EECS 16A   Designing Information Devices and Systems I
Spring 2017   Babak Ayazifar, Vladimir Stojanovic   Final Exam

Exam location: 145 Dwinelle, last SID# 2

PRINT your student ID: _____

PRINT AND SIGN your name: _____ , _____ _____
                                     (last)   (first)   (signature)

PRINT your Unix account login: ee16a-_____

PRINT your discussion section and GSI (the one you attend): _____

Name and SID of the person to your left: _____

Name and SID of the person to your right: _____

Name and SID of the person in front of you: _____

Name and SID of the person behind you: _____

Please write all your work within the box provided for a corresponding question. ONLY the work in that box will be graded. We will not be able to assign any credit for the work outside of the bounding boxes for the corresponding question.

Section 0: Pre-exam questions (2 points)

1. **Where can you write your answer? (1 pt)**

2. **What are your plans for summer break? (1 pt)**

Waiver Option

At our discretion, we have provided you with the option to waive (or opt out of) one or more parts of the exam, in exchange for a baseline partial credit. You can identify each part that has been so designated quite easily; you'll notice a blank box awaiting your initials, with the following short declaration: "By my initials in the box, I waive this part and accept 10% of the credit."

If you find that you are unable to tackle the designated part with a reasonable level of fluency, it may be to your advantage to simply indicate to us—by signing your initials in the waiver box—that you want to simply forgo the work asked of you for that part. This is designed to save you time and allow you to focus on doing your best on other parts of the exam.

For example, suppose a part that involves a proof has 10 points of credit associated with it. You have practically no idea how to carry out the proof asked of you in that part. Instead of spending time writing potentially meritless work for us to evaluate, you choose to sign your initials in the waiver box, and get 10% of the credit, which is 1 point. This saves you time.

Be aware that for each waiver box that you sign with your initials, you are, in effect, declaring the following:

> By signing my initials in the box to the left, I hereby waive my right to tackle this part of the exam. In exchange for my waiver, you will grant me 10% of the credit for this part of the problem. I acknowledge that nothing I write elsewhere for this part will be graded, and that there are no exceptions. I acknowledge, too, that the subject matter (including any result) addressed by this part may be needed in other parts of the exam, and that I may use, without penalty, elsewhere on the exam any result needed from this part.

If you choose, or otherwise fail to, sign your initials in a waiver box, you might lose anywhere from zero to the maximum number of points for that part. If you do sign the waiver box for a particular part, you will be guaranteed the 10% baseline credit for that part; in that case, no work that you show in the blank area for that part will be evaluated. No exceptions!

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3. (40 Points) Are you sleepy? Let's see what your brainwaves say ...

In this problem, you will design circuitry to record and analyze your brain signals. The technique, known as electroencephalogram (EEG), detects tiny voltage signals in your brain using small, flat metal discs (electrodes) attached to your scalp. Since you are designing a circuit for detecting drowsiness, you placed two electrodes at the front of the skull (locations 1 and 2 with potentials U_1 and U_2).

(a) (10 Points) For the circuit block given below, express $V_{1,2}$ as a function of R_4, R_5, U_1, and U_2. Assume that the op-amp supplies are $\pm 1\,\text{V}$, and $R_1 = R_2$ and $R_3 = R_4$.

Electrode Placement

Readout Circuit
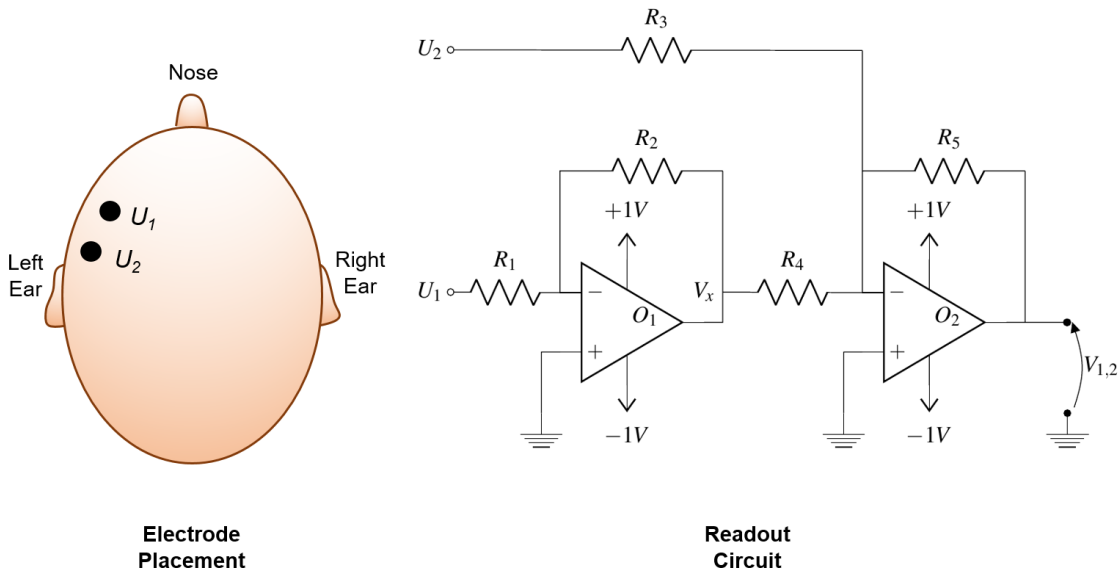
Solutions:
Left op amp is used as an inverter block:
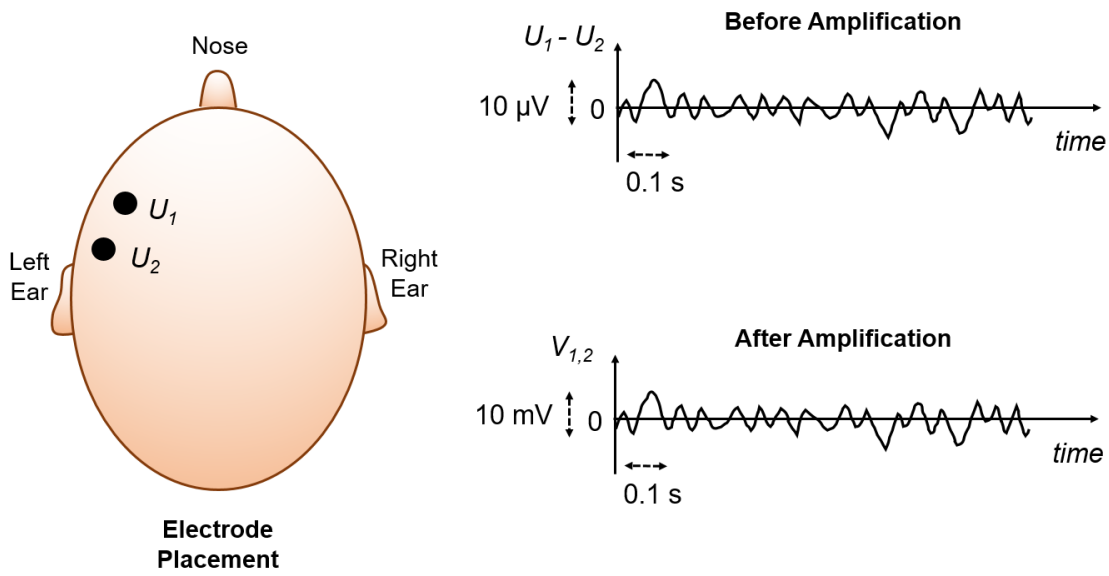
$$V_x = \frac{-R_2}{R_1}U_1$$

