Calculus - Exercises

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Linear Approximations (線性估計)

1. Find the linearization of $f(x)$ at a.
   (a) $f(x) = x^9 + 3x^2$ and $a = 1$.
   (b) $f(x) = \tan x$ and $a = 0$.
   (c) $f(x) = \sqrt[3]{x}$ and $a = 8$.
2. Find the linearization of the function $f(x) = \sqrt{x}$ at 100 and use it to approximate $\sqrt{98}$.
3. Use the linearization to approximate $\sin 0.1$.
4. Use the linearization to approximate $\sqrt[5]{32.3}$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f'(x) = 9x^8 + 6x \Longrightarrow L(x) = f(1) + f'(1)(x-1) = 4 + 15(x-1)$

   (b) $f'(x) = \sec^2 x \Longrightarrow L(x) = f(0) + f'(0)(x-0) = x$

   (c) $f'(x) = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3\sqrt[3]{x^2}}$

   $\Longrightarrow L(x) = f(8) + f'(8)(x-8) = 2 + \dfrac{1}{12}(x-8)$

2. $\because f'(x) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

   $\therefore L(x) = f(100) + f'(100)(x-100) = 10 + \dfrac{1}{20}(x-100)$

   $\Longrightarrow \sqrt{98} = f(98) \approx L(98) = 9.9$

3. Let $f(x) = \sin x \Longrightarrow f'(x) = \cos x$

   The linearization of $f(x)$ at 0 is $L(x) = f(0) + f'(0)(x-0) = x$.

   $\Longrightarrow \sin 0.1 = f(0.1) \approx L(0.1) = 0.1$

4. Let $f(x) = \sqrt[5]{x} = x^{1/5} \Longrightarrow f'(x) = \dfrac{1}{5}x^{-4/5} = \dfrac{1}{5\sqrt[5]{x^4}}$

   The linearization of $f(x)$ at 32 is

   $$L(x) = f(32) + f'(32)(x-32) = 2 + \frac{1}{80}(x-32).$$

   $\Longrightarrow \sqrt[5]{32.3} = f(32.3) \approx L(32.3) = 2.00375$

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Differentials (微分量)

1. Find the differential dy.

   (a) $y = f(x) = x^5 + 4x^3$

   (b) $y = f(x) = \sin(x^2)$

   (c) $y = f(x) = \dfrac{x^2}{1+x}$

2. Find the differential of $y = x^4 - 3x^2 + 5$ at $x = 1$.

3. Let $y = f(x) = x^3 + 6x + 1$. Compute the values of $\triangle y$ and dy if x changes from 1 to 1.1.

4. Let $y = x^4 - x^2 + 1$. Approximate $\triangle y$ if x changes from 1 to 0.8.

5. Let $y = \sqrt[3]{x}$. Approximate $\triangle y$ if x changes from 8 to 8.2.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $dy = f'(x)\,dx = (5x^4 + 12x^2)\,dx$

   (b) $dy = f'(x)\,dx = 2x\cos(x^2)\,dx$

   (c) $dy = f'(x)\,dx = \dfrac{x^2+2x}{(1+x)^2}\,dx$

2. Let $y = f(x) = x^4 - 3x^2 + 5$.
   $\Longrightarrow dy = f'(1)\,dx = \left.(4x^3 - 6x)\right|_{x=1}\,dx = -2\,dx$

3. $\triangle y = f(1.1) - f(1) = 8.931 - 8 = 0.931$
   On the other hand, $dy = f'(x)\,dx = (3x^2+6)\,dx$.
   When x changes from 1 to 1.1, we get $\triangle x = 1.1 - 1 = 0.1 = dx$.
   $\Longrightarrow dy = f'(1)\,dx = 9 \times 0.1 = 0.9$

4. Let $f(x) = x^4 - x^2 + 1 \Longrightarrow dy = f'(x)\,dx = (4x^3 - 2x)\,dx$
   When x changes from 1 to 0.8, we get $\triangle x = 0.8 - 1 = -0.2 = dx$.
   $\Longrightarrow \triangle y \approx dy = f'(1)\,dx = 2 \times (-0.2) = -0.4$

5. Let $f(x) = \sqrt[3]{x} = x^{1/3} \Longrightarrow dy = f'(x)\,dx = \dfrac{1}{3\sqrt[3]{x^2}}\,dx$
   When x changes from 8 to 8.2, we get $\triangle x = 8.2 - 8 = 0.2 = dx$.
   $\Longrightarrow \triangle y \approx dy = f'(8)\,dx = \dfrac{1}{12} \times 0.2 = \dfrac{1}{60} \approx 0.016$

