Exploring Change and Representations of Change:
Calculus Concept Connection[i]

Grade Level and Content: Pre-algebra, 7th or 8th Grade Mathematics

Big Idea: Students explore the concept of change and how change is represented in words, tables, and graphs. Students discover what size square to cut from the corners of a piece of paper to maximize the volume of the resulting box (when sides are folded up). Students create a graph of all of the student "boxes" with respect to the size of the cut square and the volume and interpret what the graph means. Students learn that Calculus can be used to find a more precise answer more quickly. Students practice interpreting other graphs in terms of the change represented.

Objectives:

Students will be able to…

1. Correctly analyze a given problem listing known information, the question asked for, and missing information.
2. Adequately develop a plan to solve a given problem.
3. Correctly construct an expression using variables and numeric values to represent the volume of a box given the dimensions (in variables and numeric values) of the cardboard to be transformed into a box and instructions for the transformation.
4. Correctly calculate the volume of a box given the dimensions including possible values for the variable.
5. Correctly construct a table and graph that represent the change in volume with respect to x-values given a Volume Table worksheet and graph paper.
6. Correctly demonstrate that they can explain and interpret results of given graphs.

Standards:

Meets standard 2.5.8.A (Grade 8 Problem Solving) which states, "Develop a plan to analyze a problem, identify the information needed to solve the problem, carry out the plan, apply estimation skills as appropriate, check whether the plan makes sense, and explain how the problem was solved in grade appropriate contexts."

Meets standard 2.8.8.E (Grade 8 Modeling) which states, "Use combinations of symbols and numbers to create expressions, equations in one or two variables, and inequalities in one variable that model problem situations."

Meets standard 2.3.8.C (Grade 8 Calculations) which states, "Calculate volume, surface area, and degrees of angles; calculate circumference and areas of circles; and use a measurement formula to solve for a missing quantity."

Meets standard 2.8.8.D (Grade 8 Functions) which states, "Create a table or graph from a functional rule."

Meets standard 2.5.8.B (Grade 8 Communication) which states, "Use precise mathematical language, notation, and representations, including numerical tables and equations, simple algebraic equations and formulas, charts, graphs, and diagrams to explain and interpret results."

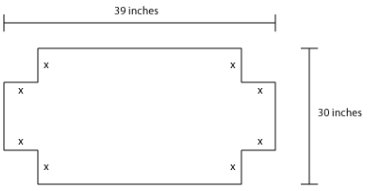
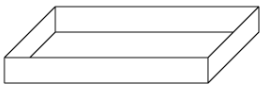
Meets standard 2.8.8.F (Grade 8 Interpret Results of Modeling) which states, "Interpret the results of solving equations in one or two variables and inequalities in one variable in the context of the situation that motivated the model.

Materials:

- 1 SMART™ Board with computer and projector
- 25 calculators
- 25 scissors
- Tape
- 25 sheets of graph paper
- 25 copies: Volume Table worksheet
- 25 copies: Graphs to Stories worksheet[ii]

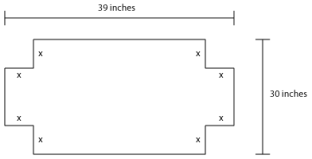
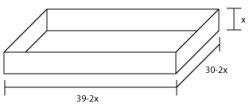
Technology: The SMART Board will be used to present the anticipatory set and the expressions, tables, and graphs that result from the discussion and calculations.

Anticipatory Set:

Objectives 1 and 2 3 minutes, 3 minutes	**Display** the following problem on the SMART Board for students to think about. Write down some ideas of how you might solve this problem: You have an enormous collection of baseball cards that you want to store in a box. Your mom does not keep any boxes around the house so you must create your own box to hold your baseball cards. You find a piece of cardboard that is 39 inches long and 30 inches wide. If you cut the same size square out of each corner, you can fold up the sides to make a box. For your 39″x30″ piece of cardboard, what size square should you cut to yield the largest volume so you can fit as many baseball cards as possible into the one box?  

