

This homework is due on Sunday, July 16, 2017, at 23:59.

Self-grades are due on Monday, July 17, 2017, at 23:59.

Submission Format

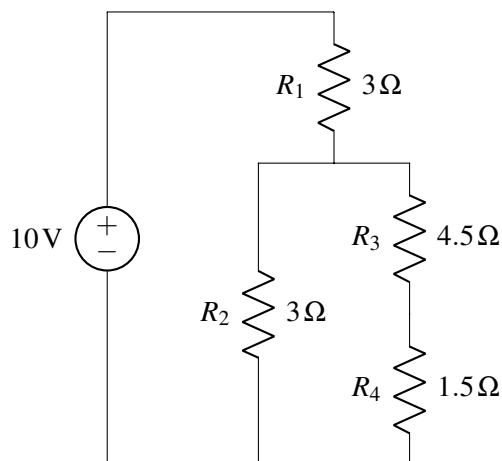
Your homework submission should consist of **one** file.

- `hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. Mechanical Circuits

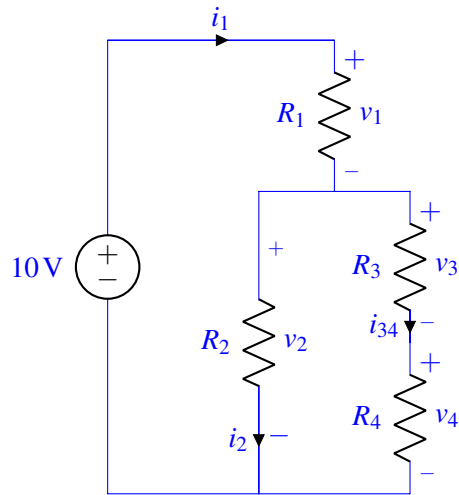
Find the voltages across and currents flowing through all of the resistors.



Solution:

Approach 1 – KCL / KVL:

First, label all the ‘junctions’ or ‘nodes’:



Now set up your KCL / KVL equations:

$$i_1 = i_2 + i_{34}$$

$$v_1 = i_1 \cdot R_1$$

$$v_2 = i_2 \cdot R_2$$

$$v_3 = i_{34} \cdot R_3$$

$$v_4 = i_{34} \cdot R_4$$

$$v_2 = v_3 + v_4 = i_{34} \cdot (R_3 + R_4)$$

$$10 = v_1 + v_2 = v_1 + v_3 + v_4$$

You can solve as a system of equations:

$$\frac{10 - v_2}{R_1} = \frac{v_2}{R_2} + \frac{v_2}{R_3 + R_4}$$

$$\frac{10}{R_1} = v_2 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3 + R_4} \right)$$

$$\frac{10}{3} = v_2 \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{4.5 + 1.5} \right) = v_2 \cdot \frac{5}{6} \Rightarrow v_2 = 4 \text{ V}$$

$$v_1 = 10 - v_2 = 6 \text{ V} \quad i_{34} = \frac{4}{4.5 + 1.5} = \frac{2}{3} \text{ A} \quad v_3 = i_{34} R_3 = 3 \text{ V} \quad v_4 = i_{34} R_4 = 1 \text{ V}$$

Alternatively, you could set it up as a matrix and use IPython to solve.

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4.5 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1.5 & 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_{34} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

This returns the array:

$$\begin{bmatrix} i_1 \\ i_2 \\ i_{34} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ \frac{4}{3} \\ 6 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

Approach 2:

We will first calculate the effective resistance seen from the voltage source to find the current supplied by the voltage source. The resistances R_3 and R_4 are in series hence have effective resistance of 6Ω . They are connected in parallel to a R_2 resistance yielding an effective resistance of

$$\left(\frac{1}{6} + \frac{1}{3}\right)^{-1} = 2\Omega.$$

This resulting effective resistance is in series to R_1 , yielding an effective resistance of 5Ω . Hence the current supplied by the voltage source is

$$\frac{10\text{V}}{5\Omega} = 2\text{A}.$$

Let us denote the voltage drop across R_i as V_i , and the current flowing through R_i and I_i . We have $I_1 = 2\text{A}$ current flowing through R_1 , hence we have $V_1 = 6\text{V}$. The remaining voltage $10\text{V} - 6\text{V} = 4\text{V}$ is across both R_2 and the sequence of resistors R_3 and R_4 . Hence, $I_2 = \frac{4\text{V}}{3\Omega} = \frac{4}{3}\text{A}$. Furthermore, $I_3 = \frac{4\text{V}}{6\Omega} = \frac{2}{3}\text{A}$. Combining all, we get the following voltages and currents:

$$\begin{aligned} I_1 &= 2\text{A}, \\ I_2 &= \frac{4}{3}\text{A}, \\ I_3 &= I_4 = \frac{2}{3}\text{A}, \\ V_1 &= 6\text{V}, \\ V_2 &= 4\text{V}, \\ V_3 &= 3\text{V}, \\ V_4 &= 1\text{V}. \end{aligned}$$

2. Cell Phone Battery

As great as smartphones are, one of their main drawbacks is that their batteries don't last a very long time. A Google Pixel, under somewhat regular usage conditions (internet, a few cat videos, etc.) uses 0.4W of power. We will model the battery as a voltage source (which, as you know, will maintain a voltage across its terminals regardless of current through it) with one caveat: they have a limited amount of charge, or capacity. When the battery runs out of charge, it no longer provides a constant voltage, and your phone dies. Typically, engineers specify battery capacity in terms of mAh , which indicates how many mA of current the battery can supply for one hour before it needs to be recharged. The Pixel's battery has a battery capacity of 2770mAh and operates at a voltage of 3.8V .

- (a) When a battery's capacity is depleted, it no longer operates as a voltage source. How long will a Pixel's full battery last under regular usage conditions?

Solution:

400mW of power at 3.8V is about 105.26mA of current. A battery that can provide 1mAh can provide 1mA for an hour, so our 2770mAh battery can source 105.26mA for $\frac{2770\text{mAh}}{105.26\text{mA}} = 26.3\text{h}$, a little more than a full day.

An alternative approach is to say 2770mAh at 3.8V is $2770\text{mAh} \cdot 3.8\text{V} = 10526\text{mWh}$. 0.4W is 400mW, so $\frac{10526\text{mWh}}{400\text{mW}} = 26.3\text{h}$ is how long the battery will last.

- (b) How many coulombs of charge does the battery contain? Recall that $1\text{C} = 1\text{A} \times 1\text{s}$, which implies that $1\text{mC} = 1\text{mAs}$. An electron has approximately $1.602 \times 10^{-19}\text{C}$ of charge. How many usable electrons worth of charge are contained in the battery when it is fully charged?

Solution:

One hour has 3600 seconds, so the battery's capacity can be written as $2770\text{mAh} \times 3600 \frac{\text{s}}{\text{h}} = 9.972 \times 10^6\text{mAs}$. To find this in coulombs, divide it by 1000 to get 9972C.

An electron has a charge of approximately $1.602 \times 10^{-19}\text{C}$, so 9972C is $\frac{9972\text{C}}{1.602 \times 10^{-19}\text{C}} \approx 6.225 \times 10^{22}$ electrons. That's a lot!

- (c) Suppose the cell phone battery is completely discharged and you want to recharge it completely. How much energy (in J) is this? Recall that a J is equivalent to a Ws.

Solution:

The battery is rated for 2770mAh at 3.8V, which gives $2770\text{mAh} \cdot 3.8\text{V} = 10526\text{mWh}$.

A joule is equivalent to a watt-second, and there are 3600 seconds in an hour, so our battery has $10526\text{mWh} \cdot 3600 \frac{\text{s}}{\text{h}} = 37893600\text{mJ}$, or 37893.6J.

- (d) Suppose PG&E charges \$0.16 per kWh. Every day, you completely discharge the battery (meaning more than typical usage) and you recharge it every night. How much will recharging cost you for the month of July (31 days)?

Solution:

2770mAh at 3.8V is $2770\text{mAh} \cdot 3.8\text{V} = 10526\text{mWh}$, or 0.010526kWh. At \$0.16 per kWh, that is $\$0.16 \cdot 0.010526$ per day, or $\$0.16 \cdot 0.010526 \cdot 31 = \0.0522 , or about 5 cents a month. Compare that to your cell phone data bill! Whew!

- (e) The battery has internal circuitry that prevents it from getting overcharged (and possibly exploding!). The circuitry is also used to transfer power into the chemical reactions that store the energy. We will model this internal circuitry as being one resistor with resistance R_{bat} , which is typically a small, non-negative resistance. Furthermore, we'll assume that all the energy dissipated across R_{bat} goes to recharging the battery. Suppose the wall plug and wire can be modeled as a 5V voltage source and 200mΩ resistor, as pictured in Figure 1. What is the power dissipated across R_{bat} for $R_{\text{bat}} = 1\text{m}\Omega$, 1Ω , and $10\text{k}\Omega$? How long will the battery take to charge for each of those values of R_{bat} ?

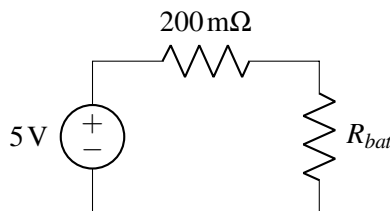


Figure 1: Model of wall plug, wire, and battery.

Solution:

The energy stored in the battery is 2770 mAh at 3.8 V, which is $2.77 \text{ Ah} \cdot 3.8 \text{ V} = 10.526 \text{ Wh}$. We can find the time to charge by dividing this energy by power in W to get time in hours.

For $R_{\text{bat}} = 1 \text{ m}\Omega$, the total resistance seen by the battery is $1 \text{ m}\Omega + 200 \text{ m}\Omega = 201 \text{ m}\Omega$ (because the wire and R_{bat} are in series), so by Ohm's law, the current is $\frac{5 \text{ V}}{0.201 \Omega} = 24.88 \text{ A}$. The voltage drop across R_{bat} is (again by Ohm's law) $24.88 \text{ A} \cdot 0.001 \Omega = 0.02488 \text{ V}$. Then power is $0.02488 \text{ V} \cdot 24.88 \text{ A} = 0.619 \text{ W}$, and the total time to charge the battery is $\frac{10.526 \text{ Wh}}{0.619 \text{ W}} = 17.00 \text{ h}$.

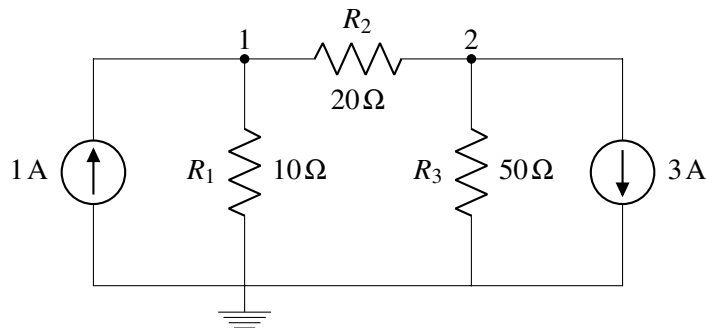
Similarly, for 1Ω , the total resistance seen by the battery is $1 \Omega + 0.2 \Omega = 1.2 \Omega$, the current through the battery is $\frac{5 \text{ V}}{1.2 \Omega} = 4.167 \text{ A}$, and the voltage across the battery is by Ohm's law $4.167 \text{ A} \cdot 1 \Omega = 4.167 \text{ V}$. Then the power is $4.167 \text{ A} \cdot 4.167 \text{ V} = 17.36 \text{ W}$, and the total time to charge the battery is $\frac{10.526 \text{ Wh}}{17.36 \text{ W}} = 0.606 \text{ h}$, about 36 min.

For $10 \text{ k}\Omega$, the total resistance seen by the battery is $10000 \Omega + 0.2 \Omega = 10000.2 \Omega$, the current through the battery is $\frac{5 \text{ V}}{10000.2 \Omega} \approx 0.5 \text{ mA}$, and the voltage across the battery is by Ohm's law $0.5 \text{ mA} \cdot 10 \text{ k}\Omega \approx 5 \text{ V}$ (up to 2 significant figures). Then the power is $5 \text{ V} \cdot 0.5 \text{ mA} = 2.5 \text{ mW}$, and the total time to charge the battery is $\frac{10.526 \text{ Wh}}{0.0025 \text{ W}} = 4210 \text{ h}$.