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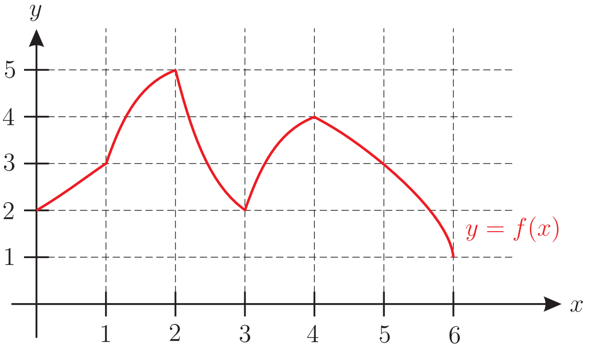
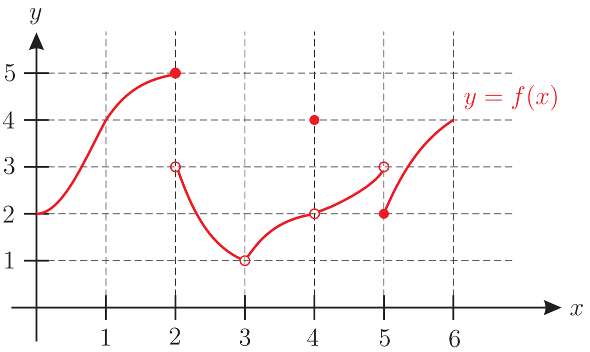
Extreme Values of Functions (函數的極值)

1. Find the extreme values and the local extreme values of the function $f(x)$ given below.

   (a)



   (b)



2. Find the extreme values of the function $f(x)$.

   (a) $f(x) = x^2 - 4x + 4$

   (b) $f(x) = -2x^2 - 12x - 11$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) The absolute maximum value is $f(2) = 5$, and the absolute minimum value is $f(6) = 1$.

   The local maximum values are $f(2) = 5$ and $f(4) = 4$, and the local minimum value is $f(3) = 2$.

   (b) The absolute maximum value is $f(2) = 5$, and there is NO absolute minimum value.

   The local maximum values are $f(2) = 5$ and $f(4) = 4$, and the local minimum value is $f(5) = 2$.

2. (a) $\because f(x) = (x-2)^2 \geq f(2) = 0$

   $\therefore$ The absolute minimum value is $f(2) = 0$, and there is NO absolute maximum value.

   (b) $\because f(x) = -2(x+3)^2 + 7 \leq f(-3) = 7$

   $\therefore$ The absolute maximum value is $f(-3) = 7$, and there is NO absolute minimum value.

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Fermat's Theorem and Critical Points (費馬引理與臨界點)

1. Find the critical point(s) of the function $f(x)$.

   (a) $f(x) = 2x^3 - 3x^2 - 36x + 5$

   (b) $f(x) = x + \dfrac{1}{x}$

   (c) $f(x) = \sqrt[3]{x}$

   (d) $f(x) = 2\cos x + \sin^2 x$

   (e) $f(x) = \dfrac{x}{x^2 - x + 1}$

2. Find the extreme values of $f(x)$ on the interval I.

   (a) $f(x) = x^4 - 2x^2 + 5$ on $I = [-2, 2]$.

   (b) $f(x) = 2x^3 - 9x^2 + 12x - 1$ on $I = [0, 3]$.

   (c) $f(x) = x\sqrt{4 - x^2}$ on $I = [-2, 2]$.

   (d) $f(x) = \sin x + \cos x$ on $I = [0, \pi]$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f(x) = 2x^3 - 3x^2 - 36x + 5$

   $\Longrightarrow f'(x) = 6x^2 - 6x - 36 = 6(x+2)(x-3)$

   $\Longrightarrow$ The critical points of f are $x = -2, 3$.

   (b) $f(x) = x + x^{-1}$

   $$\Longrightarrow f'(x) = 1 - x^{-2} = \frac{(x+1)(x-1)}{x^2}$$

   $\Longrightarrow$ The critical points of f are $x = -1, 1$.

   (c) $f(x) = x^{1/3}$

   $$\Longrightarrow f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

   $\Longrightarrow$ The critical point of f is $x = 0$.

   (d) $f(x) = 2\cos x + \sin^2 x$

   $\Longrightarrow f'(x) = -2\sin x + 2\sin x \cos x = 2\sin x(\cos x - 1)$

   $\Longrightarrow$ The critical points of f are $x = n\pi$, where n is any integer.

   (e) $$f(x) = \frac{x}{x^2 - x + 1}$$

   $$\Longrightarrow f'(x) = \frac{(x^2 - x + 1) - x(2x-1)}{(x^2 - x + 1)^2} = \frac{-(x+1)(x-1)}{(x^2 - x + 1)^2}$$

   $\Longrightarrow$ The critical points of f are $x = -1, 1$.