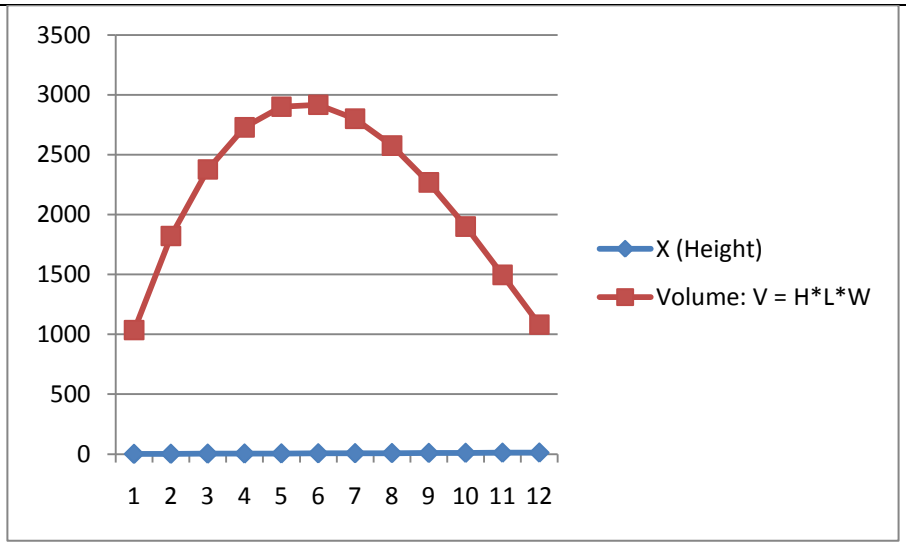
Procedure:

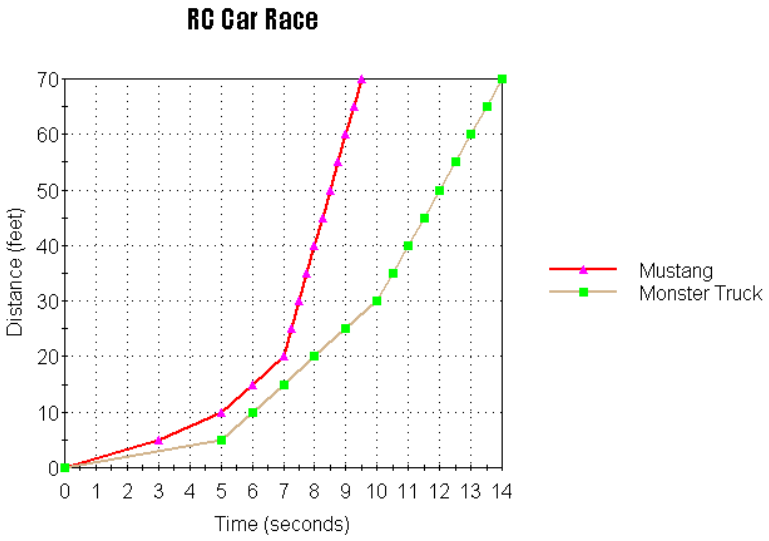
Objectives 1, 2, and 3 7 minutes, 10 minutes	Today you are going to use your knowledge of variables, volume, tables, and graphs to find the maximum volume that would result from cutting squares out of the corners of a piece of cardboard and folding up the sides to form a box. In this lesson you will get a sneak preview of some concepts from Calculus. Let's get started. First, what ideas did you come up with for solving the problem on the board?

	How would you begin solving this problem? ✓ List known information What do we know? **Write** "What we know" on the board. ✓ We know the dimensions of the cardboard. ✓ 39 x 30 inches **Write** 39 x 30 inches under "What we know." What would you do next? ✓ Determine what the problem is asking for. What question does this problem pose that we have to answer? **Write** "What is the question?" on the board. ✓ If we cut squares from each corner to make a box, what size square would give us the box with the largest volume? What next? Do we know if we have all of our information? If the problem is talking about volume, what do you think we could do to help us visualize the problem better? ✓ Formula for calculating volume What is the formula for the volume of a rectangular prism? ✓ V = length x width x height From our picture, can we write an equation that expresses the volume? What is the length if we cut a square from each corner? ✓ L=(39-2x) What is the width if we cut a square from each corner? ✓ W=(30-2x) What is the height? ✓ H=(x) **Display** illustration with formulas. *Problem:* You have an enormous collection of baseball cards that you want to store in a box. Your mom does not keep any boxes around the house so you must create your own box to hold your baseball cards. You find a piece of cardboard that is 39 inches long and 30 inches wide. If you cut the same size square out of each corner, you can fold up the sides to make a box.  

