checking nearby distributions.) In comparison, if we use all three turbines with $Q_T = 1500$ we get $Q_1 = 436.3$, $Q_2 = 466.0$, and $Q_3 = 597.7$, resulting in a total energy production of $KW_1 + KW_2 + KW_3 \approx$ 16,538.7 kW. Clearly, for this flow level it is beneficial to use only two turbines.

6. Note that an incoming flow of 3400 ft^3/s is not within the domain we established in Problem 2, so we cannot simply use our previous work to give the optimal distribution. We will need to use all three turbines, due to the capacity limitations of each individual turbine, but 3400 is less than the maximum combined capacity of 3445 ft^3/s, so we still must decide how to distribute the flows. From the graph in Problem 4, Turbine 3 produces the most power for the higher flows, so it seems reasonable to use Turbine 3 at its maximum capacity of 1225 and distribute the remaining 2175 ft^3/s flow between Turbines 1 and 2. We can again use the technique of Lagrange multipliers to determine the optimal distribution. Following the procedure we used in Problem 5, we wish to maximize $KW_1 + KW_2$ subject to the constraint $Q_1 + Q_2 = Q_T$ where $Q_T = 2175$. We can equivalently maximize

$$\begin{aligned} f(Q_1, Q_2) &= \frac{KW_1 + KW_2}{170 - 1.6 \cdot 10^{-6} Q_T^2} \\ &= \left(-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5} Q_1^2\right) + \left(-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5} Q_2^2\right) \end{aligned}$$

subject to the constraint $g(Q_1, Q_2) = Q_1 + Q_2 = Q_T$. Then we solve $\nabla f(Q_1, Q_2) = \lambda \nabla g(Q_1, Q_2) \quad \Rightarrow$ $0.1277 - 2(4.08 \cdot 10^{-5})Q_1 = \lambda$ and $0.1358 - 2(4.69 \cdot 10^{-5})Q_2 = \lambda$, thus $0.1277 - 2(4.08 \cdot 10^{-5})Q_1 = 0.1358 - 2\left(4.69 \cdot 10^{-5}\right)Q_2 \quad \Rightarrow \quad Q_1 = -99.2647 + 1.1495Q_2$. Substituting into $Q_1 + Q_2 = Q_T$ gives $-99.2647 + 1.1495Q_2 + Q_2 = 2175 \quad \Rightarrow \quad Q_2 \approx 1058.0$, and then $Q_1 \approx 1117.0$. This value for Q_1 is larger than the allowable maximum flow to Turbine 1, but the result indicates that the flow to Turbine 1 should be maximized. Thus we should recommend that the company apportion the maximum allowable flows to Turbines 1 and 3, 1110 and 1225 ft^3/s, and the remaining 1065 ft^3/s to Turbine 2. Checking nearby distributions within the domain verifies that we have indeed found the optimal distribution.

14 Review

TRUE-FALSE QUIZ

1. True. $f_y(a, b) = \lim\limits_{h \to 0} \dfrac{f(a, b + h) - f(a, b)}{h}$ from Equation 14.3.3. Let $h = y - b$. As $h \to 0$, $y \to b$. Then by substituting, we get $f_y(a, b) = \lim\limits_{y \to b} \dfrac{f(a, y) - f(a, b)}{y - b}$.

2. False. If there were such a function, then $f_{xy} = 2y$ and $f_{yx} = 1$. So $f_{xy} \neq f_{yx}$, which contradicts Clairaut's Theorem.

3. False. $f_{xy} = \dfrac{\partial^2 f}{\partial y \, \partial x}$.

4. True. From Equation 14.6.14 we get $D_{\mathbf{k}}\, f(x, y, z) = \nabla f(x, y, z) \cdot \langle 0, 0, 1 \rangle = f_z(x, y, z)$.

5. False. See Example 14.2.3.

6. False. See Exercise 14.4.46(a).

7. True. If f has a local minimum and f is differentiable at (a,b) then by Theorem 14.7.2, $f_x(a,b)=0$ and $f_y(a,b)=0$, so $\nabla f(a,b)=\langle f_x(a,b), f_y(a,b)\rangle=\langle 0,0\rangle=\mathbf{0}$.