Right op amp is used as an adder block:

$$V_{1,2} = \frac{-R_5}{R_4}V_x + \frac{-R_5}{R_3}U_2$$
$$= \frac{-R_5}{R_4}\frac{-R_2}{R_1}U_1 + \frac{-R_5}{R_3}U_2$$
$$= \frac{R_5}{R_4}(U_1 - U_2) \text{ setting } R_1 = R_2 \text{ and } R_3 = R_4$$

(b) (5 Points) Locations 1 and 2 are typical EEG electrode placement locations and provide $U_1 - U_2 = 10\,\mu V$ oscillating signal with $10\,Hz$ frequency (Fig. Before Amplification). Since this is a tiny signal, you need to amplify it. Use the circuit in part (a) to amplify the signal such that $V_{1,2}$ becomes a $10\,mV$ oscillating signal (Fig. After Amplification). Recall that $1\,mV = 1000\,\mu V$. If $R_1 = R_2 = R_3 = R_4 = 100\,\Omega$, find R_5.

Nose

$U_1 - U_2$ **Before Amplification**

$10\,\mu V$ 0

time

0.1 s

$\bullet\ U_1$

Left Ear

$\bullet\ U_2$

Right Ear

$V_{1,2}$ **After Amplification**

$10\,mV$ 0

time

0.1 s

Electrode Placement

Solutions:
Gain = 1000. $\frac{R_5}{R_4} = 1000$. Using $R_4 = 100\,\Omega$, $R_5 = 100\,k\Omega$.

(c) (15 Points) For redundancy you added three additional sets of electrodes at locations 3 to 8, which behave similar to the previous pair at locations 1 and 2. Utilizing the circuit from part (a) now you have four measurements:

$$V_{1,2} = \text{Gain} \cdot (U_1 - U_2)$$
$$V_{3,4} = \text{Gain} \cdot (U_3 - U_4)$$
$$V_{5,6} = \text{Gain} \cdot (U_5 - U_6)$$
$$V_{7,8} = \text{Gain} \cdot (U_7 - U_8)$$

Remember, you already amplified the signal to $10\,\text{mV}$ level in part (b) (Fig. Before Averaging and Amplification). With this amplification the signal now comes up to $250\,\text{mV}$ level (Fig. After Averaging and Amplification).
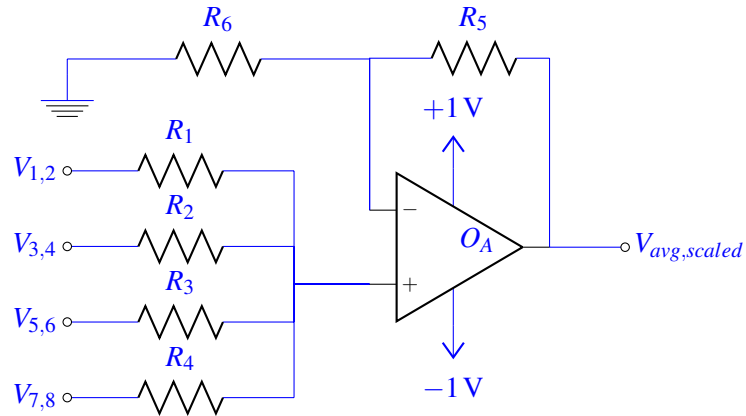
Now draw out a circuit that takes average of all four signals and amplifies the average by a factor of 25.

$$V_{avg,scaled} = 25 \cdot \frac{V_{1,2} + V_{3,4} + V_{5,6} + V_{7,8}}{4}$$

Use one op amp and six resistors for the circuit. Label all circuit elements. Provide the ratio of the resistors, e.g. $R_1 : R_2 : R_3 : R_4 : R_5 : R_6$
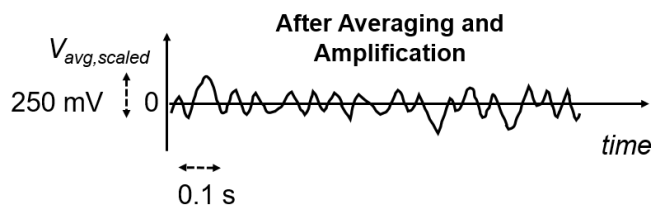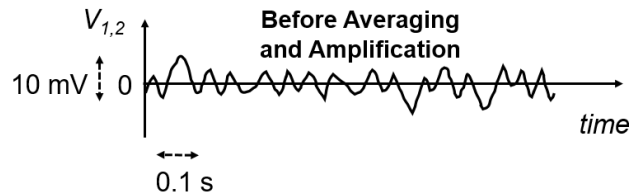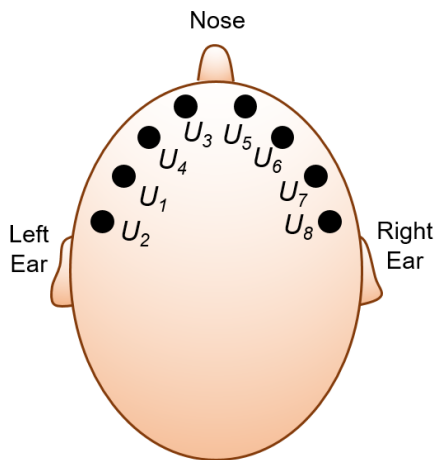
Solutions:

$R_1 - R_4$ need to be the same. Then, $1 + \frac{R_5}{R_6} = 25$, $R_5 : R_6 = 24 : 1$.

Common mistakes:
1. Using multiple op-amps, or more than 6 resistors. The question specifically says that you are allowed to use only 1 op-amp. A maximum of 5 points were awarded in this case for the rubric item "a) Adding $V_{1,2}, V_{3,4}, V_{5,6}, V_{7,8}$ (both inverting and non-inverting config)." In most cases, this was assigned if the voltages were correctly connected to an inverting op-amp, and 2 inverting op-amps were used. In very rare cases, if one or two extra resistors were used, we gave more points.

2. Writing the ratio in an expression form, as opposed to an equation form. The expression $R_1 : R_2 : R_3 : R_4 : 24R_5 : R_6$ has no meaning. It should be written as $R_1 : R_2 : R_3 : R_4 : R_5 : R_6 = 1 : 1 : 1 : 1 : 24 : 1$. We were lenient about this, and only deducted 2 points in the event that the ratio between R_5 and R_6 was ambiguous but with correct values.

