EECS 16A Designing Information Devices and Systems I
Spring 2016 Elad Alon, Babak Ayazifar MT2 Extra Practice

Exam location: Study Party in the Woz!

Note: These problems were unused from previous semesters, so we are releasing them for extra practice. The distribution of concepts is NOT indicative of the distribution of concepts on the actual midterm. These problems did *not* go through as extensive a filter before being released to students. In addition, there may be references to concepts that were not taught this semester. In particular, if you come across 'nodal analysis' it's just another name for 'Analyze a circuit to find all voltages/current flows using KCL/KVL/Ohm's Law.'

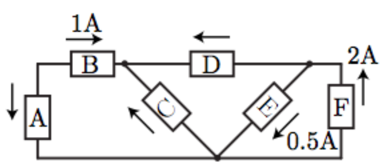
Section 1: Straightforward questions

Unless told otherwise, you must show work to get credit. There will be very little partial credit given in this section. Each problem is worth 8 points.

1. **KVL/KCL Practice**

Note: A box represents a resistor.

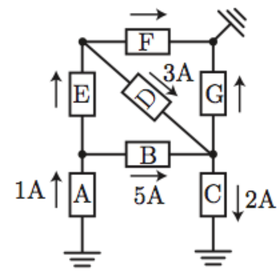
(a) Use KCL to find the values of i_a (the current through element A), i_c, and i_d for the circuit shown below. Use the arrows as the sign convention for currents. Which elements are connected in series?



Solutions: Elements A and B are connected in series, so $i_a = -1A$. By KCL on the DEF-node, $i_d = 2A - 0.5A = 1.5A$. Then by KCL on the BCD node, $i_c = -1A - 1.5A = -2.5A$.

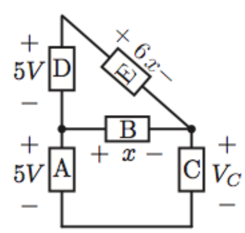
(b) Use KCL to find the values of i_e, i_f, and i_g for the circuit shown below. Which elements are connected in series?

Solutions: Applying KCL to the ABE-node, $5A + i_e = 1A$, so $i_e = -4A$. Then, applying KCL to the DEF-node, $3A + i_f = -4A$, so $i_f = -7A$. Finally, applying KCL to $BDGC$-node, $i_g + 2A = 3A + 5A$, so $i_g = 6A$. There are *no* elements connected in series here (no elements through which the same current is flowing). One might be tempted to conclude that elements G and F are connected in series. But note that this is not actually the case. There is a wire connected between node FG and the ground, meaning that the voltage at that node is set to 0. This means that some current can flow in that wire! (This is why we did not consider KCL at this node; the current flowing in that wire is a priori unknown, and we were only able to infer after solving for the other currents that it is actually $1A$ flowing into node FG.



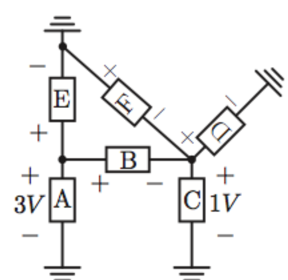
(c) In the circuit below, x denotes some unknown real number. Use KVL to find the values of v_b (the voltage difference across element B), v_c, and v_e for the circuit shown below. Which elements are connected in shunt (i.e. parallel)?

Solutions: Applying KVL to the loop DEB, we get $5-6x+x=0$, giving us $x=1$. Therefore, $v_b = x = 1V$ and $v_e = 6x = 6V$. Next, we apply KVL to the loop ABC, getting $5-1-v_c=0$, giving us $v_c = 4V$. The series combination of D and E, the series combination of A and C, and the element B are in parallel. (If you didn't put these because you interpreted the question as which individual elements were in parallel, that's fine. Give yourself full credit.) We can see this because the voltage difference across the series combination of D and E, which is $-5+6=1V$, is the same as the voltage difference across B (also $1V$).



(d) Use KVL to find the values of v_b, v_d, v_e and v_f for the circuit shown below. Which elements are connected in shunt?

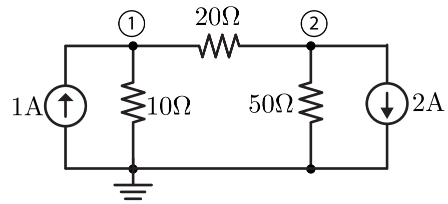
Solutions: Since the bottom ends of elements A and C are both connected to the ground, they are at the same potential. Therefore, we can apply KVL to the branch ABC as follows (another way of saying is this that ABC effectively form a loop): $3-v_b-1=0$, therefore $v_b = 2V$. Similarly, we apply KVL to the branch CD as follows: $1-v_d=0$, therefore $v_d = 1V$. and to the branch AE: $3-v_e=0$, therefore $v_e = 3V$. Finally, we apply KVL to the loop BEF: $2-3-v_f=0$, giving $v_f = -1V$. We can see that A and E are in parallel, and C, D and F are in parallel.



2. More KCL/KVL Analysis

Using techniques presented in class, label all unknown voltages (for every node/junction) and apply KCL to find them all (with respect to ground).

(a) Solve for all node voltages (with respect to ground) using KCL/KVL analysis. Verify with superposition.



Solutions:

Method 1: KCL/KVL Analysis

Applying KCL at Node 1, we get

$$\frac{0-V_1}{10}+\frac{V_2-V_1}{20}+1=0 \tag{1}$$

which gives

$$2V_1+V_1-V_2-20=0 \tag{2}$$

implying

$$3V_1-V_2=20 \tag{3}$$

Applying KCL at Node 2, we get

$$\frac{V_1-V_2}{20}+\frac{0-V_2}{50}-2=0 \tag{4}$$

which gives

$$5V_2-5V_1+2V_2+200=0 \tag{5}$$

implying

$$-5V_1+7V_2=-200 \tag{6}$$

Writing equations 3 and 6 in matrix form, we get

$$\begin{bmatrix}3 & -1\\ -5 & 7\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix}=\begin{bmatrix}20\\ -200\end{bmatrix} \tag{7}$$

Solving the system of equations, we will get $V_1=-3.75V$ and $V_2=-31.25V$.

Method 2 (verification): Superposition

First, consider the effect of only $1A$ current source on. Using current divider rule, we have

$$i_1=\frac{70}{10+70}\times 1A=0.875A \tag{8}$$

and

$$i_2 = 1A - 0.875A = 0.125A \tag{9}$$

Therefore,

$$V_1^a = i_1 \times 10\Omega = 0.875A \times 10\Omega = 8.75V \tag{10}$$

and

$$V_2^a = i_2 \times 50\Omega = 0.125A \times 50\Omega = 6.25V \tag{11}$$

Second, consider the effect of only $2A$ current source on. Using current divider rule, we have

$$i_1 = \frac{50}{50+30} \times 2A = 1.25A \tag{12}$$

and

$$i_2 = 2A - 1.25A = 0.75A \tag{13}$$

Therefore,

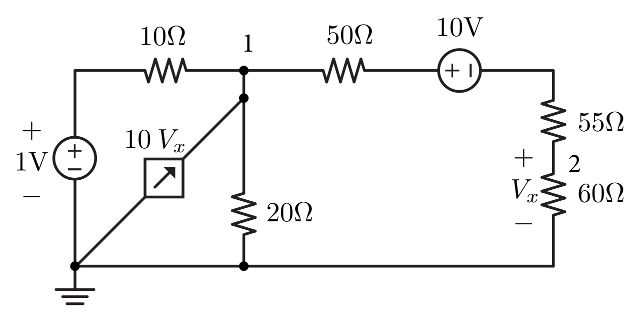
$$V_1^b = 0 - i_1 \times 10\Omega = -1.25A \times 10\Omega = -12.5V \tag{14}$$

and

$$V_2^b = 0 - i_2 \times 50\Omega = -0.75A \times 50\Omega = -37.5V \tag{15}$$

Since the circuit is linear (i.e. we have linear elements and sources), we can use the *principle of superposition* to get $V_1 = V_1^a + V_1^b$ and $V_2 = V_2^a + V_2^b$. So, we get $V_1 = 8.75V - 12.5V$ and $V_2 = 6.25V - 37.5V$. Finally, $V_1 = -3.75V$ and $V_2 = -31.25V$.
This solution agrees with the solution we obtained using nodal analysis :)

(b) Solve for all node voltages (with respect to ground) using KCL/KVL analysis.



Solutions: We assign the two nodes 1 and 2, and note that $V_x = V_2$ (because we have set the bottom wire as ground). Applying KCL at node 1 (ensuring that currents flowing into node 1 sum to zero), we get

$$\frac{1-V_1}{10} + \frac{0-V_1}{20} + \frac{10+V_2-V_1}{105} + 10V_2 = 0 \tag{16}$$

which gives

$$\frac{V_1 - 1}{2} + \frac{V_1 - 0}{4} + \frac{V_1 - V_2 - 10}{21} - 50V_2 = 0 \tag{17}$$

implying

$$67V_1 - 4204V_2 = 82 \tag{18}$$

Applying KCL to Node 2 (ensuring that currents flowing into node 2 sum to zero), we get

$$\frac{0 - V_2}{60} + \frac{-10 + V_1 - V_2}{105} = 0 \tag{19}$$

which gives

$$\frac{V_2}{4} + \frac{V_2 + 10 - V_1}{7} = 0 \tag{20}$$

implying

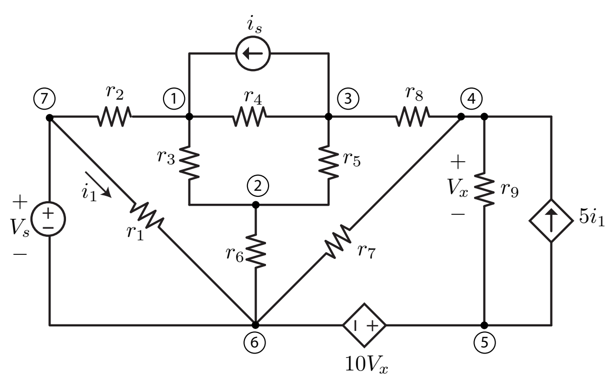
$$-4V_1 + 11V_2 = -40 \tag{21}$$

Writing the equations in matrix form, we get

$$\begin{bmatrix} 67 & -4204 \\ -4 & 11 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 82 \\ -40 \end{bmatrix} \tag{22}$$

Solving the system of equations, we will get $V_1 = 10.4023V$ and $V_2 = 0.1463V$.

(c) Setup a matrix of equations in the form $Av = b_s$ using KCL/KVL analysis. The vector $v = (v_1\ v_2 \cdots)^T$ (use the node numbers given in the schematic).



Solutions: Let's define $\vec{x} = \begin{bmatrix} V_1 & V_2 & \dots & V_7 \end{bmatrix}^T$. The nodes are as indicated in the circuit diagram. We apply KCL to:

Node 1:

$$\frac{V_2 - V_1}{r_3} + \frac{V_7 - V_1}{r_2} + i_s + \frac{V_3 - V_1}{r_4} = 0 \tag{23}$$

Node 2:

$$\frac{V_6 - V_2}{r_6} + \frac{V_1 - V_2}{r_3} + \frac{V_3 - V_2}{r_5} = 0 \tag{24}$$

Node 3:

$$\frac{V_2 - V_3}{r_5} + \frac{V_1 - V_3}{r_4} - i_s + \frac{V_4 - V_3}{r_8} = 0 \tag{25}$$

Node 4:

$$\frac{V_5 - V_4}{r_9} + \frac{V_6 - V_4}{r_7} + \frac{V_3 - V_4}{r_8} + 5\frac{(V_7 - V_6)}{r_1} = 0 \tag{26}$$

