Class 6: Option Pricing

Option Pricing

Our goal is to understand the theory and empirics of option pricing. Historically, there has been an interesting interplay between data and model building, with new data observations continually leading to new models. Prime examples of this include the jump and stochastic volatility models that were introduced in the years after the crash of 1987 and the LTCM crises in 1998.

Models used for option prices consist of two pieces: objective measure dynamics and risk-neutral dynamics. The objective measure describes the historical and predictive behavior of actual returns, whereas the risk-neutral measure is the probability measure under which option are priced. These measures are commonly parameterized by a vector of parameters, and we denote the obejctive measure parameter vector as $\Theta^{\mathbb{P}}$ and $\Theta^{\mathbb{Q}}$ as the risk-neutral parameter vector. We often think of the $\mathbb{P}$-measure specification as the "statistical" model and the $\mathbb{Q}$-measure specification as the "economic" portion of the model, as it embodies the risk premia. In terms of specification, the goal is to develop models that are capable of fitting both the observed time series of returns and also the observed panel of option prices.

Options and option pricing models are of interest not only for their practical importance, but also because option prices provide a lens to learn about market prices of jump and stochastic volatility risks. In general, it is not possible to quantify these prices of risk from only equity returns. As we saw from previous classes, the

$$dS_t = (r+\mu)S_t dt + \sigma S_t dW_t^{\mathbb{P}} + d\left(\sum\nolimits_{j=1}^{N_t^{\mathbb{P}}} S_{\tau_{j-}}\left[e^{Z_j^{\mathbb{P}}}-1\right]\right) - \lambda^{\mathbb{P}}\overline{\mu}^{\mathbb{P}} S_t dt \quad (1)$$

where

$$\mu = \mu^c + \lambda^{\mathbb{P}}\overline{\mu}^{\mathbb{P}} - \lambda^{\mathbb{Q}}\overline{\mu}^{\mathbb{Q}}$$

Z is lognormal

market price of diffusion risk

market price of jump risk

and $\overline{\mu}^j = \exp\left(\mu_z^j + \left(\sigma_z^j\right)^2/2\right) - 1$. We let μ_c denote the premium for $W_t^{\mathbb{P}}$ risk ($\mu^c = \sigma\eta$). If we estimate the parameters based solely on stock returns data, we can estimate the total premium, μ, but cannot disaggregate into its consituent components. Option prices will allow us to estimate the risk-neutral parameters, and therefore to identify the market prices of jump risk.

Before thinking about models, let's think a bit about some empirical regularities

Empirical regularities of stock return

Before developing formal models, we provide a brief review of what the data tells us about potential models and factors. We start with the information contained in the time series of underlying asset returns. Suppose we have a time series of returns, $\left\{r_t^\Delta\right\}_{t=1}^T$ where

$$r_t^\Delta = \ln\left(\frac{S_{t+\Delta}}{S_t}\right)$$

are the $\Delta-$*period* continuously-compounded returns. When analyzing continuous-time models, it is common to analyze daily returns or even intra-daily returns. Intradaily returns create a number of difficult issues that must be dealt with that include accounting for overnight periods, microstructure issues, and temporal volatility patterns. Regarding the last issue, it is well-known that stock returns tend to be more volatile near the open and close than during the middle of the day.

With these returns, it is common to estimate the unconditional moments. If we let m_Δ^k denote the m^{th} central moment, then

$$m_\Delta^k = E^{\mathbb{P}}\left[\left(r_t^\Delta - m_\Delta^1\right)^k\right],$$

for $k \geq 2$ and $m_\Delta^1 = E^{\mathbb{P}}\left[r_t^\Delta\right]$. It is common to scale higher moment estimates, which is why estimates of skewness $\left(m_\Delta^3/\left(m_\Delta^2\right)^{3/2}\right)$ and kurtosis $\left(m_\Delta^4/\left(m_\Delta^2\right)^2\right)$ are commonly used. We make formal reference to the measures to remind ourselves that the actual observed data is generated by the $\mathbb{P}$-measure specification of the model.

