Calculus - Exercises

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The Tangent Line (切線)

1. Find the equation of the tangent line to the curve at the point P.

   (a) $y = x^2 + x, \quad P(2, 6)$

   (b) $y = \sqrt{x}, \quad P(4, 2)$

   (c) $y = \dfrac{1}{x}, \quad P(3, \frac{1}{3})$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) Let $f(x) = x^2 + x$.

$$\because m_T = f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{(x+3)(x-2)}{x-2} = 5$$

$$\left(\text{or } \ m_T = f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{h(5+h)}{h} = 5 \right)$$

$$\therefore T : y - 6 = 5(x - 2)$$

(b) Let $f(x) = \sqrt{x}$.

$$\because m_T = f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{(x-4)(\sqrt{x} + 2)}$$

$$= \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4}$$

$$\left(\text{or } \ m_T = f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} \right.$$

$$\left. = \lim_{h \to 0} \frac{(\sqrt{4+h} - 2)(\sqrt{4+h} + 2)}{h(\sqrt{4+h} + 2)} = \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4} \right)$$

$$\therefore T : y - 2 = \frac{1}{4}(x - 4)$$

(c) Let $f(x) = \dfrac{1}{x}$.

$$\because m_T = f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{3 - x}{3x(x-3)} = -\frac{1}{9}$$

$$\left(\text{or } \ m_T = f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{-h}{3h(3+h)} = -\frac{1}{9} \right)$$

$$\therefore T : y - \frac{1}{3} = -\frac{1}{9}(x - 3)$$

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The Rate of Change (變化率)

1. Find (i) the average rate of change of the function $f(x)$ on $[x_1, x_2]$
(ii) the rate of change of $f(x)$ when $x = a$.

(a) $f(x) = x^2, \quad [x_1, x_2] = [3, 5], \quad a = 3$

(b) $f(x) = \sqrt{x}, \quad [x_1, x_2] = [4, 9], \quad a = 4.$

(c) $f(x) = \dfrac{1}{x+2}, \quad [x_1, x_2] = [1, 3], \quad a = 1.$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) (i) average rate of change $= \dfrac{f(5)-f(3)}{5-3} = 8$

(ii) rate of change $= f'(3) = \lim\limits_{x\to 3} \dfrac{f(x)-f(3)}{x-3} = \lim\limits_{x\to 3} \dfrac{(x+3)(x-3)}{x-3}$

$= 6$

$\left(\text{or } \ f'(3) = \lim\limits_{h\to 0} \dfrac{f(3+h)-f(3)}{h} = \lim\limits_{h\to 0} \dfrac{h(6+h)}{h} = 6 \right)$

(b) (i) average rate of change $= \dfrac{f(9)-f(4)}{9-4} = \dfrac{1}{5}$

(ii) rate of change $= f'(4) = \lim\limits_{x\to 4} \dfrac{f(x)-f(4)}{x-4} = \lim\limits_{x\to 4} \dfrac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}$

$= \lim\limits_{x\to 4} \dfrac{1}{\sqrt{x}+2} = \dfrac{1}{4}$

$\left(\text{or } \ f'(4) = \lim\limits_{h\to 0} \dfrac{f(4+h)-f(4)}{h} = \lim\limits_{h\to 0} \dfrac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)} \right.$

$\left. = \lim\limits_{h\to 0} \dfrac{1}{\sqrt{4+h}+2} = \dfrac{1}{4} \right)$

(c) (i) average rate of change $= \dfrac{f(3)-f(1)}{3-1} = -\dfrac{1}{15}$

(ii) rate of change $= f'(1) = \lim\limits_{x\to 1} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1} \dfrac{1-x}{3(x-1)(x+2)}$

$= -\dfrac{1}{9}$

$\left(\text{or } \ f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0} \dfrac{-h}{3h(3+h)} = -\dfrac{1}{9} \right)$

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The Derivative as a Function (導函數)

