A Comparative Study of Approximate Joint Diagonalization Algorithms for Blind Source Separation in Presence of Additive Noise

Serge Dégerine and Elimane Kane

***Abstract*—A comparative study of approximate joint diagonalization algorithms of a set of matrices is presented. Using a weighted least-squares criterion, without the orthogonality constraint, an algorithm is compared with an analoguous one for blind source separation (BSS). The criterion of the present algorithm is on the separating matrix while the other is on the mixing matrix. The convergence of the algorithm is proved under some mild assumptions. The performances of the two algorithms are compared with usual standard algorithms using BSS simulations results. We show that the improvement in estimating the separating matrix, resulting from the relaxation of the orthogonality restriction, can be achieved in presence of additive noise when the length of observed sequences is sufficiently large and when the mixing matrix is not close to an orthogonal matrix.**

***Index Terms*—Blind source separation (BSS), instantaneous mixture, joint diagonalization, least-squares criterion.**

I. Introduction

THE approximate joint diagonalization of a set of real m-square symmetrical matrices

$$\mathcal{M} = \{\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_J\}$$

where $\mathbf{M}_0$ is positive definite, is an essential tool in blind source separation (BSS) algorithms. The goal is to find a diagonalizer $\mathbf{B}$, called separating matrix in BSS so that the matrices $\mathbf{B}\mathbf{M}_j\mathbf{B}^{\mathrm{T}}, j = 0, \ldots, J$ are as diagonal as possible. JADE [1] and SOBI [2] algorithms use a preprocessing whitening phase: starting with any square root $\mathbf{M}_0^{1/2}$ of $\mathbf{M}_0$, when $\mathbf{M}_0$ is the observations' correlation matrix at zero-lag, the set $\mathcal{M}$ is replaced by

$$\bar{\mathcal{M}}_0 = \{\bar{\mathbf{M}}_j = \mathbf{W}\mathbf{M}_j\mathbf{W}^{\mathrm{T}}, \quad j = 0, \ldots, J\}$$

where $\mathbf{W} = \mathbf{M}_0^{-1/2}$ is called the whitening matrix. We have $\bar{\mathbf{M}}_0 = \mathbf{I}$ and, for any orthogonal matrix $\mathbf{U}$, the matrix $\mathbf{B} = \mathbf{U}\mathbf{W}$ provides an exact diagonalizer of $\mathbf{M}_0$. The problem is now to find an orthogonal diagonalizer $\mathbf{U}$ of the set $\bar{\mathcal{M}}_0$. Let $\mathrm{Off}(\mathbf{M}) = \sum_{k \neq l} m_{l,k}^2$, where $m_{l,k}$ is the (l,k)th entry of matrix $\mathbf{M}$. Then the solution is obtained by minimizing with respect to (w.r.t.) $\mathbf{U}$ the criterion

$$\mathcal{C}_{js}(\mathbf{U}; \bar{\mathcal{M}}_0) = \sum_{j=0}^{J} \mathrm{Off}(\mathbf{U}\bar{\mathbf{M}}_j\mathbf{U}^{\mathrm{T}}). \tag{1}$$

Let $\|\mathbf{M}\|_F^2 = \mathrm{tr}(\mathbf{M}\mathbf{M}^{\mathrm{T}}) = \sum_{l,k} m_{l,k}^2$ denote the squared Frobenius norm of $\mathbf{M}$. The minimization of (1) is equivalent to minimizing the criterion

$$\mathcal{C}_{\mathrm{lsu}}(\mathbf{U}, \bar{\mathcal{D}}_0; \bar{\mathcal{M}}_0) = \sum_{j=0}^{J} \|\bar{\mathbf{M}}_j - \mathbf{U}^{\mathrm{T}}\overline{\mathbf{D}}_j\mathbf{U}\|_F^2 \tag{2}$$

w.r.t. $\mathbf{U}$ and to the set of diagonal matrices [3]

$$\bar{\mathcal{D}}_0 = \{\overline{\mathbf{D}}_0 = \mathbf{I}, \overline{\mathbf{D}}_1, \ldots, \overline{\mathbf{D}}_J\}.$$

This result is obvious since

$$\mathcal{C}_{\mathrm{lsu}}(\mathbf{U}, \bar{\mathcal{D}}_0; \bar{\mathcal{M}}_0) = \sum_{j=0}^{J} \|\mathbf{U}\bar{\mathbf{M}}_j\mathbf{U}^{\mathrm{T}} - \overline{\mathbf{D}}_j\|_F^2 \tag{3}$$

and the minimum w.r.t. $\bar{\mathcal{D}}_0$ is given by

$$\hat{\overline{\mathbf{D}}}_j = \mathrm{Diag}\{\mathrm{diag}(\mathbf{U}\bar{\mathbf{M}}_j\mathbf{U}^{\mathrm{T}})\}, \quad j = 1, \ldots, J$$

where $\mathrm{diag}(\mathbf{M})$ is the vector defined by the diagonal elements of the matrix $\mathbf{M}$ and $\mathrm{Diag}(\mathbf{v})$ is the diagonal matrix whose diagonal elements are given by the vector $\mathbf{v}$.

The orthogonality constraint on $\mathbf{U}$, i.e., the whitening phase, is relaxed in [4] (see also [5] for a first presentation of this approach). Introducing some positive weights $w_0, w_1, \ldots, w_J$ in (2), this new criterion is to minimize

$$\mathcal{C}_{\mathrm{lsa}}(\mathbf{A}, \mathcal{D}; \mathcal{M}) = \sum_{j=0}^{J} w_j \|\mathbf{M}_j - \mathbf{A}\mathbf{D}_j\mathbf{A}^{\mathrm{T}}\|_F^2 \tag{4}$$

w.r.t. the mixing matrix $\mathbf{A}$ and the set $\mathcal{D} = \{\mathbf{D}_0, \ldots, \mathbf{D}_J\}$ of diagonal matrices. We propose here to relax the orthogonality constraint on $\mathbf{U}$ in (3) instead of (2). Then, the criterion consists in minimizing

$$\mathcal{C}_{\mathrm{lsb}}(\mathbf{B}, \mathcal{D}; \mathcal{M}) = \sum_{j=0}^{J} w_j \|\mathbf{B}\mathbf{M}_j\mathbf{B}^{\mathrm{T}} - \mathbf{D}_j\|_F^2 \tag{5}$$

w.r.t. the separating matrix $\mathbf{B}$ and $\mathcal{D}$. This approach was first presented in [6] (in French).

Manuscript received May 24, 2005; revised October 12, 2006. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. John J. Shynk.

The authors are with the LMC-IMAG, UMR CNRS 5523, BP 53, F-38041 Grenoble Cedex 9, France (e-mail: serge.degerine@imag.fr; elimane.kane@imag.fr).

Digital Object Identifier 10.1109/TSP.2007.893974

These weighted least-squares (WLS) criteria are not clearly justified by statistical models. In addition the maximum likelihood criterion in [7] as well as the one of mutual information in [8] show that the separating matrix $\mathbf{B}$ is a more relevant parameter than the mixing matrix $\mathbf{A}$. These two criteria lead to similar performance in BSS, but it is easier to minimize (5) than (4), and the corresponding algorithm is faster.

All these optimization problems can be solved in an iterative way using alternating direction type algorithms. The minimization of (1) is realized by successive Givens rotations [9]. It is also the case in [10] for an other criterion, without the orthogonality constraint, on a set of symmetrical positive definite matrices $\mathbf{M}_j$. The present algorithm, like the one in [4], alternates between the two following minimization stages. On stage one, the minimization of (4) or (5) is realized w.r.t. the diagonal set $\mathcal{D}$, while keeping $\mathbf{A}$ or $\mathbf{B}$ fixed. Then the solution is explicit, but simpler for (4) than for (5). On the other stage, these criteria are minimized w.r.t. $\mathbf{A}$ or $\mathbf{B}$, while keeping $\mathcal{D}$ fixed. In this stage, the minimization is realized w.r.t. each column $\mathbf{a}_k$ of $\mathbf{A}$ or row $\mathbf{b}_l$ of $\mathbf{B}$ successively, while keeping the others fixed. Then the solution is given by the eigenvector associated to the largest positive eigenvalue of some quadratic form $\mathbf{P}_k$ or to the smallest eigenvalue of an other quadratic form $\bar{\mathbf{Q}}_l$. The difference comes from the fact that $\bar{\mathbf{Q}}_l$ is always positive semidefinite while $\mathbf{P}_k$ can be negative definite.

Other approaches, using gradient methods, have been proposed for minimizing $\mathcal{C}_{\rm lsa}(\cdot)$, [11], or $\mathcal{C}_{\rm lsb}(\cdot)$, [12], [13]. During the review process, we were informed by one referee that the present approach, with the same algorithm, is in a paper that has been accepted for publication in this Transactions [14]. This algorithm, called QDIAG, is without restrictions on the separating matrix B, which can be rectangular, and on the set of matrices $\mathcal{M}$, regarding their symmetry, except for M_0 which is positive definite. Two implementations, with complexity $\mathcal{O}(Jm^3)$ or $\mathcal{O}(m^5)$, are proposed and compared with ACDC [4] and FFDIAG [13]. From a theoretical point of view, only the convergence analysis of the algorithm appears to be new here, together with the comparison of the three criteria $\mathcal{C}_{\rm lsu}(\cdot), \mathcal{C}_{\rm lsa}(\cdot)$ and $\mathcal{C}_{\rm lsb}(\cdot)$ given in this introduction. From now on, the main additional contribution of the present paper is in the simulation results where we are focused on showing the required conditions in order to improve the classical BSS methods by relaxing the orthogonality restriction.