We get the following 2 equations from inspection:

$$V_7 - V_6 = V_s \tag{27}$$

and

$$V_5 - V_6 = 10(V_4 - V_5) \implies -10V_4 + 11V_5 - V_6 = 0 \tag{28}$$

Note that we have 7 variables but only 6 equations! This means that we need to ground one of the nodes to give us another equation. It is valid to choose any node, we choose node 6 here:

$$V_6 = 0 \tag{29}$$

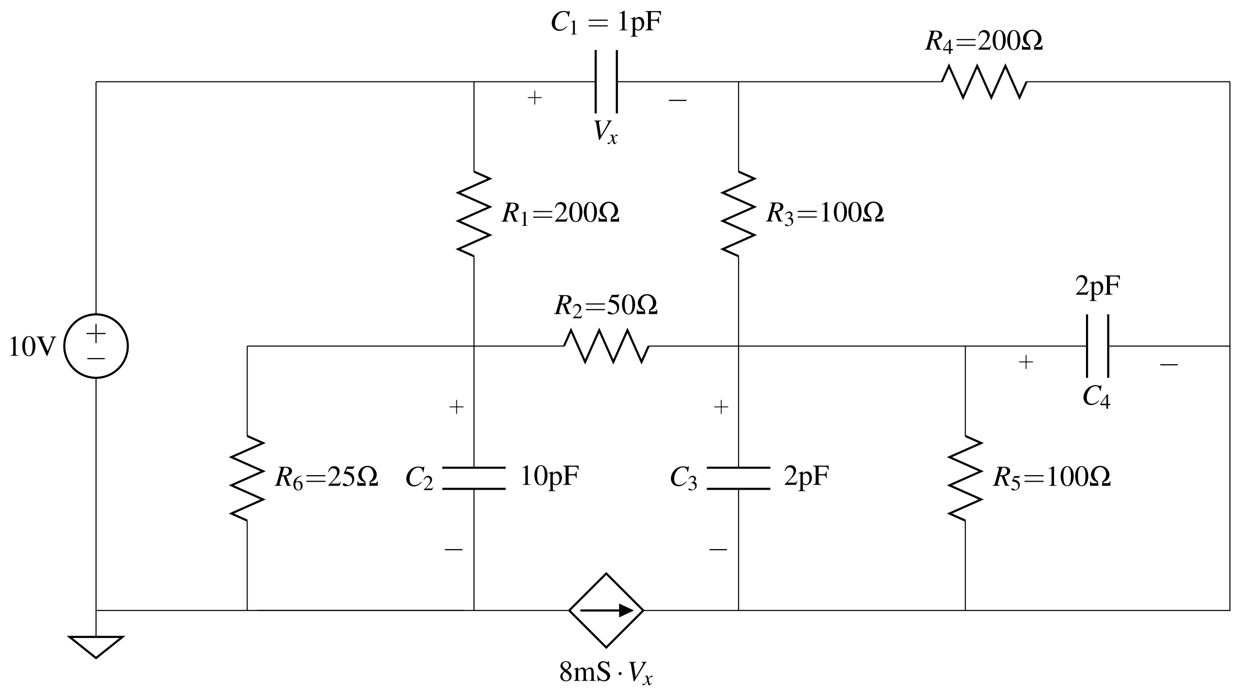
We can now write the equations in matrix form $A\vec{x} = \vec{b}$, as:

$$\begin{bmatrix} (\frac{1}{r_3}+\frac{1}{r_2}+\frac{1}{r_4}) & (\frac{-1}{r_3}) & (\frac{-1}{r_4}) & 0 & 0 & 0 & (\frac{-1}{r_2}) \\ (\frac{-1}{r_3}) & (\frac{1}{r_6}+\frac{1}{r_3}+\frac{1}{r_5}) & (\frac{-1}{r_5}) & 0 & 0 & (\frac{-1}{r_6}) & 0 \\ (\frac{-1}{r_4}) & (\frac{-1}{r_5}) & (\frac{1}{r_5}+\frac{1}{r_4}+\frac{1}{r_8}) & (\frac{-1}{r_8}) & 0 & 0 & 0 \\ 0 & 0 & (\frac{-1}{r_8}) & (\frac{1}{r_9}+\frac{1}{r_7}+\frac{1}{r_8}) & (\frac{-1}{r_9}) & (\frac{-1}{r_7}+\frac{5}{r_1}) & (\frac{-5}{r_1}) \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -10 & 11 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{bmatrix} = \begin{bmatrix} i_s \\ 0 \\ -i_s \\ 0 \\ v_s \\ 0 \\ 0 \end{bmatrix}$$

Any system equivalent to this is fine.

3. KCL/KVL Analysis + Capacitors

(a) Solve the following circuit in steady state using KCL/KVL analysis.



Solutions:

i. Identify the nodes (refer to the circuit below for our selection)

ii. Select a reference node if not set (in this case, done)

iii. Select current flow directions (arbitrary) and match them with voltage signs according to the passive sign convention (if you want to follow the solution here, the signs can be found in the circuit below)

iv. For each node with an unknown voltage write KCL in terms of currents (simple $\sum$ entering currents $=$ $\sum$ leaving currents)

v. Translate each KCL equation using Ohm's law as follows: for resistor R_{ij} connecting nodes i ($+$ voltage sign) and j ($-$ voltage sign): $I_{R_{ij}} = \frac{u_i - u_j}{R_{ij}}$

vi. Sanity check, verify that you have as many equations (KCLs translated using Ohm's law) as variables (number of unknown node voltages).

vii. Solve a system of linear equations (you can set it up in matrix form, by copying the coefficients).

By following the recipe above, we get for KCL equations (note that in steady state, currents through capacitors is equal to 0)

(1)No need for KCL: voltage known and is equal to 10V

$(2) I_1 + I_2 = I_6$

$(3) I_3 + I_5 = I_2$

$(4) I_4 = I_3$

$(5) 8\text{mS} \cdot V_x = I_4 + I_5$

When translated using node voltages–by Ohm's law mostly, except for the dependent source– we get

$$
\begin{aligned}
&(2)\frac{10\text{V}-u_2}{R_1}+\frac{u_3-u_2}{R_2}=\frac{u_2}{R_6}\\
&(3)\frac{u_4-u_3}{R_3}+\frac{u_5-u_3}{R_5}=\frac{u_3-u_2}{R_2}\\
&(4)\frac{u_5-u_4}{R_4}=\frac{u_4-u_3}{R_3}\\
&(5)8\text{mS}\cdot(10\text{V}-u_4)=\frac{u_5-u_4}{R_4}+\frac{u_5-u_3}{R_5}
\end{aligned}
$$

or if you would like to plug in the values of the resistors:

$$
\begin{aligned}
&(2)\frac{10\text{V}-u_2}{200\Omega}+\frac{u_3-u_2}{50\Omega}=\frac{u_2}{25\Omega}\\
&(3)\frac{u_4-u_3}{100\Omega}+\frac{u_5-u_3}{100\Omega}=\frac{u_3-u_2}{50\Omega}\\
&(4)\frac{u_5-u_4}{200\Omega}=\frac{u_4-u_3}{100\Omega}\\
&(5)8\text{mS}\cdot(10\text{V}-u_4)=\frac{u_5-u_4}{200\Omega}+\frac{u_5-u_3}{100\Omega}
\end{aligned}
$$

which can also be translated in matrix form as

$$
\begin{bmatrix}
-\frac{1}{200\Omega}-\frac{1}{50\Omega}-\frac{1}{25\Omega} & \frac{1}{50\Omega} & 0 & 0\\
\frac{1}{50\Omega} & -\frac{1}{50\Omega}-\frac{1}{100\Omega}-\frac{1}{100\Omega} & \frac{1}{100\Omega} & \frac{1}{100\Omega}\\
0 & \frac{1}{100\Omega} & -\frac{1}{100\Omega}-\frac{1}{200\Omega} & \frac{1}{200\Omega}\\
0 & \frac{1}{100\Omega} & -8\text{mS}+\frac{1}{200\Omega} & -\frac{1}{200\Omega}-\frac{1}{100\Omega}
\end{bmatrix}
\cdot
\begin{bmatrix} u_2\\ u_3\\ u_4\\ u_5 \end{bmatrix}
\begin{bmatrix} -\frac{10\text{V}}{200\Omega}\\ 0\\ 0\\ -8\text{mS}\cdot 10\text{V} \end{bmatrix}
$$

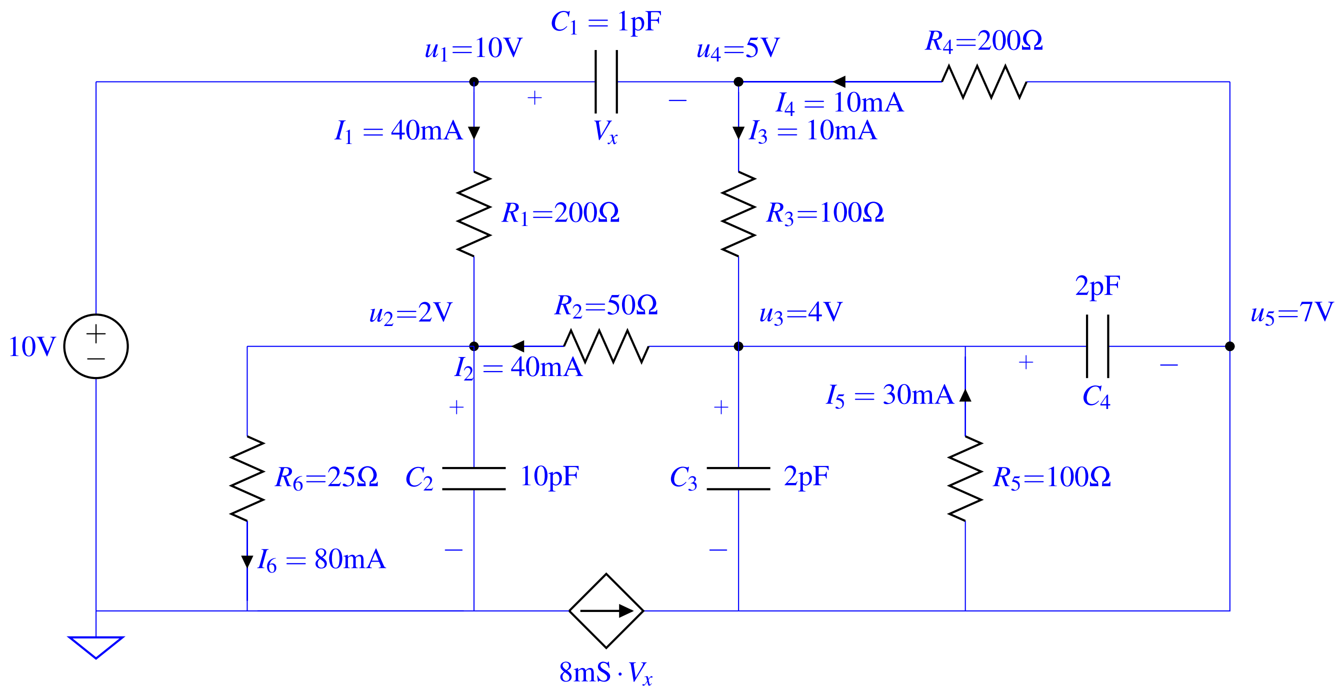
or even simplified.

which yields the solution (augmenting the known u_1 too)

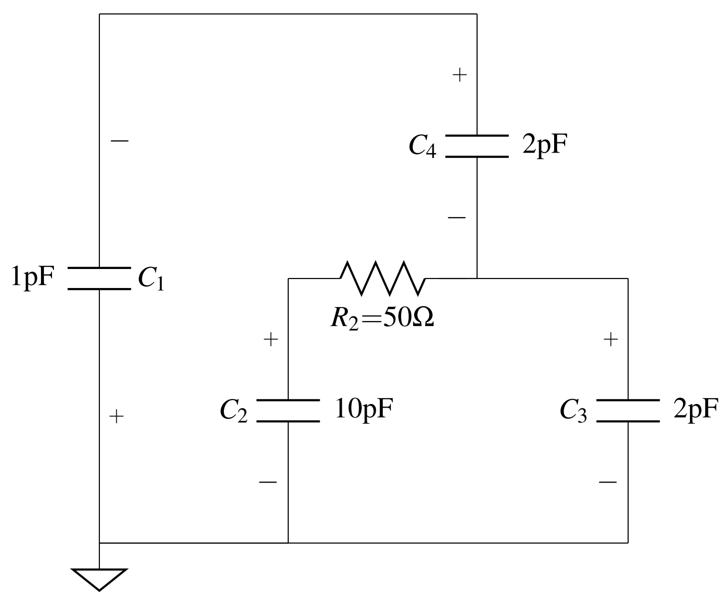
$$
\begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5 \end{bmatrix}
=
\begin{bmatrix} 10\text{V}\\ 2\text{V}\\ 4\text{V}\\ 5\text{V}\\ 7\text{V} \end{bmatrix}
$$

You can check the solution by inspection too.

The circuit with the solution and annotations is:



(b) After the circuit settles in steady-state from the previous part, disconnect the charged capacitors from the circuit and connect them in the configuration shown below. Polarity from part (a) is preserved. What are the voltages across, currents through and charge stored in each of the capacitors C_1, C_2, C_3 and C_4 in steady-state after the charge redistributes itself?



Solutions:

i. Identify the nodes (refer to the circuit below for our selection)

ii. Select a reference node if not set (in this case, done)

iii. For each node with an unknown voltage write a charge conservation equation (simple $\sum$charges on node before $\sum$charges on node after). Pay attention to the signs.

iv. Translate each equation using node voltages: $q = C \cdot (v_i - v_j)$ assuming C is connected between nodes i (with $+$ sign) and j (with $-$ sign).

v. Sanity check, verify that you have as many equations (charge conservation) as variables (number of unknown node voltages).

vi. Solve a system of linear equations (you can set it up in matrix form, by copying the coefficients).

We first note that in stead state (DC), no current flows through the capacitors. Therefore, by writing a KCL equation between C_2 and R_2 we get that there is no current flowing through the resistor either. This means that there is no voltage across this resistor and therefore we can define just one node between C_4, C_3 and C_2.