1. Find the derivative of the function $f(x)$.

   (a) $f(x) = 3x^2 + x$

   (b) $f(x) = \sqrt{x+1}$

   (c) $f(x) = \dfrac{1}{x}$

2. Determine whether or not the function $f(x)$ is differentiable at the point a.

   (a) $f(x) = |x|, \quad a = 0$

   (b) $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \quad a = 0$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{h(6x+1+3h)}{h} = 6x+1$

   (b) $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h+1}-\sqrt{x+1})(\sqrt{x+h+1}+\sqrt{x+1})}{h(\sqrt{x+h+1}+\sqrt{x+1})} \\ &= \lim_{h\to 0} \frac{1}{(\sqrt{x+h+1}+\sqrt{x+1})} = \frac{1}{2\sqrt{x+1}} \end{aligned}$$

   (c) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{-h}{xh(x+h)} = -\dfrac{1}{x^2}$

2. (a) $$\begin{cases} \lim\limits_{h\to 0^+} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^+} \dfrac{h}{h} = 1 \\ \lim\limits_{h\to 0^-} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^-} \dfrac{(-h)}{h} = -1 \end{cases}$$
   $\Longrightarrow \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ does NOT exist.
   $\Longrightarrow f(x) = |x|$ is NOT differentiable at 0.

   (b) $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h^2 \sin\frac{1}{h}}{h} = \lim\limits_{h\to 0} h\sin\dfrac{1}{h} = 0$
   (by the fact $-|h| \le h\sin\frac{1}{h} \le |h|$ for $h \neq 0$ and the squeeze theorem)
   $\Longrightarrow f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is differentiable at 0.

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Basic Differentiation Formulas (基本的微分公式)

1. Find the derivative of the function $f(x)$.

   (a) $f(x) = 3x^4 - 4x^3 + 5x^2 - x + 7$

   (b) $f(x) = \dfrac{x^5 + 2x^3 - 6}{x^3}$

   (c) $f(x) = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^2}}$

2. Evaluate the limits.

   (a) $\displaystyle\lim_{h\to 0} \frac{(1+h)^{20} - 1}{h}$

   (b) $\displaystyle\lim_{x\to 1} \frac{x^{100} - 1}{x - 1}$

   (c) $\displaystyle\lim_{h\to 0} \frac{\sqrt[3]{27+h} - 3}{h}$

3. Find the equation of the tangent line to the curve $y = 2x^3 + 4x - 8$ when $x = 1$.

4. Find the point(s) on the curve $y = 2x^3 + 3x^2 - 12x + 5$ where the tangent line is horizontal.

5. How fast is the function $f(x) = x^3 - 4x^2 + 2x - 5$ changing at $x = 3$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f'(x) = 12x^3 - 12x^2 + 10x - 1$

   (b) $f(x) = x^2 + 2 - 6x^{-3} \Longrightarrow f'(x) = 2x + 18x^{-4}$

   (c) $f(x) = x^{1/2} + x^{-2/3} \Longrightarrow f'(x) = \dfrac{1}{2}x^{-1/2} - \dfrac{2}{3}x^{-5/3}$

2. (a) Let $f(x) = x^{20} \Longrightarrow f'(x) = 20x^{19}$

   $$\Longrightarrow \lim_{h\to 0} \frac{(1+h)^{20}-1}{h} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = f'(1) = 20$$

   (b) Let $f(x) = x^{100} \Longrightarrow f'(x) = 100x^{99}$

   $$\Longrightarrow \lim_{x\to 1} \frac{x^{100}-1}{x-1} = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = f'(1) = 100$$

   (c) Let $f(x) = \sqrt[3]{x} = x^{1/3} \Longrightarrow f'(x) = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3\sqrt[3]{x^2}}$

   $$\Longrightarrow \lim_{h\to 0} \frac{\sqrt[3]{27+h}-3}{h} = \lim_{h\to 0} \frac{f(27+h)-f(27)}{h} = f'(27) = \frac{1}{27}$$

3. Let $f(x) = 2x^3 + 4x - 8 \Longrightarrow f'(x) = 6x^2 + 4 \Longrightarrow m_T = f'(1) = 10$
   Moreover, the corresponding point is $P(1, f(1)) = P(1, -2)$.
   The tangent line T is $y - (-2) = 10(x-1)$.