The paper is organized as follows. In Section II, we derive the algorithm for the minimization of $\mathcal{C}_{\rm lsb}(\cdot)$. Some properties of this criterion and convergence of the algorithm are studied in Section III. Finally, the three criteria $\mathcal{C}_{\rm lsu}(\cdot), \mathcal{C}_{\rm lsa}(\cdot)$ and $\mathcal{C}_{\rm lsb}(\cdot)$ are compared, using BSS simulation results, in Section IV.

II. Presentation of the Algorithm

Some ambiguities are well known in BSS problems. If $\mathbf{\Pi}, \mathbf{\Delta}$ and $\mathcal{E}$ designate, respectively, a permutation matrix, a diagonal matrix whose terms are positive and a sign matrix (diagonal matrix whose terms equal ± 1), the relation

$$\mathbf{A}\mathbf{D}_j\mathbf{A}^{\rm T} = (\mathbf{A}\mathbf{\Delta}\mathbf{\Pi}\mathcal{E})(\mathcal{E}\mathbf{\Pi}^{\rm T}\mathbf{\Delta}^{-1}\mathbf{D}_j\mathbf{\Delta}^{-1}\mathbf{\Pi}\mathcal{E}) \times(\mathcal{E}\mathbf{\Pi}^{\rm T}\mathbf{\Delta}\mathbf{A}^{\rm T})$$

shows clearly that $\mathbf{A}$ (or, consequently, $\mathbf{B} = \mathbf{A}^{-1}$) and $\mathcal{D}$ are not uniquely defined. Concerning the scale factor $\mathbf{\Delta}$, the problem is solved by putting $\mathbf{D}_0 = \mathbf{I}$. This corresponds to source signals with unit variance in BSS. For the permutation and sign factors, we can proceed as follows (if necessary). The sign factor $\mathcal{E}$ is chosen in such way that the first nonzero component in each column of $\mathbf{A}$ (or row of $\mathbf{B}$) is positive. Then, for the permutation factor $\mathbf{\Pi}$, the columns of $\mathbf{A}$ (or rows of $\mathbf{B}$) are arranged in decreasing order according to their first component (or to the next ones in case of equality). First, we recall briefly the algorithm in [4], for the WLS criterion on the mixing matrix $\mathbf{A}$, in order to make the comparison easier with the algorithm based on the separating matrix $\mathbf{B}$.

A. Mixing Matrix

The algorithm ACDC in [4] alternates between the two following minimization schemes.

1) Minimization With Respect to $\mathcal{D}$ or DC Phase: The minimization of (4) w.r.t. $\mathcal{D}$, when $\mathbf{A}$ is fixed, is given for $j = 0, \ldots, J$ by

$$\hat{\mathbf{D}}_j = {\rm Diag}\{[(\mathbf{A}^{\rm T}\mathbf{A}) \odot (\mathbf{A}^{\rm T}\mathbf{A})]^{-1}{\rm diag}(\mathbf{A}^{\rm T}\mathbf{M}_j\mathbf{A})\}$$

where $\odot$ denotes Hadamard's (element-wise) product.

2) Minimization With Respect to $\mathbf{A}$ or AC Phase: The minimization of (4) w.r.t. $\mathbf{A}$, when $\mathcal{D}$ is fixed, is realized by successive minimizations on the columns $\mathbf{a}_1, \ldots, \mathbf{a}_m$ of $\mathbf{A}$. The minimum of (4) w.r.t. a selected column $\mathbf{a}_k$ of $\mathbf{A}$, while keeping its other columns fixed as well as $\mathcal{D}$, comes from the following quadratic form

$$\mathbf{P}_k = \sum_{j=0}^{J} \omega_j \mathbf{D}_j(k) \left[\mathbf{M}_j - \sum_{i \neq k} \mathbf{D}_j(i)\mathbf{a}_i\mathbf{a}_i^{\rm T}\right] \tag{6}$$

where $\mathbf{D}_j(i), i = 1, \ldots, m$, are the diagonal elements of $\mathbf{D}_j$. If $\mathbf{v}$ is the unit-norm eigenvector, with a positive first nonzero component, associated to the largest eigenvalue λ of $\mathbf{P}_k$, then

$$\hat{\mathbf{a}}_k = \frac{\sqrt{\lambda}}{\sqrt{\sum_{j=0}^{J} \omega_j \mathbf{D}_j^2(k)}}\mathbf{v}$$

if $\lambda > 0$, otherwise $\hat{\mathbf{a}}_k = \mathbf{0}$.

B. Separating Matrix

Without constraint on $\mathbf{B}$ or $\mathcal{D}$ in the expression (5) of $\mathcal{C}_{\rm lsb}(\mathbf{B}, \mathcal{D}; \mathcal{M})$, the criterion is equal to zero for $\mathbf{B} = \mathbf{0}$ and $\mathbf{D}_j = \mathbf{0}, j = 0, \ldots, J$. So, as explained above, we put $\mathbf{D}_0 = \mathbf{I}$.

1) Minimization With Respect to $\mathcal{D}$: The minimization of (5) w.r.t. $\mathcal{D}$, when $\mathbf{B}$ is fixed, is straightforward

$$\hat{\mathbf{D}}_j = {\rm Diag}\{{\rm diag}(\mathbf{B}\mathbf{M}_j\mathbf{B}^{\rm T})\}, \quad j = 1, \ldots, J.$$

So, the nuisance parameter $\mathcal{D}$ in (5) can be eliminated and the minimization of the criterion is now equivalent to minimize

$$\mathcal{C}_{\rm lsb}(\mathbf{B}; \mathcal{M}) = \sum_{j=0}^{J} \omega_j {\rm Off}(\mathbf{B}\mathbf{M}_j\mathbf{B}^{\rm T}) \tag{7}$$

with the constraint $\mathrm{Diag}\{\mathrm{diag}(\mathbf{B}\mathbf{M}_0\mathbf{B}^{\mathrm{T}})\} = \mathbf{I}$. For a matter of simplicity, we use the arguments of $\mathcal{C}_{\mathrm{lsb}}(\cdot)$ to distinguish $\mathcal{C}_{\mathrm{lsb}}(\mathbf{B}, \mathcal{D}; \mathcal{M})$ from $\mathcal{C}_{\mathrm{lsb}}(\mathbf{B}; \mathcal{M})$ associated with (5) and (7).

2) Minimization With Respect to $\mathbf{B}$*:* This minimization is again realized by successive minimizations on the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_m$ defined by the rows of $\mathbf{B}$. Let us rewrite (7) as follows:

$$\mathcal{C}_{\mathrm{lsb}}(\mathbf{B}; \mathcal{M}) = \sum_{j=0}^{J} \omega_j \sum_{l=1}^{m} \sum_{i \neq l} \left(\mathbf{b}_l^{\mathrm{T}} \mathbf{M}_j \mathbf{b}_i\right)^2.$$

By keeping only the elements that depend on $\mathbf{b}_l$, we have to minimize

$$\begin{aligned} &\mathcal{C}_{\mathrm{lsb}}(\mathbf{b}_l; \mathcal{M}) \\ &\quad = \sum_{j=0}^{J} \omega_j \sum_{i \neq l} \left(\mathbf{b}_l^{\mathrm{T}} \mathbf{M}_j \mathbf{b}_i\right)^2 \\ &\quad = \mathbf{b}_l^{\mathrm{T}} \left\{ \sum_{j=0}^{J} \omega_j \mathbf{M}_j \left[\sum_{i \neq l} \mathbf{b}_i \mathbf{b}_i^{\mathrm{T}} \right] \mathbf{M}_j \right\} \mathbf{b}_l \end{aligned} \tag{8}$$

with the constraint $\mathbf{b}_l^{\mathrm{T}} \mathbf{M}_0 \mathbf{b}_l = 1$. Let $\mathbf{B}_{(l)}$ be the $(m-1) \times m$ matrix obtained by removing the lth row of $\mathbf{B}$. We have

$$\sum_{i \neq l} \mathbf{b}_i \mathbf{b}_i^{\mathrm{T}} = \mathbf{B}_{(l)}^{\mathrm{T}} \mathbf{B}_{(l)} = \mathbf{B}^{\mathrm{T}} \mathbf{B} - \mathbf{b}_l \mathbf{b}_l^{\mathrm{T}}.$$

So, the minimum of $\mathcal{C}_{\mathrm{lsb}}(\mathbf{b}_l; \mathcal{M})$,w.r.t. $\mathbf{b}_l$, is realized by the vector $\hat{\mathbf{b}}_l$ that minimizes the quadratic form

$$\mathbf{Q}_l = \sum_{j=0}^{J} \omega_j \mathbf{M}_j \mathbf{B}_{(l)}^{\mathrm{T}} \mathbf{B}_{(l)} \mathbf{M}_j$$

with the constraint $\mathbf{b}_l^{\mathrm{T}} \mathbf{M}_0 \mathbf{b}_l = 1$. Using the whitening phase as a change of variables and putting $\mathbf{B} = \bar{\mathbf{B}}\mathbf{W}$, the new solution $\hat{\bar{\mathbf{b}}}_l$ is the unit-norm eigenvector, with a positive first nonzero component, associated to the smallest eigenvalue of the new quadratic form

$$\bar{\mathbf{Q}}_l = \sum_{j=0}^{J} \omega_j \bar{\mathbf{M}}_j \bar{\mathbf{B}}_{(l)}^{\mathrm{T}} \bar{\mathbf{B}}_{(l)} \bar{\mathbf{M}}_j. \tag{9}$$

It is clear that $\bar{\mathbf{Q}}_l$ is positive semidefinite, i.e., $\mathbf{b}^{\mathrm{T}} \bar{\mathbf{Q}}_l \mathbf{b} \geq 0$ for every $\mathbf{b} \in \mathbb{R}^m$. Furthermore, looking at (8), $\hat{\bar{\mathbf{b}}}_l^{\mathrm{T}} \bar{\mathbf{Q}}_l \hat{\bar{\mathbf{b}}}_l = 0$ if and only if the solution matrix $\mathbf{B}$ obtained at this stage is such that, for every $j = 0 \ldots, J$ and $k \neq l$ (l being fixed), the (l,k)th and (k,l)th entries of $\mathbf{B}\mathbf{M}_j\mathbf{B}^{\mathrm{T}}$ are equal to zero, which is the goal (for every l) of the exact diagonalization problem.