8. False. If f is not continuous at $(2,5)$, then we can have $\lim\limits_{(x,y)\to(2,5)} f(x,y)\neq f(2,5)$. (See Example 14.2.7)

9. False. $\nabla f(x,y)=\langle 0, 1/y\rangle$.

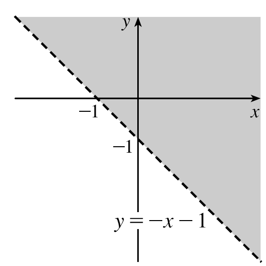
10. True. This is part (c) of the Second Derivatives Test (14.7.3).

11. True. $\nabla f=\langle \cos x, \cos y\rangle$, so $|\nabla f|=\sqrt{\cos^2 x+\cos^2 y}$. But $|\cos\theta|\le 1$, so $|\nabla f|\le\sqrt{2}$. Now $D_{\mathbf{u}}\, f(x,y)=\nabla f\cdot\mathbf{u}=|\nabla f|\,|\mathbf{u}|\cos\theta$, but $\mathbf{u}$ is a unit vector, so $|D_{\mathbf{u}}\, f(x,y)|\le\sqrt{2}\cdot 1\cdot 1=\sqrt{2}$.

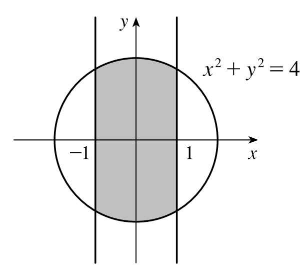
12. False. See Exercise 14.7.39.

EXERCISES

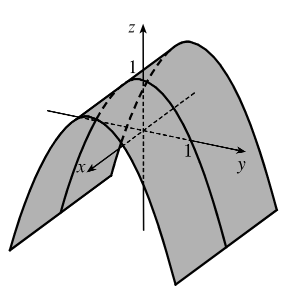
1. $\ln(x+y+1)$ is defined only when $x+y+1>0 \quad\Leftrightarrow\quad y>-x-1$, so the domain of f is $\{(x,y)\mid y>-x-1\}$, all those points above the line $y=-x-1$.



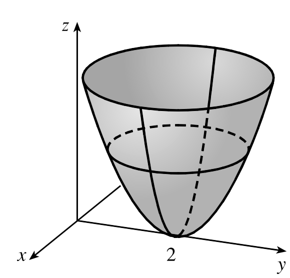
2. $\sqrt{4-x^2-y^2}$ is defined only when $4-x^2-y^2\ge 0 \quad\Leftrightarrow\quad x^2+y^2\le 4$, and $\sqrt{1-x^2}$ is defined only when $1-x^2\ge 0 \quad\Leftrightarrow\quad -1\le x\le 1$, so the domain of f is $\left\{(x,y)\mid -1\le x\le 1, -\sqrt{4-x^2}\le y\le\sqrt{4-x^2}\right\}$, which consists of those points on or inside the circle $x^2+y^2=4$ for $-1\le x\le 1$.



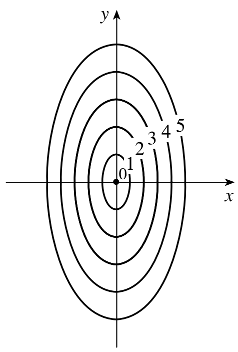
3. $z=f(x,y)=1-y^2$, a parabolic cylinder



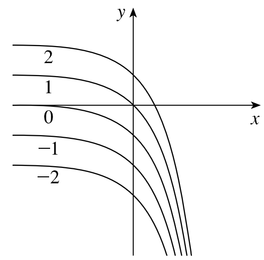
4. $z=f(x,y)=x^2+(y-2)^2$, a circular paraboloid with vertex $(0,2,0)$ and axis parallel to the z-axis



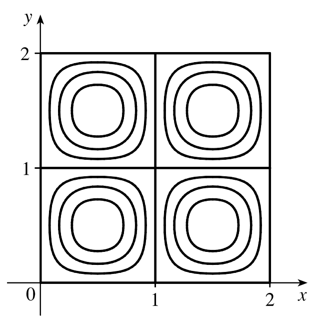
5. The level curves are $\sqrt{4x^2 + y^2} = k$ or $4x^2 + y^2 = k^2$, $k \geq 0$, a family of ellipses.



