Math 53 Lecture Notes

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Abstract

Exam Date: October 5

Topic Covered: Chapter 10, 12, 13, 14

The study of multivariable calculus starts with an investigation into parametric equation (and interesting variation of it: the polar coordinates) in $\mathbb{R}^2$. Then the study continue to build foundation for objects in $\mathbb{R}^3$ by introducing vectors and the geometry of space. With the basic understanding of vector, student could smoothly transition to vector functions, and finally, partial derivatives.

1 Variables and Dimensions

In single variable calculus, we only dealt with function in one variable. We do two major operations: differentiation and integration. For $f(x) = x^2 - x + 2$, x is the argument of the function. In cartesian coordinate system[1], which translate between algebraic expression and geometric representation, the computed value $y = f(x)$ depends on x. This function is a curve on the plane: a one-dimensional object embedded in a 2D plane. Since for $x \to y$, there is only one coordinate on the parabola, namely, x, the y coordinate is determined by the equation $y = f(x)$.

The Curve The curve itself is one dimensional. Imagine a curve in $\mathbb{R}^3$. The curve is still one dimensional because it could be just a segment of a line. We can express every point on the curve an its distance to the origin of the curve. The attributes on the curve cannot change and is only determined by one variable.

For example, a circle: $x^2 + y^2 = 0$ on the plane is still a curve: a 1D object embedded in 2D plane. Indeed, there's only one parameter to describe the curve, that is the angle θ. Then we translate the one variable to two dimensional coordinates: $x = \cos\theta, y = \sin\theta, 0 \leq \theta < 2\pi$. The angle θ is the parameter.

Parameter and Mutivariables That's the notion of parameter. For every value of intrinsic coordinate(parameter), there will be designated x and y. General functions are special case of parametric equation since for parameter t:

$$\begin{cases} x = t \\ y = f(t) \end{cases} \tag{1}$$

For 2 and 3 dimensional space, variables defined on more dimensions. $f(x, y)$ is therefore 2D object in 3D space. In MVC, we have more degree of freedoms, a whole variety of differentiations and integration; linking differentiation and integration in much more subtle ways; and playing with curves in 3D space.

2 Parametric Curve

We defined a one dimension curve by, for $\alpha \leq t \leq \beta$:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \tag{2}$$

[1] specifically, the 2D coordinate system, two independent line create a plane.

For example, a circle is defined by:

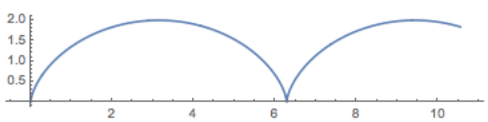
$$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases} \tag{3}$$

For example, we can expand:

$$\begin{cases} x = t^2 - t + 1 \\ y = t - 1 \end{cases} \tag{4}$$

into $x = y^2 + y + 1$.

Moreover, we can discuss the cycloid problem. How do we describe a cycloid?



$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases} \tag{5}$$

θ is auxiliary variable/parameters, and r is clearly the constant. Also, don't forget the range: $0 \leq \theta \leq 2\pi$.[2]

Slope, Derivatives For

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \tag{6}$$

Note that the tangent line is a relative notion with specific point on a curve. With parameter t, every point of curve could be described by t and t only. We have point (x_0, y_0):

$$\begin{cases} x_0 = f(t_0) \rightarrow \frac{dx}{dt} = f'(t) \\ y_0 = g(t_0) \rightarrow \frac{dy}{dt} = g'(t) \end{cases} \tag{7}$$

$$slope = \frac{g'(t_0)}{f'(t_0)}$$

at (x_0, y_0).

[2]Don't forget about the range of parameter! Especially for the test.

Using chain rule[3], we have:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Through $k = \frac{g'(t_0)}{f'(t_0)}$, we learn that 1) When $g'(t_0) = 0$ and $f'(t_0) \neq 0$, i.e. slope is 0, the tangent line is horizontal. 2) When $g'(t_0) \neq 0$ and $f'(t_0) = 0$, the line is vertical. 3) Slope > 0 means y is increasing as x is increasing. Other properties applied.

3 Integration of Parametric Curve

Area "under the curve" Derived from SVC:

$$A = \int_a^b y dx$$

We have:

$$A = \int_\alpha^\beta g(t) f'(t) dt$$

Be careful that the area could be under x axis, remember to $\int = -\text{Area}$.

