Calculus - Exercises

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Indeterminate Forms and L'Hôpital's Rule (I) (不定型式與羅必達法則 (I))

1. Find the following limits.

   (a) $\displaystyle\lim_{x\to 3}\frac{x^3-27}{x-3}$

   (b) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x}$

   (c) $\displaystyle\lim_{x\to 0}\frac{\sin(3x)}{\sin(5x)}$

   (d) $\displaystyle\lim_{x\to 0}\frac{3^x-1}{2^x-1}$

   (e) $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}$

   (f) $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x^2+x}$

   (g) $\displaystyle\lim_{x\to 0}\frac{\sin(2x)}{\tan x}$

   (h) $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{1-\cos x}$

   (i) $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}(2x)}{x}$

   (j) $\displaystyle\lim_{x\to 0}\frac{1}{x}\int_0^x\cos(4t)\,dt$

   (k) $\displaystyle\lim_{x\to 0}\frac{1}{x^3}\int_0^x\sin(t^2)\,dt$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\lim_{x\to 3}\frac{x^3-27}{x-3}\overset{L}{=}\lim_{x\to 3}\frac{3x^2}{1}=27$

(b) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x}\overset{L}{=}\lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{1}=0$

(c) $\displaystyle\lim_{x\to 0}\frac{\sin(3x)}{\sin(5x)}\overset{L}{=}\lim_{x\to 0}\frac{3\cos(3x)}{5\cos(5x)}=\frac{3}{5}$

(d) $\displaystyle\lim_{x\to 0}\frac{3^x-1}{2^x-1}\overset{L}{=}\lim_{x\to 0}\frac{3^x\cdot\ln 3}{2^x\cdot\ln 2}=\frac{\ln 3}{\ln 2}=\log_2 3$

(e) $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}\overset{L}{=}\lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-1/2}-\frac{1}{2}}{2x}\overset{L}{=}\lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-3/2}}{2}$

$\displaystyle=-\frac{1}{8}$

(f) $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x^2+x}\overset{L}{=}\lim_{x\to 0}\frac{e^x}{2x+1}=1$

(g) $\displaystyle\lim_{x\to 0}\frac{\sin(2x)}{\tan x}\overset{L}{=}\lim_{x\to 0}\frac{2\cos(2x)}{\sec^2 x}=2$

(h) $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{1-\cos x}\overset{L}{=}\lim_{x\to 0}\frac{1-\cos x}{\sin x}\overset{L}{=}\lim_{x\to 0}\frac{\sin x}{\cos x}=0$

(i) $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}(2x)}{x}\overset{L}{=}\lim_{x\to 0}\frac{\left(\frac{2}{\sqrt{1-4x^2}}\right)}{1}=2$

(j) $\displaystyle\lim_{x\to 0}\frac{1}{x}\int_0^x\cos(4t)\,dt=\lim_{x\to 0}\frac{\int_0^x\cos(4t)\,dt}{x}\overset{L}{=}\lim_{x\to 0}\frac{\cos(4x)}{1}=1$

(k) $\displaystyle\lim_{x\to 0}\frac{1}{x^3}\int_0^x\sin(t^2)\,dt=\lim_{x\to 0}\frac{\int_0^x\sin(t^2)\,dt}{x^3}\overset{L}{=}\lim_{x\to 0}\frac{\sin(x^2)}{3x^2}\overset{L}{=}\lim_{x\to 0}\frac{2x\cos(x^2)}{6x}$

$\displaystyle=\frac{1}{3}\lim_{x\to 0}\cos(x^2)=\frac{1}{3}$

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Indeterminate Forms and L'Hôpital's Rule (II) (不定型式與羅必達法則 (II))

1. Find the following limits.

   (a) $\displaystyle\lim_{x\to\infty} x^2e^{-x}$

   (b) $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

   (c) $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

   (d) $\displaystyle\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\lim_{x\to\infty} x^2e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{L}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{L}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$

