L29: Numerical Integration

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E7 Spring 2017, University of California at Berkeley

April 5, 2017

Version: release

Announcements

Lab 10 is due on April 7 at 12 pm (noon)

Question 2, test case 1: it is okay if `coefficients(4,5)` is `0`

`coefficients(4,5)` is correct if `abs(coefficients(4,5)) < 1e-15`

Today:

- Numerical Integration (Chapter 18)

Friday (April 7):

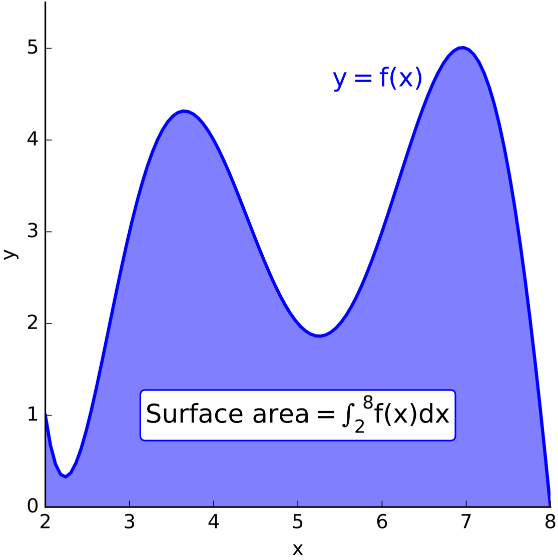
- Project discussion, tips, and recommendations
- Other discussion

Next week:

- Ordinary differential equations (Chapter 19)

Integral: area under the curve

Geometrically, the integral is equal to the "area under the curve"



Numerical integration: motivation

We know a number of analytical methods to calculate integrals
For example:

$$\int_0^4 (x^3 - x^2)dx = \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^4 = \frac{4^4}{4} - \frac{4^3}{3}$$

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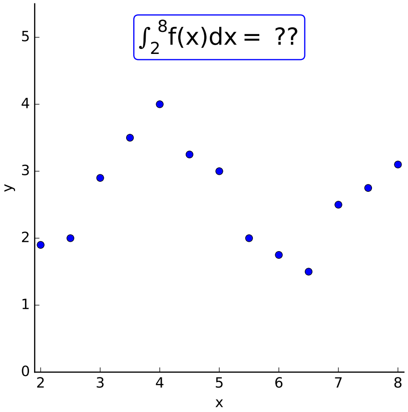

General approach to numerical integration

Objective: find an approximation of the following quantity:

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where f is a real-valued function

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2. On each of these "sub-intervals", approximate the integral using approximations of the function (*e.g.*, straight line)
3. Sum the approximated values made on each sub-interval

Method 1: Riemann integral

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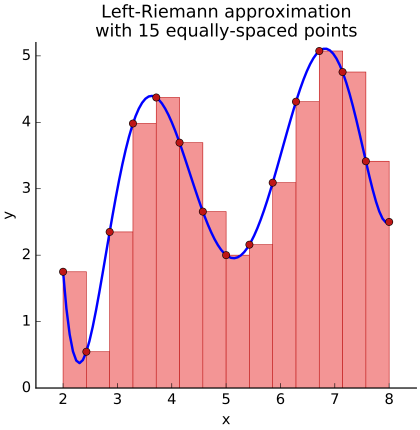

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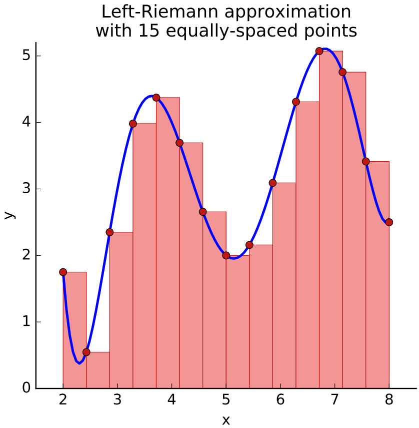

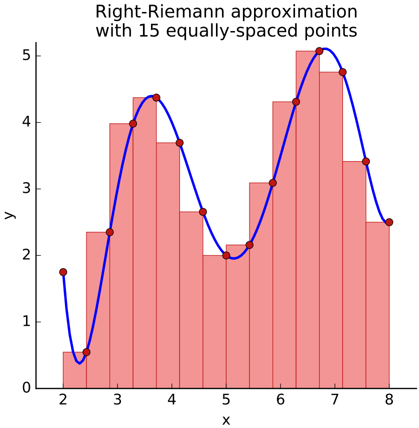

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx$$

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Method 1: Riemann integral – the math

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Left-Riemann integral:

$$= \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i)$$

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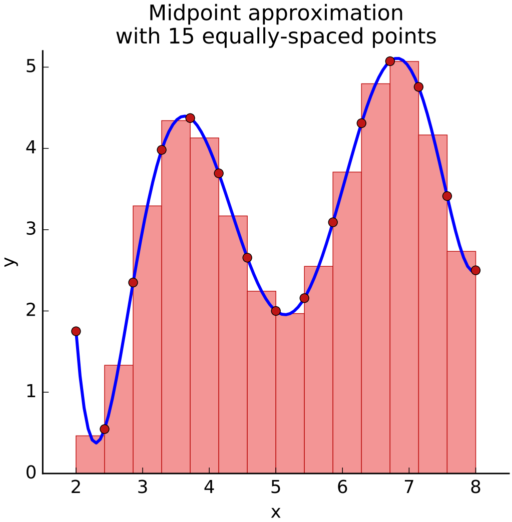
Method 2: Midpoint rule

On each interval $[x_i, x_{i+1}]$, approximate the function as a constant:

- $f(x) \approx f(\text{midpoint})$ on this interval



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$$\begin{aligned}
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&\approx \sum_{i=0}^{n-1} \text{Area of the corresponding rectangle} \\
&= \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i)
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Method 2: Midpoint rule – the math

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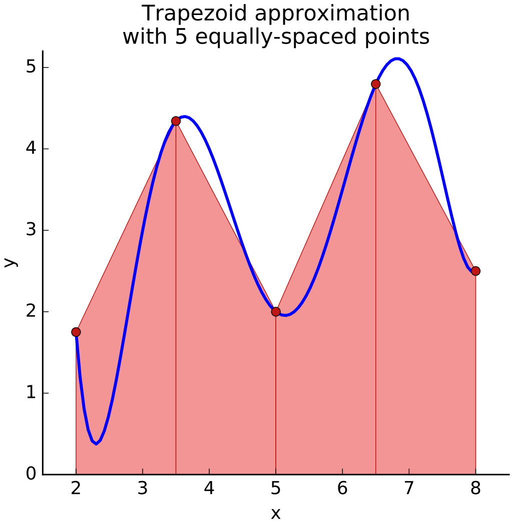
Limitation: this method requires us to calculate the value of the function at the midpoint of each interval. **We cannot use this method if we only know the value of the function at $x = x_i$, $i = \{0, 1, 2, \ldots, n\}$**

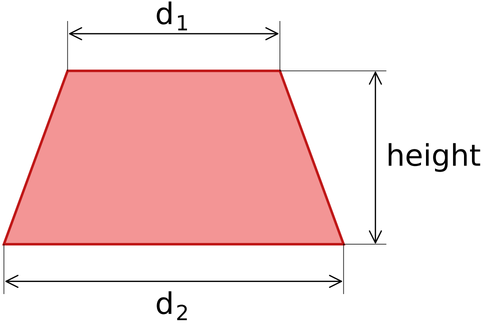
Method 3: Trapezoid rule

On each interval $[x_i, x_{i+1}]$, approximate the function as a straight line that goes through both (x_i, y_i) and (x_{i+1}, y_{i+1}):



Area of a Trapezoid

Trapezoid: a quadrilateral with two sides parallel to each other



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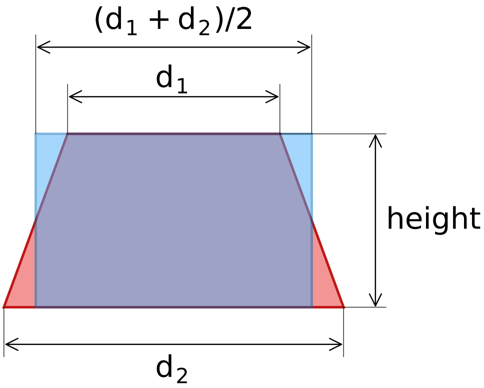
Trapezoid: a quadrilateral with two sides parallel to each other

