微積總棟員 Calculus - Exercises

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Integration by Parts (I) (分部積分法 (I))

1. Evaluate the following integrals.

   (a) $\displaystyle\int x\,2^x\,dx$

   (b) $\displaystyle\int x\ln x\,dx$

   (c) $\displaystyle\int \sin^{-1} x\,dx$

2. (a) Prove the reduction formula

$$\int (\ln x)^n\,dx = x\,(\ln x)^n - n\int (\ln x)^{n-1}\,dx \quad \text{for} \quad n \geq 1.$$

   (b) Use (a) to evaluate $\displaystyle\int (\ln x)^3\,dx$.

3. (a) Prove the reduction formula

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx \quad \text{for} \quad n \geq 1.$$

   (b) Use (a) to evaluate $\displaystyle\int x^4 e^x\,dx$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int x\,2^x\,dx = x\,\frac{2^x}{\ln 2} - \int \frac{2^x}{\ln 2}\,dx \quad \left(\text{Let } \begin{cases} u = x \\ dv = 2^x\,dx \end{cases} \Longrightarrow \begin{cases} du = dx \\ v = \frac{2^x}{\ln 2} \end{cases}\right)$

$\displaystyle = \frac{x\,2^x}{\ln 2} - \frac{1}{\ln 2}\int 2^x\,dx = \frac{x\,2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} + C$

(b) $\displaystyle\int x\,\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx \quad \left(\text{Let } \begin{cases} u = \ln x \\ dv = x\,dx \end{cases} \Longrightarrow \begin{cases} du = \frac{1}{x}\,dx \\ v = \frac{1}{2}x^2 \end{cases}\right)$

$\displaystyle = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

(c) $\displaystyle\int \sin^{-1}x\,dx = x\,\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx \quad \left(\text{Let } \begin{cases} u = \sin^{-1}x \\ dv = dx \end{cases} \Longrightarrow \begin{cases} du = \frac{1}{\sqrt{1-x^2}}\,dx \\ v = x \end{cases}\right)$

$\displaystyle\because \int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,(-2x)\,dx = -\frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw$

$\displaystyle = -\frac{1}{2}\,2\sqrt{w} + K = -\sqrt{1-x^2} + K \quad \left(\text{Let } w = 1 - x^2 \Longrightarrow dw = -2x\,dx\right)$

$\displaystyle\therefore \int \sin^{-1}x\,dx = x\,\sin^{-1}x + \sqrt{1-x^2} + C$, where $C = -K$.

2. (a) $\displaystyle\int (\ln x)^n\,dx = x\,(\ln x)^n - \int n\,(\ln x)^{n-1}\,\frac{1}{x}\,x\,dx$

$\displaystyle\left(\text{Let } \begin{cases} u = (\ln x)^n \\ dv = dx \end{cases} \Longrightarrow \begin{cases} du = n\,(\ln x)^{n-1}\,\frac{1}{x}\,dx \\ v = x \end{cases}\right)$

$\displaystyle = x\,(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

(b) $\displaystyle\int (\ln x)^3\,dx = x\,(\ln x)^3 - 3\int (\ln x)^2\,dx = x\,(\ln x)^3 - 3\left(x\,(\ln x)^2 - 2\int \ln x\,dx\right)$

$\displaystyle = x\,(\ln x)^3 - 3x\,(\ln x)^2 + 6\int \ln x\,dx$

$\displaystyle = x\,(\ln x)^3 - 3x\,(\ln x)^2 + 6\left(x\,\ln x - \int dx\right)$

$= x\,(\ln x)^3 - 3x\,(\ln x)^2 + 6x\,\ln x - 6x + C$

3. (a) $\displaystyle\int x^n e^x\,dx = x^n e^x - \int n\,x^{n-1}e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

$\displaystyle\left(\text{Let } \begin{cases} u = x^n \\ dv = e^x\,dx \end{cases} \Longrightarrow \begin{cases} du = nx^{n-1}\,dx \\ v = e^x \end{cases}\right)$

(b) $\int x^4 e^x \, dx = x^4 e^x - 4 \int x^3 e^x \, dx = x^4 e^x - 4 \left(x^3 e^x - 3 \int x^2 e^x \, dx \right)$

$$
\begin{aligned}
&= x^4 e^x - 4x^3 e^x + 12 \int x^2 e^x \, dx \\
&= x^4 e^x - 4x^3 e^x + 12 \left(x^2 e^x - 2 \int x e^x \, dx \right) \\
&= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24 \int x e^x \, dx \\
&= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24 \left(x e^x - \int e^x \, dx \right) \\
&= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24 e^x + C
\end{aligned}
$$

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Integration by Parts (II) (分部積分法 (II))

1. Evaluate the following integrals.

   (a) $\displaystyle\int (\ln x)^2\,dx$

   (b) $\displaystyle\int e^{2x}\cos(3x)\,dx$

2. Find the area between the curve $y = \tan^{-1} x$, the x-axis, and the vertical lines $y = 0$ and $y = 1$.

3. (a) Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\,\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \quad \text{for} \quad n \geq 2.$$

   (b) Use (a) to evaluate $\displaystyle\int \cos^4 x\,dx$.

4. (a) Prove the reduction formula

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx \quad \text{for} \quad n \geq 2.$$

   (b) Use (a) to evaluate $\displaystyle\int_0^{\pi/2} \sin^5 x\,dx$.

