Continuous-Time Finance HW3. Lisa He

1. $dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$

$\mu(t) = \mu S_t \quad \sigma(t) = \sigma S_t$.

a). So. $d(S_t^2) = \left(2S_t\,\mu S_t + \frac{1}{2}\sigma^2 S_t^2 \cdot 2\right)dt + \sigma S_t\, 2S_t\, dB_t$

$= (2\mu + \sigma^2)\, S_t\, dt + 2\sigma S_t^2\, dB_t$.

b) $f_x = \frac{1}{2} \cdot S_t^{-1/2}$. $f_{xx} = -\frac{1}{4} S_t^{-3/2}$.

$d(S_t^{1/2}) = \left[\frac{1}{2} S_t^{-1/2} \mu S_t + \frac{1}{2}\sigma^2 S_t^2 \cdot -\frac{1}{4} S_t^{-\frac{3}{2}}\right] dt + \sigma S_t \cdot \frac{1}{2} S_t^{-\frac{1}{2}}\, dB_t$

$= \left(\frac{1}{2}\mu S_t^{\frac{1}{2}} - \frac{1}{8}\sigma^2 S_t^{\frac{1}{2}}\right) dt + \frac{1}{2}\sigma S_t^{\frac{1}{2}}\, dB_t$

3. BM: $B_t(\mathbb{P})$ defined under $(\Omega, \mathcal{F}, \mathbb{P})$.

$dX_t = \mu X_t\,dt + \sigma X_t\, dB_t(\mathbb{P})$

want to construct $B_t(\mathbb{Q})$ s.t. $dX_t = rX_t\,dt + \sigma X_t\, dB_t(\mathbb{Q})$.

want: $\mu_t - \eta_t \sigma_t = r$. $\quad \mu_t = \mu,\ \sigma_t = \sigma$.

so $\mu - \eta_t \sigma = r$. $\quad \eta_t = \frac{\mu - r}{\sigma}$. is a constant. since. μ, r, σ are constants

Novikov condition.

$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T \eta_t^2\, dt\right)\right) = \exp \frac{1}{2}\eta_t^2 T < \infty$ since η_t is a constant.

so, we can construct: $L_t = \exp\left(\int_0^t \eta_s\, dB_s(\mathbb{P}) - \frac{1}{2}\int_0^t \eta_s^2\, ds\right)$. is a $(\mathcal{F}_t, \mathbb{P})$ martingale

define set function $\mathbb{Q}$ as follows:

$\forall A \in \mathcal{F}$, $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}(1_{[A]} L_T)$.

$1_{[\Omega]} = 1. \Rightarrow \mathbb{Q}(\Omega) = 1. \Rightarrow \mathbb{Q}$ is a probability measure.

also $\mathbb{Q}$ is equivalent to $\mathbb{P}$.

Girsanov Thm: η_t is square integrable $\int_0^T \eta_t^2\, dt = \eta_t^2 T$. we've shown L_t is a martingale.

$B_t(\mathbb{Q}) = B_t(\mathbb{P}) + \int_0^t \eta_s\, ds$. is a standard BM on $(\Omega, \mathcal{F}, \mathbb{Q})$.

$dB_t(\mathbb{Q}) = dB_t(\mathbb{P}) + \eta_t\, dt. \Rightarrow dB_t(\mathbb{P}) = dB_t(\mathbb{Q}) - \eta_t\, dt$.

substitute in, we get. $dX_t = \mu X_t\, dt + \sigma X_t\,(dB_t(\mathbb{Q}) - \eta_t\, dt)$

$\eta_t = \frac{\mu - r}{\sigma}$

$= (\mu X_t - \eta_t \sigma X_t)\, dt + \sigma X_t\, dB_t(\mathbb{Q}) = rX_t\, dt + \sigma X_t\, dB_t(\mathbb{Q})$ □