3. Nodal Analysis

Using techniques presented in class, label all unknown node potentials and apply KCL to each node to find all the node potentials.

- (a) Solve for all node potentials using nodal analysis. Verify with superposition.

**Solution:****Method 1: Nodal Analysis**

Applying KCL at Node 1, we get

$$\frac{V_1 - 0}{R_1} + \frac{V_1 - V_2}{R_2} - 1 = 0$$

$$\frac{V_1 - 0}{10} + \frac{V_1 - V_2}{20} - 1 = 0$$

which gives

$$2V_1 + V_1 - V_2 - 20 = 0$$

implying

$$3V_1 - V_2 = 20 \tag{1}$$

Applying KCL at Node 2, we get

$$\frac{V_2 - V_1}{R_2} + \frac{V_2 - 0}{R_3} + 3 = 0$$

$$\frac{V_2 - V_1}{20} + \frac{V_2 - 0}{50} + 3 = 0$$

which gives

$$5V_2 - 5V_1 + 2V_2 + 300 = 0$$

implying

$$-5V_1 + 7V_2 = -300 \quad (2)$$

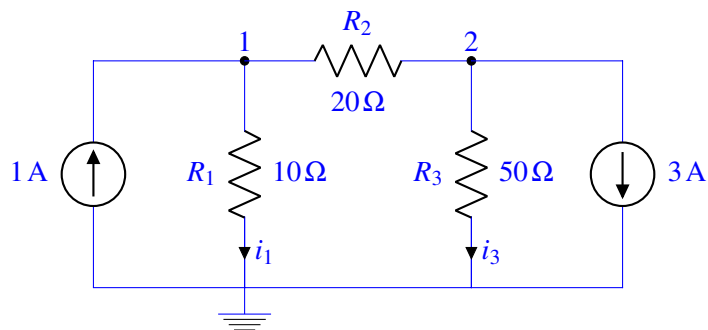
Writing equations 1 and 2 in matrix form, we get

$$\begin{bmatrix} 3 & -1 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 20 \\ -300 \end{bmatrix}$$

Solving the system of equations, we will get $V_1 = -10 \text{ V}$ and $V_2 = -50 \text{ V}$.

Method 2 (verification): Superposition

We define i_1 and i_3 as follows:



First, consider the effect of only the left 1 A current source on. Using current divider rule, we have

$$i_1 = \frac{R_2 + R_3}{R_1 + R_2 + R_3} \cdot 1 \text{ A}$$

$$i_1 = \frac{70}{10 + 70} \cdot 1 \text{ A} = 0.875 \text{ A}$$

and

$$i_3 = 1 \text{ A} - 0.875 \text{ A} = 0.125 \text{ A}$$

Therefore,

$$V_1^a = i_1 \cdot 10 \Omega = 0.875 \text{ A} \cdot 10 \Omega = 8.75 \text{ V}$$

and

$$V_2^a = i_3 \cdot 50 \Omega = 0.125 \text{ A} \cdot 50 \Omega = 6.25 \text{ V}$$

Second, consider the effect of only the right 3 A current source on. Using current divider rule, we have

$$i_1 = -\frac{R_3}{R_3 + R_2 + R_1} \cdot 3 \text{ A}$$

$$i_1 = -\frac{50}{50+30} \cdot 3 \text{ A} = -1.875 \text{ A}$$

and

$$i_3 = -3 \text{ A} + 1.875 \text{ A} = -1.125 \text{ A}$$

Therefore,

$$V_1^b = i_1 \cdot 10 \Omega = -1.875 \text{ A} \cdot 10 \Omega = -18.75 \text{ V}$$

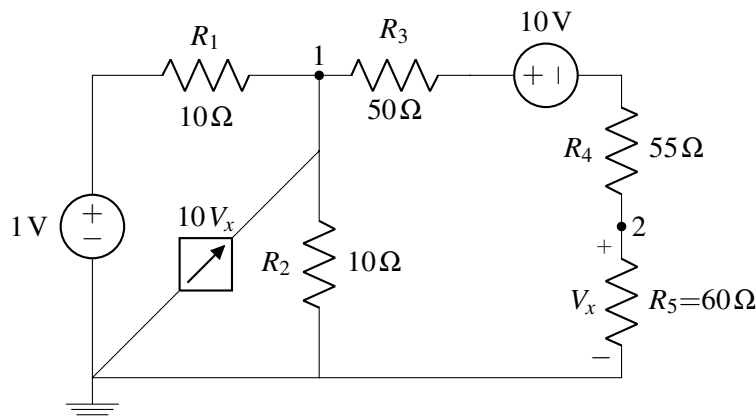
and

$$V_2^b = i_3 \cdot 50 \Omega = -1.125 \text{ A} \cdot 50 \Omega = -56.25 \text{ V}$$

Since the circuit is linear (i.e. we have linear elements and sources), we can use the *principle of superposition* to get $V_1 = V_1^a + V_1^b$ and $V_2 = V_2^a + V_2^b$. Therefore, we get $V_1 = 8.75 \text{ V} - 18.75 \text{ V}$ and $V_2 = 6.25 \text{ V} - 56.25 \text{ V}$. Finally, $V_1 = -10 \text{ V}$ and $V_2 = -50 \text{ V}$.

This solution agrees with the solution we obtained using nodal analysis.

(b) Solve for all node potentials using nodal analysis.



Solution:

We assign the two nodes 1 and 2 and note that $V_x = V_2$ (because we have set the bottom wire as ground). Applying KCL at node 1 (ensuring that currents flowing out of node 1 sum to zero), we get

$$\frac{V_1 - 1}{R_1} + \frac{V_1 - 0}{R_2} + \frac{V_1 - (10 + V_2)}{R_3 + R_4} - 10V_2 = 0$$

$$\frac{V_1 - 1}{10} + \frac{V_1 - 0}{10} + \frac{V_1 - (10 + V_2)}{105} - 10V_2 = 0$$

which gives

$$\frac{V_1 - 1}{2} + \frac{V_1 - 0}{2} + \frac{V_1 - V_2 - 10}{21} - 50V_2 = 0$$

implying

$$44V_1 - 2102V_2 = 41$$

Applying KCL to node 2 (ensuring that currents flowing into node 2 sum to zero), we get

$$\frac{V_2 - 0}{R_5} + \frac{V_2 - (V_1 - 10)}{R_3 + R_4} = 0$$

$$\frac{V_2 - 0}{60} + \frac{V_2 - (V_1 - 10)}{105} = 0$$

which gives

$$\frac{V_2}{4} + \frac{V_2 + 10 - V_1}{7} = 0$$

implying

$$-4V_1 + 11V_2 = -40$$

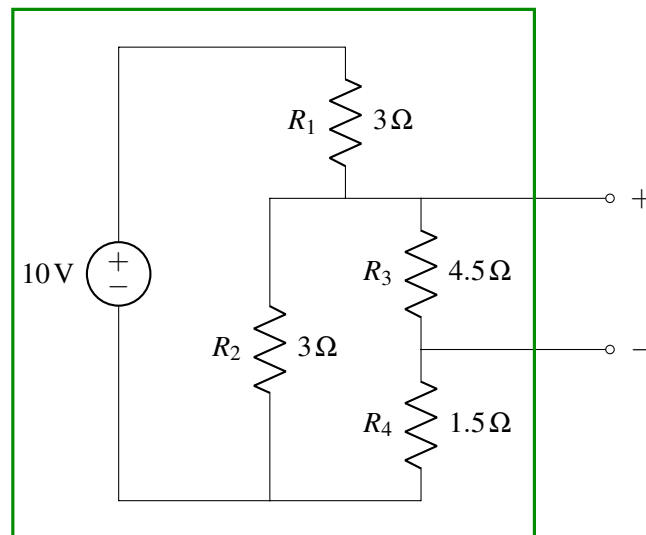
Writing the equations in matrix form, we get

$$\begin{bmatrix} 44 & -21 \\ -4 & 11 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 41 \\ -40 \end{bmatrix}$$

Solving the system of equations, we get $V_1 = 10.5539 \text{ V}$ and $V_2 = 0.2014 \text{ V}$.

4. Thévenin and Norton Equivalent Circuits

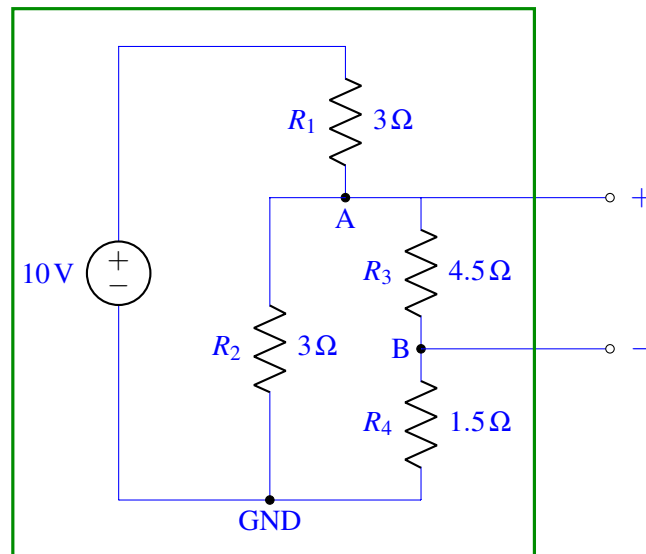
- (a) Find the Thévenin and Norton equivalent circuits seen from outside of the box.



Solution:

To find the Thévenin and Norton equivalent circuits, we are going to find the open circuit voltage between the output ports and the current flowing through the output ports when the ports are shorted.

For finding the open circuit voltage between the output ports, let us label the nodes as shown in the figure below.



First, let us begin by calculating the effective resistance between nodes A and GND. We have the 3Ω resistor in parallel to the $4.5\Omega + 1.5\Omega$ resistance. This gives an equivalent resistance of

$$\frac{1}{\frac{1}{3\Omega} + \frac{1}{4.5\Omega + 1.5\Omega}} = 2\Omega.$$

Then we see that we have a voltage divider from the positive terminal of the 10V supply. Voltage divider is made up of two resistances in series, where the resistances are 3Ω and 2Ω . This gives the voltage at node A equal to

$$V_A = 10\text{V} \frac{2\Omega}{3\Omega + 2\Omega} = 4\text{V}$$

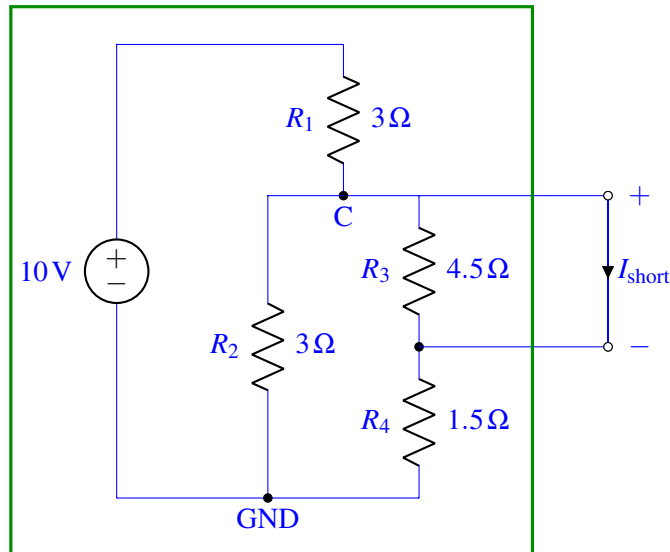
To find the voltage at node B, note that we have another voltage divider between nodes A and GND. Hence, we can find the voltage at node B as

$$V_B = V_A \frac{1.5\Omega}{4.5\Omega + 1.5\Omega} = 4\text{V} \cdot \frac{1}{4} = 1\text{V}$$

Hence the open circuit voltage seen between the output ports is equal to

$$\begin{aligned} V_{\text{open}} &= V_A - V_B \\ &= 4\text{V} - 1\text{V} \\ &= 3\text{V} \end{aligned}$$

Now let us find the short circuit current flowing through the output ports. When doing this, we get the following circuit.



