1. What is surprising to me is that on page 641, there's a $.75 decline in the equity for a change in the stock in either direction. My intuition was that the two directions would result in a decline and an increase. Also, the fact that the expected return of the stock is not included in the pricing formula didn't make intuitive sense to me. I believe the conclusions reached under so many assumptions, and the part I don't believe in is its practical applicability to the actual market in that there's extensive assumptions made that are not realistic in the real world. However, their mathematics seems right.
2. The idea of the paper is that continuous-time markets don't actually work in continuous time because correlations completely break down, that there's mechanical arbitrages and that it will create a never-ending arms race. The pictures surprise me in that the market correlations visually resemble each other so much at a frequency of an hour, but when the frequency is adjusted to 1 minute, which is not a small interval, the market correlations already have obvious visualizable differences. We use asymptotic approximation to reach a lot of our conclusions in continuous finance but when dt approaches 0, the theories break down. I don't think time is really continuous in markets, because even with timestamp, there's a limit to the unit of time we can use, such as a nanosecond, but we can never truly achieve continuous-time trading.
3.

```python
from scipy.stats import norm
import numpy
import math

# Process parameters
dt = 1/252

# Initial condition.
x = 0.0

# Number of iterations to compute.
n = 5*252

my_result = []

N = 10
# Iterate to compute the steps of the Brownian motion.
for i in range(0,N):
    x = 0.0
    for k in range(n):
        x = x + norm.rvs(scale=math.sqrt(dt))
    my_result.append(x)
print(numpy.mean(my_result))
print(numpy.var(my_result))
```

```
0.8294425660591415
7.152973488994104
```

```python
from scipy.stats import norm
import numpy

# Process parameters
dt = 1/252

# Initial condition.
x = 0.0

# Number of iterations to compute.
n = 5*252

my_result = []

N = 100
# Iterate to compute the steps of the Brownian motion.
for i in range(0,N):
    x = 0.0
    for k in range(n):
        x = x + norm.rvs(scale=math.sqrt(dt))
    my_result.append(x)
print(numpy.mean(my_result))
print(numpy.var(my_result))
```

```
-0.07749049721604062
4.748768294586135
```

```
from scipy.stats import norm
import numpy

# Process parameters
dt = 1/252

# Initial condition.
x = 0.0

# Number of iterations to compute.
n = 5*252

my_result = []

N = 1000
# Iterate to compute the steps of the Brownian motion.
for i in range(0,N):
    x = 0.0
    for k in range(n):
        x = x + norm.rvs(scale=math.sqrt(dt))
    my_result.append(x)
print(numpy.mean(my_result))
print(numpy.var(my_result))
```

```
-0.0102863304098857
5.1471781400403005
```

The results are fairly close to the mean being 0 and the variance being t=5.
I can also say that the more simulations I perform, the closer the results are to 0 and 5 respectively.

4. (a). $\Delta B_i \sim N(0, t_{i+1} - t_i)$

so, $\mathbb{E}(\Delta B_i^2) = Var(\Delta B_i) = \Delta t_i$

$\mathbb{E}(\Delta B_i^4) = 3\, var(\Delta B_i)^2 = 3 \cdot \Delta t_i^2$.

Therefore, $LHS = \mathbb{E}\left[(\Delta B_i)^4 - 2(\Delta B_i)^2 \Delta t_i + \Delta t_i^2 \,\middle|\, \mathcal{F}_{t_i}\right]$

$= 3(\Delta t_i)^2 - 2(\Delta t_i)^2 + (\Delta t_i)^2$.

$= 2(\Delta t_i)^2 = RHS.$ $\square$

(b). $LHS = \mathbb{E} \sum_{0 \le i,j \le n-1} g(B_{t_i}) \cdot [\Delta B_i^2 - \Delta t_i] \cdot g(B_{t_j}) \cdot [\Delta B_j^2 - \Delta t_j]$.

when $i \neq j$: WLOG, we can assume $i < j$.

By law of total expectation, and conditioning on $\mathcal{F}_j$. $(\mathcal{F}_i \subset \mathcal{F}_j)$

we get $\mathbb{E}\, g(B_{t_i}) \cdot g(B_{t_j}) \cdot [\Delta B_i^2 - \Delta t_i] \cdot [\Delta B_j^2 - \Delta t_j] = 0$.

as from (a) we have $\mathbb{E}\Delta B_j^2 = \Delta t_j$, and when conditioned on $\mathcal{F}_j$,

all other terms are known except ΔB_j and Δt_j

So, $LHS = \mathbb{E} \sum_{i=0}^{n-1} g(B_{t_i})^2 \cdot [\Delta B_i^2 - \Delta t_i]^2 = RHS$ $\square$

5. (a). $f(x,t) = x$. $f_x(x,t) = 1$. $f_{xx}(x,t) = 0$, $f_t(x,t) = 0$. $f(X_t) = X_t$.

integral form: $X_t = X_0 + \int_0^t 1 \cdot X_s \, dB_s = X_0 + \int_0^t X_s \, dB_s$

diff form: $dX_t = X_t \, dB_t$.

