Additional document

Application to a simple one-dimensional problem

A bar with a linear elastic behavior subjected to a tensile force F_d is considered (**Figure 1**). The bar has a length L, a section S and a Young's modulus E.

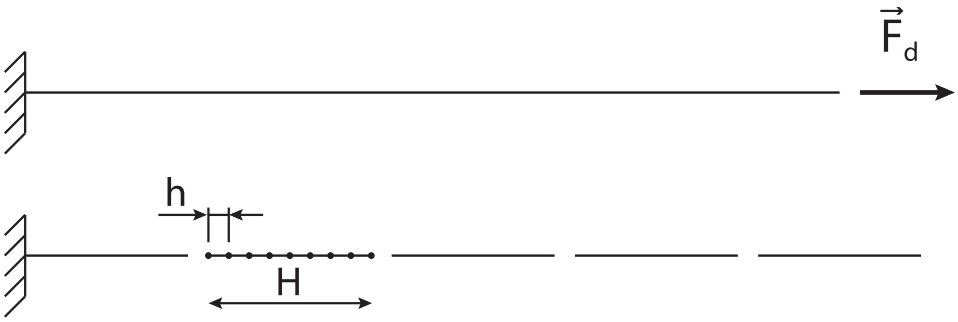

Figure 1 – Bar in tensile and decomposition into subdomains

In order to illustrate the different discretized operators, one considers the situation where the structure is partitioned into only three subdomains (**Figure 2**). Each subdomain is discretized into two linear finite elements.

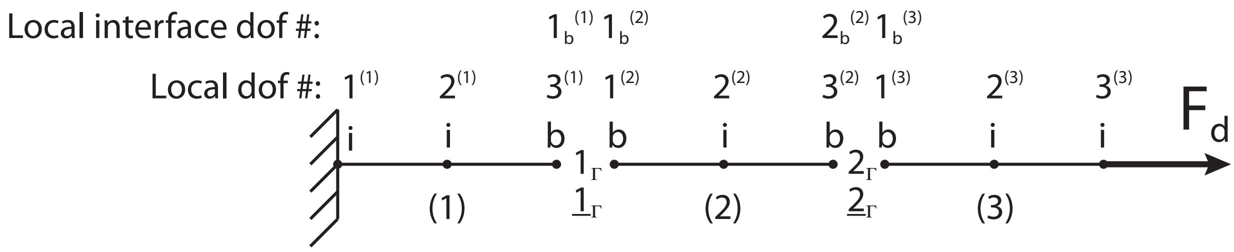

Figure 2 – Bar in tensile decomposed into three subdomains

The definition of internal (i) and boundary (b) nodes for each subdomain is also given in **Figure 2**. Stiffness matrix for a given subdomain simply reads :

$$\forall\, s \in \mathbf{S}, \quad K^{(s)} \;=\; k_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \text{ where } \quad k_0 = \frac{ES}{h} = \frac{6ES}{L} \tag{1}$$

where h refer to the length of a finite element.

1 Operators for subdomain (1)

Boundary conditions (prescribed displacement and force) are considered directly at the level of the concerned subdomains. The prescribed displacement can be accounting for by elimination or penalty method for instance. In the present case, a displacement $u_d = 0$ is prescribed at node $1^{(1)}$ of subdomain (1). In this case, one obtains :

$$\begin{pmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} u_1^{(1)} = u_d \\ u_2^{(1)} \\ u_3^{(1)} \end{pmatrix} = \begin{pmatrix} f_1^{(1)} \\ f_2^{(1)} \\ f_3^{(1)} \end{pmatrix}$$

The second and third lines of this system gives :

$$\begin{pmatrix} K_{22}^{(1)} & K_{23}^{(1)} \\ K_{32}^{(1)} & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} u_2^{(1)} \\ u_3^{(1)} \end{pmatrix} = \begin{pmatrix} f_2^{(1)} - K_{21}^{(1)}\, u_d \\ f_3^{(1)} - K_{31}^{(1)}\, u_d \end{pmatrix} = \begin{pmatrix} f_2^{(1)} \\ f_3^{(1)} \end{pmatrix} \tag{2}$$

Since $u_d = 0$ in the present case. By the penalty method, one obtains :

$$\begin{pmatrix} K_{11}^{(1)} + g & K_{12}^{(1)} & K_{13}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{pmatrix} = \begin{pmatrix} f_1^{(1)} + g\, u_d \\ f_2^{(1)} \\ f_3^{(1)} \end{pmatrix} \tag{3}$$

where g is the penalty coefficient g, which value should be chosen high in order to verify accurately the boundary condition $u_1^{(1)} = u_d$. It is clear that this approach takes into account the boundary condition in an approximate way. One can also note that a high value for the penalty coefficient can lead to ill-conditioning issue.