3. Using the gain for non-inverting op-amp to be R_2/R_1 instead of $1 + R_2/R_1$

4. Not realizing that by connecting 4 voltages with resistors to the input, the end node will have a voltage that is equal to $\frac{V_{1,2}, V_{3,4}, V_{5,6}, V_{7,8}}{4}$

5. Connecting multiple voltage inputs directly onto a node (without a resistor).

6. Flipping the value for the non-inverting op-amp gain. The gain is $1 + R_2/R_1$ where R_2 is the resistor connected directly to the output, and R_1 is the resistor connected directly to ground

7. Flipping the polarity on the op-amp. Since flipping the polarity means that the op-amp is not in non-inverting configuration, points were not awarded for being in non-inverting configuration or for the resistor ratios for $R_5 : R_6$ (even if the ratios were correct).

(d) (10 Points) One of the artifacts of drowsiness is slow rolling of the eyes, which produces an EEG as shown in the graph marked **Sleepy**. Here the amplitude is 3 times larger than the **Awake** signal. Now that you have a robust signal acquisition system in place, you can detect if the subject is drowsy using the EEG signal. Draw out a circuit to generate a positive $(+1\,\text{V})$ output if **Sleepy** and a negative $(-1\,\text{V})$ output if **Awake**.

You don't need to output a positive signal the whole time, outputting a couple of times in a second will suffice. Use $V_{avg,scaled}$, a reference voltage V_{ref}, and a single op amp to implement your circuit. What's the range of V_{ref} to distinguish between **Awake** and **Sleepy**?

Awake

Sleepy
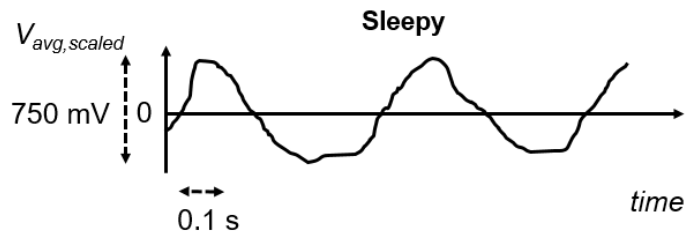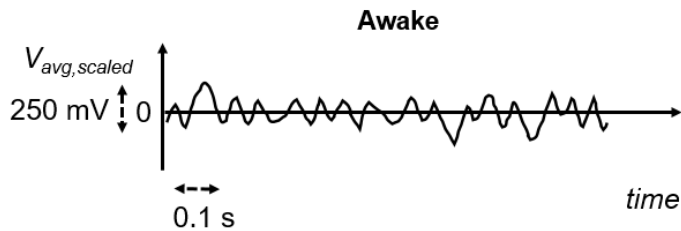
Solutions:
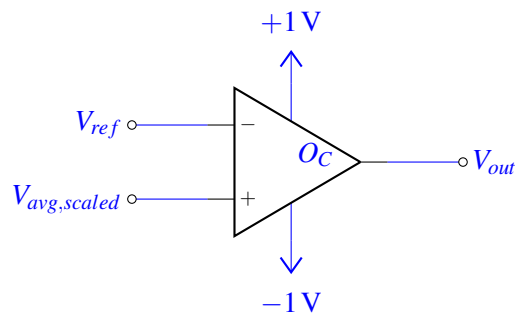
The comparator uses non-inverting configuration with $375\,\text{mV} > V_{ref} > 125\,\text{mV}$. If the average goes over max(Awake) then comparator hits the positive rail, i.e. $1\,\text{V}$ at the output, else goes to $-1\,\text{V}$. Any $375\,\text{mV} > V_{ref} > 125\,\text{mV}$ will do the job.

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4. (40 Points) Brain-on-a-Chip with 16A Neurons

Neurelic Inc, is a hot new startup building chips that emulate some of the brain functions (for example associative memory). As an intern, fresh out of 16A you get to implement the neural network circuits on this chip. The neural network consists of neurons that consist of the following blocks shown on the figure below.
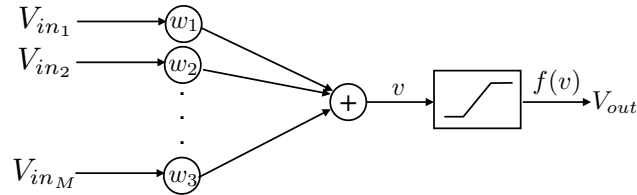

Input signals V_{in_i} are voltages from other neurons, which are multiplied by a constant weight w_i in each synapse and summed in the neuron. Each neuron also contains a nonlinear function (called a sigmoid) which is defined as

$$f(v) = \begin{cases} -1, & v \leq -1 \\ v, & -1 < v < 1 \\ +1, & v \geq +1 \end{cases}$$

where v is the internal neuron voltage after the synapse summer and $f(v)$ is the neuron voltage output.
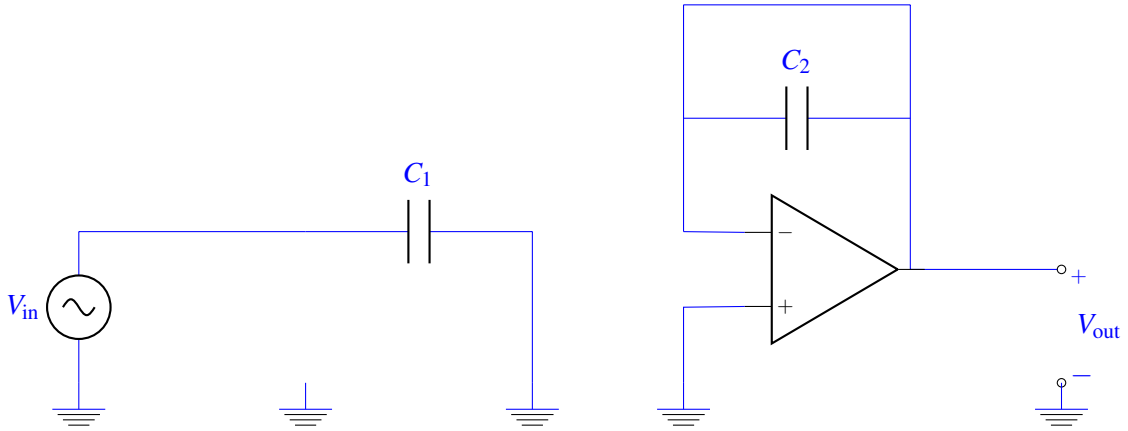
(a) Your mentor suggests that you warm-up first by analyzing the circuit below to use as neuron with a single synapse. ϕ_1 and ϕ_2 are non-overlapping clock phases that control the circuit switches.
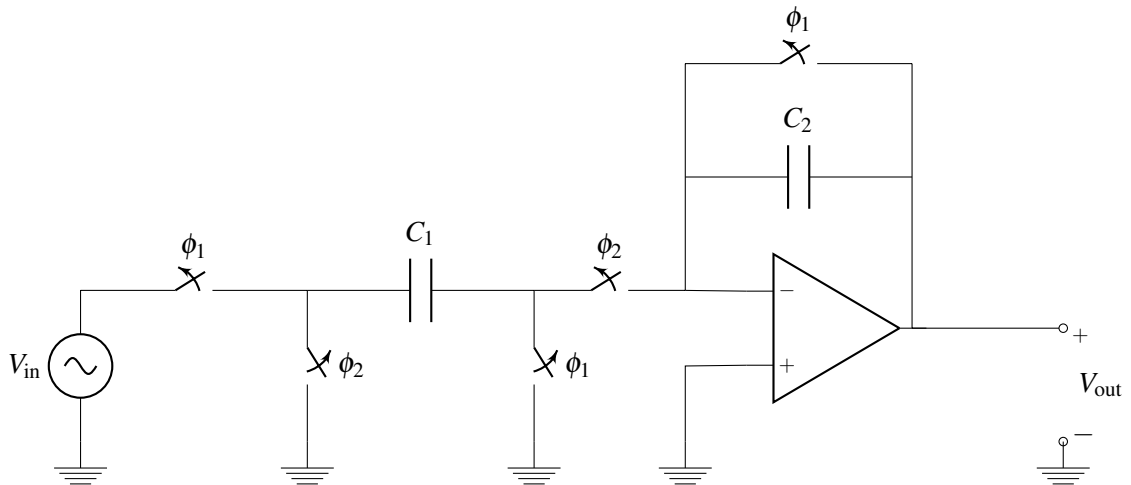
i. (5 Points) Draw an equivalent circuit during ϕ_1 and write an expression for V_{out} as a function of V_{in}, C_1 and C_2.