Unconditional moments for U.S. stock indices estimated using long-time series of daily data imply that (a) returns are negatively skewed (about -2) and (b) they have positive kurtosis (about 50). These statistics are largely driven by events such as the Crash of 1987. If unconditional annualized

fat tailed!

goes down more than up

volatility is 15%, then unconditional daily volatility is about 1%, indicating that the crash was a 22-standard deviation event. Certainly exceedingly rare if returns were generated by a normal distribution with constant parameters. There are also interesting term structure implications, that is, how the moments vary as a function of Δ. For aggregate stock index returns, the skewness and kurtosis decline as a function of Δ, approaching normality over long horizons.

Individual stock returns have different unconditional moments. The overall level of volatility is much higher, and largely because of this, there is little skewness or kurtosis. The intuition is clear: if daily volatility is 3% or 4%, then "large" movements on the order of 10% to 15% are not really outliers due to the high volatility, or at least not nearly as outlying as observations such as the Crash of 1987 were to aggregate indices.

There is also significant evidence that the moments of the return distribution are time-varying. This comes from many sources: rolling estimates of standard deviations, ARCH/GARCH models, and stochastic volatility models. In every case, it is clear that there are periods of higher and lower asset price volatility. In terms of a model, this implies that returns behave like

$$r_t^{\Delta} = \mu_t + \sqrt{V_t}\varepsilon_{t+1}.$$

ARCH and GARCH models specify that V_t is a deterministic function of past data. Stochastic volatility models, as their name indicates, specify that volatility is driven by random, unpredictable shocks according to an evolution equation.

There is a large literature developing econometric tools for estimating stochastic volatility models. We'll talk more later about how these models are estimated.

Empirical regularities of option price data

We can generically denote the observed price of an option at time t, with strike price K, with underlying price S_t and maturity time T as $O_{t,T}(K, S_t)$.

Notice that all of these variables are observed. Since the mid 1980s, options have traded on equity indices, such as the S&P 500. This provides a rich dataset across time (20 years worth of data), maturities (there are short and long-dated maturities) and strike prices (in and out-of-the-money strikes).

Analyzing option prices directly is a bit complicated because of the strong dependence of the option prices on the underlying asset values, S_t. Due to the dependence of the option prices on the underlying asset,

$$\frac{\partial O}{\partial S_t} > 0 \text{ for calls}$$
$$\frac{\partial O}{\partial S_t} < 0 \text{ for puts,}$$

as the underlying asset price fluctuates, so do the option prices, but more importantly so do the contracts that are available in the market for trading. This implies that the strike prices are changing over time, which introduces some non-stationarity in the observed prices.

Because of this, it is common to transform option prices into units that are more stationary. The main tool for this is Black-Scholes implied volatility. The Black-Scholes formula for a call option is given by

$$C\left(t, T, r, S_t, K, \sigma\right) = S_t N\left(d_1\right) - e^{-rT} K N\left(d_1 - \sigma\sqrt{T}\right),$$