3) LSB Algorithm: Starting with $\bar{\mathbf{B}}_0 = \mathbf{I}$, or with any other initialization matrix $\bar{\mathbf{B}}_0$, the LSB algorithm provides a sequence of matrices $\{\bar{\mathbf{B}}_n, n \geq 0\}$ by changing successively one selected row of the current matrix with the above process, i.e., $\bar{\mathbf{B}}_{n+1}$ differs from $\bar{\mathbf{B}}_n$ only through one row $\mathbf{b}_l$ (or is equal to $\bar{\mathbf{B}}_n$). Notice that, in each iteration, the constraint $\mathrm{Diag}\{\mathrm{diag}(\bar{\mathbf{B}}_n \bar{\mathbf{B}}_n^{\mathrm{T}})\} = \mathbf{I}$ is automatically fulfiled.

We will see in Section III-B that the convergence of the algorithm, i.e., an existing limit $\hat{\bar{\mathbf{B}}}$ for the sequence $\{\bar{\mathbf{B}}_n, n \geq 0\}$, is guaranteed with very mild assumptions. So, the stopping rule of the algorithm can operate with the change in the criterion $\mathcal{C}_{\mathrm{lsb}}(\cdot)$ or in the parameter $\bar{\mathbf{B}}$.

Let us recall that, when $\hat{\bar{\mathbf{B}}}$ is given by the algorithm, the solution of the problem is $\hat{\mathbf{B}} = \hat{\bar{\mathbf{B}}}\mathbf{W}$. Here, the whitening phase is only used to simplify the computations in the algorithm. However, this shows that this approach is equivalent to replacing the orthogonality constraint $\mathbf{U}\mathbf{U}^{\mathrm{T}} = \mathbf{I}$, in (1) or (3), by $\mathrm{Diag}\{\mathrm{diag}(\mathbf{U}\mathbf{U}^{\mathrm{T}})\} = \mathbf{I}$, which is less restrictive.

III. Properties of the Criterion and Convergence of the Algorithm

A. Existence, Invariance, and Uniqueness of the Solution

1) Existence: Criteria (1), (4), and (5) are minimized by zero. This lower bound is reached if and only if the set of matrices in $\mathcal{M}$ are jointly diagonalizable. Almost surely, this property is not satisfied in the BSS problems, except for $J = 1$ or in an asymptotic way. These criteria are continuous and can be restricted to compact sets, without loss of generality. So, the existence of a solution is guaranteed.

2) Invariance: The natural invariance property which is desirable here is the following: for any nonsingular matrix $\mathbf{R}$, the solution $\tilde{\mathbf{B}}$ associated to the set $\tilde{\mathcal{M}} = \{\tilde{\mathbf{M}} = \mathbf{R}\mathbf{M}_j\mathbf{R}^{\mathrm{T}}, j = 0, \ldots, J\}$ is given by $\tilde{\mathbf{B}} = \mathbf{B}\mathbf{R}^{-1}$ (i.e., $\tilde{\mathbf{A}} = \mathbf{R}\mathbf{A}$), where $\mathbf{B}$ is the solution associated to $\mathcal{M}$.

This property is clearly satisfied by the criterion $\mathcal{C}_{\mathrm{lsb}}(\cdot)$ since, in (5), we have

$$\|\mathbf{B}\mathbf{M}_j\mathbf{B}^{\mathrm{T}} - \mathbf{D}_j\|_F^2 = \|\mathbf{B}\mathbf{R}^{-1}\mathbf{R}\mathbf{M}_j\mathbf{R}^{\mathrm{T}}(\mathbf{B}\mathbf{R}^{-1})^{\mathrm{T}} - \mathbf{D}_j\|_F^2.$$

It is also the case for $\mathcal{C}_{\mathrm{lsu}}(\cdot)$ through the whitening phase, in which the whitening matrix $\mathbf{W}$ is replaced by $\mathbf{W}\mathbf{R}^{-1}$. On the other hand, this property is not true for $\mathcal{C}_{\mathrm{lsa}}(\cdot)$.

3) Uniqueness: As discussed at the beginning of Section II, nonuniqueness coming from the ambiguities in the factors $\mathbf{\Pi}, \mathbf{\Delta}$ and $\mathcal{E}$ can be easily eliminated by selecting one set of parameters. Nevertheless, it is not difficult to select a set $\mathcal{M}$ of matrices which are exactly diagonalizable by a continuous set of matrices $\mathbf{B}$. For instance, if $\mathbf{M}_j = \mathbf{A}\mathbf{D}_j\mathbf{A}^{\mathrm{T}}, j = 0, \ldots, J$ and, for $k \neq l$, the diagonal entries of $\mathbf{D}_j$ satisfy $\mathbf{D}_j(k) = \mathbf{D}_j(l)$ for all j, then we can replace the columns $\mathbf{a}_k$ and $\mathbf{a}_l$ of $\mathbf{A}$ by $[\cos\theta\mathbf{a}_k + \sin\theta\mathbf{a}_l]$ and $[\cos\theta\mathbf{a}_l - \sin\theta\mathbf{a}_k], \theta \in [0, \pi]$, without modifying the matrices $\mathbf{M}_j$. For example, this situation occurs when two sources have proportional spectra in the BSS problem, when $\mathbf{M}_j$ is the cross-correlation matrix of lag j.

In practice, we can suppose that the global minimum is unique, especially when the elements of $\mathcal{M}$ come from observed data, like in BSS problems. However, the difficulty in the algorithms comes from the existence of local minima. It is not difficult to observe such a phenomenon in simulations by using different starting values in the algorithms.

B. Convergence of the Algorithm

Convergence of these algorithms is difficult to prove, even if the global minimum exists. Nevertheless, alternating direction type methods are preferable to gradient methods, since criteria

always decrease and are bounded below by zero, so they are convergent. However, in general, this does not imply convergence of parameters. This point can be detailed as follows.

Let $\bar{\mathcal{B}}^*$ be the set of accumulation points of a sequence $\bar{\mathcal{B}} = \{\bar{\mathbf{B}}_n, n \geq 0\}$ given by the algorithm. If $\bar{\mathcal{B}}$ is a bounded sequence and, in addition, $\|\bar{\mathbf{B}}_{n+1} - \bar{\mathbf{B}}_n\|_F^2$ goes to zero, then, using [15, Theorem 26.1, p. 173], this sequence is convergent if $\bar{\mathcal{B}}^*$ is not a continuum (a continuum is a closed set which cannot be decomposed into two disjoint closed sets).

$\bar{\mathcal{B}}$ is a bounded sequence, since $\mathrm{Diag}\{\mathrm{diag}(\bar{\mathbf{B}}_n\bar{\mathbf{B}}_n^{\mathrm{T}})\} = \mathbf{I}$ implies $\|\bar{\mathbf{B}}_n\|_F^2 = m$. Now, $\bar{\mathbf{B}}_{n+1}$ is obtained by replacing a row $\bar{\mathbf{b}}_l$ of $\bar{\mathbf{B}}_n$ by the eigenvector, with a positive first nonzero component, $\hat{\bar{\mathbf{b}}}_l$ associated to the smallest eigenvalue λ_1 of $\bar{\mathbf{Q}}_l$. Then, we have

$$\|\bar{\mathbf{B}}_{n+1} - \bar{\mathbf{B}}_n\|_F^2 = \|\hat{\bar{\mathbf{b}}}_l - \bar{\mathbf{b}}_l\|_F^2$$

and

$$\mathcal{C}_{\mathrm{lsb}}(\bar{\mathbf{B}}_n; \bar{\mathcal{M}}_0) - \mathcal{C}_{\mathrm{lsb}}(\bar{\mathbf{B}}_{n+1}; \bar{\mathcal{M}}_0) = \bar{\mathbf{b}}_l^{\mathrm{T}}\bar{\mathbf{Q}}_l\bar{\mathbf{b}}_l - \hat{\bar{\mathbf{b}}}_l^{\mathrm{T}}\bar{\mathbf{Q}}_l\hat{\bar{\mathbf{b}}}_l.$$

Let $\bar{\mathbf{Q}}_l = \mathbf{V}\Lambda\mathbf{V}^{\mathrm{T}}$ be the eigendecomposition of $\bar{\mathbf{Q}}_l$ with $\mathbf{\Lambda} = \mathrm{Diag}\{(\lambda_1, \ldots, \lambda_m)^{\mathrm{T}}\}$ and $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_m)^{\mathrm{T}}$, where the eigenvalues are arranged in decreasing order, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. We have $\hat{\bar{\mathbf{b}}}_l = \mathbf{v}_1$ and $\hat{\bar{\mathbf{b}}}_l^{\mathrm{T}}\bar{\mathbf{Q}}_l\hat{\bar{\mathbf{b}}}_l = \lambda_1$. Putting $\bar{\mathbf{b}}_l = \sum_{i=1}^m \alpha_i\mathbf{v}_i$, it can be seen that

$$\|\bar{\mathbf{B}}_{n+1} - \bar{\mathbf{B}}_n\|_F^2 = 2(1 - \alpha_1)$$

and

$$\mathcal{C}_{\mathrm{lsb}}(\bar{\mathbf{B}}_n; \bar{\mathcal{M}}_0) - \mathcal{C}_{\mathrm{lsb}}(\bar{\mathbf{B}}_{n+1}; \bar{\mathcal{M}}_0) \geq \left(1 - \alpha_1^2\right)(\lambda_2 - \lambda_1).$$

So, the convergence of the criterion $\mathcal{C}_{\mathrm{lsb}}(\bar{\mathbf{B}}_n; \bar{\mathcal{M}}_0)$ implies that α_1 converges to ± 1 if $(\lambda_2 - \lambda_1)$ remains bounded below by a strictly positive number (separating condition). The convention used in the definition of $\hat{\bar{\mathbf{b}}}$ shows that α_1 tends to $+1$ and then $\|\bar{\mathbf{B}}_{n+1} - \bar{\mathbf{B}}_n\|_F^2$ goes to zero.