Starting form either (2) or (3), the resulting primal Schur complement for subdomain (1) (condensation on the interface boundary degree of freedom $1_b^{(1)} (= 3^{(1)})$ is respectively given by :

$$\begin{cases} S_p^{(1)} &= K_{33}^{(1)} - K_{32}^{(1)} K_{22}^{(1)^{-1}} K_{23}^{(1)} = \dfrac{k_0}{2} \\ S_p^{(1)} &= K_{33}^{(1)} - \begin{pmatrix} K_{31}^{(1)} & K_{32}^{(1)} \end{pmatrix} \begin{pmatrix} K_{11}^{(1)} + g & K_{12}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} \end{pmatrix}^{-1} \begin{pmatrix} K_{13}^{(1)} \\ K_{23}^{(1)} \end{pmatrix} \\ &= k_0 - \begin{pmatrix} 0 & -k_0 \end{pmatrix} \begin{pmatrix} k_0 + g & -k_0 \\ -k_0 & 2k_0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -k_0 \end{pmatrix} = \dfrac{g\, k_0}{2g + k_0} \end{cases}$$

It can be seen that when g tends to a high value, one obtains the same primal Schur complement $S_p^{(1)} = \dfrac{k_0}{2}$ for the two techniques, which corresponds to the equivalent stiffness of two linear finite element with stiffness k_0 in serial as expected. The dual Schur complement are consequently ($S_p^{(1)}$ is invertible here), for the two cases :

$$S_d^{(1)} = S_p^{(1)+} = S_p^{(1)-1} = \frac{2}{k_0} \quad \text{or} \quad \frac{2g + k_0}{g\, k_0}$$

There are consequently no rigid body modes for subdomain (1) : $R_b^{(1)} = \emptyset$.

Since there are no external forces on subdomain (1), the primal right-hand side for both cases is given by :

$$b_p^{(1)} = 0 \quad \text{or} \quad \frac{g\, k_0\, u_d}{2g + k_0} = 0 \quad \text{since} \quad u_d = 0$$

2 Operators for subdomain (2)

Subdomain (2) is a floating subdomain. One easily obtains the primal Schur complement :

$$S_p^{(2)} = \begin{pmatrix} K_{11}^{(2)} & K_{13}^{(2)} \\ K_{31}^{(2)} & K_{33}^{(2)} \end{pmatrix} - \begin{pmatrix} K_{12}^{(2)} \\ K_{32}^{(2)} \end{pmatrix} K_{22}^{(2)-1} \begin{pmatrix} K_{21}^{(2)} & K_{23}^{(2)} \end{pmatrix} = \frac{k_0}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The dual Schur complement (which is not unique) is given by :

$$S_d^{(2)} = S_p^{(2)+} = \frac{2}{k_0} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or (for instance)} \quad \frac{1}{2k_0} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

There is one rigid body modes :

$$R_b^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The primal right-hand side is reduced to : $b_p^{(2)} = (0, 0)^T$ (no external forces).

3 Operators for subdomain (3)

A force F_d is prescribed to the node $3^{(3)}$ of the subdomain (3). This force is taken into account at the subdomain level through the primal right-hand side :

$$b_p^{(3)} = f_1^{(3)} - \begin{pmatrix} K_{12}^{(3)} & K_{13}^{(3)} \end{pmatrix} \begin{pmatrix} K_{22}^{(3)} & K_{23}^{(3)} \\ K_{32}^{(3)} & K_{33}^{(3)} \end{pmatrix}^{-1} \begin{pmatrix} f_2^{(3)} \\ f_3^{(3)} = F_d \end{pmatrix} = F_d$$

since F_d is the only force applied on the subdomain (3).

