L20: Solving Systems of Linear Algebraic Equations

(when solving them is possible)

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Version: release

Announcements

Lab 07 is due on March 10 at 12 pm (noon)

Today:

- Systems of linear algebraic equations (Chapter 12)
  - Review/learn some linear algebra concepts
  - How many solutions are there?
  - How to find a solution when there is at least one?
  - How to find a reasonable approximation when there are no solutions?

Friday:

- Solve engineering and physics problems using:
  - Root finding
  - Solving systems of linear algebraic equations

Review of matrix multiplication: theory

Consider a m_1 by n_1 real matrix A and a m_2 by n_2 real matrix B

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$$C_{i,j} = \sum_{k=1}^{k=n_1} A_{i,k} B_{k,j}$$

- **Matrix multiplication is not commutative**, meaning that $A \times B$ is not necessarily equal to $B \times A$. If fact, sometimes $A \times B$ is defined but $B \times A$ is not

Review of matrix multiplication: example

$$A = \begin{bmatrix} 5 & 0 & 1 & 2 \\ -1 & 4 & -2 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 5 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

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$$C = A \times B = \begin{bmatrix} \boxed{29} & & \\ & & \end{bmatrix}$$

$$C_{1,1} \quad = \quad 5 \times 6 \ + \ 0 \times 2 \ + \ 1 \times (-1) \ + \ 2 \times 0 \quad = \quad 29$$

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$$C = A \times B = \begin{bmatrix} 29 & \boxed{13} & \\ & & \end{bmatrix}$$

$$C_{1,2} = 5 \times 0 + 0 \times 1 + 1 \times 5 + 2 \times 4 = 13$$

Review of matrix multiplication: example

$$A = \begin{bmatrix} \boxed{5 \quad 0 \quad 1 \quad 2} \\ -1 & 4 & -2 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 5 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} 29 & 13 & \boxed{18} \\ & & \end{bmatrix}$$

$$C_{1,3} \quad = \quad 5 \times 2 \ + \ 0 \times 3 \ + \ 1 \times 0 \ + \ 2 \times 4 \quad = \quad 18$$

Review of matrix multiplication: example

$$A = \begin{bmatrix} 5 & 0 & 1 & 2 \\ \boxed{-1 \quad 4 \quad -2 \quad 9} \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 5 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$$C = A \times B = \begin{bmatrix} 29 & 13 & 18 \\ \boxed{4} & & \end{bmatrix}$$

$$C_{2,1} = -1 \times 6 + 4 \times 2 + -2 \times (-1) + 9 \times 0 = 4$$

Review of matrix multiplication: example

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$$C = A \times B = \begin{bmatrix} 29 & 13 & 18 \\ 4 & \boxed{30} & \end{bmatrix}$$

$$C_{2,2} = -1 \times 0 + 4 \times 1 + -2 \times 5 + 9 \times 4 = 30$$

Review of matrix multiplication: example

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$$C = A \times B = \begin{bmatrix} 29 & 13 & 18 \\ 4 & 30 & \boxed{46} \end{bmatrix}$$

$$C_{2,3} \quad = \quad -1 \times 2 \ + \ 4 \times 3 \ + \ -2 \times 0 \ + \ 9 \times 4 \quad = \quad 46$$

Linear combinations: definition

Consider n vectors: $v_1, v_2, \dots, v_n$. A **non-zero** vector u is a **linear combination** of vectors $v_1, v_2, \dots, v_n$ if and only if there exist scalars $a_1, a_2, \dots, a_n$ ($\in \mathbb{R}$ or $\mathbb{C}$) such that:

$$u = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

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For example, in the example below, u is a linear combination of vectors v_1 and v_2:

$$\begin{aligned} v_1 &= (4,\ 6,\ 1) \\ v_2 &= (3,\ 0,\ 7) \\ u &= 3v_1 - 2v_2 = (6,\ 18,\ -11) \end{aligned}$$

Linear combinations: practice question

Consider the following vectors:

$$\begin{aligned} u &= (1,\ 1,\ 5,\ 10,\ 2) \\ v &= (-1,\ -1,\ 95,\ 90,\ -2) \\ w &= (0,\ 0,\ 1,\ 1,\ 0) \\ x &= (1,\ 1,\ -95,\ -90,\ 0) \end{aligned}$$

Is one of the vectors above a linear combination of the other vectors?