Notice that a matrix $\bar{\mathbf{B}}^*$ is an accumulation point of a sequence $\bar{\mathcal{B}}$ if and only if each of its rows $\bar{\mathbf{b}}_l^*, l = 1, \ldots, m$, is an eigenvector associated to the smallest eigenvalue of the corresponding quadratic form $\bar{\mathbf{Q}}_l^*$ defined by (9). Finally, the algorithm is convergent if this set of matrices is not a continuum and if the quadratic forms $\bar{\mathbf{Q}}_l$ used in the algorithm satisfy the separating condition on their lowest eigenvalues.

IV. BSS Simulation Results

In this section, we compare the LSB algorithm with the ACDC algorithm of [4] (denoted LSA for convenience here) for BSS problems. More precisely, the goal is to detail the situations where these algorithms could improve the performances of the standard LSU algorithms (JADE [1] and SOBI [2]). The LSB algorithm implemented here corresponds to the QDIAG algorithm with complexity $\mathcal{O}(Jm^3)$ of [14]. Notice that simulation results reported in [14] are essentially devoted to the comparison of QDIAG with ACDC and FFDIAG [13] in terms of rate of convergence and computational efficiency.

A. Experimental Setting

We consider an instantaneous mixture of $m = 3$ sources with additive noise

$$\begin{aligned}\mathbf{X}(t) &= \mathbf{A}_0\mathbf{S}(t) + \delta\Delta_0\mathbf{w}(t)\\ &= \mathbf{Y}(t) + \mathbf{n}(t), \quad t = 1, \ldots T\end{aligned}$$

and the true mixing matrix used in [4]

$$\mathbf{A}_0 = \begin{bmatrix} 1.9 & 3.0 & -0.5 \\ -0.2 & 0.4 & -0.1 \\ -0.2 & -0.3 & 0.9 \end{bmatrix}.$$

The components $\mathbf{S}_k(\cdot)$ of $\mathbf{S}(\cdot)$ are real-valued zero mean processes with unit variances and are independent of each other, so $\mathrm{Var}\{\mathbf{S}(t)\} = \mathbf{I}$. The components $\mathbf{w}_k(\cdot)$ of $\mathbf{w}(\cdot)$ are real-valued white Gaussian noises with unit variances and $\mathbf{w}(\cdot)$ is independent of $\mathbf{S}(\cdot)$. The matrix Δ_0 is diagonal with $\Delta_0^2 = \mathrm{Diag}\{\mathrm{diag}(\mathbf{A}_0\mathbf{A}_0^{\mathrm{T}})\}$. So, δ is a scalar used to select the noise level, since we have $\mathrm{Var}\{\mathbf{n}_k(t)\} = \delta^2\mathrm{Var}\{\mathbf{Y}_k(t)\}$ for each k, while $\mathrm{nl} = 10\log_{10}\delta^2$ is the noise level in decibel. The covariance matrix of $\mathbf{X}(\cdot)$ satisfies

$$\mathbf{R}_{\mathbf{X}}(0) = \mathbb{E}\{\mathbf{X}(t)\mathbf{X}(t)^{\mathrm{T}}\} = \mathbf{A}_0\mathbf{A}_0^{\mathrm{T}} + \delta^2\Delta_0^2.$$

Notice that the separating matrix $\mathbf{B}_0 = \mathbf{A}_0^{-1}$ is an exact diagonalizer of $\mathbf{R}_{\mathbf{X}}(0)$ in the noiseless case $\delta = 0$, but also when $\mathbf{A}_0$ is an orthogonal matrix. We will see below that, in these situations, LSU algorithms could not be improved. Let us recall that these algorithms estimate the separating matrix in the form $\hat{\mathbf{B}}_{\mathrm{lsu}} = \hat{\mathbf{U}}_{\mathrm{js}}\hat{\mathbf{W}}$ using criterion (1) for the orthogonal matrix $\hat{\mathbf{U}}_{\mathrm{js}}$. So, the whitening matrix $\hat{\mathbf{W}}$ should be an estimate of $\mathbf{W}$ satisfying $\mathbf{W}\mathbf{A}_0\mathbf{A}_0^{\mathrm{T}}\mathbf{W}^{\mathrm{T}} = \mathbf{I}$.

1) Whitening Phase: The covariance matrix $\mathbf{R}_{\mathbf{X}}(0)$ is estimated by its empirical counterpart

$$\hat{\mathbf{R}}_{\mathbf{X}}(0) = \frac{1}{T}\sum_{t=1}^{T}\mathbf{X}(t)\mathbf{X}(t)^{\mathrm{T}}$$

which is symmetrical positive definite and gives the matrix $\mathbf{M}_0$ of $\mathcal{M}$. The whitening matrix $\hat{\mathbf{W}} = \hat{\mathbf{R}}_{\mathbf{X}}(0)^{-1/2}$ is defined by $\hat{\mathbf{W}} = \mathbf{D}^{-1}\mathbf{V}^{\mathrm{T}}$, from the eigendecomposition $\hat{\mathbf{R}}_{\mathbf{X}}(0) = \mathbf{V}\mathbf{D}^2\mathbf{V}^{\mathrm{T}}$, and the whitened signal $\hat{\mathbf{X}}^w(\cdot)$ is given by $\hat{\mathbf{X}}^w(t) = \hat{\mathbf{W}}\mathbf{X}(t), t = 1, \ldots, T$.

Now, the other matrices $\bar{\mathbf{M}}_1, \ldots, \bar{\mathbf{M}}_J$ of $\bar{\mathcal{M}}_0$ are selected in two distinct ways in order to illustrate the comparison with JADE or SOBI. However, these matrices are always defined from statistics computed on the whitened signal $\hat{\mathbf{X}}^w(\cdot)$.

2) Cumulant Matrices: JADE exploits the statistical independence of the sources using the fourth-order cumulants. So, for each k, $\mathbf{S}_k(t), t = 1, \ldots, T$ are independent random variables uniformly distributed between $-\sqrt{3}$ and $\sqrt{3}$ (thus having zero mean and unit variance) as in [4]. Matrices $\bar{\mathbf{M}}_j, j = 1, \ldots, J$, of $\bar{\mathcal{M}}_0$, with $J = m(m+1)/2 = 6$, are defined by the following empirical cumulant matrices of $\hat{\mathbf{X}}^w(\cdot)$. For $1 \leq l \leq k \leq m$ and $j = (k-1) + l$, $\bar{\mathbf{M}}_j = \widehat{\mathrm{Cum}}_{k,l}(\hat{\mathbf{X}}^w)$ with

$$\widehat{\mathrm{Cum}}_{k,l}(\hat{\mathbf{X}}^w) = \left[\widehat{\mathrm{Cum}}\left\{\hat{\mathbf{X}}_k^w, \hat{\mathbf{X}}_l^w, \hat{\mathbf{X}}_i^w, \hat{\mathbf{X}}_j^w\right\}\right]_{i=1,\ldots,m}^{j=1,\ldots,m}. \quad (10)$$

Since the empirical covariance matrix of $\hat{\mathbf{X}}^w$ is equal to $\mathbf{I}$, we have

$$\widehat{\mathrm{Cum}}\left\{\hat{\mathbf{X}}_k^w,\hat{\mathbf{X}}_l^w,\hat{\mathbf{X}}_i^w,\hat{\mathbf{X}}_j^w\right\}=\frac{1}{T}\sum_{t=1}^{T}\hat{\mathbf{X}}_k^w(t)\hat{\mathbf{X}}_l^w(t)\hat{\mathbf{X}}_i^w(t)\hat{\mathbf{X}}_j^w(t)-\delta_{k,l}\delta_{i,j}-\mathbf{e}_k(i)\mathbf{e}_l(j)-\mathbf{e}_l(i)\mathbf{e}_k(j)$$

where $\delta_{k,l}$ is the Kronecker symbol, $\delta_{k,l}=1$ if $k=l$, else 0 and $[\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_m]=\mathbf{I}$.

Notice that, although $\hat{\mathbf{X}}^w(t)=\hat{\mathbf{W}}\mathbf{X}(t)$, the cumulant matrix $\widehat{\mathrm{Cum}}_{k,l}(\hat{\mathbf{X}}^w)$ is not equal to $\hat{\mathbf{W}}\widehat{\mathrm{Cum}}_{k,l}(\mathbf{X})\hat{\mathbf{W}}^{\mathrm{T}}$.

