L31: Ordinary Differential Equations

Part 1: Introduction to time-stepping methods

Lucas A. J. Bastien

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Announcements

Lab 11 is due on April 14 at 12 pm (noon)

Lab 11 is significantly shorter than most previous labs

Use the opportunity to:

- Get a lot of points on lab 11!
- Work on your project!

Project Beta Test is due on April 14 at 12 pm (noon)

Today:

- Ordinary differential equations (Chapter 19)

Wednesday

- Ordinary differential equations – Part 2
  (Review part 1 before lecture)

Friday

- Ordinary differential equations – Part 3
  (Review parts 1 and 2 before lecture)

What is an ordinary differential equation?

We have three words to define:

1. Ordinary
2. Differential
3. Equation

What is an equation?

The "**equa**" part of "**equa**tion" means "**equal**"

Equation:
Equality between two quantities involving one or more unknowns

Solving an equation:
Find the values of the unknowns that satisfy the equality

Example where the unknown is called x:

$$x^2 - 10 = 3x$$

5 is a solution of this equation because $5^2 - 10 = 3 \times 5$

-2 is a solution of this equation because $(-2)^2 - 10 = 3 \times (-2)$

2 is not a solution of this equation because $2^2 - 10 \neq 3 \times 2$

What is an ordinary differential equation?

A **differential** equation is an equation:

- **where the unknown is a function**; and
- that involves the function and one or more of its **derivatives**

The **order** of the equation is the order of the highest derivative involved in the equation

For example, below is a first-order differential equation:

$$y' = 2y$$

$y : t \mapsto e^{2t}$ is a solution because $y'(t) = (e^{2t})' = 2e^{2t} = 2y(t)$ for all t

$y : t \mapsto \sin(t)$ is not a solution because $(\sin)' = cos \neq 2\sin$

An **ordinary** differential equation is a differential equation where the unknown is a function of one variable only

Practice question

Consider the following ODE where the unknown is y:

$$3y' + 10y = y''$$

Is the function y defined by $y(t) = 2e^{5t} + e^{-2t}$ a solution to this ODE?

(A) yes

(B) no

For all t:

$y'(t) = 10e^{5t} - 2e^{-2t}$

$y''(t) = 50e^{5t} + 4e^{-2t}$

$3y'(t) + 10y(t) = 30e^{5t} - 6e^{-2t} + 20e^{5t} + 10e^{-2t} = 50e^{5t} + 4e^{-2t}$

Numerical "solutions" of ODEs

Analytical solution: exact solution, derived by hand

Numerical "solution": approximate solution estimated using numerical methods. I use the quotes around "solution" to remind you that the solution is approximate (and is therefore technically not a solution)

You should definitely know the analytical solution of the following ODE:

$$y' = ay + b$$

where the unknown is y and where a and b are real constants. Solution:

$$y(t) = ke^{at} - \frac{b}{a}$$

where k is any real constant (there are an infinite number of solutions)

General versus particular solution

Often, an ODE has an infinite number of solutions (see example in previous slide)

One can specify n "constraints" (where n is the order of the ODE) to choose a particular solution. For example:

$$y' = 2y - 10$$

Has an infinite number of solutions:

$$y(t) = ke^{2t} + 5$$

where k is any real constant

$$y' = 2y - 10$$
$$y(t = 0) = 3$$

has only one solution:

$$y(t) = -2e^{2t} + 5$$

Initial value problem: ODE with n constraints that specify the value of the function and/or its derivatives at a single point

Example of initial value problem

Radioactive decay of ^{14}C atoms (used in carbon dating):

$$\frac{dN}{dt} = -kN$$
$$N(t = 0) = N_0$$

- $N(t)$: number of ^{14}C atoms in the sample at time t
- N_0: number of ^{14}C atoms in the sample at time $t = 0$
- k: ^{14}C radioactivity constant