   (b) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim_{x\to 0^+} \frac{e^x - 1 - x}{xe^x - x} \overset{L}{=} \lim_{x\to 0^+} \frac{e^x - 1}{e^x + xe^x - 1}$
   $\overset{L}{=} \lim_{x\to 0^+} \frac{e^x}{2e^x + xe^x} = \frac{1}{2}$

   (c) $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - \sin x}{x \sin x} \overset{L}{=} \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x}$
   $\overset{L}{=} \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = 0$

   (d) $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1} \frac{x - 1 - \ln x}{x\ln x - \ln x} \overset{L}{=} \lim_{x\to 1} \frac{1 - \frac{1}{x}}{\ln x + 1 - \frac{1}{x}}$
   $= \lim_{x\to 1} \frac{x-1}{x\ln x + x - 1} \overset{L}{=} \lim_{x\to 1} \frac{1}{\ln x + 2} = \frac{1}{2}$

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Indeterminate Forms and L'Hôpital's Rule (III) (不定型式與羅必達法則 (III))

1. Find the following limits.

   (a) $\lim\limits_{x \to 0^+} x^{\sin x}$

   (b) $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^{4x}$

   (c) $\lim\limits_{x \to \infty} (\ln x)^{1/x}$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) Let $y = x^{\sin x}$. $\Longrightarrow \ln y = \ln x^{\sin x} = \sin x\, \ln x$

$$\because \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \sin x\, \ln x = \lim_{x\to 0^+} \frac{\ln x}{\csc x} \overset{L}{=} \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{-\csc x\, \cot x}$$
$$= \lim_{x\to 0^+} \left(-\tan x \cdot \frac{\sin x}{x}\right) = -\left(\lim_{x\to 0^+} \tan x\right)\left(\lim_{x\to 0^+} \frac{\sin x}{x}\right)$$
$$= -0 \times 1 = 0$$

$$\therefore \lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\left(\lim\limits_{x\to 0^+} \ln y\right)} = e^0 = 1$$

(b) $$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{4x} = \lim_{x\to\infty} \left[\left(1+\frac{1}{x}\right)^{x}\right]^4 = \left[\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x}\right]^4 = e^4$$

(c) Let $y = (\ln x)^{1/x}$. $\Longrightarrow \ln y = \ln(\ln x)^{1/x} = \dfrac{1}{x}\, \ln(\ln x)$

$$\because \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(\ln x)}{x} \overset{L}{=} \lim_{x\to\infty} \frac{\left(\frac{\left(\frac{1}{x}\right)}{\ln x}\right)}{1} = \lim_{x\to\infty} \frac{1}{x\ln x} = 0$$

$$\therefore \lim_{x\to\infty} (\ln x)^{1/x} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\left(\lim\limits_{x\to\infty} \ln y\right)} = e^0 = 1$$

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Improper Integrals of Type I (第一類型瑕積分)

1. Determine whether the improper integral is convergent or divergent. Find its value if it is convergent.

   (a) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

   (b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

   (c) $\displaystyle\int_0^\infty \cos x\,dx$

   (d) $\displaystyle\int_{-\infty}^0 xe^x\,dx$

   (e) $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

   (f) $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^x}\,dx$

2. Let f be continuous on $[0,\infty)$. The Laplace transform (拉普拉斯變換) of f is the function F defined by

$$F(s)=\int_0^\infty e^{-st}f(t)\,dt.$$

   The domain of F is the set of all real numbers s such that the improper integral is convergent. Find the Laplace transform F of the following functions and write down their domains.