(b). $g(X_t) = X_t^2$. $\mu(s) = 0$. , $\sigma(s) = X_s$.

$g_x = 2x$ $g_{xx} = 2$. $g_t = 0$

integral form: $X_t^2 = X_0^2 + \int_0^t \cancel{\tfrac{1}{2}} \cdot X_s^2 \cdot \cancel{2} \cdot ds + \int_0^t X_s \cdot 2X_s \cdot dB_s$.

$X_t^2 = X_0^2 + \int_0^t X_s^2 ds + \int_0^t 2X_s^2 dB_s$.

diff form: $d(X_t^2) = \frac{1}{2} X_t^2 \cdot 2 \cdot dt + X_t 2 X_t dB_t$

$d(X_t^2) = X_t^2 dt + 2X_t^2 dB_t$

(c). $g(X_t) = e^{X_t}$, $g_x = e^x$. $g_{xx} = e^x$. $g_t = 0$

integral form: $e^{X_t} = e^{X_0} + \int_0^t \frac{1}{2} \cdot X_s^2 \cdot e^{X_s} ds + \int_0^t X_s e^{X_s} dB_s$.

diff form: $d(e^{X_t}) = \frac{1}{2} X_t^2 e^{X_t} dt + X_s e^{X_t} dB_t$

5 (d) $g(X_t) = \log(X_t)$. $g_x = \frac{1}{x}$: $g_{xx} = -\frac{1}{x^2}$. $g_t = 0$ $\sigma(s) = X_s$.

integral form: $\log(X_t) = \log(X_0) - \int_0^t \frac{1}{2} \cdot \cancel{X_s^2} \frac{1}{\cancel{X_s^2}} ds + \int_0^t X_s \cdot \frac{1}{X_s} dB_s$

$\log(X_t) = \log(X_0) - \frac{t}{2} + \int_0^t 1\, dB_s$

diff form: $d(\log(X_t)) = \frac{1}{2} X_t^2 (-\frac{1}{X_t^2}) dt + X_s \cdot \frac{1}{X_s} dB_t$

$= -\frac{1}{2} dt + dB_t$.

(e). $g(X_t, t) = e^{-rt} X_t$. $g_x = e^{-rt}$. $g_{xx} = 0$. $g_t = -e^{-rt} \cdot x r$

integral form: $e^{-rt} X_t = X_0 + \int_0^t -X_s r \cdot e^{-rs} ds + \int_0^t X_s \cdot e^{-rs} dB_s$

diff form: $d(e^{-rt} X_t) = -e^{-rt} X_t \cdot r\, dt + X_t e^{-rt} dB_t$.

6. $g(B_t, t) = B_t^3$. $g_t = 0$ $g_x = 3x^2$. $g_{xx} = 6x$.

$B_t^3 = B_0^3 + \int_0^t 3 \cdot B_s^2 dB_s + \int_0^t 3 \cdot B_s\, ds$

$= \int_0^t 3 \cdot B_s^2 dB_s + \int_0^t 3 B_s ds$

7. let $Y_t = \int_0^t s \cdot dB_s$. $g(s) = s$. is bdd and piecewise continuous on $[0,t]$.

and we partition $[0,t]$ by $0 \le t_0 \le t_1 \le \dots \le t_n = t$.

a LH Riemann Sum becomes $\sum_{j=1}^n g(t_{j-1})(B_{t_j} - B_{t_{j-1}})$

$= \sum_{j=1}^n t_{j-1} (B_{t_j} - B_{t_{j-1}}) = I_t^{(n)}$. (notation)

We know that $B_{t_j} - B_{t_{j-1}} \sim N(0, t_j - t_{j-1})$.

$\mathbb{E}(I_t^{(n)}) = 0$. and since the increments of BM are independent, we get

$var(I_t^{(n)}) = \sum_{j=1}^n t_{j-1}^2 \mathbb{E}(B_{t_j} - B_{t_{j-1}})^2 = \sum_{j=1}^n t_{j-1}^2 \cdot var(B_{t_j} - B_{t_{j-1}})$.

$= \sum_{j=1}^n t_{j-1}^2 (t_j - t_{j-1}) \to \int_0^t s^2 ds = \frac{t^3}{3}$.

