This homework is due February 16, 2015, at Noon.

Optional Problems: We **do not** grade these problems. Nevertheless, you are responsible for learning the subject matter within their scope.
Bonus Problems: We **do** grade these problems. Doing them will provide an unspecified amount of extra credit; not doing them will not affect your homework grade negatively. We will specify if the problem is in or out of scope.

1. **Homework process and study group**

Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework?
Working in groups of 3-5 will earn credit for your participation grade.

Solution: I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on Problem 5 so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.

2. **Mechanical Inverses (Optional)**

For each of the following matrices, state whether the inverse exists. If so, find the inverse. If not, prove that no inverse exists. **Solve these by hand, do NOT use numpy!**

(a) $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ (g) $\begin{bmatrix} -1 & 1 & -0.5 \\ 1 & 1 & -0.5 \\ 0 & 1 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Solution:
Write an augmented matrix with the identity matrix on the right, and use elementary row operations until the original matrix on the left becomes the identity matrix.

(a)

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \qquad \text{R1-}\tfrac{1}{2}\text{R2} \rightarrow \text{R1} \left[\begin{array}{cc|cc} 0 & 1 & 1 & -\frac{1}{2} \\ 2 & 0 & 0 & 1 \end{array}\right]$$

$$\tfrac{1}{2}\text{R2} \rightarrow \text{R1, R1} \rightarrow \text{R2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right] \qquad \text{Inverse exists: } \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

(b)

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \qquad \text{R1+2R2} \to \text{R1} \left[\begin{array}{cc|cc} 5 & 0 & 1 & 2 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

$$\tfrac{1}{5}\text{R1} \to \text{R1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & \frac{2}{5} \\ 1 & -1 & 0 & 1 \end{array}\right] \qquad \text{-R2+R1} \to \text{R2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & \frac{1}{5} & -\frac{3}{5} \end{array}\right]$$

$$\text{Inverse exists} \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{-3}{5} \end{bmatrix}$$

(c)

$$\left[\begin{array}{cc|cc} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 1 \end{array}\right] \qquad \text{R2+}\sqrt{3}\text{R1} \to \text{R2} \left[\begin{array}{cc|cc} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 & 0 \\ 0 & -2 & \sqrt{3} & 1 \end{array}\right]$$

$$\text{-2R1} \to \text{R1, -}\tfrac{1}{2}\text{R2} \to \text{R2} \left[\begin{array}{cc|cc} 1 & \sqrt{3} & -2 & 0 \\ 0 & 1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{array}\right] \qquad \text{R1-}\sqrt{3}\text{R3} \to \text{R1} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{array}\right]$$

$$\text{Inverse exists} \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

(d)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \qquad 0.5\text{R2} \to \text{R2, } 0.5\text{R3} \to \text{R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.5 \end{array}\right]$$

$$\text{Inverse exists} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

(e)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \qquad \text{R2+R3} \rightarrow \text{R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\text{R3} \rightarrow \text{R2, R2} \rightarrow \text{R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \qquad \text{Inverse exists} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(f) Inverse does not exist. $v_3 = -v_2$. Since the columns of the matrix are not linearly independent, the inverse does not exist.

(g)

$$\left[\begin{array}{ccc|ccc} -1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \qquad \text{R1+R2} \rightarrow \text{R1} \left[\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 1 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\text{R1-2R3} \rightarrow \text{R1} \left[\begin{array}{ccc|ccc} 0 & 0 & -3 & 1 & 1 & -2 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \qquad \text{R3=R3+}\tfrac{1}{3}\text{R1} \left[\begin{array}{ccc|ccc} 0 & 0 & -3 & 1 & 1 & -2 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right]$$

$$\text{R2-R3} \rightarrow \text{R2} \left[\begin{array}{ccc|ccc} 0 & 0 & -3 & 1 & 1 & -2 \\ 1 & 0 & -\frac{1}{2} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \qquad -\tfrac{1}{3}\text{R1} \rightarrow \text{R1} \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -\frac{1}{2} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right]$$

$$\text{R2+}\tfrac{1}{2}\text{R1} \rightarrow \text{R2} \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right]$$

$$\text{R2} \rightarrow \text{R1, R3} \rightarrow \text{R2, R1} \rightarrow \text{R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

$$\text{Inverse exists.} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(h) Inverse does not exist. $v_1 + v_2 + v_3 = v_4$. Since the columns of the matrix are not linearly independent, the inverse does not exist.

3. Cubic Polynomials (Optional)

Consider the set of real-valued canonical polynomials given below:

$$\varphi_0(t) = 1 \quad \varphi_1(t) = t \quad \varphi_2(t) = t^2 \quad \varphi_3(t) = t^3,$$

where $t \in [a,b] \subset \mathbb{R}$.

