# L07: Matrices

## The basics

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# Announcements

**Lab 02 is due on February 3 at 12 pm**

bCourses "Pages":

- FAQ for lab 02: published
- "Required and useful functions": in construction
- "Common error messages and their causes": in construction

Today:

- Matrices (what are they?)
- Matrix addition, multiplication, exponentiation, inverse, transpose

Friday:

- Nested if-statements
- Comparing floating point numbers
- Practice questions

# Matrices: definitions

A **real matrix** is a 2-dimensional array of real numbers. For example:

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A 1 by $n$ matrix is also called a **row vector**. For example:

$$\begin{bmatrix} 2 & 4 & 5 & 7 \end{bmatrix}$$

A $n$ by 1 matrix is also called a **column vector**

$$\begin{bmatrix} 0 \\ 6 \\ 7 \\ 3 \end{bmatrix}$$

# Matrices: definitions (continued)

A **square matrix** is an $n$ by $n$ matrix. For example:

$$\begin{bmatrix} 4 & 5 & 4 & 0 \\ 1 & 6 & 3 & 2 \\ 1 & 7 & 9 & 8 \\ 10 & 5 & 6 & 9 \end{bmatrix}$$

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**Identity matrices** are square matrices whose:

- diagonal terms are all 1; and
- off-diagonal terms are all 0; and

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For example:

the 2 by 2 identity matrix $\rightarrow$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\uparrow$ the 3 by 3 identity matrix

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\leftarrow$ the 4 by 4 identity matrix

# Matrix operations: addition and multiplication by a scalar

Matrix addition is element-wise addition. For example:

$$\begin{bmatrix} -2 & -4 & 5 \\ 0 & 1 & 9 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 0 \\ 0 & 8 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 5 \\ 0 & 9 & 7 \end{bmatrix}$$

Multiplication by a scalar: multiply each element by the scalar. For example:

$$7 \times \begin{bmatrix} -2 & -4 & 5 \\ 0 & 1 & 9 \end{bmatrix} = \begin{bmatrix} -14 & -28 & 35 \\ 0 & 7 & 63 \end{bmatrix}$$

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  "**The inner dimensions of the matrices must be equal**"

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$$C_{i,j} = \sum_{k=1}^{k=n_1} A_{i,k} B_{k,j}$$

- **Matrix multiplication is not commutative**, meaning that $A \times B$ is not necessarily equal to $B \times A$. If fact, sometimes $A \times B$ is defined but $B \times A$ is not

# Matrix multiplication: example

$$A = \begin{bmatrix} 5 & 0 & 1 & 2 \\ -1 & 4 & -2 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 5 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

# Matrix multiplication: example

$$A = \begin{bmatrix} \boxed{5 \quad 0 \quad 1 \quad 2} \\ -1 \quad 4 \quad -2 \quad 9 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 5 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} \boxed{29} & & \\ & & \end{bmatrix}$$

$$C_{1,1} \quad = \quad 5 \times 6 \ + \ 0 \times 2 \ + \ 1 \times (-1) \ + \ 2 \times 0 \quad = \quad 29$$

# Matrix multiplication: example

$$A = \begin{bmatrix} \boxed{5 & 0 & 1 & 2} \\ -1 & 4 & -2 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & \boxed{0} & 2 \\ 2 & 1 & 3 \\ -1 & 5 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} 29 & \boxed{13} & \\ & & \end{bmatrix}$$

$$C_{1,2} \quad = \quad 5 \times 0 \ + \ 0 \times 1 \ + \ 1 \times 5 \ + \ 2 \times 4 \quad = \quad 13$$

# Matrix multiplication: example

$$A = \begin{bmatrix} \boxed{5 \quad 0 \quad 1 \quad 2} \\ -1 \quad 4 \quad -2 \quad 9 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 5 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} 29 & 13 & \boxed{18} \\ & & \end{bmatrix}$$

$$C_{1,3} \quad = \quad 5 \times 2 \; + \; 0 \times 3 \; + \; 1 \times 0 \; + \; 2 \times 4 \quad = \quad 18$$

# Matrix multiplication: example

$$A = \begin{bmatrix} 5 & 0 & 1 & 2 \\ -1 & 4 & -2 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 5 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} 29 & 13 & 18 \\ \boxed{4} & & \end{bmatrix}$$

$$C_{2,1} \quad = \quad -1 \times 6 \ + \ 4 \times 2 \ + \ -2 \times (-1) \ + \ 9 \times 0 \quad = \quad 4$$

# Matrix multiplication: example

$$A = \begin{bmatrix} 5 & 0 & 1 & 2 \\ \boxed{-1 \quad 4 \quad -2 \quad 9} \end{bmatrix} \qquad B = \begin{bmatrix} 6 & \boxed{\begin{matrix} 0 \\ 1 \\ 5 \\ 4 \end{matrix}} & \begin{matrix} 2 \\ 3 \\ 0 \\ 4 \end{matrix} \\ \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} 29 & 13 & 18 \\ 4 & \boxed{30} & \end{bmatrix}$$

$$C_{2,2} \quad = \quad -1 \times 0 \; + \; 4 \times 1 \; + \; -2 \times 5 \; + \; 9 \times 4 \quad = \quad 30$$

# Matrix multiplication: example

$$A = \begin{bmatrix} 5 & 0 & 1 & 2 \\ -1 & 4 & -2 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 5 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} 29 & 13 & 18 \\ 4 & 30 & \boxed{46} \end{bmatrix}$$

$$C_{2,3} \quad = \quad -1 \times 2 \ + \ 4 \times 3 \ + \ -2 \times 0 \ + \ 9 \times 4 \quad = \quad 46$$

# Matrix: practice question

$A$ is a 5 by 10 matrix and $B$ is a 7 by 5 matrix. Which of the following statements are true?

