L31: Ordinary Differential Equations

Part 1: Introduction to time-stepping methods

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Version: revision01

Announcements

Lab 11 is due on April 14 at 12 pm (noon)

Lab 11 is significantly shorter than most previous labs

Use the opportunity to:

- Get a lot of points on lab 11!
- Work on your project!

Project Beta Test is due on April 14 at 12 pm (noon)

Today:

- Ordinary differential equations (Chapter 19)

Wednesday

- Ordinary differential equations – Part 2
  (Review part 1 before lecture)

Friday

- Ordinary differential equations – Part 3
  (Review parts 1 and 2 before lecture)

What is an ordinary differential equation?

We have three words to define:

1. Ordinary
2. Differential
3. Equation

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2 is not a solution of this equation because $2^2 - 10 \neq 3 \times 2$

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An **ordinary** differential equation is a differential equation where the unknown is a function of one variable only

Practice question

Consider the following ODE where the unknown is y:

$$3y' + 10y = y''$$

Is the function y defined by $y(t) = 2e^{5t} + e^{-2t}$ a solution to this ODE?

(A) yes

(B) no

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$3y'(t) + 10y(t) = 30e^{5t} - 6e^{-2t} + 20e^{5t} + 10e^{-2t} = 50e^{5t} + 4e^{-2t}$

Numerical “solutions” of ODEs

Analytical solution: exact solution, derived by hand

Numerical “solution”: approximate solution estimated using numerical methods. I use the quotes around “solution” to remind you that the solution is approximate (and is therefore technically not a solution)

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You should definitely know the analytical solution of the following ODE:

$$y' = ay + b$$

where the unknown is y and where a and b are real constants. Solution:

$$y(t) = ke^{at} - \frac{b}{a}$$

where k is any real constant (there are an infinite number of solutions)

General versus particular solution

Often, an ODE has an infinite number of solutions (see example in previous slide)

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has only one solution:

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Initial value problem: ODE with n constraints that specify the value of the function and/or its derivatives at a single point

Example of initial value problem

Radioactive decay of ^{14}C atoms (used in carbon dating):

$$\frac{dN}{dt} = -kN$$
$$N(t=0) = N_0$$

- $N(t)$: number of ^{14}C atoms in the sample at time t
- N_0: number of ^{14}C atoms in the sample at time $t = 0$
- k: ^{14}C radioactivity constant

Analytical solution: $N(t) = N_0 e^{-kt}$

Numerical methods for "solving" initial value problems

We will learn methods to "solve" **first-order initial value problems**

Notation:

Generic first-order initial value problem (unknown is y, a function of t):

$$y' = F(t, y) \qquad \text{(ODE)}$$

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General approach: estimate the function's value at discrete small intervals (*i.e.* estimate the function at points $t_0, t_1, t_2, \dots$), starting from the known value, **assuming that the slope is constant over each interval:**

$$y(t_{i+1}) = y(t_i) + \text{slope} \times \Delta t_i$$

where $\Delta t_i = (t_{i+1} - t_i)$ is the spacing

Different methods: different approximations for the slope

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At each time step, assume that the slope is equal to the slope evaluated at the beginning of the time step

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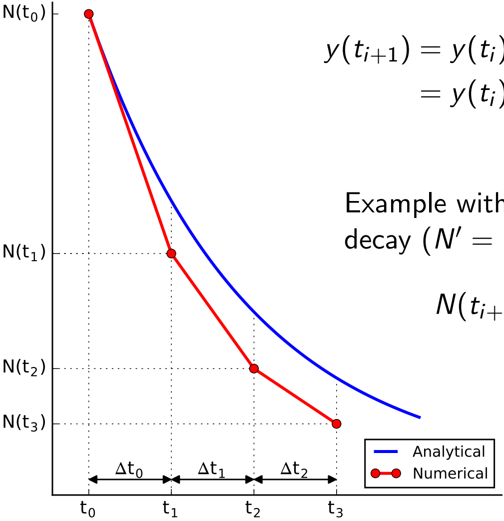

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Practice question

A parachutist falls with vertical downward velocity v according to:

$$v' = g - \frac{c}{m} v$$

$$v(t = 0) = 0$$

- $g = 10\ \mathrm{m\ s^{-2}}$ (acceleration of gravity)
- $m = 70\ \mathrm{kg}$ (mass of the parachutist)
- $c = 14\ \mathrm{kg\ s^{-1}}$ (measures drag)

What is the parachutist's velocity at $t = 2\ \mathrm{s}$, approximated using the explicit Euler method and two time steps of $t = 1\ \mathrm{s}$ each?

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Practice at home (highly recommended):

Same question with other methods seen in the next slides

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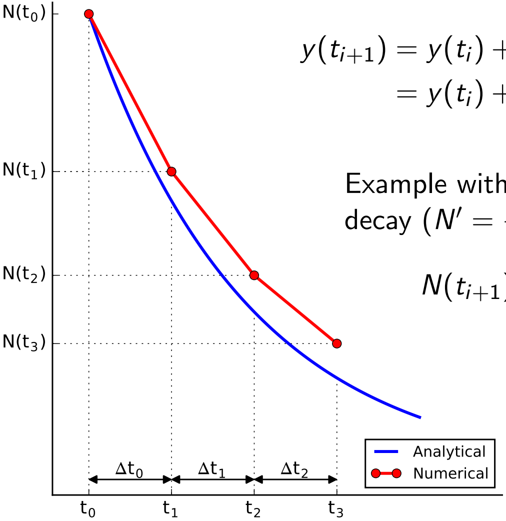

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Midpoint method

Two-step method:

1. Take a half-step using the explicit Euler method to estimate $y^\star(t_{i+1/2})$
2. Use $y^\star(t_{i+1/2})$ to estimate the slope at the midpoint

Notation: $t_{i+1/2} = t_i + \Delta t_i/2$

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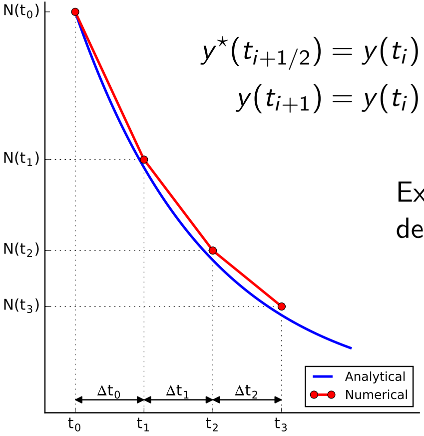

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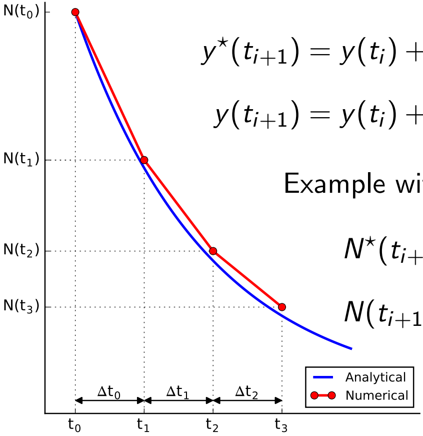

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Example with ^{14}C radioactive decay ($N' = -kN$):

$$N^\star(t_{i+1}) = N(t_i) - kN(t_i)\Delta t_i$$
$$N(t_{i+1}) = N(t_i) - k\frac{N(t_i) + N^\star(t_{i+1})}{2}\Delta t_i$$

Summary of the methods