Area of a trapezoid: height $\times$ average of the lengths of the bases



Method 3: Trapezoid rule – the math

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx$$
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$$

If the points are equally spaced, with spacing Δx:

$$
= \frac{\Delta x}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right)
$$

See the textbook for the derivation

Method 4: Simpson's rule

- Group the sub-intervals $[x_i, x_{i+1}]$ two by two
- On each of these pairs of intervals:
  - Fit a parabola
  - Calculate (analytically) the corresponding area under the parabola

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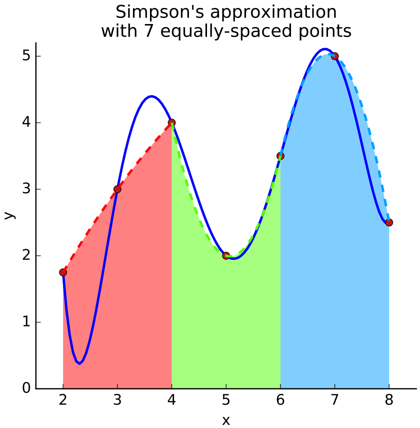

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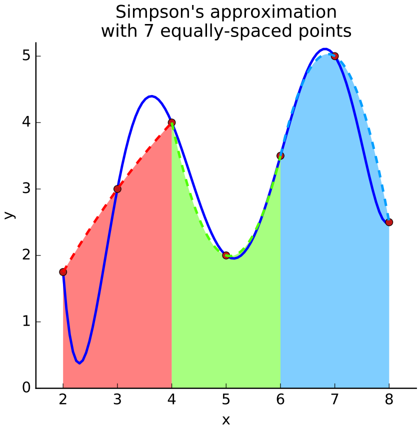

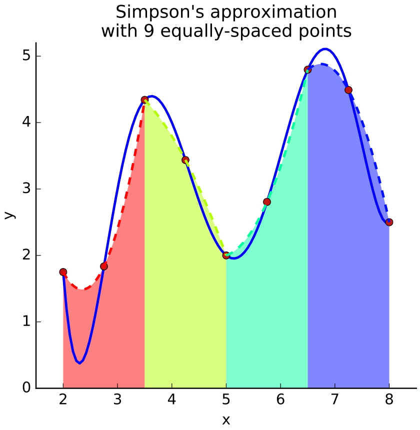

Method 4: Simpson's rule – the math

$$\int_a^b f(x)dx = \sum_{i=0}^{n/2-1} \int_{x_{2i}}^{x_{2i+2}} f(x)dx$$
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If the points are equally spaced, with spacing Δx:

$$= \frac{\Delta x}{3}\left(f(x_0) + 4 \sum_{\substack{i=1 \\ i \text{ is odd}}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i \text{ is even}}}^{n-2} f(x_i) + f(x_n) \right)$$

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Limitation: the number of sub-intervals $[x_i, x_{i+1}]$ must be even (*i.e.* the number of data points must be odd)

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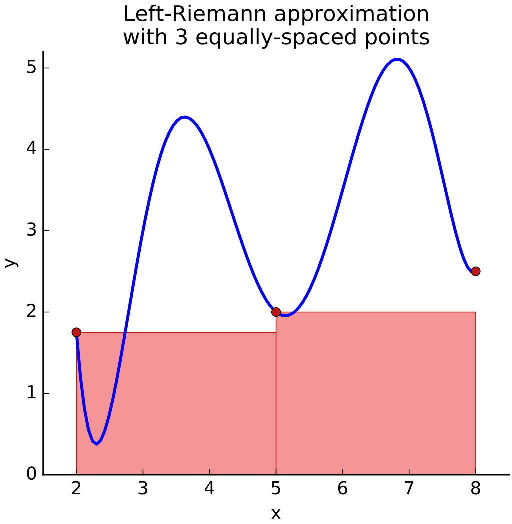
Increase the number of sub-intervals

In general, integral approximations become more accurate as the division of the interval into sub-intervals is refined



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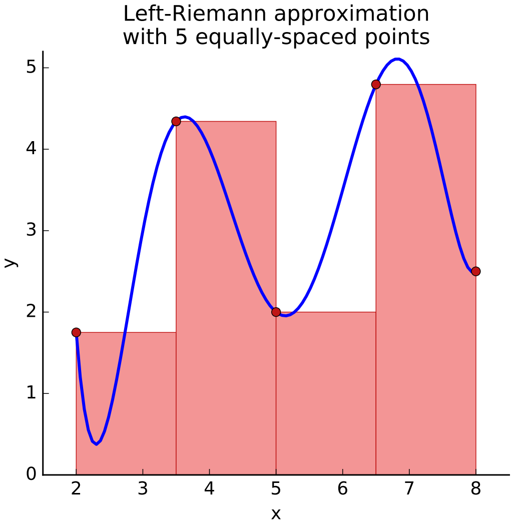

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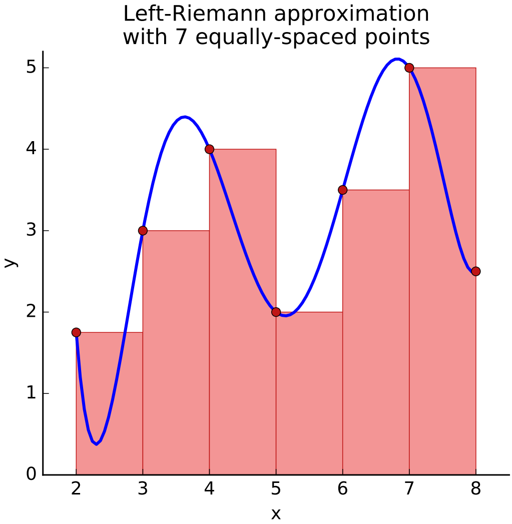

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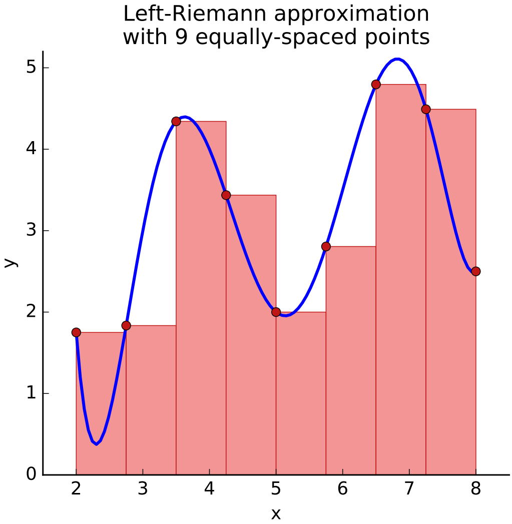

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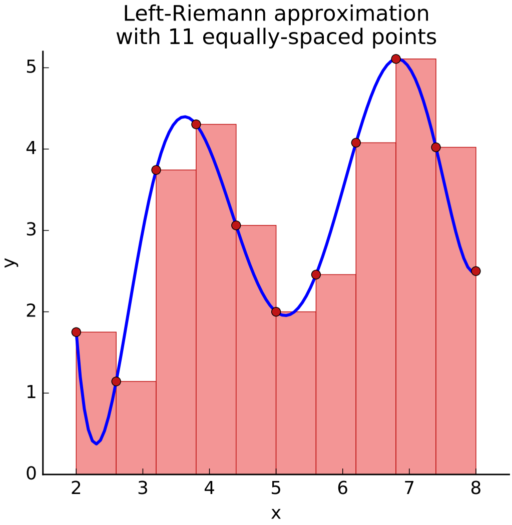

