微積總棟員 Calculus - Exercises

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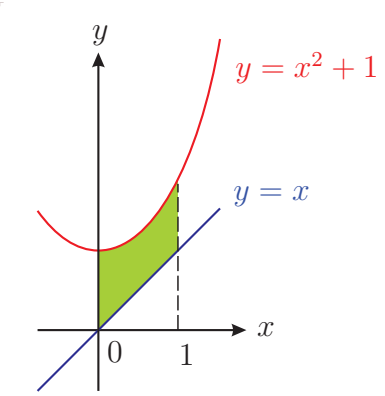
The Area between Curves (I) (曲線所圍區域的面積 (I))

1. Find the area of the region bounded by $y = x^2 + 1$, $y = x$, $x = 0$ and $x = 1$.
2. Find the area between the curves $y = 2 - x^2$ and $y = -x$.
3. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$.

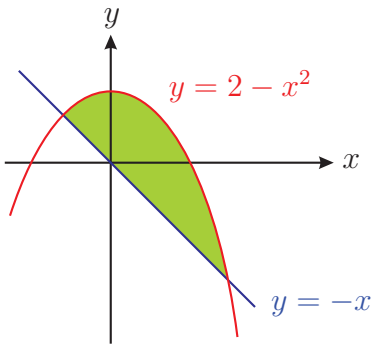
Solution [註：本解答僅提示重點，請自行補足細節流程。]

1.



$$\text{area} = \int_0^1 [(x^2+1) - x]\,dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + x\right]_0^1 = \frac{5}{6}$$

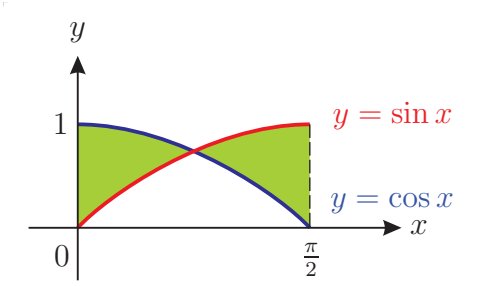
2.



Note that $y = 2 - x^2 = -x \iff x^2 - x - 2 = 0 \iff x = -1$ or 2.

$$\text{area} = \int_{-1}^2 [(2-x^2) - (-x)]\,dx = \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\right]_{-1}^2 = \frac{9}{2}$$

3.



Note that for $0 < x < \pi/2$, $y = \sin x = \cos x \iff x = \pi/4$.

$$\text{area} = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2} = 2\sqrt{2} - 2$$

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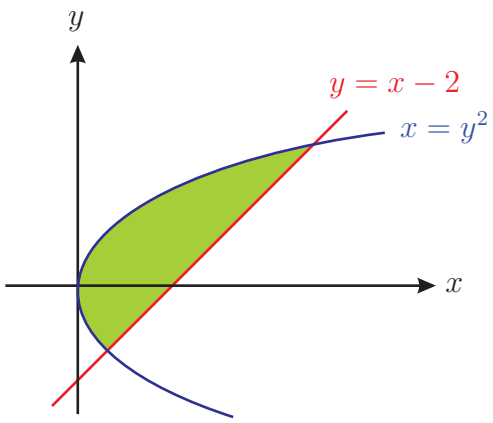
The Area between Curves (II) (曲線所圍區域的面積 (II))

1. Find the area between the curves $y = x - 2$ and $x = y^2$.
2. Find the area between the curves $x = y^2 - 4y$ and $x = 2y - y^2$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

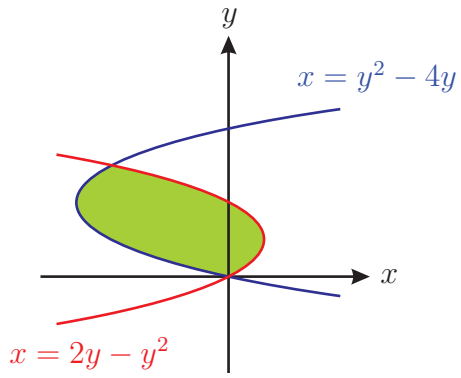
1.



Note that $x = y + 2 = y^2 \iff y^2 - y - 2 = 0 \iff y = -1$ or 2.

$$\text{area} = \int_{-1}^{2} [(y+2) - y^2]\, dy = \left[-\frac{1}{3}y^3 + \frac{1}{2}y^2 + 2y \right]_{-1}^{2} = \frac{9}{2}$$

2.



Note that $x = y^2 - 4y = 2y - y^2 \iff 2y^2 - 6y = 0 \iff y = 0$ or 3.

$$\text{area} = \int_{0}^{3} [(2y - y^2) - (y^2 - 4y)]\, dy = \left[-\frac{2}{3}y^3 + 3y^2 \right]_{0}^{3} = 9$$

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The Average of a Function (函數的平均)

1. Find the average of the function $f(x)$ on the given interval.
   (a) $f(x) = \sqrt{x}$ on $[0, 4]$
   (b) $f(x) = \cos^6 x \sin x$ on $[0, \pi]$
2. Find the number t such that the average of the function $f(x) = 4x^3 - 6x^2 - 2x$ on the interval $[0, t]$ is equal to -2.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\bar{f} = \dfrac{1}{4}\displaystyle\int_0^4 \sqrt{x}\,dx = \frac{1}{4}\left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{4}{3}$

   (b) Let $u = \cos x \Longrightarrow du = -\sin x\,dx$ & $\begin{cases} u(\pi) = -1 \\ u(0) = 1 \end{cases}$. We get

   $$\bar{f} = \frac{1}{\pi}\int_0^\pi \cos^6 x \sin x\,dx = -\frac{1}{\pi}\int_1^{-1} u^6\,du = -\frac{1}{\pi}\left[\frac{1}{7}u^7\right]_1^{-1} = \frac{2}{7\pi}$$

2. $\bar{f} = \dfrac{1}{t}\displaystyle\int_0^t (4x^3 - 6x^2 - 2x)\,dx = \frac{1}{t}\left[x^4 - 2x^3 - x^2\right]_0^t = t^3 - 2t^2 - t = -2$
   $\Longrightarrow (t+1)(t-1)(t-2) = 0 \Longrightarrow t = 1$ or 2. $(t > 0)$

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Applications of the Integral (積分的應用) [綜合練習]

1. Find the area between the curves $y = 2 - x^2$ and $y = -x$.
2. Find the area between the curves $y = x - 2$ and $x = y^2$.
3. Find the area between the curves $x = y^2 - 4y$ and $x = 2y - y^2$.
4. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$.
5. Find the average of the function $f(x) = \cos^6 x \sin x$ on $[0, \pi]$.
6. Find the number t such that the average of the function $f(x) = 4x^3 - 6x^2 - 2x$ on the interval $[0, t]$ is equal to -2.