L26: Taylor Series

And introduction to the final programming project

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Version: release

Announcements

Lab 09 is due on March 24 at 12 pm (noon)

Today:

- One more thing about interpolation (see updated slides of L25)
- Taylor series (chapter 15)
- Introduction to the final programming project

Friday:

- Pseudo-random numbers
- Discussion

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Next Week: Spring break!

- Get some rest, have some fun, see friends and family
- Get a head start on lab 10 and on your E7 project

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After Spring break:

- Numerical differentiation (Chapter 17)
- Numerical integration (Chapter 18)

Taylor series

Consider a real-valued function that is C^∞ (*i.e.* infinitely differentiable) over some interval I. Pick a point $a \in I$. Then, for any value x of this interval I:

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \\ &= f(a) + \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}$$

$f^{(k)}$ is the k^{th} derivative of f

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The equation above is the **Taylor series of f centered at a**. **It gives an expression for $f(x)$ as a function of x and of the values of the function and its derivatives at a**

Truncated Taylor series

$$f(x) = f(a) + \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
$$\approx f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

One often uses truncated Taylor series "with m terms" (i.e. the constant term plus the first $m-1$ terms of the infinite sum above) to approximate functions

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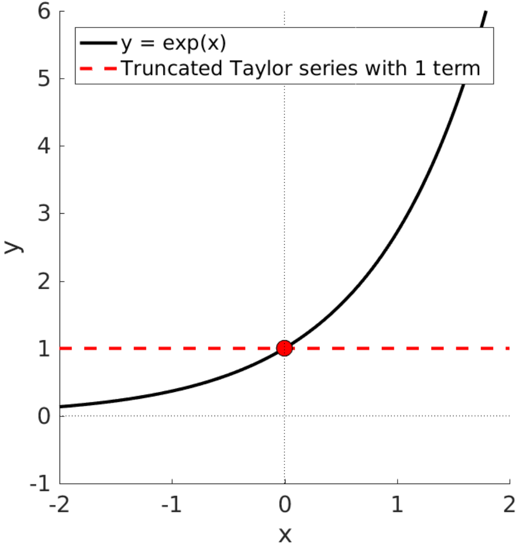
In general:

- The closer x is to the center point a, the smaller the error
- When using more terms of the series (*i.e.* higher n), the above approximation holds for a larger range of x away from the center point a (see figures in following slides)

Function: $x \mapsto \exp(x)$, **center point** $a = 0$



Truncated Taylor series: examples

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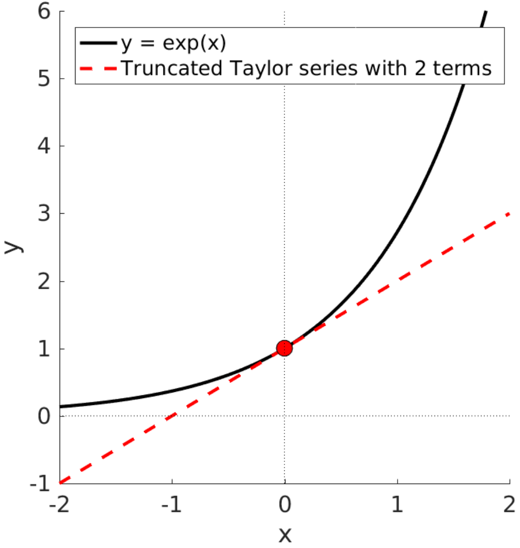

