The Fundamental Theorem of Arithmetic;
Divisibility and Prime Factorization[i]

Grade Level and Content: Pre-algebra, 7th or 8th Grade Mathematics

Big Idea: Students will learn the Fundamental Theorem of Arithmetic and how to identify the prime factorization of a number. Students will reason and solve problems related to prime factorization for practice and thought development.

Objectives:

Students will be able to …

1. Correctly analyze a given problem: list known information, the information to be found, and missing information.
2. Correctly develop a plan to solve a given problem.
3. Correctly explain the Fundamental Theorem of Arithmetic.
4. Correctly identify the prime factorization of any given number.
5. Correctly identify rectangular arrangements from the prime factorization of a given number.

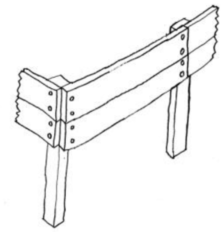
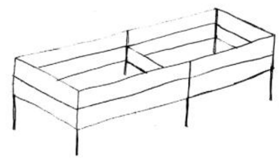
Standards: Foundational concept for standard 2.1.8.E (Grade 8 Number Theory) which states, "Apply concepts of prime and composite numbers to calculate GCFs (Greatest Common Factor) and LCNs (Least Common Multiple) of numbers."

Materials:

- 1 SMART™ Board with computer and projector
- 25 copies: Divisibility Rules Quick Reference

Technology: The SMART Board will be used to present the anticipatory set, the lesson material (prime factorization), and practice problems. The SMART Board enables the teacher and the students to work out solutions to problems together.

Anticipatory Set:

Objectives 1 and 2 3 minutes, 3 minutes	**Display** the following illustration on the SMART Board along with the question for students to think about. Write down some ideas of how you might solve this problem: Your neighbor has hired you to help him make a rectangular, raised garden bed for his vegetable garden. He has 140 plants that must be planted 8 inches apart from each other. He will pay you extra if you determine the dimensions of the garden in addition to helping with construction. The only criterion to consider is that he has to be able to access all of the plants for weeding and other care, which means the width should not exceed 4 feet. How many rectangular arrangements of 140 plants are possible? Which ones can be used for a garden bed that cannot be more than 4 feet in width? You must be able to justify your selections to your neighbor. **Note:** The plants take up minimal space; therefore, add **no** more space for them beyond the 8 inches noted above.[ii]   

Procedure:

Objectives 1 and 2 3 minutes, 6 minutes	What ideas did you come up with for solving the problem on the board? How would you begin solving this problem? ✓ List known information What do we know? **Write** "What we know" on the board with student responses underneath. ✓ We know there are 140 plants ✓ We know the plants have to be 8 inches apart ✓ We know the plants take up minimal space so we only use the 8 inches to calculate dimensions ✓ We know the garden is going to be rectangular ✓ We know the width <u>cannot be more than</u> 4 feet What would you do next?

	✓ Determine what the problem is asking for. What question does this problem pose that we have to answer? **Write** "What is the question?" on the SMART Board. ✓ How many rectangular arrangements of 140 plants are possible? ✓ Which rectangular arrangements meet the maximum width criterion of 4 feet? Do we have all of the information we need to solve the problem? ✓ Yes Let's see if we can answer the first question. How many rectangular arrangements of 140 plants are possible? **Display** on the SMART Board: How many rectangular arrangements of 140 plants are possible? Does this sound familiar? Where did you have a problem like this? ✓ Yesterday's lesson, homework, puzzles Yesterday we arranged 24 tiles into different rectangles based on the factor pairs. We also worked on Shikaku puzzles where we arranged rectangles on a grid based on the specified number of squares and a few rules. What is different about the rectangles we created yesterday and the one in this problem? ✓ The number in this problem is bigger
Objectives 3 and 4 7 minutes, 13 minutes	Today we are going to talk about finding factors of big numbers like the one in this problem. To find all of the factors of 140, we have to identify its **prime factorization**. Can anyone guess what I mean by prime factorization? What is a prime number? ✓ A number with exactly two factors, 1 and itself The term "factorization" is new to you, but factor is not. What is a factor? We just did this yesterday. ✓ A factor is a smaller part of another number. Now can you tell me what a prime factor is? ✓ A number that is prime and that is a smaller part of another number. There is a theorem that will help us understand prime factorization. **Display** on the SMART Board: The *Fundamental Theorem of Arithmetic* says that you can take any integer greater than 1 and write it as a product of prime numbers. The set of prime numbers that you find for any given number is unique to that number. Let's look at a small number first so we can get a grasp of this idea before we tackle the prime factorization of 140.