   (a) $f(t)=e^{3t}$

   (b) $f(t)=t$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty}\int_1^t x^{-3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}x^{-2}\right]_1^t$

$\displaystyle = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2}\right) = \frac{1}{2} \quad \text{(convergent)}$

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_1^t x^{-1/2}\,dx = \lim_{t\to\infty}\left[2x^{1/2}\right]_1^t$

$\displaystyle = \lim_{t\to\infty}\left(2\sqrt{t}-2\right) = \infty \quad \text{(divergent)}$

(c) $\displaystyle\int_0^\infty \cos x\,dx = \lim_{t\to\infty}\int_0^t \cos x\,dx = \lim_{t\to\infty}[\sin x]_0^t$

$\displaystyle = \lim_{t\to\infty}\sin t$ does NOT exist. (divergent)

(d) $\displaystyle\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx = \lim_{t\to-\infty}\left([xe^x]_t^0 - \int_t^0 e^x\,dx\right)$

(Integration by Parts)

$\displaystyle = \lim_{t\to-\infty}\left(-t\,e^t-1+e^t\right) = -0-1+0 = -1 \quad \text{(convergent)}$

$\displaystyle\left(\lim_{t\to-\infty} t\,e^t = 0 \text{ by L'Hôpital's Rule}\right)$

(e) $\displaystyle\because \int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\left[\tan^{-1}x\right]_0^t$

$\displaystyle = \lim_{t\to\infty}\tan^{-1}t = \frac{\pi}{2}$ and

$\displaystyle\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\left[\tan^{-1}x\right]_t^0$

$\displaystyle = \lim_{t\to-\infty}\left(-\tan^{-1}t\right) = \frac{\pi}{2}$

$\displaystyle\therefore \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}+\frac{\pi}{2} = \pi$

(convergent)

(f) $\displaystyle\because \int_0^\infty \frac{e^x}{1+e^x}\,dx = \lim_{t\to\infty}\int_0^t \frac{e^x}{1+e^x}\,dx = \lim_{t\to\infty}\int_2^{1+e^t}\frac{1}{u}\,du$

(Let $u = 1+e^x \Longrightarrow du = e^x\,dx$)

$\displaystyle = \lim_{t\to\infty}[\ln|u|]_2^{1+e^t} = \lim_{t\to\infty}\left(\ln(1+e^t)-\ln 2\right) = \infty \quad \text{(divergent)}$

$\displaystyle\therefore \int_{-\infty}^\infty \frac{e^x}{1+e^x}\,dx$ is divergent.

2. (a) $\because F(s) = \displaystyle\int_0^{\infty} e^{-st} e^{3t}\, dt = \lim_{b\to\infty} \int_0^b e^{(3-s)t}\, dt = \lim_{b\to\infty} \left[\frac{1}{3-s} e^{(3-s)t} \right]_{t=0}^{t=b}$

$$= \lim_{b\to\infty} \frac{1}{3-s} \left(e^{(3-s)b} - 1 \right) = \begin{cases} \frac{1}{s-3} & \text{if } s > 3, \quad \text{(convergent)} \\ \infty & \text{if } s < 3. \quad \text{(divergent)} \end{cases}$$

$\therefore F(s) = \dfrac{1}{s-3}$ on $(3, \infty)$.

(b) $\because F(s) = \displaystyle\int_0^{\infty} t\, e^{-st}\, dt = \lim_{b\to\infty} \int_0^b t\, e^{-st}\, dt$

$$= \lim_{b\to\infty} \left(\left[-\frac{t}{s} e^{-st} \right]_{t=0}^{t=b} + \frac{1}{s} \int_0^b e^{-st}\, dt \right) \qquad \text{(Integration by Parts)}$$

$$= \lim_{b\to\infty} \left(-\frac{b}{s\, e^{sb}} - \frac{1}{s^2\, e^{sb}} + \frac{1}{s^2} \right) = \begin{cases} \frac{1}{s^2} & \text{if } s > 0, \quad \text{(convergent)} \\ \infty & \text{if } s < 0. \quad \text{(divergent)} \end{cases}$$

$\therefore F(s) = \dfrac{1}{s^2}$ on $(0, \infty)$.

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Tests for Improper Integrals (瑕積分的審斂法)

1. Determine whether the improper integral is convergent or divergent.

   (a) $\displaystyle\int_1^\infty \frac{1}{x^6}\,dx$

   (b) $\displaystyle\int_1^\infty \frac{1}{\sqrt[3]{x^2}}\,dx$

   (c) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

   (d) $\displaystyle\int_1^\infty \frac{3x^2+2}{x^5+4x+1}\,dx$

   (e) $\displaystyle\int_0^\infty e^{-x^2}\,dx$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int_1^\infty \frac{1}{x^6}\,dx$ is convergent $(p = 6 > 1)$ and $\displaystyle\int_1^\infty \frac{1}{x^6}\,dx = \frac{1}{6-1} = \frac{1}{5}$.

