exists a self-financin

Continuous-time Finance · HW 2  Lisa He

1. $dS_t = \mu S_t\, dt + \sigma S_t\, dB_t$. $\mu(t) = \mu S_t$ . $\sigma(t) = \sigma S_t$

(a). $g_x = e^{-rt}$. $g_{xx} = 0$. $g_s = S_t \cdot e^{-rt} \cdot (-r)$.

$df(S,t) = [-rS_t e^{-rt} + e^{-rt} \mu S_t] dt + \sigma S_t e^{-rt} dB_t$.

(b). $df(S,t) = [-\mu S_t e^{-\mu t} + e^{-\mu t} \mu S_t] dt + \sigma S_t e^{-\mu t} dB_t$

$= \sigma S_t e^{-\mu t} dB_t$.

(c). $g_x = \frac{1}{S_t}$. $g_{xx} = -\frac{1}{S_t^2}$. $g_t = 0$.

$df(S,t) = [\frac{1}{S_t} \cdot \mu S_t + \frac{1}{2}\sigma^2 \cdot S_t^2 \cdot -\frac{1}{S_t^2}] dt + \sigma S_t \cdot \frac{1}{S_t} \cdot dB_t$

$= (\mu - \frac{1}{2}\sigma^2) dt + \sigma dB_t$

2. $dX_t = -X_t \theta_t dB_t$. $X_0 = 1$.

Guess. $\log(X_t) = f(t, B_t)$

$d\log(X_t) = \frac{1}{2} \cdot \theta_t^2 \cancel{X_t^2} \cdot -\frac{1}{\cancel{X_t^2}} dt + -\cancel{X_t}\theta_t\, dB_t$

$= -\theta_t dB_t - \frac{1}{2}\theta_t^2 dt$.

integrate.

$\log(X_t) = 0 + \int_0^T -\theta_t dB_t + \int_0^T -\frac{1}{2}\theta_t^2 dt$.

$X_t = \exp \cdot \{ -\int_0^T \theta_t dB_t - \int_0^T \frac{1}{2}\theta_t^2 dt \}$.

$-\int_0^T \theta_t dB_t \sim N(0, \int_0^T \theta_t^2 dt)$

so $\mathbb{E}(X_t) = \exp\{-\int_0^T \frac{1}{2}\theta_t^2 dt\} \cdot \exp \cdot \{\frac{1}{2}\int_0^T \theta_t^2 dt\}$.

$= 1$.

state space is $\mathbb{R}^+$.

3. $dX_t = (\alpha + \beta X_t)\,dt + \sigma\,dB_t$

$= \beta\left(\frac{\alpha}{\beta} + X_t\right)dt + \sigma\,dB_t$.

define. $Y_t = \frac{\alpha}{\beta} + X_t$. $dY_t = dX_t$

$dY_t = \beta \cdot Y_t\,dt + \sigma \cdot dB_t$ $\quad \mu = \beta Y_t$ ; $\sigma = \sigma$.

By Ito, $d(e^{-\beta t} Y_t) = \left[\cancel{-\beta \cdot e^{-\beta t} \cdot Y_t + e^{-\beta t}\beta \cdot Y_t}\right]dt + \sigma \cdot e^{-\beta t}\,dB_t$

$= \sigma e^{-\beta t}\,dB_t$.

$e^{-\beta t} Y_t = Y_0 + \int_0^t \sigma e^{-\beta s}\,dB_s$

$Y_t = e^{\beta t} Y_0 + \sigma \int_0^t e^{\beta(t-s)}\,dB_s$.

$\frac{\alpha}{\beta} + X_t = e^{\beta t}\left(\frac{\alpha}{\beta} + X_0\right) + \sigma \int_0^t e^{\beta(t-s)}\,dB_s$

$X_t = \frac{\alpha}{\beta}(e^{\beta t} - 1) + e^{\beta t} X_0 + \sigma \int_0^t e^{\beta(t-s)}\,dB_s$.

