L21: Root Finding and Systems of Equations

Applications

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E7 Spring 2017, University of California at Berkeley

March 10, 2017

Version: release

Announcements

Lab 08 is due on March 17 at 12 pm (noon)

Today:

- Applications of the following techniques to physical problems:
  - Root finding (Chapter 16)
    - Settling velocity of a particle
    - Black body radiation
  - Systems of linear algebraic equations (Chapter 12)
    - Electric circuit

Next week:

- Least-square regression (chapter 13)

Root finding: practice question

Assume that we are trying to find roots of a continuous real-valued function f defined over $\mathbb{R}$. Which of the following statements are true?

(A) One needs to calculate f' to use the bisection method

(B) Both bisection and Newton-Raphson are iterative methods

(C) The Newton-Raphson method finds all the roots of f at once

(D) The bisection method finds all the roots of f at once

(E) Sometimes, the Newton-Raphson method does not converge

(F) The bisection method relies on f changing sign around a root

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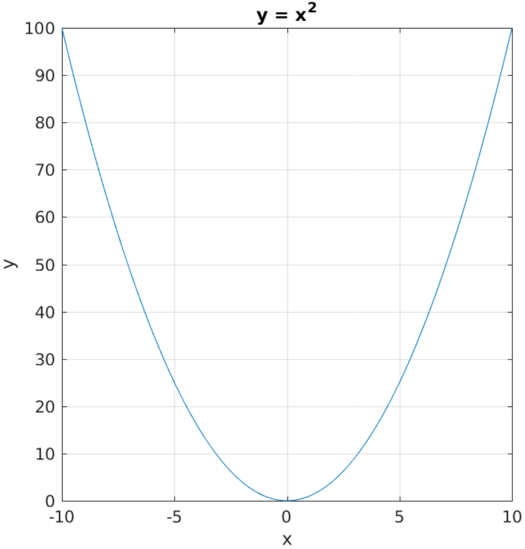
(E) Sometimes, the Newton-Raphson method does not converge

(F) The bisection method relies on f changing sign around a root

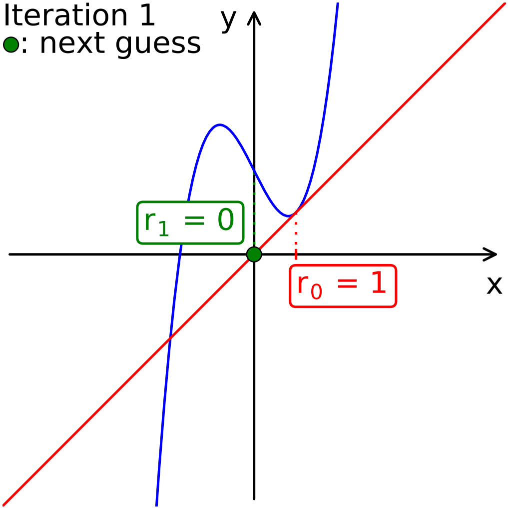
Example of a case where the bisection method fails

This function never changes sign!



Example of a case where the Newton-Raphson method fails

In the example below, the Newton-Raphson method never finds a root



Function:

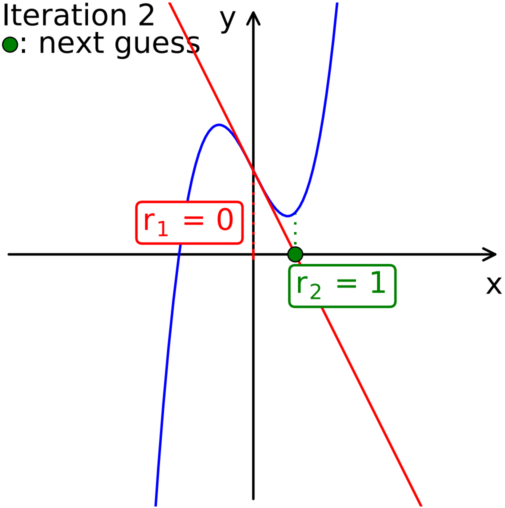
$f : x \mapsto x^3 - 2x + 2$

Initial guess:

$r_0 = 1$

i	r_i	r_{i+1}
0	1	0

In the example below, the Newton-Raphson method never finds a root



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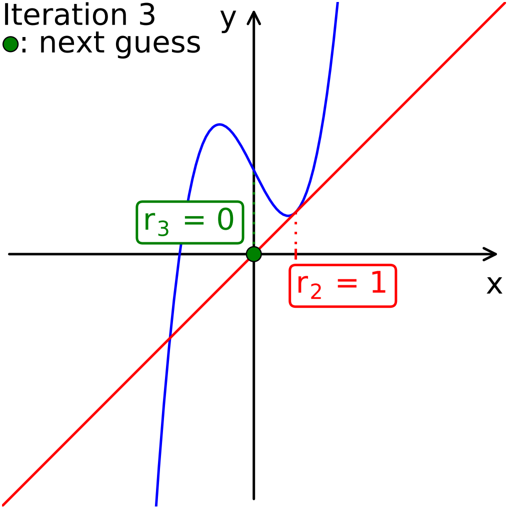

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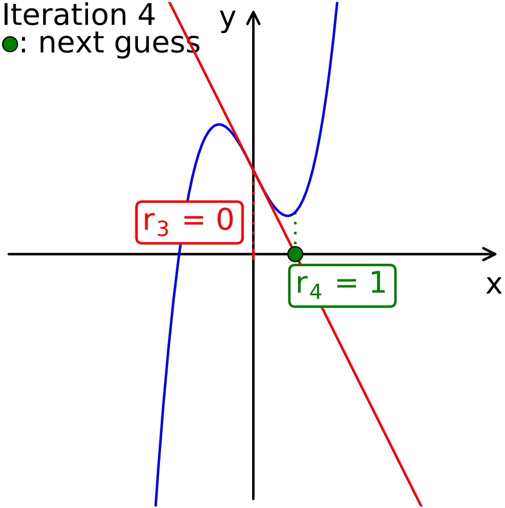

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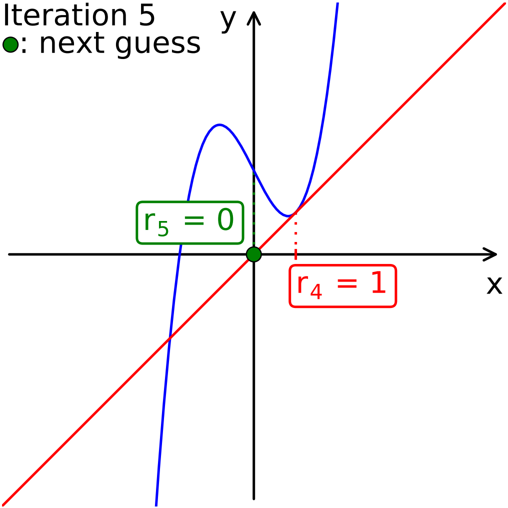

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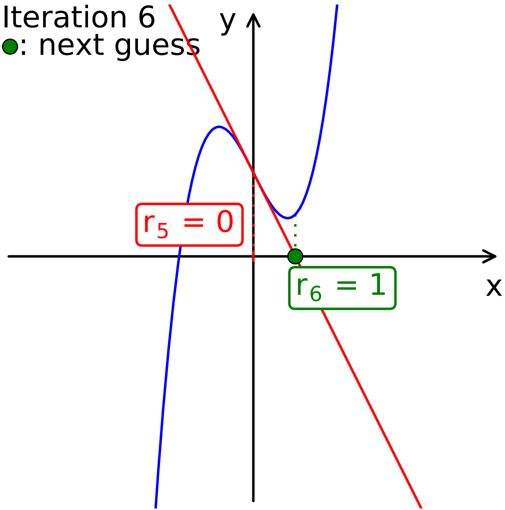

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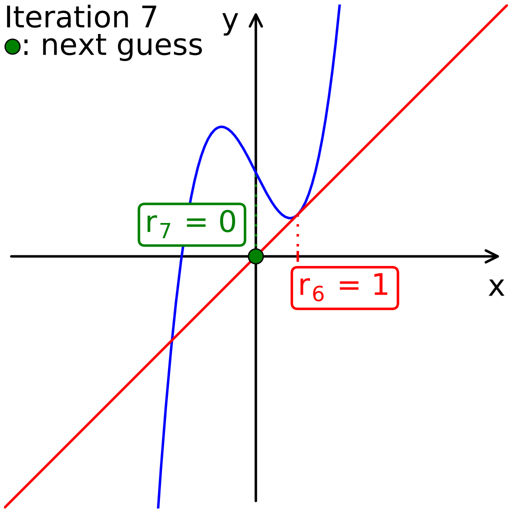

