Calculus - Exercises

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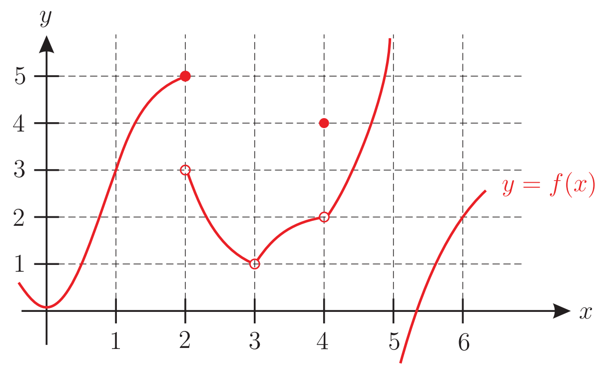
The Limit of a Function (函數的極限)

1. Find the values and limits of the function $f(x)$ given below at $a = 1, 2, 3, 4, 5, 6$.



Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f(1) = \lim_{x\to 1} f(x) = 3$

   (b) $f(2) = 5$ and $\lim_{x\to 2} f(x)$ does not exist.

   (c) $f(3)$ is undefined and $\lim_{x\to 3} f(x) = 1$

   (d) $f(4) = 4$ and $\lim_{x\to 4} f(x) = 2$

   (e) $f(5)$ is undefined and $\lim_{x\to 5} f(x)$ does not exist.

   (f) $f(6) = \lim_{x\to 6} f(x) = 2$

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Basic Formulas for Limits (極限的運算公式)

1. Suppose that $\lim_{x\to 2} f(x) = 3$, $\lim_{x\to 2} g(x) = -1$ and $\lim_{x\to 2} h(x) = 5$. Evaluate the following limits.

   (a) $\lim_{x\to 2}[2f(x) + 5g(x)]$

   (b) $\lim_{x\to 2}[f(x)]^3$

   (c) $\lim_{x\to 2}\sqrt{h(x) - 4g(x)}$

   (d) $\lim_{x\to 2}\frac{f(x)g(x)}{[h(x)]^2}$

2. Find the following limits.

   (a) $\lim_{x\to -1}(x^3 + 4x - 2)$

   (b) $\lim_{x\to 2}\frac{\sqrt{4x+3}}{x^2+x+1}$

   (c) $\lim_{x\to 3}\frac{x^2-2x-3}{x^2-x-6}$

   (d) $\lim_{x\to -2}\frac{x^2-2x-8}{x^2+3x+2}$

   (e) $\lim_{h\to 0}\frac{(2+h)^3-8}{h}$

   (f) $\lim_{x\to 8}\frac{\sqrt[3]{x}-2}{x-8}$

   (g) $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$

   (h) $\lim_{x\to 0}\frac{\sqrt{x^2+16}-4}{x^2}$

   (i) $\lim_{t\to 0}\frac{t^2}{\sqrt{t^2+9}-3}$

3. If $\lim_{x\to 1}\frac{f(x)-5}{x-1} = 8$, find $\lim_{x\to 1} f(x)$.

4. Find the number c such that $\lim_{x \to -2} \dfrac{3x^2 + cx + c + 3}{x^2 + x - 2}$ exists and evaluate this limit.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\lim_{x\to 2}[2f(x)+5g(x)] = 2\times 3 + 5\times(-1) = 1$

   (b) $\lim_{x\to 2}[f(x)]^3 = 3^3 = 27$

   (c) $\lim_{x\to 2}\sqrt{h(x)-4g(x)} = \sqrt{5-4\times(-1)} = 3$

   (d) $\lim_{x\to 2}\dfrac{f(x)g(x)}{[h(x)]^2} = \dfrac{3\times(-1)}{5^2} = -\dfrac{3}{25}$

2. (a) $\lim_{x\to -1}(x^3+4x-2) = (-1)^3 + 4\times(-1) - 2 = -7$

   (b) $\lim_{x\to 2}\dfrac{\sqrt{4x+3}}{x^2+x+1} = \dfrac{\sqrt{4\times 2+3}}{2^2+2+1} = \dfrac{\sqrt{11}}{7}$

   (c) $\lim_{x\to 3}\dfrac{x^2-2x-3}{x^2-x-6} = \lim_{x\to 3}\dfrac{(x+1)(x-3)}{(x+2)(x-3)} = \lim_{x\to 3}\dfrac{x+1}{x+2} = \dfrac{4}{5}$

   (d) $\lim_{x\to -2}\dfrac{x^2-2x-8}{x^2+3x+2} = \lim_{x\to -2}\dfrac{(x+2)(x-4)}{(x+2)(x+1)} = \lim_{x\to -2}\dfrac{x-4}{x+1} = 6$

   (e) $\lim_{h\to 0}\dfrac{(2+h)^3-8}{h} = \lim_{h\to 0}\dfrac{h(12+6h+h^2)}{h}$
   $= \lim_{h\to 0}(12+6h+h^2) = 12$

   (f) $\lim_{x\to 8}\dfrac{\sqrt[3]{x}-2}{x-8} = \lim_{x\to 8}\dfrac{(\sqrt[3]{x}-2)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)}{(x-8)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)} = \lim_{x\to 8}\dfrac{x-8}{(x-8)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)}$
   $= \lim_{x\to 8}\dfrac{1}{\sqrt[3]{x^2}+2\sqrt[3]{x}+4} = \dfrac{1}{12}$

   (g) $\lim_{x\to 9}\dfrac{\sqrt{x}-3}{x-9} = \lim_{x\to 9}\dfrac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9}\dfrac{x-9}{(x-9)(\sqrt{x}+3)}$
   $= \lim_{x\to 9}\dfrac{1}{\sqrt{x}+3} = \dfrac{1}{6}$

