Continuous-Time Finance. HW4.

Problem I (a) $f(x)=x^2$. $g_t=0$. $g_x=2x$. $g_{xx}=2$.

$\mu(s,S_s)=\mu\cdot S_s$. $\sigma=0$.

differential form:

$$df(S_t)=\left[2S_t\,\mu\cdot S_t+\tfrac{1}{2}\cancel{2\cdot 0}\right]dt+\cancel{2x\,0}+d\cdot\left(\sum_{j=1}^{N_t}S_{\tau_j}^2-S_{\tau_j-}^2\right).$$

$$=2S_t^2\mu\,dt+d\cdot\sum_{j=1}^{N_t}S_{\tau_j-}^2\left(e^{2z}-1\right).$$

integral form:

$$f(S_t)=S_0^2+\int_0^t 2S_s^2\mu\,ds+\sum_{j=1}^{N_t}S_{\tau_j-}^2\left(e^{2z}-1\right)$$

(b). $f(x)=\log(x)$ $g_t=0$. $g_x=\frac{1}{x}$. $g_{xx}=-\frac{1}{x^2}$.

$$df(S_t)=\frac{1}{S_t}\mu\cdot S_t\,dt+d\left(\sum_{j=1}^{N_t}\log(S_{\tau_j})-\log(S_{\tau_j-})\right)$$

$$=\mu dt+d\left(\sum_{j=1}^{N_t}z\right).$$

integral form:

$$\log(S_t)=\log(S_0)+\mu t+\sum_{j=1}^{N_t}z.=\log(S_0)+\mu t+zN_t$$

(c) $f(x)=e^x$ $g_t=0$ $g_x=e^x$. $g_{xx}=e^x$.

differential form:

$$d\cdot f(S_t)=e^{S_t}\mu S_t\,dt+d\left(\sum_{j=1}^{N_t}e^{S_{\tau_j}}-e^{S_{\tau_j-}}\right).$$

$$=\mu e^{S_t}S_t\,dt+d.\left(\sum_{j=1}^{N_t}e^{S_{\tau_j-}e^{z}}-e^{S_{\tau_j-}}\right)$$

integral form:

$$e^{S_t}=e^{S_0}+\int_0^t\mu e^{S_t}S_t\,dt+\sum_{j=1}^{N_t}e^{S_{\tau_j-}e^{z}}-e^{S_{\tau_j-}}.$$

2. $dS_t = \mu S_t dt + d\left(\sum_{j=1}^{N_t} S_{\tau_j^-}(e^z - 1)\right)$

N_t ~ Poisson process (λ). jump size are constant. z is constant.

(a). from ~~part~~ 1. part (b).

$\log(S_t) = \log(S_0) + \mu t + z \cdot N_t$.

$S_t = S_0 \cdot \exp\{\mu t\} \cdot (e^z)^{N_t}$.

$\frac{S_t}{S_0} = \exp\{\mu t\}(e^z)^{N_t}$.

$\log\left(\frac{S_t}{S_0}\right) = \mu t + z \cdot N_t$.

$S_t = S_0 \cdot e^{\mu t + z N_t}$

suppose the pricing kernel is $L_T = \exp((\lambda - \lambda^Q)T) \cdot \left(\frac{\lambda^Q}{\lambda}\right)^{N_T}$

let $\alpha = \log\frac{\lambda^Q}{\lambda}$ $\quad L_T = e^{(\lambda-\lambda^Q)T + \alpha N_t}$. $\quad \pi_T = L_T e^{-rT}$

suppose π_T is a valid pricing kernel

$S_0 = E_0^P(e^{-rT} S_t L_t) = E_0^P(e^{-rt} S_0 e^{\mu t + z N_t} \cdot e^{(\lambda - \lambda^Q)t + \alpha N_t})$

$= S_0 e^{-rt + \mu t + (\lambda - \lambda^Q)t} E e^{(z+\alpha)N_t}$

$= S_0 e^{-rt + \mu t + (\lambda - \lambda^Q)t + (e^{z+\alpha} - 1)\lambda T}$.

so. $-r + \mu + (\lambda - \lambda^Q) + (e^{z+\alpha} - 1)\lambda = 0$.

$\lambda^Q = \frac{\mu - r}{1 - e^z}$.