	For your 39″x30″ piece of cardboard, what size square should you cut to yield the largest volume so you can fit as many baseball cards as possible (and hopefully all) into the one box? If volume is length times width times height, what is the volume equation for this problem? ✓ V=(39-2x)(30-2x)(x) **Write** V=(39-2x)(30-2x)(x) on the board. How many unknowns do we have? How many equations do we have? ✓ Two unknowns and one equation Do we have all of the information we need to solve this problem? ✓ No What else do we need to know? ✓ We need to know what the maximum volume is so we can determine the value of x for that volume The problem does not give us the maximum volume. What could we do to figure out that information? ✓ Calculate the volume based on different values for x and see what we come up with
Objectives 4 and 5 15 minutes, 25 minutes	Let's create some 'dummy' boxes and compare them before we calculate the volume. Write down your assigned x-value as soon as I give it to you. Your group only has to calculate one volume so you want to make sure you calculate the right one. **Hand out** Graph paper (1 sheet per student), tape, and scissors. **Assign** x-values to groups. **Demonstrate** cutting out squares from each corner based on one x-value. 1. Notice that each tile on your graph paper represents one square inch. You have 30 tiles across and 39 tiles down just like the cardboard in the problem. You also have some white space around your grid. 1. First, cut off the white space. 2. Second, pick a corner and count x number of squares in from the corner. For example, if I have x=3, I count 1, 2, 3 squares. Then cut vertically into the paper along the grid line at that point counting squares as you go. Stop when you have cut past x number of squares. Finally, turn your paper and cut horizontally along the grid line until your square is completely cut from your corner. 3. Repeat step 2 for each of the remaining 3 corners.

	4. Fold up your sides and tape 5. Write your x-value on your box like this (**Demonstrate**) **Select** one student from each group to line up in front of the class with the box. Can you tell which one might have the most volume just by looking at them? Let's pick one or two and check to see if we guessed correctly. **Write** guesses on white board. Now I want you to work with your partners to calculate the volume for your assigned value of x. **Hand out** Volume Table worksheet and calculators. This worksheet has a table for you to plug in your values. When you are finished calculating your volume, fill in the appropriate row. You will share your answers with the class so everyone can fill in the entire table. You can calculate your volume in one of two ways. Does anyone know what those are? ✓ By counting the squares along each dimension and using the formula ✓ By plugging the x-value into the formula we derived You have two minutes. (**Set timer**) Go. **Display** Volume Table worksheet. Let's start with the group who calculated using x=1. What is your length, width, and volume? Now x=2,…, x=12. **Write** data in the table displayed on the SMART Board. Can anyone see from the table which x-value gives us the maximum volume? ✓ 6 Look at what we guessed. Did we guess correctly? **Display** a coordinate grid on the SMART Board in increments that are appropriate for the x-values (along x-axis) and volumes (along y-axis). Let's graph these values. If you haven't been up front, please come up now with your table. Someone from each group will plot their volume with respect to their x-value. Line up so we can move through this quickly Group "x=1," plot your point. **Repeat** for each group until all values are plotted. Now if I connect the dots, you can see how the volume changes with respect to the x-values. **Connect** the dots to reveal the graph.

	
Objective 6 5 minutes, 30 minutes	Let's look more closely at our graph. Take a minute to discuss with your partner what is happening in this graph. Consider these questions while you chat. **Display** and read the following on the SMART Board. How quickly or slowly do the values change with respect to the x-values? Does the volume increase more quickly before the maximum volume is reached or after it is reached? You have two minutes. (**Set timer**) Go. Okay, what are some interpretations you came up with? ✓ The volume changes more quickly before the maximum is reached. The rate of change slows after the graph passes the maximum. Have any of you heard of Calculus? Calculus allows us to study change. This problem is a popular Calculus problem because the volume changes with respect to the x-value. You should feel good that you solved a Calculus problem. But let's consider what our answer would be if we used Calculus. I can calculate the x-value that yields the maximum volume. **Connect** calculator to the SMART Board and **calculate**: plug in $V=4x^3 - 138x^2 + 1170x$. Solve for x in equation $12x^2 – 276x + 1170 = 0$ (derivative). **Display** the graph on the SMART Board. My answer is 5.6, and here is the graph. Our answer was a bit off from this one. What do you think accounts for the discrepancy between the answers we came up with manually and the answer I calculated using Calculus and my calculator? ✓ We didn't plug in enough values for x. We used only integers. It took a long time for us to reach our conclusion and we were still not super accurate. That should not discourage you, however. Half of the task in solving a problem like this involves understanding the concept behind it.