   (h) $\lim_{x\to 0}\dfrac{\sqrt{x^2+16}-4}{x^2} = \lim_{x\to 0}\dfrac{(\sqrt{x^2+16}-4)(\sqrt{x^2+16}+4)}{x^2(\sqrt{x^2+16}+4)}$
   $= \lim_{x\to 0}\dfrac{x^2}{x^2(\sqrt{x^2+16}+4)} = \lim_{x\to 0}\dfrac{1}{\sqrt{x^2+16}+4} = \dfrac{1}{8}$

   (i) $\lim_{t\to 0}\dfrac{t^2}{\sqrt{t^2+9}-3} = \lim_{t\to 0}\dfrac{t^2(\sqrt{t^2+9}+3)}{(\sqrt{t^2+9}-3)(\sqrt{t^2+9}+3)} = \lim_{t\to 0}\dfrac{t^2(\sqrt{t^2+9}+3)}{t^2}$
   $= \lim_{t\to 0}(\sqrt{t^2+9}+3) = 6$

3. $\lim_{x\to 1} f(x) = \lim_{x\to 1}\left[\frac{f(x)-5}{x-1}\times(x-1)+5\right]$
$$= \left(\lim_{x\to 1}\frac{f(x)-5}{x-1}\right)\times\left(\lim_{x\to 1}(x-1)\right)+\left(\lim_{x\to 1}5\right) = 8\times 0+5=5$$

4. Suppose $\lim_{x\to -2}\frac{3x^2+cx+c+3}{x^2+x-2} = L$. We have that
$$\lim_{x\to -2}(3x^2+cx+c+3) = \lim_{x\to -2}\left(\frac{3x^2+cx+c+3}{x^2+x-2}\times(x^2+x-2)\right)$$
$$= \left(\lim_{x\to -2}\frac{3x^2+cx+c+3}{x^2+x-2}\right)\times\left(\lim_{x\to -2}(x^2+x-2)\right) = L\times 0 = 0$$
$$\Longrightarrow 3\times(-2)^2+c\times(-2)+c+3 = 15-c=0 \Longrightarrow c=15$$
In addition,
$$\lim_{x\to -2}\frac{3x^2+cx+c+3}{x^2+x-2} = \lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2}$$
$$= \lim_{x\to -2}\frac{3(x+3)(x+2)}{(x-1)(x+2)} = \lim_{x\to -2}\frac{3(x+3)}{(x-1)} = -1$$

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The Squeeze Theorem (夾擠定理)

1. Evaluate $\lim\limits_{x\to 0} x^4 \cos\dfrac{1}{x^2}$.
2. If $8x-9 \le f(x) \le 4x^2-8x+7$ for $x \ge 0$, find $\lim\limits_{x\to 2} f(x)$.
3. Evaluate $\lim\limits_{x\to 0} f(x)$ if

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is a rational number (有理數)}, \\ 0 & \text{if } x \text{ is an irrational number (無理數)}. \end{cases}$$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. $\because$ For $x \neq 0$, $-1 \leq \cos\dfrac{1}{x^2} \leq 1 \Longrightarrow -x^4 \leq x^4\cos\dfrac{1}{x^2} \leq x^4$.

   In addition, we have $\lim\limits_{x\to 0}(-x^4) = \lim\limits_{x\to 0} x^4 = 0$.

   $\therefore \lim\limits_{x\to 0} x^4\cos\dfrac{1}{x^2} = 0$ by the squeeze theorem.

2. $\because 8x - 9 \leq f(x) \leq 4x^2 - 8x + 7$ for $x \geq 0$ and

   $\lim\limits_{x\to 2}(8x-9) = \lim\limits_{x\to 2}(4x^2 - 8x + 7) = 7$.

   $\therefore \lim\limits_{x\to 2} f(x) = 7$ by the squeeze theorem.

3. $\because 0 \leq f(x) \leq x^2$ for all real number x and $\lim\limits_{x\to 0} 0 = \lim\limits_{x\to 0} x^2 = 0$.

   $\therefore \lim\limits_{x\to 0} f(x) = 0$ by the squeeze theorem.

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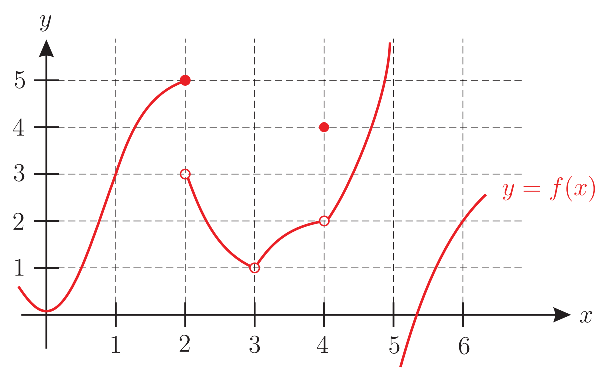
One-Sided Limits (單邊極限)

1. Find $\lim\limits_{x\to a^-} f(x)$, $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a} f(x)$ when $a = 1, 2, 3, 4, 5, 6$.