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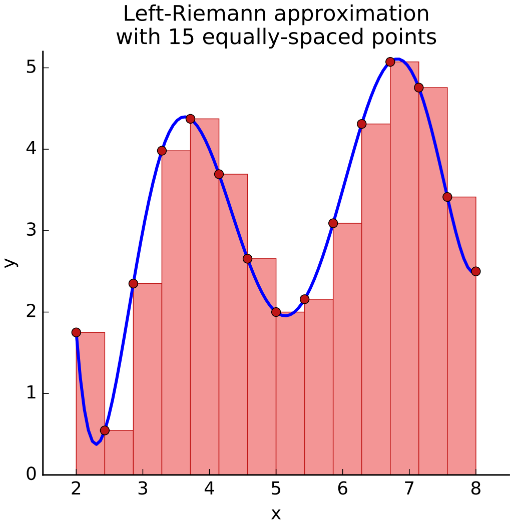

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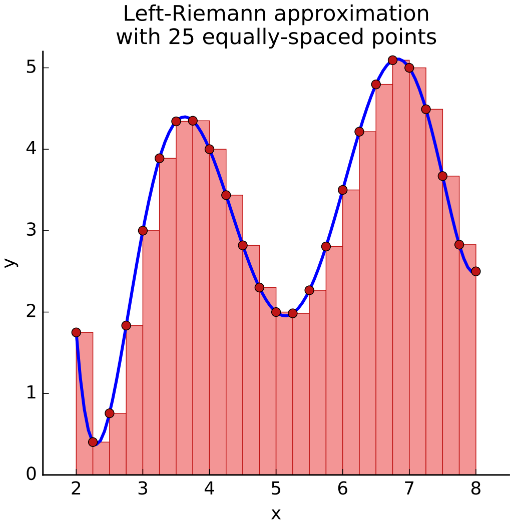

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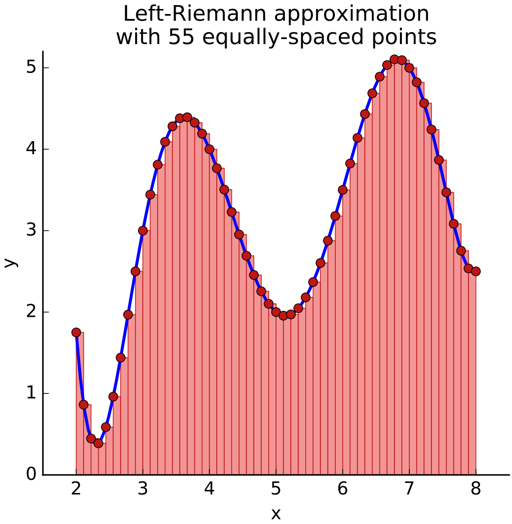

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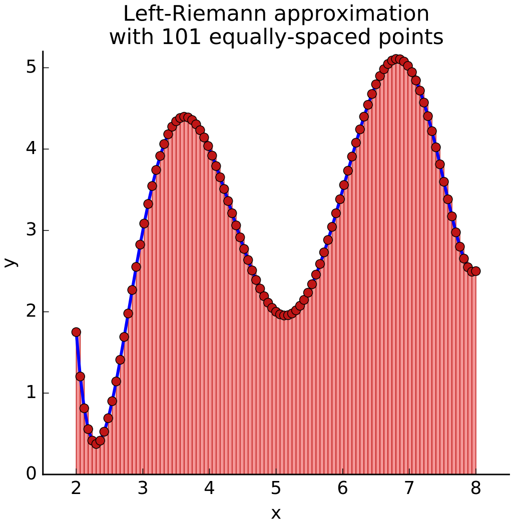

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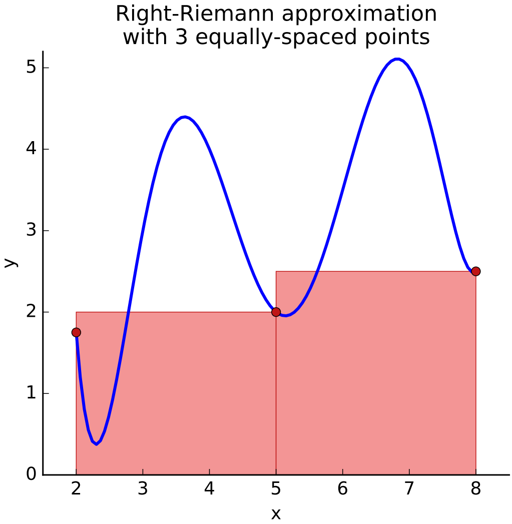

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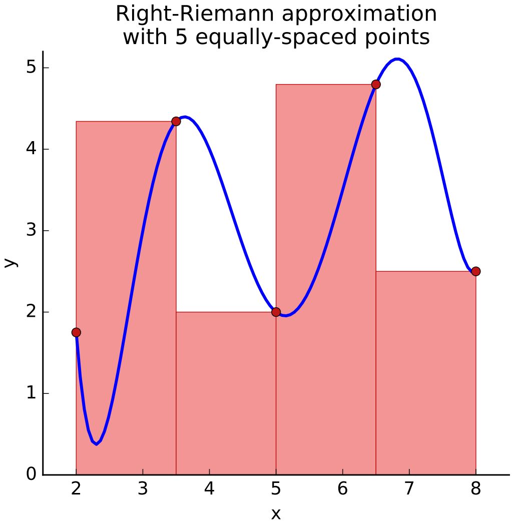

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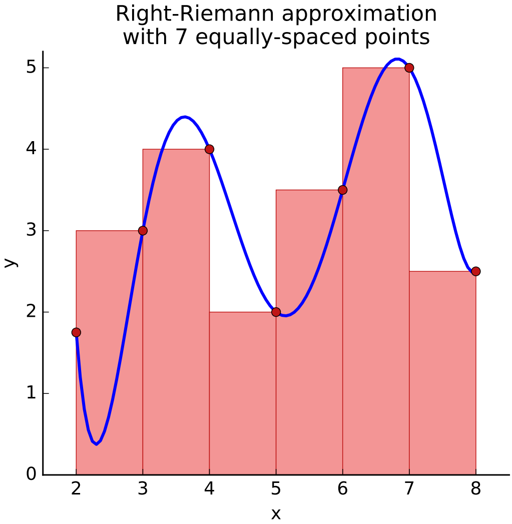

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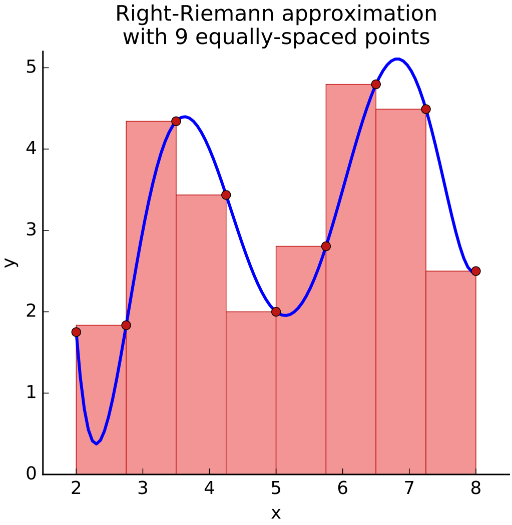

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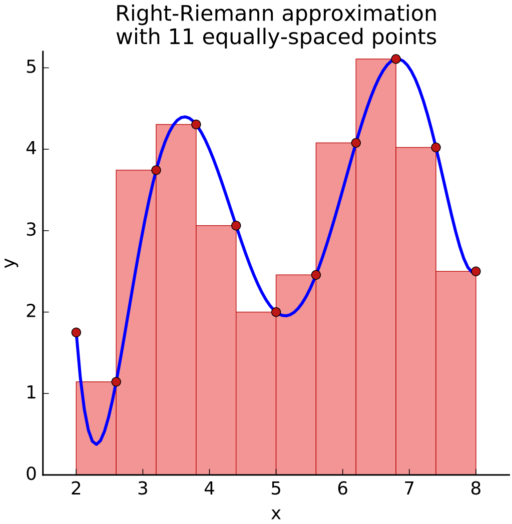