4. Let $f(x) = 2x^3 + 3x^2 - 12x + 5$
   $\Longrightarrow m_T = f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1)$
   The tangent line T is horizontal $\Longleftrightarrow m_T = 0 \Longleftrightarrow x = -2$ or 1
   $\Longleftrightarrow$ The point P is $(-2, f(-2)) = (-2, 25)$ or $(1, f(1)) = (1, -2)$.

5. $\because f'(x) = 3x^2 - 8x + 2 \quad \therefore$ The rate of change is $f'(3) = 5$.

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The Normal Line (法線)

1. Find the equations of the tangent and normal lines to the curve when $x = a$.

   (a) $y = 3x^2 + 5x - 2, \quad a = 2$

   (b) $y = x^5 - x^4 + 2x^3 - 6x + 8, \quad a = -1$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) Let $f(x) = 3x^2 + 5x - 2$.
   The corresponding point P is $(2, f(2)) = (2, 20)$.
   $\because f'(x) = 6x + 5$
   $$\therefore \begin{cases} m_T = f'(2) = 17 & \Longrightarrow T : y - 20 = 17(x - 2) \\ m_N = -\dfrac{1}{f'(2)} = -\dfrac{1}{17} & \Longrightarrow N : y - 20 = -\dfrac{1}{17}(x - 2) \end{cases}$$
   (b) Let $f(x) = x^5 - x^4 + 2x^3 - 6x + 8$.
   The corresponding point P is $(-1, f(-1)) = (-1, 10)$.
   $\because f'(x) = 5x^4 - 4x^3 + 6x^2 - 6$
   $$\therefore \begin{cases} m_T = f'(-1) = 9 & \Longrightarrow T : y - 10 = 9(x - (-1)) \\ m_N = -\dfrac{1}{f'(-1)} = -\dfrac{1}{9} & \Longrightarrow N : y - 10 = -\dfrac{1}{9}(x - (-1)) \end{cases}$$

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Product and Quotient Rules (微分的乘法與除法公式)

1. Find the derivative of the function $f(x)$.

   (a) $f(x) = (x^3 + 2x + 1)(x^2 - 1)$

   (b) $f(x) = (x^3 + \sqrt{x})(\dfrac{2}{x^2} - \dfrac{1}{x^4})$

   (c) $f(x) = \dfrac{x-1}{x+1}$

   (d) $f(x) = \dfrac{x^2 + x + 1}{\sqrt[3]{x}}$

   (e) $f(x) = \dfrac{x^2 + x + 1}{x^2 + 4x + 2}$

2. Suppose $f(2) = 3$, $g(2) = 5$, $f'(2) = -1$ and $g'(2) = 7$. Find the following values.
   (a) $(f+g)'(2)$ (b) $(fg)'(2)$ (c) $(f/g)'(2)$ (d) $(g/f)'(2)$

3. Suppose $F(x) = \dfrac{f(x)}{x^2+1}$, $f(-1) = -3$ and $f'(-1) = 2$. Evaluate $F'(-1)$.

4. Find the equations of the tangent and normal lines to the curve $y = \dfrac{\sqrt{x}}{x+1}$ at the point $(4, \frac{2}{5})$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f'(x) = (3x^2+2)(x^2-1) + (x^3+2x+1)(2x)$

   (b) $f(x) = (x^3 + x^{1/2})(2x^{-2} - x^{-4})$

   $\Longrightarrow f'(x) = (3x^2+\frac{1}{2}x^{-1/2})(2x^{-2}-x^{-4})+(x^3+x^{1/2})(-4x^{-3}+4x^{-5})$