$S_t = S_0 \cdot e^{\mu t + z \cdot N_t^Q}$

$C(t, S_0) = E^Q\left[e^{-rt}(S_t - K)^+\right]$

$= E^Q\left[e^{-rt}(S_0 e^{\mu t + z N_t^Q} - K)^+\right]$

$= \sum_{j=0}^{\infty} E^Q\left[e^{-rt}(S_0 e^{\mu t + zj} - K)^+\right] \cdot P(N_t^Q = j)$

$= \sum_{j=0}^{\infty} E^P\left[e^{-rt}(S_0 e^{\mu t + zj} - K)^+\right] \frac{\left(\frac{\mu - r}{1 - e^z} t\right)^j}{j!} e^{-\frac{\mu - r}{1 - e^z} t}$

$= \sum_{j=0}^{\infty} e^{-rt}(S_0 e^{\mu t + zj} - K)^+ \frac{\left(\frac{\mu - r}{1 - e^z} t\right)^j}{j!} e^{-\frac{\mu - r}{1 - e^z} t}$

$$\mu = r + (1-e^{z})\lambda^{\mathbb{Q}}$$

$$= \sum_{j=0}^{\infty} \left[e^{-rt} (S_0 e^{(r+(1-e^{z})\lambda^{\mathbb{Q}})t + zj} - K)^+ \right] \cdot \frac{(\lambda^{\mathbb{Q}} t)^j}{j!} e^{-\lambda^{\mathbb{Q}} t}.$$

(b). $z = -0.1$ $S_0 = 100$. $K = 100$. $r = 0.05$. $t = 1$. $\lambda^{\mathbb{Q}} = 1$.

$$S = S_0 \cdot e^{(r + (1-e^{z})\lambda^{\mathbb{Q}})t + zj} - K \geq 0$$

$$0.05 + (1 - e^{-0.1}) - 0.1 j \geq 0$$

$$j \leq 1.45.$$

So.

$$C = \sum_{j=0}^{1} e^{-0.05} \cdot (S_0 e^{r + (1-e^{-0.1}) - 0.1 j} - K) \cdot \frac{1}{j!} e^{-1}$$

$$= e^{-1.05} \, 100 \cdot (e^{0.05 + 1 - e^{-0.1}} + e^{0.05 + 1 - e^{-0.1} - 0.1} - 2)$$

$$= 10.934.$$

By Put-Call parity.

$$C + K e^{-rt} = P + S_0.$$

$$P = 10.934 + 100 e^{-0.05} - 100 = 6.057$$

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Problem 3. (a) By Ito's Product Rule

$$dY_t = d(X_t^1 X_t^2) = X_t^1 dX_t^2 + X_t^2 dX_t^1 + dX_t^1 dX_t^2.$$

$$= X_t^1 \mu_t^2 dt + X_t^1 \sigma_t^2 dB_t^1 + X_t^1 \sigma_t^{21} dB_t^2 + X_t^2 \mu_t^1 dt + X_t^2 \sigma_t^1 dB_t^1$$
$$+ \sigma_t^{12} X_t^2 dB_t^2 + \sigma_t^1 \sigma_t^2 dB_t^1 dB_t^1 + \sigma_t^1 \sigma_t^{21} dB_t^1 dB_t^2$$
$$+ \sigma_t^2 \cdot \sigma_t^{12} dB_t^1 \cdot dB_t^2 + \sigma_t^{12} \cdot \sigma_t^{21} dB_t^2 dB_t^2.$$

$$= X_t^1 \mu_t^2 dt + X_t^1 \sigma_t^2 dB_t^1 + X_t^1 \sigma_t^{21} dB_t^2 + \sigma_t^1 \sigma_t^2 dt + \sigma_t^1 \sigma_t^{21} dt$$
$$+ X_t^2 \mu_t^1 dt + X_t^2 \sigma_t^1 dB_t^1 + \sigma_t^{12} X_t^2 dB_t^2 + \sigma_t^2 \sigma_t^{12} dt + \sigma_t^{12} \sigma_t^{21} dt.$$

$$= (X_t^1 \mu_t^2 + X_t^2 \mu_t^1 + \sigma_t^1 \sigma_t^2 + \sigma_t^1 \sigma_t^{21} + \sigma_t^2 \sigma_t^{12} + \sigma_t^{12} \sigma_t^{21}) dt +$$
$$(X_t^1 \sigma_t^2 + X_t^2 \sigma_t^1) dB_t^1 + (X_t^1 \sigma_t^{21} + \sigma_t^{12} X_t^2) dB_t^2.$$