4. $d\left(\frac{S_t^i}{\beta_t}\right) = \frac{dS_t^i \beta_t - S_t^i d\beta_t}{\beta_t^2} = \frac{\mu_i S_t^i dt + S_t^i(\sigma_i^1 dB_t^1 + \sigma_i^2 dB_t^2 + \sigma_i^3 dB_t^3) - S_t^i r dt}{\beta_t}$

$= \frac{S_t^i}{\beta_t}\left[(\mu_i - r)dt + \sigma_i^1 dB_t^1 + \sigma_i^2 dB_t^2 + \sigma_i^3 dB_t^3\right].$

Suppose $\exists$ an EMM $\mathbb{Q}$. s.t. $\left(\frac{S_t^i}{\beta_t}\right)$ are martingales.

if such a probability measure $\mathbb{Q}$ exists, we can write

$B_t(\mathbb{P}) = B_t(\mathbb{Q}) - \int_0^t \eta_s ds$

$dB_t(\mathbb{P}) = dB_t(\mathbb{Q}) - \eta_t dt$

So. $d\left(\frac{S_t^i}{\beta_t}\right) \cdot \frac{\beta_t}{S_t^i} = (\mu_i - r)dt + \sigma_i^1(dB_t^1(\mathbb{Q}) - \eta_t^1 dt) + \sigma_i^2(dB_t^2(\mathbb{Q}) - \eta_t^2 dt)$

$+ \sigma_i^3 (dB_t^3(\mathbb{Q}) - \eta_t^3 dt)$

$= \left[(\mu_i - r) - \sigma_i^1\eta_t^1 - \sigma_i^2\eta_t^2 - \sigma_i^3\eta_t^3\right]dt + \sigma_i^1 dB_t^1(\mathbb{Q}) + \sigma_i^2 dB_t^2(\mathbb{Q}) + \sigma_i^3 dB_t^3(\mathbb{Q})$

since $\frac{S_t^i}{\beta_t}$ is a martingale.

$\mu_i - r = \sigma_i^1\eta_t^1 + \sigma_i^2\eta_t^2 + \sigma_i^3\eta_t^3$.

$$\begin{pmatrix}\mu_1 - r\\ \mu_2 - r\end{pmatrix} = \begin{pmatrix}\sigma_1^1 & \sigma_1^2 & \sigma_1^3\\ \sigma_2^1 & \sigma_2^2 & \sigma_3^3\end{pmatrix}\cdot\begin{pmatrix}\eta_t^1\\ \eta_t^2\\ \eta_t^3\end{pmatrix}.$$

Since there's 3 unknowns and 2 equations,

we get either zero solution or infinite solutions.

any $(\eta_t^1, \eta_t^2, \eta_t^3)$ that satisfies the above equation is a solution.

and $B_t(\mathbb{Q}) = B_t(\mathbb{P}) + \int_0^t \eta_s ds$.

5. $$d\left(\frac{S_t^i}{\beta_t}\right) = \frac{dS_t^i\,\beta_t - S_t^i\,d\beta_t}{\beta_t^2} = \frac{S_t^i\left[\mu_i dt + \sigma_i dB_t - r dt\right]}{\beta_t}$$

Suppose $\exists$ an EMM $\mathbb{Q}$ s.t. $\frac{S_t^i}{\beta_t}$ are martingales.

then $dB_t(\mathbb{P}) = dB_t(\mathbb{Q}) - \eta_t dt$

so $$d\left(\frac{S_t^i}{\beta_t}\right) = \frac{S_t^i}{\beta_t}\left[(\mu_i - r)dt + \sigma_i\left(dB_t(\mathbb{Q}) - \eta_t dt\right)\right]$$

$$= \frac{S_t^i}{\beta_t}\left[(\mu_i - r - \eta_t\sigma_i)dt + \sigma_i dB_t(\mathbb{Q}).\right]$$

since $d\left(\frac{S_t^i}{\beta_t}\right)$ is a martingale.

$$\mu_i - r - \eta_t\sigma_i = 0.$$

$$\begin{pmatrix}\mu_1 - r\\ \mu_2 - r\end{pmatrix} = \begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}\eta_t.$$

if. $\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$. $(\sigma_1 \neq 0, \sigma_2 \neq 0.)$

then we have a solution. $\eta_t = \frac{\mu_1 - r}{\sigma_1}$.

if. $\mu_1 - r = \mu_2 - r = \sigma_1 = \sigma_2 = 0$. we have infinite solutions.