Solutions: $V_{\text{out}} = 0$

ii. (5 Points) Draw an equivalent circuit during ϕ_2 and write an expression for V_{out} as a function of V_{in}, C_1 and C_2.

Solutions:

$$V_{\text{in}}C_1 = V_{\text{out}}C_2$$
$$V_{\text{out}} = \frac{C_1}{C_2}V_{\text{in}}$$

(b) (5 Points) Write an equation for V_{out} during ϕ_2 as a function of V_{in} for $C_1 = C_2$ and op-amp supply voltages of $\pm 1\,\text{V}$. Briefly explain how this circuit implements the sigmoid function.

Solutions: From part (a)(ii) we know $V_{\text{out}} = \frac{C_1}{C_2}V_{\text{in}}$. Setting $C_1 = C_2$, we find $V_{\text{out}} = V_{\text{in}}$. Because of the rails of the op amp, once the V_{in} exceeds $1\,\text{V}$, the output will be $1\,\text{V}$. From this, we see the circuit implements the sigmoid function:

$$V_{\text{out}} = \begin{cases} -1, & V_{\text{in}} \leq -1 \\ V_{\text{in}}, & -1 < V_{\text{in}} < 1 \\ +1, & V_{\text{in}} \geq +1 \end{cases}$$

(c) Then, your mentor shows you the following neuron circuit, which can realize both positive and negative synapse weight and create $V_{out} = w_1 V_{in}$ in ϕ_2.
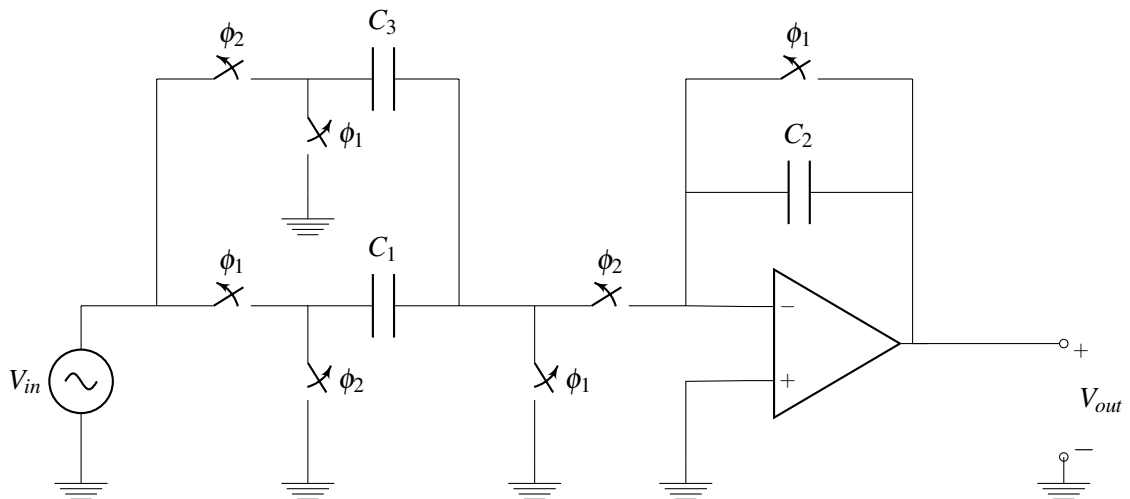
i. (5 Points) Draw an equivalent circuit during ϕ_1 and write an expression for V_{out} as a function of V_{in}, C_1, C_2, and C_3.
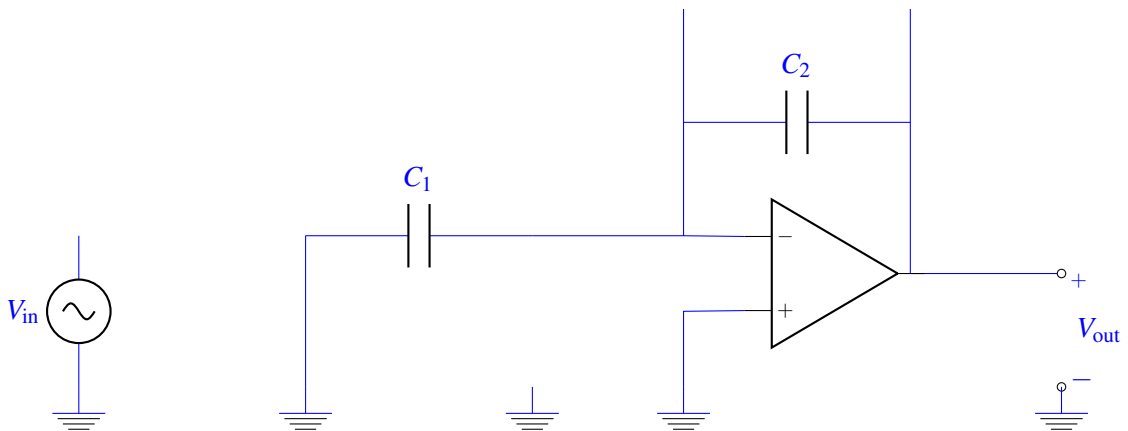
Solutions:
In phase 1, $V_{\text{out}} = 0$.
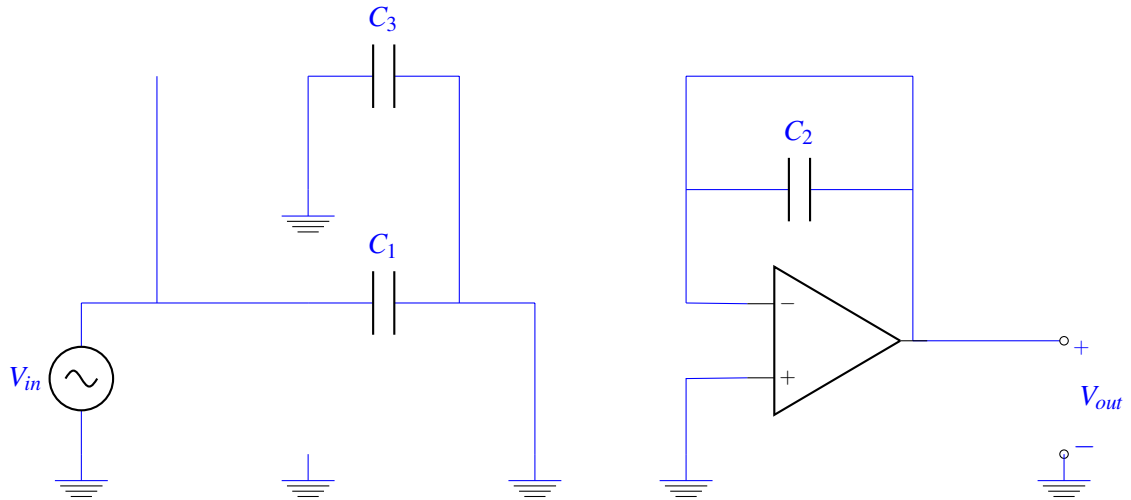
ii. (5 points) Draw an equivalent circuit during ϕ_2 and write an expression for V_{out} as a function of V_{in}, C_1, C_2, and C_3.