Note that when we short the output terminals, the voltages at the nodes change, this is why we changed the label of the node below the resistor R_1 . Since there is a short circuit parallel to the resistor R_3 , there will be no current flowing through it, hence we have

$$I_{\text{short}} = I_{R_4}$$

To find this current, let us find the equivalent resistance due to R_2 being connected parallel to R_4 when we short the output ports. We have 3Ω parallel to 1.5Ω , which gives an equivalent resistance

$$\frac{1}{\frac{1}{3\Omega} + \frac{1}{1.5\Omega}} = 1\Omega$$

We again have a voltage divider between the positive side of the 10 V supply and the ground. Using this voltage divider, we calculate the voltage at node C as

$$V_C = 10\text{ V} \frac{1\Omega}{3\Omega + 1\Omega} = 2.5\text{ V}$$

Hence, we see that the voltage across the resistor R_4 is equal to 2.5 V. Using Ohm's law, we get

$$I_{R_4} = \frac{2.5\text{ V}}{1.5\Omega} = \frac{5}{3}\text{ A}$$

Since we have $I_{\text{short}} = I_{R_4}$, we have

$$I_{\text{short}} = I_{R_4}$$

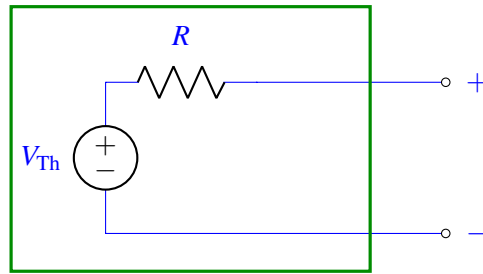
Summarizing the results, we have

$$\begin{aligned} V_{\text{open}} &= 3\text{ V} \\ I_{\text{short}} &= \frac{5}{3}\text{ A} \end{aligned}$$

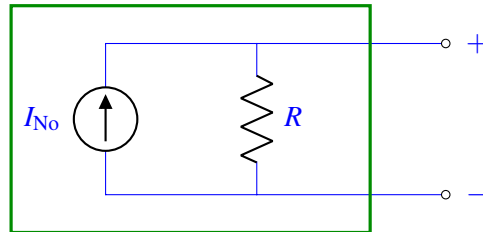
This gives

$$R_{\text{Th}} = \frac{V_{\text{open}}}{I_{\text{short}}} = \frac{9}{5}\Omega$$

Hence the Thévenin equivalent circuit is given by

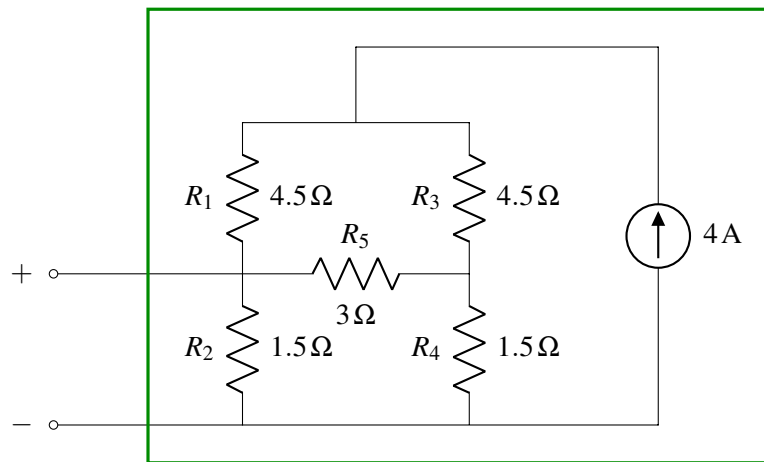


where $R = R_{Th}$ and $V_{Th} = V_{open}$, and the Norton equivalent circuit is given by



where $R = R_{Th}$ and $I_{No} = I_{short}$.

- (b) Find the Thévenin and Norton equivalent circuits seen from outside of the box.



Solution:

As with the previous part of this question, to find the Thévenin and Norton equivalent circuits we are going to find the open circuit voltage between the output ports and the current flowing through the output ports when the ports are shorted.

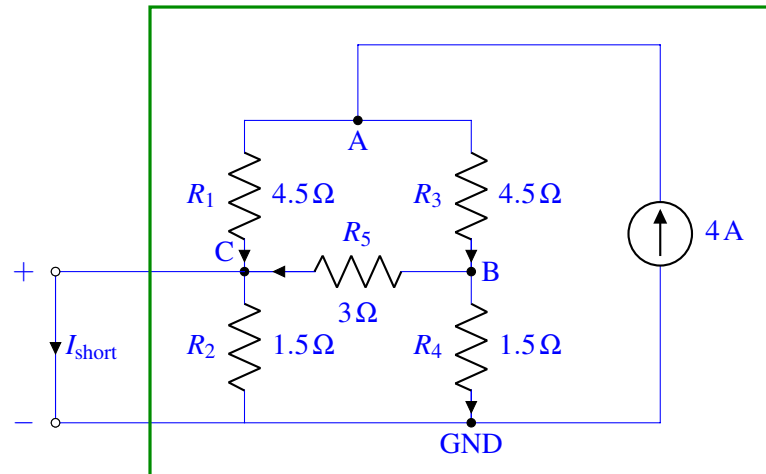
Let us first find the open circuit voltage between the output ports. In the solutions of homework 6, we have seen 3 different approaches for finding the voltages at each node of this same circuit. One of those approaches was to use the symmetry in the circuit to see there will be no current flowing through R_5 . Then, we have a current divider where each branch has the same resistance, hence the current 4 A divides equally between the left and right branches. Hence we have

$$I_{R_1} = I_{R_2} = I_{R_3} = I_{R_4} = 2 \text{ A}$$

This gives us the voltage across R_2 , equivalently the open circuit voltage between the output terminals, equal to

$$V_{open} = 2 \text{ A} \times 1.5 \Omega = 3 \text{ V}$$

Now let us find the short circuit current across the output terminals. Let us find this using nodal analysis on the resulting circuit when we short the output ports. To help do the analysis, let us label the nodes as shown in the figure below.



Now what are the unknown node voltages? We do not know the voltage at node A and B. On the other hand, because node C is connected by a short circuit to GND, we know its voltage is equal to the GND which we set as 0; hence voltage at node C is not an unknown. Next, because there is a short circuit across resistor R_2 , there will be no current flowing through it.

Let us write KCL at the nodes

$$4\text{ A} = I_{R_1} + I_{R_3} \quad (\text{Node A})$$

$$I_{R_3} = I_{R_4} + I_{R_5} \quad (\text{Node B})$$

$$I_{\text{short}} = I_{R_1} + I_{R_5} \quad (\text{Node C})$$

Now let us relate the currents I_{R_1} , I_{R_2} , I_{R_3} , I_{R_4} and I_{R_5} to node voltages using Ohm's law. We have

$$I_{R_1} = \frac{V_A - V_C}{R_1} = \frac{V_A}{4.5\Omega}$$

since $V_C = 0\text{ V}$ because it is connected to the ground by short circuit. Furthermore, we have

$$\begin{aligned} I_{R_2} &= 0\text{ A}, \\ I_{R_3} &= \frac{V_A - V_B}{R_3} = \frac{V_A - V_B}{4.5\Omega} \\ I_{R_4} &= \frac{V_B}{R_4} = \frac{V_B}{1.5\Omega}, \\ I_{R_5} &= \frac{V_B - V_C}{R_5} = \frac{V_B}{3\Omega} \end{aligned}$$

Plugging these into the first two KCL equations, we get

$$\begin{aligned} 4\text{ A} &= \frac{V_A}{4.5\Omega} + \frac{V_A - V_B}{4.5\Omega} \\ \frac{V_A - V_B}{4.5\Omega} &= \frac{V_B}{1.5\Omega} + \frac{V_B}{3\Omega} \end{aligned}$$

These equations are solved by

$$\begin{aligned} V_A &= 9.9 \text{ V}, \\ V_B &= 1.8 \text{ V} \end{aligned}$$

Using the KCL at node C, we get

$$\begin{aligned} I_{\text{short}} &= I_{R_1} + I_{R_5} \\ &= \frac{V_A}{4.5 \Omega} + \frac{V_B}{3 \Omega} \\ &= \frac{9.9}{4.5 \Omega} + \frac{1.8}{3 \Omega} \\ &= 2.8 \text{ A} \end{aligned}$$

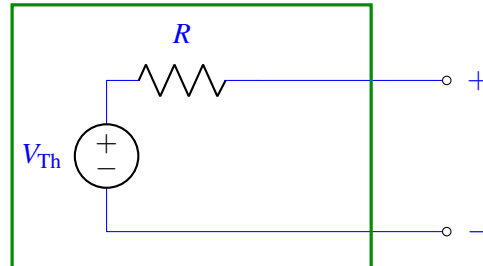
Summarizing the results, we have

$$\begin{aligned} V_{\text{open}} &= 3 \text{ V} \\ I_{\text{short}} &= 2.8 \text{ A} \end{aligned}$$

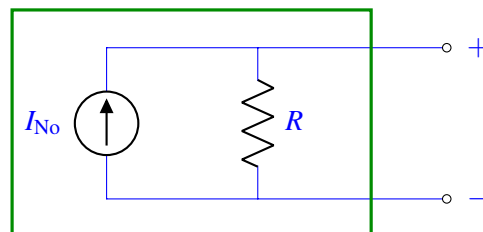
This gives

$$\begin{aligned} R_{\text{Th}} &= \frac{V_{\text{open}}}{I_{\text{short}}} \\ &= \frac{15}{14} \Omega \end{aligned}$$

Hence the Thévenin equivalent circuit is given by



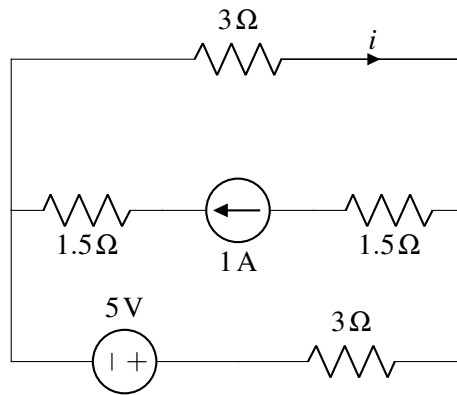
where $R = R_{\text{Th}}$ and $V_{\text{Th}} = V_{\text{open}}$, and the Norton equivalent circuit is given by



where $R = R_{\text{Th}}$ and $I_{\text{No}} = I_{\text{short}}$.

5. Nodal Analysis Or Superposition?

Solve for the current through the 3Ω resistor, marked as i , using superposition. Verify using nodal analysis. You can use IPython to solve the system of equations if you wish. Where did you place your ground, and why?



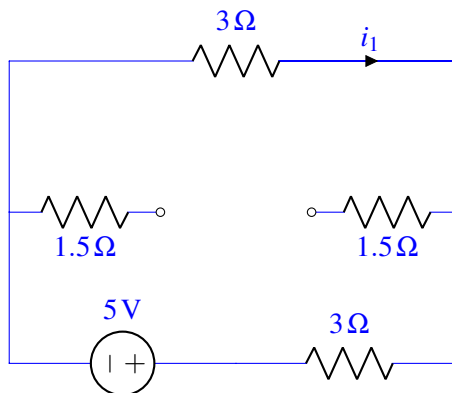
Solution:

$$i = -\frac{1}{3} \text{ A}$$

Method 1: Superposition

Consider the circuits obtained by:

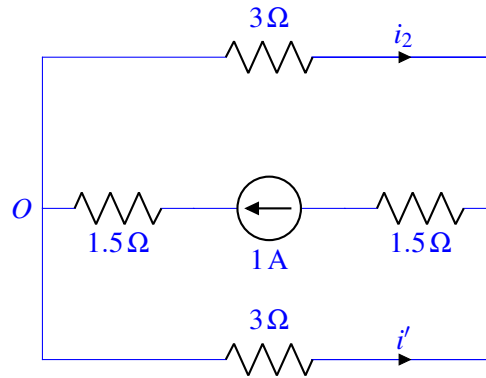
- (a) Turning off the 1 A current source:



In the above circuit, no current is going to flow through the middle branch, as it is an open circuit. Thus this is just a 5 V voltage source connected to two 3 Ω resistors in series so

$$i_1 = -\frac{5}{6} \text{ A}$$

- (b) Turning off the 5 V voltage source:



In the above circuit, notice that the 3Ω resistors are in parallel and therefore form a current divider. Since the values of the resistances are equal, the current flowing through them will also be equal, that is $i_2 = i'$. Applying KCL to node O , we get

$$1 - i_2 - i' = 0$$

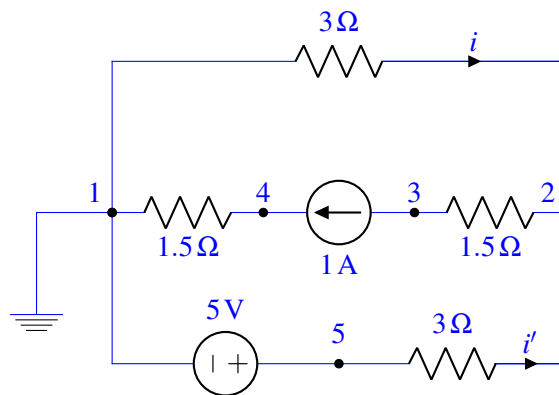
which gives us

$$i_2 = \frac{1}{2} \text{ A}$$

Now, applying the principle of superposition, we have $i = i_1 + i_2 = -\frac{5}{6} \text{ A} + \frac{1}{2} \text{ A} = -\frac{1}{3} \text{ A}$.