2. Find the following limits.

   (a) $\lim\limits_{x\to 4^-} [\![x]\!]$

   (b) $\lim\limits_{x\to 1} [\![x]\!]$

   (c) $\lim\limits_{x\to \pi} [\![x]\!]$

   (d) $\lim\limits_{x\to 3^-} x[\![x]\!]$

   (e) $\lim\limits_{x\to 0^+} \dfrac{[\![x]\!]}{x}$

   (f) $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

   (g) $\lim\limits_{x\to 2} \dfrac{x^2 - x - 2}{|x-2|}$

3. Find $\lim\limits_{x\to 1} g(x)$ if $g(x) = \begin{cases} 3x^2 + x - 2 & \text{if } x < 1, \\ x^2 - 2x + 3 & \text{if } x \geq 1. \end{cases}$

4. If $-x^2 + 4x - 3 \leq f(x) \leq x^2 - 2x + 1$ for $x > 2$, find $\lim\limits_{x\to 2^+} f(x)$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\begin{cases} \lim\limits_{x\to 1^-} f(x) = 3 \\ \lim\limits_{x\to 1^+} f(x) = 3 \end{cases} \Longrightarrow \lim\limits_{x\to 1} f(x) = 3$

   (b) $\begin{cases} \lim\limits_{x\to 2^-} f(x) = 5 \\ \lim\limits_{x\to 2^+} f(x) = 3 \end{cases} \Longrightarrow \lim\limits_{x\to 2} f(x)$ does not exist.

   (c) $\begin{cases} \lim\limits_{x\to 3^-} f(x) = 1 \\ \lim\limits_{x\to 3^+} f(x) = 1 \end{cases} \Longrightarrow \lim\limits_{x\to 3} f(x) = 1$

   (d) $\begin{cases} \lim\limits_{x\to 4^-} f(x) = 2 \\ \lim\limits_{x\to 4^+} f(x) = 2 \end{cases} \Longrightarrow \lim\limits_{x\to 4} f(x) = 2$

   (e) $\begin{cases} \lim\limits_{x\to 5^-} f(x) \text{ does not exist.} \\ \lim\limits_{x\to 5^+} f(x) \text{ does not exist.} \end{cases} \Longrightarrow \lim\limits_{x\to 5} f(x)$ does not exist.

   (f) $\begin{cases} \lim\limits_{x\to 6^-} f(x) = 2 \\ \lim\limits_{x\to 6^+} f(x) = 2 \end{cases} \Longrightarrow \lim\limits_{x\to 6} f(x) = 2$

2. (a) $\lim\limits_{x\to 4^-} [\![x]\!] = \lim\limits_{x\to 4^-} 3 = 3$

   (b) $\because \begin{cases} \lim\limits_{x\to 1^-} [\![x]\!] = \lim\limits_{x\to 1^-} 0 = 0 \\ \lim\limits_{x\to 1^+} [\![x]\!] = \lim\limits_{x\to 1^+} 1 = 1 \end{cases} \quad \therefore \lim\limits_{x\to 1} [\![x]\!]$ does not exist.

   (c) $\lim\limits_{x\to \pi} [\![x]\!] = \lim\limits_{x\to \pi} 3 = 3$

   (d) $\lim\limits_{x\to 3^-} x[\![x]\!] = \lim\limits_{x\to 3^-} x \cdot 2 = 6$

   (e) $\lim\limits_{x\to 0^+} \dfrac{[\![x]\!]}{x} = \lim\limits_{x\to 0^+} \dfrac{0}{x} = \lim\limits_{x\to 0^+} 0 = 0$

   (f) $\because \begin{cases} \lim\limits_{x\to 0^-} \dfrac{|x|}{x} = \lim\limits_{x\to 0^-} \dfrac{(-x)}{x} = \lim\limits_{x\to 0^-} (-1) = -1 \\ \lim\limits_{x\to 0^+} \dfrac{|x|}{x} = \lim\limits_{x\to 0^+} \dfrac{x}{x} = \lim\limits_{x\to 0^+} 1 = 1 \end{cases}$

   $\therefore \lim\limits_{x\to 0} \dfrac{|x|}{x}$ does not exist.

(g) $\because \begin{cases} \displaystyle\lim_{x\to 2^-} \frac{x^2-x-2}{|x-2|} = \lim_{x\to 2^-} \frac{(x-2)(x+1)}{-(x-2)} = \lim_{x\to 2^-} (-(x+1)) = -3 \\ \displaystyle\lim_{x\to 2^+} \frac{x^2-x-2}{|x-2|} = \lim_{x\to 2^+} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2^+} (x+1) = 3 \end{cases}$

$\therefore \displaystyle\lim_{x\to 1} \frac{x^2-x-2}{|x-2|}$ does not exist.

3. $\because \begin{cases} \displaystyle\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} (3x^2+x-2) = 2 \\ \displaystyle\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (x^2-2x+3) = 2 \end{cases}$

$\therefore \displaystyle\lim_{x\to 1} g(x) = 2.$

4. $\because -x^2+4x-3 \le f(x) \le x^2-2x+1$ for $x > 2$ and

$\displaystyle\lim_{x\to 2^+} (-x^2+4x-3) = \lim_{x\to 2^+} (x^2-2x+1) = 1.$

$\therefore \displaystyle\lim_{x\to 2^+} f(x) = 1$ by the squeeze theorem.

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Two Important Limits (重要的極限公式)

1. Find the following limits.

   (a) $\lim_{x\to 0}\frac{\sin(5x)}{4x}$

   (b) $\lim_{\theta\to 0}\frac{\sin(6\theta)}{\sin(2\theta)}$

   (c) $\lim_{x\to 0}\frac{\sin(x^2)}{x}$

   (d) $\lim_{\theta\to 0}\frac{\tan(3\theta)}{\sin(2\theta)}$

   (e) $\lim_{x\to 0}\frac{\tan(3x)}{\tan(5x)}$

   (f) $\lim_{t\to 0}\frac{\cos(4t)-1}{3t}$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{4x}=\lim_{x\to 0}\left(\frac{\sin(5x)}{5x}\times\frac{5}{4}\right)=1\times\frac{5}{4}=\frac{5}{4}$

   (b) $\displaystyle\lim_{\theta\to 0}\frac{\sin(6\theta)}{\sin(2\theta)}=\lim_{\theta\to 0}\left(\frac{\sin(6\theta)}{6\theta}\times\frac{2\theta}{\sin(2\theta)}\times\frac{6}{2}\right)=1\times 1\times 3=3$