   (c) Show that for positive integer k,

$$\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)}\,\frac{\pi}{2}$$

   and

$$\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2\cdot 4\cdot 6\cdots(2k)}{3\cdot 5\cdot 7\cdots(2k+1)}.$$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int (\ln x)^2\,dx = x\,(\ln x)^2 - \int 2\ln x\,dx = x\,(\ln x)^2 - 2\int \ln x\,dx$

$$\left(\text{Let } \begin{cases} u = (\ln x)^2 \\ dv = dx \end{cases} \Longrightarrow \begin{cases} du = \frac{2\ln x}{x}\,dx \\ v = x \end{cases}\right)$$

$$\because \int \ln x\,dx = x\,\ln x - \int dx = x\,\ln x - x + K \quad \left(\text{Let } \begin{cases} \bar{u} = \ln x \\ d\bar{v} = dx \end{cases} \Longrightarrow \begin{cases} d\bar{u} = \frac{1}{x}\,dx \\ \bar{v} = x \end{cases}\right)$$

$$\therefore \int (\ln x)^2\,dx = x\,(\ln x)^2 - 2x\,\ln x + 2x + C, \text{ where } C = -2K.$$

(b) $\displaystyle\int e^{2x}\cos(3x)\,dx = \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\int e^{2x}\sin(3x)\,dx$

$$\left(\text{Let } \begin{cases} u = e^{2x} \\ dv = \cos(3x)dx \end{cases} \Longrightarrow \begin{cases} du = 2e^{2x}\,dx \\ v = \frac{1}{3}\sin(3x) \end{cases}\right)$$

$$\because \int e^{2x}\sin(3x)\,dx = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\int e^{2x}\cos(3x)\,dx$$

$$\left(\text{Let } \begin{cases} \bar{u} = e^{2x} \\ d\bar{v} = \sin(3x)\,dx \end{cases} \Longrightarrow \begin{cases} d\bar{u} = 2e^{2x}\,dx \\ \bar{v} = -\frac{1}{3}\cos(3x) \end{cases}\right)$$

$$\therefore \int e^{2x}\cos(3x)\,dx = \frac{1}{3}e^{2x}\sin(3x) + \frac{2}{9}e^{2x}\cos(3x) - \frac{4}{9}\int e^{2x}\cos(3x)\,dx$$

$$\Longrightarrow \frac{13}{9}\int e^{2x}\cos(3x)\,dx = \frac{1}{3}e^{2x}\sin(3x) + \frac{2}{9}e^{2x}\cos(3x) + K$$

$$\Longrightarrow \int e^{2x}\cos(3x)\,dx = \frac{3}{13}e^{2x}\sin(3x) + \frac{2}{13}e^{2x}\cos(3x) + C,$$

where $C = \dfrac{9}{13}K$.

2. Since $\tan^{-1} x \geq 0$ on $[0, 1]$, we get

$$\text{area} = \int_0^1 \tan^{-1} x\,dx = \left[x\,\tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx.$$

$$\left(\text{Let } \begin{cases} u = \tan^{-1} x \\ dv = dx \end{cases} \Longrightarrow \begin{cases} du = \frac{1}{1+x^2}\,dx \\ v = x \end{cases}\right)$$

$$\because \int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\,2x\,dx = \frac{1}{2}\int_1^2 \frac{1}{w}\,dw$$

$$= \frac{1}{2}\left[\ln|w|\right]_1^2 = \frac{1}{2}\ln 2 \quad \left(\text{Let } w = 1 + x^2 \Longrightarrow dw = 2x\,dx \text{ and } \begin{cases} w(1) = 2 \\ w(0) = 1 \end{cases}\right)$$

$$\therefore \text{area} = \int_0^1 \tan^{-1} x\,dx = \tan^{-1} 1 - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

3. (a) $\displaystyle\int \cos^n x\,dx = \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,\sin^2 x\,dx$

$$\left(\text{Let } \begin{cases} u = \cos^{n-1} x \\ dv = \cos x\,dx \end{cases} \Longrightarrow \begin{cases} du = -(n-1)\cos^{n-2} x\,\sin x\,dx \\ v = \sin x \end{cases}\right)$$

$$= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,(1-\cos^2 x)\,dx$$

$$= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

$$\Longrightarrow n\int \cos^n x\,dx = \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx$$

$$\Longrightarrow \int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\,\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

(b) $\displaystyle\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\,\sin x + \frac{3}{4}\int \cos^2 x\,dx$

$$= \frac{1}{4}\cos^3 x\,\sin x + \frac{3}{4}\left(\frac{1}{2}\cos x\,\sin x + \frac{1}{2}\int dx\right)$$

$$= \frac{1}{4}\cos^3 x\,\sin x + \frac{3}{8}\cos x\,\sin x + \frac{3}{8}x + C$$

4. (a) $\displaystyle\int_0^{\pi/2} \sin^n x\,dx = \left[-\sin^{n-1} x\,\cos x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2} x\,\cos^2 x\,dx$

$$\left(\text{Let } \begin{cases} u = \sin^{n-1} x \\ dv = \sin x\,dx \end{cases} \Longrightarrow \begin{cases} du = (n-1)\sin^{n-2} x\,\cos x\,dx \\ v = -\cos x \end{cases}\right)$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x\,(1-\sin^2 x)\,dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x\,dx - (n-1)\int_0^{\pi/2} \sin^n x\,dx$$

$$\Longrightarrow n\int_0^{\pi/2} \sin^n x\,dx = (n-1)\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$\Longrightarrow \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

(b) $\displaystyle\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x\,dx = \frac{4}{5}\,\frac{2}{3}\int_0^{\pi/2} \sin x\,dx$

$$= \frac{8}{15}\left[-\cos x\right]_0^{\pi/2} = \frac{8}{15}$$

(c) i. $\displaystyle\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{2k-1}{2k}\int_0^{\pi/2} \sin^{2k-2} x\,dx$

$$= \frac{2k-1}{2k}\,\frac{2k-3}{2k-2}\int_0^{\pi/2} \sin^{2k-4} x\,dx$$

$$= \frac{2k-1}{2k}\,\frac{2k-3}{2k-2}\,\frac{2k-5}{2k-4}\int_0^{\pi/2} \sin^{2k-6} x\,dx$$

$$= \frac{2k-1}{2k}\,\frac{2k-3}{2k-2}\,\frac{2k-5}{2k-4}\cdots\frac{3}{4}\,\frac{1}{2}\int_0^{\pi/2} dx$$

$$= \frac{2k-1}{2k}\,\frac{2k-3}{2k-2}\,\frac{2k-5}{2k-4}\cdots\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2} = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)}\,\frac{\pi}{2}$$