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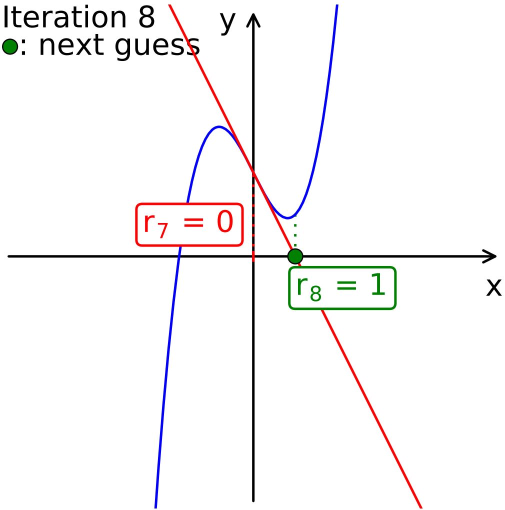

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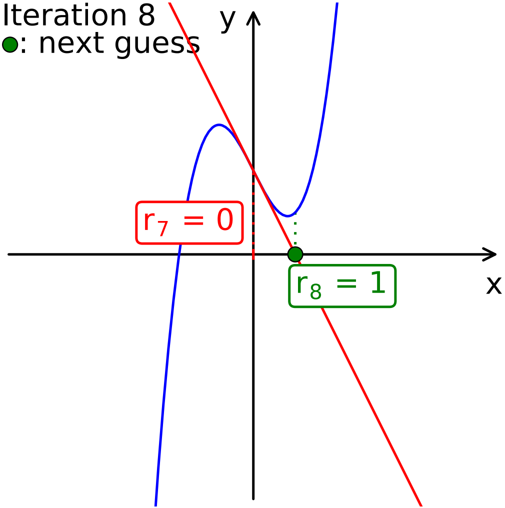

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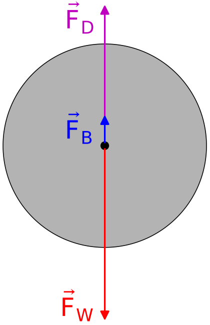
...	...	...

Application of root finding: particle gravitational settling

Particle gravitational settling: particle "falling" due to gravity



External forces acting on the particle:

d: particle diameter; ρ_p: density of particle; ρ_f: density of fluid
v: velocity of particle (settling velocity); C_D: drag coefficient; g: acceleration of gravity
Reference: Nazaroff & Alvarez-Cohen (2001), *Environmental Engineering Science*, John Wiley & Sons, Inc.

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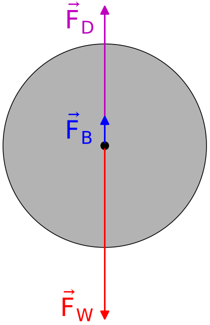

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$$F_W = \frac{\pi}{6} d^3 \rho_p g$$

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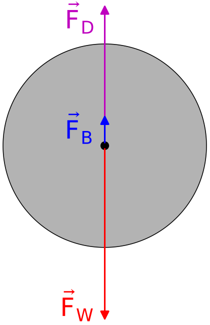

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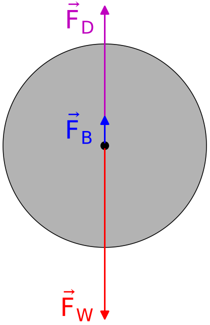

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- **Drag:** vertical upward, magnitude:

$$F_D = \frac{\pi}{4} d^2 (\frac{1}{2} \rho_f v^2) C_D$$

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Application of root finding: particle gravitational settling