Solutions:
In phase 2, $V_{\text{out}} = \frac{C_1 - C_3}{C_2} V_{\text{in}}$.
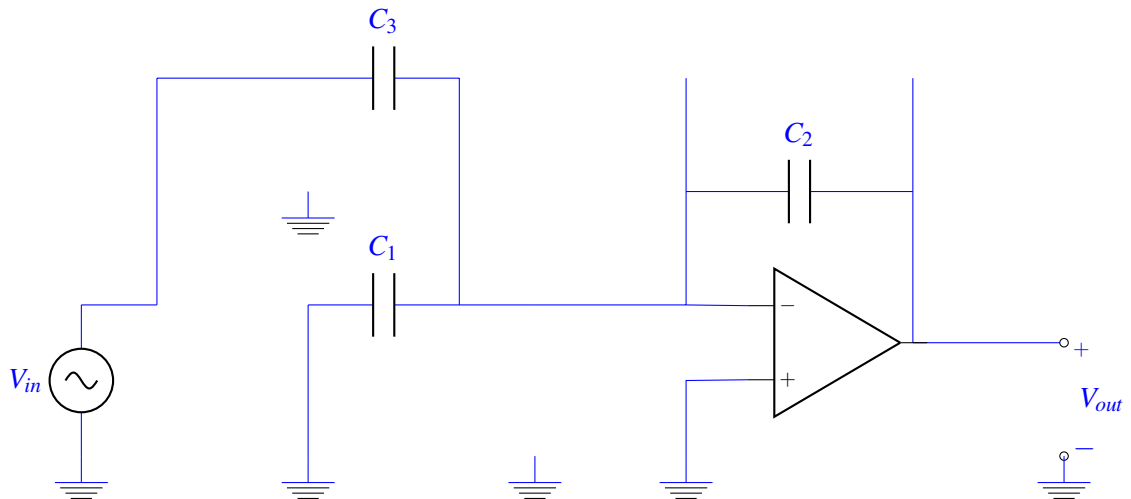
(d) (15 Points) Now it is your turn to implement a neuron that realizes the following function $V_{out} = w_1 V_{in_1} + w_2 V_{in_2}$. Draw the circuit, such that $w_1 = 1/2$ and $w_2 = -1/4$. Label all circuit elements appropriately. You should use a single op-amp and as many capacitors and switches as you need. All capacitors must be of size C_{unit}. Assume that the op-amp power supplies are $\pm 1V$ (no need to draw them in the circuit). The circuit should operate in 2 phases, with $V_{out} = w_1 V_{in_1} + w_2 V_{in_2}$ in the second phase (ϕ_2), and reset in ϕ_1.

Solutions:
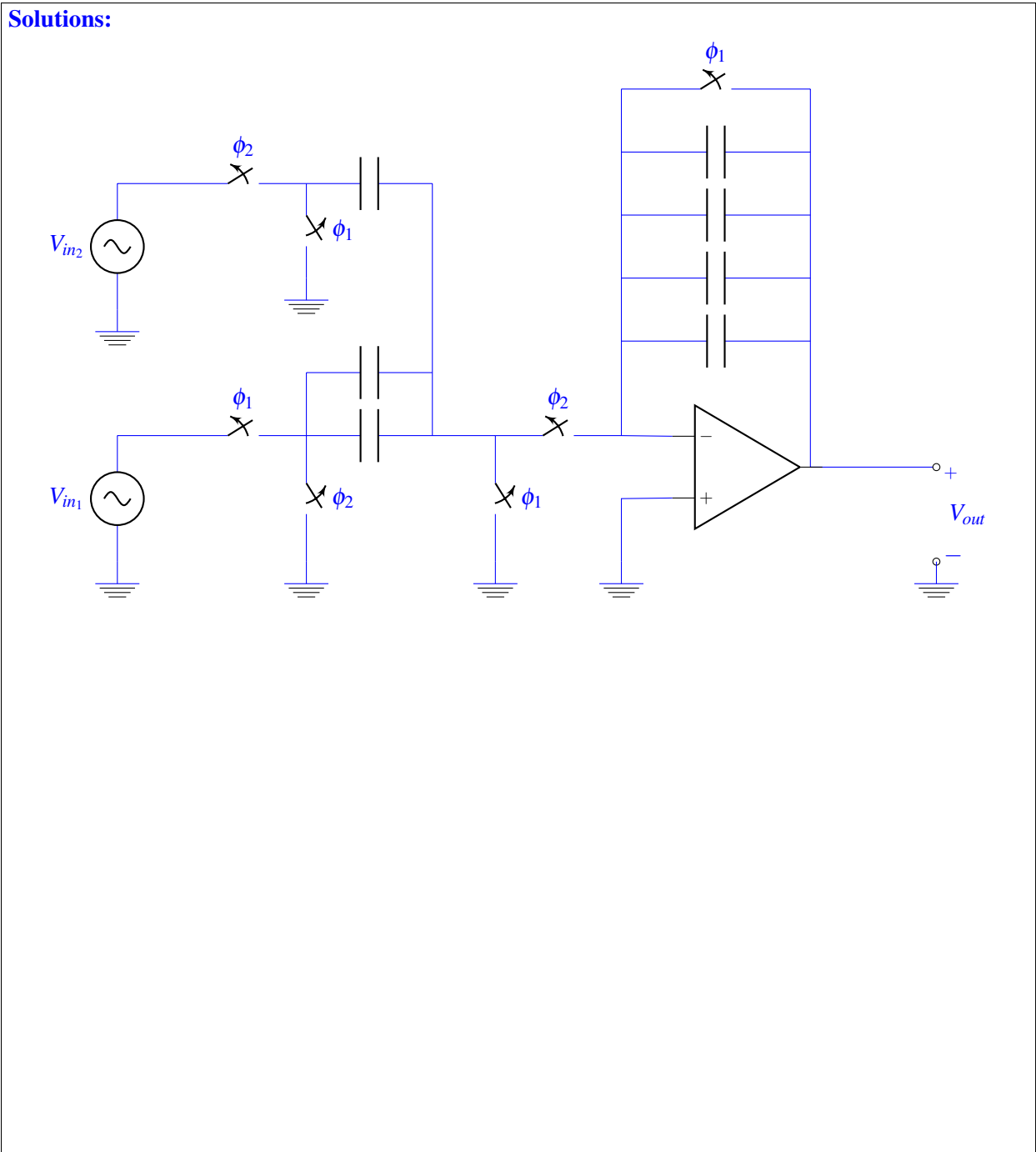
5. (40 Points) Lateration with Linear Systems of Equations