   (c) $f'(x) = \dfrac{1\cdot(x+1)-(x-1)\cdot 1}{(x+1)^2}$

   (d) $f(x) = x^{5/3} + x^{2/3} + x^{-1/3} \Longrightarrow f'(x) = \frac{5}{3}x^{2/3} + \frac{2}{3}x^{-1/3} - \frac{1}{3}x^{-4/3}$

   (e) $f'(x) = \dfrac{(2x+1)\cdot(x^2+4x+2)-(x^2+x+1)\cdot(2x+4)}{(x^2+4x+2)^2}$

2. (a) $(f+g)'(2) = f'(2) + g'(2) = 6$

   (b) $(fg)'(2) = f'(2)g(2) + f(2)g'(2) = 16$

   (c) $\left(\dfrac{f}{g}\right)'(2) = \dfrac{f'(2)g(2)-f(2)g'(2)}{[g(2)]^2} = -\dfrac{26}{25}$

   (d) $\left(\dfrac{g}{f}\right)'(2) = \dfrac{g'(2)f(2)-g(2)f'(2)}{[f(2)]^2} = \dfrac{26}{9}$

3. $F'(x) = \dfrac{f'(x)(x^2+1)-f(x)(2x)}{(x^2+1)^2}$

   $\Longrightarrow F'(-1) = \dfrac{2f'(-1)+2f(-1)}{4} = -\dfrac{1}{2}$

4. Let $f(x) = \dfrac{x^{1/2}}{x+1}$.

   $\because f'(x) = \dfrac{\frac{1}{2}x^{-1/2}\cdot(x+1)-x^{1/2}\cdot 1}{(x+1)^2}$

   $$\therefore \begin{cases} m_T = f'(4) = -\dfrac{3}{100} & \Longrightarrow T: y-\dfrac{2}{5} = -\dfrac{3}{100}(x-4) \\ m_N = -\dfrac{1}{f'(4)} = \dfrac{100}{3} & \Longrightarrow N: y-\dfrac{2}{5} = \dfrac{100}{3}(x-4) \end{cases}$$

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Derivatives of Trigonometric Functions (三角函數的導數)

1. Find the derivative of the function $f(x)$.

   (a) $f(x) = 2\sin x + \tan x$

   (b) $f(x) = \dfrac{x^2 + \cos x}{\tan x}$

   (c) $f(x) = x^4 \sec x$

   (d) $f(x) = 3x^2(\cot x + \csc x)$

   (e) $f(x) = \dfrac{x \sin x}{x^3 + 1}$

2. Evaluate the limits.

   (a) $\displaystyle\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \frac{1}{\sqrt{2}}}{x - \frac{\pi}{4}}$

   (b) $\displaystyle\lim_{h \to 0} \frac{\cos(\frac{\pi}{3} + h) - \frac{1}{2}}{h}$

3. Find the equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\frac{\pi}{2}, \pi)$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f'(x) = 2\cos x + \sec^2 x$

   (b) $f'(x) = \dfrac{(2x - \sin x)(\tan x) - (x^2 + \cos x)(\sec^2 x)}{\tan^2 x}$

   (c) $f'(x) = 4x^3 \sec x + x^4 \sec x\ \tan x$

   (d) $f'(x) = 6x(\cot x + \csc x) + 3x^2(-\csc^2 x - \csc x\ \cot x)$

   (e) $f'(x) = \dfrac{(x\sin x)'(x^3+1) - (x\sin x)(3x^2)}{(x^3+1)^2}$

   $= \dfrac{(1 \cdot \sin x + x\cos x)(x^3+1) - (x\sin x)(3x^2)}{(x^3+1)^2}$

2. (a) Let $f(x) = \sin x \Longrightarrow f'(x) = \cos x$

   $\Longrightarrow \lim\limits_{x \to \frac{\pi}{4}} \dfrac{\sin x - \frac{1}{\sqrt{2}}}{x - \frac{\pi}{4}} = \lim\limits_{x \to \frac{\pi}{4}} \dfrac{f(x) - f(\frac{\pi}{4})}{x - \frac{\pi}{4}} = f'(\frac{\pi}{4}) = \dfrac{1}{\sqrt{2}}$