Following the recipe, we get the following equations (superscript b means before and a means after)

$$(1) - q_1^b + q_4^b = -q_1^a + q_4^a$$
$$(2) q_2^b + q_3^b - q_4^b = q_2^a + q_3^a - q_4^a$$

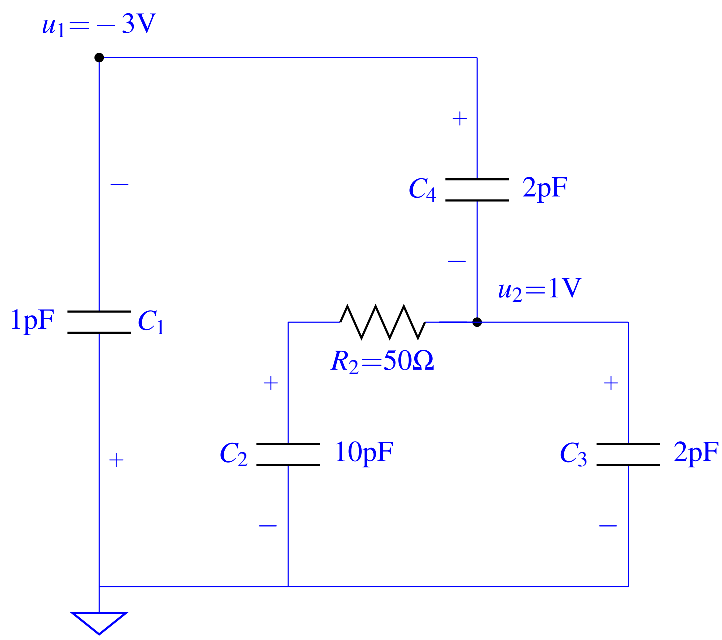
Which translates to

$$(1) - (10\text{V} - 5\text{V})C_1 + (4\text{V} - 7\text{V})C_4 = -(0 - u_1)C_1 + (u_1 - u_2)C_4$$
$$(2) 2\text{V}C_2 + (4\text{V} - 7\text{V})C_3 - (4\text{V} - 7\text{V})C_4 = u_2C_2 + u_2C_3 - (u_1 - u_2)C_4$$

By plugging in the capacitor values and solving you get:

$$u_1 = -3\text{V}$$
$$u_2 = 1\text{V}$$

The reference circuit:

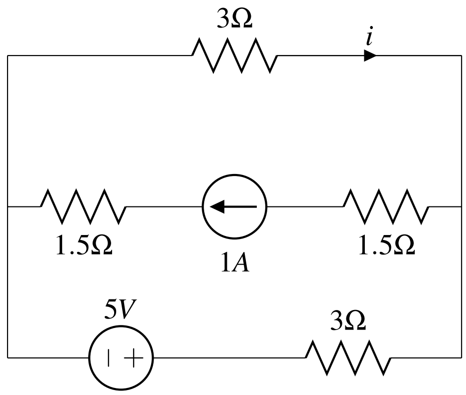


This means that the charges on the "positive" (+) plates of the capacitors are

$$q_1 = -3\text{pC}$$
$$q_2 = 10\text{pC}$$
$$q_3 = 2\text{pC}$$
$$q_4 = -8\text{pC}$$

4. KCL/KVL Or Superposition?

(a) Solve for the current through the 3Ω resistor, marked as i, using superposition. Verify using nodal analysis. You can use IPython to solve the system of equations if you wish. Where did you place your ground, and why?

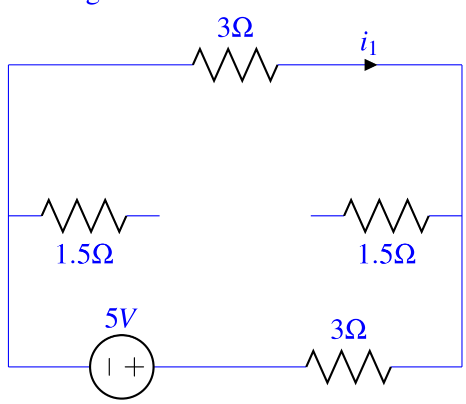


Solutions: $i = \frac{-1}{3}A$.

Method 1: Superposition

Consider the circuits obtained by:

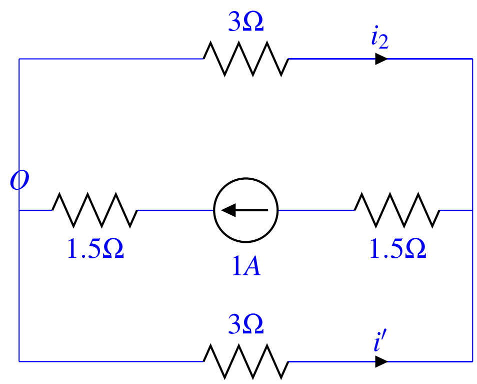
i. Turning off the $1A$ current source:



In the above circuit, no current is going to flow through the middle branch, as it is an open circuit. Thus this is just a $5V$ voltage source connected to two 3Ω resistors in series so

$$i_1 = -\frac{5}{6}A \tag{30}$$

ii. Turning off the $5V$ voltage source:



In the above circuit, notice that the 3Ω resistors are in parallel and therefore form a current divider. Since the values of the resistances are equal, the current flowing through them will also be equal, that is $i_2 = i'$. Applying KCL to node O, we get

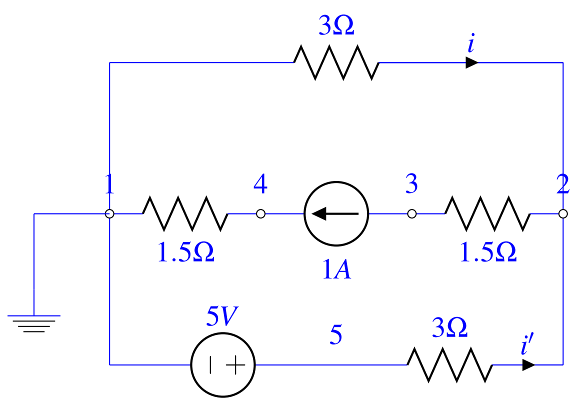
$$1 - i_2 - i' = 0 \tag{31}$$

which gives us

$$i_2 = \frac{1}{2}A \tag{32}$$

Now, applying the principle of superposition, we have $i = i_1 + i_2 = -\frac{5}{6} + \frac{1}{2} = -\frac{1}{3}A$.

Method 2: Nodal Analysis First, let's identify and label the nodes on the circuit. (Note that the numbers are arbitrary.) We also ground node 1. (Our choice of ground is arbitrary (whatever you chose is fine) but node 1 is a convenient choice because node 5 will turn out to be 5V from the voltage source, the voltage at node 4 can be calculated quickly from the current source and the 1.5Ω resistor using Ohm's law, and i can be calculated quickly once we know the voltage at node 2.)



First, we write KCL at each node. At nodes 1 and 2, we get the same equation.

$$i + i' = 1A \tag{33}$$

At nodes 3 and 4, we get the trivial equation

$$1A = 1A. \tag{34}$$

We now write the voltage drops across the circuit elements in terms of the currents using Ohm's law or in terms of known voltages

$$V_5 - V_1 = 5 \tag{35}$$
$$V_5 - V_2 = i'(3\Omega) \tag{36}$$
$$V_2 - V_3 = 1A(1.5\Omega) \tag{37}$$
$$V_4 - V_1 = 1A(1.5\Omega) \tag{38}$$
$$V_2 - V_1 = -i(3\Omega) \tag{39}$$

Since we've chosen node 1 as ground ($V_1 = 0$), we can rewrite the equations involving V_1 which gives us values for V_5 and V_4.

$$V_5 = 5 \tag{40}$$
$$V_4 = 1A(1.5\Omega) \tag{41}$$
$$V_2 = -i(3\Omega) \tag{42}$$

We next combine the KCL equations and the Ohm's Law equations to solve for V_2 and V_3. (We don't actually need to solve for V_3 once we know V_2 but the calculation is easy.)

$$-\frac{V_2}{3\Omega} + \frac{5V - V_2}{3\Omega} = 1A \tag{43}$$
$$\Rightarrow \qquad V_2 = 1V \tag{44}$$

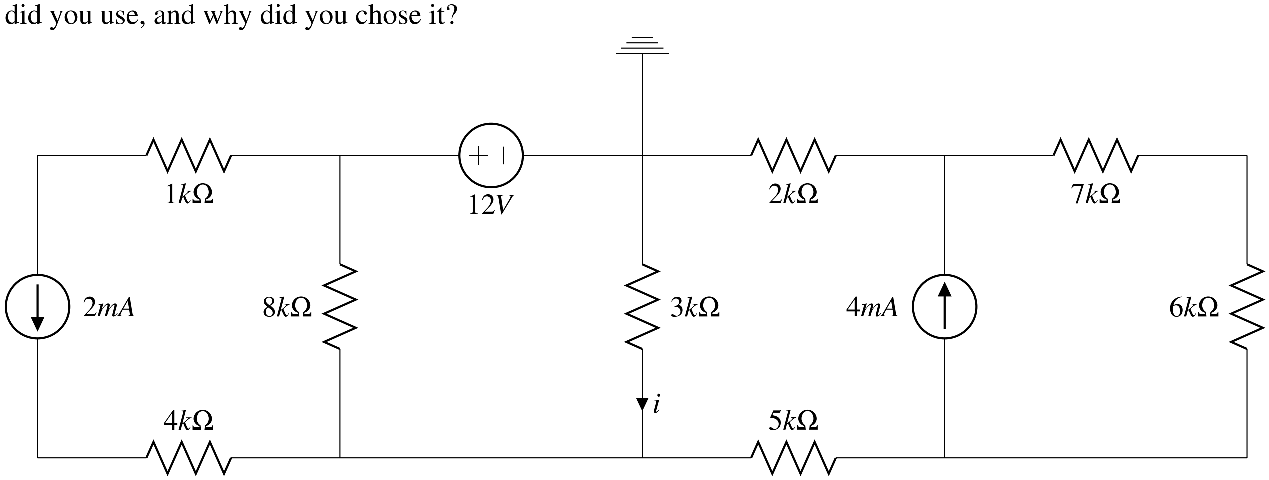
and

$$V_3 = V_2 - 1A(1.5\Omega) \tag{45}$$
$$V_3 = -0.5V \tag{46}$$

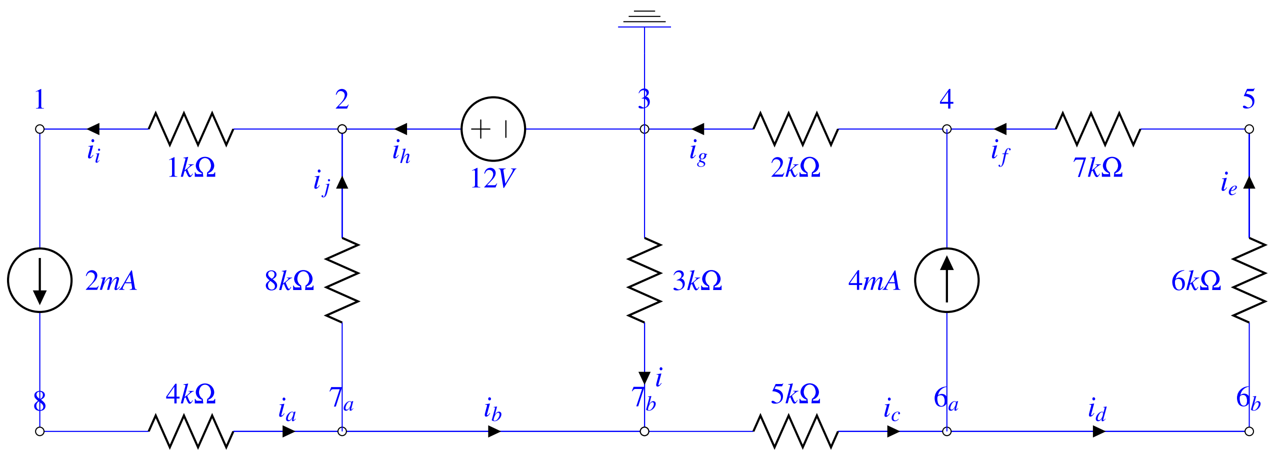
Finally, we use V_2 to solve for i

$$i = -\frac{V_2}{3\Omega} = -\frac{1}{3}A \tag{47}$$

(b) Solve for the current through the $3k\Omega$ resistor, marked as i, using whatever method you want. This problem is more mechanically tedious than the rest, so we recommend using iPython. Which method did you use, and why did you chose it?



Solutions: **Method 1: Nodal Analysis** First, let's identify and label the nodes on the circuit. One of the nodes has already been grounded for us. Note that vertices connected by short circuits are marked with same number – ask yourself why!