$\lim_{n\to\infty} \mathbb{E}(I_t^{(n)}) = 0$ $\lim_{n\to\infty} var(I_t^{(n)}) = \frac{t^3}{3}$.

construct $I_t = \lim_{n\to\infty} I_t^{(n)}$. then we know $\mathbb{E}(I_t) = 0$ and $var(I_t) = \frac{t^3}{3}$

since disjoint Brownian increments are indep. and normally distributed and that $I_t^{(n)}$ is a sum of disjoint Brownian increments, $I_t^{(n)}$ is normally distributed.

So. $I_t^{(n)} \sim N(0, \sum_{j=1}^n t_{j-1}^2 (t_j - t_{j-1}))$

7. (cont'd) $I_t^{(n)}$ converges in distr to I_t.

so $I_t \sim N(0, \frac{t^3}{3})$

so $\int_0^t s\,dB_s \sim N(0, \frac{t^3}{3})$.

so $\int_0^t s\,dB_s$ is a normal distr. with mean 0 and variance $\frac{t^3}{3}$.

8. a. $B_t - B_0 = B_t - 0 = B_t$ is normally distr. $\sim N(0, t)$.

$\sigma B_t \sim N(0, \sigma^2 t)$.

so $X_t \sim N(\mu t, \sigma^2 t)$

mean is μt. variance is $\sigma^2 t$.

b. $P(t,x,y) = \mathbb{P}(X_t = y \mid X_0 = x)$.

$= \mathbb{P}(\sigma(B_t - B_0) = y - x - \mu t)$. since $\sigma(B_t - B_0) \sim N(0, \sigma^2 t)$.

$$= \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(y-x-\mu t)^2}{2\sigma^2 t}}.$$

verify "Backward"

$$\frac{\partial p}{\partial t} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-x-\mu t)^2}{2\sigma^2 t}} \left[-\frac{1}{2} t^{-\frac{3}{2}} + \frac{1}{\sqrt{t}} \left(\frac{(y-x)^2}{2\sigma^2 t^2} - \frac{\mu^2}{2\sigma^2} \right) \right]$$

$$\frac{\partial p}{\partial x} = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y-x-\mu t)^2}{2\sigma^2 t}} \frac{1}{\sqrt{t}} \cdot \frac{(y-x-\mu t)}{\sigma^2 t}$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-x-\mu t)^2}{2\sigma^2 t}} \frac{1}{\sqrt{t}} \left[\frac{(y-x-\mu t)^2}{\sigma^4 t^2} - \frac{1}{\sigma^2 t} \right]$$

so it suffices to show.

$$-\frac{1}{2t} + \frac{(y-x)^2}{2\sigma^2 t^2} - \frac{\mu^2}{2\sigma^2} = \frac{(y-x-\mu t)\mu}{\sigma^2 t} + \frac{1}{2} \left[\frac{(y-x-\mu t)^2}{\sigma^2 t^2} - \frac{1}{t} \right]$$

$$LHS - RHS = \frac{\cancel{(y-x)^2}}{2\sigma^2 t^2} - \frac{\mu^2}{2\sigma^2} - \frac{(y-x)\mu - \mu^2 t}{\sigma^2 t} - \frac{\cancel{(y-x)^2} - 2(y-x)\mu t + \mu^2 t^2}{2\sigma^2 t^2}$$

$$= -\frac{\mu^2}{2\sigma^2} - \frac{(y-x)\mu}{\sigma^2 t} + \frac{\mu^2}{\sigma^2} + \frac{(y-x)\mu}{\sigma^2 t} - \frac{\mu^2}{2\sigma^2} = 0. \quad \square$$

verify "Forward":

$$\frac{\partial p}{\partial y} = \frac{1}{\sqrt{2\pi\sigma^2 t}} \cdot e^{-\frac{(y-x-\mu t)^2}{2\sigma^2 t}} \left[-\frac{y-x-\mu t}{\sigma^2 t}\right]$$

$$\frac{\partial^2 p}{\partial y^2} = \frac{1}{\sqrt{2\pi\sigma^2 t}} \cdot e^{-\frac{(y-x-\mu t)^2}{2\sigma^2 t}} \left[\frac{(y-x-\mu t)^2}{\sigma^4 t^2} - \frac{1}{\sigma^2 t}\right]$$

so it suffices to show.

