L07: Matrices

The basics

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Announcements

Lab 02 is due on February 3 at 12 pm

bCourses “Pages”:

- FAQ for lab 02: published
- “Required and useful functions”: in construction
- “Common error messages and their causes”: in construction

Today:

- Matrices (what are they?)
- Matrix addition, multiplication, exponentiation, inverse, transpose

Friday:

- Nested if-statements
- Comparing floating point numbers
- Practice questions

Matrices: definitions

A **real matrix** is a 2-dimensional array of real numbers. For example:

$$\begin{bmatrix} 2 & 4 & 5 \\ 0 & 1 & 9 \end{bmatrix} \leftarrow \text{a 2 by 3 matrix}$$

A ***m* by *n* matrix** ($m, n \in \mathbb{N}$) is a matrix with m rows and n columns

A 1 by n matrix is also called a **row vector**. For example:

$$\begin{bmatrix} 2 & 4 & 5 & 7 \end{bmatrix}$$

A n by 1 matrix is also called a **column vector**

$$\begin{bmatrix} 0 \\ 6 \\ 7 \\ 3 \end{bmatrix}$$

Matrices: definitions (continued)

A **square matrix** is an n by n matrix. For example:

$$\begin{bmatrix} 4 & 5 & 4 & 0 \\ 1 & 6 & 3 & 2 \\ 1 & 7 & 9 & 8 \\ 10 & 5 & 6 & 9 \end{bmatrix}$$

Identity matrices are square matrices whose:

- diagonal terms are all 1; and
- off-diagonal terms are all 0; and

For example:

the 2 by 2 identity matrix $\rightarrow$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\leftarrow$ the 4 by 4 identity matrix

$\uparrow$

the 3 by 3 identity matrix

Matrix operations: addition and multiplication by a scalar

Matrix addition is element-wise addition. For example:

$$\begin{bmatrix} -2 & -4 & 5 \\ 0 & 1 & 9 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 0 \\ 0 & 8 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 5 \\ 0 & 9 & 7 \end{bmatrix}$$

Multiplication by a scalar: multiply each element by the scalar. For example:

$$7 \times \begin{bmatrix} -2 & -4 & 5 \\ 0 & 1 & 9 \end{bmatrix} = \begin{bmatrix} -14 & -28 & 35 \\ 0 & 7 & 63 \end{bmatrix}$$

Matrix operations: matrix multiplication

Consider a m_1 by n_1 real matrix A and a m_2 by n_2 real matrix B

- The matrix multiplication of A by B (call it $C = A \times B$) is defined if and only if $n_1 = m_2$. We say that:
  "**The inner dimensions of the matrices must be equal**"

- The result is the m_1 by n_2 matrix C such that:

Element in i^{th} row and j^{th} column

$$C_{i,j} = \sum_{k=1}^{k=n_1} A_{i,k} B_{k,j}$$

- **Matrix multiplication is not commutative**, meaning that $A \times B$ is not necessarily equal to $B \times A$. If fact, sometimes $A \times B$ is defined but $B \times A$ is not

Matrix multiplication: example

$$A = \begin{bmatrix} 5 & 0 & 1 & 2 \\ -1 & 4 & -2 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 5 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} 29 & 13 & 18 \\ 4 & 30 & 46 \end{bmatrix}$$

$$\begin{aligned}
C_{1,1} &= 5 \times 6 + 0 \times 2 + 1 \times (-1) + 2 \times 0 = 29 \\
C_{1,2} &= 5 \times 0 + 0 \times 1 + 1 \times 5 + 2 \times 4 = 13 \\
C_{1,3} &= 5 \times 2 + 0 \times 3 + 1 \times 0 + 2 \times 4 = 18 \\
C_{2,1} &= -1 \times 6 + 4 \times 2 + -2 \times (-1) + 9 \times 0 = 4 \\
C_{2,2} &= -1 \times 0 + 4 \times 1 + -2 \times 5 + 9 \times 4 = 30 \\
C_{2,3} &= -1 \times 2 + 4 \times 3 + -2 \times 0 + 9 \times 4 = 46
\end{aligned}$$

Matrix: practice question

A is a 5 by 10 matrix and B is a 7 by 5 matrix. Which of the following statements are true?