6. The level curves are $e^x + y = k$ or $y = -e^x + k$, a family of exponential curves.



7.



8. (a) The point $(3, 2)$ lies partway between the level curves with z-values 50 and 60, and it appears that $(3, 2)$ is about the same distance from either level curve. So we estimate that $f(3, 2) \approx 55$.

(b) At the point $(3, 2)$, if we fix y at $y = 2$ and allow x to vary, the level curves indicate that the z-values decrease as x increases, so $f_x(3, 2)$ is negative. In other words, if we start at $(3, 2)$ and move right (in the positive x-direction), the contours show that our path along the surface $z = f(x, y)$ is descending.

(c) Both $f_y(2, 1)$ and $f_y(2, 2)$ are positive, because if we start from either point and move in the positive y-direction, the contour map indicates that the path is ascending. But the level curves are closer together in the y-direction at $(2, 1)$ than at $(2, 2)$, so the path is steeper (the z-values increase more rapidly) at $(2, 1)$ and hence $f_y(2, 1) > f_y(2, 2)$.

9. f is a rational function, so it is continuous on its domain. Since f is defined at $(1, 1)$, we use direct substitution to evaluate the limit: $\displaystyle\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2 + 2y^2} = \frac{2(1)(1)}{1^2 + 2(1)^2} = \frac{2}{3}$.

10. As $(x, y) \to (0, 0)$ along the x-axis, $f(x, 0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x, y) \to 0$ along this line. But $f(x, x) = 2x^2/(3x^2) = \frac{2}{3}$, so as $(x, y) \to (0, 0)$ along the line $x = y$, $f(x, y) \to \frac{2}{3}$. Thus the limit doesn't exist.

11. (a) $T_x(6, 4) = \displaystyle\lim_{h\to 0} \frac{T(6 + h, 4) - T(6, 4)}{h}$, so we can approximate $T_x(6, 4)$ by considering $h = \pm 2$ and using the values given in the table: $T_x(6, 4) \approx \dfrac{T(8, 4) - T(6, 4)}{2} = \dfrac{86 - 80}{2} = 3$,

$T_x(6,4) \approx \dfrac{T(4,4)-T(6,4)}{-2} = \dfrac{72-80}{-2} = 4$. Averaging these values, we estimate $T_x(6,4)$ to be approximately 3.5°C/m. Similarly, $T_y(6,4) = \lim\limits_{h\to 0} \dfrac{T(6,4+h)-T(6,4)}{h}$, which we can approximate with $h = \pm 2$:

$T_y(6,4) \approx \dfrac{T(6,6)-T(6,4)}{2} = \dfrac{75-80}{2} = -2.5$, $T_y(6,4) \approx \dfrac{T(6,2)-T(6,4)}{-2} = \dfrac{87-80}{-2} = -3.5$. Averaging these values, we estimate $T_y(6,4)$ to be approximately -3.0°C/m.

(b) Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so by Equation 14.6.9, $D_{\mathbf{u}}\,T(6,4) = \nabla T(6,4)\cdot \mathbf{u} = T_x(6,4)\,\frac{1}{\sqrt{2}} + T_y(6,4)\,\frac{1}{\sqrt{2}}$. Using our estimates from part (a), we have $D_{\mathbf{u}}\,T(6,4) \approx (3.5)\,\frac{1}{\sqrt{2}} + (-3.0)\,\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \approx 0.35$. This means that as we move through the point $(6,4)$ in the direction of $\mathbf{u}$, the temperature increases at a rate of approximately 0.35°C/m.