The differential for $e^{-\beta t} X_t$ is as follows:

$dX_t = (\alpha + \beta X_t)\,dt + \sigma\,dB_t$

$\mu = \alpha + \beta X_t$. $\sigma = \sigma$.

$d f(x,t) = \left[-\beta \cdot e^{-\beta t} X_t + e^{-\beta t}(\alpha + \beta X_t)\right]dt + \sigma \cdot e^{-\beta t}\,dB_t$

$= \alpha e^{-\beta t} X_t\,dt + \sigma e^{-\beta t}\,dB_t$.

4.(a). Suppose $f(t,x) = e^{x}$.

$f_t = 0 \quad f_x = e^x \quad f_{xx} = e^x$.

By. Feynman-Kac.

$$f_t + f_x \mu + \frac{1}{2}\sigma^2 f_{xx} = 0 + \mu e^x + \frac{1}{2}\sigma^2 e^x = (\mu + \frac{1}{2}\sigma^2) e^x.$$

$$r(x,t) = \mu + \frac{1}{2}\sigma^2$$

$$f(x,T) = g(x) = e^x.$$

then $f(t,x) = e^x = \mathbb{E}_t^{\mathbb{P}}[e^{-\int_t^T r(s)ds} \cdot g(X_T) | X_t = x]$

$$e^x \cdot e^{(\mu+\frac{1}{2}\sigma^2)(T-t)} = \mathbb{E}_t^{\mathbb{P}}[e^{X_T} | X_t = x]$$

(b) Guess $f(t,x) = \exp\{x + A(t,T)\}$.

By Dynkin's Formula,

$$f \cdot \frac{\partial A}{\partial t} + f \cdot \mu + \frac{1}{2}\sigma^2 f = 0.$$

$$\frac{\partial A}{\partial t} = -\mu - \frac{1}{2}\sigma^2$$

$$A(t,T) = (\mu + \frac{1}{2}\sigma^2)(T-t).$$

so $f(t,x) = \exp\{x + (\mu + \frac{1}{2}\sigma^2)(T-t)\}$

so the PDE is. $df(t,X_t) = \sigma \cdot \exp\{x + (\mu+\frac{1}{2}\sigma^2)(T-t)\} dB_t$.

$dX_t = \mu dt + \sigma dB_t$ boundary condition: $f(T,x) = g(x) = e^x$.

4.(c) Guess $f(t,x) = x + A(t,T)$.

$$\frac{\partial A}{\partial t} = -\mu. \quad A(t,T) = \mu(T-t) \qquad dX_t = \mu\, dt + \sigma\, dB_t.$$

$$f(t,x) = x + \mu\cdot(T-t)$$

PDE is $df(t,x) = \sigma\, dB_t$.

boundary condition $f(T,x) = g(x) = x$.

5. (a). $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$.

$$X_t = x + \int_0^t \mu(X_t)dt + \int_0^t \sigma(X_t)dB_t.$$

$$\mathbb{E}(X_t) = x + \mathbb{E}\int_0^t \mu(X_t)dt + 0 \quad \text{since } \mathbb{E}dB_t = 0.$$

$$\frac{d}{d\tau}\mathbb{E}_t(X_\tau) = \lim_{\tau\to t}\mathbb{E}\frac{X_\tau - X_t}{\tau - t} = \lim_{\tau\to t}\mathbb{E}\frac{\int_t^\tau \mu(X_t)dt + \int_t^\tau \sigma(X_t)dB_t}{\tau - t}$$

$$= \lim_{\tau\to t}\mathbb{E}\frac{\int_t^\tau \mu(X_t)dt}{\tau - t} + \lim_{\tau\to t}\mathbb{E}\frac{\int_t^\tau \sigma(X_t)dB_t}{\tau - t}.$$

$= \mathbb{E}\mu(X_\tau) + 0$ since $\mathbb{E}(dB_t) = 0$.

$= \mathbb{E}\mu(X_\tau)$

So $\frac{d}{d\tau}\mathbb{E}_t(X_\tau)\Big|_{\tau=t} = \mathbb{E}\mu(X_t) = \mu_t$ w/ probability 1.