(a) Show that the set of all cubic polynomials

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3,$$

where $t \in [a,b]$ and the coefficients p_k are real scalars, forms a vector space. What is the dimension of this vector space? Explain.

Solution:

No escape (scaling) property:
$\alpha p(t) = \alpha p_0 + \alpha p_1 t + \alpha p_2 t^2 + \alpha p_3 t^3$
The function above is, itself, a cubic polynomial, so the no escape property holds.

No escape (addition) property:
Let us define two cubic polynomials:
$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$
$q(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3$
$p(t) + q(t) = (p_0 + q_0) + (p_1 + q_1)t + (p_2 + q_2)t^2 + (p_3 + q_3)t^3$
The function above is also a cubic polynomial, so this no escape property also holds.

It can be trivially shown from the properties of scalar multiplication and addition that the remaining properties of vector spaces hold for the set of all p(t) :

Commutativity: $p(t) + q(t) = q(t) + p(t)$
Associativity of vector addition: $(p(t) + q(t)) + r(t) = p(t) + (q(t) + r(t))$
Additive identity: There exists 0 in the set of all p(t) such that for all $p(t)$, $0 + p(t) = p(t) + 0 = p(t)$
Existence of inverse: For every p(t), there is element $-p(t)$ such that $p(t) + -p(t)$=0
Associativity of scalar multiplication: $c(d(p(t))) = (cd)p(t)$
Distributivity of scalar sums: $(c+d)p(t) = cp(t) + dp(t)$
Distributivity of vector sums: $c(p(t) + q(t)) = cp(t) + cq(t)$
Scalar multiplication identity: There is $1p(t) = p(t)$

Since all properties of a vector space hold, the set of cubic polynomials is a vector space. The dimension of this space is 4 because there are 4 degrees of freedom that can be varied.

(b) Show that every real-valued cubic polynomial

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$

defined over the interval $[a,b]$ can be written as a linear combination of the canonical polynomials $\varphi_0(t)$, $\varphi_1(t)$, $\varphi_2(t)$, and $\varphi_3(t)$. In particular, show that

$$p(t) = \vec{c}^{\mathsf{T}} \vec{\varphi}(t),$$

where

$$\vec{c}^{\mathsf{T}} = [c_0\ c_1\ c_2\ c_3]$$

is a vector of appropriately chosen coefficients, and

$$\varphi(t) = \begin{bmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{bmatrix} = \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}.$$

Solution: $\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = c_0 + c_1 t + c_2 t^2 + c_3 t^3$

(c) Determine the coefficients c_k for $k = 0, 1, 2, 3$. That is to say, for the setup above determine what values would be in the c vector.

Solution: Comparing $p(t)$ to $\vec{c}^T \vec{\varphi}$, we see:

$c_0 = p_0\ c_1 = p_1\ c_2 = p_2\ c_3 = p_3$

(d) True or False? The canonical polynomials $\varphi_k(t) = t^k$, for $k = 0, 1, 2, 3$, constitute a basis for the vector space of real-valued cubic polynomials defined over the interval $[a, b]$. Briefly justify your answer

Solution: True, the canonical polynomials constitute a basis. This is because they are linearly independent and span the space of possible polynomials.

(e) A curve is a continuous mapping from the real line to $\mathbb{R}^N$. A cubic Bézier curve—used extensively in computer graphics—is a type of curve that uses as a basis the following special subset of what are more broadly known as *Bernstein polynomials*:

$$\beta_0(t) = (1-t)^3, \quad \beta_1(t) = 3t(1-t)^2, \quad \beta_2(t) = 3t^2(1-t), \quad \text{and } \beta_3(t) = t^3.$$

Prove that the Bernstein polynomials $\beta_k(t)$ defined above do indeed form a basis. To do this, show that any real-valued polynomial

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$

can be expressed as a linear combination of the polynomials $\beta_k(t)$, and determine the coefficients in that linear combination. In particular, determine the coefficients in the expansion

$$p(t) = \widehat{p}_0\, \beta_0(t) + \widehat{p}_1 \beta_1(t) + \widehat{p}_2 \beta_2(t) + \widehat{p}_3 \beta_3(t)$$

.

Hint: Determine a matrix R such that

$$\vec{\beta}(t) = R\, \vec{\varphi}(t),$$

where

$$\vec{\beta}(t) = \begin{bmatrix} \beta_0(t) \\ \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \end{bmatrix} = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}$$

and show that R is invertible. Determine its inverse R^{-1}, from which you can determine the coefficients $\widehat{p}_k$.