Alternatively, we can use Definition 14.6.2: $D_{\mathbf{u}}\,T(6,4) = \lim\limits_{h\to 0} \dfrac{T\left(6 + h\,\frac{1}{\sqrt{2}}, 4 + h\,\frac{1}{\sqrt{2}}\right) - T(6,4)}{h}$,

which we can estimate with $h = \pm 2\sqrt{2}$. Then $D_{\mathbf{u}}\,T(6,4) \approx \dfrac{T(8,6)-T(6,4)}{2\sqrt{2}} = \dfrac{80-80}{2\sqrt{2}} = 0$,

$D_{\mathbf{u}}\,T(6,4) \approx \dfrac{T(4,2)-T(6,4)}{-2\sqrt{2}} = \dfrac{74-80}{-2\sqrt{2}} = \dfrac{3}{\sqrt{2}}$. Averaging these values, we have $D_{\mathbf{u}}\,T(6,4) \approx \frac{3}{2\sqrt{2}} \approx 1.1^\circ\text{C/m}$.

(c) $T_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[T_x(x,y)\right] = \lim\limits_{h\to 0} \dfrac{T_x(x,y+h)-T_x(x,y)}{h}$, so $T_{xy}(6,4) = \lim\limits_{h\to 0} \dfrac{T_x(6,4+h)-T_x(6,4)}{h}$ which we can estimate with $h = \pm 2$. We have $T_x(6,4) \approx 3.5$ from part (a), but we will also need values for $T_x(6,6)$ and $T_x(6,2)$. If we use $h = \pm 2$ and the values given in the table, we have

$T_x(6,6) \approx \dfrac{T(8,6)-T(6,6)}{2} = \dfrac{80-75}{2} = 2.5$, $T_x(6,6) \approx \dfrac{T(4,6)-T(6,6)}{-2} = \dfrac{68-75}{-2} = 3.5$.

Averaging these values, we estimate $T_x(6,6) \approx 3.0$. Similarly,

$T_x(6,2) \approx \dfrac{T(8,2)-T_x(6,2)}{2} = \dfrac{90-87}{2} = 1.5$, $T_x(6,2) \approx \dfrac{T(4,2)-T(6,2)}{-2} = \dfrac{74-87}{-2} = 6.5$.

Averaging these values, we estimate $T_x(6,2) \approx 4.0$. Finally, we estimate $T_{xy}(6,4)$:

$T_{xy}(6,4) \approx \dfrac{T_x(6,6)-T_x(6,4)}{2} = \dfrac{3.0-3.5}{2} = -0.25$, $T_{xy}(6,4) \approx \dfrac{T_x(6,2)-T_x(6,4)}{-2} = \dfrac{4.0-3.5}{-2} = -0.25$.

Averaging these values, we have $T_{xy}(6,4) \approx -0.25$.

12. From the table, $T(6,4) = 80$, and from Exercise 11 we estimated $T_x(6,4) \approx 3.5$ and $T_y(6,4) \approx -3.0$. The linear approximation then is

$$T(x,y) \approx T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4) \approx 80 + 3.5(x-6) - 3(y-4) = 3.5x - 3y + 71$$

Thus at the point $(5, 3.8)$, we can use the linear approximation to estimate $T(5,3.8) \approx 3.5(5) - 3(3.8) + 71 \approx 77.1^\circ\text{C}$.

13. $f(x,y) = (5y^3 + 2x^2y)^8 \quad\Rightarrow\quad f_x = 8(5y^3+2x^2y)^7(4xy) = 32xy(5y^3+2x^2y)^7$,

$f_y = 8(5y^3+2x^2y)^7(15y^2+2x^2) = (16x^2+120y^2)(5y^3+2x^2y)^7$

14. $g(u,v)=\dfrac{u+2v}{u^2+v^2} \quad\Rightarrow\quad g_u=\dfrac{(u^2+v^2)(1)-(u+2v)(2u)}{(u^2+v^2)^2}=\dfrac{v^2-u^2-4uv}{(u^2+v^2)^2}$,

$$g_v=\frac{(u^2+v^2)(2)-(u+2v)(2v)}{(u^2+v^2)^2}=\frac{2u^2-2v^2-2uv}{(u^2+v^2)^2}$$