Analytical solution: $N(t) = N_0 e^{-kt}$

Numerical methods for "solving" initial value problems

We will learn methods to "solve" **first-order initial value problems**

Notation:
Generic first-order initial value problem (unknown is y, a function of t):

$$y' = F(t, y) \qquad \text{(ODE)}$$
$$y(t = t_0) = y_0 \qquad \text{(Initial condition)}$$

General approach: estimate the function's value at discrete small intervals (*i.e.* estimate the function at points $t_0, t_1, t_2, \ldots$), starting from the known value, **assuming that the slope is constant over each interval:**

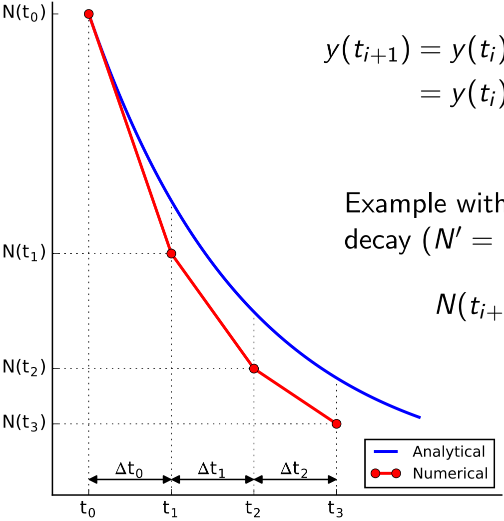
$$y(t_{i+1}) = y(t_i) + \text{slope} \times \Delta t_i$$

where $\Delta t_i = (t_{i+1} - t_i)$ is the spacing

Different methods: different approximations for the slope

Explicit Euler method

At each time step, assume that the slope is equal to the slope evaluated at the beginning of the time step



$$
\begin{aligned}
y(t_{i+1}) &= y(t_i) + y'(t_i)\Delta t_i \\
&= y(t_i) + F(t_i, y(t_i))\Delta t_i
\end{aligned}
$$

Example with ^{14}C radioactive decay ($N' = -kN$):

$$
\begin{aligned}
N(t_{i+1}) &= N(t_i) + N'(t_i)\Delta t_i \\
&= N(t_i) - kN(t_i)\Delta t_i
\end{aligned}
$$

Practice question

A parachutist falls with vertical downward velocity v according to:

$$v' = g - \frac{c}{m}v$$
$$v(t=0) = 0$$

- $g = 10\ \mathrm{m\ s^{-2}}$ (acceleration of gravity)
- $m = 70\ \mathrm{kg}$ (mass of the parachutist)
- $c = 14\ \mathrm{kg\ s^{-1}}$ (measures drag)

What is the parachutist's velocity at $t = 2\ \mathrm{s}$, approximated using the explicit Euler method and two time steps of $t = 1\ \mathrm{s}$ each?

$$\begin{aligned} v(t = 1\ \mathrm{s}) &\approx v(t=0) + v'(t=0) \times 1\ \mathrm{s} \\ &= v(t=0) + \left[g - \frac{c}{m}v(t=0)\right] \times 1\ \mathrm{s} = 10\ \mathrm{m\ s^{-1}} \end{aligned}$$

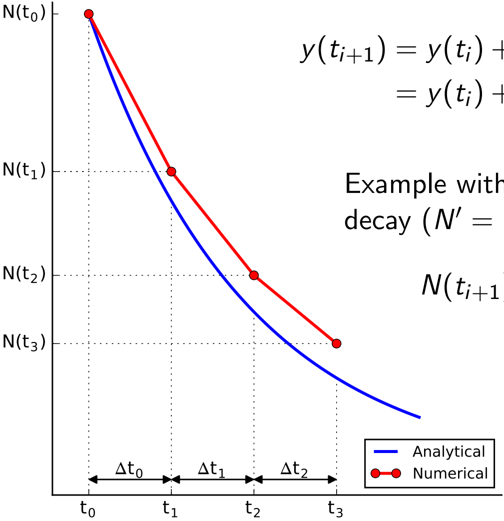
$$\begin{aligned} v(t = 2\ \mathrm{s}) &\approx v(t=1) + v'(t=1) \times 1\ \mathrm{s} \\ &= v(t=1) + \left[g - \frac{c}{m}v(t=1)\right] \times 1\ \mathrm{s} = 18\ \mathrm{m\ s^{-1}} \end{aligned}$$