Now, let's write KCL at all nodes:

1:

$$i_i = 2mA \tag{48}$$

2:

$$i_h + i_j = i_i \tag{49}$$

3:

$$i_g = i_h + i \tag{50}$$

4:

$$4 + i_f = i_g \tag{51}$$

5:

$$i_e = i_f \tag{52}$$

6a:

$$i_c = 4 + i_d \tag{53}$$

6b:

$$i_d = i_e \tag{54}$$

6 (6a+6b):

$$i_c = 4mA + i_e \tag{55}$$

7a:

$$i_a = i_b + i_j \tag{56}$$

7b:

$$i_b + i = i_c \tag{57}$$

7 (7a + 7b):

$$i_a + i = i_c + i_j \tag{58}$$

8:

$$2mA = i_a \tag{59}$$

Now, let's write Ohm's Law for all the currents that are flowing through resistors in terms of the node voltages:

$$\begin{aligned}
i_a &= \frac{V_8 - V_7}{4k\Omega} \\
i_c &= \frac{V_7 - V_6}{5k\Omega} \\
i_e &= \frac{V_6 - V_5}{6k\Omega} \\
i_f &= \frac{V_5 - V_4}{7k\Omega} \\
i_g &= \frac{V_4 - V_3}{2k\Omega} \\
i_i &= \frac{V_2 - V_1}{1k\Omega} \\
i_j &= \frac{V_7 - V_2}{8k\Omega} \\
i &= \frac{V_3 - V_7}{3k\Omega}
\end{aligned}$$

Putting it together, we get the equations for node voltages (note that we set $V_3 = 0$ and $V_2 = 12V$ by inspection, and because nodes 2 and 3 are connected to voltage sources, we do not retain the KCL equations for those nodes.)

$$\begin{aligned}
\frac{V_2 - V_1}{1k\Omega} &= 2mA \\
V_2 - V_3 &= 12V \\
V_3 &= 0V \\
4mA + \frac{V_5 - V_4}{7k\Omega} &= \frac{V_4 - V_3}{2k\Omega} \\
\frac{V_6 - V_5}{6k\Omega} &= \frac{V_5 - V_4}{7k\Omega} \\
\frac{V_7 - V_6}{5k\Omega} &= 4mA + \frac{V_6 - V_5}{6k\Omega} \\
\frac{V_8 - V_7}{4k\Omega} + \frac{V_3 - V_7}{3k\Omega} &= \frac{V_7 - V_6}{5k\Omega} + \frac{V_7 - V_2}{8k\Omega} \\
2mA &= \frac{V_8 - V_7}{4k\Omega}
\end{aligned}$$

We can see that this is a system of 8 equations in 8 variables, $V_1, V_2, V_3, ..., V_8$. We can express the system of equations as a matrix, and solve using IPython. This gives us the following solution: $V_1 = 10V, V_2 = 12V, V_3 = 0V, V_4 = 5.3770V, V_5 = -3.8033V, V_6 = -11.6721V, V_7 = 1.7705V, V_8 = 9.7705V$. Finally, $i = \frac{V_3 - V_7}{3k\Omega} = \frac{-1.7705}{3k\Omega} = -0.5902mA$.

Method 2: Superposition (and then Nodal Analysis)

Another method we can use is to make use of the superposition principle and analyze the circuit with each voltage or current source separately. For each source, we will zero out the other sources and solve for the portion of i that comes from the individual source we are considering, i_{12V}, i_{2mA}, and i_{4mA} respectively. We then compute the full value of i by adding them up.

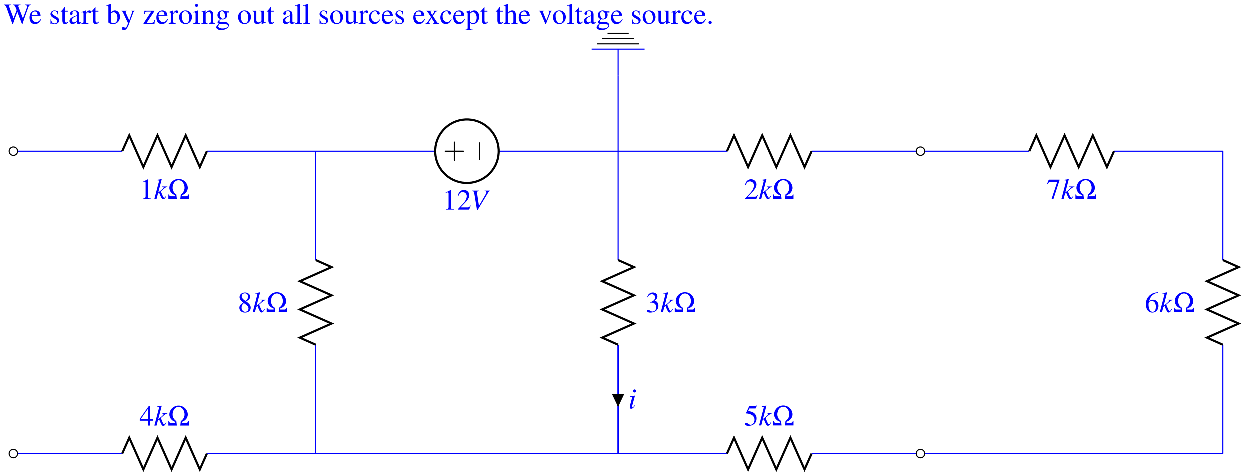
$$i = i_{12V} + i_{2mA} + i_{4mA} \tag{60}$$

We zero out voltage sources by treating them like a short circuit (a wire with no resistance – 0 voltage jump) and we zero out current sources by treating them like an open circuit (a break in the circuit – 0 current flowing).

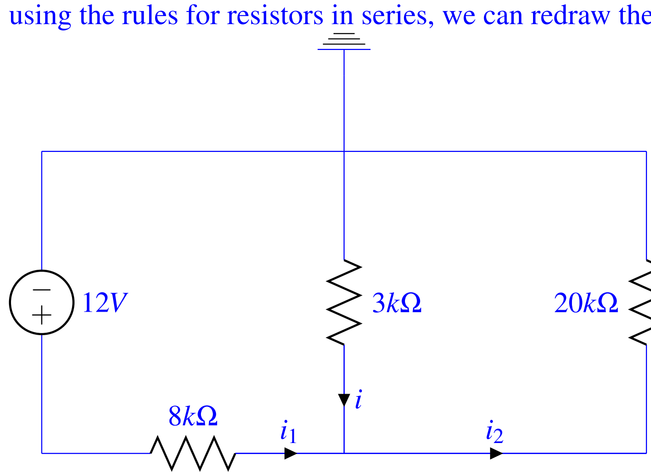
The advantage of using superposition is that we can often simplify the circuit significantly and just use equivalent resistance rules and Ohm's law rather than a full nodal analysis to calculate the quantity that we're interested in.

- $12V$ **Voltage Source**

  We start by zeroing out all sources except the voltage source.



  If the current sources are zeroed out, no current will flow through the $1k\Omega$ resistor and the $4k\Omega$ resistor. Noting this and rewriting the $2k\Omega$, $7k\Omega$, $6k\Omega$, and $5k\Omega$ resistors as an equivalent resistor using the rules for resistors in series, we can redraw the above circuit as



Using the equivalent resistance rules again and Ohm's law, we get that

$$
\begin{aligned}
12V &= i_1\left(8k\Omega + (3k\Omega||20k\Omega)\right) && (61)\\
&= i_1\left(8k\Omega + \left(\frac{1}{3k\Omega} + \frac{1}{20k\Omega}\right)^{-1}\right) && (62)\\
&= i_1\left(8k\Omega + \left(\frac{20}{60k\Omega} + \frac{3}{60k\Omega}\right)^{-1}\right) && (63)\\
&= i_1\left(8 + \frac{60}{23}\right)k\Omega && (64)\\
& && (65)
\end{aligned}
$$

We know that the voltage drop across the $3k\Omega$ and $20k\Omega$ must be the same and from KCL we know that $i_1 + i = i_2$. Thus we have

$$-i(3k\Omega) = i_2(20k\Omega) \tag{66}$$

Substituting we get

$$-i(3k\Omega) = (i_1 + i)(20k\Omega) \tag{67}$$

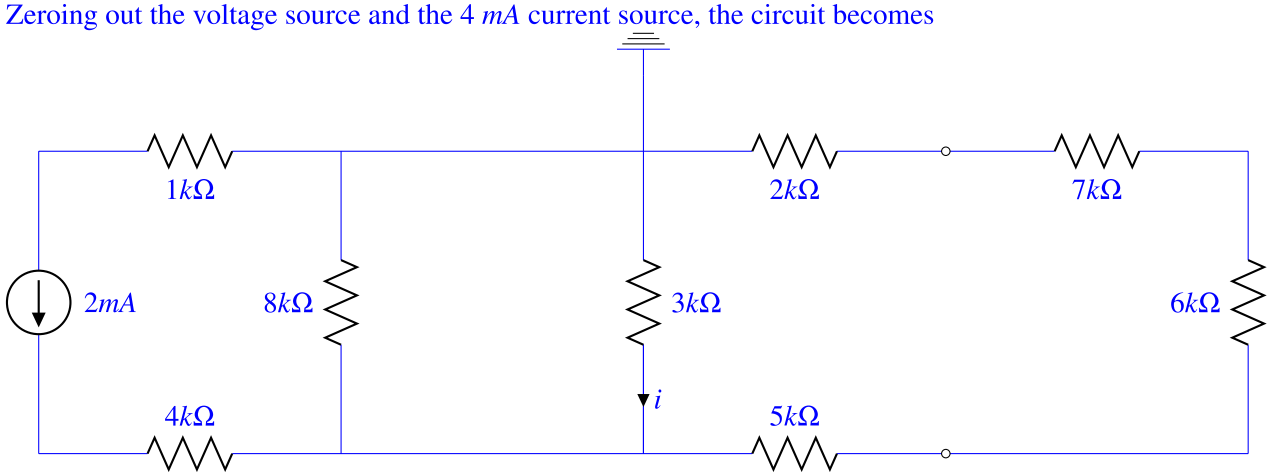
Solving for i we get

$$
\begin{aligned}
-i(3k\Omega + 20k\Omega) &= i_1(20k\Omega) && (68)\\
-i(3k\Omega + 20k\Omega) &= \left(\frac{12V}{\left(8+\frac{60}{23}\right)k\Omega}\right)(20k\Omega) && (69)
\end{aligned}
$$

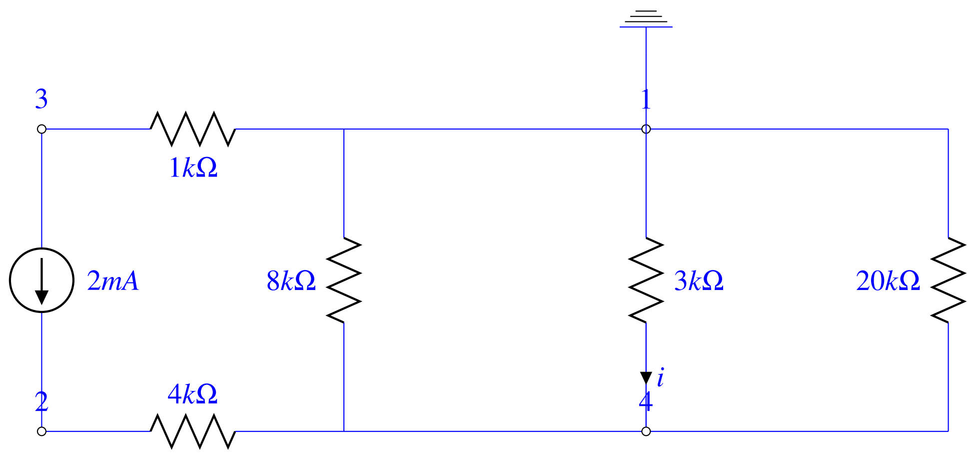
Thus we have that the portion of i from the voltage source, i_{12V}, is

$$i_{12V} = -0.9836\ mA \tag{70}$$

- $2mA$ **Current Source**
  Zeroing out the voltage source and the 4 mA current source, the circuit becomes



Redrawing the circuit applying the series equivalent resistance rules and labeling nodes, we get



The easiest way to calculate i is to calculate the voltage drop from node 4 to node 1. Once we know this, we can calculate i using Ohm's law

$$V_4 - V_1 = (-i)3k\Omega \tag{71}$$

A simple way to calculate $V_4 - V_1$ is to use Ohm's law and the equivalent resistance of the three resistors in parallel: the $8k\Omega$, $3k\Omega$, and $20k\Omega$ resistors. We know the total current traveling through these three resistors is 2 mA. Using Ohm's law, we have

$$V_4 - V_1 = 2mA\left(8k\Omega||3k\Omega||20k\Omega\right) \tag{72}$$

$$= 2mA\left(\frac{1}{8k\Omega} + \frac{1}{3k\Omega} + \frac{1}{20k\Omega}\right)^{-1} \tag{73}$$

$$= 2mA\left(\frac{15}{120k\Omega} + \frac{40}{120k\Omega} + \frac{6}{120k\Omega}\right)^{-1} \tag{74}$$

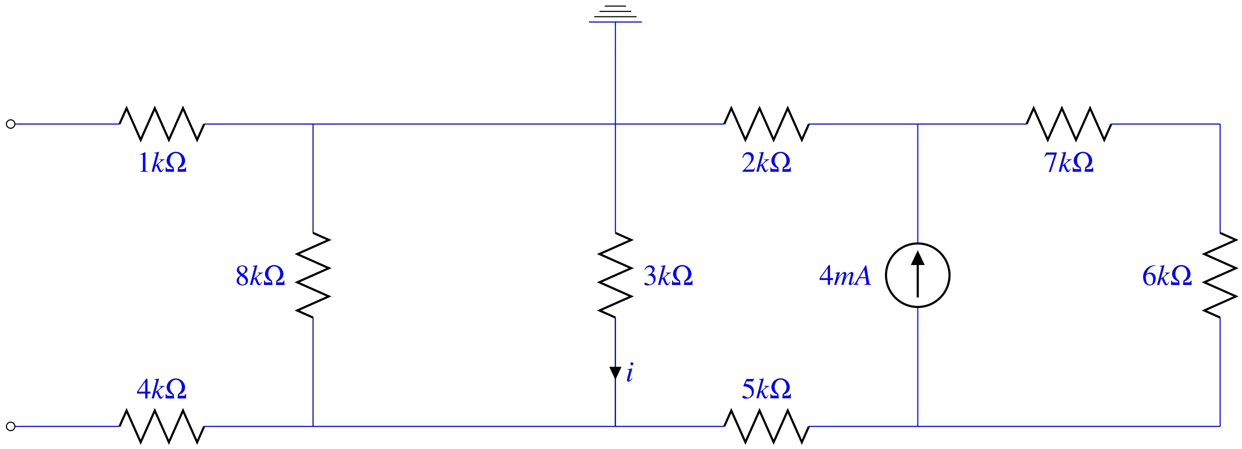
$$= 2mA\left(\frac{120k\Omega}{61}\right) \tag{75}$$

Now that we know $V_4 - V_1$, we use Ohm's Law again this time across the $3k\Omega$ resistor to calculate the portion of i from the $2mA$ current source.