2. (a) $f(x) = x^4 - 2x^2 + 5 \Longrightarrow f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$

   $\Longrightarrow$ The critical points of f in $(-2, 2)$ are $x = -1, 0, 1$.

   $\because f(-2) = 13, f(-1) = 4, f(0) = 5, f(1) = 4$ and $f(2) = 13$.

   $\therefore f$ has an absolute maximum value 13 at $x = -2, 2$ and an absolute minimum value 4 at $x = -1, 1$.

   (b) $f(x) = 2x^3 - 9x^2 + 12x - 1$

   $\Longrightarrow f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2)$

   $\Longrightarrow$ The critical points of f in $(0, 3)$ are $x = 1, 2$.

   $\because f(0) = -1, f(1) = 4, f(2) = 3$ and $f(3) = 8$.

   $\therefore f$ has an absolute maximum value 8 at $x = 3$ and an absolute minimum value -1 at $x = 0$.

(c) $f(x) = x(4-x^2)^{1/2}$

$\Longrightarrow f'(x) = (4-x^2)^{1/2} - x^2(4-x^2)^{-1/2} = \dfrac{-2(x+\sqrt{2})(x-\sqrt{2})}{\sqrt{4-x^2}}$

$\Longrightarrow$ The critical points of f in $(-2,2)$ are $x = -\sqrt{2}, \sqrt{2}$.

$\because f(-2) = 0, f(-\sqrt{2}) = -2, f(\sqrt{2}) = 2$ and $f(2) = 0$.

$\therefore f$ has an absolute maximum value 2 at $x = \sqrt{2}$ and an absolute minimum value -2 at $x = -\sqrt{2}$.

(d) $f(x) = \sin x + \cos x \Longrightarrow f'(x) = \cos x - \sin x$

$\Longrightarrow$ The critical point of f in $(0,\pi)$ is $x = \frac{\pi}{4}$.

$\because f(0) = 1, f(\frac{\pi}{4}) = \sqrt{2}$ and $f(\pi) = -1$.

$\therefore f$ has an absolute maximum value $\sqrt{2}$ at $x = \frac{\pi}{4}$ and an absolute minimum value -1 at $x = \pi$.

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The Mean Value Theorem (均值定理)

1. Suppose that $f(1) = -2$ and $f'(x) \leq 3$ for all real number x. How large can $f(5)$ possible be?
2. Suppose that $1 \leq f'(x) \leq 6$ for any real number x. Show that $3 \leq f(7) - f(4) \leq 18$.
3. Show that $|\sin x - \sin y| \leq |x - y|$ for all real numbers x and y.
4. Show that the equation

$$x^3 + x - 1 = 0 \tag{1}$$

has exactly one real root.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. $\because f(x)$ is continuous on $[1,5]$ and is differentiable on $(1,5)$.

   $\therefore$ By the mean value theorem, there exists a number $c \in (1,5)$ such that $f'(c) = \dfrac{f(5)-f(1)}{5-1}$.

   $\Longrightarrow f(5)+2 = 4f'(c) \le 12 \Longrightarrow f(5) \le 10.$

2. $\because f(x)$ is continuous on $[4,7]$ and is differentiable on $(4,7)$.

   $\therefore$ By the mean value theorem, there exists a number $c \in (4,7)$ such that $f'(c) = \dfrac{f(7)-f(4)}{7-4}$.

   $\Longrightarrow f(7)-f(4) = 3f'(c) \Longrightarrow 3 \le f(7)-f(4) \le 18.$

3. Define $f(x) = \sin x$ on $\mathbb{R}$. We may assume that $y < x$.

   $\because f(x)$ is continuous on $[y,x]$ and is differentiable on (y,x) with $f'(x) = \cos x$.

   $\therefore$ By the mean value theorem, there exists a number $c \in (y,x)$ such that $f'(c) = \dfrac{f(x)-f(y)}{x-y}$.

   $\Longrightarrow |\sin x - \sin y| = |f(x)-f(y)| = |f'(c)||x-y| = |\cos c||x-y|$
   $\le |x-y|$.

4. Define $f(x) = x^3 + x - 1$. We have that $f(x)$ is continuous on $[0,1]$.

   $\because f(0) = -1 < 0$ and $f(1) = 1 > 0$.

   $\therefore$ By the intermediate value theorem, there exists a number $c \in (0,1)$ such that $f(c) = 0$.

   $\Longrightarrow c^3 + c - 1 = 0 \quad \Longrightarrow c$ is a root of (1).