Assume steady-state (velocity v constant), so $F_W = F_B + F_D$:

$$4d(\rho_p - \rho_f)g/3 - \rho_f v^2 C_D = 0$$

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Drag coefficient:

$$C_D = \begin{cases} 24/R_e & \text{if } R_e \leqslant 0.3 \\ (24/R_e) \times (1 + 0.14 R_e^{0.7}) & \text{if } 0.3 < R_e < 1000 \\ 0.445 & \text{if } R_e \geqslant 1000 \end{cases}$$

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R_e: Reynolds number $\rightarrow$ measures importance of inertia versus viscosity

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Objective: write a function that calculates the settling velocity v

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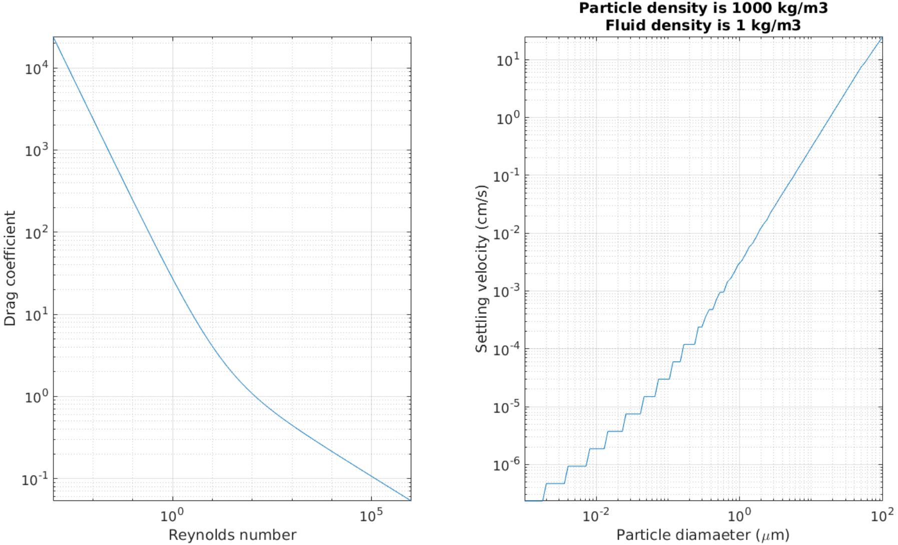
Application of root finding: particle gravitational settling

Solution: see function `my_settling_velocity.m`



Application of root finding: black body radiation

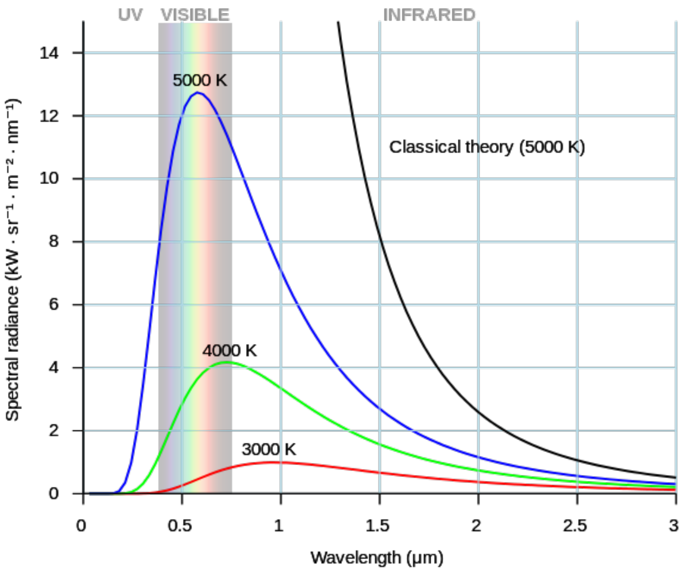
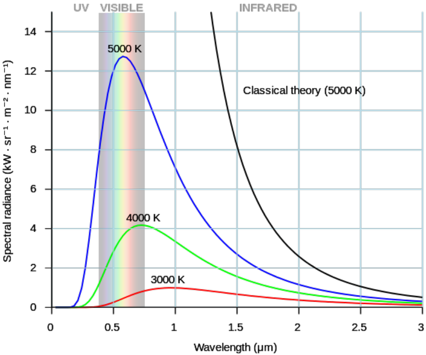


Image (worldwide public domain) retrieved from https://en.wikipedia.org/wiki/File:Black_body.svg on March 10th 2017