The primal Schur complement is given by :

$$\begin{aligned} S_p^{(3)} &= K_{11}^{(3)} - \begin{pmatrix} K_{12}^{(3)} & K_{13}^{(3)} \end{pmatrix} \begin{pmatrix} K_{22}^{(3)} & K_{23}^{(3)} \\ K_{32}^{(3)} & K_{33}^{(3)} \end{pmatrix}^{-1} \begin{pmatrix} K_{21}^{(3)} \\ K_{31}^{(3)} \end{pmatrix} \\ &= k_0 - \begin{pmatrix} -k_0 & 0 \end{pmatrix} \begin{pmatrix} 2k_0 & -k_0 \\ -k_0 & k_0 \end{pmatrix}^{-1} \begin{pmatrix} -k_0 \\ 0 \end{pmatrix} = 0 \end{aligned}$$

The kernel of $S_p^{(3)}$ contains only one rigid body mode : $R_b^{(3)} = 1$. Since, one has :

$$\left\{ \begin{aligned} u_b^{(3)} &= S_p^{(3)^+} (b_p^{(3)} + \lambda_b^{(3)}) + R_b^{(3)} \alpha_b^{(3)} \\ R_b^{(3)^T} (b_p^{(3)} + \lambda_b^{(3)}) &= 0 \end{aligned} \right.$$

The second line of this system with $R_b^{(3)} = 1$ leads to $b_p^{(3)} + \lambda_b^{(3)} = 0$ and, consequently, $\lambda_b^{(3)} = -b_p^{(3)} = -F_d$ so that subdomain (3) is balanced, which is expected for this isostatic one-dimensional bar problem. In this case, the first line leads to :

$$u_b^{(3)} = R_b^{(3)} \alpha_b^{(3)}$$

One deduces that the dual Schur complement is $S_d^{(3)} = S_p^{(3)^+} = 0$.

4 Assembly operators and concatenated operators

The primal assembly operators for each substructure are given by :

$$A^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

where the number of columns of $A^{(s)}$ corresponds to the number of interface/boundary degree of freedom for subdomain (s) and the number of lines is equal to the total number of (primal) relations for the interface skeleton. Here, one has only two of them : 1_Γ and 2_Γ (see **Figure 2**). The concatenated primal assembly operator is :

$$\mathbb{A}^{\diamondsuit} = \begin{pmatrix} A^{(1)} & A^{(2)} & A^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The dual assembly operators for each substructure are given by :

$$\underline{A}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \underline{A}^{(2)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \underline{A}^{(3)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{5}$$

where the number of columns of $\underline{A}^{(s)}$ corresponds to the number of interface/boundary degree of freedom for subdomain (s) and the number of lines is equal to the total number of (dual) relations for the interface skeleton. Here, one has only two of them : $\underline{1}_\Gamma$ and $\underline{2}_\Gamma$ (see **Figure 2**). Note that here, for this simple one-dimensional problem, the numbers of primal and dual relations are equal. The concatenated dual assembly operator is :

$$\underline{\mathbb{A}}^{\diamondsuit} = \begin{pmatrix} \underline{A}^{(1)} & \underline{A}^{(2)} & \underline{A}^{(N)} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Note that one easily verifies that :

$$\underline{\mathbb{A}}^{\Diamond}\, {\mathbb{A}^{\Diamond}}^{T} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{A}^{\Diamond}\, {\mathbb{A}^{\Diamond}}^{T} = diag(multiplicity) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \tag{6}$$

The concatenated primal and dual Schur complements are defined by :

$$S_p^{\Diamond} = \begin{pmatrix} S_p^{(1)} & 0 & 0 \\ 0 & S_p^{(2)} & 0 \\ 0 & 0 & S_p^{(3)} \end{pmatrix} = \frac{k_0}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \frac{k_0}{2} \begin{pmatrix} \frac{2g}{2g+k_0} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and :

$$S_d^{\Diamond} = \begin{pmatrix} S_d^{(1)} & 0 & 0 \\ 0 & S_d^{(2)} & 0 \\ 0 & 0 & S_d^{(3)} \end{pmatrix} = \frac{2}{k_0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \frac{2}{k_0} \begin{pmatrix} \frac{2g+k_0}{2g} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The concatenated primal and dual right-hand sides are :

$$b_p^{\Diamond} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ F_d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{g\, k_0\, u_d}{2g+k_0} \\ 0 \\ 0 \\ F_d \end{pmatrix}$$

and :

$$b_d^{\Diamond} = S_d^{\Diamond}\, b_p^{\Diamond} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} u_d \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The matrix of concatenated rigid body modes reads :

$$R_b^{\Diamond} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where the number of columns corresponds to the total number of rigid body modes for all the subdomains (no rigid body modes for subdomain (1)) and the number of lines corresponds to the total number of interface degrees of freedom (b) for all the subdomains.