Practice at home (highly recommended):

Same question with other methods seen in the next slides

Implicit Euler method

At each time step, assume that the slope is equal to the slope evaluated at the end of the time step



$$\begin{aligned} y(t_{i+1}) &= y(t_i) + y'(t_{i+1})\Delta t_i \\ &= y(t_i) + F(t_{i+1}, y(t_{i+1}))\Delta t_i \end{aligned}$$

Example with ^{14}C radioactive decay ($N' = -kN$):

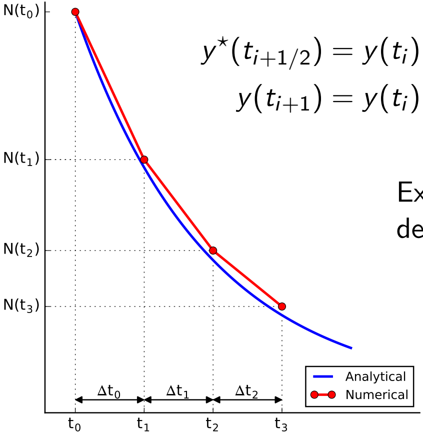
$$\begin{aligned} N(t_{i+1}) &= N(t_i) + N'(t_{i+1})\Delta t_i \\ &= N(t_i) - kN(t_{i+1})\Delta t_i \end{aligned}$$

$$N(t_{i+1}) = \frac{N(t_i)}{1 + k\Delta t_i}$$

Midpoint method

Two-step method:

1. Take a half-step using the explicit Euler method to estimate $y^\star(t_{i+1/2})$
2. Use $y^\star(t_{i+1/2})$ to estimate the slope at the midpoint



$$y^\star(t_{i+1/2}) = y(t_i) + y'(t_i)\Delta t_i/2$$
$$y(t_{i+1}) = y(t_i) + F(t_{i+1/2}, y^\star(t_{i+1/2}))\Delta t_i$$

Example with ^{14}C radioactive decay ($N' = -kN$):

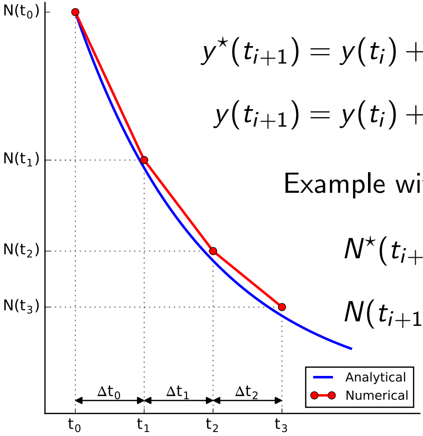
$$N^\star(t_{i+1/2}) = N(t_i) - kN(t_i)\Delta t_i/2$$
$$N(t_{i+1}) = N(t_i) - kN^\star(t_{i+1/2})\Delta t_i$$

Notation: $t_{i+1/2} = t_i + \Delta t_i/2$

Heun's method

Two-step method:

1. Take a step using the explicit Euler method to estimate $y^\star(t_{i+1})$
2. Estimate the slope as the average of the slope at t_i and t_{i+1}



$$y^\star(t_{i+1}) = y(t_i) + y'(t_i)\Delta t_i$$

$$y(t_{i+1}) = y(t_i) + \frac{F(t_i, y(t_i)) + F(t_{i+1}, y^\star(t_{i+1}))}{2}\Delta t_i$$

Example with ^{14}C radioactive decay ($N' = -kN$):

$$N^\star(t_{i+1}) = N(t_i) - kN(t_i)\Delta t_i$$

$$N(t_{i+1}) = N(t_i) - k\frac{N(t_i) + N^\star(t_{i+1})}{2}\Delta t_i$$

Summary of the methods