Method 2: Nodal Analysis

First, let's identify and label the nodes on the circuit. (Note that the numbers are arbitrary.) We also ground node 1. (Our choice of ground is arbitrary (whatever you chose is fine) but node 1 is a convenient choice because node 5 will turn out to be 5V from the voltage source, the voltage at node 4 can be calculated quickly from the current source and the 1.5Ω resistor using Ohm's law, and i can be calculated quickly once we know the voltage at node 2.)



First, we write KCL at each node. At nodes 1 and 2, we get the same equation.

$$i + i' = 1 \text{ A}$$

At nodes 3 and 4, we get the trivial equation

$$1 \text{ A} = 1 \text{ A}.$$

We now write the voltage drops across the circuit elements in terms of the currents using Ohm's law or in terms of known voltages

$$\begin{aligned}V_5 - V_1 &= 5 \text{ V} \\V_5 - V_2 &= i'(3\Omega) \\V_2 - V_3 &= 1 \text{ A}(1.5\Omega) \\V_4 - V_1 &= 1 \text{ A}(1.5\Omega) \\V_2 - V_1 &= -i(3\Omega)\end{aligned}$$

Since we've chosen node 1 as ground ($V_1 = 0 \text{ V}$), we can rewrite the equations involving V_1 which gives us values for V_5 and V_4 .

$$\begin{aligned}V_5 &= 5 \text{ V} \\V_4 &= 1 \text{ A}(1.5\Omega) \\V_2 &= -i(3\Omega)\end{aligned}$$

We next combine the KCL equations and the Ohm's Law equations to solve for V_2 and V_3 . (We don't actually need to solve for V_3 once we know V_2 but the calculation is easy.)

$$\begin{aligned}-\frac{V_2}{3\Omega} + \frac{5\text{V} - V_2}{3\Omega} &= 1 \text{ A} \\ \implies V_2 &= 1 \text{ V}\end{aligned}$$

and

$$\begin{aligned}V_3 &= V_2 - 1 \text{ A}(1.5\Omega) \\V_3 &= -0.5 \text{ V}\end{aligned}$$

Finally, we use V_2 to solve for i :

$$i = -\frac{V_2}{3\Omega} = -\frac{1}{3} \text{ A}$$

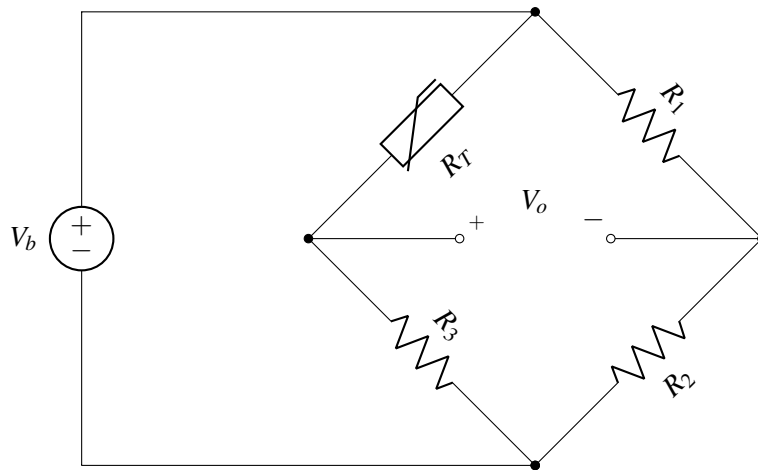
6. Thermistor

Thermistors for sensing temperature consist of sintered metal oxide that exhibits an exponential decrease in electrical resistance with increasing temperature. In semiconductors, electrical conductivity is due to the charge carriers in the conduction band. If the temperature is increased, some electrons are promoted from the valence band into the conduction band, and the conductivity also increases.

The relationship between resistance R and temperature T is given by:

$$R_T(T) = R(T_0) \exp \left(\beta \left(\frac{1}{T} - \frac{1}{T_0} \right) \right), \quad (3)$$

where T is in degrees kelvin, T_0 is the reference temperature, and β is the temperature coefficient of the material. To sense temperature, thermistors are used in a bridge circuit shown below:



The temperature response of a thermistor is given in the table below:

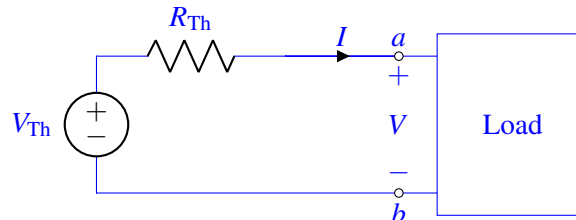
Table 1: Resistance vs. Temperature data for the thermistor

Temperature ($^{\circ}\text{C}$)	-50	-40	-30	-20	-10	0	10	20	30	40	50
R_T (k Ω)	117.2	65.2	38.8	23.8	15.2	10	6.8	4.7	3.4	2.5	1.8

- (a) For the thermistor bridge circuit, find the Thevenin equivalent circuit.

Solution:

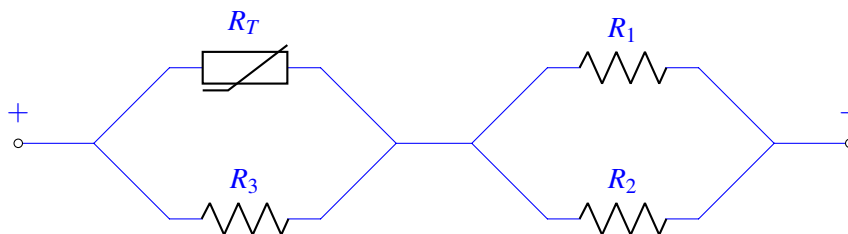
Thevenin equivalent circuit:



Thevenin equivalent voltage, $V_{Th} = V_{oc} = V_o$

$$V_o = V_+ - V_- = V_b \left(\frac{R_3}{R_T + R_3} - \frac{R_2}{R_1 + R_2} \right)$$

Thevenin equivalent resistance, R_{Th} :



$$R_{Th} = R_T \parallel R_3 + R_1 \parallel R_2$$

- (b) Find V_o and from there derive an equation for R_T as a function of V_o , V_b , and the other resistances.

Solution:

$$V_o = V_+ - V_- = V_b \left(\frac{R_3}{R_T + R_3} - \frac{R_2}{R_1 + R_2} \right)$$

$$R_T = R_3 \frac{V_b R_1 - V_o (R_1 + R_2)}{V_b R_2 + V_o (R_1 + R_2)}$$

- (c) If $R_T = R_1 = R_2 = R_3$, what will be the output of the bridge circuit? Assuming $R_1 = R_2 = R_3$, then from the Resistance vs. Temperature data for the thermistor, comment on the bridge output if the temperature rises and vice versa.

Solution:

If $R_T = R_1 = R_2 = R_3$, the bridge is balanced, therefore $V_+ = V_-$. This results in $V_o = 0$.

If the temperature increases R_T will decrease, that means $V_+ > V_-$ because R_1, R_2, R_3 do not have temperature dependence. So $V_o > 0V$. On the other hand, if the temperature decreases, R_T will increase resulting in $V_+ < V_-$ and $V_o < 0V$.

- (d) If $R_2 = R_3$, find what value of $\alpha = \frac{R_3}{R_T}$ (the relation between R_3 and R_T) provides the largest bridge sensitivity to temperature [$\frac{dQ}{d\alpha} = 0$]? The bridge sensitivity, Q is defined as,

$$Q = \frac{dV_o}{dT} = R_T \frac{dV_o}{dR_T} \frac{1}{R_T} \frac{dR_T}{dT}$$

Hint: Both equation 3 and the bridge circuit equation are required for this question.

Solution:

From the thermistor relationship,

$$\frac{1}{R_T} \frac{dR_T}{dT} = \frac{-\beta}{T^2}$$

From the bridge relationship,

$$V_o = V_b \left(\frac{R_3}{R_T + R_3} \right) - V_-$$

$$R_T \frac{dV_o}{dR_T} = - \frac{V_b R_3 R_T}{(R_3 + R_T)^2} = - \frac{V_b \alpha}{(1 + \alpha)^2}$$

Combining previous equations,

$$Q = \frac{V_b \alpha \beta}{(1 + \alpha)^2 T^2}$$

For maximizing Q , we need to set $\frac{dQ}{d\alpha} = 0$ and find that the maximum occurs at $\alpha = 1$.

$$\frac{dQ}{d\alpha} = \frac{V_b \beta (1 - \alpha)}{T^2 (1 + \alpha)^3} = 0$$

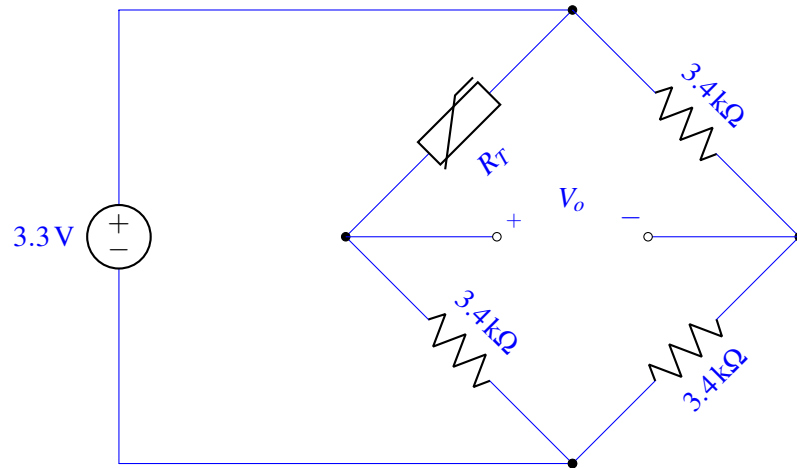
Therefore, $R_3 = R_T$ will maximize sensitivity of the bridge.

- (e) Using the relation between R_3 and R_T , design a bridge circuit that will provide highest sensitivity at 30°C . Draw your circuit and justify your design choices for R_1 , R_2 , and R_3 [$V_b = 3.3\text{ V}$].

Solution:

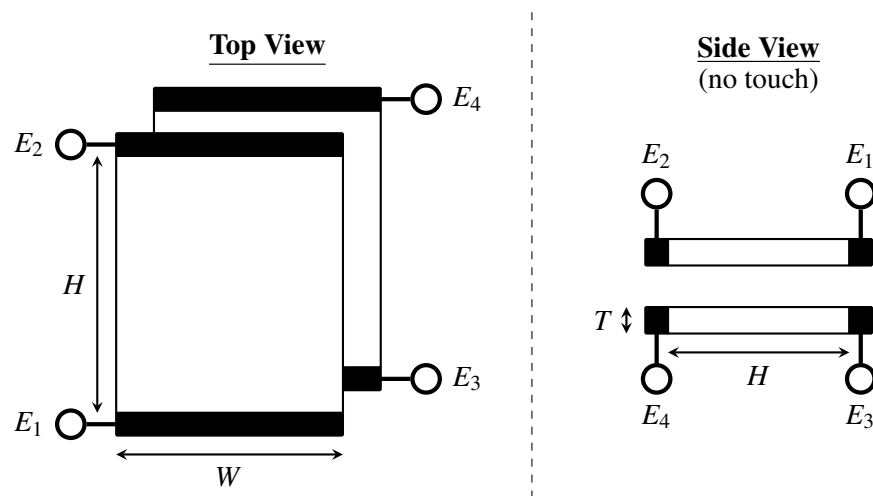
At 30°C , $R_T = 3.4\text{ k}\Omega$. Therefore, $R_2 = R_3 = 3.4\text{ k}\Omega$ will provide the highest bridge sensitivity. We choose $R_1 = R_3$ so that the bridge is balanced, and $V_o = 0\text{ V}$ at 30°C . We prefer having a balanced bridge, so that V_o will wiggle around 0 V for varying temperatures.

