Calculus - Exercises

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One-to-One Functions (一對一函數)

1. Determine whether or not the function $f(x)$ is an one-to-one function.

   (a) $f(x) = \sin x$ on $(-\infty, \infty)$.

   (b) $f(x) = \sqrt{x} + 1$ on $[0, \infty)$.

   (c) $f(x) = x^2$ on $(-\infty, \infty)$.

   (d) $f(x) = |x|$ on $(-\infty, \infty)$.

   (e) $f(x) = \dfrac{x+2}{x-1}$ on $(1, \infty)$.

   (f) $f(x) = \tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

<u>Solution</u> [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\because f(0) = f(2\pi) = 0$

   $\therefore f(x) = \sin x$ is NOT an one-to-one function on $(-\infty, \infty)$.

   (b) <u>Method I</u>: Suppose $x_1, x_2 \in [0, \infty)$ satisfying $f(x_1) = f(x_2)$.

   We get that $\sqrt{x_1} + 1 = \sqrt{x_2} + 1. \Longrightarrow \sqrt{x_1} = \sqrt{x_2} \Longrightarrow x_1 = x_2$

   $f(x) = \sqrt{x} + 1$ is an one-to-one function on $[0, \infty)$.

   <u>Method II</u>: $f(x) = \sqrt{x} + 1 \Longrightarrow f'(x) = \dfrac{1}{2\sqrt{x}} > 0$ on $(0, \infty)$.

   $\Longrightarrow f(x)$ is increasing on $(0, \infty)$.

   $\Longrightarrow f(x)$ is an one-to-one function on $[0, \infty)$.

   (c) $\because f(1) = f(-1) = 1$

   $\therefore f(x) = x^2$ is NOT an one-to-one function on $(-\infty, \infty)$.

   (d) $\because f(1) = f(-1) = 1$

   $\therefore f(x) = |x|$ is NOT an one-to-one function on $(-\infty, \infty)$.

   (e) $f(x) = \dfrac{x+2}{x-1} \Longrightarrow f'(x) = \dfrac{-3}{(x-1)^2} < 0$ on $(1, \infty)$.

   $\Longrightarrow f(x)$ is decreasing on $(1, \infty)$.

   $\Longrightarrow f(x)$ is an one-to-one function on $(1, \infty)$.

   (f) $f(x) = \tan x \Longrightarrow f'(x) = \sec^2 x > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

   $\Longrightarrow f(x)$ is increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

   $\Longrightarrow f(x)$ is an one-to-one function on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

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Inverse Functions (反函數)

1. Find the inverse function $f^{-1}(x)$, and verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.

   (a) $f(x) = \sqrt{x} + 1$ on $[0, \infty)$.

   (b) $f(x) = \dfrac{x+2}{x-1}$ on $(1, \infty)$.

   (c) $f(x) = (2-x)^3$ on $(-\infty, \infty)$.

   (d) $f(x) = x^2 - 4x$ on $[2, \infty)$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) Note that $f'(x) = \dfrac{1}{2\sqrt{x}} > 0$ on $(0, \infty)$.

   $\Longrightarrow f(x)$ is increasing on $(0, \infty)$.

   $\Longrightarrow f(x)$ is an one-to-one function on $[0, \infty)$. $\Longrightarrow f^{-1}(x)$ exists.

   Let $y = f(x) = \sqrt{x} + 1$. $\Longrightarrow f^{-1}(y) = x = (y-1)^2$

   We get that $f^{-1}(x) = (x-1)^2$ on $[1, \infty)$ and

   $f^{-1}(f(x)) = f^{-1}(\sqrt{x}+1) = \big((\sqrt{x}+1)-1\big)^2 = x$,

   $f(f^{-1}(y)) = f((y-1)^2) = \sqrt{(y-1)^2} + 1 = y$.

   (b) Note that $f'(x) = \dfrac{-3}{(x-1)^2} < 0$ on $(1, \infty)$.

   $\Longrightarrow f(x)$ is decreasing on $(1, \infty)$.

   $\Longrightarrow f(x)$ is an one-to-one function on $(1, \infty)$. $\Longrightarrow f^{-1}(x)$ exists.

   Let $y = f(x) = \dfrac{x+2}{x-1}$. $\Longrightarrow f^{-1}(y) = x = \dfrac{y+2}{y-1}$

   We get that $f^{-1}(x) = \dfrac{x+2}{x-1}$ on $(1, \infty)$ and

   $f^{-1}(f(x)) = f^{-1}(\frac{x+2}{x-1}) = \dfrac{\left(\frac{x+2}{x-1}\right)+2}{\left(\frac{x+2}{x-1}\right)-1} = x$,

   $f(f^{-1}(y)) = f(\frac{y+2}{y-1}) = \dfrac{\left(\frac{y+2}{y-1}\right)+2}{\left(\frac{y+2}{y-1}\right)-1} = y$.

   (c) Note that $f'(x) = -3(2-x)^2 < 0$ on $(-\infty, 2)$ and $(2, \infty)$.

   $\Longrightarrow f(x)$ is decreasing on $(-\infty, \infty)$.

   $\Longrightarrow f(x)$ is an one-to-one function on $(-\infty, \infty)$.

   $\Longrightarrow f^{-1}(x)$ exists.

   Let $y = f(x) = (2-x)^3$. $\Longrightarrow f^{-1}(y) = x = 2 - \sqrt[3]{y}$

   We get that $f^{-1}(x) = 2 - \sqrt[3]{x}$ on $(-\infty, \infty)$ and

   $f^{-1}(f(x)) = f^{-1}((2-x)^3) = 2 - \sqrt[3]{(2-x)^3} = x$,

   $f(f^{-1}(y)) = f(2-\sqrt[3]{y}) = \big(2-(2-\sqrt[3]{y})\big)^3 = y$.

   (d) Note that $f'(x) = 2x - 4 > 0$ on $(2, \infty)$.