$$-\cancel{\frac{1}{2t}} + \frac{(y-x)^2}{2\sigma^2 t^2} - \frac{\mu^2}{2\sigma^2} = \frac{\mu(y-x-\mu t)}{\sigma^2 t} + \frac{(y-x-\mu t)^2}{2\sigma^2 t^2} - \cancel{\frac{1}{2t}}$$

$$(LHS-RHS)\, 2\sigma^2 t^2 = (y-x)^2 - \mu^2 t^2 - 2\mu(y-x-\mu t)t - (y-x)^2 - \mu^2 t^2 + 2\mu t(y-x)$$

$$= 0 \quad \square$$

9. $$\frac{\partial d_1}{\partial t} = \frac{\log(\frac{S}{K})}{2\sigma}(T-t)^{-\frac{3}{2}} + \left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)\cdot\frac{1}{2}\cdot(T-t)^{-\frac{1}{2}}$$

$$= \frac{d_1}{2(T-t)} - \frac{1}{\sqrt{T-t}}\cdot\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)$$

$$\frac{\partial d_2}{\partial t} = \frac{d_2}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)$$

N' is the standard normal distr. so $N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}}$$

$$\frac{\partial C}{\partial t} = S\cdot N'(d_1)\cdot\frac{\partial d_1}{\partial t} - Ke^{-r(T-t)}\left[r\cdot N(d_2) + N'(d_2)\frac{\partial d_2}{\partial t}\right]$$

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

$$\frac{\partial C}{\partial S} = N(d_1) + \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{1}{\sigma\sqrt{T-t}} - Ke^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{1}{S\sigma\sqrt{T-t}}$$

$$\frac{\partial^2 C}{\partial S^2} = N'(d_1)\frac{\partial d_1}{\partial S} + \frac{1}{\sqrt{2\pi}\cdot\sigma\sqrt{T-t}}\cdot e^{-\frac{d_1^2}{2}}\cdot(-d_1)\frac{\partial d_1}{\partial S} + Ke^{-r(T-t)}\cdot\frac{1}{\sqrt{2\pi}\, S^2\sigma\sqrt{T-t}}\cdot$$

$$\left[e^{-\frac{d_2^2}{2}}\cdot d_2\cdot\frac{\partial d_2}{\partial S}\cdot S + e^{-\frac{d_2^2}{2}}\right]$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}\cdot\frac{1}{S\sigma\sqrt{T-t}} - \frac{d_1 e^{-d_1^2/2}}{S\sqrt{2\pi}\,\sigma^2(T-t)} + Ke^{-r(T-t)} e^{-\frac{d_2^2}{2}}\cdot$$

$$\left[\frac{d_2}{\sqrt{2\pi}\, S^2\sigma^2(T-t)} + \frac{1}{\sqrt{2\pi}\cdot S^2\sigma\sqrt{T-t}}\right]$$

$$\text{LHS} = \cancel{S\cdot\frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left[\frac{d_1}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)\right]} - Ke^{-r(T-t)}\left[r\cdot N(d_2)\right.$$

$$\left. + \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}}\left(\frac{d_2}{2(T-t)} - \frac{1}{\sqrt{T-t}}\cdot\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\right)\right] + r\cdot S\cdot N(d_1)\cdot$$

$$+ \cancel{e^{-\frac{d_1^2}{2}}\frac{rS}{\sigma\sqrt{2\pi}\sqrt{T-t}}} - \frac{rKe^{-r(T-t)}\, e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}\cdot\sigma\sqrt{T-t}} + \cancel{e^{-\frac{d_1^2}{2}}\frac{S\sigma}{\sqrt{2\pi}\sqrt{T-t}\, 2}}$$

$$\cancel{-\frac{d_1 e^{-\frac{d_1^2}{2}} S}{\sqrt{2\pi}(T-t)2}} + Ke^{-r(T-t)} e^{-\frac{d_2^2}{2}}\left[\frac{d_2}{\sqrt{2\pi}\, 2\cdot(T-t)} + \frac{\sigma}{\sqrt{2\pi}\, 2\cdot\sqrt{T-t}}\right]$$

$$= r\cdot S\cdot N(d_1) + Ke^{-r(T-t)}\left[-rN(d_2)\right] + Ke^{-r(T-t)}\cdot e^{-\frac{d_2^2}{2}}\left[\cancel{-\frac{1}{\sqrt{2\pi}}\frac{d_2}{2(T-t)}} + \cancel{\frac{1}{\sqrt{2\pi}\sqrt{T-t}}\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)}\right.$$

$$\left.\cancel{-\frac{r}{\sqrt{2\pi}\,\sigma\sqrt{T-t}}} + \cancel{\frac{d_2}{\sqrt{2\pi}\, 2(T-t)}} + \cancel{\frac{\sigma}{\sqrt{2\pi}\, 2\sqrt{T-t}}}\right] = r\cdot S N(d_1) - r\cdot Ke^{-r(T-t)} N(d_2) = \text{RHS} \quad \square$$

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