3) Covariance Matrices: SOBI exploits the second-order statistics of colored sources having different spectra. So, for each $k,\mathbf{S}_k(t),t=1,\ldots,T$ is a sequence of a zero-mean stationary Gaussian $\mathrm{AR}(p_k)$ process with unit variance. We use the same models as in [7] characterized by the orders $p_1=2,p_2=3$ and $p_3=4$ and the following partial autocorrelation coefficients:

$$\{0.4,0.25;.0.6,0.8,-0.5;0.5,0.7,-0.5,0.4\}.$$

Matrices $\bar{\mathbf{M}}_j,j=1,\ldots,J=6$ of $\bar{\mathcal{M}}_0$ are defined by the following empirical covariance matrices of lag j of $\hat{\mathbf{X}}^w(\cdot)$:

$$\bar{\mathbf{M}}_j=\frac{1}{2}\left[\hat{\mathbf{R}}_j^w+\hat{\mathbf{R}}_j^{w\mathbf{t}}\right],\quad \hat{\mathbf{R}}_j^w=\frac{1}{T}\sum_{t=j+1}^{T}\mathbf{X}^w(t)\mathbf{X}^w(t-j)^{\mathrm{T}}. \tag{11}$$

Notice that, here, we have $\hat{\mathbf{R}}_j^w=\hat{\mathbf{W}}\hat{\mathbf{R}}_j\hat{\mathbf{W}}^{\mathrm{T}}$, where $\hat{\mathbf{R}}_j$ is the covariance matrix of lag j of $\mathbf{X}(\cdot)$.

4) Stopping Rule: We use the same stopping rule in the three algorithms, LSU, LSA, and LSB, given by the relative variation of criteria (2), (4), and (5) after each sweep, with the same set $\bar{\mathcal{M}}_0$. For instance, in LSB, we take

$$\frac{|\mathcal{C}_{\mathrm{lsb}}(\tilde{\bar{\mathbf{B}}},\tilde{\bar{\mathcal{D}}}_0;\bar{\mathcal{M}}_0)-\mathcal{C}_{\mathrm{lsb}}(\bar{\mathbf{B}},\bar{\mathcal{D}}_0;\bar{\mathcal{M}}_0)|}{|\mathcal{C}_{\mathrm{lsb}}(\bar{\mathbf{B}},\bar{\mathcal{D}}_0;\bar{\mathcal{M}}_0)|}<10^{-4}$$

where $\tilde{\bar{\mathbf{B}}}$ comes from the modification of the m rows of $\bar{\mathbf{B}}$. In LSA, one sweep corresponds to one DC phase, for $\bar{\mathcal{D}}_0$, and m AC phases for the modification of the m columns of $\bar{\mathbf{A}}$. In LSU, one sweep corresponds to $m(m-1)/2$ Givens rotations. The maximum number of sweeps is also limited to 1000.

5) Performance Index: In all experiments reported below, the algorithms are applied on the set $\bar{\mathcal{M}}_0$, where $\bar{\mathbf{M}}_0=\mathbf{I}$ and $\bar{\mathbf{M}}_j,j=1,\ldots,J$ are given by (10) or (11). Then, the estimated separating matrix $\hat{\mathbf{B}}$ is $\hat{\mathbf{U}}\hat{\mathbf{W}},\hat{\bar{\mathbf{B}}}\hat{\mathbf{W}}$ and $\hat{\bar{\mathbf{A}}}^{-1}\hat{\mathbf{W}}$ for LSU, LSB, and LSA, respectively. We have $\hat{\mathbf{U}}\hat{\mathbf{W}}\hat{\mathbf{R}}_{\mathbf{X}}(0)[\hat{\mathbf{U}}\hat{\mathbf{W}}]^{\mathrm{T}}=\mathbf{I}$ and $\mathrm{Diag}\{\mathrm{diag}(\hat{\bar{\mathbf{B}}}\hat{\mathbf{W}}\hat{\mathbf{R}}_{\mathbf{X}}(0)[\hat{\bar{\mathbf{B}}}\hat{\mathbf{W}}]^{\mathrm{T}})\}=\mathbf{I}$. So, the empirical covariance matrix $\hat{\mathbf{R}}_{\hat{\mathbf{S}}}(0)$ of $\hat{\mathbf{S}}(t)=\hat{\mathbf{B}}\mathbf{X}(t)$ satisfies $\hat{\mathbf{R}}_{\hat{\mathbf{S}}}(0)=\mathbf{I}$ in LSU and $\mathrm{Diag}\{\mathrm{diag}(\hat{\mathbf{R}}_{\hat{\mathbf{S}}}(0))\}=\mathbf{I}$ in LSB. This normalization is coherent with $\mathrm{Var}\{\mathbf{S}(t)\}=\mathbf{I}$ and the matrix $\hat{\mathbf{C}}=\hat{\mathbf{B}}\mathbf{A}_0$ should be nearly a matrix of the form $\mathbf{\Pi}\mathcal{E}$ in the noiseless case. This property is no longer true in presence of additive noise. Furthermore, there is no normalization on the columns of $\hat{\bar{\mathbf{A}}}$ in LSA. In order to eliminate the three ambiguities $\mathbf{\Pi},\mathcal{E}$ and $\mathbf{\Delta}$, the performance index used in the sequel is the interference to signal ratio (ISR) measured as follows. In each experiment (and for each algorithm), the permutation matrix $\mathbf{\Pi}$ and the sign matrix $\mathcal{E}$ are chosen in such way that the diagonal elements of $\mathcal{E}\mathbf{\Pi}\hat{\mathbf{C}}$ are positive with maximal sum. Then, each row of $\mathcal{E}\mathbf{\Pi}\hat{\mathbf{C}}$ is divided by its diagonal component. The resulting matrix is noted $\bar{\mathbf{C}}$. Using R independent trials, the ISR is estimated by

$$\mathrm{ISR}=\frac{1}{R}\sum_{r=1}^{R}\sum_{l\neq k}\bar{\mathbf{C}}_r^2(l,k).$$

The number R of trials depends on the sequence length T with $RT=50\times10^6$ and selected values of T are 2500, 5000, 10 000, 50 000, 100 000, 500 000, and 1 000 000.

B. Experimental Results

Numerous experiments have been conducted. Those reported below have been selected in order to illustrate the more relevant facts. The weights are identical in LSA and LSB with $\omega_j=1,j=1,\ldots,J$ and three distinct values for ω_0. By using a heavy weight $\omega_0=100$, we force the separating matrix to be an "exact" diagonalizer of $\hat{\mathbf{R}}_{\mathbf{X}}(0)$ as in LSU. In order to relax this constraint, a very small weight should be chosen. This choice can be adapted to the sequence length T. From our experiments, we have selected, in an approximative way, $\omega_0=1/\sqrt{T}$ for comparison with JADE (cumulant matrices) and $\omega_0=1/10\sqrt{T}$ for comparison with SOBI (covariance matrices). The third value $\omega_0=1$ illustrates the neutral situation. Without other indication, the output of LSU is used as starting value for LSA and LSB.

1) Comparison With Jade: Please recall that the separating matrix given by JADE [9] is $\hat{\mathbf{B}}_{\mathrm{lsu}}=\hat{\mathbf{U}}\hat{\mathbf{W}}$, where $\hat{\mathbf{W}}$ is the whitening matrix defined in IV–A1 and $\hat{\mathbf{U}}$ is the orthogonal matrix solution of criterion (1) applied on the set $\bar{\mathcal{M}}_0$ of cumulant matrices given by (10). Then, the separating matrices given by LSB and LSA are, respectively, $\hat{\mathbf{B}}_{\mathrm{lsb}}=\hat{\bar{\mathbf{B}}}\hat{\mathbf{W}}$ and $\hat{\mathbf{B}}_{\mathrm{lsa}}=\hat{\bar{\mathbf{A}}}^{-1}\hat{\mathbf{W}}$, where $\hat{\bar{\mathbf{B}}}$ and $\hat{\bar{\mathbf{A}}}$ are the solutions of criteria (4) and (5) using the same set $\bar{\mathcal{M}}_0$, with $\hat{\mathbf{U}}$ and $\hat{\mathbf{U}}^{\mathrm{T}}$ as starting values.

It is not surprising here that the performance of JADE cannot be improved when the true separating matrix $\mathbf{B}_0$ is an exact diagonalizer of the true covariance matrix $\mathbf{R}_{\mathbf{X}}(0)$. So, three conditions are necessary in order to observe better performances with LSA or LSB. The noise level nl must be important, the mixing matrix $\mathbf{A}_0$ must be sufficiently different from an orthogonal matrix and the sequence length T must be very large for estimating cumulants with good accuracy. Notice that the performance of JADE can be improved, in the noiseless case, by using asymptotically optimal generalized weights (ω_j is a random matrix instead of a scalar) in LSA applied to the output of JADE [16].

Let us begin by general comments which are common to most simulations reported hereafter.

- The performances of LSA and LSB are generally identical, so further comments will be given only for LSB.
- The performances of JADE and LSB with $w_0=100$ are identical.
- By looking at the relative performances of LSB with respect to the weight ω_0, we can see that the search of an optimal weight is a difficult task, because this weight depends on the sequence length T, the noise level nl and the

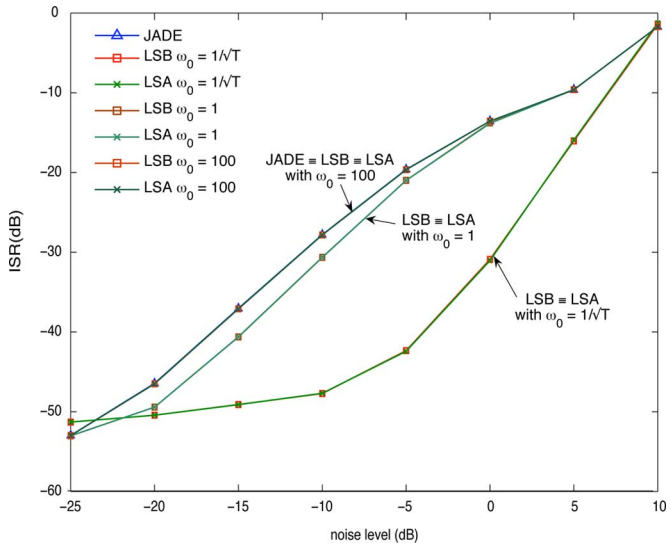

Fig. 1. ISR versus noise level with mixing matrix $\mathbf{A}_0$ and sequence length $T = 10^6$.