   (c) $\displaystyle\lim_{x\to 0}\frac{\sin(x^2)}{x}=\lim_{x\to 0}\left(\frac{\sin(x^2)}{x^2}\times x\right)=1\times 0=0$

   (d) $\displaystyle\lim_{\theta\to 0}\frac{\tan(3\theta)}{\sin(2\theta)}=\lim_{\theta\to 0}\left(\frac{\left(\frac{\sin(3\theta)}{\cos(3\theta)}\right)}{\sin(2\theta)}\right)=\lim_{\theta\to 0}\left(\frac{\sin(3\theta)}{3\theta}\times\frac{2\theta}{\sin(2\theta)}\times\frac{1}{\cos(3\theta)}\times\frac{3}{2}\right)$

   $\displaystyle=1\times 1\times 1\times\frac{3}{2}=\frac{3}{2}$

   (e) $\displaystyle\lim_{x\to 0}\frac{\tan(3x)}{\tan(5x)}=\lim_{x\to 0}\frac{\left(\frac{\sin(3x)}{\cos(3x)}\right)}{\left(\frac{\sin(5x)}{\cos(5x)}\right)}=\lim_{x\to 0}\left(\frac{\sin(3x)}{3x}\times\frac{5x}{\sin(5x)}\times\frac{\cos(5x)}{\cos(3x)}\times\frac{3}{5}\right)$

   $\displaystyle=1\times 1\times 1\times\frac{3}{5}=\frac{3}{5}$

   (f) $\displaystyle\lim_{t\to 0}\frac{\cos(4t)-1}{3t}=\lim_{t\to 0}\left(\frac{\cos(4t)-1}{4t}\times\frac{4}{3}\right)=0\times\frac{4}{3}=0$

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Infinite Limits and Vertical Asymptotes (無窮大的極限值與鉛直漸近線)

1. Find the following limits.

   (a) $\displaystyle\lim_{x\to 4^+} \frac{2x}{x-4}$

   (b) $\displaystyle\lim_{x\to 3^-} \frac{x-5}{x-3}$

   (c) $\displaystyle\lim_{x\to -1^-} \frac{x+4}{x+1}$

   (d) $\displaystyle\lim_{x\to 1^+} \frac{x-3}{x^2+x-2}$

   (e) $\displaystyle\lim_{x\to 2^-} \frac{x^2-4x+4}{x^2-x-2}$

   (f) $\displaystyle\lim_{x\to 1} \frac{x+2}{x^2-2x+1}$

   (g) $\displaystyle\lim_{x\to -2^+} \frac{x^2+2x-3}{x^2+3x+2}$

   (h) $\displaystyle\lim_{\theta\to \frac{\pi}{2}^-} \tan\theta$

   (i) $\displaystyle\lim_{\theta\to \frac{\pi}{2}^+} \tan\theta$

2. Find the vertical asymptote(s) of the curve $y = \dfrac{x+1}{x^2+x-6}$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\because x-4>0$ for $x>4$, $\lim\limits_{x\to 4^+}(x-4)=0$ and $\lim\limits_{x\to 4^+}(2x)=8$.

   $\therefore \lim\limits_{x\to 4^+}\dfrac{2x}{x-4}=\infty$

   (b) $\because x-3<0$ for $x<3$, $\lim\limits_{x\to 3^-}(x-3)=0$ and $\lim\limits_{x\to 3^-}(x-5)=-2$.

   $\therefore \lim\limits_{x\to 3^-}\dfrac{x-5}{x-3}=\infty$

   (c) $\because x+1<0$ for $x<-1$, $\lim\limits_{x\to -1^-}(x+1)=0$ and $\lim\limits_{x\to -1^-}(x+4)=3$.

   $\therefore \lim\limits_{x\to -1^-}\dfrac{x+4}{x+1}=-\infty$

   (d) $\because x^2+x-2=(x-1)(x+2)>0$ for $x>1$, $\lim\limits_{x\to 1^+}(x^2+x-2)=0$ and $\lim\limits_{x\to 1^+}(x-3)=-2$.

   $\therefore \lim\limits_{x\to 1^+}\dfrac{x-3}{x^2+x-2}=-\infty$

   (e) $\lim\limits_{x\to 2^-}\dfrac{x^2-4x+4}{x^2-x-2}=\lim\limits_{x\to 2^-}\dfrac{(x-2)^2}{(x+1)(x-2)}=\lim\limits_{x\to 2^-}\dfrac{x-2}{x+1}=0$

   (f) $\because x^2-2x+1=(x-1)^2>0$ for $x\neq 1$, $\lim\limits_{x\to 1}(x^2-2x+1)=0$ and $\lim\limits_{x\to 1}(x+2)=3$.

   $\therefore \lim\limits_{x\to 1}\dfrac{x+2}{x^2-2x+1}=\infty$

   (g) $\because x^2+3x+2=(x+2)(x+1)<0$ for $-2<x<-1$, $\lim\limits_{x\to -2^+}(x^2+3x+2)=0$ and $\lim\limits_{x\to -2^+}(x^2+2x-3)=-3$.

   $\therefore \lim\limits_{x\to -2^+}\dfrac{x^2+2x-3}{x^2+3x+2}=\infty$

   (h) $\because \cos\theta>0$ for $0<\theta<\frac{\pi}{2}$, $\lim\limits_{\theta\to \frac{\pi}{2}^-}\cos\theta=0$ and $\lim\limits_{\theta\to \frac{\pi}{2}^-}\sin\theta=1$.