Any other value of R_1 will be accepted, as long as you describe how this will affect V_o .



7. Multitouch Resistive Touchscreen

In this problem, we will look at a simplified version of the multitouch resistive touchscreen. In particular, rather than measuring the position of two potential touch points in both dimensions (i.e. a pair of coordinates (x_1, y_1) and (x_2, y_2) corresponding to two touch positions), let's think about a version where we are interested in measuring only the vertical position of the two touch points (i.e. y_1 and y_2). Therefore, unlike the touchscreens we looked at in class, both of the resistive plates (i.e. both the top and the bottom plate) would have conductive strips placed along their top and bottom edges, as shown below.



- (a) Assuming that both of the plates are made out of a material with $\rho = 1 \Omega\text{m}$ and that the dimensions of the plates are $W = 3 \text{ cm}$, $H = 12 \text{ cm}$, and $T = 0.5 \text{ mm}$, with no touches at all, what is the resistance between terminals E_1 and E_2 (which would be the same as the resistance between terminals E_3 and E_4)?

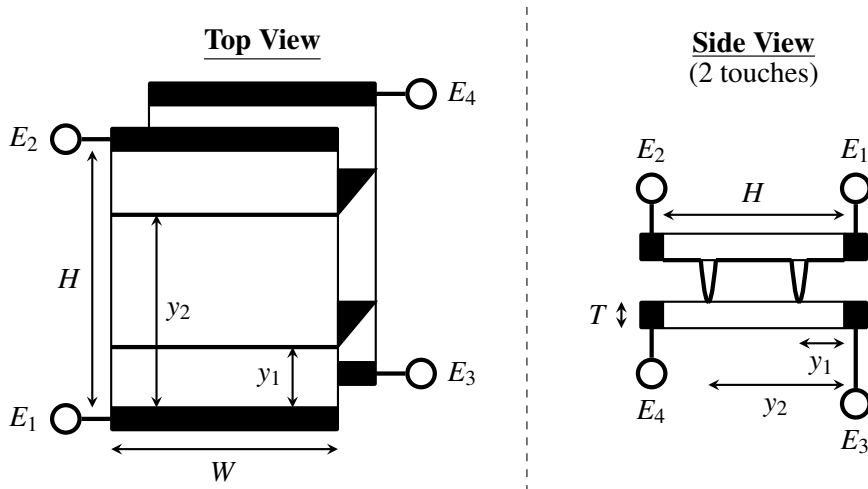
Solution:

$$R = \rho \cdot \frac{L}{A} \implies R_{E1-E2} = \rho \left(\frac{H}{W \cdot T} \right)$$

$$R_{E1-E2} = 1 \Omega\text{m} \left(\frac{12 \times 10^{-2} \text{ m}}{3 \times 10^{-2} \cdot 0.5 \times 10^{-3} \text{ m}} \right)$$

$$R_{E1-E2} = 8 \text{ k}\Omega$$

- (b) Now let's look at what happens when we have two touch points. Let's assume that at wherever height the touch occurs, a perfect contact is made between the top plate and the bottom plate along the entire width of the plates (i.e. you don't have to worry about any lateral resistors), but that otherwise none of the electrical characteristics of the plates change. Defining the bottom of the plate as being $y = 0 \text{ cm}$ (i.e. a touch at E_1 would be at $y = 0 \text{ cm}$), let's assume that the two touches happen at $y_1 = 3 \text{ cm}$ and $y_2 = 7 \text{ cm}$ and that your answer to part (a) was $5 \text{ k}\Omega$ (which may or may not be the right answer). Draw a model with 6 resistors that captures the electrical connections between E_1 , E_2 , E_3 , and E_4 and calculate their resistances. Note that for clarity, the system has been redrawn below to depict this scenario.

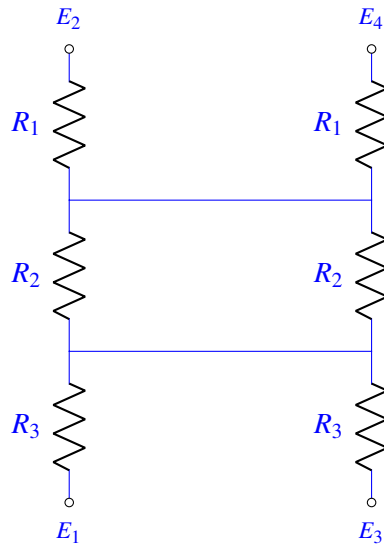


Solution:

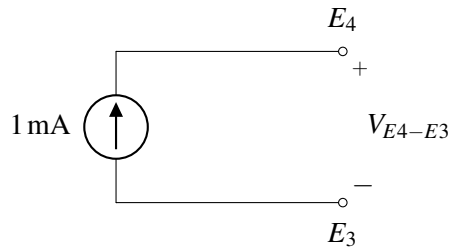
$$R_3 = \frac{3 \text{ cm}}{12 \text{ cm}} \cdot R_{E2-E1} = 1.25 \text{ k}\Omega$$

$$R_2 = \frac{7 \text{ cm} - 3 \text{ cm}}{12 \text{ cm}} \cdot R_{E2-E1} = 1.667 \text{ k}\Omega$$

$$R_1 = \frac{12 \text{ cm} - 7 \text{ cm}}{12 \text{ cm}} \cdot R_{E2-E1} = 2.0833 \text{ k}\Omega$$



- (c) Using the same assumptions as part (b), if you drove terminals E_3 and E_4 with a 1 mA current source (as shown below) but left terminals E_1 and E_2 open-circuited, what is the voltage you would measure across $E_4 - E_3$ (i.e. V_{E4-E3})?



Solution:

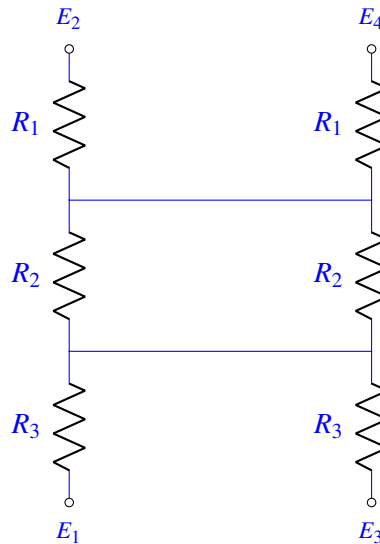
The equivalent resistance between $E_4 - E_3$ is

$$\begin{aligned}
 R_{E4-E3} &= R_1 + R_2 \parallel R_2 + R_3 = R_1 + \frac{R_2}{2} + R_3 \\
 &= 1.25\text{k}\Omega + \frac{1.667\text{k}\Omega}{2} + 2.0833\text{k}\Omega \\
 &\approx 4.167\text{k}\Omega
 \end{aligned}$$

$$V_{E4-E3} = 1\text{ mA} \cdot R_{E4-E3} \implies V_{E4-E3} = 4.167\text{ V}$$

- (d) Now let's try to generalize the situation by assuming that the two touches can happen at any two arbitrary points y_1 and y_2 , but with y_1 defined to always be less than y_2 (i.e. y_1 is always the bottom touch point). Leaving the setup the same as in part (c) except for the arbitrary y_1 and y_2 , by measuring only the voltage between E_4 and E_3 , what information can you extract about the two touch positions? Please be sure to provide an equation relating V_{E4-E3} to y_1 and y_2 as a part of your answer, and note that you may want to redraw the model from part (b) to help you with this.

Solution:



For general

$$R_3 = \frac{y_1}{12 \text{ cm}} \cdot 5 \text{ k}\Omega$$

$$R_2 = \frac{y_2 - y_1}{12 \text{ cm}} \cdot 5 \text{ k}\Omega$$

$$R_1 = \frac{12 \text{ cm} - y_2}{12 \text{ cm}} \cdot 5 \text{ k}\Omega$$

$$R_{E4-E3} = R_1 + \frac{R_2}{2} + R_3 = \left(12 \text{ cm} - y_2 + \frac{y_2 - y_1}{2} + y_1 \right) \cdot \frac{5 \text{ k}\Omega}{12 \text{ cm}}$$

$$= \left(12 \text{ cm} + \frac{y_1}{2} - \frac{y_2}{2} \right) \cdot \frac{5 \text{ k}\Omega}{12 \text{ cm}}$$

So

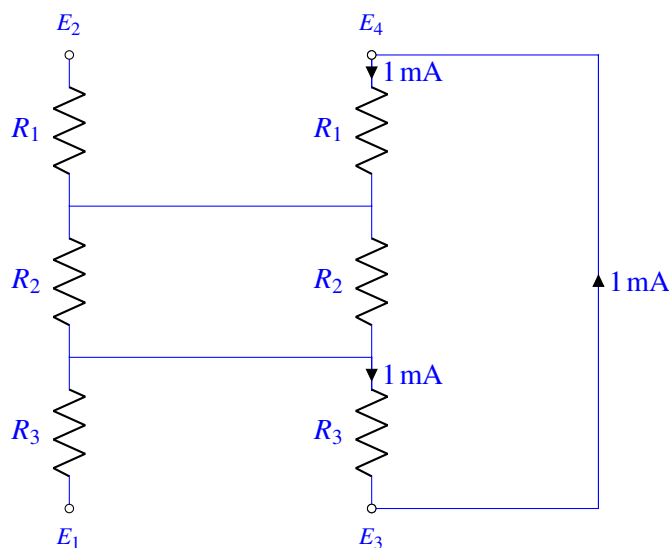
$$V_{E4-E3} = \frac{12 \text{ cm} - \frac{y_2 - y_1}{2}}{12 \text{ cm}} \cdot 5 \text{ V}$$

This means that by measuring V_{E4-E3} , we can only measure the distance between the two touch points ($y_2 - y_1$).

- (e) One of your colleagues claims that by measuring the appropriate voltages, not only can they extract what both y_1 and y_2 are in this system, but they can even do so by formulating a system of three independent voltage equations related to y_1 and y_2 . As we will see later, this will allow us to gain some robustness to noise in the voltage measurements.

In order to facilitate this, write equations relating V_{E4-E2} and V_{E1-E3} to y_1 and y_2 . (The third voltage we'll use is V_{E4-E3} , which you should have already derived an equation for in the previous part of the problem.)

Solution:



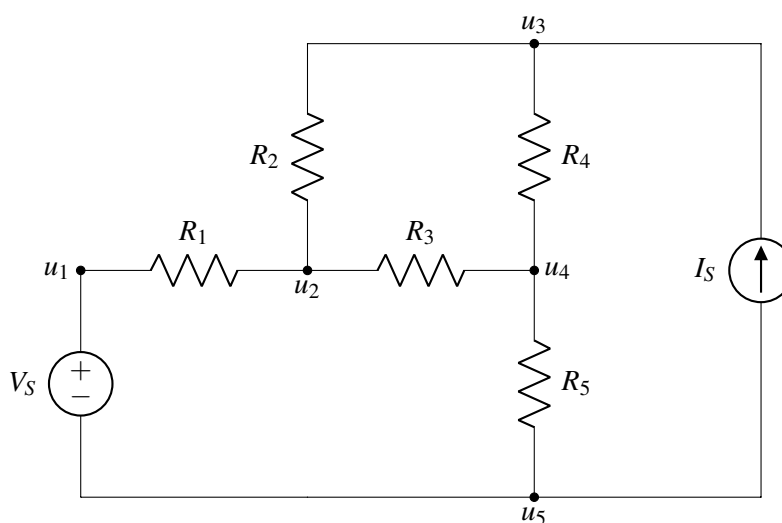
$$V_{E4-E2} = I \cdot R_1 = \frac{12 \text{ cm} - y_2}{12 \text{ cm}} \cdot 5 \text{ V}$$

$$V_{E1-E3} = I \cdot R_3 = \frac{y_1}{12 \text{ cm}} \cdot 5 \text{ V}$$

8. SPICE-y Circuits

In the 1970s, Laurence Nagel and his advisor Donald Pederson at UC Berkeley created a circuit simulation software called Simulation Program with Circuit Emphasis or SPICE. Today, SPICE is the industry standard for circuit simulation. DC analysis in SPICE is performed using linear algebra tools you are familiar with. In this problem, we will explore the linear algebra behind SPICE.