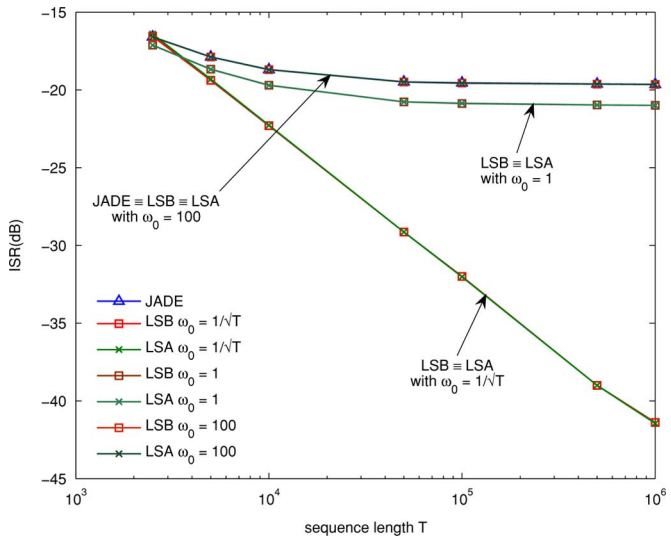

Fig. 2. ISR versus sequence length with mixing matrix $\mathbf{A}_0$ and noise level $\mathrm{nl} = -5$ dB.

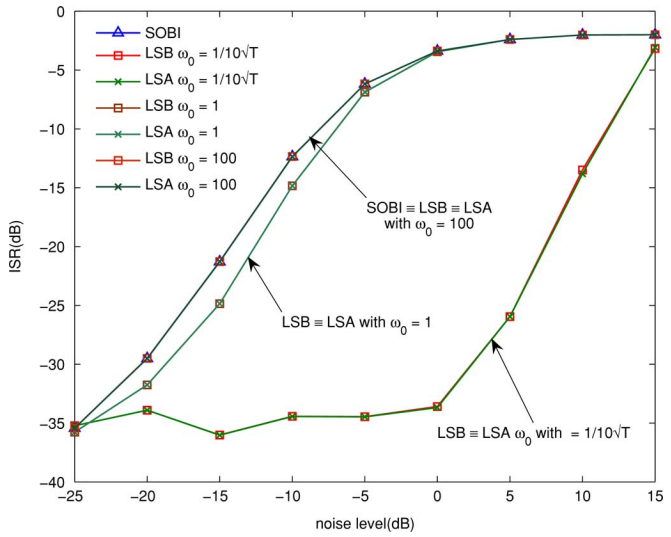

Fig. 3. ISR versus noise level with mixing matrix $\mathbf{A}_0$ and sequence length $T = 10^6$.

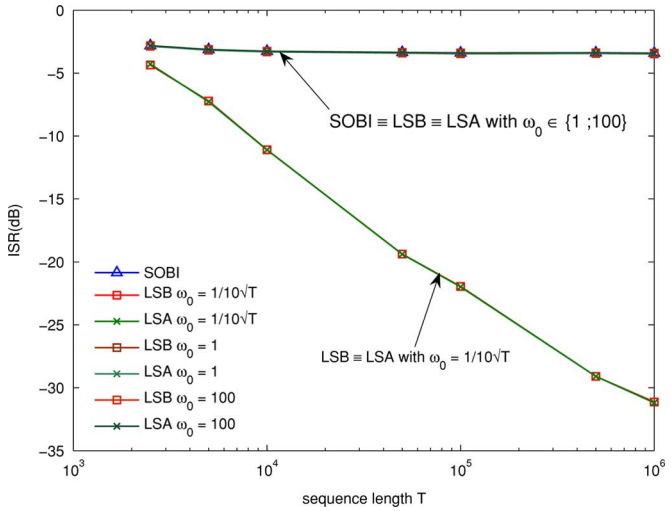

Fig. 4. ISR versus sequence length with mixing matrix $\mathbf{A}_0$ and noise level $\mathrm{nl} = 0$ dB.

separating matrix A_0. Nevertheless, since the goal is to improve the performance of JADE, this can be done only with $w_0 = 1/\sqrt{T}$.

Fig. 1 shows that, for $T = 10^6$, LSB is better than JADE when the noise level nl is between -20 dB and 5 dB, with a maximum gain of 22.7 dB for $\mathrm{nl} = -5$ dB. However, as it can be seen on Fig. 2, even for $\mathrm{nl} = -5$ dB, this advantage needs $T \geq 5000$. For $T = 5000$, the ISR values in dB are -17.88, -19.38 and -19.31 for JADE, LSB, and LSA, respectively, with $\omega_0 = 1/\sqrt{T}$. Furthermore, JADE is preferable when the mixing matrix is orthogonal. This is illustrated in Fig. 5 where the mixing matrix is the orthogonal matrix $\mathbf{V}_0$ of the eigendecomposition $\mathbf{A}_0 = \mathbf{V}_0\mathbf{D}_0\mathbf{V}_0^{\mathrm{T}}$.

2) Comparison With SOBI: The cumulant matrices in $\bar{\mathcal{M}}_0$ are now replaced by the covariance matrices given by (11). The conclusions are very similar with those given in the comparison with JADE. For $T = 10^6$, Fig. 3 shows that LSB and LSA are better than SOBI when the noise level nl is between -20 dB and 10 dB, with a maximum gain of 30.2 dB for $\mathrm{nl} = 0$ dB. In Fig. 4, for $\mathrm{nl} = 0$ dB, we observe that this amelioration needs $T \geq 2500$. For $T = 2500$, the ISR values in dB are -2.82, -4.36 and -4.29 for SOBI, LSB and LSA, respectively, with $\omega_0 = 1/10\sqrt{T}$. In general, the gain is more important here because the covariance matrices of different lags are estimated with the same accuracy than $\mathbf{R}_{\mathbf{X}}(0)$ contrary to the cumulant matrices.

Notice that Figs. 2 and 4 illustrate the fact that the whitening phase in LSU algorithms imposes a limit on the reachable performance [17].

3) Further Comments: Now, we illustrate other relevant facts regarding the algorithms LSA and LSB. The first one is the influence of the starting value, the second is the choice of the set of cumulant matrices and the third is the difficulties in such comparisons regarding the choice of the mixing matrix and the performance index.

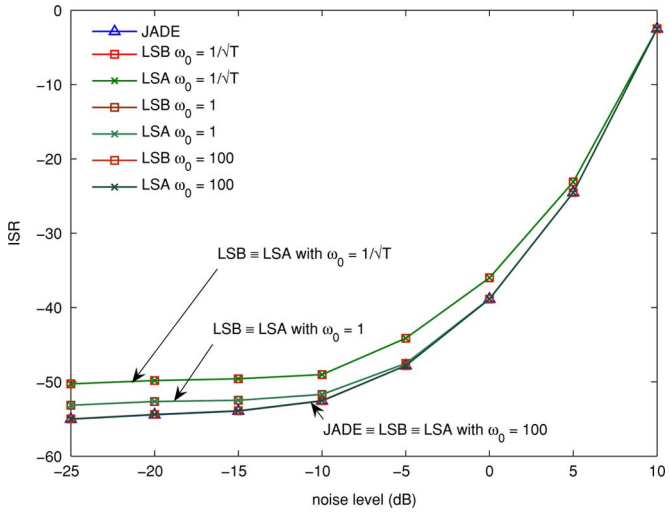

Fig. 5. ISR versus noise level with the orthogonal mixing matrix $\mathbf{V}_0$ and sequence length $T = 10^6$

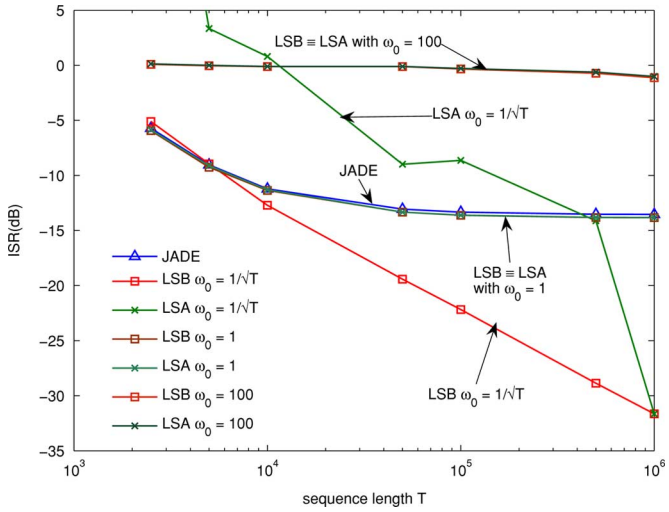

Fig. 6. ISR versus sequence length with mixing matrix $\mathbf{A}_0$ and noise level $\mathrm{nl} = 0$ dB. Here, the identity matrix is used as starting value for LSA and LSB.

Instead of the output of JADE, the identity matrix $\mathbf{I}$ is now used as starting value in LSA and LSB. Fig. 6 shows analogous results of those reported in Fig. 2, but with $\mathrm{nl} = 0$ dB. We observe that the starting value does not affect LSB, when $\omega_0 = 1/\sqrt{T}$ or $\omega_0 = 1$, as well as LSA when $\omega_0 = 1$. On the other hand, LSA with $\omega_0 = 1/\sqrt{T}$ gives very bad results (value of ISR for $T = 2500$ is 33.91 dB). In fact, these results, except for $T = 10^6$, are very different from one set of trials to another. When $\omega_0 = 100$, LSA and LSB have the same behaviour: the results, which are now very bad and far away from those of JADE, are independent of the sequence length T. Moreover, in this case, the algorithms are not convergent (the number of sweeps reaches 1000) for about 60% of trials to 90% when T goes from 2500 to 10^6. Please notice that this dependence on the starting value proves that criteria (4) and (5) can have local minima.