Otherwise, there's no solution.

hw3prob2

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```
calcuP <- function(K){
r<- 0.05
sig <-.25

S0<-100
```

```
mT<-1/12
d1 <- (log(S0/K)+(r+sig**2/2)*mT)/(sig*sqrt(mT))
d2<- d1 - sig*sqrt(mT)
P<- -S0*pnorm(-d1)+K*exp(-r*mT)*pnorm(-d2)

return(P)}
print(calcuP(60))

## [1] 3.6202e-13

print(calcuP(70))

## [1] 3.219925e-07

print(calcuP(80))

## [1] 0.001435591

print(calcuP(90))

## [1] 0.1922204

print(calcuP(100))

## [1] 2.669393

sensitosig <-function(sig){
r<- 0.05
K<-100

S0<-100
mT<-1/12
d1 <- (log(S0/K)+(r+sig**2/2)*mT)/(sig*sqrt(mT))
d2<- d1 - sig*sqrt(mT)
P<- -S0*pnorm(-d1)+K*exp(-r*mT)*pnorm(-d2)

return(P)
}
print(sensitosig(0.1))

## [1] 0.9532625

print(sensitosig(0.15))

## [1] 1.523818

print(sensitosig(0.2))

## [1] 2.096267

print(sensitosig(0.25))

## [1] 2.669393
```

```
print(sensitosig(0.3))

## [1] 3.242768

print(sensitosig(0.35))

## [1] 3.816191

print(sensitosig(0.4))

## [1] 4.389547
```

K = 60, P60 = 3.6202e-13 K = 70, P70 = 3.219925e-07 K = 80, P80 = 0.001435591 K = 90, P90 = 0.1922204 K = 100, P100 = 2.669393

When sigman changes the put price changes a lot, so it is sensitive to sigma

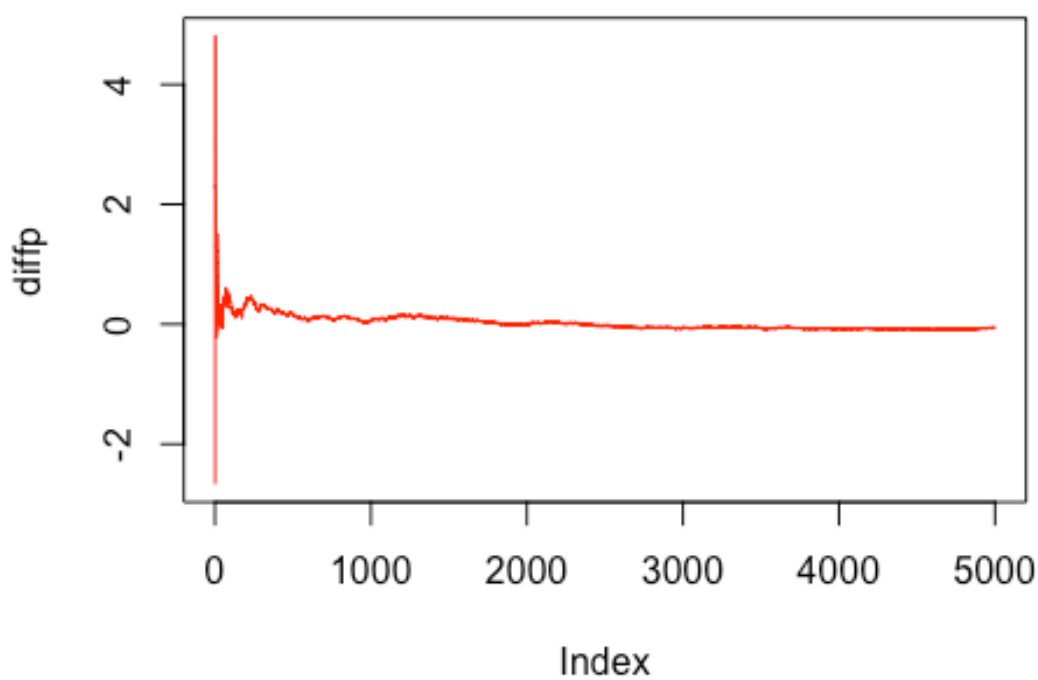
(c) [2.669393*0.99 ,2.669393*1.01]=[2.642699,2.696087]