Arc Length Derived from SVC:

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

and the real essence of arc length formula is really the distance formula $z = \sqrt{x^2 + y^2}$:

$$L = \int_\alpha^\beta \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

Area of a revolution surface Same reasoning:

$$S = \int_\alpha^\beta 2\pi g(t) \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

[3]It literally looks like dx cancelled out. But no. See section 10 for clear definition of differentials

4 Polar Coordinate

Formal definition:

$$r\text{: Distance between origins and the point}$$

$$\theta\text{: Angle between the line connecting point and origin with positive x axis}$$

$$P \to (r_0, \theta_0)$$

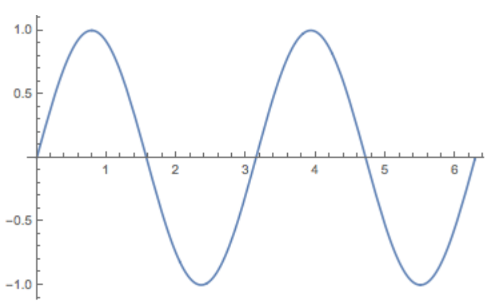
We also define polar function:

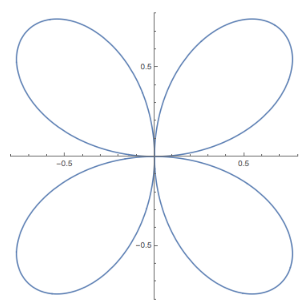
$$r = F(\theta)$$

$$\begin{cases} x_0 = r_0 \cos(\theta_0) = F(\theta_0)\cos(\theta_0) \\ y_0 = r_0 \sin(\theta_0) = F(\theta_0)\sin(\theta_0) \end{cases} \tag{8}$$

In sense, we can see polar function is parametric. In a sense, it is a "higher order function", that one function calls or define another function.

We graph polar function through two different system, cartesian and polar:





Now we have the x and y. We can get the derivative, tangent line, and arc length through formulas from section 4. However, area:

$$A = \frac{1}{2} f(\theta)^2 \delta\theta$$

$$= \int_\alpha^\beta \frac{1}{2} f(\theta)^2 de$$

One more thing, the transformation between polar and cartesian is easy: just remember few basic rules:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ x^2 + y^2 = r^2 \\ \theta = \tan \frac{y}{x} \end{cases} \tag{9}$$

5 Vectors

3D Space In 1D, a line is drawn. Every geometric object on this line is assigned an algebraic value.
In 2D, a plane is drawn. Every point on the plane is imposed a coordinate. A collection of numbers as addresses of points on plane fully describe $\mathbb{R}^2$
In 3D, a space is drawn. We use perspective to represent or visualize the 3D space on our plane/paper.

Discussion on orientation and degree of freedom in different dimensions In 1D, there are two orientation for the line. Left to right or right to left. It cannot be rotated nor switched. In 2D, there are two orientation for the plane. Either x-y or y-x. These two orientations are independent and exclusive. In 3D, there are also only two orientations. Remember to use right hand rule.

Properties For point $P(x_0, y_0, z_0)$ and $p(x_1, y_1, z_1)$:

$$d(P, Q) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}$$

Unfold this, we have expression of sphere in 3D:

$$(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2 = r^2$$

Vectors How do you really define vector? Object that has magnitude and direction. Note: just same direction and magnitude, not location! Vector is not static. It is about the movement, trajectory, the process. There could be many "lines" represent the same vector. Nevertheless, there is a numeric way to represent vector in cartesian coordinate system.
Given $\vec{a}$, first arrange so that initial point is $O = (0, 0)$ and the end point is $P(x_0, y_0)$. We could denote: $\vec{a} =< x_0, y_0 >$. Note that P is the end of the process $\vec{a}$, not the movement itself. Same in $\mathbb{R}^3$

Dot Product Result is a scaler.
Geometric Way:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$|\vec{a}| = \sqrt{x_0^2 + y_0^2 + z_0^2}$$

Algebraic Way:

$$\vec{a} =< x_0, y_0, z_0 >$$

$$\vec{b} =< x_1, y_1, z_1 >$$

$$\vec{a} \cdot \vec{b} = x_0 x_1 + y_0 y_1 + z_0 z_1$$

Cross Product Get a vector, direction is perpendicular to both.
Geometric Way:

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

Algebraic Way:

$$|\vec{a} \times \vec{b}| = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \end{bmatrix}$$

6 Vector in 3D Space

Line in 3D For vector $\vec{PQ}$ on the line to be expressed, and $\vec{PQ} = t\vec{v}$:

$$\vec{r} = \vec{v_0} + t\vec{v}$$

We can parameterize it:

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases} \tag{10}$$

OR:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Note: This representation is not unique, we could choose another point $p' =< x_0', y_0', z_0' >$ instead of P and another vector $\vec{v}$ which must be proportional to $\vec{v}$.

Discussion on describing object and its relation to equation General rule: In N dimensional space, to describe S, we need (n-d) equation.
In 3D, n = 3:

To describe a point ($d = 0$), $n - d = 3$. You need to impose three equations.
For a line, $n - d = 2$. Two equations
For a plane, just one equation will suffice.

Equation of a (flat) plane We use the normal vector, to pinpoint a plane. We have point $P(0, 0, 0), Q(x, y, z)$ and $\vec{n} =< a, b, c >$

$$\vec{PQ} = \vec{r} - \vec{r_0}$$

$$\text{Since } \vec{PQ} \perp \vec{n} \Rightarrow \vec{PQ} \cdot \vec{n} = 0$$

$$< x - x_0, y - y_0, z - z_0 > \cdot < a, b, c >= 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

7 Surfaces in 3D Space

See textbook Chapter 12.6.
General rule: Building upon the conics in 2D, you should be able to infer its corresponding 2D surface in 3D.