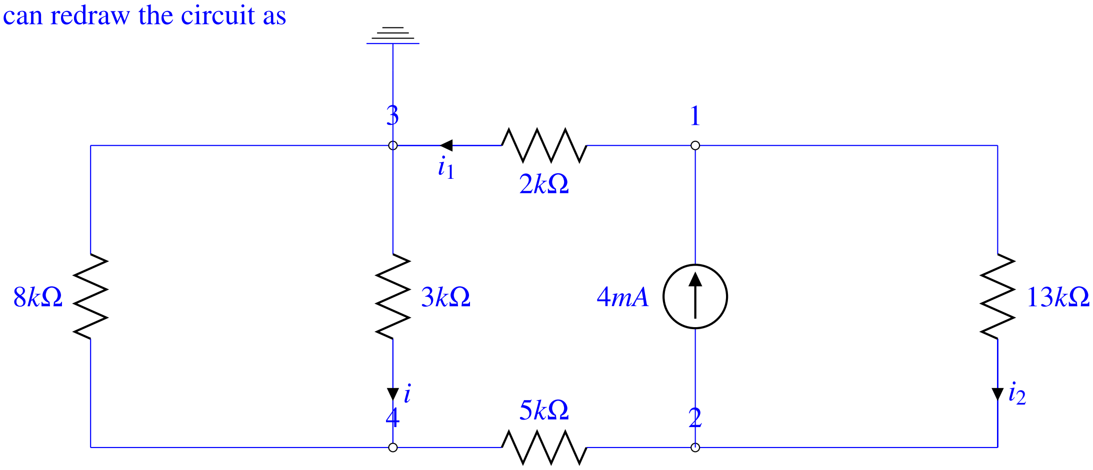
$$i_{2mA} = \frac{V_1 - V_4}{3k\Omega} = -\frac{1}{3k\Omega}2mA\left(\frac{120k\Omega}{61}\right) = -1.3115mA \tag{76}$$

$$\tag{77}$$

- $4mA$ **Current Source**
  Finally, we zero out the voltage source and the $2mA$ current source to solve for the portion of i from the $4mA$ current source.



Noticing again that no current will flow through the $1k\Omega$ and $4k\Omega$ resistors, using the series equivalent resistance rules on the $7k\Omega$ and $6k\Omega$ resistors, and labeling currents and nodes, we can redraw the circuit as



From KCL, we have that

$$i_1 + i_2 = 4mA \tag{78}$$

We know that the voltage drop from node 1 to node 2 is the same through the left half of the circuit and the right half. Using Ohm's law and equivalent resistance rules we get that

$$V_1 - V_2 = i_1\Big(2k\Omega + 5k\Omega + \big(8k\Omega || 3k\Omega\big)\Big) \tag{79}$$

and that

$$V_1 - V_2 = i_2(13k\Omega) \tag{80}$$

Substituting using the above 3 equations, we can solve for i_1.

$$(4mA - i_1)(13k\Omega) = i_1\Big(2k\Omega + 5k\Omega + \big(8k\Omega || 3k\Omega\big)\Big) \tag{81}$$

$$4mA(13k\Omega) = i_1\Big(13k\Omega + 2k\Omega + 5k\Omega + \big(8k\Omega || 3k\Omega\big)\Big) \tag{82}$$

$$4mA(13k\Omega) = i_1\left(13k\Omega + 2k\Omega + 5k\Omega + \left(\frac{1}{8k\Omega} + \frac{1}{3k\Omega}\right)^{-1}\right) \tag{83}$$

$$4mA(13k\Omega) = i_1\left(13k\Omega + 2k\Omega + 5k\Omega + \left(\frac{24k\Omega}{11}\right)\right) \tag{84}$$

We also need to know the voltage drop from node 3 to node 4. Again using the equivalent resistance rules, we have that

$$V_3 - V_4 = i_1(8k\Omega||3k\Omega) \tag{85}$$
$$V_3 - V_4 = i_1\left(\frac{24k\Omega}{11}\right) \tag{86}$$

Finally the portion of i from the $4mA$ current source, i_{4mA}, can be calculated using Ohm's law and $V_3 - V_4$.

$$i_{4mA} = \frac{V_3 - V_4}{3k\Omega} \tag{87}$$
$$= \frac{1}{3k\Omega}\left(\frac{24k\Omega}{11}\right)i_1 \tag{88}$$
$$= \frac{1}{3k\Omega}\left(\frac{24k\Omega}{11}\right)4mA(13k\Omega)\left(13k\Omega + 2k\Omega + 5k\Omega + \frac{24k\Omega}{11}\right)^{-1} \tag{89}$$
$$= 1.7049mA \tag{90}$$

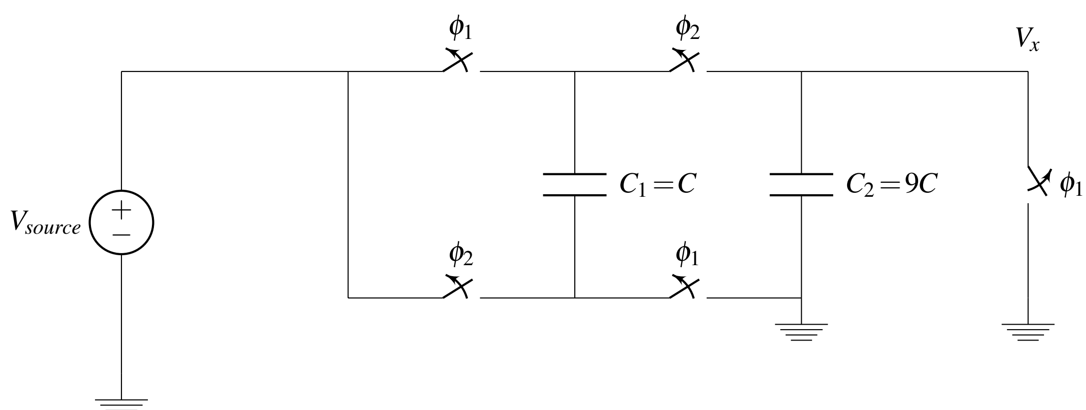
Finally we solve for i. Note that as expected we get the same value that we got using nodal analysis.

$$i = i_{12V} + i_{2mA} + i_{4mA} \tag{91}$$
$$= -0.9836mA - 1.3115mA + 1.7049mA \tag{92}$$
$$= -0.5902mA \tag{93}$$

Note: This problem is way longer and more tedious than anything that you will face on an exam in 16a. However, it is important for you to see that analyzing circuits can be done mechanically. This is why we wanted you to do this on the homework.

5. Capacitor Charge Sharing

Consider the following circuit:



In the first phase, all of the switches labeled ϕ_1 will be closed and all switches labeled ϕ_2 will be open. In the second phase, all switches labeled ϕ_1 open and all switches labeled ϕ_2 close.

(a) Draw polarity (+ and - signs) on the two capacitors C_1 and C_2. (It doesn't matter which terminal you label $+$ or $-$; just remember to keep these consistent through phases 1 and 2!)

Solutions:

i. Label all the voltages across the capacitors. Choose whichever direction you want for each capacitor(for example, you can keep the $+$ sign on the top plate and the $-$ sign on the bottom plate of each capacitor, meaning that there is a voltage drop from top to bottom). Just make sure you stay consistent across phases.

ii. Draw the circuit in each phase. Keep the polarities of the voltages across capacitors consistent in all phases!

iii. Look at the circuit in the first phase. First, determine the voltages across each capacitor according to the polarities you have defined. Then, use $Q = CV$ to determine the charge on *each plate on each capacitor*. (Make sure to emphasize the magnitude of charge on each plate. This is important if you want to explain charge sharing conceptually.)

iv. Look at the circuit in the second phase. Determine where the charge is conserved. (Guideline for this: If a plate of a capacitor, or plates of multiple capacitors that are connected to each other in one of the phases, are "floating" in the second phase, meaning that it/they are not connected to any voltage source or ground, then the total charge on the plate(s) must be conserved.)

v. Write the equation for charge conservation. Hence, determine the voltages across the capacitors. Finally, calculate the charge stored in each capacitor in the second phase.

Now, let's do the solution for part a) (step i. of the recipe). One way of marking the polarities is $+$ on the top plate and $-$ on the bottom plate of both C_1 and C_2. Let's call the voltage drop across C_1 V_{C_1} and across C_2 V_{C_2}.

(b) Draw the circuit in the first phase and in the second phase. Keep your polarity in part (a) in mind.

Solutions: (step ii. of the recipe) In phase 1, all the switches marked as ϕ_1 are closed and switches marked as ϕ_2 are open. In phase 2, all the switches marked as ϕ_2 are closed and switches marked as ϕ_1 are open. Draw both the circuits separately, side by side, with the switches in their respective positions.

(c) Find the voltages and charges on C_1 and C_2 in the first phase. Be sure to keep the polarities of the voltages the same!

Solutions: (step iii. of the recipe) In phase 1,

$$V_{C_1,1} = V_{source} - 0 = V_{source} \tag{94}$$

and

$$V_{C_2,1} = 0 - 0 = 0 \tag{95}$$

Next, we find the charge on each capacitor:

$$Q_{C_1,1} = V_{C_1,1}C_1 = CV_{source} \tag{96}$$

Note that the positive plate has a charge of $+CV_{source}$ coulombs, while the negative plate has a charge of $-CV_{source}$ coulombs. and

$$Q_{C_2,2} = V_{C_2,1}C_2 = 0 \tag{97}$$

(d) Now, in the second phase, find the voltage V_x.

Solutions: (step iv. of the recipe) Where is charge conserved? To answer this, look at the top plates of C_1 and C_2. In phase 2, they are both "floating" because they are not connected to V_{source} or ground. And in phase 1, they are not connected to each other, but in phase 2, they are connected by the switch. Therefore, in phase 2, the charges on the top plates of C_1 and C_2 will be *shared*, or distributed, because they simply cannot go anywhere else. The total charge will remain the same as in phase 1. Let's find the voltages across C_1 and C_2 in phase 2 (same polarities as phase 1!):

$$V_{C_1,2} = V_x - V_{source} \tag{98}$$

and

$$V_{C_2,2} = V_x \tag{99}$$

. Now, let's find the charge stored in top plates of C_1 and C_2:

$$Q_{C_1,2} = C(V_x - V_{source}) \tag{100}$$

and

$$Q_{C_2,2} = 9CV_x \tag{101}$$

Next, let's write the equation for charge conservation (step v. of the recipe):

$$Q_{C_1,1} + Q_{C_2,1} = Q_{C_1,2} + Q_{C_2,2} \tag{102}$$

giving

$$CV_{source} + 0 = C(V_x - V_{source}) + 9CV_x \tag{103}$$

which results in

$$V_x = \frac{V_{source}}{5} \tag{104}$$

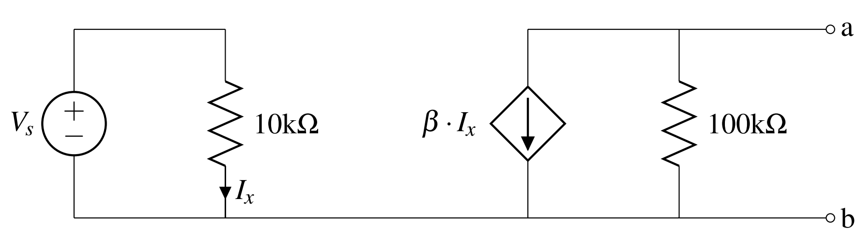
(e) (BONUS) If capacitor C_2 did not exist (i.e., had a capacitance of 0F), what would the voltage V_x be?

Solutions: We could always go back to the equations above, plug in $C_2 = 0$, and derive $V_x = 2V_{source}$. It might be worthwhile to go over what this means for the circuit, though. If $C_2 = 0F$, the capacitor is actually an open circuit. (*Why?*) So we can pretend, as the question says, that C_2 does not exist. In phase 1, as before, C_1 has a voltage drop of V_{source} across it (from top to bottom), and is charged up to CV_{source}. Now, in phase 2, the top plate of C_1 is left dangling (floating). This means that the charge on the top plate of C_1 is going to be the same, and therefore, so is the charge on the bottom plate. We will therefore get

$$V_x = V_{source} - (-V_{source}) = 2V_{source} \tag{105}$$

6. Equivalence

Find the Thévenin and Norton equivalents of the following circuit across the terminals a and b (in terms of V_s and β). Note that the current source is dependent on the current I_x.