```
mT<-1/12
delta <- 1/252
totT<- mT/delta
r<- 0.05
sig <-.25

S0<-100
mT<-1/12
K<-100
diffp <-c()
sefunc<- c()
maefunc<-c()

numSim<-5000
Psim<-c()
psum <-0
for ( n in 1:numSim){
simX <- c()
simX[1]<- 100


for (i in 2:totT){

  simX[i]<- simX[i-1]+r*simX[i-1]*delta + sig*simX[i-1]*sqrt(delta)*rnorm(1)
}
Psim[n] <- max(0,K-simX[totT])*exp(-r*mT)
psum<- psum+ Psim[n]

Pestimate<- ( psum/ n)
diffp[n]<-Pestimate- 2.669393
if(abs(diffp[n])<2.669393*0.01){
  print(n)
```

```
}
}
```

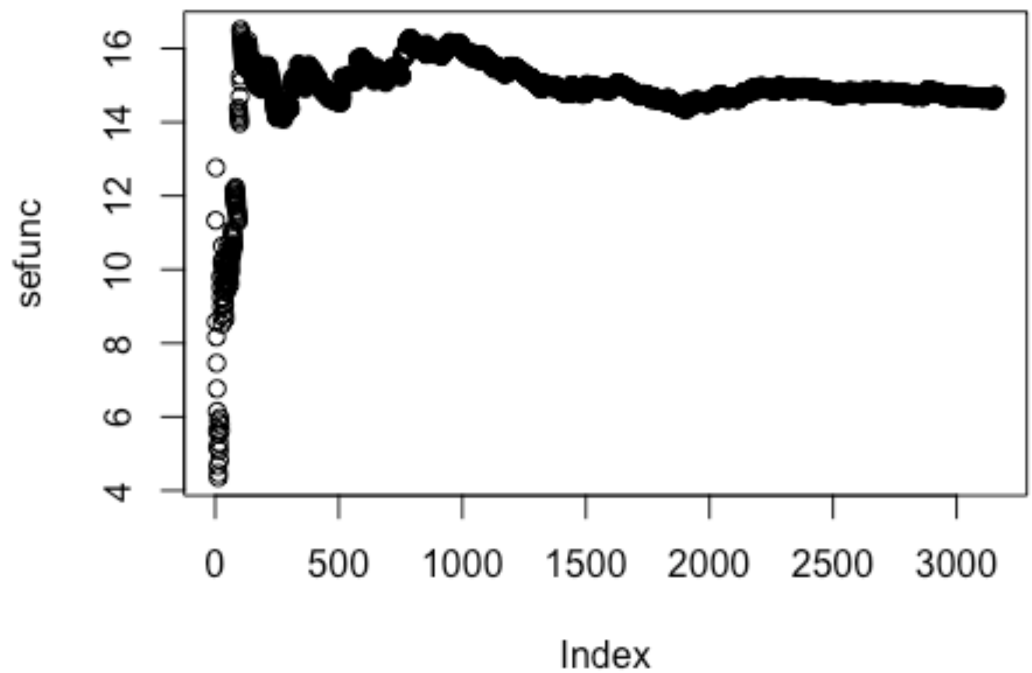
Need 3158 simulations at least