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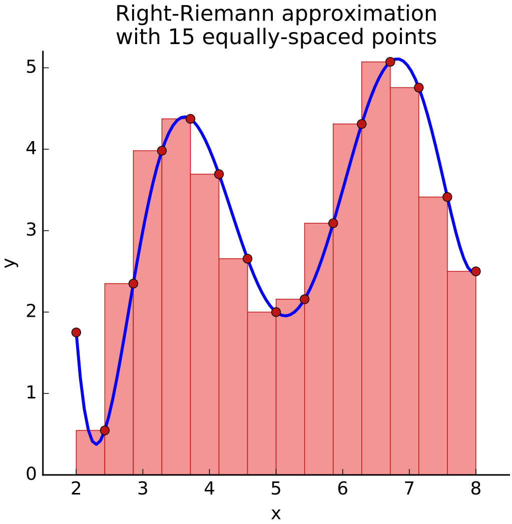

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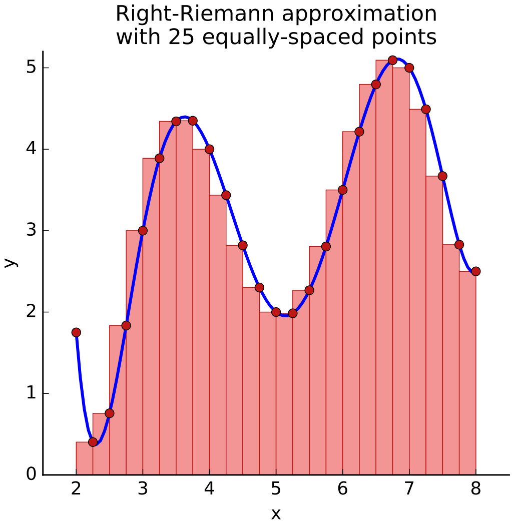

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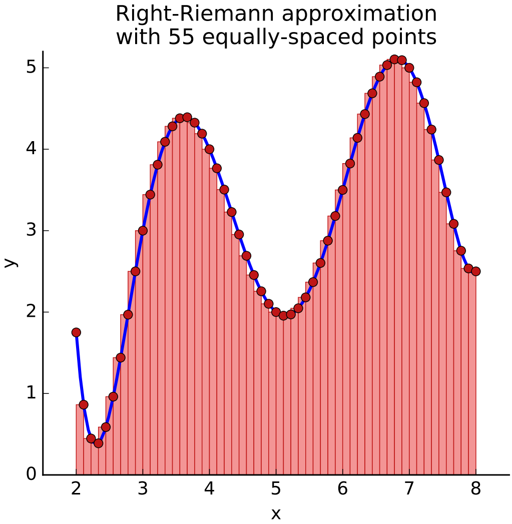

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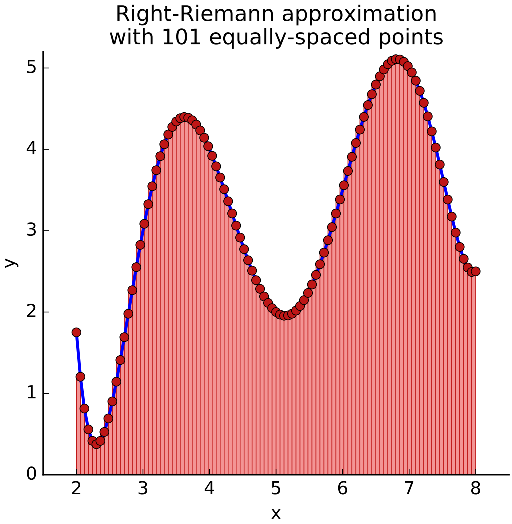

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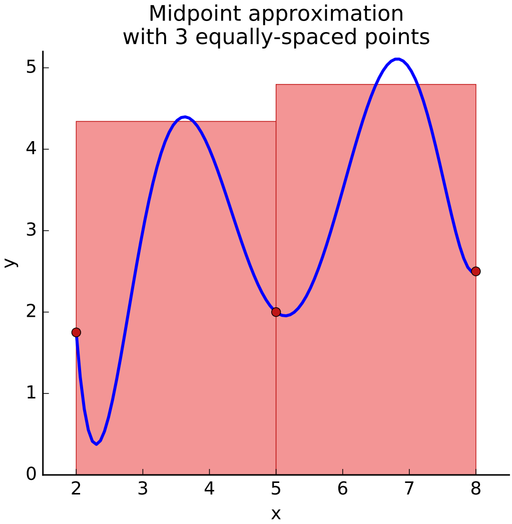

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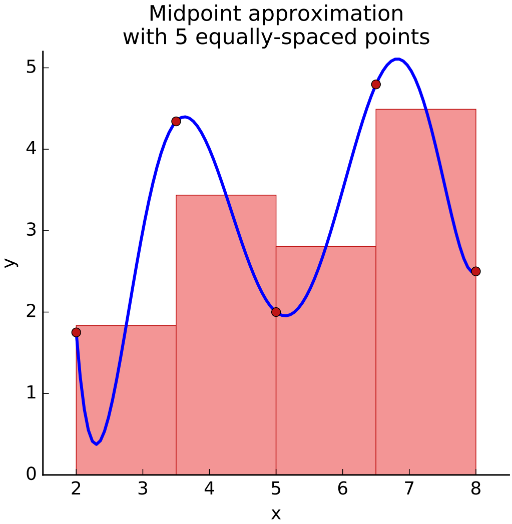

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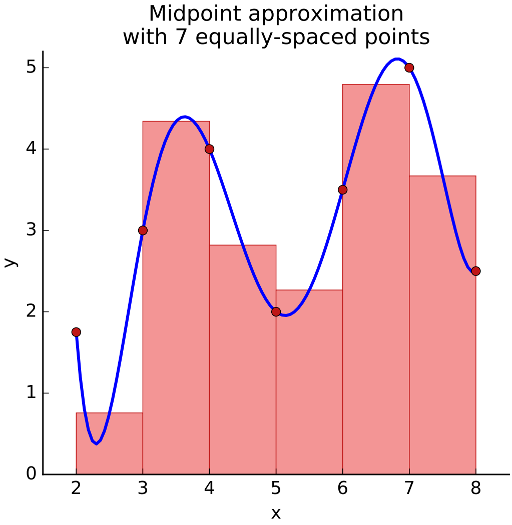

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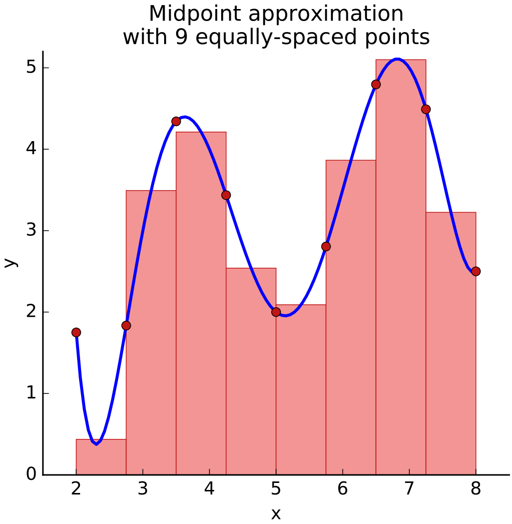

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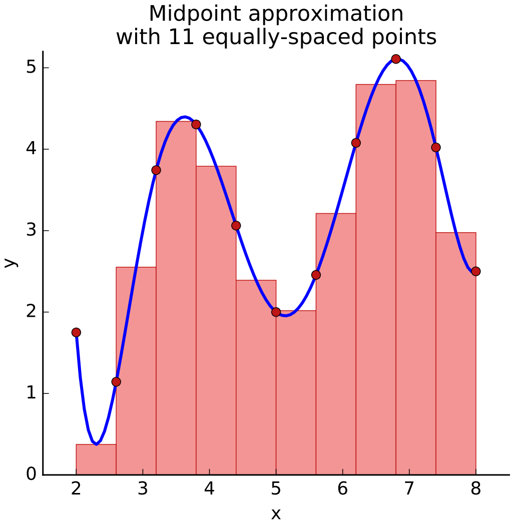

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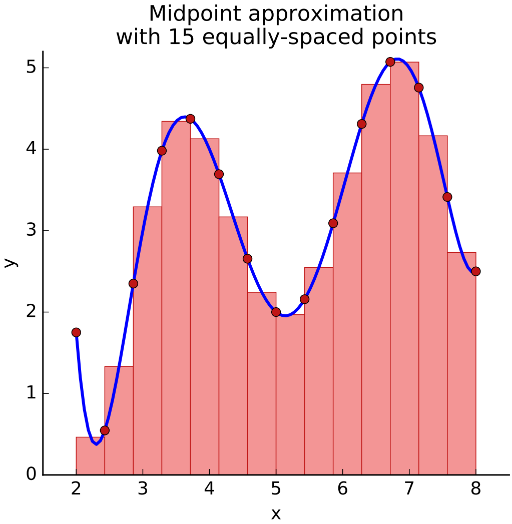