   $\Longrightarrow f(x)$ is increasing on $(2, \infty)$.

   $\Longrightarrow f(x)$ is an one-to-one function on $[2, \infty)$. $\Longrightarrow f^{-1}(x)$ exists.

   Let $y = f(x) = x^2 - 4x$. $\Longrightarrow x^2 - 4x - y = 0$

$\implies f^{-1}(y) = x = 2 + \sqrt{4+y} \quad (x \geq 2 \text{ by assumption.})$
We get that $f^{-1}(x) = 2 + \sqrt{4+x}$ on $[-4, \infty)$ and
$f^{-1}(f(x)) = f^{-1}(x^2 - 4x) = 2 + \sqrt{4 + (x^2 - 4x)} = x$,
$f(f^{-1}(y)) = f(2+\sqrt{4+y}) = (2+\sqrt{4+y})^2 - 4(2+\sqrt{4+y}) = y$.

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Properties of Inverse Functions (反函數的基本性質)

1. Find the derivative of the inverse function $f^{-1}(x)$ at the point b.

   (a) $f(x) = x^3 - 3x^2 + 1$ on $[2, \infty)$; $b = 1$

   (b) $f(x) = 3x + \cos(2x)$ on $(-\infty, \infty)$; $b = \frac{3\pi}{4}$

   (c) $f(x) = \tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$; $b = \sqrt{3}$

   (d) $f(x) = \sqrt{x^2 + 6x + 16}$ on $[-3, \infty)$; $b = 4$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f(x) = x^3 - 3x^2 + 1 \Longrightarrow f'(x) = 3x^2 - 6x$

   $\because f(3) = 1$

   $\therefore (f^{-1})'(1) = \dfrac{1}{f'(3)} = \dfrac{1}{9}$ by Inverse Function Theorem.

   (b) $f(x) = 3x + \cos(2x) \Longrightarrow f'(x) = 3 - 2\sin(2x)$

   $\because f(\frac{\pi}{4}) = \frac{3\pi}{4}$

   $\therefore (f^{-1})'(\frac{3\pi}{4}) = \dfrac{1}{f'(\frac{\pi}{4})} = 1$ by Inverse Function Theorem.

   (c) $f(x) = \tan x \Longrightarrow f'(x) = \sec^2 x$

   $\because f(\frac{\pi}{3}) = \sqrt{3}$

   $\therefore (f^{-1})'(\sqrt{3}) = \dfrac{1}{f'(\frac{\pi}{3})} = \dfrac{1}{4}$ by Inverse Function Theorem.

   (d) $f(x) = \sqrt{x^2 + 6x + 16}$

   $\Longrightarrow f'(x) = \dfrac{1}{2}(x^2 + 6x + 16)^{1/2}(2x + 6) = \dfrac{x + 3}{\sqrt{x^2 + 6x + 16}}$

   $\because f(0) = 4$

   $\therefore (f^{-1})'(4) = \dfrac{1}{f'(0)} = \dfrac{4}{3}$ by Inverse Function Theorem.

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Formulas for the Natural Exponential Function (自然指數函數的基本公式)

1. Find the derivative of the function $f(x)$.
   (a) $f(x) = \sin(e^x)$
   (b) $f(x) = \dfrac{e^x}{x^2+1}$
   (c) $f(x) = e^{x^2+3x}$
   (d) $f(x) = \tan(3e^{4x})$
   (e) $f(x) = e^{5x}\sin(x^2)$

2. Evaluate the integral $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$.

3. Find the area of the region bounded by $y = e^x$, $y = 1$, $x = 0$ and $x = 1$.

4. Find the linearization of the function $f(x) = e^x$ at 0 and use it to approximate $e^{0.02}$.

5. Find $f(x)$ if $f''(x) = 1 - e^{2x}$, $f(0) = 0$ and $f'(0) = -1$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f'(x) = \cos(e^x) \cdot e^x = e^x \cos(e^x)$

   (b) $f'(x) = \dfrac{e^x(x^2+1) - e^x \cdot 2x}{(x^2+1)^2} = \dfrac{e^x(x-1)^2}{(x^2+1)^2}$

   (c) $f'(x) = e^{x^2+3x} \cdot (2x+3) = (2x+3)e^{x^2+3x}$

   (d) $f'(x) = \sec^2(3e^{4x}) \cdot 3e^{4x} \cdot 4 = 12e^{4x}\sec^2(3e^{4x})$

   (e) $f'(x) = \left(e^{5x} \cdot 5\right)\sin(x^2) + e^{5x}\left(\cos(x^2) \cdot 2x\right) = e^{5x}(5\sin(x^2) + 2x\cos(x^2))$

2. (a) Let $u = \dfrac{1}{x} \Longrightarrow du = -\dfrac{1}{x^2}\,dx$. We get

   $$\int \frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{1/x} + C$$

3. area $= \displaystyle\int_0^1 (e^x - 1)\,dx = [e^x - x]_0^1 = e - 2$

4. $\because f'(x) = e^x$

   $\therefore L(x) = f(0) + f'(0)(x-0) = 1 + 1 \cdot x = 1 + x$

   $\Longrightarrow e^{0.02} = f(0.02) \approx L(0.02) = 1.02$

5. $f'(x) = \displaystyle\int (1 - e^{2x})\,dx = x - \frac{1}{2}e^{2x} + C$

   $\because f'(0) = -\dfrac{1}{2} + C = -1 \quad \therefore C = -\dfrac{1}{2}$

   $\Longrightarrow f'(x) = x - \dfrac{1}{2}e^{2x} - \dfrac{1}{2}$

   $\Longrightarrow f(x) = \displaystyle\int (x - \frac{1}{2}e^{2x} - \frac{1}{2})\,dx = \frac{1}{2}x^2 - \frac{1}{4}e^{2x} - \frac{1}{2}x + D$