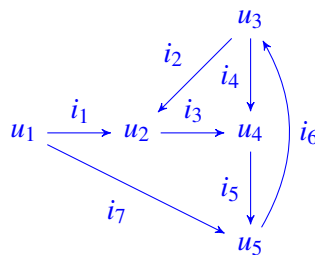
Throughout the problem, we will be referring to the circuit below.



- (a) A circuit designer inputs a circuit to SPICE in the form of a netlist. SPICE then translates the netlist into an incidence matrix. Translate the circuit above into a directed graph and write the edge-node

incidence matrix \mathbf{F} for the graph. \mathbf{F} should have dimensions $M \times N$, where M is the number of branch currents (which is equal to the number of components) and N is the number of node potentials.

Solution:



The incidence matrix for the above graph is as follows:

$$\mathbf{F} = \begin{array}{c} \begin{array}{c} \text{node} \rightarrow \\ \text{current} \downarrow \end{array} \\ \begin{array}{c} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{array} \end{array} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

- (b) SPICE now has a representation for the circuit, in the form of an incidence matrix. SPICE represents the current through each of the elements as one vector \vec{i} , whose entries correspond to the currents in all branches. Find the product $\mathbf{F}^T \vec{i}$ and show that $\mathbf{F}^T \vec{i} = \vec{0}$ represents the KCL equations for this circuit.

Solution:

Let the vector \vec{i} represent the flows through each edge. The product $\mathbf{F}^T \vec{i}$ is shown below:

$$\mathbf{F}^T \vec{i} = \begin{array}{c} \begin{array}{c} \text{component} \rightarrow \\ \text{node} \downarrow \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \end{array} \begin{bmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = \begin{array}{c} \text{node} \downarrow \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \end{array} \begin{bmatrix} i_1 + i_7 \\ -i_1 - i_2 + i_3 \\ i_2 + i_4 - i_6 \\ -i_3 - i_4 + i_5 \\ -i_5 + i_6 - i_7 \end{bmatrix}$$

- (c) Now let's include our knowledge about the current source in our KCL equation. Rewrite $\mathbf{F}^T \vec{i} = \vec{0}$ as $\tilde{\mathbf{F}}^T \tilde{\vec{i}} = -I_s \vec{c}$. A quick note on the notation here: $\tilde{\mathbf{F}}^T$ is \mathbf{F}^T with column \vec{c} removed. Column \vec{c} is the column of \mathbf{F}^T that corresponds to the branch current that has the current source. $\tilde{\vec{i}}$ is the same as the current vector \vec{i} but has the branch current corresponding to the current source removed.

Solution:

In the incidence matrix, \vec{c} is the 6th column of \mathbf{F}^T , so the new KCL equation, with the current source

I_s included, looks like:

$$\tilde{\mathbf{F}}^T \tilde{\mathbf{i}} = \begin{array}{c} \text{component} \rightarrow \\ \text{node} \downarrow \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix} \end{array} \begin{bmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_7 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_7 \end{bmatrix} = \begin{array}{c} \text{node} \downarrow \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix} \end{array} \begin{bmatrix} 0 \\ 0 \\ I_s \\ 0 \\ -I_s \end{bmatrix} = -I_s \vec{c}$$

- (d) SPICE then assigns each node a potential and represents these potentials in a vector \vec{u} . Show that the multiplication $\vec{v} = \mathbf{F}\vec{u}$ represents the voltages across all components in the circuit.

Solution:

$$\mathbf{F} = \begin{array}{c} \text{node} \rightarrow \\ \text{component} \downarrow \\ \begin{matrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{matrix} \end{array} \begin{array}{c} u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{array}{c} \text{component} \downarrow \\ \begin{matrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{matrix} \end{array} \begin{bmatrix} u_1 - u_2 \\ u_3 - u_2 \\ u_2 - u_4 \\ u_3 - u_4 \\ u_4 - u_5 \\ u_5 - u_3 \\ u_1 - u_5 \end{bmatrix} = \begin{array}{c} \text{component} \downarrow \\ \begin{matrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{matrix} \end{array} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}$$

Here, we see our “passive sign convention” in action. Each voltage is the difference in potential from where the current is coming from to where the current is going. Cool!

- (e) Now let’s look at our vector of voltages \vec{v} . First of all, there is a voltage source in our circuit with its voltage corresponding to $u_1 - u_5 = V_s$. Second of all, there are resistors in our circuit. By Ohm’s law, the voltage across a resistor is equal to the current through the resistor multiplied by the resistance, i.e. $v_R = i_R R$. Third of all, we have a current source in our circuit, which means that we cannot immediately say anything about the voltage between u_3 and u_5 (we’ll see later on that this is a-ok). Write a vector \vec{d} that satisfies the equality $\mathbf{F}\vec{u} = \vec{v} = \vec{d}$, where \vec{d} is a vector containing source voltages, products of resistor currents and resistances, and v_{cs} for the voltage across the current source.

Solution:

$$\mathbf{F}\vec{u} = \begin{array}{c} \text{component} \downarrow \\ \begin{matrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{matrix} \end{array} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix} = \begin{array}{c} \text{component} \downarrow \\ \begin{matrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{matrix} \end{array} \begin{bmatrix} i_1 R_1 \\ i_2 R_2 \\ i_3 R_3 \\ i_4 R_4 \\ i_5 R_5 \\ v_{cs} \\ V_s \end{bmatrix}$$

- (f) Now, write the vector \vec{d} as a sum $\vec{d} = \mathbf{R}\tilde{\mathbf{i}} + \vec{v}_s$. In this expression, $\tilde{\mathbf{i}}$ is the vector of all currents in the circuit with the current source current removed, same as before. \vec{v}_s is a vector that only contains voltages across sources (and zeroes everywhere else), and \mathbf{R} is a diagonal matrix with resistances along the diagonal. Rearrange this equation to $\mathbf{F}\vec{u} - \mathbf{R}\tilde{\mathbf{i}} = \vec{v}_s$ and write all the matrices and vectors explicitly.

Note: Be careful with the entries of \mathbf{R} that correspond to the voltage source and the current source. Recall that a voltage source supplies a specified voltage between its terminals regardless of the current through it, and that in \vec{d} we have the entry for the voltage across the current source set to v_{cs} .

Solution:

$$\mathbf{F}\vec{u} = \mathbf{R}\vec{i} + \vec{v}_s$$

$\begin{array}{c} \text{node} \rightarrow \\ \text{component} \downarrow \end{array}$	u_1	u_2	u_3	u_4	u_5		$\begin{array}{c} \text{current} \rightarrow \\ \text{component} \downarrow \end{array}$	i_1	i_2	i_3	i_4	i_5	i_7		$\begin{array}{c} \text{component} \downarrow \end{array}$	
i_1	1	-1	0	0	0	$=$	i_1	R_1	0	0	0	0	0	$+$	i_1	0
i_2	0	-1	1	0	0		i_2	0	R_2	0	0	0	0		i_2	0
i_3	0	1	0	-1	0		i_3	0	0	R_3	0	0	0		i_3	0
i_4	0	0	1	-1	0		i_4	0	0	0	R_4	0	0		i_4	0
i_5	0	0	0	1	-1		i_5	0	0	0	0	R_5	0		i_5	0
i_6	0	0	-1	0	1		i_6	0	0	0	0	0	0		i_6	v_{cs}
i_7	1	0	0	0	-1		i_7	0	0	0	0	0	0		i_7	V_s

This can be rearranged to:

$$\mathbf{F}\vec{u} - \mathbf{R}\vec{i} = \vec{v}_s$$

$\begin{array}{c} \text{node} \rightarrow \\ \text{component} \downarrow \end{array}$	u_1	u_2	u_3	u_4	u_5		$\begin{array}{c} \text{current} \rightarrow \\ \text{component} \downarrow \end{array}$	i_1	i_2	i_3	i_4	i_5	i_7		$\begin{array}{c} \text{component} \downarrow \end{array}$	
i_1	1	-1	0	0	0	$=$	i_1	R_1	0	0	0	0	0	$=$	i_1	0
i_2	0	-1	1	0	0		i_2	0	R_2	0	0	0	0		i_2	0
i_3	0	1	0	-1	0		i_3	0	0	R_3	0	0	0		i_3	0
i_4	0	0	1	-1	0		i_4	0	0	0	R_4	0	0		i_4	0
i_5	0	0	0	1	-1		i_5	0	0	0	0	R_5	0		i_5	0
i_6	0	0	-1	0	1		i_6	0	0	0	0	0	0		i_6	v_{cs}
i_7	1	0	0	0	-1		i_7	0	0	0	0	0	0		i_7	V_s

- (g) At this point, you have two matrix equations (from parts (f) and (c)) in terms of two unknown vectors: the potentials \vec{u} and the remaining currents not defined by the current source \vec{i} . Combine these two matrix equations into a single, giant system of equations $\mathbf{A}\vec{x} = \vec{b}$. \vec{b} should be a vector whose only non-zero entries are the source voltages (V_s and v_{cs}) and the current of the current source I_s (pay attention to the signs). \vec{x} should be a vector that combines \vec{u} and \vec{i} . The formats of the different elements are:

$$\mathbf{A} = \begin{bmatrix} \text{Left-hand side from part (f)} \\ \text{Left-hand side from part (c)} \end{bmatrix}, \vec{x} = \begin{bmatrix} \vec{u} \\ \vec{i} \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} \text{Right-hand side from part (f)} \\ \text{Right-hand side from part (c)} \end{bmatrix}$$

Solution:

\mathbf{A}

$$\vec{x} = \vec{b}$$

$$\left[\begin{array}{c|c} \mathbf{F} & -\mathbf{R} \\ \hline \mathbf{0} & \mathbf{F}^T \end{array} \right]$$

$$\vec{x} = \vec{b}$$

unkonwn → part ↓	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3	i_4	i_5	i_7	u_1	u_2	u_3	u_4	u_5	i_1	i_2	i_3
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$$\hat{\mathbf{A}} \hat{\mathbf{x}} = \vec{b}$$

unkown → part ↓	u_1	u_2	u_3	u_4	i_1	i_2	i_3	i_4	i_5	i_7		part	
(f)	1	-1	0	0	$-R_1$	0	0	0	0	0	$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_{cs} \\ V_s \\ 0 \\ 0 \\ I_s \\ 0 \end{bmatrix}$	(f)	0
(f)	0	-1	1	0	0	$-R_2$	0	0	0	0		(f)	0
(f)	0	1	0	-1	0	0	$-R_3$	0	0	0		(f)	0
(f)	0	0	1	-1	0	0	0	$-R_4$	0	0		(f)	0
(f)	0	0	0	1	0	0	0	0	$-R_5$	0		(f)	0
(f)	0	0	-1	0	0	0	0	0	0	0		(f)	v_{cs}
(f)	1	0	0	0	0	0	0	0	0	0		(f)	V_s
(c)	0	0	0	0	1	0	0	0	0	1		(c)	0
(c)	0	0	0	0	-1	-1	1	0	0	0		(c)	0
(c)	0	0	0	0	0	1	0	1	0	0		(c)	I_s
(c)	0	0	0	0	0	0	-1	-1	1	0	(c)	0	
(c)	0	0	0	0	0	0	0	0	-1	-1	(c)	$-I_s$	

- (i) At this point we should have 10 unknowns and 12 equations. That is, we have 2 more equations than unknowns. Therefore, we have the freedom to get rid of two equations that are redundant to create a new system of linear equations $\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{b}}$. This is good news because we remember having one equation from part (f) that was particularly hard to solve. That is the equation corresponding to the voltage across the current source v_{cs} . We will eliminate this equation by taking its corresponding row from $\hat{\mathbf{A}}$ and its corresponding right-hand side (v_{cs}) from \vec{b} . We are allowed to eliminate another equation.² We note that the sum of all equations from part (c) sum to $0 = 0$, so that one of them can be taken out (this will always be the case because the rows of \mathbf{A}^T are the columns of \mathbf{A} and they always sum to $\vec{0}$, as discussed earlier). Choose an equation from these to eliminate and remove its corresponding row from $\hat{\mathbf{A}}$ and its corresponding right-hand side from \vec{b} . After these edits, write the new system of linear equations $\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{b}}$ explicitly.