   (b) Let $f(x) = \cos x \Longrightarrow f'(x) = -\sin x$

   $\Longrightarrow \lim\limits_{h \to 0} \dfrac{\cos(\frac{\pi}{3} + h) - \frac{1}{2}}{h} = \lim\limits_{h \to 0} \dfrac{f(\frac{\pi}{3} + h) - f(\frac{\pi}{3})}{h} = f'(\frac{\pi}{3}) = -\dfrac{\sqrt{3}}{2}$

3. Let $f(x) = 2x\sin x$.

   $\because f'(x) = 2\sin x + 2x\cos x$

   $\therefore m_T = f'(\frac{\pi}{2}) = 2 \Longrightarrow T : y - \pi = 2(x - \frac{\pi}{2})$

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Higher Derivatives (高階導數)

1. Evaluate $f''(x)$.
   (a) $f(x) = x^4 - 2x^3 + x^2 - 6x + 7$
   (b) $f(x) = \sin x - \cos x$
   (c) $f(x) = \sec x$

2. Find $\dfrac{d^3y}{dx^3}$ if $y = x^5 - 2x^4 + x^3 - 6x + \cos x$.

3. Evaluate $\dfrac{d^{35}}{dx^{35}}(\cos x)$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f'(x) = 4x^3 - 6x^2 + 2x - 6 \implies f''(x) = 12x^2 - 12x + 2$

   (b) $f'(x) = \cos x + \sin x \implies f''(x) = -\sin x + \cos x$

   (c) $f'(x) = \sec x\ \tan x \implies f''(x) = (\sec x\ \tan x)\tan x + \sec x\,(\sec^2 x)$

2. $\dfrac{dy}{dx} = 5x^4 - 8x^3 + 3x^2 - 6 - \sin x$

   $\implies \dfrac{d^2y}{dx^2} = 20x^3 - 24x^2 + 6x - \cos x$

   $\implies \dfrac{d^3y}{dx^3} = 60x^2 - 48x + 6 + \sin x$

3. Let $f(x) = \cos x$.

   $$\implies \begin{cases} f^{(4k)}(x) = \cos x \\ f^{(4k+1)}(x) = -\sin x \\ f^{(4k+2)}(x) = -\cos x \\ f^{(4k+3)}(x) = \sin x \end{cases}$$

   $\because 35 \div 4 = 8...3 \quad \therefore \dfrac{d^{35}}{dx^{35}}(\cos x) = \sin x$

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The Chain Rule (鏈鎖律)

1. Suppose $y = \sin u$ and $u = x^3 - x + 1$. Find $\dfrac{dy}{dx}$.
2. Evaluate $\dfrac{d}{dx}\left(\sqrt[3]{x^2 + x + 5}\right)$.
3. Suppose $f'(4) = 3$ and $H(x) = f(x^2)$. Find $H'(2)$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \cos u \cdot (3x^2 - 1) = \cos(x^3 - x + 1) \cdot (3x^2 - 1)$

2. Let $y = \sqrt[3]{u} = u^{1/3}$ and $u = x^2 + x + 5$.
$$\Longrightarrow \frac{d}{dx}\left(\sqrt[3]{x^2 + x + 5}\right) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{3}u^{-2/3} \cdot (2x + 1)$$
$$= \frac{1}{3}(x^2 + x + 5)^{-2/3} \cdot (2x + 1)$$