```
plot(diffp, type = "l",col ="red")
```

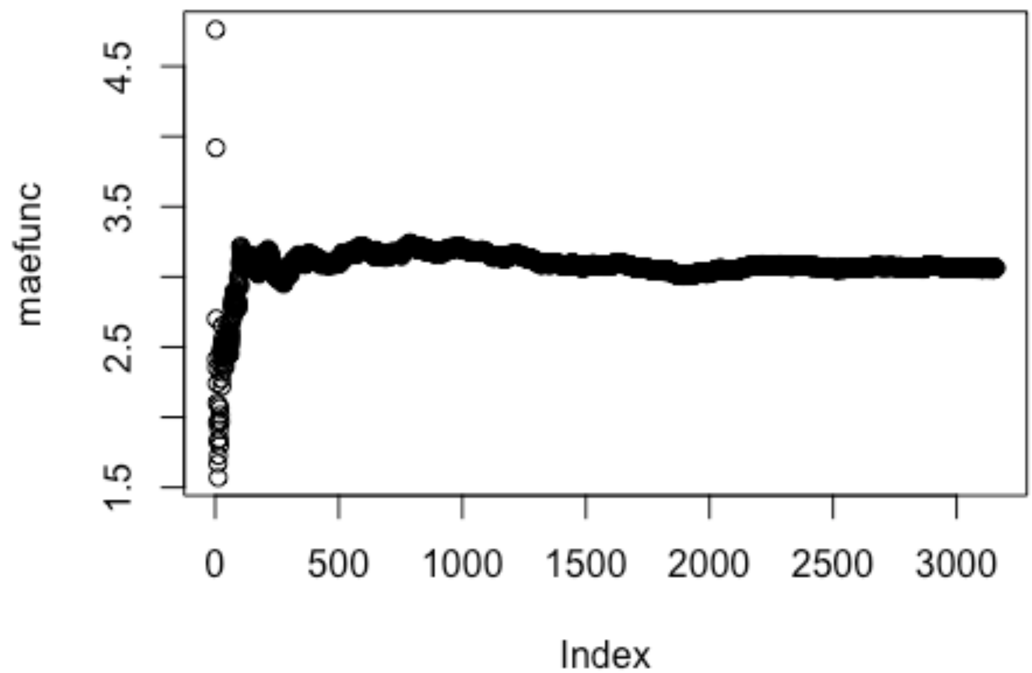



```
delta <- 1/252
totT<- mT/delta
r<- 0.05
sig <-.25

S0<-100
mT<-1/12
K<-100
diffp <-c()
sefunc<-c()
maefunc<-c()

numSim<-3158
Psim<-c()
psum <-0
for ( n in 1:numSim){
```

```r
simX <- c()
simX[1]<- 100


for (i in 2:totT){

  simX[i]<- simX[i-1]+r*simX[i-1]*delta + sig*simX[i-1]*sqrt(delta)*rnorm(1)
}
Psim[n] <- max(0,K-simX[totT])*exp(-r*mT)
psum<- psum+ Psim[n]

Pestimate<- ( psum/ n)
diffp[n]<-Pestimate- 2.669393

SEsum <-0
MAEsum<-0
for ( j in 1:n){
  SEsum<- SEsum+(Psim[j]-Pestimate)^2
  MAEsum<- MAEsum+ abs(Psim[j]-Pestimate)
}
sefunc[n] <- SEsum/(n-1)

maefunc[n] <-MAEsum / (n -1)