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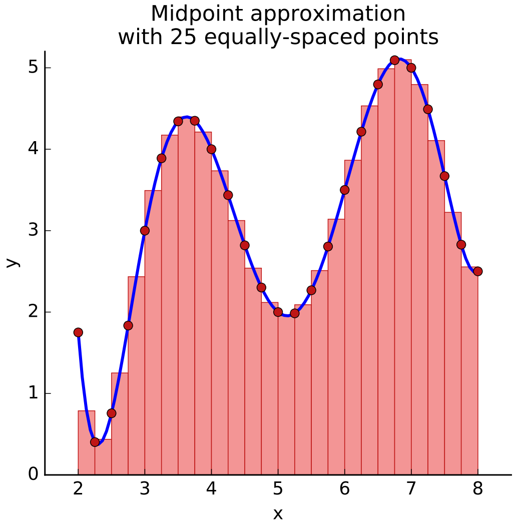

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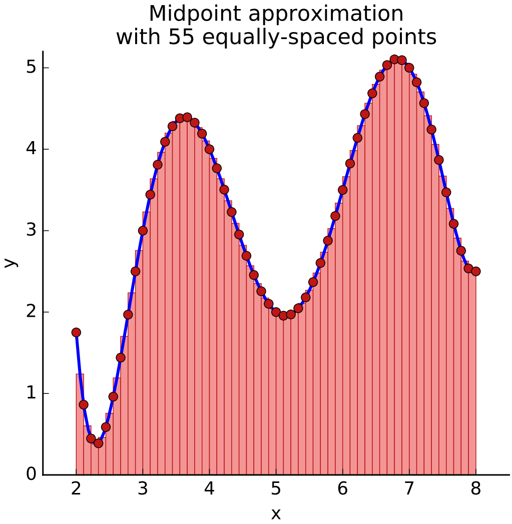

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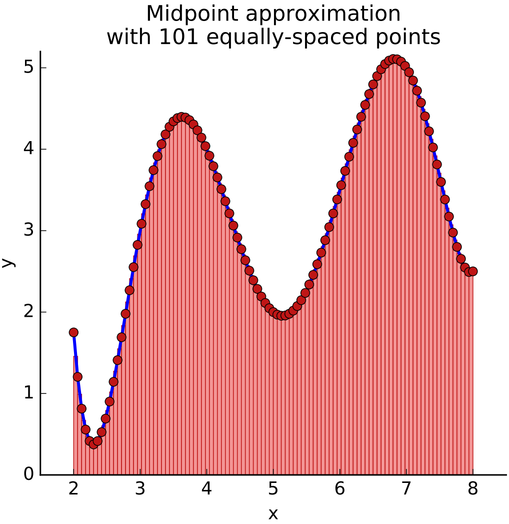

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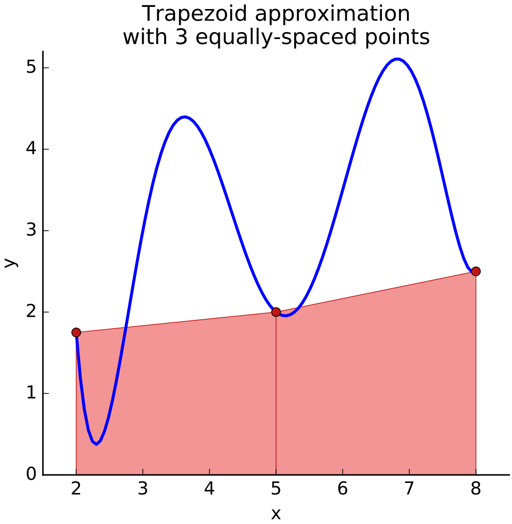

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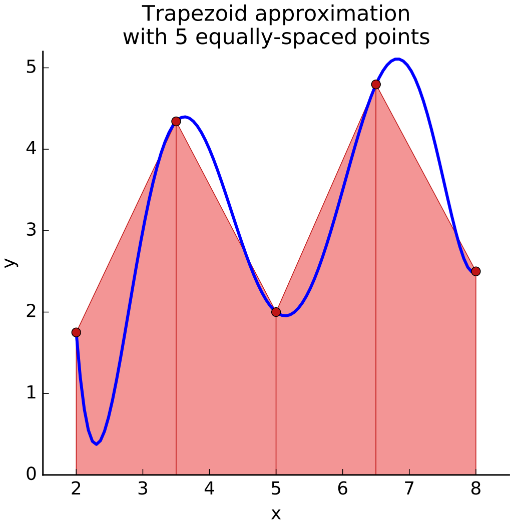

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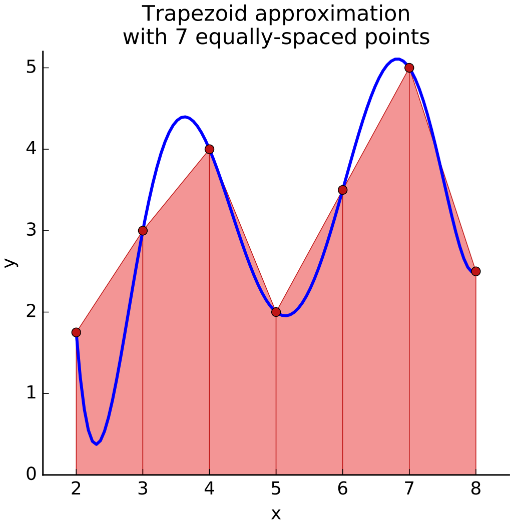

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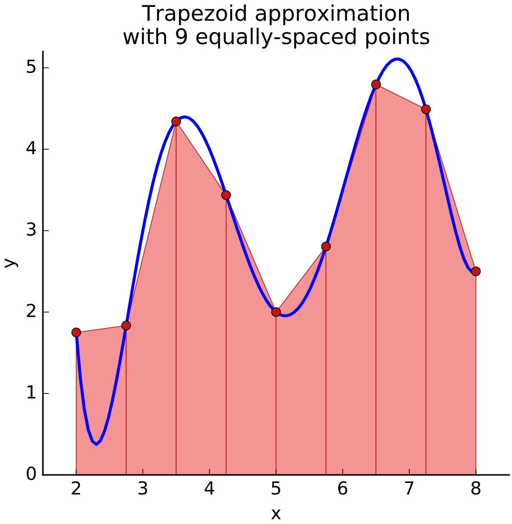

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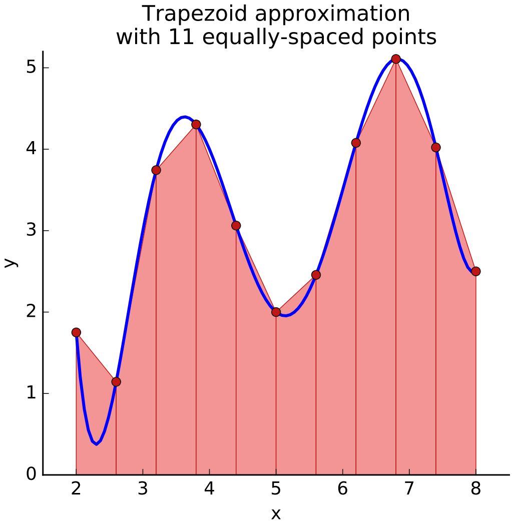

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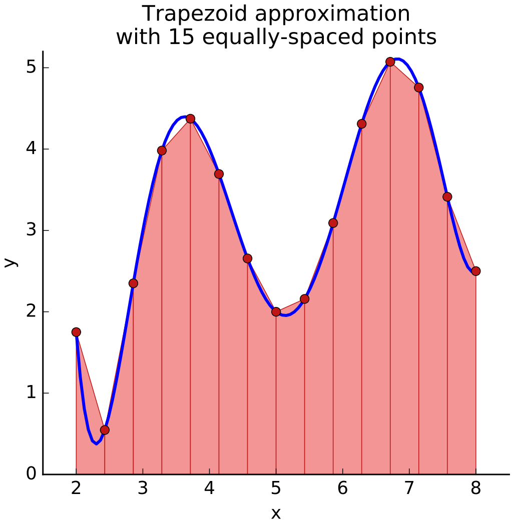