3. Let $g(x) = x^2$.
$$\Longrightarrow H'(x) = \frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\,g'(x) = f'(x^2) \cdot 2x$$
$$\Longrightarrow H'(2) = 4f'(4) = 12$$

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Applications of the Chain Rule (鏈鎖律的應用)

1. Find the derivative of the function $f(x)$.

   (a) $f(x) = (3x^2 + 5x - 1)^7$

   (b) $f(x) = \sqrt{\dfrac{x-1}{2x+1}}$

   (c) $f(x) = (x^2+1)^5(x^3 - 4x^2 + 3)^9$

   (d) $f(x) = \sqrt{x + \sqrt{x}}$

   (e) $f(x) = \tan(x^2 + 5)$

   (f) $f(x) = \sin(\sqrt{x})$

   (g) $f(x) = \cos^5(x^3 + 1)$

2. How fast is the function $f(x) = \left(\dfrac{2x-1}{x^2+x+1}\right)^3$ changing at $x = 1$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f'(x) = 7(3x^2 + 5x - 1)^6(6x + 5)$

   (b) $f(x) = \left(\dfrac{x-1}{2x+1}\right)^{1/2} \implies f'(x) = \dfrac{1}{2}\left(\dfrac{x-1}{2x+1}\right)^{-1/2}\left(\dfrac{1\cdot(2x+1)-(x-1)\cdot 2}{(2x+1)^2}\right)$

   (c) $f'(x) = \left[5(x^2+1)^4(2x)\right](x^3-4x^2+3)^9+(x^2+1)^5\left[9(x^3-4x^2+3)^8(3x^2-8x)\right]$

   (d) $f(x) = \left(x + x^{1/2}\right)^{1/2} \implies f'(x) = \dfrac{1}{2}\left(x + x^{1/2}\right)^{-1/2}\left(1 + \dfrac{1}{2}x^{-1/2}\right)$

   (e) $f'(x) = \sec^2(x^2+5)\cdot(2x)$

   (f) $f(x) = \sin(x^{1/2}) \implies f'(x) = \cos(x^{1/2})\cdot\dfrac{1}{2}x^{-1/2}$

   (g) $f'(x) = 5\cos^4(x^3+1)\cdot\left(-\sin(x^3+1)\cdot(3x^2)\right)$

2. $\because f'(x) = 3\left(\dfrac{2x-1}{x^2+x+1}\right)^2\left(\dfrac{2\cdot(x^2+x+1)-(2x-1)(2x+1)}{(x^2+x+1)^2}\right)$

   $\therefore$ The rate of change is $f'(1) = \dfrac{1}{9}$.

Calculus - Exercises

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Implicit Differentiation (隱函數微分法)

1. Evaluate $\dfrac{dy}{dx}$.

   (a) $\sin(xy) = xy$

   (b) $x^3 + 2x^2 + 3x^2y + 5y^2 = 0$

2. Find the equation of the tangent line to the curve $x^2 + xy + y^2 = 3$ at $(1, 1)$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\sin(xy) = xy \overset{\frac{d}{dx}}{\Longrightarrow} \cos(xy) \cdot \left(1 \cdot y + x \cdot \dfrac{dy}{dx}\right) = \left(1 \cdot y + x \cdot \dfrac{dy}{dx}\right)$

$\Longrightarrow (x\cos(xy) - x)\dfrac{dy}{dx} = y - y\cos(xy) \Longrightarrow \dfrac{dy}{dx} = \dfrac{y - y\cos(xy)}{x\cos(xy) - x}$

(b) $x^3 + 2x^2 + 3x^2y + 5y^2 = 0$

$\overset{\frac{d}{dx}}{\Longrightarrow} 3x^2 + 4x + 3\left(2x \cdot y + x^2 \cdot \dfrac{dy}{dx}\right) + 10y\dfrac{dy}{dx} = 0$