Pf of Ito's Product Rule.

$$d(xy) = \frac{\partial(xy)}{\partial x} dx + \frac{\partial(xy)}{\partial y} dy + \frac{1}{2} \frac{\partial^2(xy)}{\partial x^2} (dx)^2 + \frac{\partial^2(xy)}{\partial x \partial y} dx dy + \frac{1}{2} \frac{\partial^2(xy)}{\partial y^2} (dy)^2.$$

$$= y\,dx + x\,dy + 0 + 1\,dx dy + 0$$

(b) similarly, $d\left(\frac{x}{y}\right) = \frac{\partial(\frac{x}{y})}{\partial x} dx + \frac{\partial(\frac{x}{y})}{\partial y} dy + \frac{1}{2} \frac{\partial^2(\frac{x}{y})}{\partial x^2} (dx)^2 + \frac{\partial^2(\frac{x}{y})}{\partial x \partial y} dx dy + \frac{1}{2} \frac{\partial^2(\frac{x}{y})}{\partial y^2} (dy)^2.$

$$= \frac{dx}{y} - \frac{x}{y^2} dy + 0 - \frac{dx dy}{y^2} + \frac{x (dy)^2}{y^3}.$$

$$dY_t = d\left(\frac{X_t^1}{X_t^2}\right) = \frac{dX_t^1}{X_t^2} - \frac{X_t^1 dX_t^2}{(X_t^2)^2} - \frac{dX_t^1 dX_t^2}{(X_t^2)^2} + \frac{(X_t^1)(dX_t^2)^2}{(X_t^2)^3}.$$

$$= \frac{\mu_t^1}{X_t^2} dt + \frac{\sigma_t^1}{X_t^2} dB_t^1 + \frac{\sigma_t^{12}}{X_t^2} dB_t^2 - \frac{X_t^1}{(X_t^2)^2} \mu_t^2 dt - \frac{X_t^1}{(X_t^2)^2} \sigma_t^2 dB_t^1 - \frac{X_t^1}{(X_t^2)^2} \sigma_t^{21} dB_t^2$$

$$- \frac{\sigma_t^1 \sigma_t^2 dt + \sigma_t^1 \sigma_t^{21} dt + \sigma_t^{12} \sigma_t^2 dt + \sigma_t^{21} \sigma_t^{12} dt}{(X_t^2)^2} + \frac{\left[X_t^1 (\sigma_t^2)^2 + X_t^1 (\sigma_t^{21})^2 + 2X_t \sigma_t^2 \sigma_t^{21}\right] dt}{(X_t^2)^3}$$

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Problem 3b. (cont'd)

$$d\left(\frac{X_t^1}{X_t^2}\right) = \left[\frac{\mu_t^1}{X_t^2} - \frac{X_t^1\mu_t^2 + \sigma_t^1\sigma_t^2 + \sigma_t^1\sigma_t^{21} + \sigma_t^{12}\sigma_t^2 + \sigma_t^{21}\sigma_t^{12}}{(X_t^2)^2} + \frac{X_t^1(\sigma_t^2)^2 + X_t^1(\sigma_t^{21})^2 + 2X_t^1\sigma_t^2\sigma_t^{21}}{(X_t^2)^3}\right]dt$$

$$+ \left(\frac{\sigma_t^1}{X_t^2} - \frac{X_t^1\sigma_t^2}{(X_t^2)^2}\right)dB_t^1 + \left(\frac{\sigma_t^{12}}{X_t^2} - \frac{X_t^1\sigma_t^{21}}{(X_t^2)^2}\right)dB_t^2.$$

3.(c) $X_t = f(t, X_t^1, X_t^2)$.