15. $F(\alpha,\beta)=\alpha^2\ln(\alpha^2+\beta^2) \quad\Rightarrow\quad F_\alpha=\alpha^2\cdot\dfrac{1}{\alpha^2+\beta^2}\,(2\alpha)+\ln(\alpha^2+\beta^2)\cdot 2\alpha=\dfrac{2\alpha^3}{\alpha^2+\beta^2}+2\alpha\ln(\alpha^2+\beta^2)$,

$$F_\beta=\alpha^2\cdot\frac{1}{\alpha^2+\beta^2}\,(2\beta)=\frac{2\alpha^2\beta}{\alpha^2+\beta^2}$$

16. $G(x,y,z)=e^{xz}\sin(y/z) \quad\Rightarrow\quad G_x=ze^{xz}\sin(y/z),\ G_y=e^{xz}\cos(y/z)(1/z)=(e^{xz}/z)\cos(y/z)$,

$G_z=e^{xz}\cdot\cos(y/z)(-y/z^2)+\sin(y/z)\cdot xe^{xz}=e^{xz}\left[x\sin(y/z)-(y/z^2)\cos(y/z)\right]$

17. $S(u,v,w)=u\arctan(v\sqrt{w}) \quad\Rightarrow\quad S_u=\arctan(v\sqrt{w}),\ S_v=u\cdot\dfrac{1}{1+(v\sqrt{w})^2}\,(\sqrt{w})=\dfrac{u\sqrt{w}}{1+v^2w}$,

$$S_w=u\cdot\frac{1}{1+(v\sqrt{w})^2}\left(v\cdot\tfrac{1}{2}w^{-1/2}\right)=\frac{uv}{2\sqrt{w}\,(1+v^2w)}$$

18. $C=1449.2+4.6T-0.055T^2+0.00029T^3+(1.34-0.01T)(S-35)+0.016D \quad\Rightarrow$

$\partial C/\partial T=4.6-0.11T+0.00087T^2-0.01(S-35)$, $\partial C/\partial S=1.34-0.01T$, and $\partial C/\partial D=0.016$. When $T=10$, $S=35$, and $D=100$ we have $\partial C/\partial T=4.6-0.11(10)+0.00087(10)^2-0.01(35-35)\approx 3.587$, thus in 10°C water with salinity 35 parts per thousand and a depth of 100 m, the speed of sound increases by about 3.59 m/s for every degree Celsius that the water temperature rises. Similarly, $\partial C/\partial S=1.34-0.01(10)=1.24$, so the speed of sound increases by about 1.24 m/s for every part per thousand the salinity of the water increases. $\partial C/\partial D=0.016$, so the speed of sound increases by about 0.016 m/s for every meter that the depth is increased.

19. $f(x,y)=4x^3-xy^2 \quad\Rightarrow\quad f_x=12x^2-y^2,\ f_y=-2xy,\ f_{xx}=24x,\ f_{yy}=-2x,\ f_{xy}=f_{yx}=-2y$

20. $z=xe^{-2y} \quad\Rightarrow\quad z_x=e^{-2y},\ z_y=-2xe^{-2y},\ z_{xx}=0,\ z_{yy}=4xe^{-2y},\ z_{xy}=z_{yx}=-2e^{-2y}$

21. $f(x,y,z)=x^ky^lz^m \quad\Rightarrow\quad f_x=kx^{k-1}y^lz^m,\ f_y=lx^ky^{l-1}z^m,\ f_z=mx^ky^lz^{m-1},\ f_{xx}=k(k-1)x^{k-2}y^lz^m$,

$f_{yy}=l(l-1)x^ky^{l-2}z^m,\ f_{zz}=m(m-1)x^ky^lz^{m-2},\ f_{xy}=f_{yx}=klx^{k-1}y^{l-1}z^m, f_{xz}=f_{zx}=kmx^{k-1}y^lz^{m-1}$,

$f_{yz}=f_{zy}=lmx^ky^{l-1}z^{m-1}$

22. $v=r\cos(s+2t) \quad\Rightarrow\quad v_r=\cos(s+2t),\ v_s=-r\sin(s+2t),\ v_t=-2r\sin(s+2t),\ v_{rr}=0,\ v_{ss}=-r\cos(s+2t)$,