   Assume that equation (1) has two distinct roots c_1 and c_2 with $c_1 < c_2$.

   $\because f(x)$ is continuous on $[c_1,c_2]$ and is differentiable on (c_1,c_2), and $f(c_1) = f(c_2) = 0$.

   $\therefore$ By Rolle's theorem, there exists a number $d \in (c_1,c_2)$ such that $f'(d) = 0$.

   But $f'(x) = 3x^2 + 1 \ge 1$, a contradiction.

Consequently, (1) has exactly one real root.

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Increasing and Decreasing Functions (遞增與遞減函數)

1. Find interval(s) on which the function f is increasing or decreasing.
   (a) $f(x) = 2x^3 + 3x^2 - 36x + 2$
   (b) $f(x) = x^4 - 2x^2$
   (c) $f(x) = 3x^4 + 8x^3$
   (d) $f(x) = x\sqrt{16 - x^2}$ on $[-4, 4]$.
   (e) $f(x) = \sin x + \cos x$ on $[0, 2\pi]$.
2. Show that $\sqrt{1+x} < 1 + \dfrac{1}{2}x$ for $x > 0$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f(x) = 2x^3 + 3x^2 - 36x + 2$

    $\Longrightarrow f'(x) = 6x^2 + 6x - 36 = 6(x-2)(x+3)$

    $\because f'(x) > 0$ on $(-\infty, -3)$ and $(2, \infty)$.

    $\therefore f$ is increasing on $(-\infty, -3)$ and $(2, \infty)$.

    $\because f'(x) < 0$ on $(-3, 2)$.

    $\therefore f$ is decreasing on $(-3, 2)$.

   (b) $f(x) = x^4 - 2x^2$

    $\Longrightarrow f'(x) = 4x^3 - 4x = 4x(x+1)(x-1)$

    $\because f'(x) > 0$ on $(-1, 0)$ and $(1, \infty)$.

    $\therefore f$ is increasing on $(-1, 0)$ and $(1, \infty)$.

    $\because f'(x) < 0$ on $(-\infty, -1)$ and $(0, 1)$.

    $\therefore f$ is decreasing on $(-\infty, -1)$ and $(0, 1)$.

   (c) $f(x) = 3x^4 + 8x^3$

    $\Longrightarrow f'(x) = 12x^3 + 24x^2 = 12x^2(x+2)$

    $\because f'(x) > 0$ on $(-2, 0)$ and $(0, \infty)$.

    $\therefore f$ is increasing on $(-2, \infty)$.

    $\because f'(x) < 0$ on $(-\infty, -2)$.

    $\therefore f$ is decreasing on $(-\infty, -2)$.

   (d) $f(x) = x\sqrt{16 - x^2} = x(16 - x^2)^{1/2}$

    $$\Longrightarrow f'(x) = (16-x^2)^{1/2} - x^2(16-x^2)^{-1/2} = \frac{-2(x+2\sqrt{2})(x-2\sqrt{2})}{\sqrt{16-x^2}}$$

    $\because f'(x) > 0$ on $(-2\sqrt{2}, 2\sqrt{2})$.

    $\therefore f$ is increasing on $(-2\sqrt{2}, 2\sqrt{2})$.

    $\because f'(x) < 0$ on $(-4, -2\sqrt{2})$ and $(2\sqrt{2}, 4)$.

    $\therefore f$ is decreasing on $(-4, -2\sqrt{2})$ and $(2\sqrt{2}, 4)$.

   (e) $f(x) = \sin x + \cos x$

    $\Longrightarrow f'(x) = \cos x - \sin x$

    $\because f'(x) > 0$ on $(0, \frac{\pi}{4})$ and $(\frac{5\pi}{4}, 2\pi)$.

    $\therefore f$ is increasing on $(0, \frac{\pi}{4})$ and $(\frac{5\pi}{4}, 2\pi)$.

$\because f'(x) < 0$ on $(\frac{\pi}{4}, \frac{5\pi}{4})$.

$\therefore f$ is decreasing on $(\frac{\pi}{4}, \frac{5\pi}{4})$.

2. Let $f(x) = \sqrt{1+x} - 1 - \dfrac{1}{2}x = (1+x)^{1/2} - 1 - \dfrac{1}{2}x$ on $[0, \infty)$.

$\Longrightarrow f'(x) = \dfrac{1}{2}(1+x)^{-1/2} - \dfrac{1}{2}$

$\Longrightarrow f''(x) = -\dfrac{1}{4}(1+x)^{-3/2} = -\dfrac{1}{4\sqrt{(1+x)^3}} < 0$ on $(0, \infty)$.