}

plot(sefunc)
```

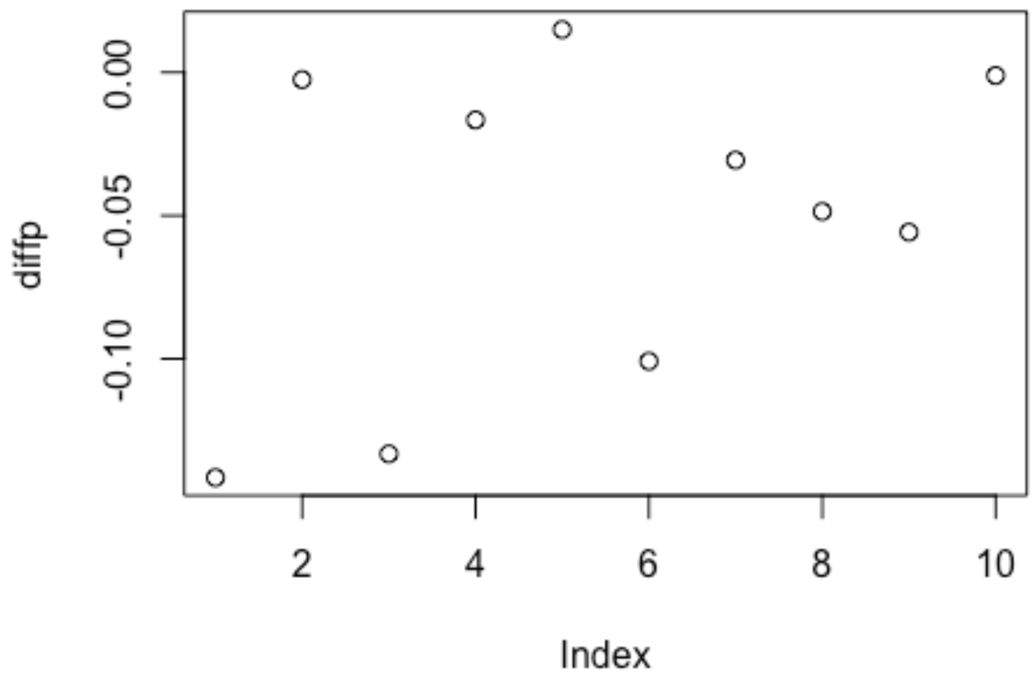



```r
plot(maefunc)
```

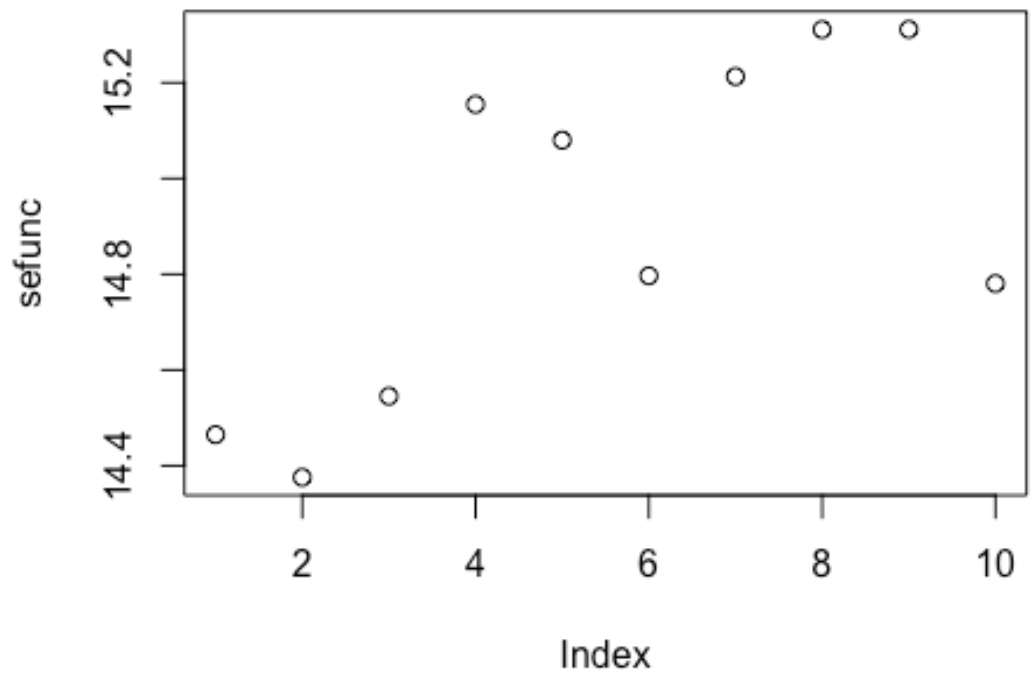



```
r<- 0.05
sig <-.25

S0<-100
mT<-1/12
K<-100
diffp <-c()
sefunc<-c()
maefunc<-c()