   $\because f(0) = -\dfrac{1}{4} + D = 0 \quad \therefore D = \dfrac{1}{4}$

   $\Longrightarrow f(x) = \dfrac{1}{2}x^2 - \dfrac{1}{4}e^{2x} - \dfrac{1}{2}x + \dfrac{1}{4}$

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The Natural Logarithmic Function (自然對數函數)

1. Solve the following equations.
   (a) $e^{x^2+1} = 3$
   (b) $\ln(x^3 - 5) = 4$
   (c) $e^{\sqrt{x}} = 7$
   (d) $\ln(\sqrt{x} - 1) = 3$
   (e) $e^{-0.01x} = 1000$
   (f) $e^{2x} + 2 = 3e^x$

2. Simplify the following expressions.
   (a) $\ln 30 - \ln 5 + \ln 7$
   (b) $\ln 14 - \ln 15 + 3\ln 3 - \ln 2 + 2\ln 5$
   (c) $\ln(x^2 - 1) - \ln(x^2 + x) + \ln x$

3. Decompose the following expressions.
   (a) $\ln 360$
   (b) $\ln \dfrac{63}{20}$
   (c) $\ln \dfrac{(2x+1)^2 \sqrt[3]{x^2+1}}{x^2 - 1}$

4. Evaluate the integral $\displaystyle\int_0^{\ln 5} e^{2x}\, dx$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $e^{x^2+1}=3 \Longrightarrow \ln(e^{x^2+1})=x^2+1=\ln 3 \Longrightarrow x^2=\ln 3-1$
$\Longrightarrow x=\pm\sqrt{\ln 3-1}$
(b) $\ln(x^3-5)=4 \Longrightarrow e^{\ln(x^3-5)}=x^3-5=e^4 \Longrightarrow x^3=e^4+5$
$\Longrightarrow x=\sqrt[3]{e^4+5}$
(c) $e^{\sqrt{x}}=7 \Longrightarrow \ln(e^{\sqrt{x}})=\sqrt{x}=\ln 7 \Longrightarrow x=(\ln 7)^2$
(d) $\ln(\sqrt{x}-1)=3 \Longrightarrow e^{\ln(\sqrt{x}-1)}=\sqrt{x}-1=e^3 \Longrightarrow \sqrt{x}=e^3+1$
$\Longrightarrow x=(e^3+1)^2$
(e) $e^{-0.01x}=1000 \Longrightarrow \ln(e^{-0.01x})=-0.01x=\ln 1000=3\ln 10$
$\Longrightarrow x=-300\ln 10$
(f) Let $A=e^x$ and we get that $e^{2x}+2=3e^x \Longrightarrow A^2+2=3A$
$\Longrightarrow A^2-3A+2=0 \Longrightarrow (A-1)(A-2)=0 \Longrightarrow e^x=1$ or $e^x=2$
$\Longrightarrow x=\ln 1=0$ or $x=\ln 2$

2. (a) $\ln 30-\ln 5+\ln 7=\ln\left(\dfrac{30\times 7}{5}\right)=\ln 42$
(b) $\ln 14-\ln 15+3\ln 3-\ln 2+2\ln 5=\ln 14-\ln 15+\ln 3^3-\ln 2+\ln 5^2$
$=\ln\left(\dfrac{14\times 27\times 25}{15\times 2}\right)=\ln 315$
(c) $\ln(x^2-1)-\ln(x^2+x)+\ln x=\ln\dfrac{(x^2-1)x}{(x^2+x)}=\ln(x-1)$

3. (a) $\ln 360=\ln\left(2^3\times 3^2\times 5\right)=\ln 2^3+\ln 3^2+\ln 5=3\ln 2+2\ln 3+\ln 5$
(b) $\ln\dfrac{63}{20}=\ln\left(\dfrac{3^2\times 7}{2^2\times 5}\right)=\ln 3^2+\ln 7-\ln 2^2-\ln 5$
$=2\ln 3+\ln 7-2\ln 2-\ln 5$
(c) $\ln\dfrac{(2x+1)^2\sqrt[3]{x^2+1}}{x^2-1}=\ln\dfrac{(2x+1)^2(x^2+1)^{1/3}}{(x+1)(x-1)}$
$=\ln(2x+1)^2+\ln(x^2+1)^{1/3}-\ln(x+1)-\ln(x-1)$
$=2\ln(2x+1)+\dfrac{1}{3}\ln(x^2+1)-\ln(x+1)-\ln(x-1)$

4. Let $u=2x \Longrightarrow du=2\,dx$ & $\begin{cases} u(\ln 5)=2\ln 5 \\ u(0)=0 \end{cases}$. We get
$$\int_0^{\ln 5} e^{2x}\,dx=\frac{1}{2}\int_0^{2\ln 5} e^u\,du=\frac{1}{2}\left[e^u\right]_0^{2\ln 5}=\frac{1}{2}(25-1)=12$$

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Formulas for the Natural Logarithmic Function (自然對數函數的基本公式)

1. Find the derivative of the function $f(x)$.

   (a) $f(x) = \ln(x^2)$

   (b) $f(x) = \ln(4x^3 + 5x + 1)$

   (c) $f(x) = \ln(2x + 3)^4$

   (d) $f(x) = \ln(\cos(x^3))$

   (e) $f(x) = \ln|x^3 + 2x^2 + 5|$

   (f) $f(x) = \ln(x + \sin x)$

   (g) $f(x) = \ln(4x^2 + 3)$

   (h) $f(x) = \ln|\tan(5x)|$

   (i) $f(x) = x^2 \ln|x|$

   (j) $f(x) = (\ln x)^3$

   (k) $f(x) = \ln|\sec x + \tan x|$

   (l) $f(x) = \dfrac{\ln x}{1 + \ln x}$

   (m) $f(x) = \ln(|\ln x|)$

   (n) $f(x) = \ln \dfrac{x + 1}{x^2 - 2x}$

   (o) $f(x) = \ln \dfrac{x^3\sqrt{2x + 1}}{(3x - 1)^2}$

2. Evaluate $\dfrac{dy}{dx}$ if $\ln(xy) = e^{x+y}$.

3. Evaluate the integrals.

   (a) $\displaystyle\int \frac{3x^2 + 2}{2x^3 + 4x + 1}\,dx$

   (b) $\displaystyle\int \frac{e^x}{1 + e^x}\,dx$

   (c) $\displaystyle\int \frac{1}{x\ln(x^3)}\,dx$

(d) $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx$

(e) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{3\cos\theta}{2+\sin\theta}\,d\theta$

