	25	$800.00	$20,000.00
Internet Subscription (60/mo for 2 mo)	25	$120.00	$3,000.00
Microsoft Excel	25	$60.00	$1,500.00
Geometer's Sketchpad (school bundle)	1	$1,600.00	$1,600.00
TI-84 Graphing Calculators	25	$70.00	$1,750.00
Paper and Photocopies			
Photocopies (per copy)	1250	$0.15	$187.50
Graph Paper (printable sheets)	100	$0.20	$20.00
Tools and Manipulatives			
Protractors	25	$3.75	$93.75
Compasses	25	$4.00	$100.00
Rulers	25	$0.29	$7.25
Scissors	25	$3.00	$75.00
Measuring Tape	10	$7.00	$70.00
Legos Block Set	1	$15.00	$15.00
Square Nuts	25	$0.05	$1.25
Hexagonal Nuts	25	$0.05	$1.25
Clear Tape	3	$2.00	$6.00
		Total:	**$32,346.00**

*Lessons could be modified to use less expensive equipment such as a Mimio®

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