(b).

$$\mathbb{E}(X_t) = x + \mathbb{E}\int_0^t \mu(X_t)dt.$$

$$[\mathbb{E}(X_t)]^2 = x^2 + 2x\,\mathbb{E}\int_0^t \mu(X_t)dt.$$

$$X_t^2 = x^2 + \left[\int_0^t \sigma(X_t)dB_t\right]^2 + 2x\int_0^t \mu(X_t)dt + 2x\int_0^t \sigma(X_t)dB_t.$$

$$\mathbb{E}(X_t^2) = x^2 + \mathbb{E}\int_0^t \sigma^2(X_t)dt + 2x\,\mathbb{E}\int_0^t \mu(X_t)dt$$

$$\mathbb{E}(X_t^2) - [\mathbb{E}(X_t)]^2 = \mathbb{E}\int_0^t \sigma^2(X_t)dt.$$

$$\frac{d}{d\tau}\mathrm{var}_t(X_\tau) = \lim_{\tau\to t}\frac{\mathbb{E}(X_\tau)^2 - [\mathbb{E}(X_\tau)]^2 - \mathbb{E}(X_t)^2 + [\mathbb{E}(X_t)]^2}{\tau - t}$$

$$= \lim_{\tau\to t}\frac{\mathbb{E}\int_t^\tau \sigma^2(X_t)dt}{\tau - t} = \mathbb{E}\sigma^2(X_\tau).$$

So $\frac{d}{d\tau}\mathrm{var}_t(X_\tau)\Big|_{\tau=t} = \mathbb{E}\sigma^2(X_t) = \sigma_t^2$ w/ probability 1.

regularization condition: assume that $\mathbb{E}\left[\left(\int_0^t \sigma_t^2 dt\right)^{1/2}\right] < \infty$. so $\int\sigma(X_t)dB_t$ is a martingale

so. $\mathrm{var}\left(\int_0^t \sigma(X_t)dB_t\right) = \mathbb{E}\left(\int_0^t \sigma^2(X_t)dt\right)$. by Prop 5B in Duffie

$\mathrm{var}(X_t) = \mathrm{var}\left(\int_0^t \sigma(X_t)dB_t\right)$. so $\frac{d}{d\tau}\mathrm{var}_t(X_\tau) = \lim_{\tau\to t}\frac{\mathbb{E}\int_t^\tau \sigma^2(X_t)dt}{\tau - t} = \mathbb{E}\sigma^2(X_\tau)$

so. $\frac{d}{d\tau}\mathrm{var}_t(X_\tau)\Big|_{\tau=t} = \mathbb{E}(\sigma^2(X_t)) = \sigma_t^2$ w/ P 1

6. (a). match coefficients:

$$\begin{cases} \dfrac{\partial C}{\partial t} + \mu S_t \dfrac{\partial C}{\partial S} + \dfrac{\sigma^2 S_t^2}{2} \dfrac{\partial^2 C}{\partial S^2} = a_t \mu S_t + b_t \beta_t r & ① \\ \sigma \cancel{S_t} \cdot \dfrac{\partial C}{\partial S} = a_t \sigma \cancel{S_t} \end{cases}$$

$\Rightarrow a_t = \dfrac{\partial C}{\partial S}$ plug into ①.

we get. $b_t = \dfrac{\dfrac{\partial C}{\partial t} + \dfrac{\sigma^2 S_t^2}{2} \cdot \dfrac{\partial^2 C}{\partial S^2}}{\beta_t r}$.

plug $a_t$, $b_t$ into. $a_t S_t + b_t \beta_t$

$$= \frac{\partial C}{\partial S} . S_t + \frac{\dfrac{\partial C}{\partial t} + \dfrac{\sigma^2 S_t^2}{2} \dfrac{\partial^2 C}{\partial S^2}}{r}.$$

Call this $C_t$.

so $C_t \cdot r = \dfrac{\partial C}{\partial S} S_t r + \dfrac{\partial C}{\partial t} + \dfrac{\sigma^2 S_t^2}{2} \dfrac{\partial^2 C}{\partial S^2}$. □ B-M-S. PDE