   $\therefore \lim\limits_{\theta\to \frac{\pi}{2}^-}\tan\theta=\lim\limits_{\theta\to \frac{\pi}{2}^-}\dfrac{\sin\theta}{\cos\theta}=\infty$

   (i) $\because \cos\theta<0$ for $\frac{\pi}{2}<\theta<\pi$, $\lim\limits_{\theta\to \frac{\pi}{2}^+}\cos\theta=0$ and $\lim\limits_{\theta\to \frac{\pi}{2}^+}\sin\theta=1$.

   $\therefore \lim\limits_{\theta\to \frac{\pi}{2}^+}\tan\theta=\lim\limits_{\theta\to \frac{\pi}{2}^+}\dfrac{\sin\theta}{\cos\theta}=-\infty$

2. Since $x^2+x-6=0 \Longrightarrow (x+3)(x-2)=0$, we conclude that the vertical asymptotes may be $x=-3$ or $x=2$.

(i) $\because \lim_{x\to -3^+} \dfrac{x+1}{x^2+x-6} = \lim_{x\to -3^+} \dfrac{x+1}{(x+3)(x-2)} = \infty$

$\left(\text{or } \lim_{x\to -3^-} \dfrac{x+1}{x^2+x-6} = \lim_{x\to -3^-} \dfrac{x+1}{(x+3)(x-2)} = -\infty\right)$

$\therefore x = -3$ is a vertical asymptote.

(ii) $\because \lim_{x\to 2^+} \dfrac{x+1}{x^2+x-6} = \lim_{x\to 2^+} \dfrac{x+1}{(x+3)(x-2)} = \infty$

$\left(\text{or } \lim_{x\to 2^-} \dfrac{x+1}{x^2+x-6} = \lim_{x\to 2^-} \dfrac{x+1}{(x+3)(x-2)} = -\infty\right)$

$\therefore x = 2$ is a vertical asymptote.

From (i) and (ii), the vertical asymptotes are $x = -3$ and $x = 2$.

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Limits at Infinity and Horizontal Asymptotes (無窮遠處的極限與水平漸近線)

1. Find the following limits.

   (a) $\displaystyle\lim_{x\to\infty}\frac{4x^2+2x+3}{5x^2-x+4}$

   (b) $\displaystyle\lim_{x\to-\infty}\frac{2x^3-x^2+5}{x^3+3x+1}$

   (c) $\displaystyle\lim_{x\to\infty}\frac{6x+5}{7x^2+4x+3}$

   (d) $\displaystyle\lim_{x\to\infty}\frac{x^3-x+1}{3x^2+2}$

   (e) $\displaystyle\lim_{x\to-\infty}\frac{3x}{\sqrt{x^2+5}}$

   (f) $\displaystyle\lim_{x\to\infty}(\sqrt{x^2+1}-x)$

   (g) $\displaystyle\lim_{x\to\infty}x\sin\frac{1}{x}$

   (h) $\displaystyle\lim_{x\to\infty}\frac{\sin x}{x}$

2. Find the horizontal and vertical asymptotes of the following curves.

   (a) $\displaystyle y=\frac{x^2+x-6}{x^2-x-2}$

   (b) $\displaystyle y=\frac{x+3}{\sqrt{4x^2+2x+1}}$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\lim_{x\to\infty}\frac{4x^2+2x+3}{5x^2-x+4}=\lim_{x\to\infty}\frac{4+\frac{2}{x}+\frac{3}{x^2}}{5-\frac{1}{x}+\frac{4}{x^2}}=\frac{4}{5}$

   (b) $\displaystyle\lim_{x\to-\infty}\frac{2x^3-x^2+5}{x^3+3x+1}=\lim_{x\to-\infty}\frac{2-\frac{1}{x}+\frac{5}{x^3}}{1+\frac{3}{x^2}+\frac{1}{x^3}}=2$

   (c) $\displaystyle\lim_{x\to\infty}\frac{6x+5}{7x^2+4x+3}=\lim_{x\to\infty}\frac{\frac{6}{x}+\frac{5}{x^2}}{7+\frac{4}{x}+\frac{3}{x^2}}=0$

   (d) $\displaystyle\lim_{x\to\infty}\frac{x^3-x+1}{3x^2+2}=\lim_{x\to\infty}\frac{x-\frac{1}{x}+\frac{1}{x^2}}{3+\frac{2}{x^2}}=\infty$

   (e) $\displaystyle\lim_{x\to-\infty}\frac{3x}{\sqrt{x^2+5}}=\lim_{x\to-\infty}\frac{-3}{\sqrt{1+\frac{5}{x^2}}}$ (since $\sqrt{x^2}=-x$ for $x<0$)
   $=-3$

   (f) $\displaystyle\lim_{x\to\infty}(\sqrt{x^2+1}-x)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x}$
   $\displaystyle=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=0$

   (g) $\displaystyle\lim_{x\to\infty}x\sin\frac{1}{x}=\lim_{t\to0^+}\frac{\sin t}{t}=1$
   (Let $t=\frac{1}{x}$. We get $x=\frac{1}{t}$ and $x\to\infty\Longleftrightarrow t\to0^+$)

   (h) $\because$ For $x>0$, $-1\le\sin x\le1\Longrightarrow -\dfrac{1}{x}\le\dfrac{\sin x}{x}\le\dfrac{1}{x}$.
   In addition, we have $\displaystyle\lim_{x\to\infty}(-\frac{1}{x})=\lim_{x\to\infty}\frac{1}{x}=0$.
   $\displaystyle\therefore\lim_{x\to\infty}\frac{\sin x}{x}=0$ by the squeeze theorem.