4. Find the area under the curve $y = \dfrac{4\ln x}{x}$ from 1 to 2.

<u>Solution</u> [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f'(x) = \dfrac{2x}{x^2} = \dfrac{2}{x}$

   (b) $f'(x) = \dfrac{12x^2+5}{4x^3+5x+1}$

   (c) $f'(x) = \dfrac{4(2x+3)^3 \cdot 2}{(2x+3)^4} = \dfrac{8}{2x+3}$

   (d) $f'(x) = \dfrac{-\sin(x^3)\cdot 3x^2}{\cos(x^3)} = -3x^2\tan(x^3)$

   (e) $f'(x) = \dfrac{3x^2+4x}{x^3+2x^2+5}$

   (f) $f'(x) = \dfrac{1+\cos x}{x+\sin x}$

   (g) $f'(x) = \dfrac{8x}{4x^2+3}$

   (h) $f'(x) = \dfrac{\sec^2(5x)\cdot 5}{\tan(5x)} = 5\cot(5x)\,\sec^2(5x)$

   (i) $f'(x) = 2x\ln|x| + x^2 \cdot \dfrac{1}{x} = x(2\ln|x|+1)$

   (j) $f'(x) = 3(\ln x)^2 \cdot \dfrac{1}{x} = \dfrac{3(\ln x)^2}{x}$

   (k) $f'(x) = \dfrac{\sec x\,\tan x + \sec^2 x}{\sec x + \tan x} = \dfrac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$

   (l) $f'(x) = \dfrac{\frac{1}{x}(1+\ln x) - \ln x \cdot \frac{1}{x}}{(1+\ln x)^2} = \dfrac{1}{x(1+\ln x)^2}$

   (m) $f'(x) = \dfrac{(\frac{1}{x})}{\ln x} = \dfrac{1}{x\ln x}$

   (n) $f(x) = \ln\dfrac{x+1}{x^2-2x} = \ln(x+1) - \ln x - \ln(x-2)$

   $\Longrightarrow f'(x) = \dfrac{1}{x+1} - \dfrac{1}{x} - \dfrac{1}{x-2}$

   (o) $f(x) = \ln\dfrac{x^3\sqrt{2x+1}}{(3x-1)^2} = 3\ln x + \dfrac{1}{2}\ln(2x+1) - 2\ln(3x-1)$

   $\Longrightarrow f'(x) = \dfrac{3}{x} + \dfrac{1}{2x+1} - \dfrac{6}{3x-1}$

2. $\ln(xy) = \ln x + \ln y = e^{x+y} \overset{\frac{d}{dx}}{\Longrightarrow} \dfrac{1}{x} + \dfrac{1}{y} \cdot \dfrac{dy}{dx} = e^{x+y}\left(1 + \dfrac{dy}{dx}\right)$

$\Longrightarrow \left(\dfrac{1}{y} - e^{x+y}\right)\dfrac{dy}{dx} = -\dfrac{1}{x} + e^{x+y} \Longrightarrow \dfrac{dy}{dx} = \dfrac{-\frac{1}{x} + e^{x+y}}{\frac{1}{y} - e^{x+y}} = \dfrac{-y + xye^{x+y}}{x - xye^{x+y}}$

3. (a) Let $u = 2x^3 + 4x + 1 \Longrightarrow du = (6x^2 + 4)\,dx$. We get

$$\int \frac{3x^2+2}{2x^3+4x+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}\ln|2x^3+4x+1| + C$$

(b) Let $u = 1 + e^x \Longrightarrow du = e^x\,dx$. We get

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|1+e^x| + C = \ln(1+e^x) + C$$

(c) Let $u = \ln(x^3) \Longrightarrow du = \dfrac{3}{x}\,dx$. We get

$$\int \frac{1}{x\ln(x^3)}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|\ln(x^3)| + C$$

(d) Let $u = x^2 + 1 \Longrightarrow du = 2x\,dx$ & $\begin{cases} u(1) = 2 \\ u(0) = 1 \end{cases}$. We get

$$\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u}\,du = \frac{1}{2}\left[\ln|u|\right]_1^2 = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

(e) Let $u = 2 + \sin\theta \Longrightarrow du = \cos\theta\,d\theta$ & $\begin{cases} u(\frac{\pi}{2}) = 3 \\ u(-\frac{\pi}{2}) = 1 \end{cases}$. We get

$$\int_{-\pi/2}^{\pi/2} \frac{3\cos\theta}{2+\sin\theta}\,d\theta = 3\int_1^3 \frac{1}{u}\,du = 3\left[\ln|u|\right]_1^3 = 3(\ln 3 - \ln 1) = 3\ln 3$$

4. Let $u = \ln x \Longrightarrow du = \dfrac{1}{x}\,dx$ & $\begin{cases} u(2) = \ln 2 \\ u(1) = \ln 1 = 0 \end{cases}$. We get

$$\text{area} = \int_1^2 \frac{4\ln x}{x}\,dx = \int_0^{\ln 2} 4u\,du = \left[2u^2\right]_0^{\ln 2} = 2(\ln 2)^2$$

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Integrals of Trigonometric Functions (三角函數的積分公式)