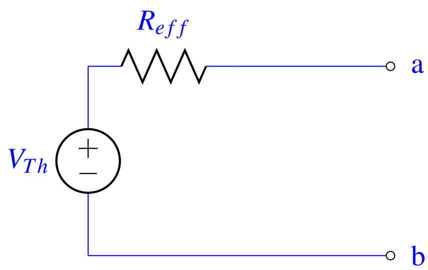


Figure 1: Thévenin equivalent

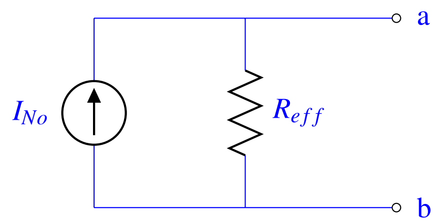


Figure 2: Norton equivalent

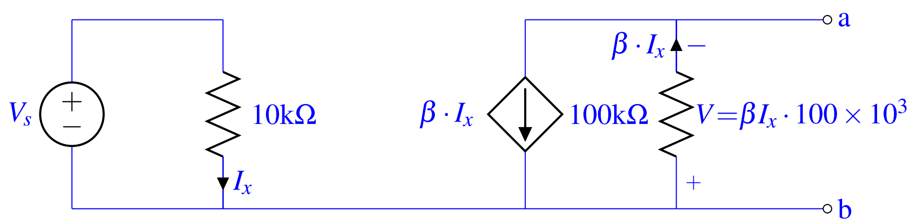


Figure 3: Step 1

Solutions: To calculate the Thévenin equivalent, the recipe is:

(a) Calculate the output voltage in open circuit condition (no load resistor or equivalently infinite load resistance). Denote this as V_{Th}.

(b) Calculate the output current in short circuit condition (load has 0 resistance). Denote this as I_{No}.

(c) Calculate the effective resistance $R_{eff} = \frac{V_{Th}}{I_{No}}$.

The Thévenin equivalent would be the circuit in Figure **??**. The Norton equivalent would be the circuit in Figure **??**.

Applying this recipe to our circuit. First, calculate V_{Th} as in Figure **??**. To get $V_{Th} = -\beta I_x \cdot 100 \times 10^3$ but since $I_x = \frac{V_s}{10 \times 10^3}$ we get $V_{Th} = -\beta \frac{V_s \cdot 100 \times 10^3}{10 \times 10^3} = -10\beta V_s$.

Second, calculate I_{No} as in Figure **??**. To get $I_{No} = -\beta I_x = -\beta \frac{V_s}{10 \times 10^3}$. From this we know that $R_{eff} =$ 100kΩ.

Which yields the equivalent circuits in Figures **??** and **??**

To demonstrate the notion of equivalence, consider the circuit in Figure **??**. If we solve it straight up, we get $V_l = -\beta I_x (R_l \mid\mid 100\text{k}\Omega) = -\beta \frac{V_s}{10 \times 10^3} \cdot (R_l \mid\mid 100\text{k}\Omega)$ and $i_l = \frac{V_l}{R_l} = \frac{-\beta \frac{V_s}{10 \times 10^3} \cdot (R_l \mid\mid 100\text{k}\Omega)}{R_l} = -\beta \frac{V_s}{10 \times 10^3} \cdot \frac{100\text{k}\Omega}{100\text{k}\Omega + R_l}$.

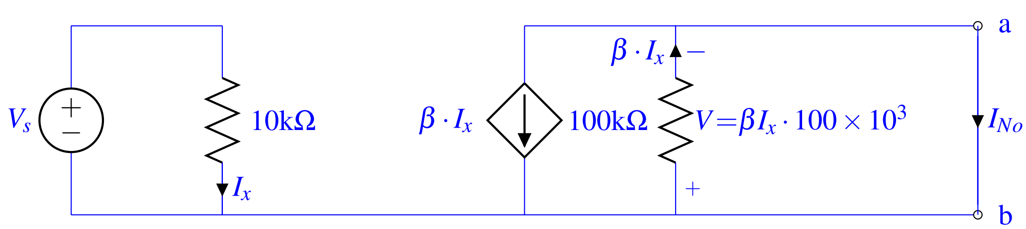
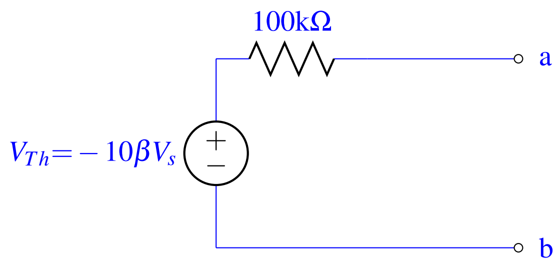


Figure 4: Step 2



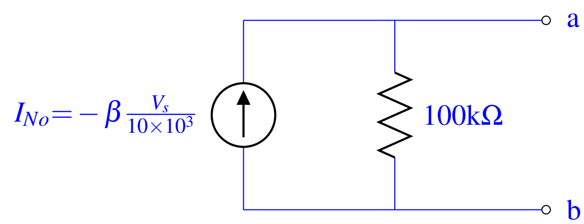


Figure 5: Thévenin equivalent for the circuit in the problem

Figure 6: Norton equivalent for the circuit in the problem

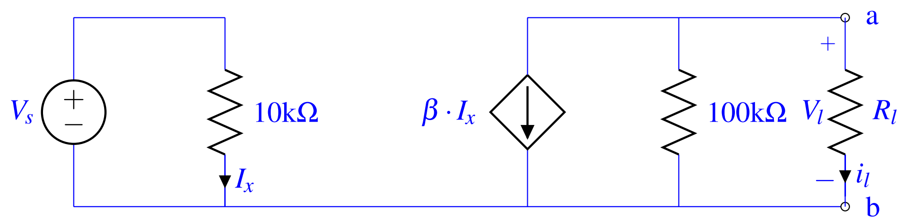
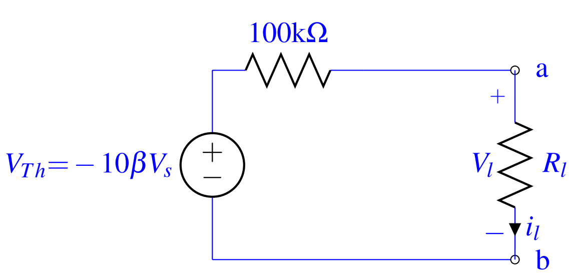


Figure 7: Circuit with load R_l

If we solve the circuit in Figure **??**, we get (from voltage divider, last discussion) $V_l = V_{Th}\frac{R_l}{R_l+100\text{k}\Omega} = -10\beta V_s\frac{R_l}{R_l+100\text{k}\Omega} = -10\beta V_s\frac{R_l}{R_l+100\text{k}\Omega}\frac{100\times10^3}{100\times10^3} = -\beta\frac{V_s}{10\times10^3}\frac{100\times10^3\cdot R_l}{R_l+100\text{k}\Omega} = -\beta\frac{V_s}{10\times10^3}\cdot(R_l \mid\mid 100\text{k}\Omega)$, and therefore the current will be the same (same voltage, same resistor).

Now if we solve the circuit in Figure **??**, we get (from current divider, either derive it or note that we kind of derived it when we computed i_l from Figure **??**) $i_l = I_{No}\frac{100\text{k}\Omega}{100\text{k}\Omega+R_l} = -\beta\frac{V_s}{10\times10^3}\frac{100\text{k}\Omega}{100\text{k}\Omega+R_l}$, and therefore the voltage will be the same (same current, same resistor).

What exactly are we doing when we find the equivalent circuits? Imagine you are given a box with two terminals. The value of an equivalent circuit is that we can completely describe the behavior of that box without knowing what is inside. In addition, actually calculating the Thévenin and Norton equivalent voltage and current can be done with only the two terminals. How would we measure the voltage? Simply take a voltmeter and measure the voltage across the terminals. How would we measure the current? Simply connect the terminals together and measure the current through this short. The resistance is a bit more subtle. If we look at the equivalent circuits, you could either SHORT the voltage source in the Thévenin equivalent, or OPEN the current source in the Norton equivalent. Then, you must apply a test voltage source at the output



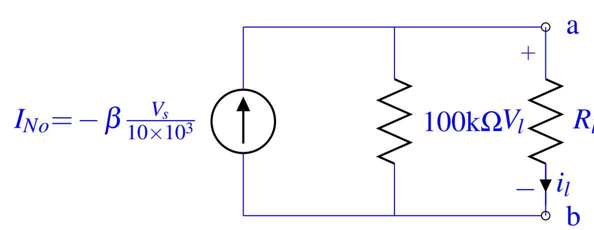


Figure 8: Thévenin equivalent with load

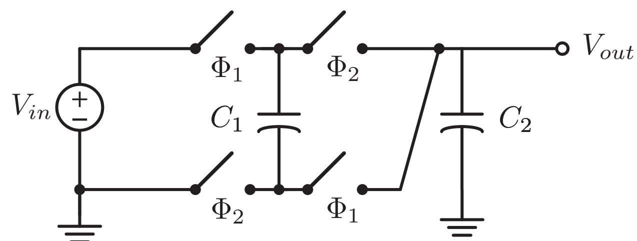
Figure 9: Norton equivalent with load

and measure the current supplied by this source. Alternatively, you can just divide the equivalent voltage source by the equivalent current source. The Thévenin and Norton equivalents must be consistent (with the obvious exception that the Norton equivalent burns power while the Thévenin equivalent does not...)

Section 2: Free-form Problems

7. DC-DC Voltage Divider

As we have learned in class, one of the reasons for using AC voltages is that we can easily transform the voltage (step up or step down) using transformers. Unfortunately, such circuits do not work at DC and we need to come up with other ways of dividing DC voltages. We have learned about resistive dividers, but we found issues such as inefficiencies. An alternative circuit, a capacitive charge pump, is shown below. It relies on two switches which are activated in sequence, first switch Φ_1 is closed (during this period Φ_2 switches are open), and next Φ_2 closes and Φ_1 is opened. In practice this is done periodically but for this problem we will analyze each phase separately. Note that V_{in} is a DC voltage.



(a) During phase Φ_1, calculate the voltage across and charge stored by each capacitor C_1 and C_2.

Solutions: Since the two capacitors are connected in series, the same current flows through each one (by KCL), so they both charge to the same value. We can model the system as two capacitors in series, so the amount of charge is given by

$$q_1 = q_2 = q = C_{eff}V_{in} = \left(\frac{C_1C_2}{C_1+C_2}\right)V_{in}$$

$$v_1 = q_1/C_1 = \left(\frac{C_2}{C_1+C_2}\right)V_{in}$$

$$v_2 = q_2/C_2 = \left(\frac{C_1}{C_1+C_2}\right)V_{in}$$

(b) During phase Φ_2, calculate the output voltage V_{out} and show that it is a fraction of the input voltage V_{in}.

Solutions: In phase 2, the two capacitors are placed in parallel. Note that even if they are charged to different voltages at the end of phase 1, at the end of phase 2 their voltages must equalize, since they are placed in parallel. In this scenario, similar to putting two batteries in parallel, a large current will from from one capacitor to the other. Nevertheless, the total charge in the system does not change, so at the in phase 2, we have two capacitors in parallel holding a charge of $q_{tot} = q_1 + q_2$, and so the output voltage is given by

$$V_{out} = \frac{q_{tot}}{C_1+C_2} = \frac{2\left(\frac{C_1C_2}{C_1+C_2}\right)V_{in}}{C_1+C_2}$$

Simplifying

$$V_{out} = \frac{2C_1C_2}{(C_1+C_2)^2}V_{in} < V_{in}$$

(c) For the special case of $C_1 = C_2$, calculate the output voltage and the efficiency of the system. To calculate the efficiency, calculate the energy stored in the capacitors during the end of phase Φ_1 and Φ_2.

Solutions: For this special case, each capacitor is charged to the same voltage in phase 1 and so when they are connected in parallel, there is no charge transfer between the capacitors, and the output is exactly half of the input voltage. The derivation above confirms that

$$V_{out} = \frac{2C^2}{(2C)^2}V_{in} = \frac{V_{in}}{2}$$

The energy stored by C_1 and C_2 at the end of phase 1 is given by

$$E_1 = E_2 = \frac{1}{2}C\left(\frac{V_{in}}{2}\right)^2$$

$$E_{t,1} = 2\frac{1}{2}CV_{in}^2\frac{1}{4} = \frac{1}{4}CV_{in}^2$$

At the end of phase 2, the total energy is stored in two parallel capacitors $C_1 + C_2 = 2C$

$$E_{t,2} = \frac{1}{2}2C\left(\frac{V_{in}}{2}\right)^2 = E_{t,1}$$

which means that no energy is lost in dividing the voltage. If we repeat this calcualtion in the general case, we will find that unless the division ratio is set at two, there is energy loss. Where does the energy go ? In practice most switches have resistance and so it goes into heating the switches. What if the switch is ideal ??