2. (a) $\displaystyle\because\lim_{x\to\pm\infty}\frac{x^2+x-6}{x^2-x-2}=\lim_{x\to\pm\infty}\frac{1+\frac{1}{x}-\frac{6}{x^2}}{1-\frac{1}{x}-\frac{2}{x^2}}=1$
   $\therefore y=1$ is the only horizontal asymptote.
   Since $x^2-x-2=0\Longrightarrow(x+1)(x-2)=0$, we conclude that the vertical asymptotes may be $x=-1$ or $x=2$.
   $\displaystyle\because\lim_{x\to-1^-}\frac{x^2+x-6}{x^2-x-2}=\lim_{x\to-1^-}\frac{(x+3)(x-2)}{(x+1)(x-2)}=\lim_{x\to-1^-}\frac{x+3}{x+1}=-\infty$
   $\displaystyle\left(\text{or }\lim_{x\to-1^+}\frac{x^2+x-6}{x^2-x-2}=\lim_{x\to-1^+}\frac{(x+3)(x-2)}{(x+1)(x-2)}=\lim_{x\to-1^+}\frac{x+3}{x+1}=\infty\right)$

and $\displaystyle\lim_{x\to 2}\frac{x^2+x-6}{x^2-x-2}=\lim_{x\to 2}\frac{(x+3)(x-2)}{(x+1)(x-2)}=\lim_{x\to 2}\frac{x+3}{x+1}=\frac{5}{3}$

$\therefore x=-1$ is the only vertical asymptote.

(b) $\because \displaystyle\lim_{x\to\infty}\frac{x+3}{\sqrt{4x^2+2x+1}}=\lim_{x\to\infty}\frac{1+\frac{3}{x}}{\sqrt{4+\frac{2}{x}+\frac{1}{x^2}}}=\frac{1}{2}$

(since $\sqrt{x^2}=x$ for $x>0$)

and $\displaystyle\lim_{x\to-\infty}\frac{x+3}{\sqrt{4x^2+2x+1}}=\lim_{x\to-\infty}\frac{-1-\frac{3}{x}}{\sqrt{4+\frac{2}{x}+\frac{1}{x^2}}}=-\frac{1}{2}$

(since $\sqrt{x^2}=-x$ for $x<0$)

$\therefore y=\pm\dfrac{1}{2}$ are the horizontal asymptotes.

$\because 4x^2+2x+1>0$ for all real number x.

$\Longrightarrow f(x)=\dfrac{x+3}{\sqrt{4x^2+2x+1}}$ is continuous on $\mathbb{R}=(-\infty,\infty)$.

$\therefore$ There is no vertical asymptote.

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Continuous Functions (連續函數)

1. Determine whether the function f is continuous at the point a. If f is discontinuous at a, show that f has a removable discontinuity, a jump discontinuity or an infinity discontinuity there.

   (a) $f(x) = \begin{cases} \dfrac{x^2+x-6}{x-2} & \text{if } x \neq 2, \\ 5 & \text{if } x = 2, \end{cases}$ and $a = 2$.

   (b) $f(x) = \begin{cases} \dfrac{x^2-2x-3}{x^2-x-2} & \text{if } x \neq -1, \\ 2 & \text{if } x = -1, \end{cases}$ and $a = -1$.

   (c) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$ and $a = 0$.

   (d) $f(x) = \begin{cases} \sqrt{x^2+5}-1 & \text{if } x \leq 2, \\ x^2-x+1 & \text{if } x > 2, \end{cases}$ and $a = 2$.

   (e) $f(x) = \begin{cases} x^2-2x-2 & \text{if } x < 3, \\ -x^2+x+7 & \text{if } x \geq 3, \end{cases}$ and $a = 3$.

2. For what value of c is the function f continuous on $(-\infty, \infty)$?
   $$f(x) = \begin{cases} cx^2+x & \text{if } x < 2, \\ x^2+4x+c & \text{if } x \geq 2. \end{cases}$$

3. For what values of a and b is the function f continuous on $(-\infty, \infty)$?
   $$f(x) = \begin{cases} -ax^2-x-a & \text{if } x < -1, \\ ax^2+bx+6 & \text{if } -1 \leq x \leq 2, \\ 3x^2-bx-b & \text{if } x > 2. \end{cases}$$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\because \lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2} = \lim\limits_{x\to 2} \dfrac{(x-2)(x+3)}{x-2}$
$= \lim\limits_{x\to 2}(x+3) = 5 = f(2)$
$\therefore f$ is continuous at 2.

(b) $\because \lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} \dfrac{x^2-2x-3}{x^2-x-2} = \lim\limits_{x\to -1} \dfrac{(x+1)(x-3)}{(x+1)(x-2)}$
$= \lim\limits_{x\to -1} \dfrac{x-3}{x-2} = \dfrac{4}{3} \neq 2 = f(-1)$
$\therefore f$ has a removable discontinuity at -1.

(c) $\because \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{1}{x^2} = \infty$
$\therefore f$ has an infinity discontinuity at 0.

(d) $\because \begin{cases} \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \left(\sqrt{x^2+5}-1\right) = 2 \\ \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x^2-x+1) = 3 \end{cases}$
$\therefore f$ has a jump discontinuity at 2.

(e) $\because \begin{cases} \lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (x^2-2x-2) = 1 \\ \lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (-x^2+x+7) = 1 \end{cases}$
$\Longrightarrow \lim\limits_{x\to 3} f(x) = 1 = f(3)$
$\therefore f$ is continuous at 3.

2. $\because f$ is continuous at 2.
$\therefore \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = f(2) \Longrightarrow 4c+2 = 4+8+c \Longrightarrow c = \dfrac{10}{3}$.

