一、填空题

(1)0.7

(2)$\frac{1}{6}$

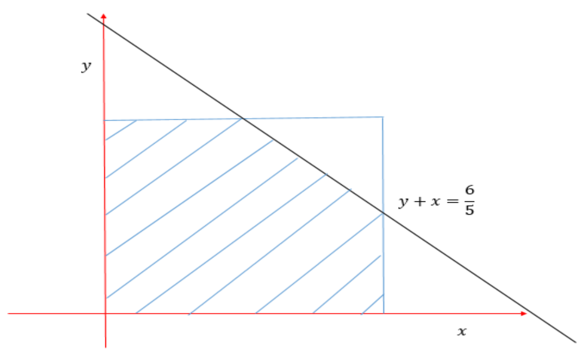
(3)$\frac{13}{48}$

解. X 从 $\{1,2,3,4\}$ 中等可能的取值,Y 在 $\{1,\cdots,X\}$ 中等可能的取值. 因此,

$$P(Y=2)=P(X=2)P(Y=2|X=2)$$
$$+P(X=3)P(Y=2|X=3)+P(X=4)P(Y=2|X=4)$$
$$\Rightarrow \frac{1}{4}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{4}=\frac{13}{48}.$$

(4)$\frac{17}{25}$

解. 随机变量 X,Y 服从 $(0,1)$ 上的均匀分布. 则积分区域如图所示,



$$P(X+Y<\frac{6}{5})=\int_0^{\frac{1}{5}}\int_0^1 1dydx+\int_{\frac{1}{5}}^1\int_0^{\frac{6}{5}-x}1dydx=\frac{17}{25}.$$

(5)$\begin{cases}\frac{1}{8}, & y\in(2,10)\\ 0, & \text{其它}\end{cases}$

(6)$\frac{1}{4\sqrt{y}}$.

(7)$1-F(\frac{1-y}{4})$.

(8)$\begin{cases}1-e^{-\lambda y}, & y\in[2,+\infty)\\ 0, & \text{其它}\end{cases}$

解. 取 $y \in (2, \infty)$, 则

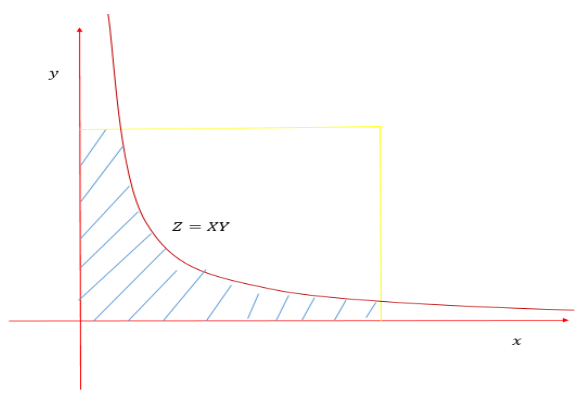
$$
\begin{aligned}
F_Y(y) &= P(\max(X,2) \le y) \\
&= P(X \le 2)P(\max(X,2) \le y | X \le 2) + P(X > 2)P(\max(X,2) \le y | X > 2) \\
&= P(X \le 2)P(2 \le y | X \le 2) + P(X > 2)P(X < y | X > 2) \\
&= (1 - e^{-2\lambda}) + e^{-2\lambda}\frac{e^{-2\lambda} - e^{-\lambda y}}{e^{-2\lambda}} \\
&= 1 - e^{-\lambda y}.
\end{aligned}
$$

(9)$\frac{1}{2}f_Y(z) + \frac{1}{2}f_Y(z-2)$.

(10)$\frac{3^6}{\Gamma(6)}$.

(11)$\frac{1}{2}$.

解. 由题可知 $X \sim U(0,1), Y|X = x \sim U(0, \frac{1}{x})$. 积分区域如图所示,



$$f(x,y) = f_X(x) \cdot f_{Y|X}(y|x) = x, \quad 0 < y < \frac{1}{x}.$$

$$
f_Y(y) = \begin{cases} \int_0^{\frac{1}{y}} x dx = \frac{1}{2y^2}, & y > 1 \\ \int_0^1 x dx = \frac{1}{2}, & 0 < y \le 1 \\ 0, & \text{其它} \end{cases}
$$

$$\Rightarrow P(Y < 1) = \frac{1}{2}.$$

(12)$\frac{147}{512}$.