1. Evaluate the integrals.

   (a) $\displaystyle\int x^2 \csc(4x^3)\,dx$

   (b) $\displaystyle\int \cot(3x+1)\,dx$

   (c) $\displaystyle\int x\sec(x^2)\,dx$

   (d) $\displaystyle\int_0^{\pi/6} \tan(2x)\,dx$

2. Find $f(x)$ if $f''(x) = \sec^2 x$, $f(0) = 0$ and $f'(0) = 1$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) Let $u = 4x^3 \Longrightarrow du = 12x^2\,dx$. We get
$$\int x^2 \csc(4x^3)\,dx = \frac{1}{12}\int \csc u\,du = \frac{1}{12}\ln|\csc u - \cot u| + C$$
$$= \frac{1}{12}\ln|\csc(4x^3) - \cot(4x^3)| + C$$
   (b) Let $u = 3x + 1 \Longrightarrow du = 3\,dx$. We get
$$\int \cot(3x+1)\,dx = \frac{1}{3}\int \cot u\,du = \frac{1}{3}\ln|\sin u| + C$$
$$= \frac{1}{3}\ln|\sin(3x+1)| + C$$
   (c) Let $u = x^2 \Longrightarrow du = 2x\,dx$. We get
$$\int x\sec(x^2)\,dx = \frac{1}{2}\int \sec u\,du = \frac{1}{2}\ln|\sec u + \tan u| + C$$
$$= \frac{1}{2}\ln|\sec(x^2) + \tan(x^2)| + C$$
   (d) Let $u = 2x \Longrightarrow du = 2\,dx$ & $\begin{cases} u(\frac{\pi}{6}) = \frac{\pi}{3} \\ u(0) = 0 \end{cases}$. We get
$$\int_0^{\pi/6} \tan(2x)\,dx = \frac{1}{2}\int_0^{\pi/3} \tan u\,du = \frac{1}{2}\left[\ln|\sec u|\right]_0^{\pi/3}$$
$$= \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

2. $f'(x) = \displaystyle\int \sec^2 x\,dx = \tan x + C$

   $\because f'(0) = 0 + C = 1 \quad \therefore C = 1$

   $\Longrightarrow f'(x) = \tan x + 1$

   $\Longrightarrow f(x) = \displaystyle\int (\tan x + 1)\,dx = \ln|\sec x| + x + D$

   $\because f(0) = 0 + 0 + D = 0 \quad \therefore D = 0$

   $\Longrightarrow f(x) = \ln|\sec x| + x$

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Derivatives of General Exponential Functions (一般型指數函數的導數)

1. Find the derivative of the function $f(x)$.
   (a) $f(x) = x^3 + 5^x$
   (b) $f(x) = 2^{3x^2+4x+1}$
   (c) $f(x) = 4\cos(7^x)$
   (d) $f(x) = \dfrac{x^2}{2^x + 5}$
   (e) $f(x) = \ln(3^x + 8)$
2. Evaluate the integrals.
   (a) $\displaystyle\int 4^x\, dx$
   (b) $\displaystyle\int 10^{2x+3}\, dx$
   (c) $\displaystyle\int_{-1}^{3} 2^x\, dx$
   (d) $\displaystyle\int_{1}^{4} \frac{2^{\sqrt{x}}}{\sqrt{x}}\, dx$
3. Find the area under the curve $y = 3^x$ from 1 to 3.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f'(x) = 3x^2 + 5^x \ln 5$

   (b) $f'(x) = 2^{3x^2+4x+1} \ln 2 \cdot (6x+4)$

   (c) $f'(x) = -4\sin(7^x) \cdot 7^x \ln 7$

   (d) $f'(x) = \dfrac{2x(2^x+5) - x^2(2^x \ln 2)}{(2^x+5)^2}$

   (e) $f'(x) = \dfrac{3^x \ln 3}{3^x + 8}$

2. (a) $\displaystyle\int 4^x\,dx = \frac{4^x}{\ln 4} + C$

   (b) Let $u = 2x+3 \Longrightarrow du = 2\,dx$. We get

   $$\int 10^{2x+3}\,dx = \frac{1}{2}\int 10^u\,du = \frac{1}{2}\,\frac{10^u}{\ln 10} + C = \frac{10^{2x+3}}{2\ln 10} + C$$

   (c) $\displaystyle\int_{-1}^{3} 2^x\,dx = \left[\frac{2^x}{\ln 2}\right]_{-1}^{3} = \frac{8}{\ln 2} - \frac{\frac{1}{2}}{\ln 2} = \frac{15}{2\ln 2}$

   (d) Let $u = \sqrt{x} \Longrightarrow du = \dfrac{1}{2\sqrt{x}}\,dx$ & $\begin{cases} u(4) = 2 \\ u(1) = 1 \end{cases}$. We get

   $$\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int_1^2 2^u\,du = 2\,\left[\frac{2^u}{\ln 2}\right]_1^2 = \frac{4}{\ln 2}$$

3. area $= \displaystyle\int_1^3 3^x\,dx = \left[\frac{3^x}{\ln 3}\right]_1^3 = \frac{24}{\ln 3}$

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Derivatives of General Logarithmic Functions (一般型對數函數的導數)

1. Find the derivative of the function $f(x)$.

   (a) $f(x) = \log_6 |x|$

   (b) $f(x) = x^3 \log_2(x^2 + 1)$

   (c) $f(x) = (\log_4 x)^3$

   (d) $f(x) = \log_7 |\sin x|$

   (e) $f(x) = 7^x \log_3 x$

   (f) $f(x) = \dfrac{\log_{10} x}{x^2 + 3}$

2. Evaluate the integrals.

   (a) $\displaystyle\int \frac{(\log_5 x)^2}{x}\, dx$

   (b) $\displaystyle\int_{10}^{100} \frac{1}{x \log_{10} x}\, dx$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f(x) = \log_6 |x| \Longrightarrow f'(x) = \dfrac{1}{x \ln 6}$