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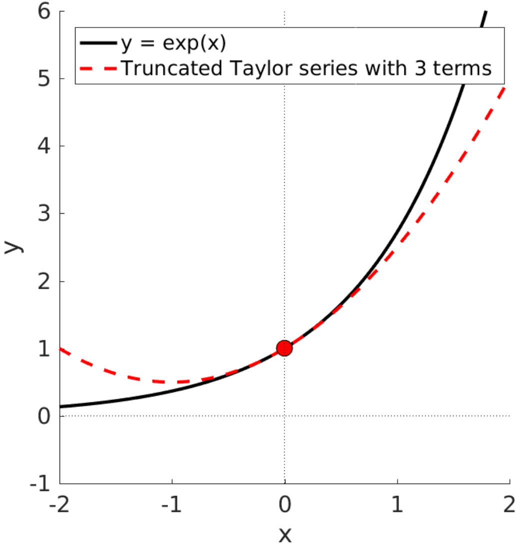

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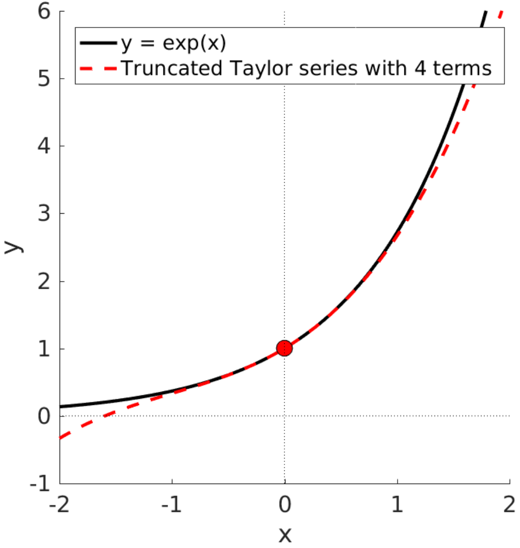

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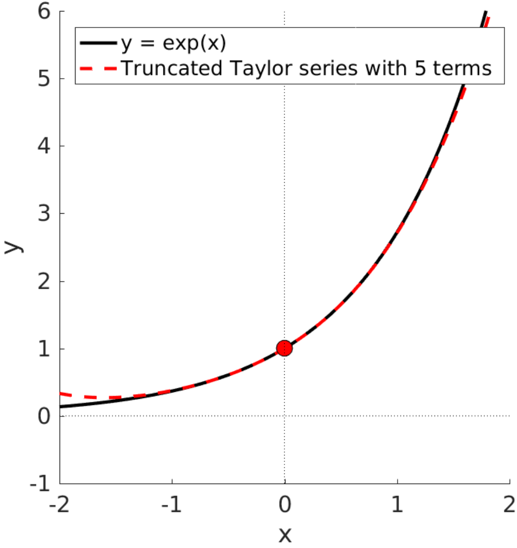

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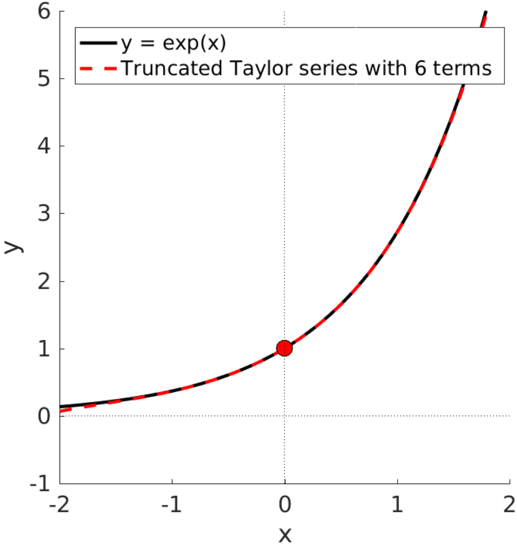