$$dX_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t^1}dX_t^1 + \frac{\partial f}{\partial X_t^2}dX_t^2 + \cancel{\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2} + \frac{1}{2}\frac{\partial^2 f}{(\partial X_t^1)^2}(dX_t^1)^2$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial {X_t^2}^2}(dX_t^2)^2 + \cancel{\frac{\partial^2 f}{\partial t\,\partial X_t^1}dt\,dX_t^1} + \cancel{\frac{\partial^2 f}{\partial t\,\partial X_t^2}dt\,dX_t^2} + \frac{\partial^2 f}{\partial X_t^1\,\partial X_t^2}dX_t^1 dX_t^2.$$

$$= f_t\,dt + f_{x_1}(\mu_t^1 dt + \sigma_t^1 dB_t^1 + \sigma_t^{12}dB_t^2) + f_{x_2}(\mu_t^2 dt + \sigma_t^2 dB_t^1 + \sigma_t^{21}dB_t^2)$$

$$+ \frac{1}{2}f_{x_1x_1}((\sigma_t^1)^2 + {\sigma_t^{12}}^2 + 2\sigma_t^1\sigma_t^{12})dt + \frac{1}{2}f_{x_2x_2}({\sigma_t^2}^2 + {\sigma_t^{21}}^2 + 2\sigma_t^2\sigma_t^{21})dt$$

$$+ f_{x_1x_2}(\sigma_t^1\sigma_t^2 + \sigma_t^1\sigma_t^{21} + \sigma_t^{12}\sigma_t^2 + \sigma_t^{12}\sigma_t^{21})dt$$

$$= [f_t + f_{x_1}\mu_t^1 + f_{x_2}\mu_t^2 + \frac{1}{2}f_{x_1x_1}((\sigma_t^1)^2 + {\sigma_t^{12}}^2 + 2\sigma_t^1\sigma_t^{12}) + \frac{1}{2}f_{x_2x_2}({\sigma_t^2}^2 + {\sigma_t^{21}}^2 + 2\sigma_t^2\sigma_t^{21})$$

$$+ f_{x_1x_2}(\sigma_t^1\sigma_t^2 + \sigma_t^1\sigma_t^{21} + \sigma_t^{12}\sigma_t^2 + \sigma_t^{12}\sigma_t^{21})]dt + (f_{x_1}\sigma_t^1 + f_{x_2}\sigma_t^2)dB_t^1$$

$$+ (f_{x_1}\sigma_t^{12} + f_{x_2}\sigma_t^{21})dB_t^2.$$

3.(d). Suppose there exists a self financing strategy (a_t, b_t, c_t): for X_t^1, β_t, X_t^2.

β_t is money market account. $d\beta_t = r\beta_t dt$.

$Y_t = a_t X_t^1 + b_t\beta_t + c_t X_t^2$. $\Rightarrow dY_t = a_t dX_t^1 + b_t d\beta_t + c_t dX_t^2$.

$$dY_t = a_t(\mu_t^1 dt + \sigma_t^1 dB_t^1 + \sigma_t^{12}dB_t^2) + b_t r\beta_t dt + c_t(\mu_t^2 dt + \sigma_t^2 dB_t^1 + \sigma_t^{21}dB_t^2)$$

$$= (a_t\mu_t^1 + b_t r\beta_t + c_t\mu_t^2)dt + (a_t\sigma_t^1 + c_t\sigma_t^2)dB_t^1 + (a_t\sigma_t^{12} + \sigma_t^{21}c_t)dB_t^2.$$

match the coefficients:

$$\begin{cases} a_t\mu_t^1 + b_t r\beta_t + c_t\mu_t^2 = f_t + f_{x_1}\mu_t^1 + f_{x_2}\mu_t^2 + \frac{1}{2}f_{x_1x_1}({\sigma_t^1}^2 + {\sigma_t^{12}}^2 + 2\sigma_t^1\sigma_t^{12}) + \frac{1}{2}f_{x_2x_2}({\sigma_t^2}^2 + {\sigma_t^{21}}^2 + 2\sigma_t^2\sigma_t^{21}) \\ \qquad + f_{x_1x_2}(\sigma_t^1\sigma_t^2 + \sigma_t^1\sigma_t^{21} + \sigma_t^{12}\sigma_t^2 + \sigma_t^{12}\sigma_t^{21}) \\ a_t\sigma_t^1 + c_t\sigma_t^2 = f_{x_1}\sigma_t^1 + f_{x_2}\sigma_t^2 \\ a_t\sigma_t^{12} + c_t\sigma_t^{21} = f_{x_1}\sigma_t^{12} + f_{x_2}\sigma_t^{21} \end{cases}$$

$$\left.\begin{matrix} a_t\sigma_t^1 + c_t\sigma_t^2 = f_{x_1}\sigma_t^1 + f_{x_2}\sigma_t^2 \\ a_t\sigma_t^{12} + c_t\sigma_t^{21} = f_{x_1}\sigma_t^{12} + f_{x_2}\sigma_t^{21} \end{matrix}\right\} \Rightarrow \begin{matrix} a_t = f_{x_1} \\ c_t = f_{x_2} \end{matrix}$$