$\Longrightarrow f'(x)$ is decreasing on $[0, \infty)$.

$\Longrightarrow f'(x) < f'(0) = 0$ on $(0, \infty)$.

$\Longrightarrow f(x)$ is decreasing on $[0, \infty)$.

$\Longrightarrow f(x) < f(0) = 0$ on $(0, \infty)$.

$\Longrightarrow \sqrt{1+x} < 1 + \dfrac{1}{2}x$ for $x > 0$.

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First Derivative Test (一階導數判別法)

1. Find the local extreme values of the function $f(x)$.

   (a) $f(x) = 3x^4 + 8x^3 + 5$

   (b) $f(x) = x + \dfrac{1}{x}$

   (c) $f(x) = \dfrac{x^2}{x^2+1}$

   (d) $f(x) = \sqrt[3]{x}(x-4)$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f(x) = 3x^4 + 8x^3 + 5$

$\Longrightarrow f'(x) = 12x^3 + 24x^2 = 12x^2(x+2)$

$\Longrightarrow$ The critical points of f are $x = -2, 0$.

$\because f'(x) > 0$ on $(-2, 0)$ and $(0, \infty)$.

$\therefore$ By first derivative test, f has NO local extreme value at $x = 0$.

$\because f'(x) < 0$ on $(-\infty, -2)$ and $f'(x) > 0$ on $(-2, 0)$.

$\therefore$ By first derivative test, f has a local minimum value $f(-2) = -11$ at $x = -2$.

(b) $f(x) = x + \dfrac{1}{x} = x + x^{-1}$

$\Longrightarrow f'(x) = 1 - x^{-2} = \dfrac{(x+1)(x-1)}{x^2}$

$\Longrightarrow$ The critical points of f are $x = -1, 1$.

$\because f'(x) < 0$ on $(0, 1)$ and $f'(x) > 0$ on $(1, \infty)$.

$\therefore$ By first derivative test, f has a local minimum value $f(1) = 2$ at $x = 1$.

$\because f'(x) > 0$ on $(-\infty, -1)$ and $f'(x) < 0$ on $(-1, 0)$.

$\therefore$ By first derivative test, f has a local maximum value $f(-1) = -2$ at $x = -1$.

(c) $f(x) = \dfrac{x^2}{x^2+1}$

$\Longrightarrow f'(x) = \dfrac{(2x)(x^2+1) - x^2(2x)}{(x^2+1)^2} = \dfrac{2x}{(x^2+1)^2}$

$\Longrightarrow$ The critical point of f is $x = 0$.

$\because f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$.

$\therefore$ By first derivative test, f has a local minimum value $f(0) = 0$ at $x = 0$.

(d) $f(x) = \sqrt[3]{x}(x-4) = x^{4/3} - 4x^{1/3}$

$\Longrightarrow f'(x) = \dfrac{4}{3}x^{1/3} - \dfrac{4}{3}x^{-2/3} = \dfrac{4(x-1)}{3\sqrt[3]{x^2}}$

$\Longrightarrow$ The critical points of f are $x = 0, 1$.

$\because f'(x) < 0$ on $(0, 1)$ and $f'(x) > 0$ on $(1, \infty)$.

$\therefore$ By first derivative test, f has a local minimum value

$f(1) = -3$ at $x = 1$.

$\because f'(x) < 0$ on $(-\infty, 0)$ and $(0, 1)$.

$\therefore$ By first derivative test, f has No local extreme value at $x = 0$.

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Concavity and Inflection Points (圖形的凹性與反曲點)

1. Find interval(s) on which the graph of the function f is concave up or concave down. In addition, find the inflection point(s) of the graph of f.

   (a) $f(x) = x^3 + 12x^2 - 7x + 3$

   (b) $f(x) = x^4 + 2x^3 + 3$

   (c) $f(x) = x^4 - 2x^3 - 12x^2 + 3x + 5$

   (d) $f(x) = 3x^5 - 5x^4 + 2x + 1$

   (e) $f(x) = x + \dfrac{1}{x}$ on $(-\infty, 0) \cup (0, \infty)$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f(x) = x^3 + 12x^2 - 7x + 3 \Longrightarrow f'(x) = 3x^2 + 24x - 7$
   $\Longrightarrow f''(x) = 6x + 24 = 6(x+4)$
   $\because f''(x) > 0$ on $(-4, \infty)$.
   $\therefore$ The graph of f is concave up on $(-4, \infty)$.
   $\because f''(x) < 0$ on $(-\infty, -4)$.
   $\therefore$ The graph of f is concave down on $(-\infty, -4)$.
   The inflection point is $(-4, f(-4)) = (-4, 159)$.