What are two factors of 24? Remember the rectangles we created with 24 tiles?

- ✓ 6 and 4
- ✓ 8 and 3
- ✓ 12 and 2
- ✓ 24 and 1

We are going to create a factor tree to help us break the number 24 into its prime parts. We don’t want to use the 1 in our tree. The number 1 is not prime. What is the requirement for a number to be prime?

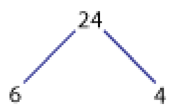
- ✓ It has to have exactly two factors

How many factors does the number 1 have?

- ✓ One factor

Let’s start with the factors 6 and 4. When we create a factor tree, we place the number at the top and two of its factors underneath it like this:

Display on the SMART Board:



Our goal is to continue the tree until all of the numbers on the lowest branches are prime numbers. Are 6 and 4 prime numbers?

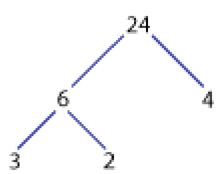
- ✓ No

How can we break six down further?

- ✓ 2 and 3

We add two branches for the 2 and the 3 like this:

Display on the SMART Board:

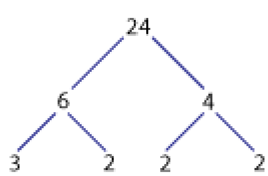


How about the 4? How can it be broken down?

- ✓ 2 and 2

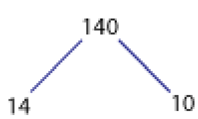
We add two more branches for the two 2s like this:

Display on the SMART Board:



Are the numbers on our lowest branches prime?

- ✓ Yes

	The prime factors of 24 are three times two times two times two. **Write** 3 * 2 * 2 * 2 on the SMART Board. Can we simplify these in any way? How do we write a number that is multiplied by itself? ✓ 2 squared Okay, how do we write three 2s? ✓ 2 cubed So our list of prime factors is "three times two cubed." **Write** $3 * 2^3$ on the SMART Board. When you write a number as a product of prime numbers, you are writing the **prime factorization** of that number. Three times two cubed is the prime factorization of 24.
Objective 4 10 minutes, 23 minutes	Now let's write the prime factorization of 140. Wow, that is a big number. This might be a good time to review divisibility rules. **Hand out** Divisibility Rules Quick Reference. This is a list of all of the divisibility rules that you have learned since fourth grade. Keep these in your notebook for quick reference. Where would you start? Can you tell me one factor of 140 using your divisibility rules instead of your calculator? ✓ Answers will vary – 10 or 5 or 2 How do you know it is divisible by 10? (**or** 5 or 2 depending on what student(s) say) ✓ A number is divisible by 10 when the digit in the ones place is 0 Great. Now we can start our tree. What number, when multiplied by 10, equals 140? ✓ 14 So if I write 140 on the board like this: **Write** 140 on the board. What do I do next? ✓ Draw two branches, one for the factor 10 and one for the factor 14 **Display** on the SMART Board:  Are we finished? Can we factor further? Let's look at the number 14. What numbers are factors of 14?