where

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

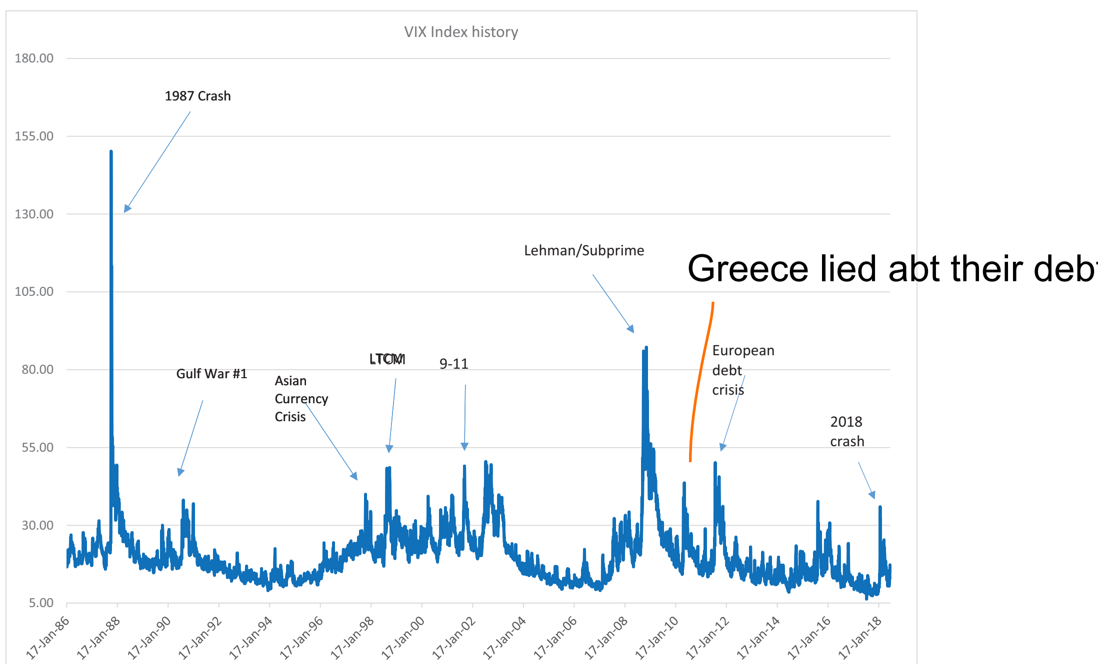
where $N(x)$ is the normal CDF. Since option prices and volatility are one-to-one, one can use a market option price to invert the option pricing formula and find the volatility parameter, σ^{IV}, that generates a Black-Scholes price that equals the market price.

If we let $C_{t,T}^{Mar}\left(K, S_t\right)$ denote the market price of an option, then

$$C_{t,T}^{Mar}\left(K, S_t\right) = C\left(t, T, r, S_t, K, \sigma_t^{IV}\right).$$

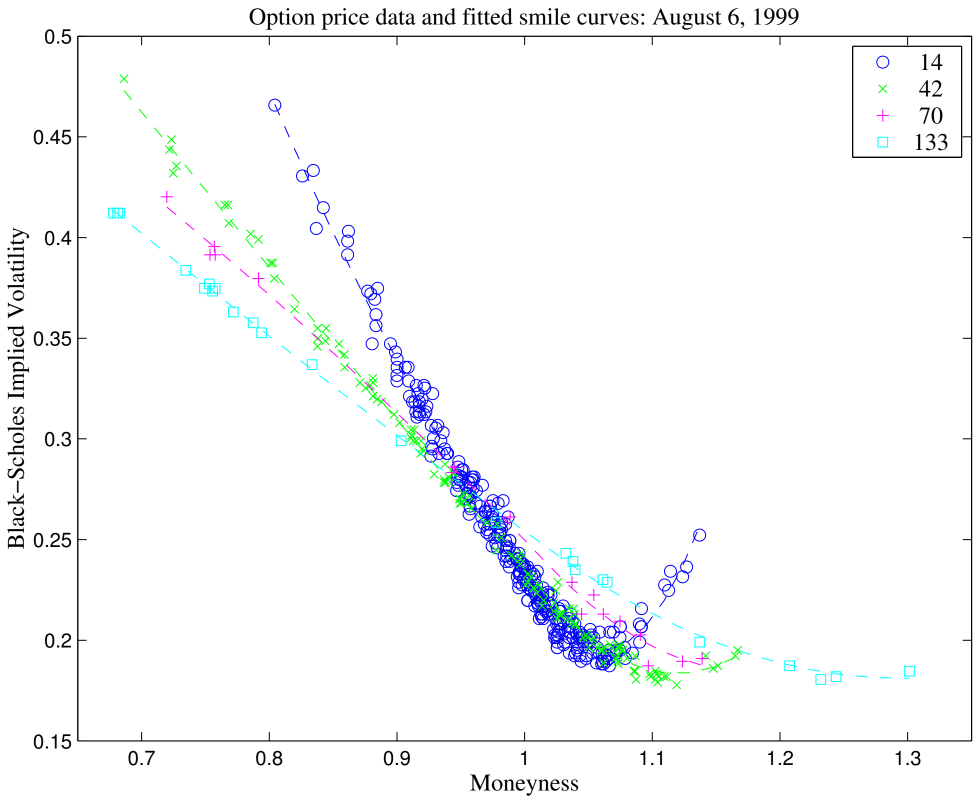
The implied volatility can potentially be a function of the option characteristics, (K, t, T). A plot of σ_t^{IV} as a function of t is the time series of implied volatility, a plot of σ_t^{IV} as a function of T is the term structure of implied volatility, and a plot of σ_t^{IV} as a function of K is called the cross-section of implied volatility (which is commonly called the volatility smile). Implied volatilities provide an intuitive metric for understanding the information contained in option prices.