   (b) $f(x) = x^4 + 2x^3 + 3 \Longrightarrow f'(x) = 4x^3 + 6x^2$
   $\Longrightarrow f''(x) = 12x^2 + 12x = 12x(x+1)$
   $\because f''(x) > 0$ on $(-\infty, -1)$ and $(0, \infty)$.
   $\therefore$ The graph of f is concave up on $(-\infty, -1)$ and $(0, \infty)$.
   $\because f''(x) < 0$ on $(-1, 0)$.
   $\therefore$ The graph of f is concave down on $(-1, 0)$.
   The inflection points are $(-1, f(-1)) = (-1, 2)$ and
   $(0, f(0)) = (0, 3)$.

   (c) $f(x) = x^4 - 2x^3 - 12x^2 + 3x + 5 \Longrightarrow f'(x) = 4x^3 - 6x^2 - 24x + 3$
   $\Longrightarrow f''(x) = 12x^2 - 12x - 24 = 12(x+1)(x-2)$
   $\because f''(x) > 0$ on $(-\infty, -1)$ and $(2, \infty)$.
   $\therefore$ The graph of f is concave up on $(-\infty, -1)$ and $(2, \infty)$.
   $\because f''(x) < 0$ on $(-1, 2)$.
   $\therefore$ The graph of f is concave down on $(-1, 2)$.
   The inflection points are $(-1, f(-1)) = (-1, -7)$ and
   $(2, f(2)) = (0, -37)$.

   (d) $f(x) = 3x^5 - 5x^4 + 2x + 1 \Longrightarrow f'(x) = 15x^4 - 20x^3 + 2$
   $\Longrightarrow f''(x) = 60x^3 - 60x^2 = 60x^2(x-1)$
   $\because f''(x) > 0$ on $(1, \infty)$.
   $\therefore$ The graph of f is concave up on $(1, \infty)$.
   $\because f''(x) < 0$ on $(-\infty, 0)$ and $(0, 1)$.
   $\therefore$ The graph of f is concave down on $(-\infty, 0)$ and $(0, 1)$.

The inflection point is $(1, f(1)) = (1, 1)$.

(e) $f(x) = x + x^{-1} \Longrightarrow f'(x) = 1 - x^{-2}$

$\Longrightarrow f''(x) = 2x^{-3} = \dfrac{2}{x^3}$

$\because f''(x) > 0$ on $(0, \infty)$.

$\therefore$ The graph of f is concave up on $(0, \infty)$.

$\because f''(x) < 0$ on $(-\infty, 0)$.

$\therefore$ The graph of f is concave down on $(-\infty, 0)$.

There is NO inflection point.

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Second Derivative Test (二階導數判別法)

1. Find the local extreme values of the function $f(x)$.

   (a) $f(x) = x^3 + 3x^2 + 4$

   (b) $f(x) = x + \dfrac{1}{x}$ on $(-\infty, 0) \cup (0, \infty)$

   (c) $f(x) = \dfrac{x^2}{x-1}$ on $(-\infty, 1) \cup (1, \infty)$

   (d) $f(x) = \sqrt{x} - \sqrt[4]{x}$ on $[0, \infty)$

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $f(x) = x^3 + 3x^2 + 4$

$\Longrightarrow f'(x) = 3x^2 + 6x = 3x(x+2)$

$\Longrightarrow$ The critical points of f are $x = -2, 0$.

In addition, $f''(x) = 6x + 6$.

$\because f''(-2) = -6 < 0$

$\therefore$ By second derivative test, f has a local maximum value $f(-2) = 8$ at $x = -2$.

$\because f''(0) = 6 > 0$

$\therefore$ By second derivative test, f has a local minimum value $f(0) = 4$ at $x = 0$.

(b) $f(x) = x + \dfrac{1}{x} = x + x^{-1}$

$\Longrightarrow f'(x) = 1 - x^{-2} = \dfrac{(x+1)(x-1)}{x^2}$

$\Longrightarrow$ The critical points of f are $x = -1, 1$.

In addition, $f''(x) = 2x^{-3} = \dfrac{2}{x^3}$.

$\because f''(-1) = -2 < 0$

$\therefore$ By second derivative test, f has a local maximum value $f(-1) = -2$ at $x = -1$.

$\because f''(1) = 2 > 0$

$\therefore$ By second derivative test, f has a local minimum value $f(1) = 2$ at $x = 1$.

(c) $f(x) = \dfrac{x^2}{x-1}$

$\Longrightarrow f'(x) = \dfrac{(2x)(x-1) - x^2}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$

$\Longrightarrow$ The critical points of f are $x = 0, 2$.