   (b) $f(x) = x^3 \log_2(x^2+1)$

   $$\Longrightarrow f'(x) = 3x^2 \log_2(x^2+1) + x^3 \frac{2x}{(x^2+1)\ \ln 2}$$
   $$= 3x^2\ \log_2(x^2+1) + \frac{2x^4}{(x^2+1)\ \ln 2}$$

   (c) $f(x) = (\log_4 x)^3 \Longrightarrow f'(x) = 3(\log_4 x)^2 \cdot \dfrac{1}{x\ \ln 4} = \dfrac{3(\log_4 x)^2}{x\ \ln 4}$

   (d) $f(x) = \log_7 |\sin x| \Longrightarrow f'(x) = \dfrac{\cos x}{\sin x\ \ln 7}$

   (e) $f(x) = 7^x\ \log_3 x \Longrightarrow f'(x) = 7^x\ \ln 7\ \log_3 x + 7^x \dfrac{1}{x\ \ln 3}$

   (f) $f(x) = \dfrac{\log_{10} x}{x^2+3} \Longrightarrow f'(x) = \dfrac{\frac{1}{x\ \ln 10}(x^2+3) - \log_{10} x \cdot (2x)}{(x^2+3)^2}$

2. (a) Let $u = \log_5 x \Longrightarrow du = \dfrac{1}{x\ \ln 5}\, dx$. We get

   $$\int \frac{(\log_5 x)^2}{x}\, dx = \ln 5 \int u^2\, du = \ln 5 \cdot \frac{1}{3} u^3 + C = \frac{\ln 5}{3} (\log_5 x)^3 + C$$

   (b) Let $u = \log_{10} x \Longrightarrow du = \dfrac{1}{x\ \ln 10}\, dx\ \&\ \begin{cases} u(100) = 2 \\ u(10) = 1 \end{cases}$. We get

   $$\int_{10}^{100} \frac{1}{x\ \log_{10} x}\, dx = \ln 10 \int_1^2 \frac{1}{u}\, du = \ln 10\ [\ln |u|]_1^2$$
   $$= \ln 10\,(\ln 2 - \ln 1) = \ln 10 \cdot \ln 2$$

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Logarithmic Differentiation (對數微分法)

1. Find the derivative of the function $f(x)$.

   (a) $f(x) = \dfrac{(x^3+7)^2}{\sqrt[3]{2x+5}}$

   (b) $f(x) = \dfrac{x^5(x^2+4x+1)}{(x^3+2)\sqrt[4]{x-7}}$

   (c) $f(x) = (2x+1)^x$

   (d) $f(x) = x^{\sin x}$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f(x) = \dfrac{(x^3+7)^2}{\sqrt[3]{2x+5}} \Longrightarrow \ln f(x) = 2\ln(x^3+7) - \dfrac{1}{3}\ln(2x+5)$

$$\overset{\frac{d}{dx}}{\Longrightarrow} \frac{f'(x)}{f(x)} = 2\cdot\frac{3x^2}{x^3+7} - \frac{1}{3}\cdot\frac{2}{2x+5} = \frac{6x^2}{x^3+7} - \frac{2}{6x+15}$$

$$\Longrightarrow f'(x) = f(x)\left(\frac{6x^2}{x^3+7} - \frac{2}{6x+15}\right) = \frac{(x^3+7)^2}{\sqrt[3]{2x+5}}\left(\frac{6x^2}{x^3+7} - \frac{2}{6x+15}\right)$$

(b) $f(x) = \dfrac{x^5(x^2+4x+1)}{(x^3+2)\sqrt[4]{x-7}}$

$$\Longrightarrow \ln f(x) = 5\ln x + \ln(x^2+4x+1) - \ln(3x+2) - \frac{1}{4}\ln(x-7)$$

$$\overset{\frac{d}{dx}}{\Longrightarrow} \frac{f'(x)}{f(x)} = 5\cdot\frac{1}{x} + \frac{2x+4}{x^2+4x+1} - \frac{3}{3x+2} - \frac{1}{4}\cdot\frac{1}{x-7}$$

$$= \frac{5}{x} + \frac{2x+4}{x^2+4x+1} - \frac{3}{3x+2} - \frac{1}{4x-28}$$

$$\Longrightarrow f'(x) = f(x)\left(\frac{5}{x} + \frac{2x+4}{x^2+4x+1} - \frac{3}{3x+2} - \frac{1}{4x-28}\right)$$

$$= \frac{x^5(x^2+4x+1)}{(x^3+2)\sqrt[4]{x-7}}\left(\frac{5}{x} + \frac{2x+4}{x^2+4x+1} - \frac{3}{3x+2} - \frac{1}{4x-28}\right)$$

(c) $f(x) = (2x+1)^x \Longrightarrow \ln f(x) = x\ln(2x+1)$

$$\overset{\frac{d}{dx}}{\Longrightarrow} \frac{f'(x)}{f(x)} = 1\cdot\ln(2x+1) + x\cdot\frac{2}{2x+1} = \ln(2x+1) + \frac{2x}{2x+1}$$

$$\Longrightarrow f'(x) = f(x)\left(\ln(2x+1) + \frac{2x}{2x+1}\right) = (2x+1)^x\left(\ln(2x+1) + \frac{2x}{2x+1}\right)$$

(d) $f(x) = x^{\sin x} \Longrightarrow \ln f(x) = \sin x\,\ln x$

$$\overset{\frac{d}{dx}}{\Longrightarrow} \frac{f'(x)}{f(x)} = \cos x\,\ln x + \sin x\cdot\frac{1}{x} = \cos x\,\ln x + \frac{\sin x}{x}$$

$$\Longrightarrow f'(x) = f(x)\left(\cos x\,\ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x\,\ln x + \frac{\sin x}{x}\right)$$

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Inverse Trigonometric Functions (反三角函數)