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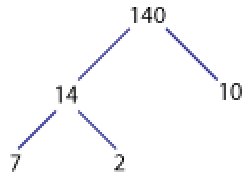
- ✓ 7 and 2

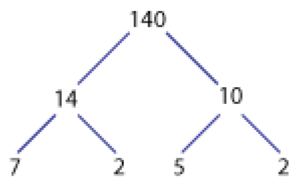
So we can write this:



Now let's consider the number 10. What numbers are factors of ten?

- ✓ 5 and 2

So we can write this:



Are we finished? Can we factor further?

What are the factors of 7? Of 5? Of 2?

- ✓ 7 and 1
- ✓ 5 and 1
- ✓ 2 and 1

How many factors does 7, 5, and 2 each have? What do we call numbers that have only two factors?

- ✓ Prime numbers

So the bottom of our tree is what?

- ✓ The prime factorization of 140

And we write this as (7)(2)(5)(2). Can we simplify this? What can we do with the 2s?

- ✓ $(7)(5)(2)^2$

Now work on some problems with your partner. You can use your divisibility rules.

Display on the SMART Board:

Identify the prime factorization for these numbers:

1. 51
2. 96
3. 123

You have 3 minutes. **Set Timer.** Go.

Walk around the room, observe, and answer questions.

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Objective 5

5 minutes,

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Now let's look at how the prime factorization can help us create different rectangular arrangements from the same number.

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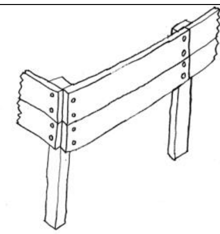
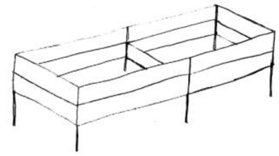
28 minutes	Take a look at the 24-tile rectangles we did yesterday. The factor pairs that served as the dimensions of each rectangle could have been derived from the prime factors. Here is the prime factorization again: **Display** on the SMART Board: Prime factorization of 24: (3)(2)(2)(2) Factor pairs of 24 (excluding 24 and 1): ✓ 6 X 4 ✓ 8 X 3 ✓ 12 X 2 How can you arrange the prime factor to get the 6 X 4 factor pair for a rectangle? There is a property of multiplication that will help us know what to do. Do you remember the Commutative Property of Multiplication? ✓ When we multiply numbers, we can multiply in any order. So to get 6 times 4 we can multiply 3 times 2 to get 6 and then 2 times 2 to get 4? **Write** the equations on the SMART Board. ✓ (3 x 2)=6 and (2 x 2)=4 and (6 x 4)=24 "*Student*," can you tell me how we can derive the factor pair 8 times 3 from the prime factors? ✓ (2 x 2 x 2)=8 and (8 x 3)=24 Can we find a general rule for what we just did? ✓ If we multiply the prime factors together to form two numbers, we have two factors or one set of factor pairs that can be the dimensions of a rectangle For the prime factors of 140, what are the possible rectangles? Here is the prime factorization again: **Write** the prime factorization of 140 on the SMART Board: (5)(7)(2)(2) **Write** rectangular dimensions of 140 on the SMART Board: ✓ 35 X 4 (5 x 7)=35 and (2 x 2)=4 ✓ 14 X 10 (5 x 2)=10 and (7 x 2)=14 ✓ 20 X 7 (5 x 2 x 2)=20 and (7) ✓ 28 X 5 (7 x 2 x 2)=28 and (5) ✓ 70 x 2 (7 x 5 x 2)=70 and (2)
Objectives 1 and 2 9 minutes, 37 minutes	Let's look at our problem again: **Display** problem on the SMART Board.

Your neighbor has hired you to help him make a rectangular, raised garden bed for his vegetable garden. He has 140 plants that must be planted 8 inches apart from each other. He will pay you extra if you determine the dimensions of the garden in addition to helping with construction. The only criterion to consider is that he has to be able to access all of the plants for weeding and other care, which means the width should not exceed 4 feet. How many rectangular arrangements of 140 plants are possible? Which ones can be used for a garden bed that cannot be more than 4 feet in width? You must be able to justify your selections to your neighbor.