1b). $\int a_t \, dS_t = \int \dfrac{\partial C}{\partial S_t} . dS_t = \int N(d_1) \, dS_t$.

$$= \int N(d_1) \mu S_t \, dt + \int \sigma S_t \cdot N(d_1) \cdot dB_t.$$

WTS: $E \int N(d_1) \mu S_t dt < \infty$ and. $E\left(\int \sigma^2 S^2 N(d_1)^2 dt\right) < \infty$

it suffices to show. $E.\left(\int_0 S_t^2 N(d_1)^2 dt\right) < \infty$

we know $0 \le N(d_1) \le 1$, so $E \int S_t^2 N(d_1)^2 dt \le E \int S_t^2 dt < \infty$

$\int b_t \, d\beta_t = \int \left(\dfrac{\partial C}{\partial t} + \dfrac{\sigma^2 S_t^2}{2} \dfrac{\partial^2 C}{\partial S^2}\right) dt = -\int_0^T K e^{-rt} N(d_2) \, dt < \infty$.

so. both integrals are bdd and well defined.

# Hw2prob7

Lisa He

```
r<- 0.05
sig <-.25
K<-100
S0<-100
```

```r
mT<-1/4
d1 <- (log(S0/K)+(r+sig**2/2)*mT)/(sig*sqrt(mT))
d2<- d1 - sig*sqrt(mT)
C<- S0*pnorm(d1)-K*exp(-r*mT)*pnorm(d2)
```

C is 5.5984

```r
maxN <-1000000
mcest <-c()
for (N in seq(from=1000,to=maxN+1000,by=10000)){
ST<- S0*exp((r-.5*sig^2)*mT+sig*rnorm(N,0,sqrt(mT))) - K
sumC <-0
for( i in 1:N){
  if (ST[i] >0){
    sumC <- sumC + ST[i]*exp(-r*mT)
  }
}
mcest[N]<- sumC/N
}

for (N in seq(from=1000,to=maxN+1000,by=10000)){
 if (mcest[N]>5.5884&mcest[N]<5.6084) {
   print(N)
 }
}
```

```
## [1] 71000
## [1] 81000
## [1] 131000
## [1] 141000
## [1] 161000
## [1] 171000
## [1] 201000
## [1] 211000
## [1] 221000
## [1] 231000
## [1] 241000
## [1] 261000
## [1] 271000
## [1] 331000
## [1] 341000
## [1] 361000
## [1] 371000
## [1] 381000
## [1] 391000
## [1] 401000
## [1] 421000
## [1] 431000
## [1] 451000
## [1] 481000
## [1] 491000
```

```
## [1] 501000
## [1] 511000
## [1] 521000
## [1] 531000
## [1] 541000
## [1] 571000
## [1] 581000
## [1] 591000
## [1] 631000
## [1] 641000
## [1] 651000
## [1] 701000
## [1] 721000
## [1] 741000
## [1] 751000
## [1] 761000
## [1] 771000
## [1] 781000
## [1] 791000
## [1] 801000
## [1] 811000
## [1] 821000
## [1] 831000
## [1] 841000
## [1] 851000
## [1] 861000
## [1] 881000
## [1] 891000
## [1] 901000
## [1] 911000
## [1] 921000
## [1] 931000
## [1] 941000
## [1] 951000
## [1] 971000
## [1] 981000
## [1] 991000
```

We need at least 71000 simulations to get a close enough estimate.

```
N<-71000
mcestc <-c()
ST<- S0*exp((r-.5*sig^2)*mT+sig*rnorm(N,0,sqrt(mT))) - K
sumC <-0
for( i in 1:N){
  if (ST[i] >0){
    sumC <- sumC + ST[i]*exp(-r*mT)
  }
  if (ST[i]<0){
    ST[i]<-0
```

```r
  }
}
mcestc<- sumC/N
se<- sqrt(var(ST)/N)
print(se)

## [1] 0.03174086
```

when doing 710000 simulations, the standard error is 0.03154979