(A) Yes

(B) No

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$$v = 100w - u \quad \text{and} \quad u = 100w - v \quad \text{and} \quad w = \frac{1}{100}u + \frac{1}{100}v$$

Linear independence and rank of a matrix

A set of **non-zero** vectors $v_1, v_2, \ldots, v_n$ are **linearly independent** if and only if we cannot write any of these vectors as a linear combination of the other vectors

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The rank of an $m \times n$ matrix A, often noted $\mathrm{rank}(A)$ measures "how linearly independent" the rows of the matrix are. More precisely, it is the largest integer r such that r of the rows of A are linearly independent

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Note: $0 \leqslant \mathrm{rank}(A) \leqslant m$ and $0 \leqslant \mathrm{rank}(A) \leqslant n$

Rank of a matrix: examples

$$A = \begin{bmatrix} 1 & 1 & 5 & 10 & 2 \\ -1 & -1 & 95 & 90 & -2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & -95 & -90 & 0 \end{bmatrix}$$

$$\mathrm{rank}(A) =$$

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Row 2 is a linear combination of rows 1 and 3

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Therefore:

$$\mathrm{rank}(A) = 3$$

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Therefore:

$$\mathrm{rank}(B) = 4$$

Practice: solve the following systems of two equations and two unknowns:

$$
\begin{aligned}
x + 2 &= 2y \\
2x - 3 &= y
\end{aligned}
$$

Introduction to systems: two equations and two unknowns

Practice: solve the following systems of two equations and two unknowns:

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Unique solution:
$x = 8/3$ and $y = 7/3$

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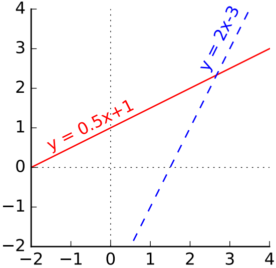

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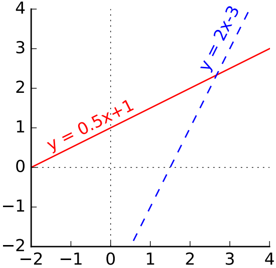

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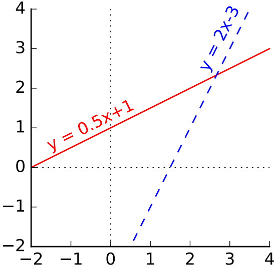

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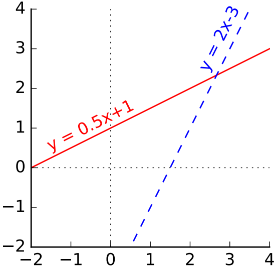

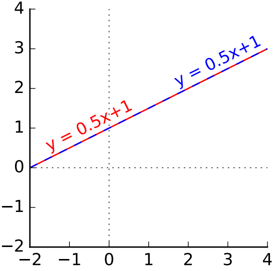

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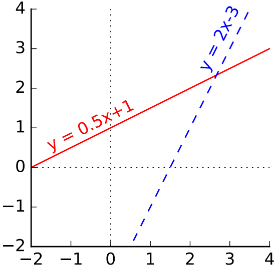

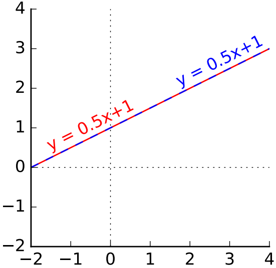

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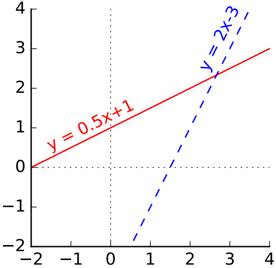