3. $\because f$ is continuous at -1.
$\therefore \lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^+} f(x) = f(-1)$
$\Longrightarrow -a+1-a = a-b+6 \Longrightarrow 3a-b=-5$ .... (1)
$\because f$ is continuous at 2.
$\therefore \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = f(2)$
$\Longrightarrow 4a+2b+6 = 12-2b-b \Longrightarrow 4a+5b=6$ .... (2)
From (1) and (2), we get that $a=-1$ and $b=2$.

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Formulas for Continuous Functions (連續函數的運算)

1. Find the following limits.

   (a) $\lim_{x\to\pi} \cos(x + \sin x)$

   (b) $\lim_{x\to 2} \left|\dfrac{x^2 - 5x + 6}{x^2 - 4}\right|$

   (c) $\lim_{x\to -2} \sin\left(\dfrac{x^2 + 4x + 4}{x^2 + 3x + 2}\right)$

   (d) $\lim_{x\to \frac{\pi}{4}^+} \sqrt[3]{\tan |x|}$

   (e) $\lim_{x\to 4} \dfrac{\sqrt{x} + 2}{3|x| - x^2}$

2. Find intervals on which the following functions are continuous.

   (a) $f(x) = \sqrt{x^2 - 1} + x + 2$

   (b) $f(x) = \dfrac{\sqrt{x} + 3x}{x - 2}$

   (c) $f(x) = \dfrac{x - 1}{x^2 - 2x - 3}$

   (d) $f(x) = \sin(\cos(\sin x))$

   (e) $f(x) = \dfrac{\sin x}{x^2 + 2x + 3}$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\lim_{x\to\pi}\cos(x+\sin x)=\cos(\pi+\sin\pi)=\cos\pi=-1$

   (b) $\lim_{x\to 2}\left|\frac{x^2-5x+6}{x^2-4}\right|=\left|\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4}\right|=\left|\lim_{x\to 2}\frac{(x-2)(x-3)}{(x-2)(x+2)}\right|$
   $=\left|\lim_{x\to 2}\frac{x-3}{x+2}\right|=\left|-\frac{1}{4}\right|=\frac{1}{4}$

   (c) $\lim_{x\to -2}\sin\left(\frac{x^2+4x+4}{x^2+3x+2}\right)=\sin\left(\lim_{x\to -2}\frac{x^2+4x+4}{x^2+3x+2}\right)$
   $=\sin\left(\lim_{x\to -2}\frac{(x+2)^2}{(x+1)(x+2)}\right)=\sin\left(\lim_{x\to -2}\frac{x+2}{x+1}\right)=\sin 0=0$

   (d) $\lim_{x\to\frac{\pi}{4}^+}\sqrt[3]{\tan|x|}=\sqrt[3]{\tan|\frac{\pi}{4}|}=\sqrt[3]{1}=1$

   (e) $\lim_{x\to 4}\frac{\sqrt{x}+2}{3|x|-x^2}=\frac{\sqrt{4}+2}{3\times|4|-4^2}=-1$

2. (a) f is continuous on its domain $\{x\in\mathbb{R}:x^2-1\geq 0\}$
   $=\{x\in\mathbb{R}:x\leq -1 \text{ or } x\geq 1\}=(-\infty,-1]\cup[1,\infty)$.

   (b) f is continuous on its domain $\{x\in\mathbb{R}:x\geq 0 \text{ and } x-2\neq 0\}$
   $=\{x\in\mathbb{R}:x\geq 0 \text{ and } x\neq 2\}=[0,2)\cup(2,\infty)$.

   (c) f is continuous on its domain $\{x\in\mathbb{R}:x^2-2x-3\neq 0\}$
   $=\{x\in\mathbb{R}:x\neq -1,3\}=(-\infty,-1)\cup(-1,3)\cup(3,\infty)$.

   (d) f is continuous on its domain $\mathbb{R}=(-\infty,\infty)$.

   (e) f is continuous on its domain $\mathbb{R}=(-\infty,\infty)$.
   (since $x^2+2x+3>0$ for all real number x)

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The Intermediate Value Theorem (中間值定理)

1. Show that there is a root of the equation
$$\cos x = x \tag{1}$$
on $(0, 1)$.

2. Use Bolzano's Theorem to show that there is a root of the equation
$$x^3 + x - 1 = 0 \tag{2}$$
between 0 and 1.

3. A number c is called a fixed point (固定點) of a function $f(x)$ if $f(c) = c$.

   (a) Suppose $f(x)$ is a continuous function satisfying $0 \leq f(x) \leq 1$ on $[0, 1]$. Show that $f(x)$ has a fixed point in $[0, 1]$.

   (b) Show that the statement in (a) is not always true if the interval $[0, 1]$ is replaced by $(0, 1)$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. Define $f(x) = \cos x - x$. Then $f(x)$ is continuous on $[0, 1]$.
   $\because f(0) = \cos 0 - 0 = 1 > 0$ and $f(1) = \cos 1 - 1 < 0$.
   $\therefore$ By the intermediate value theorem, there exists a number
   $c \in (0, 1)$ such that $f(c) = 0$.
   $\Longrightarrow \cos c - c = 0$
   $\Longrightarrow c$ is a root of (1).

2. Define $f(x) = x^3 + x - 1$. We have that $f(x)$ is continuous on $[0, 1]$.
   $\because f(0)f(1) = (-1) \cdot 1 = -1 < 0$.
   $\therefore$ By Bolzano's Theorem, there exists a number $c \in (0, 1)$ such that
   $f(c) = 0$.
   $\Longrightarrow c^3 + c - 1 = 0$
   $\Longrightarrow c$ is a root of (2).

