Homework 4: Due Thursday March 8

1. Consider the simplest jump model,

$$dS_t = \mu S_t dt + d\left(\sum_{j=1}^{N_t} S_{\tau_j -}\left(e^Z - 1\right)\right)$$

   where N_t is a Poisson process with intensity λ and we assume the jump sizes are constant. This, if you recall, implies that $\Delta S_{\tau_j} = S_{\tau_j} - S_{\tau_j -} = S_{\tau_j -}\left(e^Z - 1\right)$ and that $S_{\tau_j} = S_{\tau_j -} e^Z$. Apply Ito's lemma to write in integrated and differential form: $X_t = f\left(S_t\right)$ for

   (a) $f(x) = x^2$

   (b) $f(x) = \log(x)$

   (c) $f(x) = e^x$

2. Consider the pricing of options in the setting of the model in previous question.

   (a) What is the no-arbitrage value of a call option struck at K? Provide a closed form formula for the option price.

   (b) Assume that $Z = -10\%$, $S_0 = 100$, and $K = 100$. What is the numerical value of a call option and put option struck at K?

3. Consider the following two processes:

$$dX_t^1 = \mu_t^1 dt + \sigma_t^1 dB_t^1 + \sigma_t^{12} dB_t^2$$
$$dX_t^2 = \mu_t^2 dt + \sigma_t^2 dB_t^1 + \sigma_t^{21} dB_t^2$$

   where the Brownian motions are independent and N_t^i has stochastic intensity λ_t^i.

   (a) Write a stochastic differential equation for $Y_t = X_t^1 X_t^2$.

(b) Write a stochastic differential equation for $Y_t = \frac{X_t^1}{X_t^2}$.

(c) Assume there is a derivative on X_t^1 and X_t^2. Use Ito's lemma to characterize the increments in $X_t = f\left(t, X_t^1, X_t^2\right)$.

(d) Assume that X_t^1 and X_t^2 are traded assets and consider a self-financing portfolio in the risk-free asset and the two risky assets. Can you find a self-financing portfolio that perfectly replicates the cash flows of the derivative in the previous question? Provide formulas for the portfolio weights in the self-financing strategy.

(e) Construct an equivalent martingale measure and provide a PDE for the price of the derivative.