In the lab, we used a microphone and several transmitters to find the microphone position. In this problem, we'll set up the system of equations you need in order to uniquely determine the microphone position. We will use 5 beacons for the setup on this exam.

All beacons transmit their unique signal at time 0 that each arrive at the microphone at different time T_m depending on the microphone position. Let $\tau_m = T_m - T_0$ be the time difference of arrival between beacon m and beacon 0.

(a) (8 Points) Using the fact that $R_m = vT_m$, where R_m is the distance of the microphone from beacon m and v is the speed of the audio signals in air, show that we can write:

$$0 = v\tau_m + 2R_0 + \frac{R_0^2 - R_m^2}{v\tau_m} \tag{1}$$

where τ_m is the time difference of arrival for beacon 0 and beacon m.

☐ By my initials in the box, I waive this part and accept 10% of the credit.

Solutions: We can write:

$$v\tau_m = R_m - R_0$$

Next, we re-arrange and square to get:

$$R_m^2 = (v\tau_m + R_0)^2$$

$$R_m^2 = (v\tau_m)^2 + R_0^2 + 2(v\tau_m)R_0$$

$$0 = (v\tau_m)^2 + 2(v\tau_m)R_0 + R_0^2 - R_m^2$$

$$0 = v\tau_m + 2R_0 + \frac{R_0^2 - R_m^2}{v\tau_m}$$

QED

(b) (8 Points) We introduce coordinates based on beacon 0. Beacon 0 is located at $(0,0)$, beacon m is located at known positions (x_m, y_m). We try to find the microphone position in the sensor plane, given by (x,y). We see that R_m, the distance of the microphone from beacon m, is a function of the microphone position:

$$R_m = \sqrt{(x-x_m)^2 + (y-y_m)^2} \tag{2}$$

By plugging the relationship from eq. 2 into eq. 1 above for R_0, R_1, and R_2, we can write a system of equations with two equations and two unknowns:

$$0 = v\tau_1 + 2\sqrt{x^2+y^2} + \frac{2x_1 x - x_1^2 + 2y_1 y - y_1^2}{v\tau_1} \tag{3}$$

$$0 = v\tau_2 + 2\sqrt{x^2+y^2} + \frac{2x_2 x - x_2^2 + 2y_2 y - y_2^2}{v\tau_2} \tag{4}$$

What is the minimum number of beacons you can use to create this system of equations? Justify why you cannot use linear algebra methods to solve the system of equations in this part.

Solutions: In order to create this system of equations, we would need 3 beacons: beacon 0 and 1 to generate coefficients in the first equation, and beacon 0 and 2 to generate coefficients in the second equation.

As written, the system of equations requires us to evaluate the square root of the sum of the squares of our unknowns. This operation cannot be expressed by in a linear form. In effect,

$$\sqrt{x^2+y^2}$$

cannot be written as

$$k_x x + k_y y + k_c$$

This needs to be true in order to write a matrix equation, $\mathbf{A}\vec{x} = \vec{b}$ where $\vec{x} = [x,y]^T$.

(c) (8 Points) We can use extra information to make the system linear. Subtract Eq. 3 from Eq. 4. This is a linear equation, and should have the form of

$$ax + by + c = 0 \tag{5}$$

How many linear equations of this form can we construct with 3 beacons, and how many unknowns do we have? Can you use this system to find a unique solution for (x, y) using linear algebra methods? Why or why not?

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Solutions: We can start with the equations given in the previous parts.

$$0 = v\tau_1 + 2\sqrt{x^2 + y^2} + \frac{2x_1 x - x_1^2 + 2y_1 y - y_1^2}{v\tau_1}$$

$$0 = v\tau_2 + 2\sqrt{x^2 + y^2} + \frac{2x_2 x - x_2^2 + 2y_2 y - y_2^2}{v\tau_2}$$

$$0 = v\tau_2 - v\tau_1 + 2\sqrt{x^2 + y^2} - 2\sqrt{x^2 + y^2} + \frac{2x_2 x - x_2^2 + 2y_2 y - y_2^2}{v\tau_2} - \frac{2x_1 x - x_1^2 + 2y_1 y - y_1^2}{v\tau_1}$$

$$a = \frac{2x_2}{v\tau_2} - \frac{2x_1}{v\tau_1}$$

$$b = \frac{2y_2}{v\tau_2} - \frac{2y_1}{v\tau_1}$$

$$c = v\tau_2 - v\tau_1 + \frac{x_1^2 + y_1^2}{v\tau_1} - \frac{x_2^2 + y_2^2}{v\tau_2}$$

With three beacons, we can only write one linear equation, but we have two unknowns.

$$ax + by + c = 0$$

Therefore, we cannot find a unique solution for *x* and *y*, since all solutions along the line described by this equation are valid.

(d) (8 Points) For our microphone lateration system, we can use 5 beacons. Given 5 beacons, how many unique equations of the form $ax + by + c = 0$ can you write? (You do not need to write out the expressions for a, b, and c.) Write a matrix equation, $\mathbf{A}\vec{x} = \vec{b}$, in terms of a_n, b_n, and c_n, where the subscript n corresponds to one of the unique equations. (Also, use: $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.)

Solutions: With 5 total beacons, we can write 3 unique sets of coefficients, a_n, b_n, and c_n.
Therefore, setup of $\mathbf{A}$ and $\vec{b}$ will be:

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} -c_1 \\ -c_2 \\ -c_3 \end{bmatrix}$$

Written out,

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \vec{b} = \begin{bmatrix} -c_1 \\ -c_2 \\ -c_3 \end{bmatrix}$$

(e) (8 Points) There is always some error in every measurement. In the lateration case, the biggest error is probably in measuring the time of arrival for different beacons. This error can result in a situation in which $\mathbf{A}\vec{x} = \vec{b}$ has no solutions. Find a matrix expression for $\hat{\vec{x}}$, an estimate for $\vec{x}$ that minimizes $\|\mathbf{A}\vec{x} - \vec{b}\|$.