ii. $$\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2k}{2k+1}\int_0^{\pi/2} \sin^{2k-1} x\,dx$$

$$= \frac{2k}{2k+1}\,\frac{2k-2}{2k-1}\int_0^{\pi/2} \sin^{2k-3} x\,dx$$

$$= \frac{2k}{2k+1}\,\frac{2k-2}{2k-1}\,\frac{2k-4}{2k-3}\int_0^{\pi/2} \sin^{2k-5} x\,dx$$

$$= \frac{2k}{2k+1}\,\frac{2k-2}{2k-1}\,\frac{2k-4}{2k-3}\cdots\frac{4}{5}\,\frac{2}{3}\int_0^{\pi/2} \sin x\,dx$$

$$= \frac{2k}{2k+1}\,\frac{2k-2}{2k-1}\,\frac{2k-4}{2k-3}\cdots\frac{4}{5}\,\frac{2}{3}\,\left[-\cos x\right]_0^{\pi/2}$$

$$= \frac{2k}{2k+1}\,\frac{2k-2}{2k-1}\,\frac{2k-4}{2k-3}\cdots\frac{4}{5}\,\frac{2}{3}\,1 = \frac{2\cdot 4\cdot 6\cdots(2k)}{3\cdot 5\cdot 7\cdots(2k+1)}$$

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Trigonometric Integrals (I) (三角函數的積分 (I))

1. Evaluate the following integrals.

   (a) $\displaystyle\int \sin^3 x\ \cos^6 x\, dx$

   (b) $\displaystyle\int \sin^2 x\ \cos^7 x\, dx$

   (c) $\displaystyle\int \sin^5 x\ \cos^3 x\, dx$

   (d) $\displaystyle\int \frac{\sin x\ \cos x}{\sin^2 x + 2}\, dx$

   (e) $\displaystyle\int \cot x\ \csc^3 x\, dx$

   (f) $\displaystyle\int_0^{\pi} \sin^3 x\, dx$

   (g) $\displaystyle\int_0^{\pi/2} \sin^2 x\ \cos^2 x\, dx$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int \sin^3 x\, \cos^6 x\, dx = \int \sin^2 x\, \cos^6 x\, \sin x\, dx = -\int (1-\cos^2 x) \cos^6 x\, (-\sin x)\, dx$

$\displaystyle= -\int (1-u^2)u^6\, du \qquad (\text{Let } u = \cos x \Longrightarrow du = -\sin x\, dx)$

$\displaystyle= \int (-u^6 + u^8)\, du = -\frac{1}{7}u^7 + \frac{1}{9}u^9 + C = -\frac{1}{7}\cos^7 x + \frac{1}{9}\cos^9 x + C$

(b) $\displaystyle\int \sin^2 x\, \cos^7 x\, dx = \int \sin^2 x\, (\cos^2 x)^3\, \cos x\, dx$

$\displaystyle= \int \sin^2 x\, (1-\sin^2 x)^3\, \cos x\, dx = \int u^2(1-u^2)^3\, du$

$(\text{Let } u = \sin x \Longrightarrow du = \cos x\, dx)$

$\displaystyle= \int (u^2 - 3u^4 + 3u^6 - u^8)\, du = \frac{1}{3}u^3 - \frac{3}{5}u^5 + \frac{3}{7}u^7 - \frac{1}{9}u^9 + C$

$\displaystyle= \frac{1}{3}\sin^3 x - \frac{3}{5}\sin^5 x + \frac{3}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C$

(c) <u>Method I</u>:

$\displaystyle\int \sin^5 x\, \cos^3 x\, dx = \int \sin^5 x\, \cos^2 x\, \cos x\, dx = \int \sin^5 x\, (1-\sin^2 x) \cos x\, dx$

$\displaystyle= \int u^5(1-u^2)\, du \qquad (\text{Let } u = \sin x \Longrightarrow du = \cos x\, dx)$

$\displaystyle= \int (u^5 - u^7)\, du = \frac{1}{6}u^6 - \frac{1}{8}u^8 + C$

$\displaystyle= \frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x + C$

<u>Method II</u>:

$\displaystyle\int \sin^5 x\, \cos^3 x\, dx = \int \sin^4 x\, \cos^3 x\, \sin x\, dx = -\int (1-\cos^2 x)^2 \cos^3 x\, (-\sin x)\, dx$

$\displaystyle= -\int (1-u^2)^2\, u^3\, du \qquad (\text{Let } u = \cos x \Longrightarrow du = -\sin x\, dx)$

$\displaystyle= \int (-u^3 + 2u^5 - u^7)\, du = -\frac{1}{4}u^4 + \frac{1}{3}u^6 - \frac{1}{8}u^8 + C$

$\displaystyle= -\frac{1}{4}\cos^4 x + \frac{1}{3}\cos^6 x - \frac{1}{8}\cos^8 x + C$

(d) $\displaystyle\int \frac{\sin x\, \cos x}{\sin^2 x + 2}\, dx = \frac{1}{2}\int \frac{1}{\sin^2 x + 2}\, 2\sin x\, \cos x\, dx$

$(\text{Let } u = \sin^2 x + 2 \Longrightarrow du = 2\sin x\, \cos x\, dx)$

$$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(\sin^2 x + 2) + C$$

(e) $$\int \cot x\, \csc^3 x\, dx = \int \frac{\cos x}{\sin^4 x}\,dx = \int \frac{1}{u^4}\,du$$

(Let $u = \sin x \Longrightarrow du = \cos x\, dx$)

$$= -\frac{1}{3u^3} + C = -\frac{1}{3\sin^3 x} + C = -\frac{1}{3}\csc^3 x + C$$

(f) $$\int_0^\pi \sin^3 x\, dx = \int_0^\pi \sin^2 x\, \sin x\, dx = -\int_0^\pi (1-\cos^2 x)(-\sin x)\,dx$$

$$\left(\text{Let } u = \cos x \Longrightarrow du = -\sin x\, dx \text{ and } \left\{ \begin{array}{l} u(\pi) = -1 \\ u(0) = 1 \end{array} \right.\right)$$

$$= -\int_1^{-1} (1-u^2)\,du = -\left[u - \frac{1}{3}u^3\right]_1^{-1} = \frac{4}{3}$$