	I hope this gets you at least a little excited about taking Calculus someday. Calculus gave us our x-value more quickly and precisely. You do not need to know the details of the calculation to appreciate how quickly and accurately I was able to find the answer using Calculus. You also do not need to know the details to understand the concept behind them. Congratulations!
8 minutes, 38 minutes	Let's look at another graph. **Hand out** Graphs to Stories worksheet. **Display** car race graph on the SMART Board.  Look at the RC Car Race graph. This is a race between a mustang and a monster truck. What is the first question that comes to mind? ✓ Who won the race? Look at the distance. **Point** to graph. The race ended at 70 feet from the starting point. How long did it take the mustang to reach 70 feet? ✓ About 9 ½ seconds How long did it take the truck? ✓ 14 seconds Who won the race? ✓ The Mustang won the race Can you tell me what is happening at these two points, (5,10) and (7,20), when the direction of the line changes? **Point** to (5,10) and (7,20) graph? ✓ The mustang is increasing speed or accelerating Does the mustang accelerate more or less quickly than the truck? ✓ More quickly How did you determine that the mustang is faster at changing speed?

- ✓ The line representing the mustang is steeper than the line representing the truck

Okay, now let's look at some hard data. Your problem asks you to calculate the speed of the monster truck and mustang at 4, 5, 6, 7, 8, and 9 seconds into the race. In the interest of time, I have completed a table for you. Take notes on this because there is another graph that asks for similar calculations.

Display the table and reveal gradually to follow discussion.

Mustang			Monster Truck		
Time (sec)	Distance (feet)	Speed (ft/sec)	Time (sec)	Distance (feet)	Speed (ft/sec)
4	8	2	4	3	0.75
5	5	1	5	5	1
6	15	2.5	6	10	1.66667
7	20	2.85714	7	15	2.14286
8	20	2.5	8	20	2.5
9	60	6.66667	9	25	2.77778
Total Distance… 52 feet In… 5 seconds With an acceleration of… 4.666667 feet/second In… 5 seconds			Total Distance… 22 feet In… 5 seconds With an acceleration of… 2.027778 feet/second In… 5 seconds		

This table shows the speed at the 6 points in question. Do you see how I derived the speed? I used the formula "speed equals distance divided by time."

These 6 points represent 5 seconds of the race. How far did each car travel in those 5 seconds? How do you calculate that?

- ✓ Subtract the distance at 4 seconds from the distance at 9 seconds

Reveal the "Total distance…in…" portion of the table.

How fast did each car accelerate in those 5 seconds? How do you calculate that?

- ✓ Subtract the speed at 4 seconds from the speed at 9 seconds

Reveal the "With an acceleration of…in…" portion of the table.

This graph is another example of a Calculus problem. With Calculus, I could easily calculate the acceleration or deceleration at any given point along either path.

Closing:

1 minute, 39 minutes	**Homework:** Write your interpretation of the other three graphs. Read the problem for questions to ask when looking at the graph and for ideas to consider when interpreting the graph.