In addition, $f''(x) = \dfrac{(2x-2)(x-1)^2 - (x^2-2x)\cdot 2(x-1)}{(x-1)^4}$

$= \dfrac{2}{(x-1)^3}$.

$\because f''(0) = -2 < 0$

$\therefore$ By second derivative test, f has a local maximum value

$f(0) = 0$ at $x = 0$.

$\because f''(2) = 2 > 0$

$\therefore$ By second derivative test, f has a local minimum value

$f(2) = 4$ at $x = 2$.

(d) $f(x) = \sqrt{x} - \sqrt[4]{x} = x^{1/2} - x^{1/4}$

$\implies f'(x) = \dfrac{1}{2}x^{-1/2} - \dfrac{1}{4}x^{-3/4} = \dfrac{2\sqrt[4]{x} - 1}{4\sqrt[4]{x^3}}$

$\implies$ The critical point of f is $x = \dfrac{1}{16}$.

In addition, $f''(x) = -\dfrac{1}{4}x^{-3/2} + \dfrac{3}{16}x^{-7/4} = \dfrac{-4\sqrt[4]{x} + 3}{16\sqrt[4]{x^7}}$.

$\because f''(\frac{1}{16}) = 8 > 0$

$\therefore$ By second derivative test, f has a local minimum value

$f(\frac{1}{16}) = -\dfrac{1}{4}$ at $x = \dfrac{1}{16}$.

<u>Remark</u>: Try to solve above problems by applying first derivative test.

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Antiderivatives and Indefinite Integrals (反導函數與不定積分)

1. Find the antiderivative of $f(x)$.

   (a) $f(x) = x^3$

   (b) $f(x) = \sec^2 x$

   (c) $f(x) = \sqrt{x}$

2. Evaluate the following indefinite integrals.

   (a) $\displaystyle\int 2\,dx$

   (b) $\displaystyle\int \sec x\ \tan x\,dx$

   (c) $\displaystyle\int 6x^2\,dx$

3. Find $f(x)$ if $f'(x) = 4x$ and $f(1) = 3$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\because \dfrac{d}{dx}(\dfrac{1}{4}x^4) = x^3$

   $\therefore$ The antiderivative of $f(x) = x^3$ is $\dfrac{1}{4}x^4 + C$.

   (b) $\because \dfrac{d}{dx}(\tan x) = \sec^2 x$

   $\therefore$ The antiderivative of $f(x) = \sec^2 x$ is $\tan x + C$.

   (c) $\because \dfrac{d}{dx}(\dfrac{2}{3}x^{3/2}) = x^{1/2} = \sqrt{x}$

   $\therefore$ The antiderivative of $f(x) = \sqrt{x}$ is $\dfrac{2}{3}x^{3/2} + C$.

2. (a) $\displaystyle\int 2\,dx = 2x + C$

   (b) $\displaystyle\int \sec x\ \tan x\,dx = \sec x + C$

   (c) $\displaystyle\int 6x^2\,dx = 2x^3 + C$

3. $\because \dfrac{d}{dx}(2x^2) = 4x = f'(x)$

   $\therefore f(x) = 2x^2 + C$

   $\because f(1) = 2 + C = 3 \Longrightarrow C = 1$

   $\therefore f(x) = 2x^2 + 1$

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Formulas for Indefinite Integrals (不定積分的基本公式)

1. Evaluate the following indefinite integrals.

   (a) $\displaystyle\int (4x^3 + 6x - 5)\,dx$

   (b) $\displaystyle\int (3x^2 - 4x + \sin x)\,dx$

   (c) $\displaystyle\int \frac{7x^5 - 4x^3 + 6x + 1}{2\sqrt{x}}\,dx$

   (d) $\displaystyle\int \left(12x + \frac{1}{x^2} - \frac{6}{x^3}\right)\,dx$

   (e) $\displaystyle\int \left(3\sin x - \cos x + 5\sec^2 x\right)\,dx$

2. Find $f(x)$ if $f''(x) = 12x^2 - 6x + 6$, $f'(1) = 8$ and $f(1) = 10$.

3. Find $g(x)$ if $g''(x) = 12x^2 + 24x$, $g(1) = 6$ and $g(-1) = 0$.

Solution [註：本解答僅提示重點，請自行補足細節流程。]

1. (a) $\displaystyle\int (4x^3+6x-5)\,dx = x^4+3x^2-5x+C$

   (b) $\displaystyle\int (3x^2-4x+\sin x)\,dx = x^3-2x^2-\cos x+C$

   (c) $\displaystyle\int \frac{7x^5-4x^3+6x+1}{2\sqrt{x}}\,dx = \int \left(\frac{7}{2}x^{9/2}-2x^{5/2}+3x^{1/2}+\frac{1}{2}x^{-1/2}\right)\,dx$