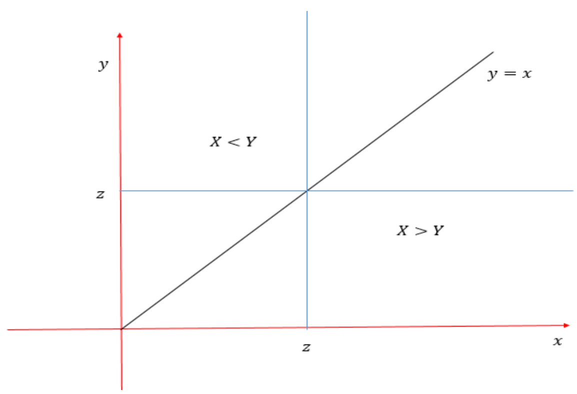
(13)$\frac{3}{8}$.

(14)$\begin{cases} 1-e^{-2x}, & x>0 \\ 0, & \text{其它} \end{cases}$

(15)$e(2\lambda)$.

解. 方法一：取 $z \in (0,+\infty)$, 则

$$\begin{aligned} F_Z(z) &= P(\min(X,Y) \le z) \\ &= P(X \le Y)P(\min(X,Y) \le z|X \le Y) + P(X > Y)P(\min(X,Y) \le z|X > y) \\ &= P(X \le Y)P(X \le z|X \le Y) + P(X > Y)P(Y \le z|X > Y) \end{aligned}$$



$$P(X \le Y) = \int_0^{+\infty}\int_0^{y} e^{-\lambda x-\lambda y}dxdy = \frac{1}{2}.$$

$$P(X > Y) = \int_0^{+\infty}\int_x^{+\infty} e^{-\lambda x-\lambda y}dydx = \frac{1}{2}.$$

$$P(X \le z|X \le Y) = \frac{P(X \le z, X \le Y)}{P(X \le Y)} = \frac{\int_0^z\int_x^{+\infty} e^{-\lambda x-\lambda y}dydx}{\frac{1}{2}} = 1-e^{-2\lambda z}.$$

$$P(Y \le z|X > Y) = \frac{P(Y \le z, Y < X)}{P(Y < X)} = \frac{\int_0^z\int_y^{+\infty} e^{-\lambda x-\lambda y}dxdy}{\frac{1}{2}} = 1-e^{-2\lambda z}.$$

$$\Rightarrow F_Z(z) = 1-e^{-2\lambda z}.$$

方法二: 取 $z \in (0, +\infty)$, 则

$$\begin{aligned}F(z) &= P(\min(X,Y) \leq z) = 1 - P(\min(X,Y) > z) = 1 - P(X > z, Y > z)\\ &= 1 - P(X > z)P(Y > z) = 1 - [1 - F_X(z)][1 - F_Y(z)]\\ &= 1 - e^{2\lambda}\end{aligned}$$

二、解答题

1.

解. 设配备有 k 个工人.X 表示某一时刻发生故障的设备数量. 那么, 由题意可知

$$P(X > k) < 0.01 \Leftrightarrow P(X \leq k) \geq 0.99$$

$$P(X \leq k) = \sum_{i=1}^{k} \binom{k}{i} (0.02)^i (0.98)^{k-i}$$

利用泊松定理得

$$P(X \leq k) = \sum_{i=1}^{k} \frac{(200 \times 0.02)^i}{i!} e^{-200 \times 0.02}$$

经计算可得 $k = 9$.

2.

解. 由题意可知系统的寿命 $T = \min(T_1, T_2, \cdots, T_4)$. 取 $t \in (0, +\infty)$, 则

$$F(t) = P(\min(T_1, T_2, \cdots, T_4) \leq t) = 1 - \Pi_{i=1}^{4} P(T_i > t) = 1 - e^{-2t}.$$

$$\Rightarrow f(t) = \begin{cases} 2e^{-2t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

3.

解. (1)X 与 Y 的边缘分布律为

$$X \sim \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

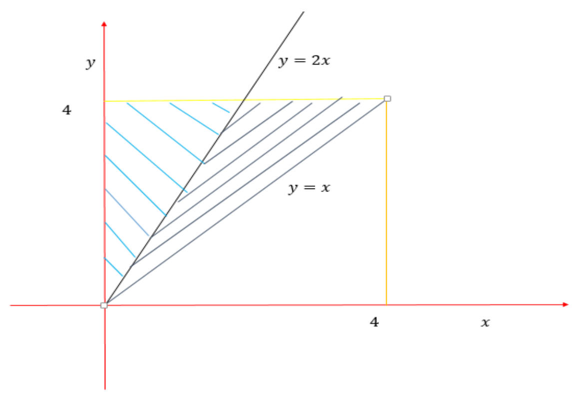
$$Y \sim \begin{pmatrix} 1 & 2 & 3 \\ 0.3 & 0.45 & 0.25 \end{pmatrix}$$

(2)

$$Y|X = 2 \sim \begin{pmatrix} 1 & 2 & 3 \\ 0.4 & 0.5 & 0.1 \end{pmatrix}$$

$$F_{Y|X}(y|x=2)=\begin{cases}0, & y<1\\ 0.4, & 1\le y<2\\ 0.9, & 2\le y<3\\ 1, & y\ge 3\end{cases}$$

4.