Note: The plants take up minimal space; therefore, add **no** more space for them beyond the 8 inches noted above.





Before we go any further, let's look again at what the problem is asking.

Display the SMART Board screen on which "What is the question?" is written from earlier in the lesson.

- ✓ How many rectangular arrangements of 140 plants are possible?
- ✓ Which rectangular arrangement(s) meet the maximum width criterion of 4 feet?

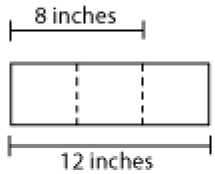
We found the rectangular arrangements, but we have not determined which ones meet the 4 foot maximum width requirement.

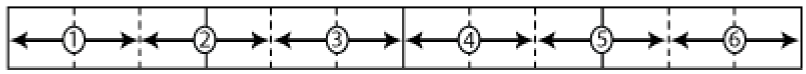
Does anyone have an idea of where to begin solving this problem?

Focusing-interaction Pattern[iii] – Listen to student responses and derive an answer by guiding student thinking and communication (not by demonstrating a pre-established solution). Possible solutions with questions for anticipated student responses follow:

Possible solutions:

Draw a model: Draw 4 boxes where each represents one foot, divide each 'foot' into thirds so that each third represents 4 inches, color 2 boxes to represent 8 inches, continue to color two boxes at a time until there are no boxes or only one box left. Factor pairs with at least one factor less than or equal to the maximum number of plants will work as the dimensions of the bed. For each qualifying factor pair, calculate the dimensions by multiplying each factor by 8 inches.





Rectangular arrangement must have one factor that is less than or equal to 6 plants. Possible arrangements of plants include 35 x 4, 28 x 5, or 70 x 2.

Possible dimensions include:

(35 x 4) requires a 280 X 32 inch bed
(28 x 5) requires a 224 X 40 inch bed
(70 x 2) requires a 560 X 16 inch bed

Calculation A: Calculate the maximum number of plants that will fit along the width of a 4 foot bed and compare that number to the factor pairs. Factor pairs with at least one factor less than or equal to the maximum number of plants will work as the dimensions of the bed. For each qualifying factor pair, calculate the dimensions by multiplying each factor by 8 inches.

Width is (4 feet x 12 inches) or 48 inches with 8 inches per plant
48/8=6
Six plants will fit along the width of a 4 foot bed.
Rectangular arrangement must have one factor that is less than or equal to 6 plants. Possible arrangements of plants include 35 x 4, 28 x 5, or 70 x 2.

Possible dimensions include:

(35 x 4) requires a 280 X 32 inch bed
(28 x 5) requires a 224 X 40 inch bed
(70 x 2) requires a 560 X 16 inch bed

Calculation B: For each factor pair, calculate the dimensions that would be needed to arrange plants in that way by multiplying each factor by 8 inches. Dimensions with at least one number that does not exceed 48 inches will work as the dimensions for the bed.

(35 x 4) requires a 280 X 32 inch bed
(14 x 10) requires a 112 X 80 inch bed
(20 x 7) requires a 200 X 56 inch bed
(28 x 5) requires a 224 X 40 inch bed
(70 x 2) requires a 560 X 16 inch bed

Possible dimensions include:

(35 x 4) requires a 280 X 32 inch bed
(28 x 5) requires a 224 X 40 inch bed
(70 x 2) requires a 560 X 16 inch bed

	Questions for anticipated[iv] problems/responses:

Question	Reasoning
Explain to me where you are going with this train of thought.	You are unsure of where they are going or you are unsure if they know where they are going.
Can you tell me how you got that answer?	Students skipped one or more steps or called out an answer that needs an explanation.
What do the boxes represent (modeling)?	Students don't have a solid grasp of how to use modeling and their models are confusing or confusing them.
How many inches in a foot?	Students forget to convert to inches.
Eight inches is what fraction of 12 inches?	Students struggle with blocking off their 12 inch models to represent the 8 inches required for each plant.
What is the unit of measurement?	Students forget to convert from feet to inches or vice versa.
The factor pairs of 140 indicate the arrangement of plants within the bed. What do we have to do to get the dimensions of each rectangular arrangement?	Students forget to multiply the factor pairs by 8 inches to get the dimensions.