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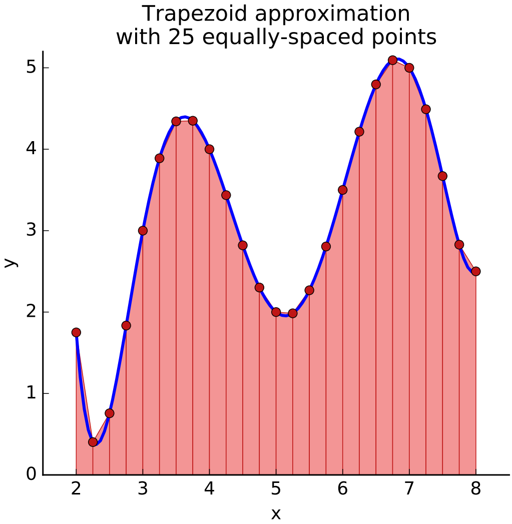

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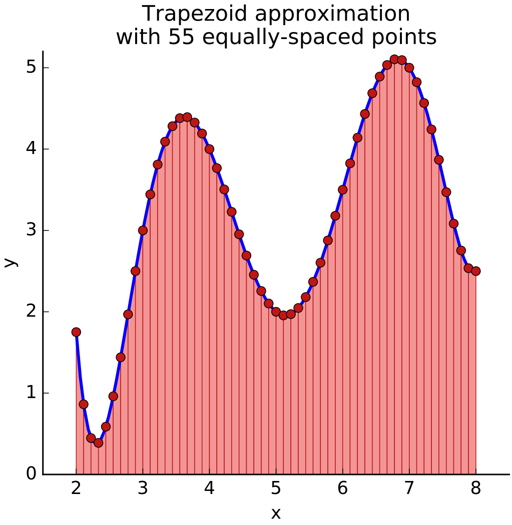

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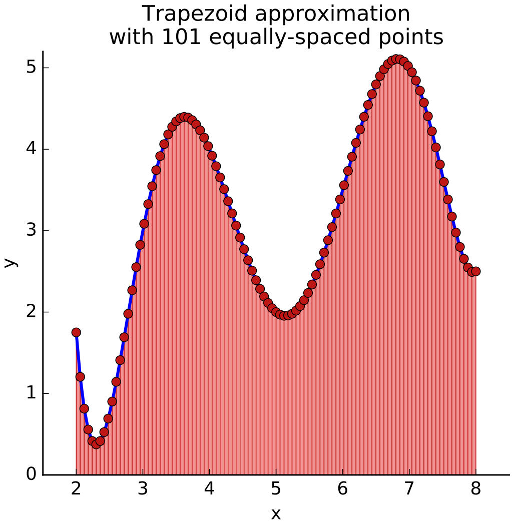

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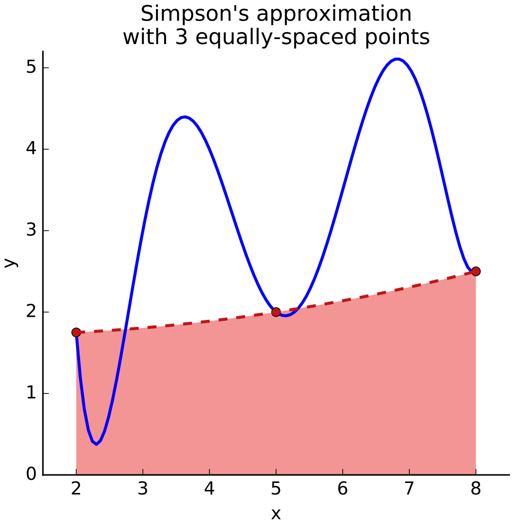

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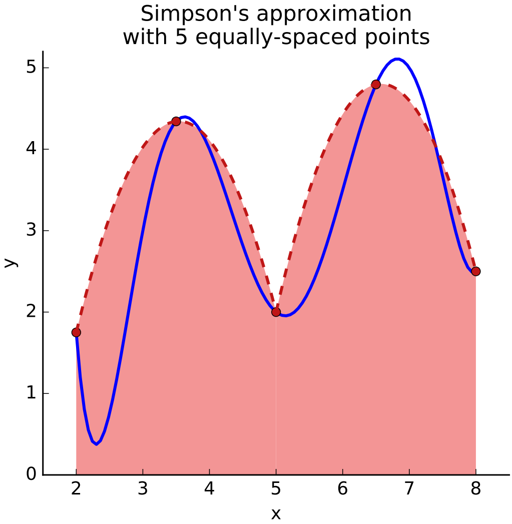

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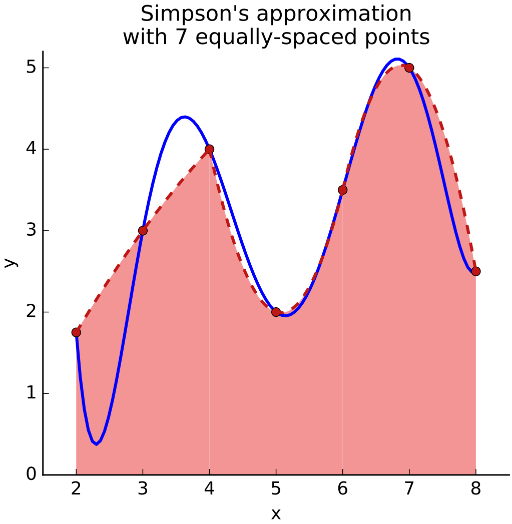

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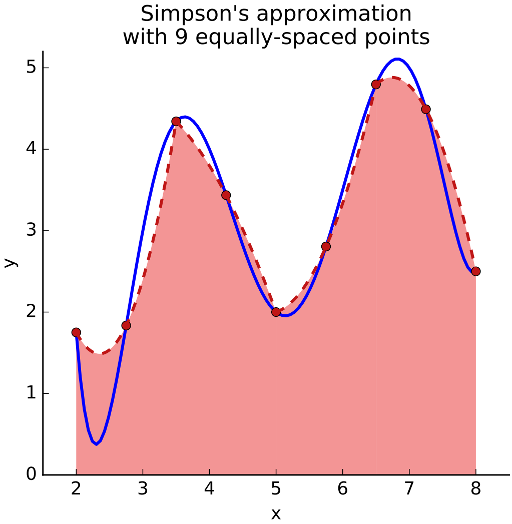

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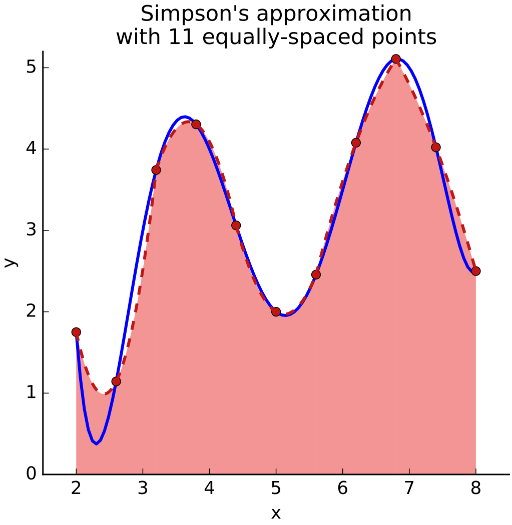

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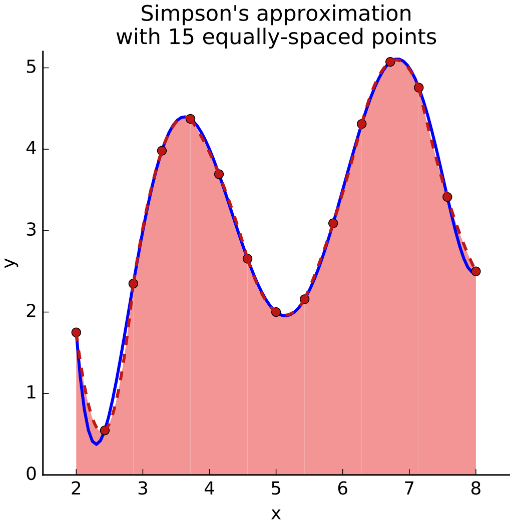