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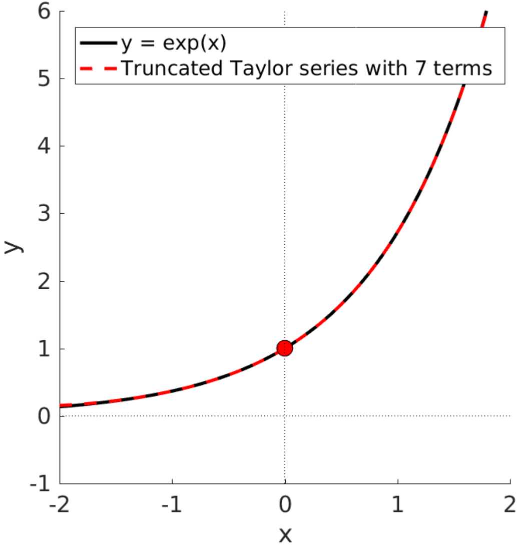

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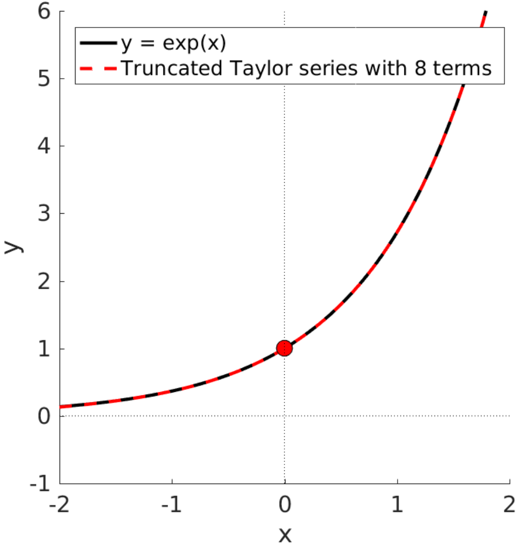

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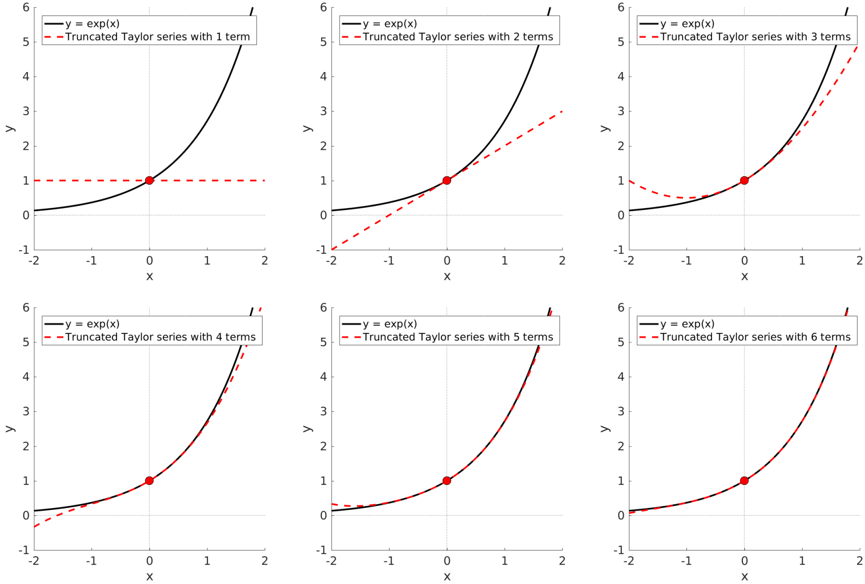
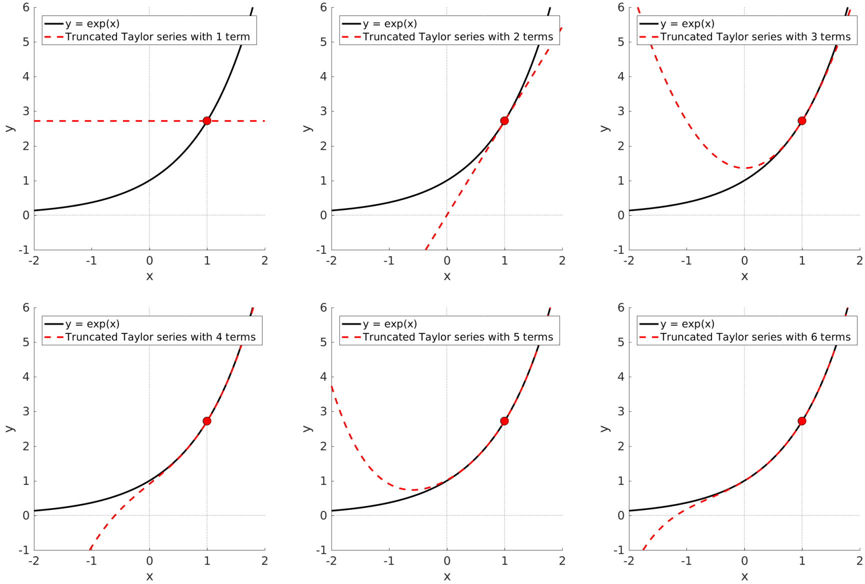