Before discussing models, let's look at some pictures.



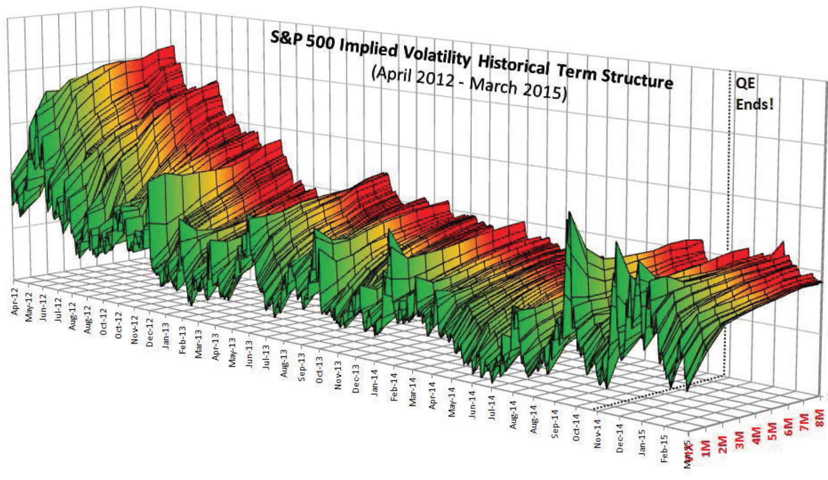
The VIX index (ATM implied volatility)

at the money option



$\frac{K}{S}$

The implied volatility smile



S&P 500 Implied Volatility Historical Term Structure
(April 2012 - March 2015)
QE
Ends!
35
30
25
20
15
10
Apr-12
May-12
Jun-12
Jul-12
Aug-12
Aug-12
Oct-12
Oct-12
Nov-12
Dec-12
Jan-13
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Mar-13
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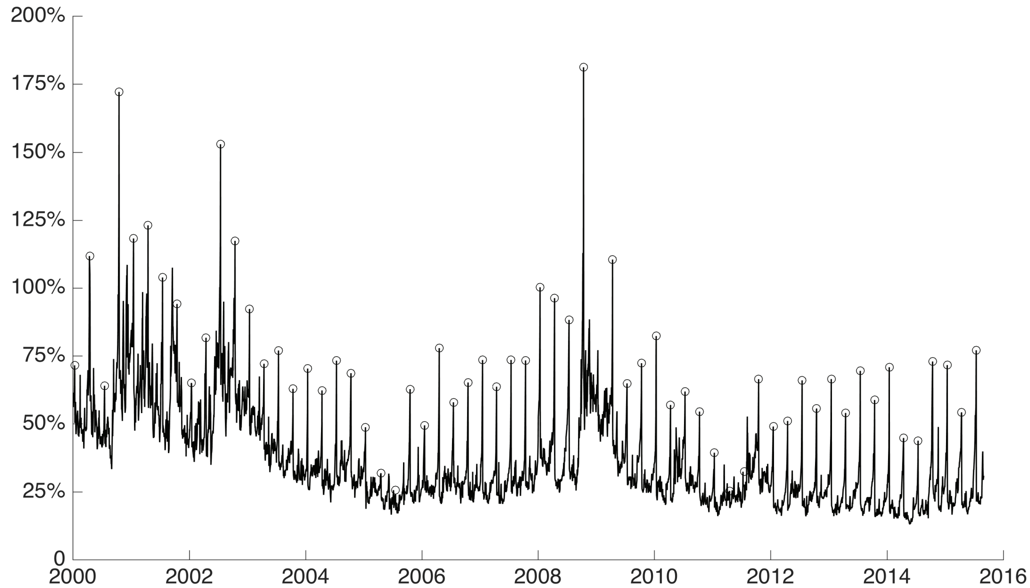
6M
7M
8M



At-the-money implied volatility for Intel. Circles indicate earnings dates

Let's first consider the example of earnings announcements.