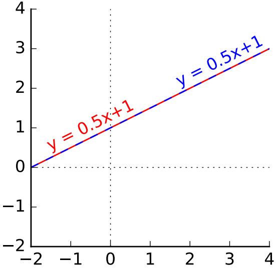

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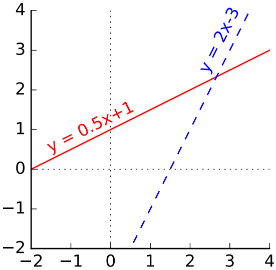

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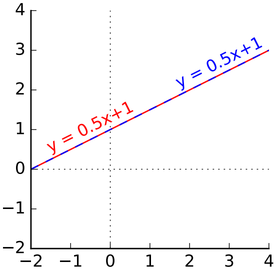
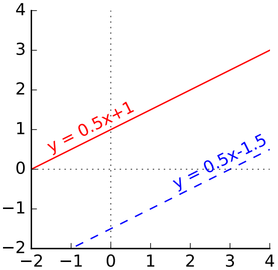


$$x + 2 = 2y$$
$$x - 3 = 2y$$

No solution



More practice: solve this system of equations

Solve this system of 5 equations with 5 unknowns:

$$
\begin{aligned}
&5x_1 + 8x_2 + x_3 + 2x_4 + x_5 = 6 \\
&3x_1 + 4x_2 + 10x_3 + 3x_4 + 7x_5 = 3 \\
&9x_1 + 2x_2 + 10x_3 + 9x_4 + 8x_5 = 8 \\
&4x_1 + 4x_2 + 6x_3 + 7x_5 = 2 \\
&x_1 + x_2 + 4x_5 = 7
\end{aligned}
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\end{aligned}
$$

Actually, solving this system of equations by hand is quite tedious.
Today, we will learn how to solve systems of linear algebraic equations using Matlab, instead of doing it by hand!

More practice: write the previous system in matrix form

$$
\begin{aligned}
5x_1 + 8x_2 + x_3 + 2x_4 + x_5 &= 6 \\
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x_1 + x_2 + 4x_5 &= 7
\end{aligned}
$$

$$\boxed{\rightarrow Ax = b}$$

with:

$$
A = \begin{bmatrix} 5 & 8 & 1 & 2 & 1 \\ 3 & 4 & 10 & 3 & 7 \\ 9 & 2 & 10 & 9 & 8 \\ 4 & 4 & 6 & 0 & 7 \\ 1 & 1 & 0 & 0 & 4 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ 3 \\ 8 \\ 2 \\ 7 \end{bmatrix}
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You absolutely need to be able to write a system of linear algebraic equations in matrix form, when given to you as separate equations (and vice-versa). This skill is needed for several topics, including least-square regression and interpolation

Linear algebraic equations: definitions and notation

A system of linear algebraic equations is a system of m equations ($m \geqslant 1$) that can be written in the following form:

$$
\begin{aligned}
&a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\
&a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\
&\ldots \\
&a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m
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where:

- The n **unknowns** $x_1, x_2, \ldots, x_n$ are **scalars** ($\in \mathbb{R}$ or $\mathbb{C}$)

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Note that m **(number of equations) can be different from** n **(number of unknowns)**

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\ldots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m
\end{aligned}
$$

can be written in matrix form as: $Ax = b$

- A is the system's matrix (size: $m \times n$)
- x is the vector of unknowns (size: $n \times 1$)
- b is the system's "right-hand side" (size: $m \times 1$)

$$
A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

Example of a system of non-linear equations

The equations of the following system are **not** linear:

$$
\begin{aligned}
2x^2 + 4y + 7z &= 0 \\
x^3 + 4y^7 &= -1 \\
x + 4y^{10} + z^3 &= 5
\end{aligned}
$$