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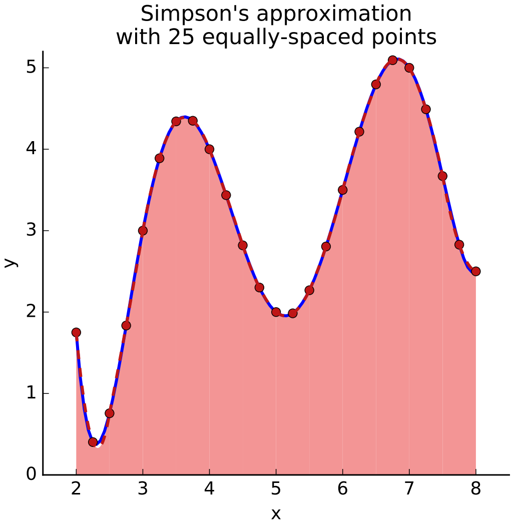

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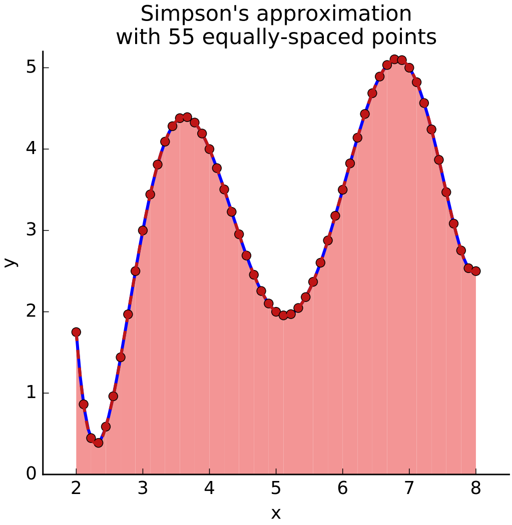

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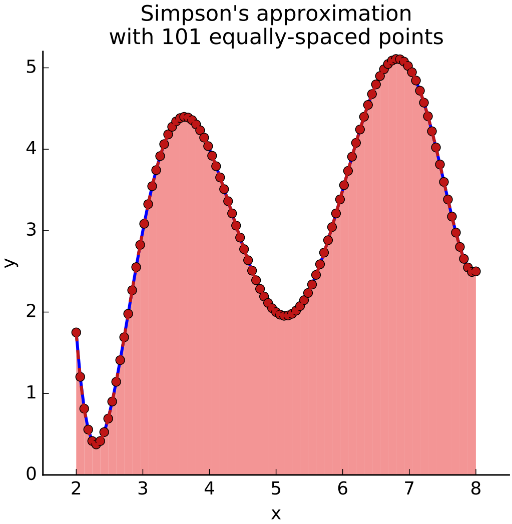
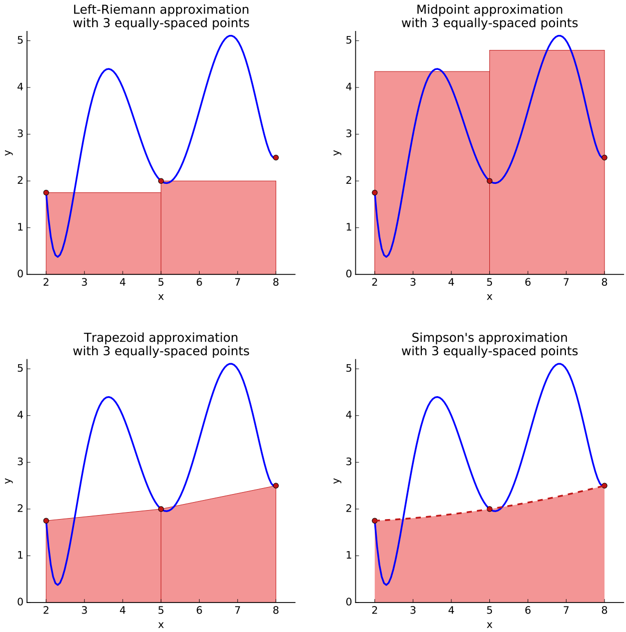


Comparison of different integral approximations



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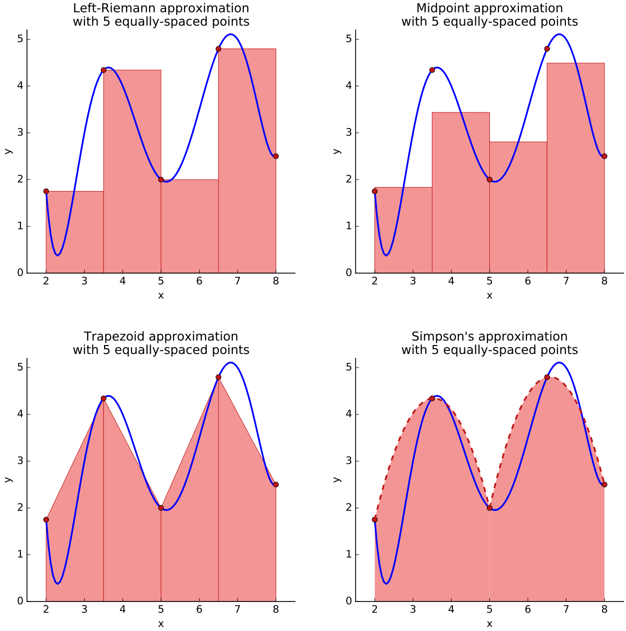

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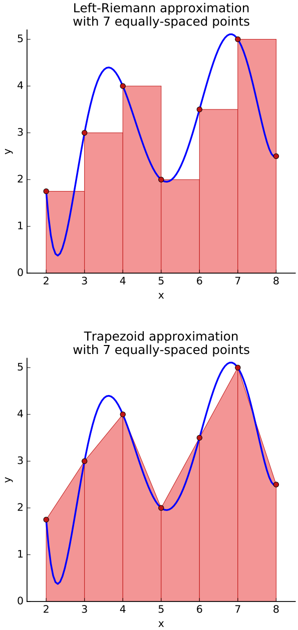

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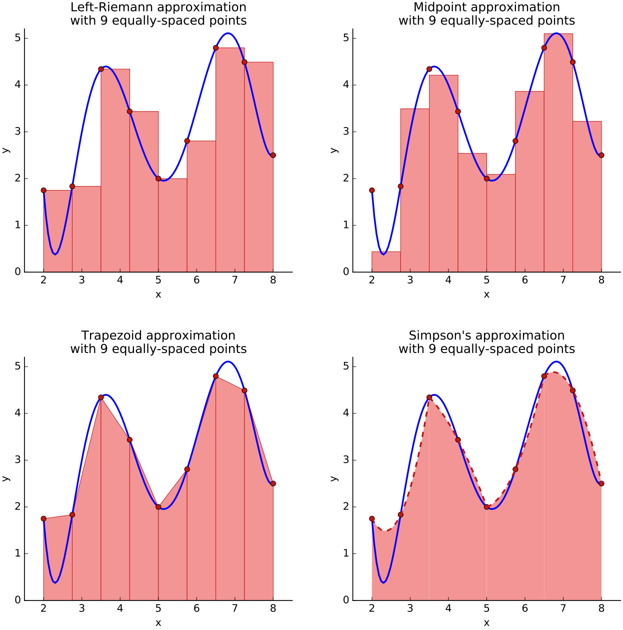