Solutions: In general, if we have $\mathbf{A}\vec{x} = \vec{b}$ with no solutions for $\vec{x}$, we can find a least-squares minimization solution by multiplying both sides of the matrix equation by $\mathbf{A}^T$ from the left:

$$\mathbf{A}^T\mathbf{A}\hat{\vec{x}} = \mathbf{A}^T\vec{b}$$

$$\hat{\vec{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$$

6. (40 Points) Constrained Least-Squares Optimization

In this problem, you'll go through a process of guided discovery to solve the following optimization problem:

Consider a matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$, of full column rank, where $M > N$. Determine a unit vector $\vec{x}$ that minimizes $\|\mathbf{A}\vec{x}\|$, where $\|\cdot\|$ denotes the 2-norm—that is,

$$\|\mathbf{A}\vec{x}\|^2 \triangleq \langle \mathbf{A}\vec{x}, \mathbf{A}\vec{x} \rangle = (\mathbf{A}\vec{x})^T \mathbf{A}\vec{x} = \vec{x}^T \mathbf{A}^T \mathbf{A}\vec{x}.$$

This is equivalent to solving the following optimization problem:

$$\text{Determine} \quad \vec{x} = \arg\min_{\vec{x}} \|\mathbf{A}\vec{x}\|^2 \quad \text{subject to the constraint} \quad \|\vec{x}\|^2 = 1.$$

This task may *seem* like solving a standard least-squares problem $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{b} = \vec{0}$, but it isn't. An important distinction is that in our problem, $\vec{x} = \vec{0}$ is *not* a valid solution, because the zero vector does not have unit length. Our optimization problem is a least squares problem with a constraint—hence the term *Constrained Least-Squares Optimization*. The constraint is that the vector $\vec{x}$ must lie on the unit sphere in $\mathbb{R}^N$. You'll tackle this problem in a methodical, step-by-step fashion.

Let $(\lambda_1, \vec{v}_1), \ldots, (\lambda_N, \vec{v}_N)$ denote the eigenpairs (i.e., eigenvalue/eigenvector pairs) of $\mathbf{A}^T \mathbf{A}$. Assume that the eigenvalues are all real and indexed in an ascending fashion—that is,

$$\lambda_1 \leq \cdots \leq \lambda_N.$$

Assume, too, that each eigenvector has been normalized to have unit length—that is, $\|\vec{v}_k\| = 1$ for all $k \in \{1, \ldots, N\}$.

(a) (10 Points) Show that $0 < \lambda_1$.

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By my initials in the box, I waive this part and accept 10% of the credit.

Solutions:
Consider $\|\mathbf{A}\vec{v}\|^2$ for eigenvector $\vec{v}$, with eigenvalue λ.

$$\|\mathbf{A}\vec{v}\|^2 = \vec{v}^T \mathbf{A}^T \mathbf{A}\vec{v}$$
$$= \vec{v}^T \lambda \vec{v}$$
$$\lambda = \frac{\|\mathbf{A}\vec{v}\|^2}{\|\vec{v}\|^2}$$

So $\lambda > 0$ since norms are positive if $\vec{v} \neq \vec{0}$. Since $\mathbf{A}$ is full rank, the numerator is never 0 unless $\vec{v} = \vec{0}$.

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(b) (10 Points) Consider two eigenpairs $(\lambda_k, \vec{v}_k)$ and $(\lambda_\ell, \vec{v}_\ell)$ corresponding to distinct eigenvalues of $\mathbf{A}^T\mathbf{A}$—that is, $\lambda_k \neq \lambda_\ell$. Prove that the corresponding eigenvectors $\vec{v}_k$ and $\vec{v}_\ell$ are orthogonal: $\vec{v}_k \perp \vec{v}_\ell$.

To help you get started, consider the two equations

$$\mathbf{A}^T\mathbf{A}\vec{v}_k = \lambda_k \vec{v}_k \tag{6}$$

and

$$\vec{v}_\ell^T \mathbf{A}^T\mathbf{A} = \lambda_\ell \vec{v}_\ell^T. \tag{7}$$

Premultiply Equation (6) by $\vec{v}_\ell^T$, postmultiply Equation (7) by $\vec{v}_k$, compare the two, and explain how one may then infer that $\vec{v}_k$ and $\vec{v}_\ell$ are orthogonal.

☐ By my initials in the box, I waive this part and accept 10% of the credit.

Solutions: Following the hint:

$$\vec{v}_\ell^T \mathbf{A}^T \mathbf{A} \vec{v}_k = \vec{v}_\ell^T \lambda_k \vec{v}_k$$
$$\vec{v}_\ell^T \mathbf{A}^T \mathbf{A} \vec{v}_k = \lambda_\ell \vec{v}_\ell^T \vec{v}_k$$

We see the two expressions on the left are equal, so set the two expressions on the right equal to each other:

$$\lambda_k \vec{v}_\ell^T \vec{v}_k = \lambda_\ell \vec{v}_\ell^T \vec{v}_k$$

If $\lambda_k \neq \lambda_l$, then the only possible solution is that $\vec{v}_\ell^T \vec{v}_k = 0$, which means $\vec{v}_\ell$ and $\vec{v}_k$ are orthogonal.

(c) (10 Points) Since the N eigenvectors of $\mathbf{A}^T\mathbf{A}$ are mutually orthogonal—and each has unit length—they form an orthonormal basis in $\mathbb{R}^N$. This means that we can express an arbitrary vector $\vec{x} \in \mathbb{R}^N$ as a linear combination of the eigenvectors $\vec{v}_1, \ldots, \vec{v}_N$, as follows:

$$\vec{x} = \sum_{n=1}^{N} \alpha_n \vec{v}_n.$$

i. (5 Points) Determine the nth coefficient α_n in terms of $\vec{x}$ and one or more of the eigenvectors $\vec{v}_1, \ldots, \vec{v}_N$.

Solutions: Since $\vec{v}_i$ are orthogonal, the coefficient α_i is the projection of $\vec{x}$ on to $\vec{v}_i$. Since $\vec{v}_i$ are all unit vectors, the projection is simply the inner product.

$$\alpha_i = \langle \vec{x}, \vec{v}_i \rangle = \vec{x}^T \vec{v}_i$$

ii. (5 Points) Suppose $\vec{x}$ is a unit-length vector (i.e., a unit vector) in $\mathbb{R}^N$. Show that

$$\sum_{n=1}^{N} \alpha_n^2 = 1.$$

Solutions: Consider $\|\vec{x}\|^2 = 1$.

$$\|\vec{x}\|^2 = \vec{x}^T \vec{x}$$

$$= \left(\sum_{i=1}^{N} \alpha_i \vec{v}_i\right)^T \left(\sum_{j=1}^{N} \alpha_j \vec{v}_j\right) = \left(\sum_{i=1}^{N} \alpha_i \vec{v}^T_i\right)\left(\sum_{j=1}^{N} \alpha_j \vec{v}_j\right)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \vec{v}^T_i \vec{v}_j$$

Now since the v_i are orthogonal we know: $\vec{v}_i^T \vec{v}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

So $\|\vec{x}\|^2 = \sum_{n=1}^{N} \alpha_n^2 = 1$.