(g) $$\int_0^{\pi/2} \sin^2 x\, \cos^2 x\, dx = \int_0^{\pi/2} (\sin x\, \cos x)^2\, dx = \int_0^{\pi/2} \left(\frac{1}{2}\sin(2x)\right)^2\, dx$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(\frac{1-\cos(4x)}{2}\right)\,dx = \frac{1}{4}\left[\frac{1}{2}x - \frac{1}{8}\sin(4x)\right]_0^{\pi/2} = \frac{\pi}{16}$$

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Trigonometric Integrals (II) (三角函數的積分 (II))

1. Evaluate the following integrals.

   (a) $\displaystyle\int \tan^7 x \sec^3 x \, dx$

   (b) $\displaystyle\int \tan^5 x \sec^4 x \, dx$

   (c) $\displaystyle\int \tan^2 x \, dx$

   (d) $\displaystyle\int \tan^6 x \sec^4 x \, dx$

   (e) $\displaystyle\int \tan^3 x \, dx$

   (f) $\displaystyle\int \cot x \csc^3 x \, dx$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int \tan^7 x \sec^3 x\,dx = \int \left(\tan^2 x\right)^3 \sec^2 x \sec x \tan x\,dx$

$$= \int \left(\sec^2 x - 1\right)^3 \sec^2 x \sec x \tan x\,dx = \int (u^2-1)^3 u^2\,du$$

(Let $u = \sec x \Longrightarrow du = \sec x \tan x\,dx$)

$$= \int (u^8 - 3u^6 + 3u^4 - u^2)\,du = \frac{1}{9}u^9 - \frac{3}{7}u^7 + \frac{3}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \frac{1}{9}\sec^9 x - \frac{3}{7}\sec^7 x + \frac{3}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

(b) Method I:

$$\int \tan^5 x \sec^4 x\,dx = \int \tan^5 x \sec^2 x \sec^2 x\,dx = \int \tan^5 x\,(1+\tan^2 x) \sec^2 x\,dx$$

$$= \int u^5\,(1+u^2)\,du = \int (u^5 + u^7)\,du = \frac{1}{6}u^6 + \frac{1}{8}u^8 + C$$

(Let $u = \tan x \Longrightarrow du = \sec^2 x\,dx$)

$$= \frac{1}{6}\tan^6 x + \frac{1}{8}\tan^8 x + C$$

Method II:

$$\int \tan^5 x \sec^4 x\,dx = \int \left(\tan^2 x\right)^2 \sec^3 x \sec x \tan x\,dx$$

$$= \int \left(\sec^2 x - 1\right)^2 \sec^3 x \sec x \tan x\,dx = \int (u^2-1)^2 u^3\,du$$

(Let $u = \sec x \Longrightarrow du = \sec x \tan x\,dx$)

$$= \int (u^7 - 2u^5 + u^3)\,du = \frac{1}{8}u^8 - \frac{1}{3}u^6 + \frac{1}{4}u^4 + C$$

$$= \frac{1}{8}\sec^8 x - \frac{1}{3}\sec^6 x + \frac{1}{4}\sec^4 x + C$$

(c) $\displaystyle\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

(d) $\displaystyle\int \tan^6 x \sec^4 x\,dx = \int \tan^6 x \sec^2 x \sec^2 x\,dx = \int \tan^6 x\,(1+\tan^2 x) \sec^2 x\,dx$

$$= \int u^6\,(1+u^2)\,du = \int (u^6 + u^8)\,du = \frac{1}{7}u^7 + \frac{1}{9}u^9 + C$$

(Let $u = \tan x \Longrightarrow du = \sec^2 x\,dx$)

$$= \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C$$

(e) $\displaystyle\int \tan^3 x\,dx = \int \tan^2 x\,\tan x\,dx = \int (\sec^2 x - 1)\,\tan x\,dx$

$$= \int \tan x\,\sec^2 x\,dx - \int \tan x\,dx = \int \tan x\,\sec^2 x\,dx - \ln|\sec x|$$

$$\because \int \tan x\,\sec^2 x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C$$

(Let $u = \tan x \Longrightarrow du = \sec^2 x\,dx$)

$$\therefore \int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C$$

(f) $\displaystyle\int \cot x\,\csc^3 x\,dx = -\int \csc^2 x\,(-\csc x\,\cot x)\,dx = -\int u^2\,du$

(Let $u = \csc x \Longrightarrow du = -\csc x\,\cot x\,dx$)

$$= -\frac{1}{3}u^3 + C = -\frac{1}{3}\csc^3 x + C$$

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Trigonometric Integrals (III) (三角函數的積分 (III))

1. Evaluate the following integrals.

   (a) $\displaystyle\int \cos(3x)\,\cos(8x)\,dx$

   (b) $\displaystyle\int \sin(6x)\,\cos(5x)\,dx$

   (c) $\displaystyle\int \sin(3x)\,\sin(7x)\,dx$

   (d) $\displaystyle\int \cos(7x)\,\sin(5x)\,dx$

   (e) $\displaystyle\int_0^{\pi/8} \cos(5x)\,\cos(3x)\,dx$

2. Show that for positive integers m and n, we have

   (a) $\displaystyle\int_{-\pi}^{\pi} \sin(mx)\,\cos(nx)\,dx = 0$

   (b) $\displaystyle\int_{-\pi}^{\pi} \sin(mx)\,\sin(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$

   (c) $\displaystyle\int_{-\pi}^{\pi} \cos(mx)\,\cos(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int \cos(3x)\,\cos(8x)\,dx = \frac{1}{2}\int (\cos(11x)+\cos(-5x))\,dx$

$\displaystyle= \frac{1}{2}\int (\cos(11x)+\cos(5x))\,dx = \frac{1}{22}\sin(11x)+\frac{1}{10}\sin(5x)+C$