Closing:

1 minute, 39 minutes	**Homework:** Application, problem solving, and reasoning problems in the book

[i] Content idea from Foresman and Wesley, 1999
[ii] Drawing from Happyfarming.com, 2010; Photo from Raised-garden-beds.com, 2010
[iii] Focusing-Interaction Pattern concept from Herbel-Eisenmann and Breyfogle, 2005
[iv] Anticipating responses concept from Wallace, 2007

Divisibility Rules Quick Reference

Number	Rule	Example
2	The digit in the ones place is 2, 4, 6, 8, or 0.	13,47**8** – yes, because the number ends with the digit 8 13,478/2 = 6,739
3	The sum of the number's digits is divisible by 3.	14,802 – yes, because 1 + 4 + 8 + 0 + 2 = 15 which is divisible by 3 14,802/3 = 4,934
4	As a number, the last two digits are divisible by 4.	20,0**32** – yes, because 32 is divisible by 4 20, 032/4 = 5,008
5	The number ends in a 5 or 0.	33,67**0** – yes, because the number ends in a 0. 33,670/5 = 6,734 30,67**5** – yes, because the number ends in a 5. 30,675/5 = 6,135
6	The number is divisible by both 2 and 3.	5,93**4** – yes, because the number ends with the digit 4 which is divisible by 2 and because 5 + 9 + 3 + 4 = 21 which is divisible by 3
7	When you double the last digit and subtract it from the rest of the number, your answer is 0 or divisible by 7. You can repeat this process with each resulting number if it is also too large to determine.	3,941 – yes, because 1 doubled is 2 and 394 – 2 = 392; 2 doubled is 4 and 39 – 4 is 35 which is divisible by 7 3,941/7 = 563
8	As a number, the last three digits are divisible by 8.	44,120 – yes, because 120 is divisible by 8; 120/8 = 15 44,120/8 = 5,515

Number	Rule	Example
9	The sum of the number's digits is divisible by 9.	45, 279 – yes, because 4 + 5 + 2 + 7 + 9 = 27 which is divisible by 9 45,279/9 = 5,031
10	The digit in the ones place is 0.	67,67**0** – yes, because the number ends in a 0. 67,670/10 = 6,767
11	When you sum every second digit and subtract all other digits, your answer is 0 or divisible by 11.	2**9**1,**8**1**9** – yes, because 9 + 8 + 9 = 26 and 26 – 2 – 1 – 1 = 22 which is divisible by 11 291,819/11 = 26,529
12	The number is divisible by both 3 and 4.	7,9**32** – yes, because 7 + 9 + 3 + 2 = 21 which is divisible by 3 and the last two digits as a number, 32, is divisible by 4 7,932/12 = 661

References

Foresman, Scott and Wesley, Addison (1999). *Middle School Math Course 3, Volume 2, Teacher's Edition*, Addison Wesley Longman, Inc. Chapters 7-9, p. T27-503

Happyfarming.com (2010) Garden Bed Drawing. Retrieved from http://happyfarming.com/images/raised_bed_plans.jpg

Herbel-Eisenmann, Beth A. and Breyfogle, M. Lynn (May 2005). Questioning Our Patterns of Questioning. *Mathematics Teaching in the Middle School* v. 10 no. 9 p. 484-489.

Raised-garden-beds.com (2010). Garden Bed Photo. Retrieved from http://qwickstep.com/search/raised-garden-bed-plans.html

Wallace, Ann (2007, May). Anticipating Student Responses to Improve Problem Solving. *Mathematics Teaching in the Middle School* v. 12 no. 9 p. 504-511.