解. (1) 利用密度函数规一性,

$$\iint f(x,y)dxdy=1\Rightarrow\int_0^4\int_x^4 cdydx=1\Rightarrow c=\frac{1}{8}.$$

(2)

$$f_X(x)=\int_x^4\frac{1}{8}dy=\frac{1}{2}-\frac{1}{8}x.$$

$$f_Y(y)=\int_0^y\frac{1}{8}dx=\frac{1}{8}y.$$

显然 $f(x,y)\neq f_X(x)\cdot f_Y(y)$, 所以 X 与 Y 不独立.

(3) 当 $0<x<y<4$ 时,

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\frac{1}{y}$$

$$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}=\frac{1}{4-x}.$$

$$f_{X|Y}(x|y)=\begin{cases}\frac{1}{y}, & 0<x<y<4\\ 0, & \text{其它}\end{cases}$$

$$f_{Y|X}(y|x)=\begin{cases}\frac{1}{4-x}, & 0<x<y<4\\ 0, & \text{其它}\end{cases}$$

(4)

$$P(X<1|Y<2)=\frac{P(X<1,Y<2)}{P(Y<2)}=\frac{\int_0^1\int_x^2\frac{1}{8}dxdy}{\int_0^2\frac{1}{8}ydx}=\frac{3}{4}.$$

从 (X,Y) 的联合密度函数可知,(X,Y) 服从均匀分布, 那么 $X|Y=2\sim U(0,2)$.

$$\Rightarrow P(X<1|Y=2)=\frac{1}{2}.$$

(5) 取 $z\in(0,4)$, 则

$$F_Z(z)=P(X+Y\le z)=\int_0^{\frac{z}{2}}\frac{1}{8}dydx=\frac{z^2}{32}.$$

取 $z\in(4,8)$, 则

$$F_Z(z)=\int_0^{z-4}\int_x^4\frac{1}{8}dydx+\int_{z-4}^{\frac{z}{2}}\int_x z-x\frac{1}{8}dydx=-\frac{1}{32}z^2+\frac{1}{2}z-1.$$

$$\Rightarrow f_Z(z)=\begin{cases}\frac{z}{16}, & z\in(0,4)\\ -\frac{z}{16}+\frac{1}{2}, & z\in(4,8)\\ 0, & \text{其它}\end{cases}$$

(6) 方法一:

$$\begin{aligned}P(\min(2X,Y)\le 2)&=P(2X\le Y)P(\min(2X,Y)\le 2|2X\le Y)+P(X>Y)P(\min(2X,Y)\le 2|X>Y)\\&=P(2X<Y)P(2X<2|2X<Y)+P(2X>Y)P(Y<2|2X>Y)\end{aligned}$$

$$P(2X<Y)=\int_0^2\int_{2x}^5\frac{1}{8}dydx=\frac{1}{2}.$$

$$P(2X>Y)=\int_0^4\int_{\frac{1}{2}y}^y\frac{1}{8}dxdy=\frac{1}{2}.$$

$$P(2X<2|2X<Y)=\frac{P(2X<2,2X<y)}{P(2X<Y)}=\frac{\int_0^1\int_{2x}^2\frac{1}{8}dydx}{\frac{1}{2}}=\frac{3}{4}.$$

$$P(Y<2|2X>Y)=\frac{P(2X>Y,Y<2)}{P(2X>Y)}=\frac{\int_0^2\int_{\frac{1}{2}y}y\frac{1}{8}dydx}{\frac{1}{2}}=\frac{1}{4}.$$

$$\Rightarrow P(\min(2X,Y)<2)=\frac{1}{2}\cdot\frac{3}{4}+\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{2}.$$

方法二:

$$
\begin{aligned}
P(\min(2X,Y)<2) &= 1-P(\min(2X,Y)\geq 2) \\
&= 1-P(2X>2,Y>2) \\
&= 1-\int_2^4\int_1^y \frac{1}{8}dxdy \\
&= \frac{1}{2}
\end{aligned}
$$