Until now, we have used the set $\bar{\mathcal{M}}_0$ of cumulant matrices associated with the whitened signal $\hat{\mathbf{X}}^w(\cdot)$ because, as it will be seen below, this set leads to the best performances. This point is

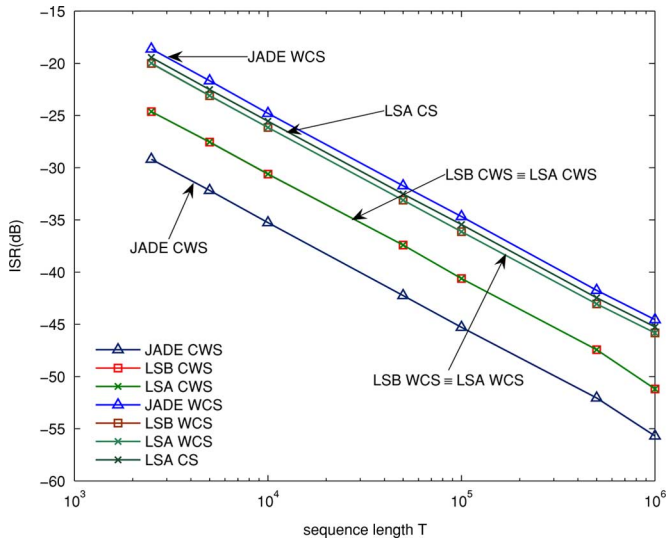

Fig. 7. ISR versus sequence length with mixing matrix $\mathbf{A}_0$, without noise and using $\omega_0 = 1/\sqrt{T}$ for LSA and LSB. Here, the comparison lays on the choice of the set $\mathcal{M}$ of cumulant matrices.

illustrated in Fig. 7 in the noiseless case and using $\omega_0 = 1/\sqrt{T}$ for LSA and LSB. In this experiment, the actual approach, noted CWS for cumulants of whitened signal, is compared with those based on the set $\mathcal{M}$ of empirical cumulant matrices of the signal $\mathbf{X}(\cdot)$. More precisely, this set is defined by $\mathbf{M}_0 = \hat{\mathbf{R}}_{\mathbf{X}}(0)$ and, for $1 \le l \le k \le m$ and $j = (k-1) + l$, $\mathbf{M}_j = \widehat{\mathrm{Cum}}_{k,l}(\mathbf{X})$, with

$$\widehat{\mathrm{Cum}}_{k,l}(\mathbf{X}) = \left[\widehat{\mathrm{Cum}}\{\mathbf{X}_k, \mathbf{X}_l, \mathbf{X}_i, \mathbf{X}_j\}\right]_{i=1,\ldots,m}^{j=1,\ldots,m}$$

where, using obvious notations for elements of the matrix $\hat{\mathbf{R}}_{\mathbf{X}} = \hat{\mathbf{R}}_{\mathbf{X}}(0)$

$$\begin{aligned}\widehat{\mathrm{Cum}}\{\mathbf{X}_k, \mathbf{X}_l, \mathbf{X}_i, \mathbf{X}_j\} &= \frac{1}{T}\sum_{t=1}^{T} \mathbf{X}_k(t)\mathbf{X}_l(t)\mathbf{X}_i(t)\mathbf{X}_j(t) \\ &\quad - \hat{\mathbf{R}}_{\mathbf{X}}(k,l)\hat{\mathbf{R}}_{\mathbf{X}}(i,j) \\ &\quad - \hat{\mathbf{R}}_{\mathbf{X}}(k,i)\hat{\mathbf{R}}_{\mathbf{X}}(l,j) \\ &\quad - \hat{\mathbf{R}}_{\mathbf{X}}(l,i)\hat{\mathbf{R}}_{\mathbf{X}}(k,j).\end{aligned}$$

Methods noted WCS come from the change of $\widehat{\mathrm{Cum}}_{k,l}(\hat{\mathbf{X}}^w)$ in CWS by $\hat{\mathbf{W}}\widehat{\mathrm{Cum}}_{k,l}(\mathbf{X})\hat{\mathbf{W}}^{\mathbf{T}}$. Methods noted CS, for cumulants of signal, are applied directly on the set $\mathcal{M}$, without whitening phase. However, using the invariance property discussed in Section III-A-2, such versions for JADE and LSB are not different from the above WCS approach. This is not true for LSA. It can be seen in Fig. 7 that the recommended approach CWS leads to the best performances for the three criteria. Notice also that JADE is preferable to LSA or LSB in the noiseless case (because $\mathbf{B}_0$ is an exact diagonalizer of $\mathbf{R}_{\mathbf{X}}(0)$). It is clear that JADE WCS is not equivalent to JADE CWS. The comparison between ACDC and JADE reported in [4] corresponds to LSA CS and JADE WCS. It is the reason why our conclusion here differs from those in [4] where ACDC is said to be better than JADE in the noiseless case. Notice that LSA with CWS, WCS,

TABLE I
RATE OF CONVERGENCE AND PERFORMANCE OF JADE, LSB, LSA, AND FFDIAG WITH $\omega_0 = 1/\sqrt{T}$, RANDOM MIXING MATRIX, AND NOISE LEVEL $\mathrm{nl} = -5$ dB

Sequence length *Number of trials*	*Method*	$JADE$	LSB	LSA	$FFDIAG$
$T = 2500$	CPU time	1.8	4.8	15.5	483.7
	ISR	30.91	15.75	20.31	27.94
$r = 20000$	score	-4.13	-4.59	-4.68	-4.56
$T = 5000$	CPU time	1.8	4.7	14.3	216.0
	ISR	38.75	23.31	18.74	31.49
$r = 10000$	score	-4.45	-5.27	-5.58	-5.36
$T = 10000$	CPU time	1.8	4.8	13.9	119.5
	ISR	14.25	16.21	29.75	17.46
$r = 5000$	score	-4.59	-5.72	-6.16	-5.95
$T = 50000$	CPU time	1.9	5.2	15.7	34.9
	ISR	14.85	13.25	37.69	6.03
$r = 1000$	score	-4.68	-6.82	-7.56	-7.31
$T = 100000$	CPU time	1.5	4.9	17.6	61.5
	ISR	-4.06	8.71	-6.99	-5.18
$r = 500$	score	-4.76	-7.66	-8.73	-8.60
$T = 500000$	CPU time	2.1	5.0	23.5	7.4
	ISR	-4.45	7.65	-8.57	-8.01
$r = 100$	score	-4.61	-7.88	-9.55	-9.94
$T = 1000000$	CPU time	1.6	5.8	25.6	6.4
	ISR	-5.10	-10.01	-9.92	-13.44
$r = 50$	score	-4.47	-9.07	-10.14	-11.59

TABLE II
RATE OF CONVERGENCE AND PERFORMANCE OF JADE, LSB, LSA, AND LS AJD WITH $\omega_0 = 1/\sqrt{T}$, RANDOM MIXING MATRIX, AND NOISE LEVEL $\mathrm{nl} = -5$ dB, IN THE CASE OF TWO SOURCES

Sequence length *Number of trials*	*Method*	$JADE$	LSB	LSA	$LS\ AJD$
$T = 100000$	CPU time	1.6	2.0	44.4	2.0
	ISR	-4.35	-4.85	-4.86	-4.17
$r = 500$	score	-9.26	-12.00	-12.13	-12.42
$T = 500000$	CPU time	1.6	2.3	44.7	1.7
	ISR	-10.56	-13.97	-14.12	-9.26
$r = 100$	score	-9.99	-13.58	-13.64	-14.66
$T = 1000000$	CPU time	1.4	2.4	56.4	1.4
	ISR	-10.26	-14.01	-14.08	-25.50
$r = 50$	score	-9.46	-14.83	-15.09	-25.43

and CS can be regarded as different suboptimal solutions of the approach in [16] using generalized weights.

One important fact, in such comparisons, is that some conclusions depend on the mixing matrix and on the performance index. This point is illustrated in Table I, corresponding to the experiment used for Fig 2, but with a random mixing matrix. For each trial, the elements of A_0 are independent random variables with the standard normal distribution. We also consider the index of performance used to the presentation of FFDIAG in [13]

$$\mathrm{score} = \frac{1}{2R}\sum_{r=1}^{R}\left[\sum_{l}\left(\sum_{k}\frac{|\hat{C}_r(l,k)|^2}{\max_i |\hat{C}_r(l,i)|^2} - 1\right)\right] + \frac{1}{2R}\sum_{r=1}^{R}\left[\sum_{l}\left(\sum_{k}\frac{|\hat{C}_r(l,k)|^2}{\max_i |\hat{C}_r(i,k)|^2} - 1\right)\right].$$

Please notice that the score is equal to the ISR if, for each matrix $\hat{C}_r$, we have

$$\max_i |\hat{C}_r(l,i)|^2 = |\hat{C}_r(l,i_l)|^2 = \max_i |\hat{C}_r(i,i_l)|^2, \quad l = 1,\ldots,m.$$

This will be the case if $\hat{C}_r$ is close to a matrix of the form $\mathbf{\Pi\Delta}\mathcal{E}$, as expected. Here, the ISR index is not relevant because its values are very different from one set of trials to another. On the other hand, the score values (also in dB) are stable and slightly better for LSA than for LSB. The mean CPU time (in milliseconds) of each trial shows that LSB is clearly faster than LSA. Notice that the CPU time of JADE must be added to those of LSA and LSB, since these algorithms use the output of JADE as starting value.

