SUPPLEMENTARY READING FOR MATH 53: DIFFERENTIAL FORMS AND THE GENERAL STOKES FORMULA

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The purpose of these notes is to outline briefly a general formalism which allows for a uniform treatment of various integrals studied in Chapter 16 of Stewart's book. More details can be found in the book by I. Madsen and J. Tornehave, *From Calculus to Cohomology*, Cambridge University Press, 1997.

1. What do we integrate?

Let Y be an m–dimensional domain in $\mathbb{R}^n$ (of course, $m \leq n$). What can we integrate over Y? Let for example $m = 1$, i.e., Y is a curve C. We have learned how to integrate the following expressions over curves in $\mathbb{R}^3$:

$$Pdx + Qdy + Rdz, \tag{1.1}$$

where P, Q, R are functions in (x, y, z). How do we perform this integration in practice? We first *parametrize* our curve: $x = x(t), y = y(t), z = z(t), a \leq t \leq b$. In other words, we identify our curve C with a *flat* one-dimensional domain: the interval $[a, b]$. Then we write

$$\int_C (Pdx + Qdy + Rdz) = \int_{[a,b]} \left(P\frac{\partial x}{\partial t}dt + Q\frac{\partial y}{\partial t}dt + r\frac{\partial z}{\partial t}dt \right)$$

(here we use the fact that $dx = \dfrac{\partial x}{\partial t}dt$, etc.). The RHS is a familiar one-dimensional integral. This is what the book calls the line integral of the vector field $\mathbf{F} = \langle P, Q, R\rangle$ over C, but as we see it is more natural to think that we really integrate not $\mathbf{F}$, but the expression (1.1).

Likewise, the general expression that can be integrated over a surface S in $\mathbb{R}^3$ (the case $m = 2$) is

$$Adydz + Bdzdx + Cdxdy, \tag{1.2}$$

where A, B, C are functions in (x, y, z). The integral of this expression over S can be computed as follows. We *parametrize* S:

$$x = x(u, v), y = y(u, v), z = z(u, v), \qquad (u, v) \in D \subset \mathbb{R}^2,$$

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that is identify S with a *flat* domain D in $\mathbb{R}^2$. After that we can write

$$\iint_S (Adydz + Bdzdx + Cdxdy) = \iint_D \left(A\frac{\partial(y,z)}{\partial(u,v)}dudv + B\frac{\partial(z,x)}{\partial(u,v)}dudv + C\frac{\partial(x,y)}{\partial(u,v)}dudv \right). \tag{1.3}$$

Here we use the change of variables formulas:

$$dxdy = \frac{\partial(x,y)}{\partial(u,v)}dudv, \qquad \text{etc.},$$

where $\frac{\partial(x,y)}{\partial(u,v)}$ denotes the familiar *Jacobian*.

Note that, according to the last formula, we have

$$dxdy = -dydx,$$

because

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{\partial(y,x)}{\partial(u,v)}.$$

The right hand side of formula (1.3) is what the book calls the *flux* of the vector field $\langle A, B, C\rangle$ (look at formula (9) of Sect. 16.7 and expand $\mathbf{r}_u \times \mathbf{r}_v$). But this formula shows that it is more natural to think that we are integrating the expression (1.2).

Finally, the most general expression that we can integrate over a solid E in $\mathbb{R}^3$ has the form

$$f(x,y,z)dxdydz, \tag{1.4}$$

and the corresponding integral is

$$\iiint_E f(x,y,z)dxdydz,$$

which is the standard triple integral. Note that we do not need to parametrize E: unlike C and S, it is *already flat* (in other words, (x, y, z) themselves are the flat coordinates).

After these examples one can guess that the general object that we should integrate over an m–dimensional domain Y in $\mathbb{R}^n$ (with coordinates $x_1, \dots, x_n$) is the sum of terms of the form

$$f(x_1,\dots,x_n)dx_{i_1}dx_{i_2}\dots dx_{i_m}, \tag{1.5}$$

where $f(x_1,\dots,x_n)$ is a function in $x_1,\dots,x_n$.

We must have exactly m factors dx_j in this expression, since we are going to integrate m times. Moreover, all dx_{i_j}'s appearing in a product $dx_{i_1}dx_{i_2}\dots dx_{i_m}$ *must be distinct*. Indeed, we don't want to integrate $dxdx$, say, over a surface S in $\mathbb{R}^3$ – the result would be 0.

So for example for $m = 2$ and $n = 3$ we have the following possibilities:

$$dxdy, dydx, dydz, dzdy, dxdz, dzdx.$$

But actually $dxdy$ is the same as $-dydx$, as we have seen above. Hence there are only 3 truly distinct elements: $dxdy, dydz, dzdx$. Therefore the sum of terms of the form

(1.5) becomes formula (1.2). Other examples are given by formulas (1.1) and (1.4) – they correspond to the cases when $m = 1, n = 3$ and $m = 3, n = 3$, respectively.

How to integrate the expression (1.5) over an m-dimensional domain Y in $\mathbb{R}^n$? In the same way as in the case $n = 3$ (see above). First, *parametrize* the domain, i.e., identify it with a *flat* domain $Z \subset \mathbb{R}^m$ with coordinates $z_1, \ldots, z_m$. This means writing each x_i as a function $x_i(z_1, \ldots, z_m)$. After that we can write

$$\int_Y f dx_{i_1} dx_{i_2} \ldots dx_{i_m} = \int_Z f \frac{\partial(x_{i_1}, x_{i_2}, \ldots, x_{i_m})}{\partial(z_1, z_2, \ldots, z_m)} dz_1 dz_2 \ldots dz_m,$$

where we have included the Jacobian of the corresponding change of variables. The RHS is a standard multiple integral, which can be evaluated as an iterated integral in the same way as we do it for $m = 2$ and $m = 3$.