- Consider a simple extension of the Black-Scholes model incorporating earning announcements:

$$S_T = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T(\mathbb{Q}) + \sum\nolimits_{j=1}^{N_T^d} Z_j\right], \tag{2}$$

  where

$$Z_j = -\frac{1}{2}\left(\sigma_j^{\mathbb{Q}}\right)^2 + \sigma_j^{\mathbb{Q}}\varepsilon$$

  and $\varepsilon \sim N(0,1)$.

- Since $W_T(\mathbb{Q})$ and $\sum_{j=1}^{N_T^d} Z_j$ are normally distributed (a non-random mixture of normal random variables is normal), continuously-compounded returns are exactly normally distributed. This implies that the price of a European call option struck at K and expiring in T_i days is given by the usual Black-Scholes formula with a modified volatility input.

- If we let $BS\left(x, \sigma_T^2, r, T_i, K\right)$ denote the usual Black-Scholes pricing formula, the modified volatility input, assuming a single jump is

$$\sigma_{t,T_i}^2 = \sigma^2 + \frac{1}{T_i}\left(\sigma_j^{\mathbb{Q}}\right)^2.$$

- What are the implications?

  1. Assuming a single announcement to maturity, the moment before an earnings release, annualized IV is

$$\sigma_{\tau_j-,T_i}^2 = \sigma^2 + T_i^{-1}\left(\sigma_j^{\mathbb{Q}}\right)^2,$$

     and after $\sigma_{\tau_j,T_i}^2 = \sigma^2$. This implies there is a discontinuous drop in IV immediately following the earnings release.

  2. As we approach an EAD, IV increases and the rate of increase is proportional to T_i^{-1}. earning annoucement

  3. Holding the number of jumps constant, the term structure of IV decreases as the maturity of the option increases.

the volatility goes up just before EAD and crashes down after the announcement

Table 4
Anticipated uncertainty (term structure estimator, by firm)

Term	Mean	Median	SE	25%	75%	Err_1	Err_2	Obs
AAPL	8.83	8.49	0.40	7.08	9.76	0	0	51
AIG	5.21	5.07	0.52	3.03	5.98	4	0	36
AMGN	5.46	4.79	0.47	3.60	6.64	4	0	40
AMZN	12.06	11.06	0.49	9.49	13.94	0	0	63
BA	4.33	4.09	0.29	3.35	5.01	2	0	27
BAC	3.96	3.31	0.47	2.28	4.69	5	0	51
C	3.34	2.90	0.21	2.29	4.48	8	0	47
CAT	5.67	5.07	0.38	4.31	6.13	0	0	46
COP	3.33	2.89	0.49	1.72	4.11	4	0	28
CSCO	8.13	7.28	0.35	6.59	9.10	0	0	60
CVX	2.53	2.52	0.19	1.85	2.94	11	0	36
DELL	6.25	5.90	0.51	4.53	7.06	4	0	28
EBAY	8.36	8.48	0.57	6.74	10.20	0	0	30
FCX	5.26	4.95	0.40	3.36	6.38	4	0	43
FSLR	14.13	14.17	0.67	11.14	16.02	1	1	27
GE	4.17	3.48	0.37	2.75	4.10	4	0	63
GOOGL	7.94	7.60	0.45	6.22	9.33	0	0	43
GS	4.98	4.00	0.41	3.17	5.93	6	1	55
IBM	5.68	5.02	0.30	4.49	6.15	0	0	63
INTC	7.04	6.27	0.36	5.31	7.66	1	0	62
JNJ	2.56	2.25	0.21	1.72	3.05	5	0	36
JPM	4.62	3.80	0.37	2.95	5.25	4	0	59
MA	6.90	6.14	0.53	4.76	8.58	0	0	27
MO	2.71	2.60	0.26	1.85	3.22	22	0	44
MRK	2.90	2.80	0.25	1.77	3.98	10	0	40
MS	6.39	4.36	0.86	3.24	7.69	5	0	31
MSFT	5.35	5.03	0.29	3.96	6.04	2	0	62
NEM	3.39	3.67	0.39	2.11	4.62	7	0	28
NFLX	14.92	13.98	0.91	11.25	18.39	0	0	27
PFE	2.99	3.21	0.18	2.16	3.73	6	0	47
PG	3.45	2.91	0.27	2.34	4.08	2	0	52
QCOM	6.78	6.15	0.32	5.35	7.53	1	0	63
SHLD	7.76	7.86	0.55	5.20	8.84	0	0	27
T	3.40	2.79	0.42	2.27	3.80	6	0	35
UPS	3.53	3.38	0.45	2.14	4.25	4	0	28
VZ	2.95	2.84	0.28	1.95	3.74	7	0	36
WFC	5.90	3.86	1.06	3.10	5.34	0	0	31
WMT	3.26	3.30	0.15	2.60	3.57	3	0	51
X	6.75	6.44	0.61	5.64	7.68	2	0	27
XOM	2.50	2.28	0.21	1.80	3.20	11	0	51
YHOO	9.96	8.87	0.62	7.09	10.33	1	0	47