(b) $\displaystyle\int \sin(6x)\,\cos(5x)\,dx = \frac{1}{2}\int (\sin(11x)+\sin x)\,dx$

$\displaystyle= -\frac{1}{22}\cos(11x)-\frac{1}{2}\cos x+C$

(c) $\displaystyle\int \sin(3x)\,\sin(7x)\,dx = -\frac{1}{2}\int (\cos(10x)-\cos(-4x))\,dx$

$\displaystyle= -\frac{1}{2}\int (\cos(10x)-\cos(4x))\,dx = -\frac{1}{20}\sin(10x)+\frac{1}{8}\sin(4x)+C$

(d) $\displaystyle\int \cos(7x)\,\sin(5x)\,dx = \frac{1}{2}\int (\sin(12x)+\sin(-2x))\,dx$

$\displaystyle= \frac{1}{2}\int (\sin(12x)-\sin(2x))\,dx = -\frac{1}{24}\cos(12x)+\frac{1}{4}\cos(2x)+C$

(e) $\displaystyle\int_0^{\pi/8} \cos(5x)\,\cos(3x)\,dx = \frac{1}{2}\int_0^{\pi/8} (\cos(8x)+\cos(2x))\,dx$

$\displaystyle= \frac{1}{2}\left[\frac{1}{8}\sin(8x)+\frac{1}{2}\sin(2x)\right]_0^{\pi/8} = \frac{1}{4\sqrt{2}}$

2. (a) If $m \neq n$, $\displaystyle\int_{-\pi}^{\pi} \sin(mx)\,\cos(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} (\sin((m+n)x)+\sin((m-n)x))\,dx$

$\displaystyle= \frac{1}{2}\left[-\frac{1}{m+n}\cos((m+n)x)-\frac{1}{m-n}\cos((m-n)x)\right]_{-\pi}^{\pi} = 0$

(Apply the fact that $\cos x$ is an even function.)

If $m = n$, $\displaystyle\int_{-\pi}^{\pi} \sin(mx)\,\cos(nx)\,dx = \int_{-\pi}^{\pi} \sin(mx)\,\cos(mx)\,dx$

$\displaystyle= \frac{1}{2}\int_{-\pi}^{\pi} \sin(2mx)\,dx = \frac{1}{2}\left[-\frac{1}{2m}\cos(2mx)\right]_{-\pi}^{\pi} = 0$

(Apply the fact that $\cos x$ is an even function.)

(b) If $m \neq n$, $\displaystyle\int_{-\pi}^{\pi} \sin(mx)\,\sin(nx)\,dx = -\frac{1}{2}\int_{-\pi}^{\pi} (\cos((m+n)x)-\cos((m-n)x))\,dx$

$\displaystyle= -\frac{1}{2}\left[\frac{1}{m+n}\sin((m+n)x)-\frac{1}{m-n}\sin((m-n)x)\right]_{-\pi}^{\pi} = 0$

If $m = n$, $\displaystyle\int_{-\pi}^{\pi} \sin(mx)\,\sin(nx)\,dx = \int_{-\pi}^{\pi} \sin^2(mx)\,dx$

$$= \int_{-\pi}^{\pi} \left(\frac{1-\cos(2mx)}{2}\right)\,dx = \left[\frac{1}{2}x - \frac{1}{4m}\sin(2mx)\right]_{-\pi}^{\pi} = \pi$$

(c) If $m \neq n$, $\displaystyle\int_{-\pi}^{\pi} \cos(mx)\,\cos(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \left(\cos((m+n)x) + \cos((m-n)x)\right)\,dx$

$$= \frac{1}{2}\left[\frac{1}{m+n}\sin((m+n)x) + \frac{1}{m-n}\sin((m-n)x)\right]_{-\pi}^{\pi} = 0$$

If $m = n$, $\displaystyle\int_{-\pi}^{\pi} \cos(mx)\,\cos(nx)\,dx = \int_{-\pi}^{\pi} \cos^2(mx)\,dx$

$$= \int_{-\pi}^{\pi} \left(\frac{1+\cos(2mx)}{2}\right)\,dx = \left[\frac{1}{2}x + \frac{1}{4m}\sin(2mx)\right]_{-\pi}^{\pi} = \pi$$

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Trigonometric Substitution (I) (三角代換法 (I))

1. Evaluate the following integrals.

   (a) $\displaystyle\int \sqrt{1-4x^2}\,dx$

   (b) $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx$

   (c) $\displaystyle\int_0^2 x^2\sqrt{4-x^2}\,dx$

   (d) $\displaystyle\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int \sqrt{1-4x^2}\,dx = \frac{1}{2}\int \cos^2\theta\,d\theta = \frac{1}{2}\int \left(\frac{1+\cos(2\theta)}{2}\right)d\theta$

$$\left(\begin{array}{l}\text{Write } y=\sqrt{1-4x^2} \Longrightarrow (2x)^2+y^2=1 \\ \left[\text{Compare with } \sin^2\theta+\cos^2\theta=1.\right] \\ \Longrightarrow \left\{\begin{array}{l} 2x=\sin\theta \Longrightarrow x=\frac{1}{2}\sin\theta \Longrightarrow dx=\frac{1}{2}\cos\theta\,d\theta \\ y=\cos\theta \end{array}\right.\end{array}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right)+C = \frac{1}{4}\theta+\frac{1}{8}2\sin\theta\cos\theta+C$$

$= \dfrac{1}{4}\sin^{-1}(2x)+\dfrac{1}{2}x\sqrt{1-4x^2}+C$ (請自行繪製輔助之直角三角形)

(b) $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{9-(2x)^2}}\,2\,dx = \frac{1}{2}\int \frac{1}{\sqrt{9-u^2}}\,du$

(Let $u=2x \Longrightarrow du=2\,dx$)

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{3}\right)+C = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)+C$$