Table I gives also the results of FFDIAG. This algorithm minimizes $\mathcal{C}_{\mathrm{lsb}}(\cdot)$ using a Newton-like method. Here, it appears to be preferable to LSA and LSB when $T \geq 500\,000$.

We have seen that the performances of LSA and LSB are similar in order to improve those of JADE or SOBI. In fact, the difference between the two methods comes from the implementation of the two algorithms. In LSB, the estimation of $\mathcal{D}$ is eliminated and the computation of $\hat{\tilde{\mathbf{b}}}_l$ is easier than the one of $\hat{\mathbf{a}}_k$. So, LSB is faster than LSA. However, the algorithm LS AJD in [18], using exact joint diagonalization for minimizing $\mathcal{C}_{\mathrm{lsa}}(\cdot)$, is faster, with better score results, in the case of two sources when T is sufficiently large (see Table II).

The author proposes to use the output of this algorithm as an initial guess when $m > 2$. This point is illustrated in Tables III

TABLE III
RATE OF CONVERGENCE AND PERFORMANCE OF JADE, LSB, LSA, AND LS AJD WITH $\omega_0 = 1/\sqrt{T}$, RANDOM MIXING MATRIX, AND NOISE LEVEL $\mathrm{nl} = -5$ dB, IN THE CASE OF THREE SOURCES

Method	*Sequence length* / *Number of trials*	$T = 10^5$ $r = 500$	$T = 5 \times 10^5$ $r = 100$	$T = 10^6$ $r = 50$
$JADE$	CPU time	1.6	3.8	2.5
	ISR	-5.73	-5.53	-5.60
	score	-4.58	-5.86	-5.17
LSB_{jade}	CPU time	5.1	4.7	5.1
	ISR	-7.00	24.63	27.39
	score	-8.62	-7.46	-8.62
LSA_{jade}	CPU time	150.7	235.9	305.0
	ISR	9.97	8.45	4.62
	score	-7.27	-9.23	-7.66
$LS\ AJD$	CPU time	2.1	1.8	2.8
	ISR	-0.93	8.44	-2.71
	score	-4.52	-5.14	-6.28
LSB_{ajd}	CPU time	3.7	4.7	3.1
	ISR	3.46	18.73	-3.04
	score	-7.28	-6.27	-5.61
LSA_{ajd}	CPU time	120.7	153.0	131.6
	ISR	-10.37	-8.89	-10.25
	score	-9.65	-9.42	-9.58

TABLE IV
RATE OF CONVERGENCE AND PERFORMANCE OF JADE, LSB, LSA, AND LS AJD WITH $\omega_0 = 1/\sqrt{T}$, RANDOM MIXING MATRIX, AND NOISE LEVEL $\mathrm{nl} = -5$ dB, IN THE CASE OF THREE SOURCES

Method	*Sequence length* / *Number of trials*	$T = 10^5$ $r = 500$	$T = 5 \times 10^5$ $r = 100$	$T = 10^6$ $r = 50$
$JADE$	CPU time	2.2	2.0	2.2
	ISR	1.42	-1.63	-6.37
	score	-5.02	-4.39	-5.37
LSB_{jade}	CPU time	5.0	4.7	4.1
	ISR	-4.29	-2.67	-0.49
	score	-7.89	-7.71	-8.97
LSA_{jade}	CPU time	122.6	193.9	196.0
	ISR	0.53	3.27	12.54
	score	-7.91	-7.53	-9.56
$LS\ AJD$	CPU time	2.6	1.9	2.2
	ISR	14.17	2.16	-8.84
	score	-4.89	-7.80	-8.82
LSB_{ajd}	CPU time	4.8	3.9	4.9
	ISR	-3.21	-2.66	-10.55
	score	-8.29	-7.72	-9.24
LSA_{ajd}	CPU time	94.9	121.5	126.4
	ISR	0.12	-1.94	-11.19
	score	-8.30	-8.56	-10.29

and IV, where the subscript jade or ajd in LSB and LSA specifies the output used as starting value. We observe that LS AJD alone is as fast as JADE and can provide better score results when $T \geq$ 500 000. As a starting value, the output of LS AJD is preferable to the one of JADE for LSA, but not for LSB. The large variation of the score from one table to the other shows that this measure strongly depends on the mixing matrix. Additional work would be necessary to clarify this point.

V. CONCLUSION

In this paper, a comparative study of approximate joint diagonalization algorithms for blind source separation in presence of additive noise has been presented, with a special emphasis on a nonorthogonal approximate joint diagonalization algorithm. It is an iterative algorithm, using alternating direction type technique, based on a weighted least-squares criterion applied to the separating matrix. Convergence property to a stationary point of the criterion is proved under mild assumptions. Simulation results show that this algorithm leads to similar performances but is faster than the ACDC algorithm using the same criterion applied on the mixing matrix. The conditions in which these algorithms could improve the performances of the usual JADE and SOBI algorithms are clarified. In our experiments, we have observed that this can be done only in presence of noise, when the length of observed sequences is sufficiently large, and with a mixing matrix not close to an orthogonal matrix.

ACKNOWLEDGMENT

The authors would like to thank the reviewers for their constructive comments and suggestions and for pointing out some useful references.

REFERENCES

[1] J.-F. Cardoso and A. Souloumiac, "Blind beamforming for non gaussian signals," *Proc. Inst. Elect Eng. F*, vol. 140, no. 6, pp. 362–370, Dec. 1993.

[2] A. Belouchrani, K. Abed-Meriam, J.-F. Cardoso, and E. Moulines, "A blind source separation technique using second-order statistics," *IEEE Trans. Signal Process.*, vol. SP-45, no. 2, pp. 434–444, Feb. 1997.
[3] M. Wax and J. Sheinvald, "A least-squares approach to joint diagonalization," *IEEE Signal Process. Lett.*, vol. 4, no. 2, pp. 52–53, Feb. 1997.
[4] A. Yeredor, "Non-orthogonal joint diagonalization in the least-squares sense with application in blind source separation," *IEEE Trans. Signal Process.*, vol. SP-50, no. 7, pp. 1545–1553, Jul. 2002.
[5] A. Yeredor, "Approximate joint diagonalization using non-orthogonal matrices," in *Proc. ICA*, Helsinki, Finland, Jun. 2000, pp. 33–38.
[6] S. Dégerine, "Sur la diagonalisation conjointe approchée par un critère des moindres carrés," presented at the GRETSI Conf., Toulouse, France, Sep. 2001.
[7] S. Dégerine and A. Zaïdi, "Separation of an instantaneous mixture of Gaussian autoregressive sources by the exact maximum likelihood approach," *IEEE Trans. Signal Process.*, vol. SP-52, no. 6, pp. 1499–1512, Jun. 2004.
[8] D. T. Pham, "Blind separation of instantaneous mixture of sources via the Gaussian mutual information criterion," *Signal Process.*, vol. 81, no. 9, pp. 850–870, 2001.
[9] J.-F. Cardoso and A. Souloumiac, "Blind beamforming for non gaussian signals," *SIAM J. Matrix Anal. Appl.*, vol. 17, no. 1, pp. 161–164, Jan. 1996.
[10] D. T. Pham, "Joint approximate diagonalization of positive definite hermitian matrices," *SIAM J. Matrix Anal. Appl.*, vol. 22, no. 4, pp. 1136–1152, 2001.
[11] A.-J. van der Veen, "Joint diagonalization via subspace fitting techniques," in *Proc. Int. Conf. Acoustics, Speech, Signal Processing*, Salt Lake City, UT, May 2001, vol. 5, pp. 2773–2776.
[12] M. Joho and H. Mathis, "Joint diagonalization of correlation matrices by using gradient methods with application to blind signal separation," in *Proc. SAM*, Rosslyn, VA, Aug. 2002, pp. 273–277.
[13] A. Ziehe, P. Laskov, G. Nolte, and K.-R. Múller, "A fast algorithm for joint diagonalization with non-orthogonal transformations and its application to blind source separation," *J. Mach. Learn. Res.*, vol. 5, pp. 777–800, 2004.
[14] R. Vollgraf and K. Obermayer, "Quadratic optimization for simultaneous matrix diagonalization," *IEEE Trans. Signal Process.*, vol. SP-54, no. 9, pp. 3270–3278, Sep. 2006.
[15] A. M. Ostrowski, *Solution of Equations in Euclidean and Banach Spaces*, 3rd ed. New York: Academic, 1973, p. 173.
[16] A. Smekhov and A. Yeredor, "Optimization of JADE using a novel optimally weighted joint diagonalization approach," in *Proc. EUSIPCO*, 2004, pp. 221–224.
[17] J.-F. Cardoso, "On the performance of orthogonal source separation algorithms," in *Proc. EUSIPCO*, 1994, pp. 776–779.
[18] A. Yeredor, "On using exact joint diagonalization for noniterative approximate joint diagonalization," *IEEE Signal Process. Lett.*, vol. 12, no. 9, pp. 645–648, Sep. 2005.

Serge Dégerine was born in Bonneville, France, in 1948. He received the Ph.D. degree in applied mathematics and the Doctorat es Sciences degree from the Joseph Fourier University, Grenoble, France, in 1975 and 1988, respectively.

From 1975 to 1994, he was an Assistant Professor with the Pierre Mendès-France University, Grenoble. Since 1994, he has been a Professor at the Joseph Fourier University. At first, his research interests were on statistics for directional data. For several years, he has been working on time series analysis, with a special emphasis on partial autocorrelation function and applications to signal processing.

Elimane Kane was born in Pikine, Sénégal, in 1976. He received the M.Sc. degree in applied mathematics degree from Joseph Fourier University, Grenoble, France, in 2001, where he is currently pursuing the Ph.D. degree in applied mathematics on the study of the noise effect in blind source separation.

His area of research is statistics with applications to signal processing.