$v_{tt}=-4r\cos(s+2t),\ v_{rs}=v_{sr}=-\sin(s+2t),\ v_{rt}=v_{tr}=-2\sin(s+2t),\ v_{st}=v_{ts}=-2r\cos(s+2t)$

23. $z=xy+xe^{y/x} \quad\Rightarrow\quad \dfrac{\partial z}{\partial x}=y-\dfrac{y}{x}e^{y/x}+e^{y/x},\ \dfrac{\partial z}{\partial y}=x+e^{y/x}$ and

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\left(y-\frac{y}{x}e^{y/x}+e^{y/x}\right)+y\left(x+e^{y/x}\right)=xy-ye^{y/x}+xe^{y/x}+xy+ye^{y/x}=xy+xy+xe^{y/x}=xy+z.$$

24. $z = \sin(x + \sin t) \Rightarrow \dfrac{\partial z}{\partial x} = \cos(x + \sin t),\ \dfrac{\partial z}{\partial t} = \cos(x + \sin t)\cos t,$

$\dfrac{\partial^2 z}{\partial x \partial t} = -\sin(x + \sin t)\cos t,\ \dfrac{\partial^2 z}{\partial x^2} = -\sin(x + \sin t)$ and

$$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x \partial t} = \cos(x + \sin t)\left[-\sin(x + \sin t)\cos t\right] = \cos(x + \sin t)(\cos t)\left[-\sin(x + \sin t)\right] = \frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2}.$$

25. (a) $z_x = 6x + 2 \Rightarrow z_x(1, -2) = 8$ and $z_y = -2y \Rightarrow z_y(1, -2) = 4$, so an equation of the tangent plane is $z - 1 = 8(x - 1) + 4(y + 2)$ or $z = 8x + 4y + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(1, -2, 1)$ is $\langle 8, 4, -1\rangle$. Then parametric equations for the normal line there are $x = 1 + 8t$, $y = -2 + 4t$, $z = 1 - t$, and symmetric equations are $\dfrac{x - 1}{8} = \dfrac{y + 2}{4} = \dfrac{z - 1}{-1}$.

26. (a) $z_x = e^x \cos y \Rightarrow z_x(0, 0) = 1$ and $z_y = -e^x \sin y \Rightarrow z_y(0, 0) = 0$, so an equation of the tangent plane is $z - 1 = 1(x - 0) + 0(y - 0)$ or $z = x + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(0, 0, 1)$ is $\langle 1, 0, -1\rangle$. Then parametric equations for the normal line there are $x = t$, $y = 0$, $z = 1 - t$, and symmetric equations are $x = 1 - z$, $y = 0$.

27. (a) Let $F(x, y, z) = x^2 + 2y^2 - 3z^2$. Then $F_x = 2x$, $F_y = 4y$, $F_z = -6z$, so $F_x(2, -1, 1) = 4$, $F_y(2, -1, 1) = -4$, $F_z(2, -1, 1) = -6$. From Equation 14.6.19, an equation of the tangent plane is $4(x - 2) - 4(y + 1) - 6(z - 1) = 0$ or, equivalently, $2x - 2y - 3z = 3$.

(b) From Equations 14.6.20, symmetric equations for the normal line are $\dfrac{x - 2}{4} = \dfrac{y + 1}{-4} = \dfrac{z - 1}{-6}$.

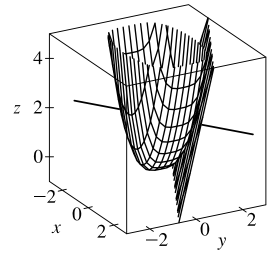
28. (a) Let $F(x, y, z) = xy + yz + zx$. Then $F_x = y + z$, $F_y = x + z$, $F_z = x + y$, so $F_x(1, 1, 1) = F_y(1, 1, 1) = F_z(1, 1, 1) = 2$. From Equation 14.6.19, an equation of the tangent plane is $2(x - 1) + 2(y - 1) + 2(z - 1) = 0$ or, equivalently, $x + y + z = 3$.