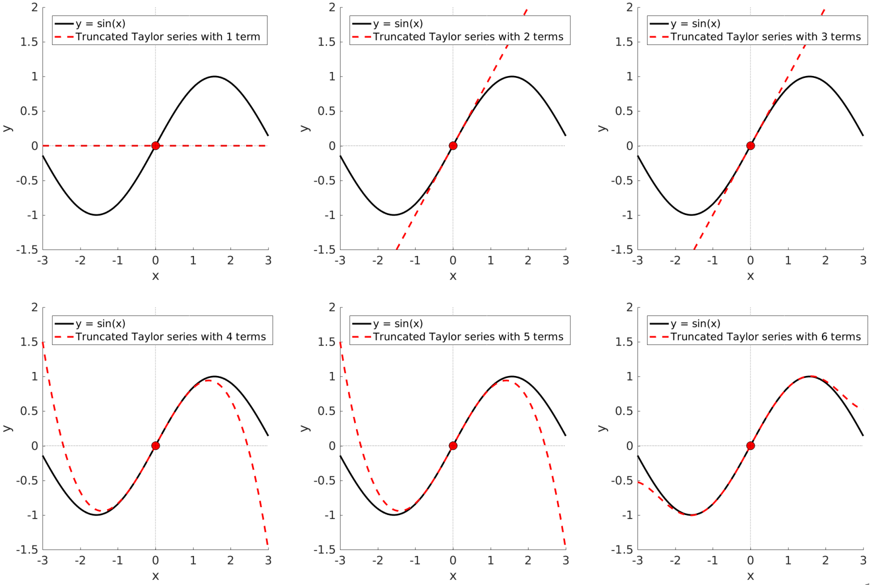
Truncated Taylor series: examples

Function: $x \mapsto \exp(x)$, center point $a = 1$



Truncated Taylor series: examples

Function: $x \mapsto \sin(x)$, **center point** $a = 0$