Application of root finding: black body radiation

Black body: a physical body that absorbs all incoming radiation (it is an idealization)

- A black body emits radiation at different wavelengths
- The emission spectrum (radiated energy as a function of wavelength λ) depends on the body's temperature T according to Planck's law:



E: spectral radiance per unit wavelength; λ: wavelength; T: temperature;
h: Planck constant; c: speed of light in vacuum; k_B: Boltzmann constant

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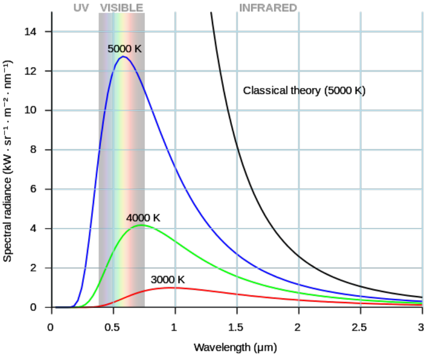

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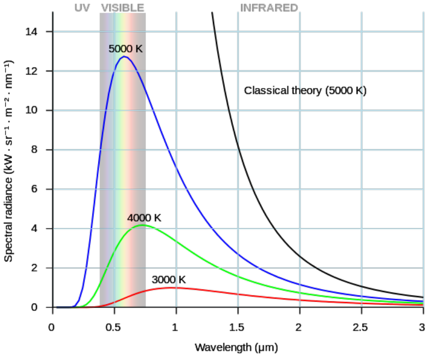
The sun is almost a black body at $T \approx 6000$ K; it emits mostly in the visible

Temperature of Earth: ≈ 273 K, it emits in the infrared that we cannot see



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Application of root finding: black body radiation

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Application of root finding: black body radiation

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Method:

1. Determine the derivative $E'(\lambda)$ of $E(\lambda)$
2. Find $\lambda_{\max}$ such that $E'(\lambda_{\max}) = 0$
   If E has a maximum at $\lambda = \lambda_{\max}$, then $E'(\lambda_{\max}) = 0$

   Important: $E'(\lambda_{\max}) = 0$ does not imply that E has a maximum at $\lambda = \lambda_{\max}$!

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4. Ideally (not done here), verify that E has indeed a maximum at $\lambda = \lambda_{\max}$

Application of root finding: black body radiation

$$E(\lambda) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp[hc/(\lambda k_B T)] - 1}$$

$$E'(\lambda) = -\frac{10hc^2}{\lambda^6} \frac{1}{\exp[hc/(\lambda k_B T)] - 1} + \frac{2hc^3}{\lambda^7 k_B T} \left(\frac{\exp[hc/(\lambda k_B T)]}{(\exp[hc/(\lambda k_B T)^2] - 1)^2} \right)$$

Application of root finding: black body radiation

Solution: see `my_black_body_wavelength.m`

Note that, according to Wien's wavelength displacement law:

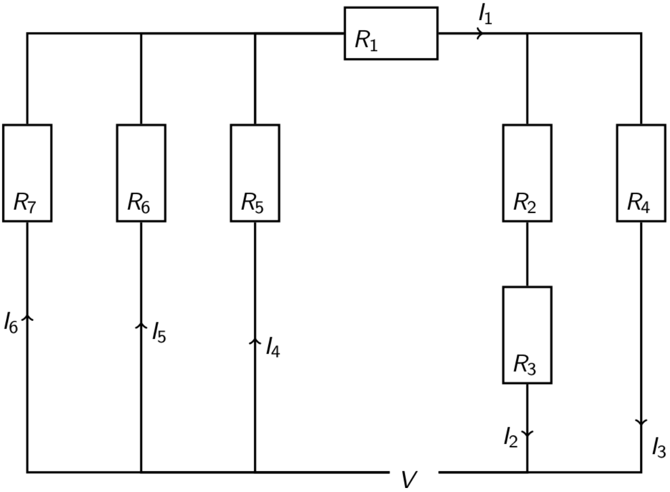
$$\lambda_{\max} = \frac{b}{T} \quad \text{with} \quad b = 0.0028977729 \text{ m K}^{\star}$$

```
>> temp = 300;
>> low = 1e-9;
>> high = 1e-5;
>> tol = 1e-3;
>> lambda_max = my_black_body_wavelength(temp, low, high, tol)
lambda_max =
   9.6592e-06

>> % Compare our result to the result given by Wien's law
>> 0.0028977729 / temp
ans =
   9.6592e-06
```

* The Wien wavelength displacement law constant was retrieved from http://physics.nist.gov on March 13th 2017

Application of systems of equations: electric circuit



Objective: Knowing the value of V (constant voltage) and the values of the resistances R_1 through R_7, what are the values of the currents I_1 through I_6?