Solution: Let us define the vectors $\vec{p} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$ and $\widehat{\vec{p}} = \begin{bmatrix} \widehat{p_0} \\ \widehat{p_1} \\ \widehat{p_2} \\ \widehat{p_3} \end{bmatrix}$. We can find $\widehat{\vec{p}}$ from $\vec{p}$ as follows:

$$\widehat{\vec{p}}^T \vec{\beta} = \vec{p}^T \varphi$$
$$\widehat{\vec{p}}^T \vec{\beta} = \vec{p}^T R^{-1} \vec{\beta}$$
$$\widehat{\vec{p}}^T = \vec{p}^T R^{-1}$$

We can find the matrix R from the definitions of φ_k and β_k.

$$\begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix} = \begin{bmatrix} 1-3t+3t^2-t^3 \\ 3t-6t^2+3t^3 \\ 3t^2-3t^3 \\ t^3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

The above matrix, R, is clearly invertible since there is a pivot in each column. We proceed to find the inverse:

$$\left[\begin{array}{cccc|cccc} 1 & -3 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -6 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\tfrac{1}{3}\text{R3+R4} \to \text{R3} \quad \left[\begin{array}{cccc|cccc} 1 & -3 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -6 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\tfrac{1}{3}\text{R2+2R3} \to \text{R2} \quad \left[\begin{array}{cccc|cccc} 1 & -3 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\text{R2-R4} \to \text{R2} \quad \left[\begin{array}{cccc|cccc} 1 & -3 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\text{R1+3R2} \to \text{R1} \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & -1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\text{R1+R4} \to \text{R1} \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\text{R1-3R3} \to \text{R1} \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\vec{p}^T R^{-1} = \begin{bmatrix} p_0 \\ p_0 + \frac{1}{3} p_1 \\ p_0 + \frac{2}{3} p_1 \\ \frac{1}{3} p_2 \end{bmatrix}$$

Thus,

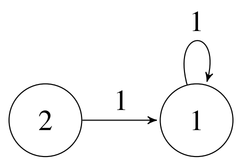
$\widehat{p_0} = p_0$

$\widehat{p_1} = p_0 + \frac{1}{3} p_1$

$\widehat{p_2} = p_0 + \frac{2}{3} p_1 + \frac{1}{3} p_2$

$\widehat{p_3} = p_0 + p_1 + p_2 + p_3$

4. Pumps Properties Proofs Part 2 (Optional)



Figure

(a) Suppose we have a pump setup as in Figure , with associated matrix A. Write out the state transition matrix A.

$$\begin{bmatrix} \quad A \quad \end{bmatrix} \begin{bmatrix} x[n] \end{bmatrix} = \begin{bmatrix} x[n+1] \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

(b) Suppose the reservoirs are initialized to have the following water levels: $x_1[0] = 0.5$, $x_2[0] = 0.5$. In a completely alternate universe the pools are initialized to have the following water levels: $x_1[0] = 0.3$, $x_2[0] = 0.7$. For both initial states, what are the water levels at timestep 1 ($x_1[1]$, $x_2[1]$)?

Solution:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} = \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(c) If you observe the pools at timestep 1, can you figure out what the initial ($x_1[0]$, $x_2[0]$) water levels were? Why or why not?

Solution:

No, at timestep 1 configuration 1 and 2 are indistinguishable. This implies the experiment inherently has linear dependence.

(d) Now generalize: if there exists a state transition matrix where two different initial state vectors lead to the same water levels/state vectors at a timestep in the future, can you recover the initial water levels? Prove your answer.

Solution: We are given that two different initial states, $x[0]$ and $y[0]$ lead to the same resulting state $x[1]$.

$A\vec{x}[0] = \vec{x}[1]$

$A\vec{y}[0] = \vec{x}[1]$

If we can recover the initial water levels, this would mean that there is some operation that can be performed on the vector representing the resulting state to yield the previous state. This operation would, by definition, be the inverse of the matrix A.

$\vec{x}[0] = A^{-1}\vec{x}[1]$

$\vec{y}[0] = A^{-1}\vec{x}[1]$

However, we can see that the above two equations contradict each other. Therefore, A is not invertible, and you cannot recover the previous state from the resulting state.

(e) Suppose there is a 2-pixel imaging experiment with a measurement matrix that happens to have the same numbers as the pump state transition matrix. In other words, the pump configuration also determines the measurements you take for a 2-pixel imaging experiment that is in all other respects completely unrelated. What do the answers to the previous subparts of this question imply about this experimental setup?

Solution: This implies that the image cannot be uniquely reconstructed from the measurement, and that two different images could yield the same measurement.

(f) Suppose there is a pump state transition matrix (and corresponding experiment matrix, similar to the last subpart) where every initial state is guaranteed to have a unique state vector in the next timestep. Consider what this statement implies about the system of linear equations represented by the state transition matrix A. Can you recover the initial state? Prove your answer, and explain what this implies about the experiment.

Solution: Since each unique input leads to a unique output, we know that the state transition matrix A is full rank. This means the matrix is invertible. This implies that, in the experiment, we can determine the previous states from the current state.

5. Your Own Problem Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student should submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?