Over- and under-determined systems

Consider a system of m linear algebraic equations with n unknowns

The system is **over-determined** if and only if $m > n$

For example:

$$\begin{aligned} 3x + 2y &= -7 \\ x + 2y &= 0 \\ -x + y &= 1 \end{aligned}$$

The system is **under-determined** if and only if $m < n$

For example:

$$\begin{aligned} 3x + 2y + z &= -7 \\ x + y - z &= 0 \end{aligned}$$

$\mathrm{rank}(A)$ **measures "how linearly independent" the rows of the matrix are.** If a row of A is a linear combination of other rows of the matrix, then:

EITHER:	**OR:**
One of the equations is redundant	**Equations are incompatible**

Rank of a matrix and implications for solutions of systems

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$$\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

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For example:

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has no solution

Solutions of systems of linear algebraic equations

Consider a system of m linear algebraic equations with n unknowns. We want to find a solution to this system

There are 3 possible cases; you must know them for E7:
(You don't need to know the linear algebra proofs)

Case 1: `rank([A,b]) == rank(A)+1`

- No solution exists

Case 2: `rank([A,b]) == rank(A)` and `rank(A) == n`

- There is one and only one solution

Case 3: `rank([A,b]) == rank(A)` and `rank(A) < n`

- There is an infinite number of solutions

Case 2: There is one and only one solution

Use Matlab's backslash operator (\):

```
>> a = [1, -2; 2, -1];
>> b = [-2; 3];
>> rank([a, b]) == rank(a)
ans =
  logical
   1
>> rank(a) == size(a, 2)
ans =
  logical
   1

>> % Solve for x
>> x = a\b
x =
    2.6667
    2.3333

>> % Verify that x is a solution (a*x should be equal to b)
>> a*x
ans =
   -2.0000
    3.0000
```

Case 3: There is an infinite number of solutions

Use Matlab's built-in function pinv to get one of these solutions

```
>> a = [0.5, -1; 1, -2];
>> b = [-1; -2];
>> rank([a, b]) == rank(a)
ans =
  logical
   1
>> rank([a, b]) == rank(a)
ans =
  logical
   1

>> % Solve for a specific solution x. pinv(a) is the pseudo-inverse
>> x = pinv(a)*b                        % of a. More detail on this
x =                                     % topic next week
   -0.4000
    0.8000

>> % Verify that x is a solution (a*x should be equal to b)
>> a*x
ans =
   -1.0000
   -2.0000
```

Solving systems of linear algebraic equations in Matlab

Case 1: No solution exists

WARNING! In this case, using Matlab's backslash operator (\) yields something that is **NOT** a solution

```
>> a = [0.5, -1; 1, -2; 2, 3];
>> b = [-2; -3; 10];
>> rank([a, b]) == rank(a) + 1
ans =
  logical
   1

>> % Try using Matlab's backslash operator
>> x = a\b
x =
    1.4857
    2.3429

>> % WARNING: x is not a solution (a*x is not equal to b) but
>> %          is a "reasonable'' approximation (see next slides)
>> a*x
ans =
   -1.6000
   -3.2000
   10.0000
```

Square error

What is the value of x calculated by Matlab in the previous example, since it is not a solution to the system? Since x is not a solution:

$$
\begin{aligned}
&a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 + \mathrm{error}_1 \\
&a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 + \mathrm{error}_2 \\
&\ldots \\
&a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m + \mathrm{error}_m
\end{aligned}
$$

The square error s is:

$$
\begin{aligned}
s &= \sum_{i=1}^{i=m} (\mathrm{error}_i)^2 \\
&= \sum_{i=1}^{i=m} (a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n - b_i)^2
\end{aligned}
$$

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  - So that positive and negative errors don't cancel out
- Why not absolute value instead of square?
  - $x \mapsto |x|$ is not differentiable at $x = 0$

When you use Matlab's backslash operator to try to solve a system that has no solution, Matlab returns the value of x (if any) that minimizes the square error (more on this topic next week)