$\Longrightarrow \left(3x^2 + 10y\right)\dfrac{dy}{dx} = -3x^2 - 4x - 6xy \Longrightarrow \dfrac{dy}{dx} = \dfrac{-3x^2 - 4x - 6xy}{3x^2 + 10y}$

2. $x^2 + xy + y^2 = 3 \overset{\frac{d}{dx}}{\Longrightarrow} 2x + \left(1 \cdot y + x \cdot \dfrac{dy}{dx}\right) + 2y\dfrac{dy}{dx} = 0$

$\Longrightarrow \dfrac{dy}{dx} = \dfrac{-2x - y}{x + 2y}$

$\Longrightarrow m_T = \left.\dfrac{dy}{dx}\right|_{(x,y)=(1,1)} = -1 \quad \Longrightarrow T : y - 1 = -(x - 1)$

Calculus - Exercises

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The Derivative (導數) [綜合練習]

1. Use the definition to find the derivative of the function $f(x)$.

   (a) $f(x) = \sqrt{x+1}$

   (b) $f(x) = \dfrac{1}{x}$

2. Find the derivative of the function $f(x)$.

   (a) $f(x) = 3x^4 - 4x^3 + 5x^2 - x + 7$

   (b) $f(x) = 2\sin x + \tan x$

   (c) $f(x) = (x^3 + \sqrt{x})(\dfrac{2}{x^2} - \dfrac{1}{x^4})$

   (d) $f(x) = x^4 \sec x$

   (e) $f(x) = \dfrac{x^2 + x + 1}{x^2 + 4x + 2}$

   (f) $f(x) = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^2}}$

   (g) $f(x) = \dfrac{x \sin x}{x^3 + 1}$

   (h) $f(x) = (3x^2 + 5x - 1)^7$

   (i) $f(x) = \sqrt{\dfrac{x-1}{2x+1}}$

   (j) $f(x) = \sin(\sqrt{x})$

   (k) $f(x) = \sqrt{x + \sqrt{x}}$

   (l) $f(x) = \cos^5(x^3 + 1)$

3. Evaluate the limits.

   (a) $\displaystyle \lim_{x \to \frac{\pi}{4}} \frac{\sin x - \frac{1}{\sqrt{2}}}{x - \frac{\pi}{4}}$

   (b) $\displaystyle \lim_{h \to 0} \frac{(1+h)^{20} - 1}{h}$

(c) $\lim_{x \to 1} \dfrac{x^{100} - 1}{x - 1}$

(d) $\lim_{h \to 0} \dfrac{\sqrt[3]{27 + h} - 3}{h}$

4. Suppose $f(2) = 3$, $g(2) = 5$, $f'(2) = -1$ and $g'(2) = 7$. Find the following values.
(a) $(f + g)'(2)$ (b) $(fg)'(2)$ (c) $(f/g)'(2)$ (d) $(g/f)'(2)$

5. Suppose $F(x) = \dfrac{f(x)}{x^2 + 1}$, $f(-1) = -3$ and $f'(-1) = 2$. Evaluate $F'(-1)$.

6. How fast is the function $f(x) = \left(\dfrac{2x - 1}{x^2 + x + 1} \right)^3$ changing at $x = 1$.

7. Find the equations of the tangent and normal lines to the curve $y = x^5 - x^4 + 2x^3 - 6x + 8$ when $x = -1$.

8. Find the point(s) on the curve $y = 2x^3 + 3x^2 - 12x + 5$ where the tangent line is horizontal.

9. Find the equation of the tangent line to the curve $x^2 + xy + y^2 = 3$ at $(1, 1)$.

10. Find $\dfrac{d^3y}{dx^3}$ if $y = x^5 - 2x^4 + x^3 - 6x + \cos x$.

11. Evaluate $\dfrac{dy}{dx}$ if $x^3 + 2x^2 + 3x^2y + 5y^2 = 0$.

12. Determine whether or not the function $f(x) = |x|$ is differentiable at 0.