This table provides the average estimate of anticipated uncertainty $\sigma_j^{\mathbb{Q}}$ using the term-structure estimator $\sigma_{term}^{\mathbb{Q}}$. We report the summary statistics over the sample period from January 2000 to August 2015 for all firms with at least 7 years of EAD data. We report the mean, (Mean), median (Median), the standard error (SE), and the lower and upper quartile (25% and 75%) of all observations without errors. Err_1 counts the number of EAD on which the hypothesis of a decreasing term structure is violated, and Err_2 counts the number of EAD on which the violations were more than 5% (i.e., $\sigma_{t,T_2} - \sigma_{t,T_1} > 0.05$). The last column provides the number of observations (Obs).

Estimates of earnings announcement jump volatility

Table 9
Earnings announcement jump risk premiums

Year	Std	$\mathbb{P}$-vol (CC)	$\mathbb{Q}$-vol mean	$\mathbb{P}$-vol (CO)	$\mathbb{Q}$-vol jump	$\mathbb{Q}$-vol median
2000–2005	0.94	7.63	9.92	6.65	7.46	7.08
2006–2010	0.92	7.22	7.74	5.60	6.77	6.15
2011–2015	0.91	7.35	6.78	6.60	6.34	5.19
Pooled	0.92	7.42	8.22	6.31	6.87	6.03

This table provides summary statistics on $\mathbb{P}$ and $\mathbb{Q}$-measure volatility on EADs. The second column (*Std*) provides the standard deviation of standardized equity returns the day after the earnings release, column $\mathbb{P}$-Vol (CC) provides the 1-day close-to-close standard deviation of returns on EADs, $\mathbb{Q}$-Vol mean provides the risk-neutral counterpart. $\mathbb{P}$-Vol (CO) and $\mathbb{Q}$-Vol jump are based on close-to-open returns and the risk-neutral jump volatility $\sigma_j^{\mathbb{Q}}$, respectively. The last column provides estimates of the median of option-implied EAD volatility under $\mathbb{Q}$.

Estimates of earnings announcement jump risk premia

Back to index options. What does the data tell us? A few things immediately

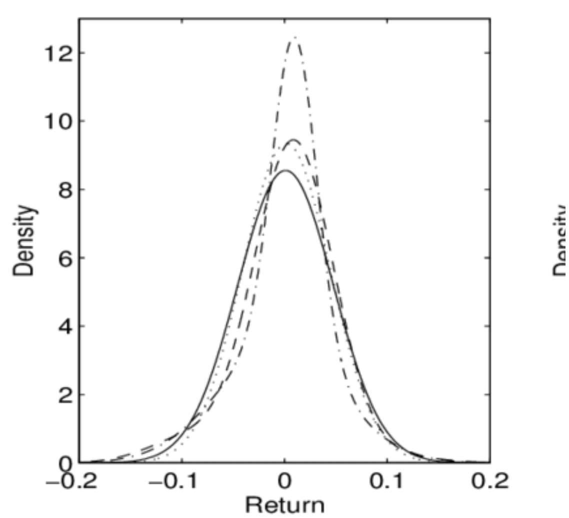
- At-the-money option implied volatility moves around a lot over time. This implies that σ isn't not constant, but is stochastic and mean-reverting.