How can you build a circuit that divides by another ratio, say 3, without incurring an efficiency loss?

(d) Assume that this circuit is used with a load represented by the current source $I_L = 10$mA. Suppose that the cycle described above repeats periodically at a rate of 10 kHz, or 10,000 times per second, with each phase Φ_1 and Φ_2 exactly 50% of each cycle. During phase Φ_2, which lasts 50μs, we wish the output voltage to not droop by more than 5mV. Specify the size of C_1 and C_2 to satisfy this constraint.

Solutions: During phase 2, the voltage on the capacitors will droop due to the load current. For a capacitor, we know that

$$I_c = C\frac{dV}{dt} = (C_1+C_2)\frac{dV}{dt} = I_L$$

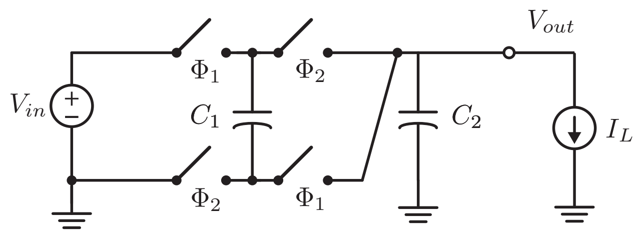
$$\frac{dV}{dt} = \frac{I_L}{C_1+C_2}$$

Since the right hand is a constant, we have

$$\Delta V = \Delta t\frac{I_L}{C_1+C_2}$$

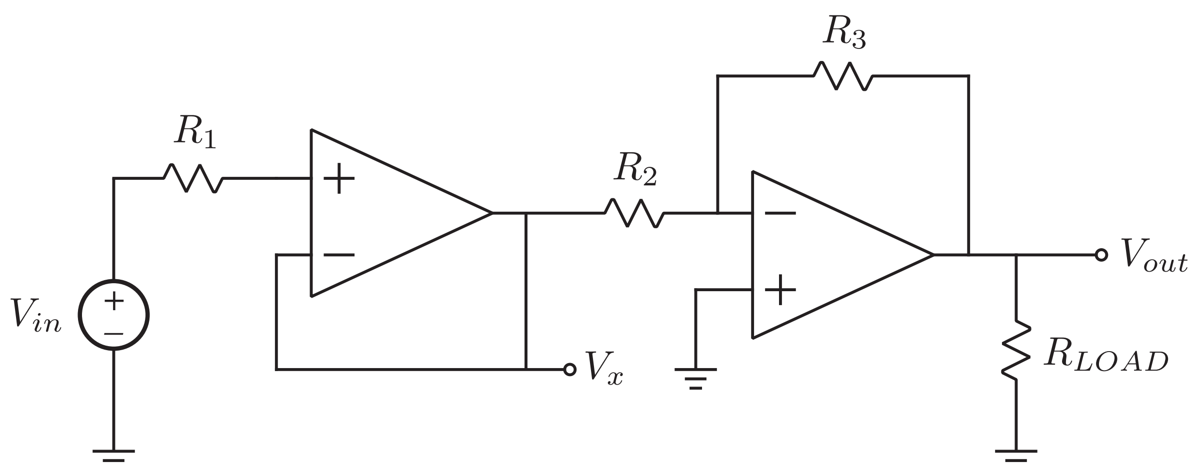
Solving for $C_1 + C_2$, we arrive at

$$C_1 + C_2 = \frac{\Delta t I_L}{\Delta V} = 10\mu\text{F}$$



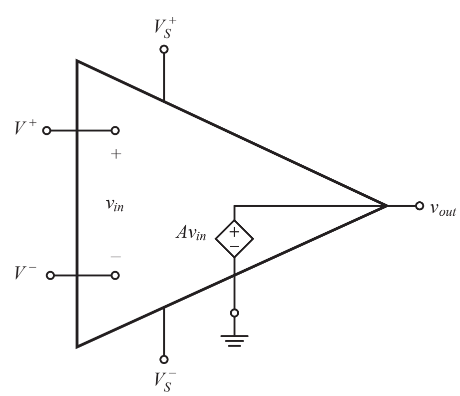
8. Thevenin's Theorem And Op-Amp Circuits!

You're given the below circuit – which cascades two opamps.



We're going to explore what this circuit does in several different ways.

(a) Let's first assume that the Golden Rules hold ($A = \infty$ for both the opamps) Calculate V_x in terms of V_{in}. Calculate V_{out} in terms of V_x, and therefore in terms of V_{in}.



Solutions: Begin with the golden rules:

- Current into both terminals of the opamp(s) is 0.
- $V_+ = V_-$ *when the opamp is connected in negative feedback.*

Solve the circuit systematically. First, label all the nodes in the circuit. Let $V_{1\pm}$ be the inverting/non-inverting inputs of the leftmost op-amp.

Applying the consequences of the Golden Rules, we get $V_{1+} = V_{in}$ and $V_{1-} = V_{1+} = V_{in}$. Putting these together, we get $V_x = V_{1-} = V_{in}$. As we saw last discussion, the first opamp forms a voltage follower, or buffer circuit.

Next, we find V_{out} in terms of V_x. As a consequence of the Golden Rules, current into input terminals of opamp 2 will also be 0. Using this knowledge, we can do simple nodal analysis to solve the circuit. Applying KCL at node $2-$:

$$i_2 - i_3 = 0 \tag{106}$$

giving us

$$i_2 = i_3 = i \tag{107}$$

Also, $V_{2-} = V_{2+} = 0$.
Now, noting from Ohm's Law that $i_2 = \frac{V_x - V_{2-}}{R_2}$ and $i_3 = \frac{V_{2-} - V_{out}}{R_3}$, we get

$$\frac{V_x - V_{2-}}{R_2} = \frac{V_{2-} - V_{out}}{R_3} \tag{108}$$

giving us

$$\frac{V_x}{R_2} = \frac{-V_{out}}{R_3} \tag{109}$$

Therefore,

$$V_{out} = -\frac{V_x R_2}{R_3} = -\frac{V_{in} R_2}{R_3} \tag{110}$$

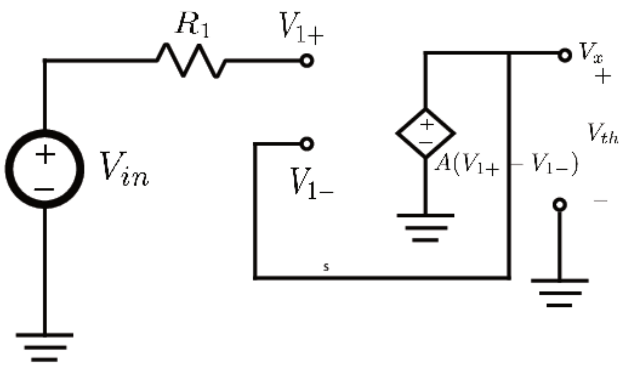
Note that the answer for V_{out} in terms of V_x did not depend on any part of the circuit to the left of node x. Talk about interfaces and opamps as black boxes a bit here (connecting to the discussion in lecture) and how we can simplify solving these sort of cascaded opamps circuits by solving for the gain in each part of the circuit.

(b) Now, suppose we remove the infinite gain assumption. The model of the op amp remains the same, but now A is finite for both opamps. This complicates our analysis...Thevenin and Norton, save us!

What is the Thevenin equivalent circuit at V_x with respect to ground *looking back*? (Imagine that you disconnect the rest of the circuit to the right of node x. Look at the rest of the circuit, and find the Thevenin voltage and equivalent resistance.)

Solutions: Since we can no longer apply all the Golden Rules, we can no longer make the simplifying assumptions we made in the previous part. We could always solve this circuit brute force using nodal analysis, but we don't want to do that here. In the next problem parts, we are going to see that finding the Thevenin equivalents of the circuits can simplify calculations, and this procedure can give us additional insight into the circuit.

We are treating the entire circuit to the left of V_x as a black box. Let's draw the part of the circuit we want to calculate the Thevenin equivalent for:



Calculating Thevenin (open circuit) voltage V_{th} Looking at the node marked x, we can see that

$$V_{th} = A(V_{1+} - V_{1-}) = V_{1-} \tag{111}$$

Also note that $V_{1+} = V_{in}$ (as there is an open circuit). Work through this to get

$$V_{th} = \frac{AV_{in}}{A+1} \tag{112}$$

Point out here that if $A \to \infty$, we would have $V_{th} = V_x = V_{in}$ as before.

Calculating Thevenin resistance R_{eq}**: Method 1** Find the Norton current I_N.

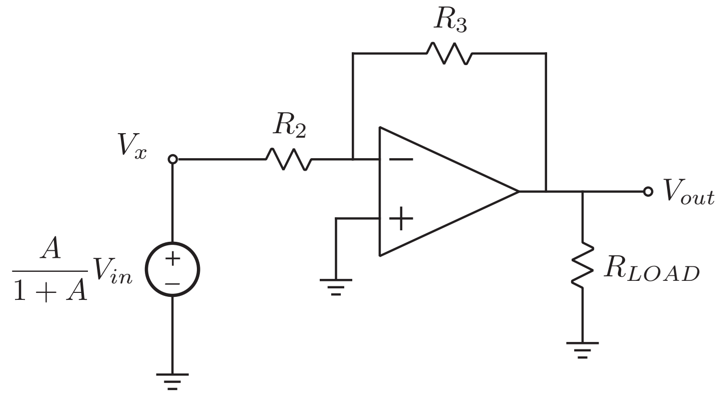
Here, you notice that the value of the VCVS will be $A(V_{1+} - V_{1-}) = AV_{in}$. We are directly connecting the VCVS to ground by a short circuit. So, $I_N = \infty$. Finally, $R_{eq} = \frac{V_{th}}{I_N} = 0\Omega$.

Calculating Thevenin resistance R_{eq}**: Method 2** As discussed in Problem 1 on this worksheet, zero out the independent source V_{in}. Then, apply V_{test} across node x and the ground node and calculate I_{test} as shown in the figure.

In this circuit, we can see that $V_{1+} = 0$ and $V_{1-} = V_{test}$. Also, $A(V_{1+} - V_{1-} = -AV_{1-}$. Therefore, the VCVS and the voltage source V_{test} are connected by a short circuit, and $I_{test} = \infty$. This gives us $R_{eq} = \frac{V_{test}}{I_{test}} = 0\Omega$.

(c) Redraw the circuit using the Thevenin equivalent you have obtained. Does it look simpler?

Solutions:



The circuit certainly does look simpler; we have abstracted out the effect of the first opamp. Notice what the first opamp circuit has done; it took a non-ideal voltage source (V_{in} with resistance R_1), and converted it to an ideal voltage source (since $R_{out} = 0$) This is why voltage followers are useful.

(d) Now, calculate the Thevenin equivalent circuit at V_{out} with respect to ground *looking back*. (Under the same finite-gain assumption as the previous part).

Solutions:

Calculating Thevenin (open circuit) voltage $V_{th} = V_{out}$:

Note that $V_{2+} = 0$. Therefore, the op-amp enforces:

$$V_{out} = A(V_{2+} - V_{2-}) = -AV_{2-} \tag{113}$$

Or re-written:

$$V_{2-} = -\frac{V_{out}}{A} \tag{114}$$

Write the node equation at node $2-$:

$$\frac{V_x - V_{2-}}{R_2} = \frac{V_{2-} - V_{out}}{R_3} \tag{115}$$

Now, plugging in our expression for V_{2-}:

$$\frac{V_x}{R_2} + \frac{V_{out}}{AR_2} = -\frac{V_{out}}{AR_3} - \frac{V_{out}}{R_3} \tag{116}$$

Re-arrange to solve for V_{out}:

$$V_{out} = -\frac{V_x}{R_2}\left[\frac{1}{AR_2} + \frac{1}{AR_3} + \frac{1}{R_3}\right]^{-1} \tag{117}$$

Now, plug in $V_x = AV_{in}/(A+1)$ from part (b) (notice, here we crucially use the fact that loading does not change V_x):

$$V_{out} = -\frac{AV_{in}}{(A+1)R_2}\left[\frac{1}{AR_2} + \frac{1}{AR_3} + \frac{1}{R_3}\right]^{-1} \tag{118}$$

Notice that taking limit $A \to \infty$ gives the expected answer.

Calculating Thevenin resistance R_{eq}: Method 1

$I_N = \infty$. Why?

This gives us, again, $R_{eq} = \frac{V_{th}}{I_N} = 0\Omega$.

Calculating Thevenin resistance R_{eq}: Method 2

$I_{test} = \infty$. Why?

This gives us $R_{eq} = \frac{V_{test}}{I_{test}} = 0\Omega$.