(c) $\displaystyle\int_0^2 x^2\sqrt{4-x^2}\,dx = 16\int_0^{\pi/2}(\sin\theta\,\cos\theta)^2\,d\theta = 16\int_0^{\pi/2}\left(\frac{1}{2}\sin(2\theta)\right)^2 d\theta$

$$\left(\begin{array}{l}\text{Write } y=\sqrt{4-x^2} \Longrightarrow x^2+y^2=4 \\ \left[\text{Compare with } (2\sin\theta)^2+(2\cos\theta)^2=4.\right] \\ \Longrightarrow \left\{\begin{array}{l} x=2\sin\theta \Longrightarrow dx=2\cos\theta\,d\theta \\ y=2\cos\theta \end{array}\right. \quad \text{and} \quad \begin{array}{c|c} x & \theta \\ \hline 2 & \frac{\pi}{2} \\ 0 & 0 \end{array}\end{array}\right)$$

$$= 4\int_0^{\pi/2}\left(\frac{1-\cos(4\theta)}{2}\right)d\theta = 4\left[\frac{1}{2}\theta-\frac{1}{8}\sin(4\theta)\right]_0^{\pi/2} = \pi$$

(d) $\displaystyle\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx = -\frac{1}{2}\int_0^3 \frac{1}{\sqrt{36-x^2}}(-2x)\,dx = -\frac{1}{2}\int_{36}^{27}\frac{1}{\sqrt{u}}\,du$

$$\left(\text{Let } u=36-x^2 \Longrightarrow du=-2x\,dx\ \&\ \left\{\begin{array}{l} u(3)=27 \\ u(0)=36 \end{array}\right.\right)$$

$$= -\frac{1}{2}\left[2\sqrt{u}\right]_{36}^{27} = 6-3\sqrt{3}$$

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Trigonometric Substitution (II) (三角代換法 (II))

1. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

   (b) $\displaystyle\int \frac{4x}{\sqrt{x^2+4}}\,dx$

   (c) $\displaystyle\int \frac{dx}{(x^2-1)^{3/2}}$

   (d) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+9}}$

   (e) $\displaystyle\int \frac{2}{25+9x^2}\,dx$

   (f) $\displaystyle\int \frac{dx}{\sqrt{x^2+4x-5}}$

   (g) $\displaystyle\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx = 2\int \tan^2\theta\,d\theta = 2\int (\sec^2\theta - 1)\,d\theta$

$$\left(\begin{array}{l} \text{Write } y=\sqrt{x^2-4} \Longrightarrow y^2+4=x^2 \\ \text{[Compare with } (2\tan\theta)^2+4=(2\sec\theta)^2.\text{]} \\ \Longrightarrow \begin{cases} x = 2\sec\theta \Longrightarrow dx = 2\sec\theta\,\tan\theta\,d\theta \\ y = 2\tan\theta \end{cases} \end{array}\right)$$

$= 2\tan\theta - 2\theta + C = \sqrt{x^2-4} - 2\sec^{-1}\left(\dfrac{x}{2}\right) + C$

(請自行繪製輔助之直角三角形)

(b) $\displaystyle\int \frac{4x}{\sqrt{x^2+4}}\,dx = \int \frac{2}{\sqrt{x^2+4}}\,2x\,dx = \int \frac{2}{\sqrt{u}}\,du$

(Let $u = x^2+4 \Longrightarrow du = 2x\,dx$)

$= 4\sqrt{u} + C = 4\sqrt{x^2+4} + C$

(c) $\displaystyle\int \frac{dx}{(x^2-1)^{3/2}} = \int \frac{\sec\theta}{\tan^2\theta}\,d\theta$

$$\left(\begin{array}{l} \text{Write } y=\sqrt{x^2-1} \Longrightarrow y^2+1=x^2 \\ \text{[Compare with } \tan^2\theta+1=\sec^2\theta.\text{]} \\ \Longrightarrow \begin{cases} x = \sec\theta \Longrightarrow dx = \sec\theta\,\tan\theta\,d\theta \\ y = \tan\theta \end{cases} \end{array}\right)$$

$\displaystyle= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C = -\frac{1}{\sin\theta} + C$

(Let $u = \sin\theta \Longrightarrow du = \cos\theta\,d\theta$)

$\displaystyle= -\csc\theta + C = -\frac{x}{\sqrt{x^2-1}} + C$

(請自行繪製輔助之直角三角形)

(d) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+9}} = \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$

$$\left(\begin{array}{l} \text{Write } y=\sqrt{x^2+9} \Longrightarrow y^2=x^2+9 \\ \text{[Compare with } (3\sec\theta)^2=(3\tan\theta)^2+9.\text{]} \\ \Longrightarrow \begin{cases} x = 3\tan\theta \Longrightarrow dx = 3\sec^2\theta\,d\theta \\ y = 3\sec\theta \end{cases} \end{array}\right)$$

$\displaystyle= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \frac{1}{u^2}\,du = -\frac{1}{9u} + C = -\frac{1}{9\sin\theta} + C$

(Let $u = \sin\theta \Longrightarrow du = \cos\theta\, d\theta$)

$$= -\frac{1}{9}\csc\theta + C = -\frac{\sqrt{x^2+9}}{9x} + C$$

(請自行繪製輔助之直角三角形)

(e) $$\int \frac{2}{25+9x^2}\,dx = \frac{2}{3}\int \frac{1}{25+(3x)^2}\,3\,dx = \frac{2}{3}\int \frac{1}{25+u^2}\,du$$

(Let $u = 3x \Longrightarrow du = 3\,dx$)

$$= \frac{2}{3}\,\frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right) + C = \frac{2}{15}\tan^{-1}\left(\frac{3x}{5}\right) + C$$

(f) $$\int \frac{dx}{\sqrt{x^2+4x-5}} = \int \frac{dx}{\sqrt{(x+2)^2-9}} = \int \sec\theta\, d\theta$$

$$\left(\begin{array}{l} \text{Write } y = \sqrt{(x+2)^2-9} \Longrightarrow y^2+9 = (x+2)^2 \\ \text{[Compare with } (3\tan\theta)^2 + 9 = (3\sec\theta)^2.] \\ \Longrightarrow \left\{\begin{array}{l} x+2 = 3\sec\theta \Longrightarrow x = -2+3\sec\theta \Longrightarrow dx = 3\sec\theta\,\tan\theta\, d\theta \\ y = 3\tan\theta \end{array}\right. \end{array}\right)$$

$$= \ln|\sec\theta + \tan\theta| + K = \ln\left|\frac{x+2}{3} + \frac{\sqrt{x^2+4x-5}}{3}\right| + K$$