1. Find the following values.

   (a) $\sin^{-1}(-\frac{1}{\sqrt{2}})$

   (b) $\arccos\frac{1}{2}$

   (c) $\arctan(-\frac{1}{\sqrt{3}})$

   (d) $\sec^{-1}2$

   (e) $\arccos(\cos\frac{2\pi}{7})$

   (f) $\tan^{-1}(\tan\frac{4\pi}{5})$

   (g) $\tan(\cos^{-1}\frac{3}{4})$

   (h) $\sin(\operatorname{arcsec} 3)$

   (i) $\sec(\arcsin x)$

   (j) $\csc(\tan^{-1}x)$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) Let $\theta = \sin^{-1}(-\dfrac{1}{\sqrt{2}}) \Longrightarrow \sin\theta = -\dfrac{1}{\sqrt{2}}$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

   $\Longrightarrow \theta = -\dfrac{\pi}{4}$

   (b) Let $\theta = \arccos\dfrac{1}{2} \Longrightarrow \cos\theta = \dfrac{1}{2}$ and $\theta \in [0, \pi] \Longrightarrow \theta = \dfrac{\pi}{3}$

   (c) Let $\theta = \arctan(-\dfrac{1}{\sqrt{3}}) \Longrightarrow \tan\theta = -\dfrac{1}{\sqrt{3}}$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

   $\Longrightarrow \theta = -\dfrac{\pi}{6}$

   (d) Let $\theta = \sec^{-1} 2 \Longrightarrow \sec\theta = 2$ and $\theta \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

   $\Longrightarrow \theta = \dfrac{\pi}{3}$

   (e) $\because \dfrac{2\pi}{7} \in [0, \pi]$ $\therefore \arccos(\cos\dfrac{2\pi}{7}) = \dfrac{2\pi}{7}$

   (f) $\tan^{-1}(\tan\dfrac{4\pi}{5}) = \tan^{-1}(\tan(\pi - \dfrac{\pi}{5})) = \tan^{-1}(\tan(-\dfrac{\pi}{5})) = -\dfrac{\pi}{5}$

   (Note that $\dfrac{4\pi}{5} \notin (-\frac{\pi}{2}, \frac{\pi}{2})$.)

   (g) Let $\theta = \cos^{-1}\dfrac{3}{4} \Longrightarrow \cos\theta = \dfrac{3}{4} \Longrightarrow \tan(\cos^{-1}\dfrac{3}{4}) = \tan\theta = \dfrac{\sqrt{7}}{3}$

   (請自行繪製輔助之直角三角形)

   (h) Let $\theta = \text{arcsec } 3 \Longrightarrow \sec\theta = 3 \Longrightarrow \sin(\text{arcsec } 3) = \sin\theta = \dfrac{\sqrt{8}}{3}$

   (請自行繪製輔助之直角三角形)

   (i) Let $\theta = \arcsin x \Longrightarrow \sin\theta = x = \dfrac{x}{1}$

   $\Longrightarrow \sec(\arcsin x) = \sec\theta = \dfrac{1}{\sqrt{1-x^2}}$

   (請自行繪製輔助之直角三角形)

   (j) Let $\theta = \tan^{-1} x \Longrightarrow \tan\theta = x = \dfrac{x}{1}$

   $\Longrightarrow \csc(\tan^{-1} x) = \csc\theta = \dfrac{\sqrt{1+x^2}}{x}$

   (請自行繪製輔助之直角三角形)

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Derivatives of Inverse Trigonometric Functions (反三角函數的導數)

1. Find the derivative of the function $f(x)$.

   (a) $f(x) = \sin^{-1}(x^2 - 3x)$

   (b) $f(x) = e^{3x}\tan^{-1} x$

   (c) $f(x) = \dfrac{\cos^{-1} x}{x^2 + 1}$

   (d) $f(x) = \sec^{-1}(e^{-x})$

   (e) $f(x) = (\cot^{-1} x)^3$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f'(x) = \dfrac{2x-3}{\sqrt{-x^4+6x^3-9x^2+1}}$

   (b) $f'(x) = 3e^{3x}\tan^{-1}x + \dfrac{e^{3x}}{1+x^2}$

   (c) $f'(x) = \dfrac{\left(-\frac{1}{\sqrt{1-x^2}}\right)(x^2+1) - \cos^{-1}x\cdot(2x)}{(x^2+1)^2}$

   $= -\dfrac{1}{(x^2+1)\sqrt{1-x^2}} - \dfrac{2x\cos^{-1}x}{(x^2+1)^2}$

   (d) $f'(x) = \dfrac{-e^{-x}}{e^{-x}\sqrt{(e^{-x})^2-1}} = -\dfrac{1}{\sqrt{e^{-2x}-1}} = -\dfrac{e^x}{\sqrt{1-e^{2x}}}$

   (e) $f'(x) = 3(\cot^{-1}x)^2\cdot\left(-\dfrac{1}{1+x^2}\right) = -\dfrac{3(\cot^{-1}x)^2}{1+x^2}$

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Indefinite Integrals involving Inverse Trigonometric Functions (反三角函數相關的不定積分)

1. Evaluate the integrals.

   (a) $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$

   (b) $\displaystyle\int \frac{1}{2x^2+2x+1}\,dx$

   (c) $\displaystyle\int \frac{1}{\sqrt{e^{2x}-16}}\,dx$

   (d) $\displaystyle\int_{\sqrt{3}}^{2} \frac{1}{\sqrt{4-x^2}}\,dx$

2. Find the area under the curve $y=\dfrac{1}{9+x^2}$ from 0 to $3\sqrt{3}$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) Let $u = \ln x \Longrightarrow du = \dfrac{1}{x}\,dx$. We get

$$\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C$$
$$= \sin^{-1}(\ln x) + C$$

(b) Let $u = x + \dfrac{1}{2} \Longrightarrow du = dx$. We get

$$\int \frac{1}{2x^2+2x+1}\,dx = \frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{1}{4}}\,dx = \frac{1}{2}\int \frac{1}{u^2+(\frac{1}{2})^2}\,du$$
$$= \frac{1}{2}\cdot 2\tan^{-1}(2u) + C = \tan^{-1}(2x+1) + C$$