5 Solution for the primal interface problem

The (assembled) quantities for the primal interface problem are :

$$\mathbf{S}_p = \mathbb{A}^{\Diamond}\, S_p^{\Diamond}\, {\mathbb{A}^{\Diamond}}^{T} = \frac{k_0}{2} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{or} \quad \frac{k_0}{2} \begin{pmatrix} 1 + \frac{2g}{2g+k_0} & -1 \\ -1 & 1 \end{pmatrix}$$

$$\mathbf{b}_p = \mathbb{A}^{\Diamond}\, b_p^{\Diamond} = \begin{pmatrix} 0 \\ F_d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{g\, k_0\, u_d}{2g+k_0} \\ F_d \end{pmatrix}$$

The solution of the primal interface $\mathbf{S}_p \mathbf{u}_b = \mathbf{b}_p$ is, as expected :

$$\mathbf{u}_b = \begin{pmatrix} \dfrac{2F_d}{k_0} \\ \\ \dfrac{4F_d}{k_0} \end{pmatrix} = \begin{pmatrix} \dfrac{F_d L}{3ES} \\ \\ \dfrac{2F_d L}{3ES} \end{pmatrix} \quad \Rightarrow \quad u_b^{\diamondsuit} = \mathbb{A}^{\diamondsuit\, T} \mathbf{u}_b = \begin{pmatrix} \dfrac{F_d L}{3ES} & \dfrac{F_d L}{3ES} & \dfrac{2F_d L}{3ES} & \dfrac{2F_d L}{3ES} \end{pmatrix}^T$$

For the penalty method, one obtains (with $u_d = 0$) :

$$\mathbf{u}_b = \begin{pmatrix} F_d \left(\dfrac{1}{g} + \dfrac{2}{k_0} \right) + u_d \\ \\ F_d \left(\dfrac{1}{g} + \dfrac{4}{k_0} \right) + u_d \end{pmatrix}$$

One deduces the solution within each subdomain for the internal (i) degrees of freedom by solving (local) Dirichlet problem :

$$u_i^{(s)} = K_{ii}^{(s)\,-1} \left(f_i^{(s)} - K_{ib}^{(s)} u_b^{(s)} \right)$$

For subdomain (1), one gets :

$$u_i^{(1)} = u_2^{(1)} = K_{22}^{(1)\,-1} \left(f_2^{(1)} - K_{23}^{(1)} u_3^{(1)} \right) = \frac{F_d}{k_0} = \frac{F_d L}{6ES}$$

and for the penalty method :

$$\begin{aligned} u_i^{(1)} &= \begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} = \begin{pmatrix} K_{11}^{(1)} + g & K_{12}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} \end{pmatrix}^{-1} \left(\begin{pmatrix} f_1^{(1)} + g\, u_d \\ f_2^{(1)} \end{pmatrix} - \begin{pmatrix} K_{13}^{(1)} \\ K_{23}^{(1)} \end{pmatrix} u_3^{(1)} \right) \\ &= \begin{pmatrix} \dfrac{F_d}{g} + u_d \\ F_d \left(\dfrac{1}{g} + \dfrac{1}{k_0} \right) + u_d \end{pmatrix} \end{aligned}$$

Note that when g tends to a high value, the solution tends to the exact solution $\left(0, \dfrac{F_d}{k_0} \right)^T$.

For subdomain (2), internal degrees of freedom are given by :

$$u_i^{(1)} = u_2^{(2)} = K_{22}^{(2)\,-1} \left(f_2^{(2)} - \begin{pmatrix} K_{21}^{(1)} & K_{23}^{(1)} \end{pmatrix} \begin{pmatrix} u_1^{(2)} \\ u_2^{(2)} \end{pmatrix} \right) = \frac{3F_d}{k_0} \quad \text{or} \quad F_d \left(\frac{1}{g} + \frac{3}{k_0} \right) + u_d$$

Finally, for subdomain (3), one gets :

$$\begin{aligned} u_i^{(1)} &= \begin{pmatrix} u_2^{(3)} \\ u_3^{(3)} \end{pmatrix} = \begin{pmatrix} K_{22}^{(3)} & K_{23}^{(3)} \\ K_{23}^{(3)} & K_{33}^{(3)} \end{pmatrix}^{-1} \left(\begin{pmatrix} f_2^{(3)} = 0 \\ f_3^{(3)} = F_d \end{pmatrix} - \begin{pmatrix} K_{21}^{(3)} \\ K_{31}^{(3)} \end{pmatrix} u_1^{(3)} \right) \\ &= \begin{pmatrix} \dfrac{5F_d}{k_0} \\ \dfrac{6F_d}{k_0} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} F_d \left(\dfrac{1}{g} + \dfrac{5}{k_0} \right) + u_d \\ F_d \left(\dfrac{1}{g} + \dfrac{6}{k_0} \right) + u_d \end{pmatrix} \end{aligned}$$