- What about the distribution of volatility? It is clearly positively skewed: it spikes upward.

- What does the volatility smile tell us? Thought: if OTM put IV is much higher than ATM call/put IV, then the OTM put options are relatively more expensive than the Black-Scholes model would imply. How to interpret? Think of the risk-neutral pricing equation:

$$P(t, T, r, S_t, K) = \int e^{-r(T-t)}(K - S_T)_+ q(S_T|S_t) dS_T$$

  If the OTM put requires a higher volatility to match the market price, it must be because the left tail of $q(S_T|S_t)$ is fatter than that of a log-normal.



Market vs. log-normal tails

Putting the pieces together, we need a model that is left-skewed and has time-varying volatility. Here is a model capturing all of these features: the equity index price, S_t, and the spot variance, V_t, jointly solve:

$$\frac{dS_t}{S_{t-}} = \left(r_t + \eta_s V_t - \lambda^{\mathbb{Q}} \overline{\mu}_s^{\mathbb{Q}}\right) dt + \sqrt{V_{t-}} dW_t^s(\mathbb{P}) + d \sum_{j=1}^{N_t(\mathbb{P})} \left(e^{Z_j^s(\mathbb{P})} - 1\right) \quad (3)$$

$$dV_t = \kappa_v \left(\theta_v - V_t\right) dt + \sigma_v \sqrt{V_{t-}} dW_t^v \left(\mathbb{P}\right) + d \left(\sum_{j=1}^{N_t(\mathbb{P})} Z_j^v \left(\mathbb{P}\right)\right),$$

where,

- $W_t^s \left(\mathbb{P}\right)$ and $W_t^v \left(\mathbb{P}\right)$ are two Brownian motions with correlation ρ;
- $\eta_s V_t$ is diffusive equity risk premium,
- $\lambda^{\mathbb{Q}} \overline{\mu}_s^{\mathbb{Q}}$ is the jump equity risk premium,
- $N_t \left(\mathbb{P}\right)$ a Poisson process with constant intensity $\lambda^{\mathbb{P}}$,
- $Z_j^s(\mathbb{P}) \sim N \left(\mu_s^{\mathbb{P}}, \left(\sigma_s^{\mathbb{P}}\right)^2\right)$ are the jumps to log-returns,
- The jump equity risk premium is:

  $$\lambda^{\mathbb{P}} \overline{\mu}_s^{\mathbb{P}} - \lambda^{\mathbb{Q}} \overline{\mu}_s^{\mathbb{Q}}$$

  is the jump component of the equity risk premium,
- $\overline{\mu}_s^{\mathbb{P}} = \exp \left(\mu_s^{\mathbb{P}} + 0.5 \left(\sigma_s^{\mathbb{P}}\right)^2\right)$ and $Z_j^v \left(\mathbb{P}\right) \sim \exp \left(\mu_v^{\mathbb{P}}\right)$ are the jumps sizes in volatility, .

How to calculate option prices?

This model is great, but how can we actually calculate option prices? It turns out for affine stochastic volatility models, one can get pretty close to 'closed forms,' at least up to a single numerical integration. Here how it works:

- The characteristic function, $\Phi(u) = E^{\mathbb{Q}}[e^[iuS_T]$, for affine models is exponential affine.
- The characteristic function is a Fourier transform of the pdf:

$$\Phi(u) = E^{\mathbb{Q}}[e^[iuS_T] = \int_{-\inf}^{\inf} e^{iux} q(x) dx$$

- By applying a Fourier inversion theorem,

$$q(x) = \frac{1}{2\pi} \int_{-\inf}^{\inf} e^{-iux} \Phi(u) du$$

  , we can recover the risk-neutral density.
- This, in turn, allows us to calculate probabilities like $\mathbb{Q}(\mathbb{S}_\mathbb{T} > \mathbb{K})$ which are needed to calculate option prices.
- In some models, the characteristic function is known analytically, in other cases, you need to solve an ODE numerically.