3. (a) Define $F(x) = f(x) - x$.
   $\because f(x)$ is continuous on $[0, 1]$.
   $\therefore F(x)$ is also continuous on $[0, 1]$.
   Moreover, we have $F(0) = f(0) - 0 \geq 0$ and $F(1) = f(1) - 1 \leq 0$.
   If $F(0) = f(0) - 0 = 0$, we get that $f(0) = 0$.
   $\Longrightarrow 0$ is a fixed point of $f(x)$.
   If $F(1) = f(1) - 1 = 0$, we get that $f(1) = 1$.
   $\Longrightarrow 1$ is a fixed point of $f(x)$.
   If $F(0) > 0$ and $F(1) < 0$, then by the intermediate value theorem, there exists a number $c \in (0, 1)$ such that $F(c) = f(c) - c = 0$.
   $\Longrightarrow f(c) = c \Longrightarrow c$ is a fixed point of $f(x)$.

   (b) Let $f(x) = x^2$ on $(0, 1)$. It is obvious that $f(x)$ is continuous and $0 \leq f(x) \leq 1$ on $(0, 1)$.
   But there is no point $c \in (0, 1)$ such that $f(c) = c^2 = c$.
   $\Longrightarrow f(x) = x^2$ has no fixed point in $(0, 1)$.

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The Limit (極限) [綜合練習]

1. Suppose that $\lim\limits_{x\to 2} f(x) = 3$, $\lim\limits_{x\to 2} g(x) = -1$ and $\lim\limits_{x\to 2} h(x) = 5$.
   Evaluate the following limits.

   (a) $\lim\limits_{x\to 2}[f(x)]^3$

   (b) $\lim\limits_{x\to 2}[2f(x) + 5g(x)]$

   (c) $\lim\limits_{x\to 2} \dfrac{f(x)g(x)}{[h(x)]^2}$

2. Find the following limits.

   (a) $\lim\limits_{x\to -1}(x^3 + 4x - 2)$

   (b) $\lim\limits_{x\to 0} \dfrac{\sin(5x)}{4x}$

   (c) $\lim\limits_{x\to \pi} \cos(x + \sin x)$

   (d) $\lim\limits_{x\to -\infty} \dfrac{2x^3 - x^2 + 5}{x^3 + 3x + 1}$

   (e) $\lim\limits_{x\to -\infty} \dfrac{3x}{\sqrt{x^2 + 5}}$

   (f) $\lim\limits_{x\to 2} \left|\dfrac{x^2 - 5x + 6}{x^2 - 4}\right|$

   (g) $\lim\limits_{\theta\to \frac{\pi}{2}^-} \tan\theta$

   (h) $\lim\limits_{x\to -2} \sin\left(\dfrac{x^2 + 4x + 4}{x^2 + 3x + 2}\right)$

   (i) $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

   (j) $\lim\limits_{t\to 0} \dfrac{t^2}{\sqrt{t^2 + 9} - 3}$

   (k) $\lim\limits_{x\to 2} \dfrac{x^2 - x - 2}{|x - 2|}$

   (l) $\lim\limits_{x\to 4^-} [\![x]\!]$

(m) $\displaystyle\lim_{x\to 4^+} \frac{2x}{x-4}$

(n) $\displaystyle\lim_{t\to 0} \frac{\cos(4t)-1}{3t}$

(o) $\displaystyle\lim_{x\to\infty} x\sin\frac{1}{x}$

(p) $\displaystyle\lim_{h\to 0} \frac{(2+h)^3-8}{h}$

(q) $\displaystyle\lim_{\theta\to 0} \frac{\sin(6\theta)}{\sin(2\theta)}$

(r) $\displaystyle\lim_{x\to 3} \frac{x^2-2x-3}{x^2-x-6}$

(s) $\displaystyle\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$

(t) $\displaystyle\lim_{x\to -2^+} \frac{x^2+2x-3}{x^2+3x+2}$

(u) $\displaystyle\lim_{x\to\infty} (\sqrt{x^2+1}-x)$

3. Find $\displaystyle\lim_{x\to 1} g(x)$ if $g(x) = \begin{cases} 3x^2+x-2 & \text{if } x < 1, \\ x^2-2x+3 & \text{if } x \geq 1. \end{cases}$

4. Find intervals on which the following functions are continuous.

(a) $f(x) = \dfrac{x-1}{x^2-2x-3}$

(b) $f(x) = \dfrac{\sqrt{x}+3x}{x-2}$

5. Determine whether the function f is continuous at the point a. If f is discontinuous at a, show that f has a removable discontinuity, a jump discontinuity or an infinity discontinuity there.

(a) $f(x) = \begin{cases} \dfrac{x^2-2x-3}{x^2-x-2} & \text{if } x \neq -1, \\ 2 & \text{if } x = -1, \end{cases}$ and $a = -1$.

(b) $f(x) = \begin{cases} x^2-2x-2 & \text{if } x < 3, \\ -x^2+x+7 & \text{if } x \geq 3, \end{cases}$ and $a = 3$.

(c) $f(x) = \begin{cases} \sqrt{x^2+5}-1 & \text{if } x \leq 2, \\ x^2-x+1 & \text{if } x > 2, \end{cases}$ and $a = 2$.

6. For what values of a and b is the function f continuous on $(-\infty, \infty)$?
$$f(x) = \begin{cases} -ax^2 - x - a & \text{if } x < -1, \\ ax^2 + bx + 6 & \text{if } -1 \le x \le 2, \\ 3x^2 - bx - b & \text{if } x > 2. \end{cases}$$

7. Evaluate $\lim_{x \to 0} x^4 \cos \dfrac{1}{x^2}$.

8. If $\lim_{x \to 1} \dfrac{f(x) - 5}{x - 1} = 8$, find $\lim_{x \to 1} f(x)$.

9. Find the horizontal and vertical asymptotes of the curve $y = \dfrac{x^2 + x - 6}{x^2 - x - 2}$.

10. Show that there is a root of the equation $\cos x = x$ on $(0, 1)$.