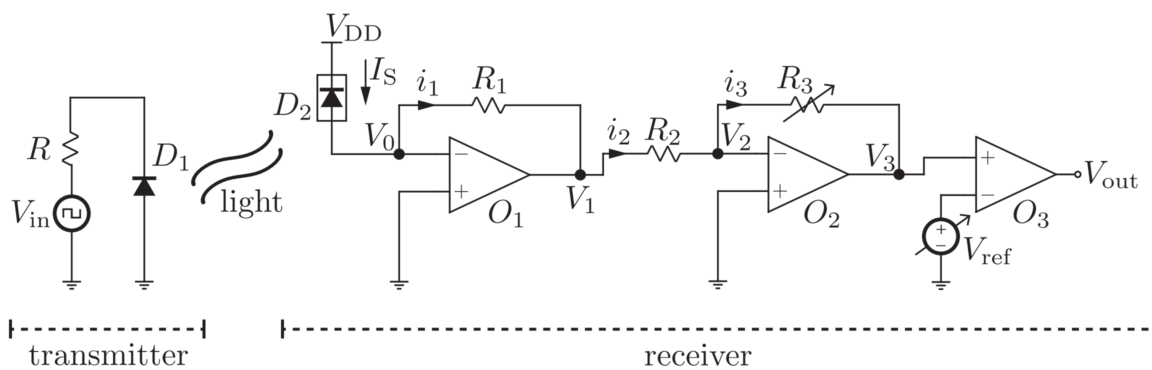
(e) Now, calculate V_{out} in the limit as $A \to \infty$. Do you get the same answer as before?

Solutions: Taking $A \to \infty$, we get $V_{out} = -\frac{R_3 V_{in}}{R_2}$. This is indeed the same answer as before.

9. Wireless communication with an LED

Note: This problem is slightly more difficult (out of scope in difficulty) than a midterm problem. It was a HW problem from Fall 2015.

In this question, we are going to analyze the system shown in the figure below. It shows a circuit that can be used as a wireless communication system using visible light (or infrared, very similar to remote controls).



The element D_1 in the transmitter is a light-emitting diode (LED in short). An LED is an element that emits light where the brightness of the light is controlled by the current flowing through it. You can recall controlling the light emitted by an LED using your MSP430 in touch screen lab part 1. In our circuit, the current across the LED, hence its brightness, can be controlled by choosing the applied voltage V_{in} and the value of the resistor R. In the receiver, the element labeled as D_2 is a reverse biased solar cell. You can recall using a reverse biased solar cell in imaging labs 1 to 3 as a light controlled current source, by I_S we denote the current supplied by the solar cell. In this circuit the LED D_1 is used as a means for transmitting information with light, and the reverse biased solar cell D_2 is used as a receiver of light to see if anything was transmitted.

Remark: In imaging lab part 3, we have talked about how non-idealities such as background light affect the performance of a system that does light measurements. In this question we assume ideal conditions, that is, there is no source of light around except for the LED.

In our system, we define two states for the transmitter, the *transmitter is sending something* when they turn on the LED, and *transmitter is not sending anything* when they turn off the LED. On the receiver side, the goal is to convert the current I_S generated by the solar cell into a voltage and amplify it so that we can read the output voltage V_{out} to see if the transmitter was sending something or not. The circuit implements this operation through a series of op-amps. It might look look complicated at first glance, but we can analyze it a section at a time.

(a) Currents i_1, i_2 and i_3 are labeled on the diagram. Assuming the Golden Rules hold, is $I_S = i_1$? $i_1 = i_2$? $i_2 = i_3$? Treat the solar cell as an ideal current source.

Solutions:

We use the op-amp golden rules, which says that in an op-amp, no current flows into or out of V_+ or V_-. Therefore we can use KCL at node V_0 and V_2 to conclude that $\boxed{I_S = i_1}$ and $\boxed{i_2 = i_3}$. However, if $I_S \neq 0$ and $R_1 \neq R_2$, then $\boxed{i_1 \neq i_2}$. This is because $V_0 = V_2 = 0V$ and V_1 is some non-zero voltage. If $R_1 \neq R_2$, then the currents flowing through them are different.

(b) Use the Golden Rules to find V_0, V_1, V_2 and V_3 in terms of I_S, R_1, R_2 and R_3.

Hint: Solve for them from left to right, and remember to use the op-amp golden rules.

Solutions:

Using the Golden rules, we know that $\boxed{V_0 = 0V}$. Using Ohm's law, we know that $V_0 - V_1 = i_1 R_1$. From the previous part we know that $i_1 = I_S$. Thus we get $\boxed{V_1 = -I_S R_1}$. Using the Golden rules again, we get $\boxed{V_2 = 0V}$. Using Ohm's law and the KCL result from the previous part, we get the following equations:

$$\begin{aligned} V_1 - V_2 &= i_2 R_2 \\ V_2 - V_3 &= i_3 R_3 \\ i_2 &= i_3 \end{aligned}$$

Solving them, we get $\boxed{V_3 = I_S R_1 R_3 / R_2}$.

(c) In the previous part, how could you check your work to gain confidence that you got the right answer?

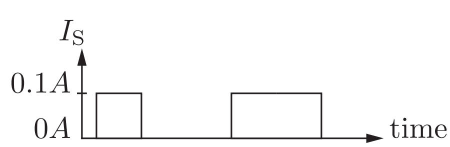
Solutions:

One sanity check is checking that your answer has the right units (Voltage = Amperes $\times$ Ohms $\times$ Ohms / Ohms = Amperes $\times$ Ohms).

Also notice that R_2 and R_3 form a voltage divider since no current flows into the negative terminal of O_2. Thus we can check that the voltage divider equation holds:

$$V_2 - V_1 = (V_3 - V_1)\frac{R_2}{R_2 + R_3}$$

(d) Now, assume that the transmitter has chosen the values of V_{in} and R to control the intensity of light emitted by LED such that when *transmitter is sending something* I_S is equal to 0.1 A, and when the *transmitter is not sending anything* I_S is equal to 0 A. The following figure shows a visual example of how this current I_S might look like as time changes (note that this is just here for helping visualizing the form of the current supplied by the solar cell).



For the receiver, suppose $V_{\text{ref}} = 2V$, $R_1 = 10\,\Omega$, $R_2 = 1000\,\Omega$, and the supply voltages of the op-amps are $V_{DD} = 5V$ and $V_{SS} = -5V$. Pick a value of R_3 such that V_{out} is V_{DD} when the *transmitter is sending something* and V_{SS} when the *transmitter is not sending anything*?

Solutions:

We want $V_{\text{out}} = V_3 - V_{\text{ref}} > 0V$ when $I_S = 0.1A$, and $V_{\text{out}} = V_3 - V_{\text{ref}} < 0V$ when $I_S = 0A$. We plug the known resistor values into the equation in the previous part to get $V_3 = I_S R_3 / 100$. When $I_S = 0$, $V_{\text{out}} = -2V < 0V$. When $I_S = 0.1A$, $V_{\text{out}} = R_3/1000 - V_{\text{ref}} > 0V$. Thus $\boxed{R_3 > 2000\Omega}$.

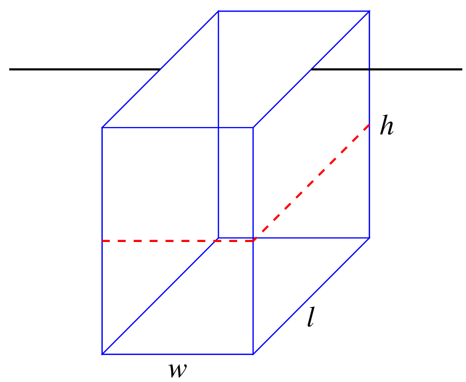
(e) In the previous part, how could you check your work to gain confidence that you got the right answer?

Solutions:

We can check if the answer makes sense. We know that O_2 serves as an inverting amplifier and V_1 is negative (since there is a voltage drop from $V_0 = 0V$), thus we want it to amplify V_1 until it is higher than V_{ref}, so R_3 should be bigger than R_2.

10. Wine Barrel Filler

You own a wine tasting place in Berkeley! You have a very elegant dispenser set up for each kind of wine. To minimize the number of bottles you use, you dispense the wine directly from refillable rectangular barrels. To make sure that the barrels never run out, you want to design a level "detector" which will send the appropiate signal to the tank of wine to pour wine into a barrel until a certain level. Two lateral faces of the barrel (opposite to each other) are coated inside with a perfectly conducting material and you have wires coming out of the barrel at the two faces. You are given that the resistivity of wine is ρ. The dimensions of the barrels (other than height) are l and w. Design a circuit to control the level of the wine. You don't want it to go below a threshold h_{min} and above a threshold h_{max}. The only commands the circuit needs to output are "Fill" and "Stop Fill". Recall the formula $R = \frac{\rho l}{A}$.



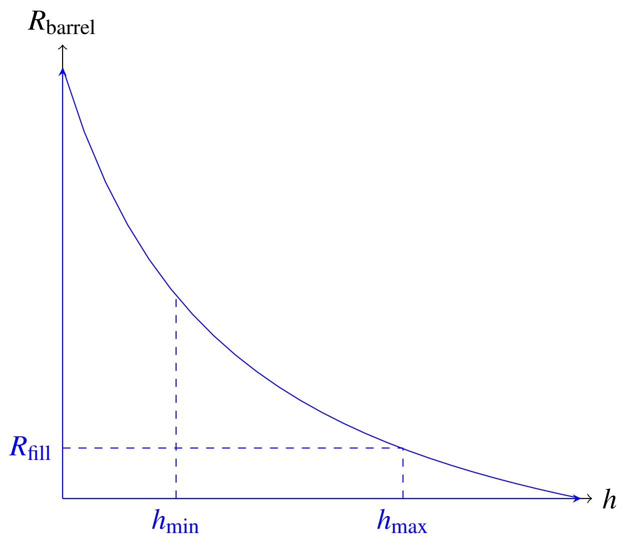
Solutions:

As the wine level gets lower, the resistance of the wine will increase because the width of the resistor (wine) decreases.

Using the dashed line as the current level of wine in the barrel and the fact that the wires run from the side spanning l, what are the dimensions we should use for the formula? w should be length and $l \cdot h$ should be the cross sectional area. The formula yields:

$$R_{\text{barrel}} = \frac{\rho L}{A} = \frac{\rho w}{lh}$$

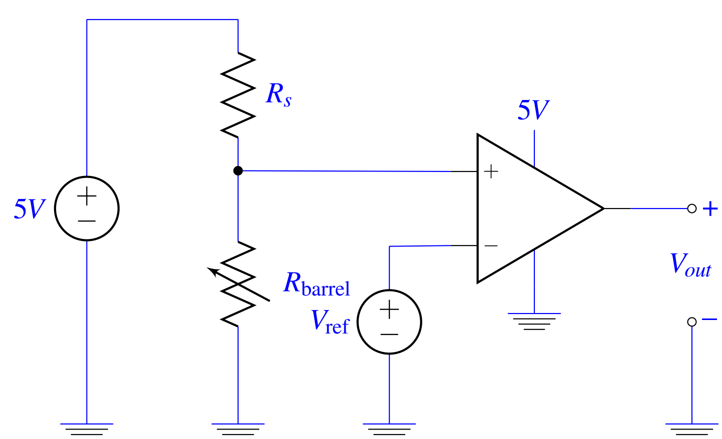
Now consider the graph of R_{barrel} as h varies:



$$R_{\text{min}} \text{ or } (R_{\text{fill}}) = \frac{\rho w}{l h_{\text{max}}}$$
$$R_{\text{max}} = \frac{\rho w}{l h_{\text{min}}}$$

Since voltage across a resistor decreases as the cross-sectional area increases (meaning as h increases), then any voltage read that is below the voltage across a barrel with h_{max} means that the barrel has $h \geq h_{\text{max}}$ and we should send the "Stop Fill" command. Otherwise, the level is below the minimum and we should send the "Fill" command.

Using what we learned about operational amplifiers being used as comparators, let's design a circuit that has a sensing resistor R_s (to make sure voltage drop across R_{barrel} is not just the voltage source's voltage), a voltage source, and an operational amplifier that takes in a feed from the voltage across the barrel and feeds it into the dispenser as a signal. Let the positive terminal of the op amp's power supply be $5V$ and the negative terminal be $0V$ or ground. Let our voltage source provide $5V$ as well.



Now what is the range of V_{in} and what is V_{ref}? If the level of wine is 0, meaning $h = 0$, then the resistance of the barrel is infinite and the voltage drop across it should be the voltage of the voltage source, $5V$. That's the maximum. The minimum however, is not 0. By Ohm's Law, voltage drop across a resistor is low if the resistance is low. The lowest resistance R_{fill}, corresponds to a $h = h_{max}$. Using Ohm's Law:

$$\begin{aligned}
I &= \frac{V_0}{R_{eq}} \\
R_{eq} &= R_s + R_{barrel} \\
I &= \frac{5}{R_s + R_{barrel}} A \\
V_{\text{ref}} &= IR_{barrel} \\
V_{\text{ref}} &= \frac{5 \times R_{barrel}}{R_s + R_{barrel}} V \\
\min V_{in} &= \frac{5 \times R_{fill}}{R_s + R_{fill}} V
\end{aligned}$$

This is the minimum V_{in} corresponding to h_{max}. If we want to send a $0V$ signal if $h \geq h_{max}$, then we want to reject $V \leq \min V_{in}$. This means V_{ref} is the lower bound of the range of V_{in}.

$$\begin{aligned}
V_{in} &\in \left[\frac{5 \times R_{fill}}{R_s + R_{fill}} V \text{ , } 5V \right] \\
V_{\text{ref}} &= \frac{5 \times R_{fill}}{R_s + R_{fill}} V
\end{aligned}$$