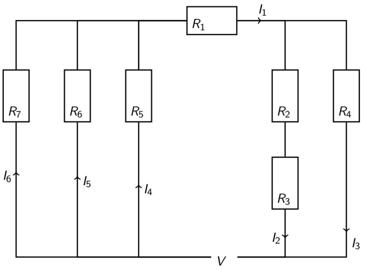
Application of systems of equations: electric circuit

Method:

1. Write a system of 6 equations and six unknowns
2. Write the system in matrix form
3. Use Matlab to solve this system

Application of systems of equations: electric circuit

Step 1: Write a system of 6 equations and six unknowns



Application of systems of equations: electric circuit

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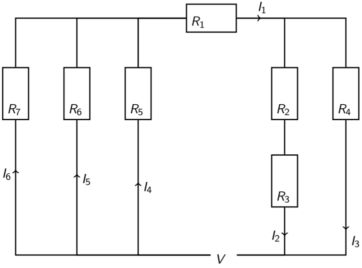

$$I_1 = I_2 + I_3$$

Application of systems of equations: electric circuit

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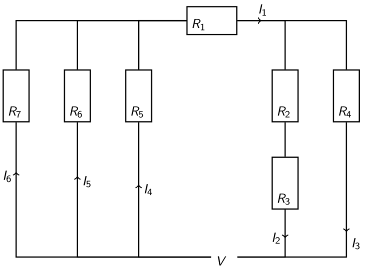

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Application of systems of equations: electric circuit

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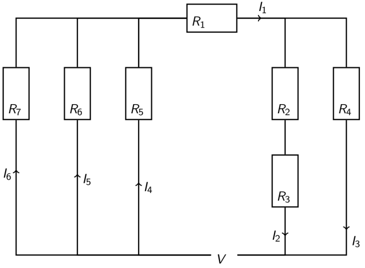

$$I_1 = I_2 + I_3$$
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$$R_4 I_3 = (R_2 + R_3) I_2$$

Application of systems of equations: electric circuit

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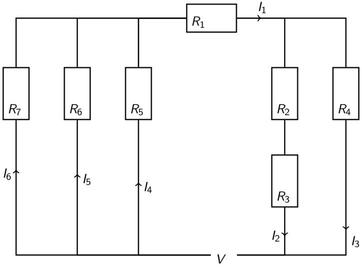

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$$R_5 I_4 + R_1 I_1 + (R_2 + R_3) I_2 = V$$

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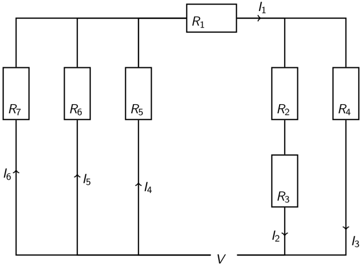

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$$R_6 I_5 = R_5 I_4$$

Application of systems of equations: electric circuit

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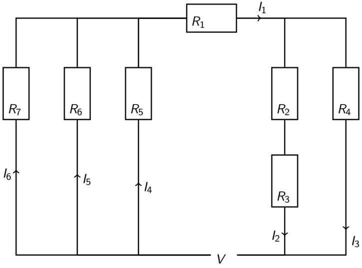