(c) Let $u = e^x \Longrightarrow du = e^x\,dx$. We get

$$\int \frac{1}{\sqrt{e^{2x}-16}}\,dx = \int \frac{1}{e^x\sqrt{e^{2x}-16}}\,e^x\,dx = \int \frac{1}{u\sqrt{u^2-16}}\,du$$
$$= \frac{1}{4}\sec^{-1}(\frac{u}{4}) + C = \frac{1}{4}\sec^{-1}(\frac{e^x}{4}) + C$$

(d) $$\int_{\sqrt{3}}^{2} \frac{1}{\sqrt{4-x^2}}\,dx = \Big[\sin^{-1}(\frac{x}{2})\Big]_{\sqrt{3}}^{2} = \sin^{-1} 1 - \sin^{-1}(\frac{\sqrt{3}}{2})$$
$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

2. area $$= \int_0^{3\sqrt{3}} \frac{1}{9+x^2}\,dx = \Big[\frac{1}{3}\tan^{-1}(\frac{x}{3})\Big]_0^{3\sqrt{3}} = \frac{1}{3}\tan^{-1}(\sqrt{3}) - \frac{1}{3}\tan^{-1} 0$$
$$= \frac{\pi}{9}$$

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Transcendental Functions (超越函數) [綜合練習]

1. Determine whether or not the function $f(x)$ is an one-to-one function.
   (a) $f(x) = x^2$ on $(-\infty, \infty)$.
   (b) $f(x) = \dfrac{x+2}{x-1}$ on $(1, \infty)$.

2. Find the inverse function $f^{-1}(x)$, and verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.
   (a) $f(x) = \sqrt{x} + 1$ on $[0, \infty)$.
   (b) $f(x) = \dfrac{x+2}{x-1}$ on $(1, \infty)$.

3. Find the derivative of the inverse function $f^{-1}(x)$ at the point b.
   (a) $f(x) = x^3 - 3x^2 + 1$ on $[2, \infty)$; $b = 1$
   (b) $f(x) = 3x + \cos(2x)$ on $(-\infty, \infty)$; $b = \frac{3\pi}{4}$

4. Simplify the following expressions.
   (a) $\ln 14 - \ln 15 + 3\ln 3 - \ln 2 + 2\ln 5$
   (b) $\log_3 60 + \log_3 21 - \log_3 70$
   (c) $\ln(x^2 - 1) - \ln(x^2 + x) + \ln x$

5. Decompose the following expressions.
   (a) $\ln 360$
   (b) $\log_{10} 0.021$
   (c) $\ln \dfrac{(2x+1)^2\sqrt[3]{x^2+1}}{x^2-1}$

6. Solve the following equations.
   (a) $\ln(x^3 - 5) = 4$
   (b) $\log_2(x^2 - 1) = 3$
   (c) $3^{x^2-x-4} = 9$

(d) $e^{-0.01x} = 1000$

(e) $4^x - 10 \cdot 2^x + 16 = 0$

7. Find the following values.

(a) $\arccos \frac{1}{2}$

(b) $\arctan(-\frac{1}{\sqrt{3}})$

(c) $\tan(\cos^{-1} \frac{3}{4})$

(d) $\sec(\arcsin x)$

8. Find the derivative of the function $f(x)$.

(a) $f(x) = x^3 + 5^x$

(b) $f(x) = 2^{3x^2+4x+1}$

(c) $f(x) = \ln(4x^3 + 5x + 1)$

(d) $f(x) = \sin(e^x)$

(e) $f(x) = \dfrac{x^5(x^2 + 4x + 1)}{(x^3 + 2)\sqrt[4]{x - 7}}$

(f) $f(x) = \dfrac{e^x}{x^2 + 1}$

(g) $f(x) = \sin^{-1}(x^2 - 3x)$

(h) $f(x) = \ln |x^3 + 2x^2 + 5|$

(i) $f(x) = 7^x \log_3 x$

(j) $f(x) = x^{\sin x}$

(k) $f(x) = (\ln x)^3$

(l) $f(x) = e^{x^2+3x}$

(m) $f(x) = \sec^{-1}(e^{-x})$

(n) $f(x) = \ln \dfrac{x^3\sqrt{2x + 1}}{(3x - 1)^2}$

(o) $f(x) = \dfrac{\log_{10} x}{x^2 + 3}$

9. Evaluate the integrals.

(a) $\displaystyle\int 10^{2x+3}\,dx$

(b) $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

(c) $\displaystyle\int x\sec(x^2)\,dx$

(d) $\displaystyle\int \frac{1}{2x^2+2x+1}\,dx$

(e) $\displaystyle\int \frac{e^x}{1+e^x}\,dx$

(f) $\displaystyle\int \frac{(\log_5 x)^2}{x}\,dx$

(g) $\displaystyle\int \frac{3x^2+2}{2x^3+4x+1}\,dx$

(h) $\displaystyle\int_{-1}^{3} 2^x\,dx$

(i) $\displaystyle\int_{0}^{\ln 5} e^{2x}\,dx$

(j) $\displaystyle\int_{\sqrt{3}}^{2} \frac{1}{\sqrt{4-x^2}}\,dx$

(k) $\displaystyle\int_{0}^{\pi/6} \tan(2x)\,dx$

(l) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{3\cos\theta}{2+\sin\theta}\,d\theta$

10. Evaluate $\dfrac{dy}{dx}$ if $\ln(xy) = e^{x+y}$.

11. Find the area under the curve $y = \dfrac{4\ln x}{x}$ from 1 to 2.

12. Find the linearization of the function $f(x) = e^x$ at 0 and use it to approximate $e^{0.02}$.

13. Find $f(x)$ if $f''(x) = 1 - e^{2x}$, $f(0) = 0$ and $f'(0) = -1$.