(A) $A + B$ is defined

(B) $B + A$ is defined

(C) $3 \times A$ is defined

(D) $A \times B$ is defined

(E) $B \times A$ is defined

Notes:

- Matrix addition is commutative, so if $A + B$ is defined, then so is $B + A$, and in this case $A + B = B + A$
- Matrix multiplication by a scalar is always defined

Consider the following matrices:

$$A = \begin{bmatrix} 4 & -1 \\ 3 & 0 \\ 6 & 8 \\ 2 & -7 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$$

Calculate $A \times B$!

$$A \times B = \begin{bmatrix} 9 & -19 \\ 6 & -12 \\ 4 & 0 \\ 11 & -29 \end{bmatrix}$$

Matrix: practice question

Consider the following matrices:

$$A = \begin{bmatrix} 4 & -1 \\ 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$$

Calculate $A \times B$ and $B \times A$!

$$A \times B = \begin{bmatrix} 9 & -19 \\ 6 & -12 \end{bmatrix}$$
$$B \times A = \begin{bmatrix} -4 & -2 \\ 5 & 1 \end{bmatrix}$$

Again, matrix multiplication is not commutative. For example, in the example above: $A \times B \neq B \times A$

Inverse of a matrix

Consider an n by n square matrix A. The inverse of A, if it exists, is the n by n square matrix A^{-1} such that $A \times A^{-1}$ is the n by n identity matrix

- The inverse of a non-square matrix never exists
- The inverse of a square matrix does not always exist
- If A^{-1} is the inverse of A, then A is the inverse of A^{-1}
  - In other words: $(A^{-1})^{-1} = A$
- The inverse of the identity matrix is the identity matrix

In the example below, A^{-1} is the inverse of A:

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 0 & 1 \\ 0.25 & -0.5 \end{bmatrix}$$

$$A \times A^{-1} = A^{-1} \times A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ $\leftarrow$ the 2 by 2 identity matrix

Matrix exponentiation

Consider an n by n square matrix A and a positive integer n. Then:

$$A^n = A \times A \times \cdots \times A \text{ (n times)}$$

For example:

$$\begin{aligned}
\begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix}^3 &= \begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -4 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 12 \\ 0 & -8 \end{bmatrix}
\end{aligned}$$

Transpose of a matrix

Consider an m by n matrix A. The transpose of A is the n by m matrix A^{T} such that $A^{\mathrm{T}}_{i,j} = A_{j,i}$

- In other words, when transposing a matrix "**the rows become the columns and the columns become the rows**"
- Note that $(A^{\mathrm{T}})^{\mathrm{T}} = A$

For example:

$$\begin{bmatrix} -5 & -8 & 7 & -1 & 6 \\ 6 & 7 & -9 & 9 & 4 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} -5 & 6 \\ -8 & 7 \\ 7 & -9 \\ -1 & 9 \\ 6 & 4 \end{bmatrix}$$

Arrays and matrix operations in Matlab

The following Matlab operators are matrix operators:

- `*` (matrix multiplication)
- `/` (multiply by the inverse or pseudo-inverse of)
- `^` (matrix exponentiation)
- `'` (transpose). Alternatively, use the function "`transpose`"

To inverse a matrix "`A`" in Matlab, use `inv(A)` or `A^(-1)`

The following Matlab operators are element-wise operators:

- `.*` (element-wise multiplication)
- `./` (element-wise division)
- `.^` (element-wise exponentiation)

Matrix addition and subtraction are the same as element-wise addition and subtraction, respectively, so the operators `.+` and `.-` do not exist

Element-wise and matrix operations between scalars are equivalent

Practice question

What will the values of the variables "c" and "d" be after executing the following commands?

```
>> a = [3, 4, 1; -2, 1, 0];
>> b = [1, 0; 1, 1; 2, -2];
>> c = ((a*b)')^2;
>> d = ((a*b)').^2;
```

c:

$$\begin{bmatrix} 79 & -10 \\ 20 & -1 \end{bmatrix}$$

d:

$$\begin{bmatrix} 81 & 1 \\ 4 & 1 \end{bmatrix}$$

Adding a single character (here: a period) can completely change the result of an expression!