   $\displaystyle= \frac{7}{11}x^{11/2}-\frac{4}{7}x^{7/2}+2x^{3/2}+x^{1/2}+C$

   (d) $\displaystyle\int \left(12x+\frac{1}{x^2}-\frac{6}{x^3}\right)\,dx = \int \left(12x+x^{-2}-6x^{-3}\right)\,dx$

   $= 6x^2-x^{-1}+3x^{-2}+C$

   (e) $\displaystyle\int \left(3\sin x-\cos x+5\sec^2 x\right)\,dx = -3\cos x-\sin x+5\tan x+C$

2. $\displaystyle f'(x) = \int (12x^2-6x+6)\,dx = 4x^3-3x^2+6x+C$

   $\because f'(1) = 7+C = 8 \quad \therefore C = 1$

   $\Longrightarrow f'(x) = 4x^3-3x^2+6x+1$

   $\displaystyle\Longrightarrow f(x) = \int (4x^3-3x^2+6x+1)\,dx = x^4-x^3+3x^2+x+D$

   $\because f(1) = 4+D = 10 \quad \therefore D = 6$

   $\Longrightarrow f(x) = x^4-x^3+3x^2+x+6$

3. $\displaystyle g'(x) = \int (12x^2+24x)\,dx = 4x^3+12x^2+C$

   $\displaystyle\Longrightarrow g(x) = \int (4x^3+12x^2+C)\,dx = x^4+4x^3+Cx+D$

   $\because \begin{cases} g(1) = 5+C+D = 6 \\ g(-1) = -3-C+D = 0 \end{cases} \qquad \therefore \begin{cases} C = -1 \\ D = 2 \end{cases}$

   $\Longrightarrow g(x) = x^4+4x^3-x+2$

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Applications of the Derivative (導數的應用) [綜合練習]

1. Find the critical point(s) of the function $f(x)$.
   (a) $f(x) = 2x^3 - 3x^2 - 36x + 5$
   (b) $f(x) = \dfrac{x}{x^2 - x + 1}$

2. Find the extreme values of $f(x)$ on the interval I.
   (a) $f(x) = x^4 - 2x^2 + 5$ on $I = [-2, 2]$.
   (b) $f(x) = x\sqrt{4 - x^2}$ on $I = [-2, 2]$.

3. Find interval(s) on which the function f is increasing or decreasing.
   (a) $f(x) = 2x^3 + 3x^2 - 36x + 2$
   (b) $f(x) = x\sqrt{16 - x^2}$ on $[-4, 4]$.

4. Find interval(s) on which the graph of the function f is concave up or concave down. In addition, find the inflection point(s) of the graph of f.
   (a) $f(x) = x^3 + 12x^2 - 7x + 3$
   (b) $f(x) = x^4 - 2x^3 - 12x^2 + 3x + 5$

5. Find the local extreme values of the function $f(x)$.
   (a) $f(x) = x^3 + 3x^2 + 4$
   (b) $f(x) = \sqrt[3]{x}(x - 4)$

6. Let $f(x) = x + \dfrac{1}{x}$.
   (a) Find the critical point(s) of the function f.
   (b) Find interval(s) on which the function f is increasing or decreasing.
   (c) Find the local extreme values of the function f.
   (d) Find interval(s) on which the graph of the function f is concave up or concave down.

(e) Find the inflection point(s) of the graph of f.

7. Evaluate the following indefinite integrals.

   (a) $\int (4x^3 + 6x - 5)\, dx$

   (b) $\int \left(12x + \frac{1}{x^2} - \frac{6}{x^3}\right)\, dx$

   (c) $\int \left(3\sin x - \cos x + 5\sec^2 x\right)\, dx$

8. Find $f(x)$ if $f''(x) = 12x^2 - 6x + 6$, $f'(1) = 8$ and $f(1) = 10$.

9. Find the linearization of $f(x) = x^9 + 3x^2$ at $a = 1$.

10. Find the linearization of the function $f(x) = \sqrt{x}$ at 100 and use it to approximate $\sqrt{98}$.

11. Find the differential dy.

    (a) $y = f(x) = x^5 + 4x^3$

    (b) $y = f(x) = \dfrac{x^2}{1+x}$

12. Let $y = f(x) = x^3 + 6x + 1$. Compute the values of $\triangle y$ and dy if x changes from 1 to 1.1.

13. Show that $|\sin x - \sin y| \leq |x - y|$ for all real numbers x and y.

14. Show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$.

15. Show that the equation

$$x^3 + x - 1 = 0 \tag{1}$$

    has exactly one real root.