[i] Idea from Barger and McCoy, 2010
[ii] Idea from Barger and McCoy, 2010

Name_____

Volume Table Worksheet

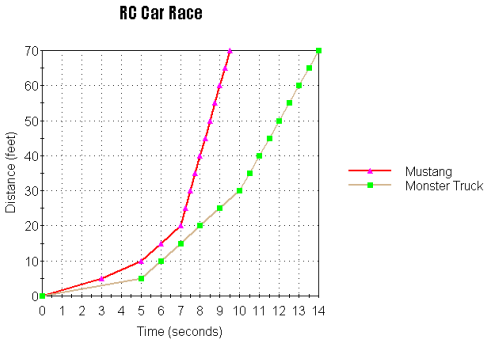
Height (x)	Length (39-2x)	Width (30-2x)	Volume H*L*W
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

Scratch Work:

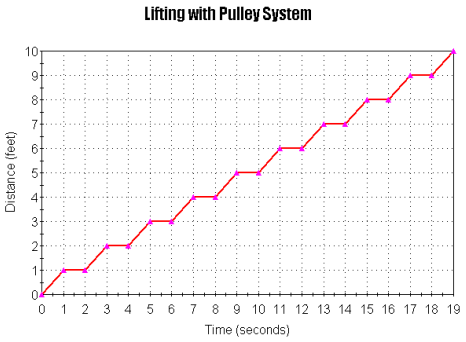
Name_____

Graphs to Stories[i]

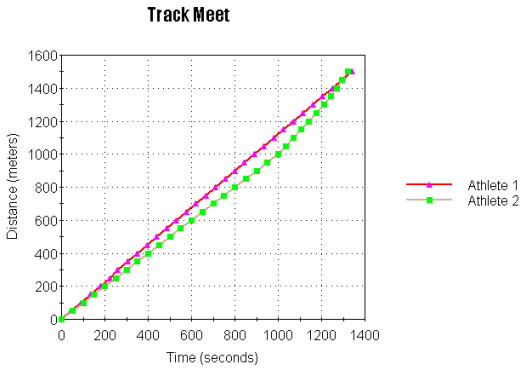
The RC Car Race graph shows the distance and time ratio of two radio controlled cars during a race. Describe what is happening with each car? Do they change speed? Calculate the speed of the Monster Truck and Mustang at 4, 5, 6, 7, 8, and 9 seconds into the race. (s = d/t) Compare which one changes speed more quickly. What is the total increase in speed for each vehicle during those 6 seconds? What is change of speed? What words do we use to describe change in speed?



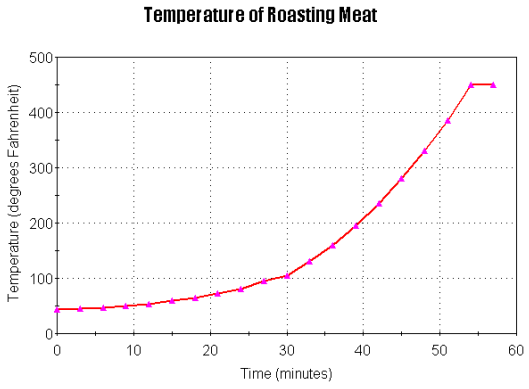
Your father asks you to help him lift some things into the loft above your garage. He developed a pulley system because it was easier to lift things by pulley than to carry them by hand on a ladder. You load the bucket with tools and watch your father pull them up. Analyze the graph and describe what is happening as your father lifts the bucket. Does he lift at a constant speed? Is there any acceleration or deceleration? What do you think the peaks mean? What do the horizontal lines mean? The diagonal lines?



Two of your friends are athletes on the track team. The Track Meet graph shows how each one did on the most recent 1500 meter middle distance track event. Describe what happened during the race. Calculate each athlete's speed at 400 meters, 1000 meters, and 1500 meters. (s = d/t) For each distance, estimate the time as best as you can from the graph. Describe the speed of each athlete in terms of change. Did athlete 1run at a constant speed or did the speed change? How about athlete 2?



Your mother needs your help with dinner. She asks you to watch her roast and to tell her when its internal temperature reaches 180 degrees Fahrenheit. You watch intently for 20 minutes and decide to watch TV for a while because the roast's temperature is increasing so slowly. You are sure that you can watch your favorite 30-minute TV show before the roast reaches 180 degrees. Based on the graph, is this an accurate assumption? Did the temperature of the roast increase at a constant rate or did it increase more quickly after you stopped watching it? Will your mother be proud of you? Will you eat the roast?



[i]Graphs created using Kids' Zone: Create a Graph NCES, 2010

References

Barger, Rita H. and McCoy, Ann C. (2010, February). Calculus in the Middle School.

Mathematics Teaching in the Middle School v. 15 no. 4 p. 348-353.