Solution:

$$\hat{\mathbf{A}} \hat{\mathbf{x}} = \hat{\mathbf{b}}$$

unkown → part ↓	u_1	u_2	u_3	u_4	i_1	i_2	i_3	i_4	i_5	i_7		part	
(f)	1	-1	0	0	$-R_1$	0	0	0	0	0	$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_s \\ 0 \\ 0 \\ 0 \\ I_s \\ 0 \end{bmatrix}$	(f)	0
(f)	0	-1	1	0	0	$-R_2$	0	0	0	0		(f)	0
(f)	0	1	0	-1	0	0	$-R_3$	0	0	0		(f)	0
(f)	0	0	1	-1	0	0	0	$-R_4$	0	0		(f)	0
(f)	0	0	0	1	0	0	0	0	$-R_5$	0		(f)	0
(f)	1	0	0	0	0	0	0	0	0	0		(f)	V_s
(c)	0	0	0	0	1	0	0	0	0	1		(c)	0
(c)	0	0	0	0	-1	-1	1	0	0	0		(c)	0
(c)	0	0	0	0	0	1	0	1	0	0		(c)	I_s
(c)	0	0	0	0	0	0	-1	-1	1	0		(c)	0

- (j) Now given the following values: $I_s = 5 \text{ mA}$, $V_s = 5 \text{ V}$, $R_1 = 4 \text{ k}\Omega$, $R_2 = 3 \text{ k}\Omega$, $R_3 = 5 \text{ k}\Omega$, $R_4 = 2 \text{ k}\Omega$, and $R_5 = 6 \text{ k}\Omega$, solve for all non-reference potentials and all currents (you may use IPython for this part).

Solution:

We first plug in the values of the resistors and the sources into our system of linear equations to get:

²We actually don't have to take a second equation out at this point. Instead, we can solve using Gaussian elimination. However, we will take a second equation out, so that we are able to invert the resultant matrix $\hat{\mathbf{A}}$ to find the unique solution in our circuit (unique because we selected a reference node).

$$\hat{\mathbf{A}} \hat{\mathbf{x}} = \hat{\mathbf{b}}$$

<small>unkonwn → part ↓</small>	u_1	u_2	u_3	u_4	i_1	i_2	i_3	i_4	i_5	i_7		<small>part</small>	
(f)	1	-1	0	0	-4000	0	0	0	0	0	$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$	(f)	0
(f)	0	-1	1	0	0	-3000	0	0	0	0		(f)	0
(f)	0	1	0	-1	0	0	-5000	0	0	0		(f)	0
(f)	0	0	1	-1	0	0	0	-2000	0	0		(f)	0
(f)	0	0	0	1	0	0	0	0	-6000	0	$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_7 \end{bmatrix}$	(f)	0
(f)	1	0	0	0	0	0	0	0	0	0		(f)	5
(c)	0	0	0	0	1	0	0	0	0	1		(c)	0
(c)	0	0	0	0	-1	-1	1	0	0	0		(c)	0
(c)	0	0	0	0	0	1	0	1	0	0	(c)	$\frac{5}{1000}$	
(c)	0	0	0	0	0	0	-1	-1	1	0	(c)	0	

We use IPython to find the solution $\hat{\mathbf{x}} = \hat{\mathbf{A}}^{-1} \hat{\mathbf{b}}$ and get

$$\hat{\mathbf{x}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_7 \end{bmatrix} = \begin{bmatrix} 5 \text{ V} \\ \frac{73}{5} \text{ V} \\ \frac{106}{5} \text{ V} \\ \frac{78}{5} \text{ V} \\ -\frac{3}{1250} \text{ A} \\ \frac{11}{5000} \text{ A} \\ -\frac{1}{5000} \text{ A} \\ \frac{7}{2500} \text{ A} \\ \frac{13}{5000} \text{ A} \\ \frac{3}{1250} \text{ A} \end{bmatrix} = \begin{bmatrix} 5 \text{ V} \\ 14.6 \text{ V} \\ 21.2 \text{ V} \\ 15.6 \text{ V} \\ -2.4 \text{ mA} \\ 2.2 \text{ mA} \\ -200 \mu\text{A} \\ 2.8 \text{ mA} \\ 2.6 \text{ mA} \\ 2.4 \text{ mA} \end{bmatrix}$$

- (k) **PRACTICE:** We showed in this problem, so far, that the SPICE method works for the given circuit. Starting from this part, we will show that the methods followed in this problem will always work (assuming the circuit doesn't have contradictions such as a voltage source in a short circuit, a current source in an open circuit, voltage sources in parallel or current sources in series). From this point onward in this problem, we will assume that we have a circuit with a total of M components (resistors, independent voltage sources and independent current sources), M_c of which are current sources. Moreover, we will assume that we have N nodes in the circuit. What are the dimensions of the incidence matrix \mathbf{F} ?

Solution:

$$\mathbf{F} \in \mathbb{R}^{M \times N}$$

- (l) **PRACTICE:** If we were to eliminate the variables in \vec{i} for current sources and move their corresponding columns from \mathbf{F}^T in $\mathbf{F}^T \vec{i} = \vec{0}$ to the right-hand side to get $\tilde{\mathbf{F}}^T \vec{i} = -(I_{s_1} \vec{c}_1 + \dots + I_{s_{M_c}} \vec{c}_{M_c}) \triangleq \vec{c}^*$ (where \vec{c}_i is the column of \mathbf{F}^T corresponding to the current sources I_{s_i}), similar to part (c), what would be the dimensions of the matrix $\tilde{\mathbf{F}}^T$?

Solution:

$\mathbf{F}^T \in \mathbb{R}^{N \times M}$, so if we eliminate M_c columns from it (corresponding to the current sources), we are left with $\tilde{\mathbf{F}}^T \in \mathbb{R}^{N \times (M - M_c)}$.

- (m) **PRACTICE:** If we were to combine the two systems of linear equations $\mathbf{F} \vec{u} - \mathbf{R} \vec{i} = \vec{v}_s$ (similar to part (b)) and the system $\tilde{\mathbf{F}}^T \vec{i} = \vec{c}^*$ (from last part) into one big system of linear equations $\mathbf{A} \vec{x} = \vec{b}$ (similar to part (g)), what would be the dimensions of the matrix \mathbf{A} ?

We can construct the matrix \mathbf{A} as follows (block matrix representation):

This will indeed satisfy

$$\left[\begin{array}{c|c} \mathbf{F} & \mathbf{R} \\ \hline \mathbf{0} & \widetilde{\mathbf{F}}^T \end{array}\right] \left[\begin{array}{c} \vec{u} \\ \vdots \\ \vdots \\ \widetilde{i} \end{array}\right] = \left[\begin{array}{c} \vec{v}_s \\ \vdots \\ \vdots \\ \vec{c}^* \end{array}\right]$$

This is because $\mathbf{F} \in \mathbb{R}^{M \times N}$, $\tilde{\mathbf{F}}^T \in \mathbb{R}^{N \times (M-M_c)}$ and $\mathbf{R} \in \mathbb{R}^{M \times (M-M_c)}$.

- Solution:**

The matrix $\hat{\mathbf{A}}$ can be created from \mathbf{A} by eliminating one column. Therefore, $\hat{\mathbf{A}} \in \mathbb{R}^{(M+N) \times (M-M_c+N-1)}$.

- (o) **PRACTICE:** Given the new dimensions of $\hat{\mathbf{A}}$, how many equations can we eliminate without risking removing linearly independent equations? Which equations would you remove (similar to part (i))? After removing these equations, would the resultant system $\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{b}}$ have a square matrix $\hat{\mathbf{A}}$?

Since $\hat{\mathbf{A}} \in \mathbb{R}^{(M+N) \times (M-M_c+N-1)}$ ($M+N$ equations and $M-M_c+N-1$ unknowns), we can remove M_c+1 equations without the risk of losing information.

We would remove one equation per current source from the equations with \vec{v}_s on their right-hand side (where they correspond to the voltages across the current sources). In addition we would remove one equation from the amended KCL equations $\tilde{\mathbf{F}}^T \tilde{\mathbf{i}} = \tilde{\mathbf{c}}^*$.

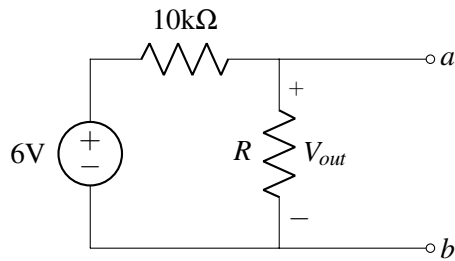
Now we get $\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{b}}$ similar to part (i), where $\hat{\mathbf{A}}$ is a square matrix of dimensions $(M - M_c + N - 1) \times (M - M_c + N - 1)$.

We see that we can always remove all the equations corresponding to the current sources from our system of linear equations without risking losing information.

9. Resistive Voltage “Regulator”

In this problem, we will design a circuit that provides an approximately constant voltage divider across a range of loads. We will use a resistive voltage divider circuit. The goal is to design a circuit that, from a source voltage of 6V, would yield an output voltage within 5% of 4V for loads in the range of 1k Ω to 100k Ω .

- (a) First, consider the resistive voltage divider in the following circuit. What resistance R would achieve a voltage V_{out} of 4V?

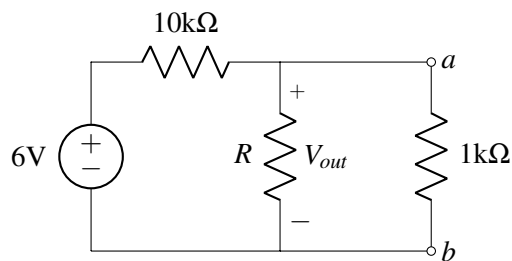


Solution:

The current in the circuit is $I = \frac{6V}{R+10k\Omega}$ which means that the output voltage can be calculated as $V_{out} = I \cdot R = 6V \cdot \frac{R}{R+10k\Omega}$. By constraining $V_{out} = 4V$, we get

$$\begin{aligned}\frac{4V}{6V} &= \frac{R}{R+10k\Omega} \\ \frac{4V}{6V} (R+10k\Omega) &= R \\ \frac{4V}{6V} \cdot 10k\Omega &= R \left(1 - \frac{4V}{6V}\right) \\ \frac{4V}{6V \left(1 - \frac{4V}{6V}\right)} \cdot 10k\Omega &= R \\ \frac{4V}{6V - 4V} \cdot 10k\Omega &= R \\ 20.00k\Omega &= R\end{aligned}$$

- (b) Now using the same resistor R as calculated in part (a), consider loading the circuit with a resistor of $1k\Omega$ as depicted in the following circuit. What is the voltage V_{out} now?

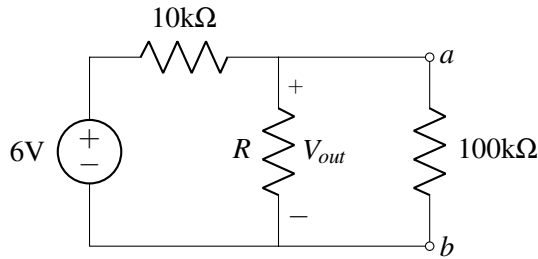


Solution:

The value from part (a) is $R = 20.00k\Omega$. This means that the effective resistance is $(20.00k\Omega \parallel 1k\Omega) = \frac{20.00 \cdot 1}{20.00 + 1} = 0.95k\Omega$. The current is therefore $I = \frac{6V}{10k\Omega + 0.95k\Omega}$, which yields:

$$\begin{aligned}V_{out} &= \frac{6V \cdot 0.95k\Omega}{10k\Omega + 0.95k\Omega} \\ V_{out} &= 0.52V\end{aligned}$$

- (c) Now using the same resistor R as calculated in part (a), consider loading the circuit with a resistor of $100k\Omega$, instead, as depicted in the following circuit. What is the voltage V_{out} now?