8 Vector Function

Curve in 3D:

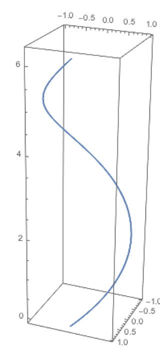
$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases} \tag{11}$$

Therefore, we have:

$$\vec{r}(t) =< f(t), g(t), h(t) >$$

Example of spiral curve For function:

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases} \tag{12}$$



Its projection on xy plane is a circle.
Now we can find the direction vector $\vec{(v)}$ at, say, $\frac{\pi}{4}$:

$$\vec{r}(t) = \vec{r_0} + \vec{v}t$$

$$\vec{r_0} =< \cos\frac{\pi}{4}, \sin\frac{\pi}{4}, \frac{\pi}{4} >=< \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4} >$$

$$\vec{v} = \vec{r}'(t_0) =< -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 >$$

Therefore, the tangent lane:

$$\vec{r}(s) =< \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4} > + < -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 > s$$

Regarding function with more than one variable Be careful with the domain.

Finding the limit of a multivariable function Rule: Approach it along various curve/equation/function, if all of them yield the same result, then the limit is the result. It not, the limit doesn't exist. Generally, you should test for two cases.

9 Derivatives

In single variable...

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \text{ if it exist.}$$

Geometrically, it is the slop of the tangent line at point x, and the linear approximation of the line at point x, the instantaneous rate of change.

Partial Derivatives As for $f(x, y)$, we take two derivatives: with respect to x and to y.

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

When we are taking f_x, we use y as a constant. Vice versa. Interestingly:

$$f_x y = f_y x$$

Tangent Plane We can see tangent plane as the plane consist of two tangent lines. Or we derive from curve:

$$y - y_0 = f'(x_0)(x - x_0)$$

$$\rightarrow$$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

10 Differentials, Tangent Planes and Linear Approxiamation

10.1 Tangent Planes

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

10.2 Linear Approximation

The resulted tangent plane $z-z_0 = a(x-x_0)+a(y-y_0)$ is particularly helpful for computing linear approximations of two variables. From previous equation

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We have

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The linear function whose graph is the tangent plane based on point $(x_0, y_0, f(x_0, y_0))$:

$$z = L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called linearization of f at (x_0, y_0). Furthermore, the approximation

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called the linear approximation or the tangent plane approximation of f at (x_0, y_0).

Now we continue to find the corresponding increment of z:

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

From the increment, we can define: If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 and \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Sometimes you can't use equation above to check the differentiability of a function. Nevertheless: If the partial derivative f_x and f_y exist near (a,b) and are continuous at (a, b), then f is differentiable at (a,b)[4]

Parameterization For point $p(2, 3)$ and function $f(x, y)$,

$$f(\delta x_p, \delta y_p) = f(< 2, 3 >)+ < \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y} > \cdot < \delta x, \delta y >$$

[4]which makes a lot of sense because if the two partial derivative (i.e. the tangent lines on x or y direction) is differentiable, the plane must be differentiable.

10.3 Differentials

10.3.1 Misconception

In single variable calculus, we often see that, for $f(x)$:

$$df = f'(x)dx$$

$$f' = dy/dx$$

Student then ask: Does f' have independent meaning? Shouldn't it just be $\frac{df}{dx}$? Wrong. f' has independent meaning because it represent the derivative of function f at point x_0.

10.3.2 Differentials in One Variable

Definition The differential of $f(x)$ at the point x_0 is the function of x defined by

$$f(x) - f(x_0) = f'(x_0)(x - x_0)$$

Note that the differential is attach to a particular point, just like the tangent line. The function depends on x_0. Now, we give $f(x) - f(x_0)$ a notation $df_{x_0}^{(x)}$. It is a function of x whoes graph is the tangent line at x_0, up to shift by $f(x_0)$:

$$df_{x_0}^{(x)} = f'(x_0)(x - x_0)$$

Furthermore, we give $x - x_0$ a notation[5]:

$$df_{x_0}^{(x)} = f'(x_0)d_{x_{x_0}}$$

Linear Approximation and Error Bound in One Variable We have

$$\begin{aligned} \Delta f &= f(x_0 + \Delta x) - f(x_0) \\ &= f'(x_0)(x - x_0) + Error \\ &= df_{x_0} + \varepsilon(x)(x - x_0) \\ where \varepsilon(x) \to 0 as x \to 0 & \end{aligned} \tag{13}$$

The linear part of Δf we call it differential. It is linear for $(x - x_0)$. [6] This is the idea of linear approximation. Since the linear term dominant.

[5]Note how all three variables depend on the starting point x_0, here we answer the misconception question: f' has independent meaning. And it depends on x_0

[6]For more information, you can look at Taylor Series at x_0 and the error bound for each following terms. As $x \to 0$, quadratic and cubic terms get smaller in a much faster rate.

Back to multivariable For the same procedure, we now have the differential of dz, which we call it total differential:

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\delta z}{\delta x}dx + \frac{\delta z}{\delta y}dy$$

The operation of three and more variables should follow the same procedure.