6 Solution for the dual interface problem

The (assembled) quantities for the primal interface problem are :

$$
\begin{aligned}
\mathbf{S}_d &= \underline{\mathbb{A}}^{\diamond}\, S_d{}^{\diamond}\, \underline{\mathbb{A}}^{\diamond\,T} = \frac{2}{k_0}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \text{or} \quad \frac{2}{k_0}\begin{pmatrix}\dfrac{2g+k_0}{2g} & 0\\ 0 & 1\end{pmatrix}\\
\boldsymbol{b}_d &= \underline{\mathbb{A}}^{\diamond}\, b_d{}^{\diamond} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}u_d\\ 0\end{pmatrix}\\
\mathbf{G} &= \underline{\mathbb{A}}^{\diamond}\, R_b{}^{\diamond} = \begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix}\\
e^{\diamond} &= R_b{}^{\diamond\,T}\, b_p{}^{\diamond} = \begin{pmatrix}0\\ F_d\end{pmatrix}
\end{aligned}
$$

Solution of dual interface problem :

$$
\begin{pmatrix}\mathbf{S}_d & \mathbf{G}\\ \mathbf{G}^T & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{\lambda}_b\\ \alpha_b{}^{\diamond}\end{pmatrix} = \begin{pmatrix}-\boldsymbol{b}_d\\ -e^{\diamond}\end{pmatrix}
$$

gives :

$$
\begin{pmatrix}\boldsymbol{\lambda}_b\\ \alpha_b{}^{\diamond}\end{pmatrix} = \begin{pmatrix}F_d\\ F_d\\ \dfrac{2F_d}{k_0}\\ \dfrac{4F_d}{k_0}\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}F_d\\ F_d\\ F_d\left(\dfrac{1}{g}+\dfrac{2}{k_0}\right)-u_d\\ F_d\left(\dfrac{1}{g}+\dfrac{4}{k_0}\right)-u_d\end{pmatrix}
$$

One deduces :

$$
\Rightarrow \lambda_b{}^{\diamond} = \underline{\mathbb{A}}^{\diamond\,T}\boldsymbol{\lambda}_b = \begin{pmatrix}F_d\\ -F_d\\ F_d\\ -F_d\end{pmatrix} \quad \text{and } \alpha_b{}^{\diamond} = \begin{pmatrix}\alpha_b^{(2)}\\ \alpha_b^{(3)}\end{pmatrix} = \begin{pmatrix}\dfrac{2F_d}{k_0}\\ \dfrac{4F_d}{k_0}\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}F_d\left(\dfrac{1}{g}+\dfrac{2}{k_0}\right)+u_d\\ F_d\left(\dfrac{1}{g}+\dfrac{4}{k_0}\right)+u_d\end{pmatrix}
$$

The interface displacement for each subdomain can be obtained subsequently by :

$$
u_b^{(s)} = S_p^{(s)+}\,(b_p^{(s)}+\lambda_b^{(s)}) + R_b^{(s)}\alpha_b^{(s)}
$$

That is to say :

$$
\begin{aligned}
u_b^{(1)} &= S_p^{(1)+}\,(b_p^{(1)}+\lambda_b^{(1)}) + R_b^{(1)}\alpha_b^{(1)} = S_d^{(1)}\lambda_b^{(1)} = \frac{2F_d}{k_0} \quad \text{or} \quad F_d\left(\frac{1}{g}+\frac{2}{k_0}\right)+u_d\\
u_b^{(2)} &= S_p^{(2)+}\,(b_p^{(2)}+\lambda_b^{(2)}) + R_b^{(2)}\alpha_b^{(2)} = \begin{pmatrix}\dfrac{2F_d}{k_0}\\ \dfrac{4F_d}{k_0}\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}F_d\left(\dfrac{1}{g}+\dfrac{2}{k_0}\right)+u_d\\ F_d\left(\dfrac{1}{g}+\dfrac{4}{k_0}\right)+u_d\end{pmatrix}\\
u_b^{(3)} &= S_p^{(3)+}\,(b_p^{(3)}+\lambda_b^{(3)}) + R_b^{(3)}\alpha_b^{(3)} = \frac{4F_d}{k_0} \quad \text{or} \quad F_d\left(\frac{1}{g}+\frac{4}{k_0}\right)+u_d
\end{aligned}
$$

which is the expected exact solution. Internal degrees of freedom in the subdomains can be obtained as explained in **Section 5**.