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt[3]{x^2}}\,dx = \int_1^\infty \frac{1}{x^{2/3}}\,dx$ is divergent $(p = \frac{2}{3} \le 1)$.

(c) $\displaystyle\because \frac{1}{x^2} \ge \frac{\sin^2 x}{x^2} \ge 0$ on $[1,\infty)$ and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent.

$(p = 2 > 1)$

$\displaystyle\therefore \int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent by the comparison test.

(d) $\displaystyle\because \lim_{x\to\infty} \frac{\left(\frac{3x^2+2}{x^5+4x+1}\right)}{\left(\frac{1}{x^3}\right)} = \lim_{x\to\infty} \frac{3x^5+2x^3}{x^5+4x+1} = 3$ and

$\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ is convergent. $(p = 3 > 1)$

$\displaystyle\therefore \int_1^\infty \frac{3x^2+2}{x^5+4x+1}\,dx$ is convergent by the limit comparison test.

(e) (i) $\because e^{-x^2}$ is continuous on $[0,1]$. $\displaystyle\therefore \int_0^1 e^{-x^2}\,dx$ exists.

(ii) $\because$ For $x \ge 1$, $e^{x^2} \ge e^x \Longrightarrow e^{-x} \ge e^{-x^2} \ge 0$ and

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}\left(-\int_{-1}^{-t} e^u\,du\right)$$

(Let $u = -x \Longrightarrow du = -\,dx$)

$$= \lim_{t\to\infty}\left(-\left[e^u\right]_{-1}^{-t}\right) = \lim_{t\to\infty}\left(-\frac{1}{e^t}+\frac{1}{e}\right) = \frac{1}{e} \text{ is convergent.}$$

$\displaystyle\therefore \int_1^\infty e^{-x^2}\,dx$ is convergent by the comparison test.

From (i) and (ii), $\displaystyle\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$ is convergent.

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Improper Integrals of Type II (第二類型瑕積分)

1. Determine whether the improper integral is convergent or divergent. Find its value if it is convergent.

   (a) $\displaystyle\int_1^5 \frac{1}{\sqrt{5-x}}\,dx$

   (b) $\displaystyle\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

   (c) $\displaystyle\int_0^1 \ln x\,dx$

   (d) $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx$

2. Find the value of p such that $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ exists. For such p, evaluate $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int_1^5 \frac{1}{\sqrt{5-x}}\,dx = \lim_{t\to 5^-}\int_1^t (5-x)^{-1/2}\,dx = \lim_{t\to 5^-}\left(-\int_4^{5-t} u^{-1/2}\,du\right)$

(Let $u = 5-x \Longrightarrow du = -\,dx$)

$\displaystyle = \lim_{t\to 5^-}\left(-\left[2u^{1/2}\right]_4^{5-t}\right) = \lim_{t\to 5^-}\left(-2\sqrt{5-t}+4\right) = 4 \quad \text{(convergent)}$

(b) $\displaystyle\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to 1^+}\int_t^9 (x-1)^{-1/3}\,dx = \lim_{t\to 1^+}\int_{t-1}^8 u^{-1/3}\,du$

(Let $u = x-1 \Longrightarrow du = dx$)

$\displaystyle = \lim_{t\to 1^+}\left[\frac{3}{2}u^{2/3}\right]_{t-1}^8 = \lim_{t\to 1^+}\left(6-\frac{3}{2}(t-1)^{2/3}\right) = 6 \quad \text{(convergent)}$

(c) $\displaystyle\int_0^1 \ln x\,dx = \lim_{t\to 0^+}\int_t^1 \ln x\,dx = \lim_{t\to 0^+}\left(\left[x\ln x\right]_t^1 - \int_t^1 dx\right)$