(請自行繪製輔助之直角三角形)

$$= \ln\left|x+2+\sqrt{x^2+4x-5}\right| + C, \text{ where } C = K - \ln 3.$$

(g) $$\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}} = \int_1^4 \frac{du}{\sqrt{u^2+9}}$$

$$\left(\text{Let } u = e^t \Longrightarrow du = e^t\,dt \ \& \ \left\{\begin{array}{l} u(\ln 4) = 4 \\ u(0) = 1 \end{array}\right.\right)$$

$$= \int_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(\frac{4}{3})} \sec\theta\, d\theta = \left[\ln|\sec\theta + \tan\theta|\right]_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(\frac{4}{3})}$$

$$\left(\begin{array}{l} \text{Write } y = \sqrt{u^2+9} \Longrightarrow y^2 = u^2+9 \\ \text{[Compare with } (3\sec\theta)^2 = (3\tan\theta)^2 + 9.] \\ \Longrightarrow \left\{\begin{array}{l} u = 3\tan\theta \Longrightarrow du = 3\sec^2\theta\, d\theta \\ y = 3\sec\theta \end{array}\right. \text{ and } \begin{array}{c|c} u & \theta \\ \hline 4 & \tan^{-1}(\frac{4}{3}) \\ 1 & \tan^{-1}(\frac{1}{3}) \end{array} \end{array}\right)$$

$$= \ln 3 - \ln\frac{\sqrt{10}+1}{3} = 2\ln 3 - \ln(\sqrt{10}+1)$$

(請自行繪製輔助之直角三角形)

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Integration by Partial Fractions (I) (部分分式積分法 (I))

1. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{8x+14}{2x^2+7x+3}\,dx$

   (b) $\displaystyle\int \frac{2x^2+3x+4}{x^3+3x^2+3x+1}\,dx$

   (c) $\displaystyle\int \frac{7x^2-21x+10}{x^3-3x^2-4x+12}\,dx$

   (d) $\displaystyle\int \frac{6x^3+4x^2+8x+1}{2x^4+2x^3+7x^2+x+3}\,dx$

   (Hint: $2x^4+2x^3+7x^2+x+3=(2x^2+1)(x^2+x+3)$)

   (e) $\displaystyle\int \frac{2x^3-3x^2+11x-5}{x^4-2x^3+5x^2-4x+4}\,dx$

   (Hint: $x^4-2x^3+5x^2-4x+4=(x^2-x+2)^2$)

   (f) $\displaystyle\int \frac{x^4+4x^3+8x^2+18x+16}{x^6+4x^4+8x^3+16x^2+32x+64}\,dx$

   (Hint: $x^6+4x^4+8x^3+16x^2+32x+64=(x^2+4)^3$)

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int \frac{8x+14}{2x^2+7x+3}\,dx = \int \left(\frac{4}{2x+1}+\frac{2}{x+3}\right)dx$

$\displaystyle= 2\int \frac{2}{2x+1}\,dx + 2\int \frac{1}{x+3}\,dx$

$= 2\ln|2x+1| + 2\ln|x+3| + C$

(b) $\displaystyle\int \frac{2x^2+3x+4}{x^3+3x^2+3x+1}\,dx = \int \left(\frac{2}{x+1}+\frac{-1}{(x+1)^2}+\frac{3}{(x+1)^3}\right)dx$

$\displaystyle= 2\int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx + 3\int \frac{1}{(x+1)^3}\,dx$

$\displaystyle= 2\ln|x+1| + (x+1)^{-1} - \frac{3}{2}(x+1)^{-2} + C$

(c) $\displaystyle\int \frac{7x^2-21x+10}{x^3-3x^2-4x+12}\,dx = \int \left(\frac{4}{x+2}+\frac{1}{x-2}+\frac{2}{x-3}\right)dx$

$\displaystyle= 4\int \frac{1}{x+2}\,dx + \int \frac{1}{x-2}\,dx + 2\int \frac{1}{x-3}\,dx$

$= 4\ln|x+2| + \ln|x-2| + 2\ln|x-3| + C$

(d) $\displaystyle\int \frac{6x^3+4x^2+8x+1}{2x^4+2x^3+7x^2+x+3}\,dx = \int \left(\frac{2x}{2x^2+1}+\frac{2x+1}{x^2+x+3}\right)dx$

$\displaystyle= \frac{1}{2}\int \frac{4x}{2x^2+1}\,dx + \int \frac{2x+1}{x^2+x+3}\,dx$

$\displaystyle= \frac{1}{2}\ln(2x^2+1) + \ln(x^2+x+3) + C$

(e) $\displaystyle\int \frac{2x^3-3x^2+11x-5}{x^4-2x^3+5x^2-4x+4}\,dx$

$\displaystyle= \int \left(\frac{2x-1}{x^2-x+2}+\frac{6x-3}{(x^2-x+2)^2}\right)dx$

$\displaystyle= \int \frac{2x-1}{x^2-x+2}\,dx + 3\int \frac{2x-1}{(x^2-x+2)^2}\,dx$

$= \ln(x^2-x+2) - 3(x^2-x+2)^{-1} + C$

(f) $\displaystyle\int \frac{x^4+4x^3+8x^2+18x+16}{x^6+4x^4+8x^3+16x^2+32x+64}\,dx$

$\displaystyle= \int \left(\frac{1}{x^2+4}+\frac{4x}{(x^2+4)^2}+\frac{2x}{(x^2+4)^3}\right)dx$