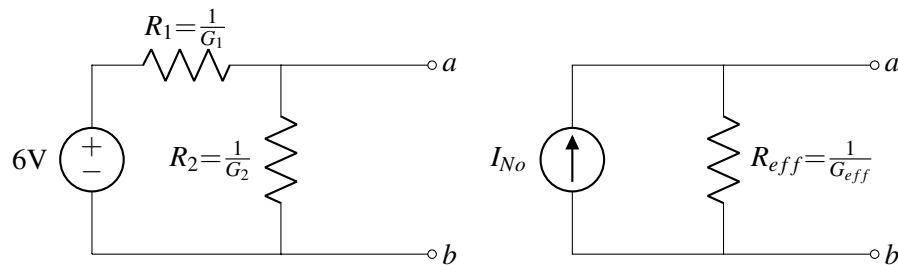
Solution:

The value from part (a) is $R = 20.00\text{k}\Omega$. This means that the effective resistance is $(20.00\text{k}\Omega \parallel 100\text{k}\Omega) = \frac{20.00 \cdot 100}{20.00 + 100} = 16.67\text{k}\Omega$. The current is therefore $I = \frac{6\text{V}}{10\text{k}\Omega + 16.67\text{k}\Omega}$, which yields:

$$V_{out} = \frac{6\text{V} \cdot 16.67\text{k}\Omega}{10\text{k}\Omega + 16.67\text{k}\Omega}$$

$$V_{out} = 3.75\text{V}$$

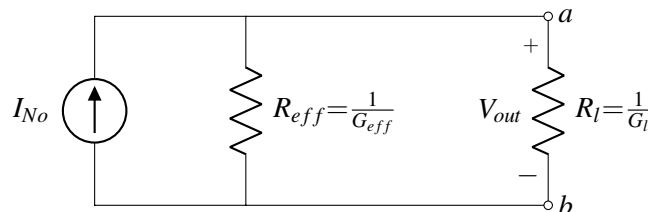
- (d) Now we would like to design a divider that would keep the voltage V_{out} regulated for loads for a range of loads R_l . By that, we would like the voltage to remain within a 5% window of 4V. That is, we would like to design the following circuit such that $3.80\text{V} \leq V_{out} \leq 4.20\text{V}$ for a range of loads R_l . As a first step, what is the Norton equivalent of the circuit on the left? Write I_{No} and G_{eff} in terms of conductance values $G_1 = \frac{1}{R_1}$ and $G_2 = \frac{1}{R_2}$.



Solution:

In open circuit, the current through the resistors is $I = \frac{6\text{V}}{R_1 + R_2}$. This means that the open circuit voltage is $V_{Th} = I \cdot R_2 = 6\text{V} \cdot \frac{R_2}{R_1 + R_2}$. We now calculate the short circuit current I_{No} . In short circuit, the current is simply $I = \frac{6\text{V}}{R_1}$. Therefore, the effective resistance is $R_{eff} = \frac{V_{Th}}{I_{No}} = \frac{6\text{V} \cdot \frac{R_2}{R_1 + R_2}}{\frac{6\text{V}}{R_1}} = \frac{R_1 \cdot R_2}{R_1 + R_2} = (R_1 \parallel R_2)$. From this, we can write $I_{No} = 6\text{V} \cdot G_1$ and $G_{eff} = G_1 + G_2$.

- (e) For the second step, using the Norton equivalent circuit you found in part (d), what is the range of G_{eff} that achieves $3.80\text{V} \leq V_{out} \leq 4.20\text{V}$ in terms of I_{No} and G_l ?



Solution:

From Ohm's law, $V_{out} \cdot (G_{eff} + G_l) = I_{No}$. This means that $V_{out} = \frac{I_{No}}{G_{eff} + G_l}$. From the bound $3.80V \leq V_{out} \leq 4.20V$, we have $3.80V \leq \frac{I_{No}}{G_{eff} + G_l} \leq 4.20V$. This yields the following two inequalities:

$$3.80V \leq \frac{I_{No}}{G_{eff} + G_l}$$

$$\frac{I_{No}}{G_{eff} + G_l} \leq 4.20V,$$

which is equivalent to

$$G_{eff} + G_l \leq \frac{I_{No}}{3.80V}$$

$$\frac{I_{No}}{4.20V} \leq G_{eff} + G_l,$$

which is equivalent to

$$G_{eff} \leq \frac{I_{No}}{3.80V} - G_l$$

$$\frac{I_{No}}{4.20V} - G_l \leq G_{eff},$$

or concisely

$$\frac{I_{No}}{4.20V} - G_l \leq G_{eff} \leq \frac{I_{No}}{3.80V} - G_l.$$

- (f) Translate the range of G_{eff} in terms of I_{No} and G_l (that you found in part (e)) into a range on G_2 in terms of G_1 and G_l .

Solution:

By plugging in the values of G_{eff} and I_{No} into $\frac{I_{No}}{4.20V} - G_l \leq G_{eff} \leq \frac{I_{No}}{3.80V} - G_l$, we get

$$\frac{6VG_1}{4.20V} - G_l \leq G_1 + G_2 \leq \frac{6VG_1}{3.80V} - G_l,$$

or equivalently

$$\frac{6VG_1}{4.20V} - G_l - G_1 \leq G_2 \leq \frac{6VG_1}{3.80V} - G_l - G_1,$$

which is equivalent to

$$G_1 \left(\frac{6V}{4.20V} - 1 \right) - G_l \leq G_2 \leq G_1 \left(\frac{6V}{3.80V} - 1 \right) - G_l.$$

- (g) Say we want to support loads in the range $1k\Omega \leq R_l \leq 100k\Omega$ with approximately constant voltage as described above (that is, $3.80V \leq V_{out} \leq 4.20V$). What is the range of G_2 in terms of G_1 now? Translate the range of G_2 in terms of G_1 into a range of R_2 in terms of R_1 .

Solution:

Note: The unit of conductance is Siemens and is denoted by S.

We plug $R_l = 1k\Omega$ (equivalently $G_l = \frac{1}{1}mS$) into the bound found above and get

$$G_1 \left(\frac{6V}{4.20V} - 1 \right) - \frac{1}{1}mS \leq G_2 \leq G_1 \left(\frac{6V}{3.80V} - 1 \right) - \frac{1}{1}mS.$$

Similarly, we plug $R_l = 100\text{k}\Omega$ (equivalently $G_l = \frac{1}{100}\text{mS}$) into the bound found above and get

$$G_1 \left(\frac{6\text{V}}{4.20\text{V}} - 1 \right) - \frac{1}{100}\text{mS} \leq G_2 \leq G_1 \left(\frac{6\text{V}}{3.80\text{V}} - 1 \right) - \frac{1}{100}\text{mS}.$$

The intersection of the above two sets of inequalities is

$$G_1 \left(\frac{6\text{V}}{4.20\text{V}} - 1 \right) - \frac{1}{100}\text{mS} \leq G_2 \leq G_1 \left(\frac{6\text{V}}{3.80\text{V}} - 1 \right) - \frac{1}{1}\text{mS}$$

In terms of R_2 that range is (by taking the reciprocal of *all* sides and using $G = \frac{1}{R}$)

$$\frac{1}{\frac{1}{R_1} \left(\frac{6\text{V}}{4.20\text{V}} - 1 \right) - \frac{1}{100\text{k}\Omega}} \geq R_2 \geq \frac{1}{\frac{1}{R_1} \left(\frac{6\text{V}}{3.80\text{V}} - 1 \right) - \frac{1}{1\text{k}\Omega}}.$$

- (h) Note that conductance is always non-negative. From the bounds on G_2 you found in the previous part, derive a bound on G_1 that ensures that G_2 is always non-negative and non-empty (that is, the whole range of possible G_2 values is non-negative and is not empty). Translate this range into a range of possible R_1 values.

Hint: In addition to the conductance being non-negative, also make sure that the range for G_2 is non-empty.

Solution:

Since conductance has to be positive, we need the range above to consist of positive values (that is, the lower bound has to be positive and the upper bound is larger than the lower bound) which yields:

$$0 \leq G_1 \left(\frac{6\text{V}}{4.20\text{V}} - 1 \right) - \frac{1}{100}\text{mS}$$

and

$$G_1 \left(\frac{6\text{V}}{4.20\text{V}} - 1 \right) - \frac{1}{100}\text{mS} \leq G_1 \left(\frac{6\text{V}}{3.80\text{V}} - 1 \right) - \frac{1}{1}\text{mS},$$

which is equivalent to

$$\frac{1}{100}\text{mS} \leq G_1 \left(\frac{6\text{V}}{4.20\text{V}} - 1 \right)$$

and

$$\frac{1}{1}\text{mS} - \frac{1}{100}\text{mS} \leq G_1 \left(\frac{6\text{V}}{3.80\text{V}} - \frac{6\text{V}}{4.20\text{V}} \right),$$

which is equivalent to

$$0.0233\text{mS} = \frac{1}{\frac{6\text{V}}{4.20\text{V}} - 1} \cdot \frac{1}{100}\text{mS} \leq G_1$$

and

$$6.5835\text{mS} = \frac{\frac{1}{1}\text{mS} - \frac{1}{100}\text{mS}}{\frac{6\text{V}}{3.80\text{V}} - \frac{6\text{V}}{4.20\text{V}}} \leq G_1.$$

The intersection of both ranges yields

$$6.5835\text{mS} \leq G_1,$$

which in terms of R_1 is equivalent to

$$R_1 \leq 0.15189\text{k}\Omega$$

- (i) Pick the values of R_1 and R_2 that achieve $3.80\text{V} \leq V_{out} \leq 4.20\text{V}$ for $1\text{k}\Omega \leq R_L \leq 100\text{k}\Omega$ while minimizing the power consumed by the voltage divider circuit in open circuit (when there is no load attached to the output). What are these values R_1 and R_2 ? How much power is consumed in this case? Calculate and report this power consumption using both the original circuit and the Norton equivalent circuit. Are the power you calculated using the original circuit and the power you calculated using the Norton equivalent circuit equal?

Solution:

The power in the original circuit (no load) can be calculated as $\frac{V^2}{R_1+R_2}$. In order to minimize the power, we should pick the largest possible $R_1 + R_2$. This can be achieved by choosing $R_1 = 0.15189\text{k}\Omega$ (the highest possible) and $R_2 = \frac{1}{\frac{1}{0.15189\text{k}\Omega} \left(\frac{6\text{V}}{4.20\text{V}} - 1 \right) - \frac{1}{100\text{k}\Omega}} = 0.35567\text{k}\Omega$ (the upper bound). The power in this case is $P = \frac{(6\text{V})^2}{0.15189\text{k}\Omega + 0.35567\text{k}\Omega} = 70.9276\text{mW}$. If we use the Norton equivalent circuit, we have $R_{eff} = 0.1064\text{k}\Omega$ which yields the power consumption $P = I_{No}^2 \cdot R_{eff} = 166.0861\text{mW}$. The powers are not equal since Thévenin and Norton equivalents don't preserve power consumption of the circuit (think about the Thévenin equivalent in open circuit: the power consumption is always 0W).

- (j) **PRACTICE:** Now using the same values R_1 and R_2 from the previous part, load the circuit with a load of $51\text{k}\Omega$. How much power is consumed by each of the three resistors, R_1 , R_2 and R_L (use the original circuit to compute the power)?

Solution:

The effective resistance in the output is $(R_2 \parallel R_L)$ which is equal to $0.3532\text{k}\Omega$. This means that the current from the source is $I = \frac{6\text{V}}{R_1 + (R_2 \parallel R_L)} = 11.8789\text{mA}$. Therefore, the power consumed by R_1 is $P_{R_1} = I^2 \cdot R_1 = 21.4330\text{mW}$. The output voltage is $V_{out} = I \cdot (R_2 \parallel R_L) = 4.1957\text{V}$, which means the power consumed by R_2 is $P_{R_2} = \frac{V_{out}^2}{R_2} = 49.4953\text{mW}$ and the power consumed by R_L is $P_{R_L} = \frac{V_{out}^2}{R_L} = 0.3452\text{mW}$ for a total power of $P = 71.2735\text{mW}$.

General note: Try doing the whole problem using resistance directly without going through conductances. You will see that the math will not be as "pretty" as the one presented here (using conductances). Sometimes using conductance yields easier derivation than the derivation using resistance.

10. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Working in groups of 3-5 will earn you credit for your participation grade.

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.