Terminology: the expression (1.5) is called a *differential form* of degree m, or an *m–form*. Note that a 0–form is just the same as a function. Thus, (1.1) is a 1–form, (1.2) is a 2–form, etc. Conclusion: *we are integrating m–forms.*

2. de Rham differential

Now we introduce an operation d that manufactures an $(m + 1)$–form, denoted by $d\omega$, out of any m–form ω. On an m–form (1.5) it acts as follows:

$$d\left(f(x_1, \ldots, x_n) dx_{i_1} dx_{i_2} \ldots dx_{i_m}\right) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j dx_{i_1} dx_{i_2} \ldots dx_{i_m}.$$

A general m–form is a sum of terms of the form (1.5). The result of application of d to a general form is just the sum of the applications of d to each of these terms (in other words, d is a linear map).

This operation is called the *de Rham differential*, after French mathematician Georges de Rham.

Examples. (1) Let ω be a 0–form, that is a function f. Then we find:

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j.$$

This is nothing but the usual differential of the function f considered as a 1–form.

(2) Let ω be a 1–form (1.1) in $\mathbb{R}^3$. Then

$$\begin{aligned} d\omega = {} & \frac{\partial P}{\partial x} dxdx + \frac{\partial P}{\partial y} dydx + \frac{\partial P}{\partial z} dzdx \\ & + \frac{\partial Q}{\partial x} dxdy + \frac{\partial Q}{\partial y} dydy + \frac{\partial Q}{\partial z} dzdy \\ & + \frac{\partial R}{\partial x} dxdz + \frac{\partial R}{\partial y} dydz + \frac{\partial R}{\partial z} dzdz. \end{aligned}$$

But remember that $dxdx = dydy = dzdz = 0$, and also that $dydx = -dxdy$, etc. Therefore we can rewrite this formula as follows:

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz$$
$$+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx$$
$$+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

We see that the functions arising here are the components of curl of the vector field $\langle P, Q, R\rangle$. Thus we obtain a very nice interpretation of curl using the de Rham differential!

(3) Let ω be a 2–form given by (1.2). Then we find in the same way as in example (2):

$$d\omega = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dxdydz.$$

The function in the RHS is nothing but div of the vector field $\langle A, B, C\rangle$.

An important property of d is that it is *nilpotent*: $d^2 = 0$ (check this!). This implies that if $\omega = d\nu$, then $d\omega = 0$. This fact is a generalization of the following facts that we have seen in our course:

$$\operatorname{curl}(\vec{\nabla} f) = 0, \qquad \operatorname{div}(\operatorname{curl} \mathbf{F}) = 0.$$

Moreover, the converse is also true (it is known as the *Poincare lemma*):

If ω is an m–form well-defined on the entire $\mathbb{R}^n$, such that $d\omega = 0$, then there exists an $(m-1)$–form ν, such that $\omega = d\nu$.

This is a generalization of the following facts:

$m = 1$. If $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F}$ is conservative, i.e., $\mathbf{F} = \vec{\nabla} f$ for some function f.
$m = 2$. If $\operatorname{div} \mathbf{E} = 0$, then $\mathbf{E} = \operatorname{curl} \mathbf{F}$ for some $\mathbf{F}$.

3. The general Stokes formula

Let Y be an m–dimensional domain in $\mathbb{R}^n$, and denote by $b(Y)$ its boundary (it is $(m-1)$–dimensional). Let ω be any $(m-1)$–form in $\mathbb{R}^n$. Then the following formula, called the *general Stokes formula*, is true:

$$\int_Y d\omega = \int_{b(Y)} \omega. \tag{3.1}$$

Here are the special cases of this formula (to see that, look at the examples above):

$m = 1, n = 1$. This is the Fundamental Theorem of Calculus.
$m = 1, n = 2$ and $m = 1, n = 3$. This is the Fundamental Theorem for line integrals.
$m = 2, n = 2$. This is Green's Theorem.
$m = 2, n = 3$. This is Stokes' Theorem.
$m = 3, n = 3$. This is the Divergence Theorem.

Thus, formula (3.1) gives us an elegant one-line formulation of the main results of Chapter 16.

It also indicates a tantalizing connection between the *geometric* operation of taking the boundary $b(Y)$ of a domain Y and the *algebraic* operation d on differential forms. According to formula (3.1), the two are dual to each.

Integration is a pairing of two objects: one geometric – the domain of integration – and one algebraic – the differential form that we are integrating. On each type of objects there is a natural operation: taking the boundary of the domain on the geometric side, and taking the de Rham differential on the algebraic side. Both operations are nilpotent: for the de Rham differential d we have $d^2 = 0$, and we also know that the boundary of the boundary of any domain is empty. The general Stokes formula (3.1) says that we can "trade" one operation for the other. It is in this sense that they are dual to each other.

This is a deep result which opens up the possibility of analyzing geometry by algebraic means. This idea is in fact one of the cornerstones of modern analysis.

If you want to read more about this, try the article "The notion of dimension in geometry and algebra" by Yu.I. Manin, available at http://arxiv.org/abs/math.AG/0502016