3d. cont'd. plug a_t, c_t into the first equation, we get:

$$b_t = \frac{f_t + \frac{1}{2} f_{x_1x_1}\left((\sigma_t^1)^2 + (\sigma_t^{12})^2 + 2\sigma_t^1 \sigma_t^2\right) + \frac{1}{2} f_{x_2x_2}\left((\sigma_t^2)^2 + (\sigma_t^{21})^2 + 2\sigma_t^2 \sigma_t^{21}\right) + f_{x_1x_2}\left(\sigma_t^1\sigma_t^2 + \sigma_t^1\sigma_t^{21} + \sigma_t^{12}\sigma_t^2 + \sigma_t^{12}\sigma_t^{21}\right)}{r\beta_t}.$$

$a_t = f_{x_1}$ $\quad$ $c_t = f_{x_2}$.

so. (a_t, b_t, c_t) as above is such a self-financing portfolio.

3e. Suppose $\exists$. $B_t^1(\mathbb{Q}) = B_t^1(\mathbb{P}) + \int_0^t \eta_s^1 ds$ that are EMM's.

$B_t^2(\mathbb{Q}) = B_t^1(\mathbb{P}) + \int_0^t \eta_s^2 ds$.

and that $\mathbb{E} \exp\left(\frac{1}{2}\int_0^T {\eta_t^1}^2 dt\right) < \infty$ and $\mathbb{E} \exp\left(\frac{1}{2}\int_0^T {\eta_t^2}^2 dt\right) < \infty$

$$dX_t^1 = \mu_t^1 dt + \sigma_t^1 \left(dB_t^1(\mathbb{Q}) - \eta_t^1 dt\right) + \sigma_t^{12}\left(dB_t^2(\mathbb{Q}) - \eta_t^2 dt\right).$$

$$dX_t^2 = \mu_t^2 dt + \sigma_t^2 \left(dB_t^1(\mathbb{Q}) - \eta_t^1 dt\right) + \sigma_t^{21}\left(dB_t^2(\mathbb{Q}) - \eta_t^2 dt\right).$$

$$dX_t^1 = \left(\mu_t^1 - \sigma_t^1 \eta_t^1 - \eta_t^2 \cdot \sigma_t^{12}\right) dt + \sigma_t^1 dB_t^1(\mathbb{Q}) + \sigma_t^{12} dB_t^2(\mathbb{Q})$$

$$dX_t^2 = \left(\mu_t^2 - \sigma_t^2 \eta_t^1 - \sigma_t^{21} \cdot \eta_t^2\right) dt + \sigma_t^2 dB_t^1(\mathbb{Q}) + \sigma_t^{21} dB_t^2(\mathbb{Q})$$

so we need:

$$\begin{cases} \mu_t^1 - rX_t^1 = \sigma_t^1 \eta_t^1 + \eta_t^2 \cdot \sigma_t^{12} \\ \mu_t^2 - rX_t^2 = \sigma_t^2 \eta_t^1 + \sigma_t^{21} \eta_t^2. \end{cases}$$

$$\begin{cases} \eta_t^1 = \dfrac{(\mu_t^1 - rX_t^1)\sigma_t^{21} - (\mu_t^2 - rX_t^2)\sigma_t^{12}}{\sigma_t^1 \sigma_t^{21} - \sigma_t^2 \sigma_t^{12}}. \\ \eta_t^2 = \dfrac{(\mu_t^1 - rX_t^1)\sigma_t^2 - (\mu_t^2 - rX_t^2)\sigma_t^{21}}{\sigma_t^2 \sigma_t^{12} - \sigma_t^1 \sigma_t^{21}}. \end{cases}$$

so we've found. EMM's. where

$dB_t^1(\mathbb{Q}) = dB_t^1(\mathbb{P}) + \eta_t^1$

$dB_t^2(\mathbb{Q}) = dB_t^2(\mathbb{P}) + \eta_t^2$. $\quad$ as indicated above.