$\displaystyle= \int \frac{1}{x^2+4}\,dx + 2\int \frac{2x}{(x^2+4)^2}\,dx + \int \frac{2x}{(x^2+4)^3}\,dx$

$$= \frac{1}{2} \tan^{-1}(\frac{x}{2}) - 2(x^2+4)^{-1} - \frac{1}{2}(x^2+4)^{-2} + C$$

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Integration by Partial Fractions (II) (部分分式積分法 (II))

1. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{2x^2+x+1}{x^3-x^2+x-1}\,dx$

   (b) $\displaystyle\int \frac{2x^3+7x^2+13x+10}{x^2+2x+3}\,dx$

   (c) $\displaystyle\int \frac{9x^2-3x+3}{x^3+1}\,dx$

   (d) $\displaystyle\int \frac{x^2+4x+5}{x^2+3x+2}\,dx$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int \frac{2x^2+x+1}{x^3-x^2+x-1}\,dx = \int \left(\frac{2}{x-1}+\frac{1}{x^2+1}\right)dx$
$\displaystyle= 2\int \frac{1}{x-1}\,dx + \int \frac{1}{x^2+1}\,dx$
$= 2\ln|x-1| + \tan^{-1}x + C$

(b) $\displaystyle\int \frac{2x^3+7x^2+13x+10}{x^2+2x+3}\,dx = \int \left(2x+3+\frac{x+1}{x^2+2x+3}\right)dx$
$\displaystyle= \int (2x+3)\,dx + \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}\,dx$
$\displaystyle= x^2+3x+\frac{1}{2}\ln(x^2+2x+3) + C$

(c) $\displaystyle\int \frac{9x^2-3x+3}{x^3+1}\,dx = \int \left(\frac{5}{x+1}+\frac{4x-2}{x^2-x+1}\right)dx$
$\displaystyle= 5\int \frac{1}{x+1}\,dx + 2\int \frac{2x-1}{x^2-x+1}\,dx$
$= 5\ln|x+1| + 2\ln(x^2-x+1) + C$

(d) $\displaystyle\int \frac{x^2+4x+5}{x^2+3x+2}\,dx = \int \left(1+\frac{2}{x+1}+\frac{-1}{x+2}\right)dx$
$\displaystyle= \int 1\,dx + 2\int \frac{1}{x+1}\,dx - \int \frac{1}{x+2}\,dx$
$= x + 2\ln|x+1| - \ln|x+2| + C$

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Techniques of Integration (積分技巧) [綜合練習]

1. Evaluate the following indefinite integrals.

   (a) $\displaystyle\int \cos(7x)\,\sin(5x)\,dx$

   (b) $\displaystyle\int \tan^3 x\,dx$

   (c) $\displaystyle\int \sqrt{1-4x^2}\,dx$

   (d) $\displaystyle\int \frac{2x^3+7x^2+13x+10}{x^2+2x+3}\,dx$

   (e) $\displaystyle\int \cot x\,\csc^3 x\,dx$

   (f) $\displaystyle\int \frac{dx}{(x^2-1)^{3/2}}$

   (g) $\displaystyle\int e^{2x}\cos(3x)\,dx$

   (h) $\displaystyle\int \sin^2 x\,\cos^7 x\,dx$

   (i) $\displaystyle\int \frac{8x+14}{2x^2+7x+3}\,dx$

   (j) $\displaystyle\int \sin(3x)\,\sin(7x)\,dx$

   (k) $\displaystyle\int \frac{dx}{\sqrt{x^2+4x-5}}$

   (l) $\displaystyle\int \tan^6 x\,\sec^4 x\,dx$

   (m) $\displaystyle\int \sin^{-1} x\,dx$

   (n) $\displaystyle\int \sin^3 x\,\cos^6 x\,dx$

   (o) $\displaystyle\int x\ln x\,dx$

(p) $\displaystyle\int \frac{2x^3 - 3x^2 + 11x - 5}{x^4 - 2x^3 + 5x^2 - 4x + 4}\,dx$
(Hint: $x^4 - 2x^3 + 5x^2 - 4x + 4 = (x^2 - x + 2)^2$)

(q) $\displaystyle\int \cot x \,\csc^3 x\,dx$

(r) $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

(s) $\displaystyle\int \tan^7 x\,\sec^3 x\,dx$

(t) $\displaystyle\int \frac{4x}{\sqrt{x^2+4}}\,dx$

(u) $\displaystyle\int \frac{9x^2 - 3x + 3}{x^3+1}\,dx$

2. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^{\pi/8} \cos(5x)\,\cos(3x)\,dx$

(b) $\displaystyle\int_0^{3} \frac{x}{\sqrt{36-x^2}}\,dx$

(c) $\displaystyle\int_0^{\pi/2} \sin^2 x\,\cos^2 x\,dx$

(d) $\displaystyle\int_0^{2} x^2\sqrt{4-x^2}\,dx$

(e) $\displaystyle\int_0^{\pi} \sin^3 x\,dx$

(f) $\displaystyle\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

3. Find the area between the curve $y = \tan^{-1} x$, the x-axis, and the vertical lines $y = 0$ and $y = 1$.

4. (a) Prove the reduction formula

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx \quad \text{for} \quad n \geq 1.$$

(b) Use (a) to evaluate $\displaystyle\int x^4 e^x\,dx$.

5. (a) Prove the reduction formula

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx \quad \text{for} \quad n \geq 2.$$

(b) Use (a) to evaluate $\displaystyle\int_0^{\pi/2} \sin^5 x \, dx$.

(c) Show that for positive integer k,

$$\int_0^{\pi/2} \sin^{2k} x \, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)} \frac{\pi}{2}$$

and

$$\int_0^{\pi/2} \sin^{2k+1} x \, dx = \frac{2 \cdot 4 \cdot 6 \cdots (2k)}{3 \cdot 5 \cdot 7 \cdots (2k+1)}.$$

6. Show that for positive integers m and n, we have

(a) $\displaystyle\int_{-\pi}^{\pi} \sin(mx) \cos(nx) \, dx = 0$

(b) $\displaystyle\int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$

(c) $\displaystyle\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$