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$$R_7 I_6 = R_6 I_5$$

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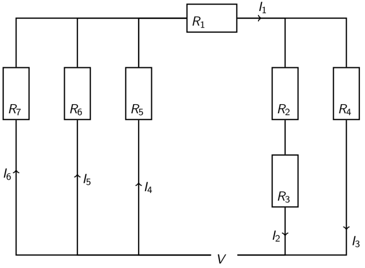
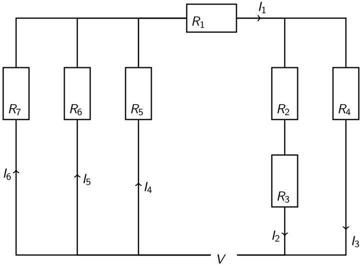


$$I_1 = I_2 + I_3$$
$$I_1 = I_4 + I_5 + I_6$$
$$R_4 I_3 = (R_2 + R_3) I_2$$
$$R_5 I_4 + R_1 I_1 + (R_2 + R_3) I_2 = V$$
$$R_6 I_5 = R_5 I_4$$
$$R_7 I_6 = R_6 I_5$$

Step 2: Write the system in matrix form

Step 1: Write a system of 6 equations and six unknowns



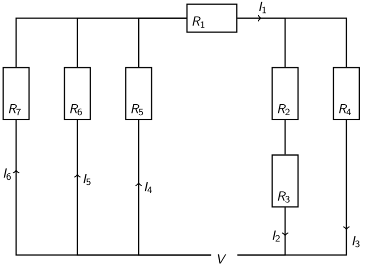
$$I_1 = I_2 + I_3$$
$$I_1 = I_4 + I_5 + I_6$$
$$R_4 I_3 = (R_2 + R_3) I_2$$
$$R_5 I_4 + R_1 I_1 + (R_2 + R_3) I_2 = V$$
$$R_6 I_5 = R_5 I_4$$
$$R_7 I_6 = R_6 I_5$$

Step 2: Write the system in matrix form (tip: write unknowns first)

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix}$$

Application of systems of equations: electric circuit

Step 1: Write a system of 6 equations and six unknowns



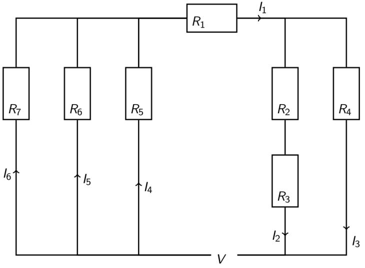
$$
\begin{aligned}
I_1 &= I_2 + I_3\\
I_1 &= I_4 + I_5 + I_6\\
R_4 I_3 &= (R_2 + R_3) I_2\\
R_5 I_4 + R_1 I_1 + (R_2 + R_3) I_2 &= V\\
R_6 I_5 &= R_5 I_4\\
R_7 I_6 &= R_6 I_5
\end{aligned}
$$

Step 2: Write the system in matrix form (tip: write unknowns first)

$$
\begin{bmatrix}
1 & -1 & -1 & 0 & 0 & 0\\
1 & 0 & 0 & -1 & -1 & -1\\
0 & -(R_2+R_3) & R_4 & 0 & 0 & 0\\
R_1 & (R_2+R_3) & 0 & R_5 & 0 & 0\\
0 & 0 & 0 & -R_5 & R_6 & 0\\
0 & 0 & 0 & 0 & -R_6 & R_7
\end{bmatrix}
\begin{bmatrix} I_1\\ I_2\\ I_3\\ I_4\\ I_5\\ I_6 \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ 0\\ V\\ 0\\ 0 \end{bmatrix}
$$

Application of systems of equations: electric circuit

Step 1: Write a system of 6 equations and six unknowns



$$
\begin{aligned}
I_1 &= I_2 + I_3 \\
I_1 &= I_4 + I_5 + I_6 \\
R_4 I_3 &= (R_2 + R_3) I_2 \\
R_5 I_4 + R_1 I_1 + (R_2 + R_3) I_2 &= V \\
R_6 I_5 &= R_5 I_4 \\
R_7 I_6 &= R_6 I_5
\end{aligned}
$$

Step 2: Write the system in matrix form (tip: write unknowns first)

$$
\begin{bmatrix}
1 & -1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & -1 & -1 \\
0 & -(R_2+R_3) & R_4 & 0 & 0 & 0 \\
R_1 & (R_2+R_3) & 0 & R_5 & 0 & 0 \\
0 & 0 & 0 & -R_5 & R_6 & 0 \\
0 & 0 & 0 & 0 & -R_6 & R_7
\end{bmatrix}
\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ V \\ 0 \\ 0 \end{bmatrix}
$$

Step 3: Solve the system using Matlab (see `my_circuit.m`)