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(d) (10 Points) Now you're well-positioned to tackle the grand challenge of this problem—determine the unit vector $\vec{x}$ that minimizes $\|\mathbf{A}\vec{x}\|$.

Note that the task is the same as finding a unit vector $\vec{x}$ that minimizes $\|\mathbf{A}\vec{x}\|^2$.

Express $\|\mathbf{A}\vec{x}\|^2$ in terms of $\{\alpha_1, \alpha_2 \ldots \alpha_N\}$, $\{\lambda_1, \lambda_2 \ldots \lambda_N\}$, and $\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_N\}$, and find an expression for $\vec{\hat{x}}$ such that $\|\mathbf{A}\vec{x}\|^2$ is minimized. You may *not* use any tool from calculus to solve this problem—so avoid differentiation of any flavor.

For the optimal vector $\vec{\hat{x}}$ that you determine—that is, the vector

$$\vec{\hat{x}} = \arg\min_{\vec{x}} \|\mathbf{A}\vec{x}\|^2 \quad \text{subject to the constraint} \quad \|\vec{x}\|^2 = 1,$$

determine a simple, closed-form expression for the minimum value

$$\min_{\|\vec{x}\|=1} \|\mathbf{A}\vec{x}\| = \|\mathbf{A}\vec{\hat{x}}\|.$$

$\boxed{}$ By my initials in the box, I waive this part and accept 10% of the credit.

Solutions: Note $\|\mathbf{A}\vec{x}\|^2 = x^T \mathbf{A}^T \mathbf{A} x$. We express $\vec{x}$ in terms of $\vec{v}_i$ (the eigenvalues of $\mathbf{A}^T \mathbf{A}$) and expand.

$$\mathbf{A}^T \mathbf{A} \vec{x} = \mathbf{A}^T \mathbf{A} \sum_{n=1}^{N} \alpha_n \vec{v}_n$$

$$= \sum_{n=1}^{N} \alpha_n \mathbf{A}^T \mathbf{A} \vec{v}_n$$

$$= \sum_{n=1}^{N} \alpha_n \lambda_n \vec{v}_n$$

Now:

$$x^T \mathbf{A}^T \mathbf{A} x = \vec{x} \sum_{n=1}^{N} \alpha_n \lambda_n \vec{v}_n$$

$$= \sum_{n=1}^{N} \alpha_n^2 \lambda_n$$

To minimize, we pick the entire weight of α of the smallest λ, i.e. we pick $\vec{x}$ to be the eigenvector $\vec{v}_i$ with the smallest eigenvalue, and the minimum value is λ_i.

EECS 16A, Spring 2017, Final Exam

7. (20 Points) Tunnel Diodes

Introduced by Leo Esaki in 1958, a *tunnel diode* is an electronic circuit element whose current-voltage (*I-V*) curve in a particular circuit (not shown) looks like the one shown in Figure (a) below—an adaptation of Figure 7.7 on p. 200 of the book *It's a Nonlinear World*, by Richard Enns, Springer, 2011. Note that in the plot, the current *I* can be thought of as a function of the voltage *V*. **For the purposes of this problem, you need <u>not</u> know <u>anything</u> about diodes.**
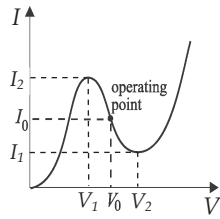
(a) I-V Curve
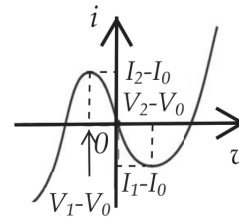
(b) Re-centered I-V Curve

As a well-trained UC Berkeley engineer, you decide to recenter the *I-V* curve to the operating point (V_0, I_0). That is, you define new variables i and v such that:

$$i = I - I_0 \quad \text{and} \quad v = V - V_0,$$

This leads to the *i-v* curve shown in Figure (b). You're charged with the task of determining the parameters that characterize the tunnel diode. In particular, you learn that:

- the diode can be characterized by the cubic polynomial

$$i = -\alpha v + \beta v^3,$$

  where $\alpha, \beta > 0$; and

- if you make M voltage current measurements (v_m, i_m) where $m = 1, 2, \ldots, M$, you can then estimate the unknown parameters α and β.

For simplicity, and without loss of generality, assume throughout this problem that the vector of voltage measurements

$$\begin{bmatrix} v_1 \\ \vdots \\ v_M \end{bmatrix}$$

has unit length. That is,

$$\sum_{m=1}^{M} v_m^2 = 1.$$

You make all the necessary measurements, and set up a set of M linear equations in two unknowns, as follows:

$$\underbrace{\begin{bmatrix} -v_1 & v_1^3 \\ \vdots & \vdots \\ -v_M & v_M^3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} i_1 \\ \vdots \\ i_M \end{bmatrix}}_{\vec{b}},$$

The columns of the matrix $\mathbf{A}$,

$$a_1 = \begin{bmatrix} -v_1 \\ \vdots \\ -v_M \end{bmatrix} \quad \text{and} \quad a_2 = \begin{bmatrix} v_1^3 \\ \vdots \\ v_M^3 \end{bmatrix},$$

are not mutually orthogonal. Orthogonalize them, and determine a QR decomposition of $\mathbf{A}$. Take as your first orthonormal vector q_1 the first column of $\mathbf{A}$, which is already of unit length. Determine completely the second orthonormal vector q_2 and the 2×2 upper-triangular matrix $\mathbf{R}$ in the decomposition $\mathbf{A} = \mathbf{QR}$.

[ ] By my initials in the box, I waive this part and accept 10% of the credit.

Solutions: Using the Gram-Schmidt process:

$$\vec{v}_1 = \vec{a}_1$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{\vec{a}_1}{1} = a_1$$

$$\vec{v}_2 = a_2 - \frac{\langle \vec{a}_1, \vec{a}_2 \rangle}{\langle \vec{a}_1, \vec{a}_1 \rangle} \vec{a}_1 = \vec{a}_2 - \langle \vec{a}_1, \vec{a}_2 \rangle \vec{a}_1 = \vec{a}_2 - \left(-\sum_{i=1}^{M} v_i^4 \right) \vec{a}_1$$

$$\vec{q}_2 = \frac{v_2}{\|v_2\|}$$

The vectors $\vec{q}_1$ and $\vec{u}_2$ form the columns of the matrix $\mathbf{Q}$. We can find the matrix $\mathbf{R}$ by finding $\mathbf{Q}^T \mathbf{A}$.

$$\mathbf{R} = \mathbf{Q}^T \mathbf{A}$$

$$\mathbf{R} = \begin{bmatrix} 1 & \vec{q}_1^T \vec{a}_2 \\ 0 & \vec{q}_2^T \vec{a}_2 \end{bmatrix}$$