Truncated Taylor series: examples

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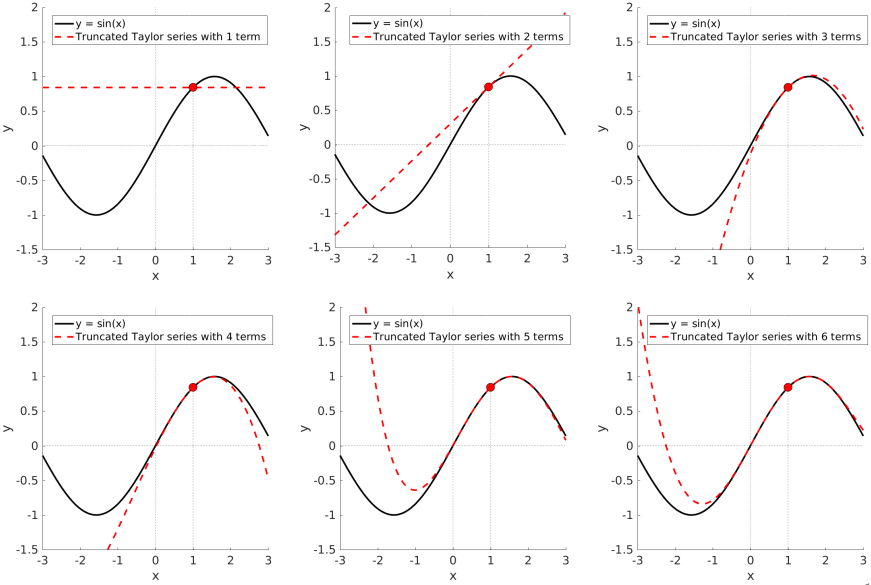
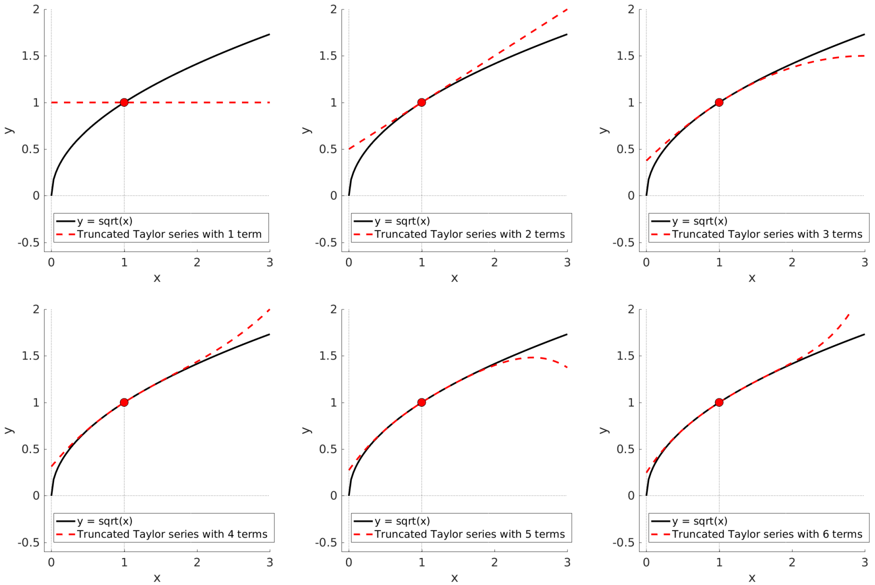