**(A)** $A + B$ is defined

**(B)** $B + A$ is defined

**(C)** $3 \times A$ is defined

**(D)** $A \times B$ is defined

**(E)** $B \times A$ is defined

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Notes:

- Matrix addition is commutative, so if $A + B$ is defined, then so is $B + A$, and in this case $A + B = B + A$
- Matrix multiplication by a scalar is always defined

# Matrix: practice question

Consider the following matrices:

$$A = \begin{bmatrix} 4 & -1 \\ 3 & 0 \\ 6 & 8 \\ 2 & -7 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$$

Calculate $A \times B$!

# Matrix: practice question

Consider the following matrices:

$$A = \begin{bmatrix} 4 & -1 \\ 3 & 0 \\ 6 & 8 \\ 2 & -7 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$$

Calculate $A \times B$!

$$A \times B = \begin{bmatrix} 9 & -19 \\ 6 & -12 \\ 4 & 0 \\ 11 & -29 \end{bmatrix}$$

# Matrix: practice question

Consider the following matrices:

$$A = \begin{bmatrix} 4 & -1 \\ 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$$

Calculate $A \times B$ and $B \times A$!

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Calculate $A \times B$ and $B \times A$!

$$A \times B = \begin{bmatrix} 9 & -19 \\ 6 & -12 \end{bmatrix}$$

$$B \times A = \begin{bmatrix} -4 & -2 \\ 5 & 1 \end{bmatrix}$$

# Matrix: practice question

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$$A \times B = \begin{bmatrix} 9 & -19 \\ 6 & -12 \end{bmatrix}$$

$$B \times A = \begin{bmatrix} -4 & -2 \\ 5 & 1 \end{bmatrix}$$

**Again, matrix multiplication is not commutative.** For example, in the example above: $A \times B \neq B \times A$

# Inverse of a matrix

Consider an $n$ by $n$ square matrix $A$. The inverse of $A$, if it exists, is the $n$ by $n$ square matrix $A^{-1}$ such that $A \times A^{-1}$ is the $n$ by $n$ identity matrix

# Inverse of a matrix

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In the example below, $A^{-1}$ is the inverse of $A$:

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 0 & 1 \\ 0.25 & -0.5 \end{bmatrix}$$

$$A \times A^{-1} = A^{-1} \times A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ $\leftarrow$ the 2 by 2 identity matrix

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- The inverse of a non-square matrix never exists
- The inverse of a square matrix does not always exist
- If $A^{-1}$ is the inverse of $A$, then $A$ is the inverse of $A^{-1}$
  - In other words: $(A^{-1})^{-1} = A$
- The inverse of the identity matrix is the identity matrix

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# Matrix exponentiation

Consider an $n$ by $n$ square matrix $A$ and a positive integer $n$. Then:

$$A^n = A \times A \times \cdots \times A \; (n \text{ times})$$

For example:

$$\begin{aligned}
\begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix}^3 &= \begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -4 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 12 \\ 0 & -8 \end{bmatrix}
\end{aligned}$$

# Transpose of a matrix

Consider an $m$ by $n$ matrix $A$. The transpose of $A$ is the $n$ by $m$ matrix $A^{\mathrm{T}}$ such that $A^{\mathrm{T}}_{i,j} = A_{j,i}$

- In other words, when transposing a matrix "**the rows become the columns and the columns become the rows**"

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For example:

$$\begin{bmatrix} -5 & -8 & 7 & -1 & 6 \\ 6 & 7 & -9 & 9 & 4 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} -5 & 6 \\ -8 & 7 \\ 7 & -9 \\ -1 & 9 \\ 6 & 4 \end{bmatrix}$$

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- In other words, when transposing a matrix "**the rows become the columns and the columns become the rows**"
- Note that $(A^{\mathrm{T}})^{\mathrm{T}} = A$

For example:

$$\begin{bmatrix} -5 & -8 & 7 & -1 & 6 \\ 6 & 7 & -9 & 9 & 4 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} -5 & 6 \\ -8 & 7 \\ 7 & -9 \\ -1 & 9 \\ 6 & 4 \end{bmatrix}$$

# Arrays and matrix operations in Matlab

The following Matlab operators are matrix operators:

- `*` (matrix multiplication)
- `/` (multiply by the inverse or pseudo-inverse of)
- `^` (matrix exponentiation)
- `'` (transpose). Alternatively, use the function "`transpose`"

To inverse a matrix "`A`" in Matlab, use `inv(A)` or `A^(-1)`

The following Matlab operators are element-wise operators:

- `.*` (element-wise multiplication)
- `./` (element-wise division)
- `.^` (element-wise exponentiation)

Matrix addition and subtraction are the same as element-wise addition and subtraction, respectively, so the operators `.+` and `.-` do not exist

Element-wise and matrix operations between scalars are equivalent

# Practice question

What will the values of the variables “c” and “d” be after executing the following commands?

```
>> a = [3, 4, 1; -2, 1, 0];
>> b = [1, 0; 1, 1; 2, -2];
>> c = ((a*b)')^2;
>> d = ((a*b)').^2;
```

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c:

$$\begin{bmatrix} 79 & -10 \\ 20 & -1 \end{bmatrix}$$

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$$\begin{bmatrix} 79 & -10 \\ 20 & -1 \end{bmatrix}$$

d:

$$\begin{bmatrix} 81 & 1 \\ 4 & 1 \end{bmatrix}$$

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>> a = [3, 4, 1; -2, 1, 0];
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$$\begin{bmatrix} 79 & -10 \\ 20 & -1 \end{bmatrix}$$

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$$\begin{bmatrix} 81 & 1 \\ 4 & 1 \end{bmatrix}$$

**Adding a single character (here: a period) can completely change the result of an expression!**