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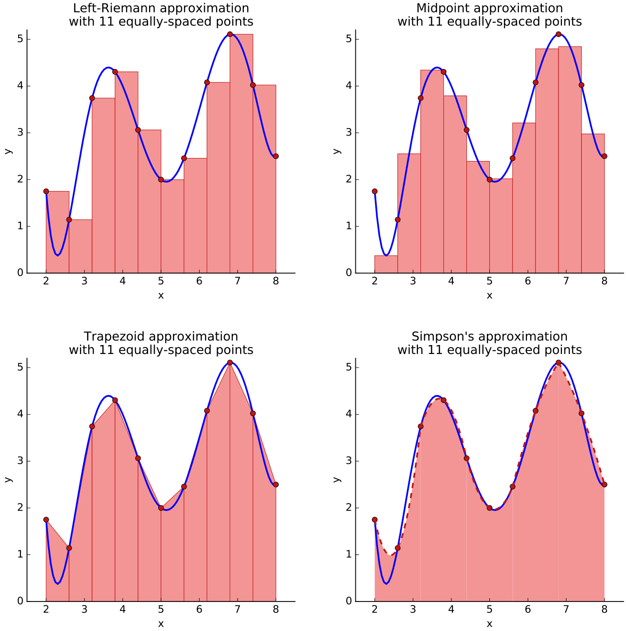

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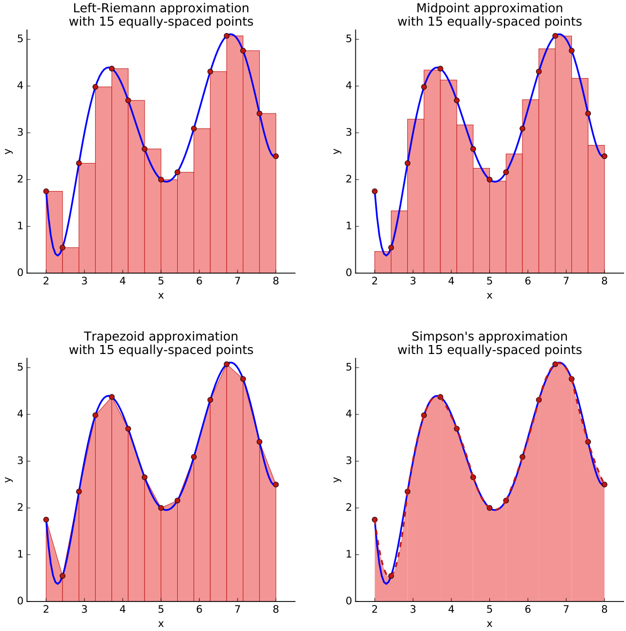

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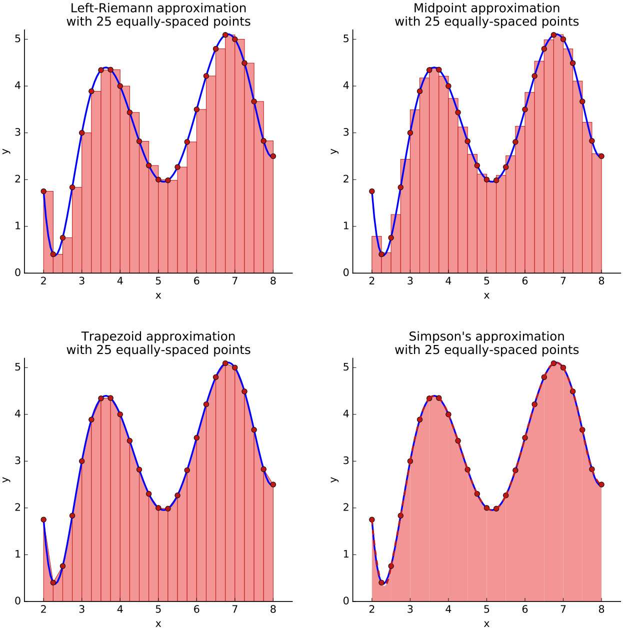

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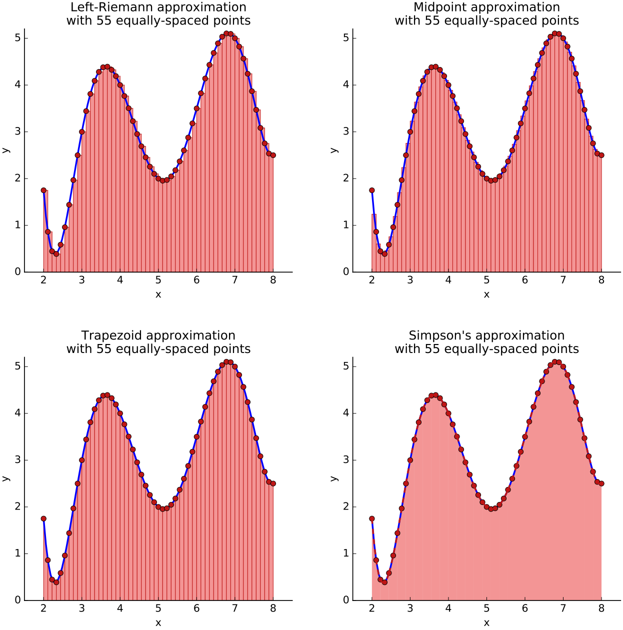

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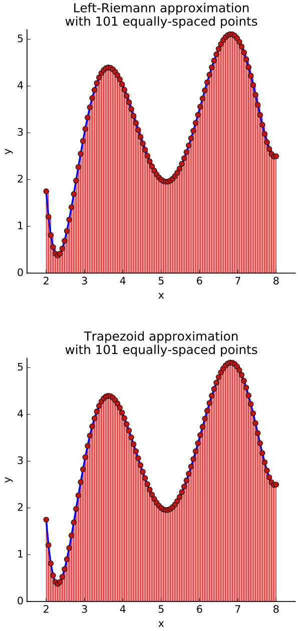

If the error made on the approximation of the integral on one of the sub-intervals $[x_i, x_{i+1}]$ is of order $\mathcal{O}((\Delta x)^m)$, then what is the order of the error made on the integral over the entire interval $[a, b]$?

Note: the explanation presented here is not a rigorous mathematical proof

Order of the error of the integral approximations

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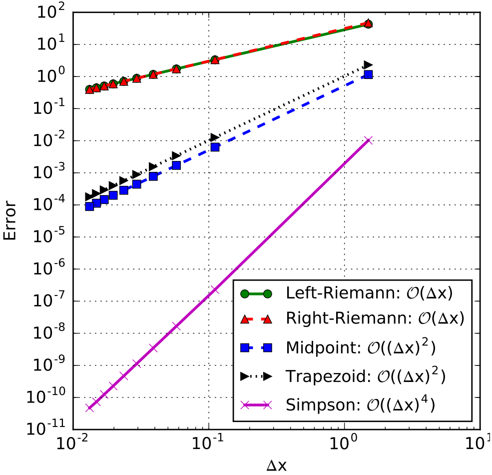
Order of the error of the integral approximations

Method	Order of the error on each sub-interval	Order of the error on the overall interval
Left-Riemann integral	$\mathcal{O}((\Delta x)^2)$	$\mathcal{O}(\Delta x)$
Right-Riemann integral	$\mathcal{O}((\Delta x)^2)$	$\mathcal{O}(\Delta x)$
Midpoint rule	$\mathcal{O}((\Delta x)^3)$	$\mathcal{O}((\Delta x)^2)$
Trapezoid rule	$\mathcal{O}((\Delta x)^3)$	$\mathcal{O}((\Delta x)^2)$
Simpson's rule	$\mathcal{O}((\Delta x)^5)$	$\mathcal{O}((\Delta x)^4)$

See the textbook for the derivations

Order of the error of the integral approximations

Error versus Δx for different integration approximations, when calculating the integral of $x \mapsto \sin(x) + x^2$ from 2 to 8



The slope of the line in a log-log plot indicates the order of the method

IMPORTANT practice question

We use a 2^{nd}- and a 4^{th}-order integration approximation to estimate the integral of a function, using equally-spaced points (spacing is Δx).

Which of the following statements are true?

(A) The error made when using the 4^{th}-order method is always smaller than the error made when using the 2^{nd}-order method

(B) On average, if we reduce Δx by a factor of 2, the error made when using the 2^{nd}-order method is divided by 4

(C) On average, if we reduce Δx by a factor of 2, the error made when using the 4^{th}-order method is divided by 4

(D) On average, if we reduce Δx by a factor of 2, the error made when using the 4^{th}-order method is divided by 16

(E) The error made when using the 4^{th}-order method is always twice as small as the error made when using the 2^{nd}-order method

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