numSim<-3158

for (k in 1:10){
delta <- 1/252/k
totT<- mT/delta

Psim<-c()
psum <-0
for ( n in 1:numSim){
simX <- c()
simX[1]<- 100
```

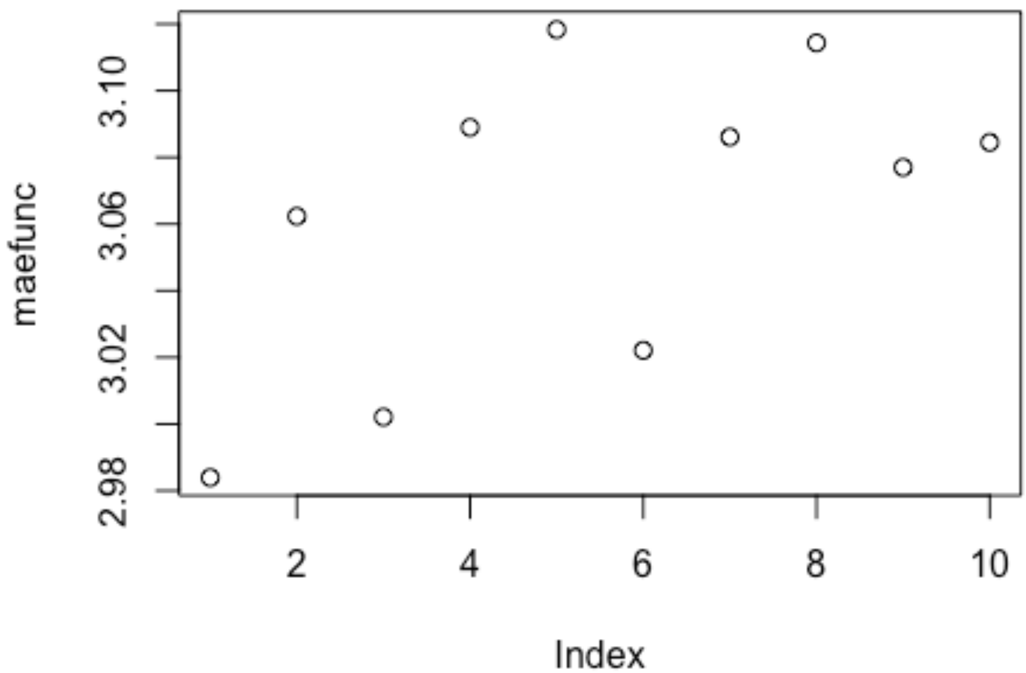
```
for (i in 2:totT){

  simX[i]<- simX[i-1]+r*simX[i-1]*delta + sig*simX[i-1]*sqrt(delta)*rnorm(1)
}
Psim[n] <- max(0,K-simX[totT])*exp(-r*mT)
psum<- psum+ Psim[n]
}




Pestimate<- ( psum/ numSim)
diffp[k]<-Pestimate- 2.669393



SEsum <-0
MAEsum<-0
for ( j in 1:n){
  SEsum<- SEsum+(Psim[j]-Pestimate)^2
  MAEsum<- MAEsum+ abs(Psim[j]-Pestimate)
}
sefunc[k] <- SEsum/(numSim-1)

maefunc[k] <-MAEsum / (numSim -1)



}
plot(diffp)
```




```
plot(sefunc)
```




```
plot(maefunc)
```



The diffp is not converging after we reduced delta to 1/2520, so number of simulations are more important. It's better to run 2 times as many simulations than to reduce delta in half