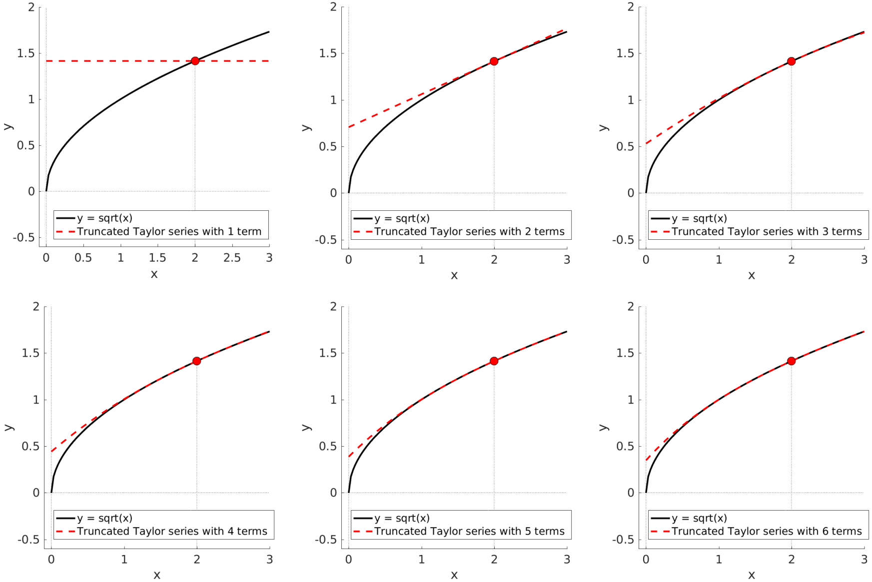
Truncated Taylor series: examples

Function: $x \mapsto \sqrt{x}$, center point $a = 1$

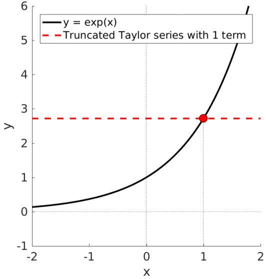


Truncated Taylor series: examples

Function: $x \mapsto \sqrt{x}$**, center point** $a = 2$



Example with function: $x \mapsto \exp(x)$ **and center point** $a = 1$



1 term:
Approximate the function as constant

$$f(x) \approx f(a)$$

Truncated Taylor series: geometrical interpretation

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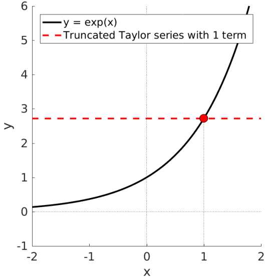
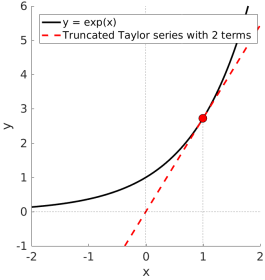




1 term:
Approximate the function as constant

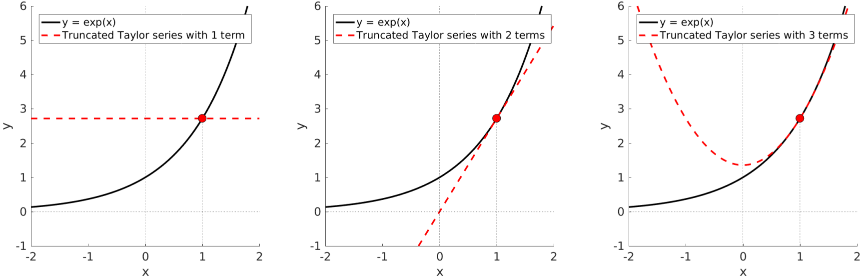
$$f(x) \approx f(a)$$

2 terms:
Approximate the function with its tangent at $x = a$

$$f(x) \approx f(a) + f'(a)(x - a)$$

Truncated Taylor series: geometrical interpretation

Example with function: $x \mapsto \exp(x)$ **and center point** $a = 1$



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More terms:
Add curvature to the approximation

"Classic" Taylor series

Maclaurin series: Taylor series with center point $a = 0$

$$\exp(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$$

$$\ln(1 - x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \qquad \ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$$

$$\cos(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cosh(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!} \qquad \sinh(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

Taylor series: practice question

Consider an object falling down vertically. At time $t = t_0 = 1$ s:

- it is **located** at height $z(t = t_0) = 35$ m
- its vertical **velocity** is $v(t = t_0) = -4 \text{ m s}^{-1}$
  (the minus sign indicates that the velocity is downward)
- its vertical **acceleration** is $a(t = t_0) = -10 \text{ m s}^{-2}$
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Estimate the height of the item at $t = 3$ s, **using a Taylor series with 1 term**

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$$\begin{aligned} h(t) &\approx h(t_0) \\ h(t = 3\text{ s}) &\approx h(t_0) \\ &= 35 \text{ m} \end{aligned}$$

This approximation is not very good: it indicates that the object has not moved!