(Integration by Parts)

$\displaystyle = \lim_{t\to 0^+}\left(-t\ln t - 1 + t\right) = -0-1+0 = -1 \quad \text{(convergent)}$

$\displaystyle\left(\lim_{t\to 0^+} t\ln t = 0 \text{ by L'Hôpital's Rule}\right)$

(d) $\displaystyle\because \int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-2}\,dx = \lim_{t\to 0^+}\left[-x^{-1}\right]_t^1 = \lim_{t\to 0^+}\left(-1+\frac{1}{t}\right)$

$= \infty \quad$ (divergent)

$\displaystyle\therefore \int_{-1}^1 \frac{1}{x^2}\,dx$ is divergent.

2. (i) If $p \le 0$, $\displaystyle\int_0^1 \frac{1}{x^p}\,dx = \int_0^1 x^{-p}\,dx = \left[\frac{1}{1-p}x^{1-p}\right]_0^1 = \frac{1}{1-p}$

(exists)

(ii) If $p = 1$, $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\left[\ln|x|\right]_t^1 = \lim_{t\to 0^+}(-\ln t)$

$= \infty \quad$ (divergent)

(iii) If $p > 0$ with $p \ne 1$, $\displaystyle\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-p}\,dx = \lim_{t\to 0^+}\left[\frac{1}{1-p}x^{1-p}\right]_t^1$

$$= \frac{1}{1-p}\lim_{t\to 0^+}\left(1-t^{1-p}\right) = \begin{cases} \infty & \text{if } p>1, \quad \text{(divergent)} \\ \frac{1}{1-p} & \text{if } 0<p<1. \quad \text{(convergent)} \end{cases}$$

From (i)-(iii), $\int_0^1 \frac{1}{x^p}\,dx$ exists $\iff p < 1$, and $\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}$ for $p < 1$.

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Further Applications (進階的應用) [綜合練習]

1. Find the following limits.

   (a) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$

   (b) $\displaystyle\lim_{x\to 3} \frac{x^3-27}{x-3}$

   (c) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)}$

   (d) $\displaystyle\lim_{x\to 0^+} x^{\sin x}$

   (e) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x-1}\right)$

   (f) $\displaystyle\lim_{x\to\infty} x^2 e^{-x}$

   (g) $\displaystyle\lim_{x\to 0} \frac{e^x-1}{x^2+x}$

   (h) $\displaystyle\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{4x}$

   (i) $\displaystyle\lim_{x\to 0} \frac{x-\sin x}{1-\cos x}$

   (j) $\displaystyle\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

   (k) $\displaystyle\lim_{x\to 0} \frac{1}{x^3}\int_0^x \sin(t^2)\,dt$

2. Determine whether the improper integral is convergent or divergent. Find its value if it is convergent.

   (a) $\displaystyle\int_1^\infty \frac{1}{x^6}\,dx$

   (b) $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

   (c) $\displaystyle\int_0^\infty \cos x\,dx$

(d) $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

(e) $\int_{-\infty}^0 xe^x\,dx$

(f) $\int_{-1}^1 \frac{1}{x^2}\,dx$

(g) $\int_1^5 \frac{1}{\sqrt{5-x}}\,dx$

(h) $\int_0^1 \ln x\,dx$

3. Determine whether the improper integral is convergent or divergent.

(a) $\int_1^\infty \frac{3x^2+2}{x^5+4x+1}\,dx$

(b) $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

(c) $\int_0^\infty e^{-x^2}\,dx$

4. Find the value of p such that $\int_0^1 \frac{1}{x^p}\,dx$ exists. For such p, evaluate $\int_0^1 \frac{1}{x^p}\,dx$.

5. Let f be continuous on $[0,\infty)$. The Laplace transform (拉普拉斯變換) of f is the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

The domain of F is the set of all real numbers s such that the improper integral is convergent. Find the Laplace transform F of the following functions and write down their domains.

(a) $f(t) = e^{3t}$

(b) $f(t) = t$