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Estimate the height of the item at $t = 3$ s, **using a Taylor series with 2 terms**

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Estimate the height of the item at $t = 3$ s, **using a Taylor series with 2 terms**

$$h(t) \approx h(t_0) + h'(t_0)(t - t_0)$$

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$$\begin{aligned} h(t) &\approx h(t_0) + h'(t_0)(t - t_0) \\ &= h(t_0) + v(t_0)(t - t_0) \end{aligned}$$

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Estimate the height of the item at $t = 3$ s, **using a Taylor series with 2 terms**

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h(t) &\approx h(t_0) + h'(t_0)(t - t_0) \\
&= h(t_0) + v(t_0)(t - t_0) \\
h(t = 3\ \mathrm{s}) &\approx 35\ \mathrm{m} - 4\ \mathrm{m\,s^{-1}} \times (3\ \mathrm{s} - 1\ \mathrm{s}) \\
&= 27\ \mathrm{m}
\end{aligned}
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$$h(t) \approx h(t_0) + h'(t_0)(t - t_0) + \frac{1}{2}h''(t_0)(t - t_0)^2$$

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&= h(t_0) + v(t_0)(t - t_0) + \frac{1}{2} a(t_0)(t - t_0)^2 \\
h(t = 3 \mathrm{~s}) &\approx 35 \mathrm{~m} - 4 \mathrm{~m~s^{-1}}(3 \mathrm{~s} - 1 \mathrm{~s}) - \frac{1}{2} \times 10 \mathrm{~m~s^{-2}} \times (3 \mathrm{~s} - 1 \mathrm{~s})^2 \\
&= 7 \mathrm{~m}
\end{aligned}
$$

E7 final programming project: objectives

Objective: Write a function that decides where your spaceship should go, depending on

- Location of scrap (worth points if picked up)
- Location of slow down areas (nebulae)
- Location of impenetrable areas
- Location and type of ghosts

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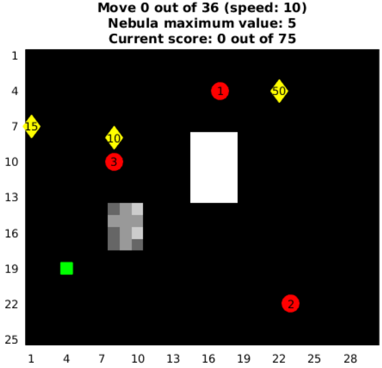
- **Have fun!**

E7 final programming project: game engine and visualizer



The function that runs the game will be provided to you:

```
function [game_stats] = e7planets_play(map, player_function)
```

- map: describes the game area (location of scrap, ghosts, ...)
  - We will provide you with sample maps
  - You can create your own maps
- player_function: **the function that you will write!**

E7 final programming project: your task

Write a function with the following header:

```
function [direction] = player_function(map)
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`map`: 1×1 `struct` array

- `map.grid`: $m \times n$ array of class `double` containing integers
  - A 0 indicates a place where you cannot go
  - A 1 indicates a place you can travel to and from
  - A number greater than 1 indicates a slow-down location. Each time you try to move away from a slow down location (`map.grid(i,j)>1`), you have 1 in `map.grid(i,j)` chances to successfully move away from this location

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- `map.player`: 1×1 `struct` array that indicates your location (previous and current) and your current score

- `map.scrap`: `struct` array that indicates the locations and values of scrap

E7 final programming project: your task

- `map.scrap`: `struct` array that indicates the locations and values of scrap
- `map.ghosts`: `struct` array that indicates the location (previous and current) and type of each ghost. Ghost types:
  - `'random'`: moves randomly
  - `'backandforth'`: moves in a straight line, back and forth
  - `'towardplayer'`: moves toward your spaceship

E7 final programming project: your task

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- `map.ghosts`: `struct` array that indicates the location (previous and current) and type of each ghost. Ghost types:
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You win the game if you pick up all the scrap within the number of allocated moves, and without being caught by a ghost

E7 final programming project: grading

Your function will be graded on **many different maps of varying size and difficulty**, for example:

- Scrap only
- Scrap and slow-down areas
- Scrap and obstacles
- Scrap, slow-down areas, and obstacles
- All of the above, with ghosts

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Recommendation: **focus on basic functionality before trying to implement advanced features**