(b) From Equations 14.6.20, symmetric equations for the normal line are $\dfrac{x - 1}{2} = \dfrac{y - 1}{2} = \dfrac{z - 1}{2}$ or, equivalently, $x = y = z$.

29. (a) Let $F(x, y, z) = x + 2y + 3z - \sin(xyz)$. Then $F_x = 1 - yz\cos(xyz)$, $F_y = 2 - xz\cos(xyz)$, $F_z = 3 - xy\cos(xyz)$, so $F_x(2, -1, 0) = 1$, $F_y(2, -1, 0) = 2$, $F_z(2, -1, 0) = 5$. From Equation 14.6.19, an equation of the tangent plane is $1(x - 2) + 2(y + 1) + 5(z - 0) = 0$ or $x + 2y + 5z = 0$.

(b) From Equations 14.6.20, symmetric equations for the normal line are $\dfrac{x - 2}{1} = \dfrac{y + 1}{2} = \dfrac{z}{5}$ or $x - 2 = \dfrac{y + 1}{2} = \dfrac{z}{5}$. Parametric equations are $x = 2 + t$, $y = -1 + 2t$, $z = 5t$.

30. Let $f(x, y) = x^2 + y^4$. Then $f_x(x, y) = 2x$ and $f_y(x, y) = 4y^3$, so $f_x(1, 1) = 2$, $f_y(1, 1) = 4$ and an equation of the tangent plane is $z - 2 = 2(x - 1) + 4(y - 1)$ or $2x + 4y - z = 4$. A normal vector to the tangent plane is $\langle 2, 4, -1\rangle$ so the normal line is given by $\dfrac{x-1}{2} = \dfrac{y-1}{4} = \dfrac{z-2}{-1}$ or $x = 1 + 2t,\ y = 1 + 4t,$ $z = 2 - t$.



31. The hyperboloid is a level surface of the function $F(x, y, z) = x^2 + 4y^2 - z^2$, so a normal vector to the surface at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 8y_0, -2z_0\rangle$. A normal vector for the plane $2x + 2y + z = 5$ is $\langle 2, 2, 1\rangle$. For the planes to be parallel, we need the normal vectors to be parallel, so $\langle 2x_0, 8y_0, -2z_0\rangle = k\,\langle 2, 2, 1\rangle$, or $x_0 = k$, $y_0 = \frac{1}{4}k$, and $z_0 = -\frac{1}{2}k$. But $x_0^2 + 4y_0^2 - z_0^2 = 4 \quad\Rightarrow\quad k^2 + \frac{1}{4}k^2 - \frac{1}{4}k^2 = 4 \quad\Rightarrow\quad k^2 = 4 \quad\Rightarrow\quad k = \pm 2$. So there are two such points: $\left(2, \frac{1}{2}, -1\right)$ and $\left(-2, -\frac{1}{2}, 1\right)$.

32. $u = \ln(1 + se^{2t}) \quad\Rightarrow\quad du = \dfrac{\partial u}{\partial s}\,ds + \dfrac{\partial u}{\partial t}\,dt = \dfrac{e^{2t}}{1 + se^{2t}}\,ds + \dfrac{2se^{2t}}{1 + se^{2t}}\,dt$

33. $f(x, y, z) = x^3\sqrt{y^2 + z^2} \quad\Rightarrow\quad f_x(x, y, z) = 3x^2\sqrt{y^2 + z^2},\ f_y(x, y, z) = \dfrac{yx^3}{\sqrt{y^2 + z^2}},\ f_z(x, y, z) = \dfrac{zx^3}{\sqrt{y^2 + z^2}},$ so $f(2, 3, 4) = 8(5) = 40$, $f_x(2, 3, 4) = 3(4)\sqrt{25} = 60$, $f_y(2, 3, 4) = \frac{3(8)}{\sqrt{25}} = \frac{24}{5}$, and $f_z(2, 3, 4) = \frac{4(8)}{\sqrt{25}} = \frac{32}{5}$. Then the linear approximation of f at $(2, 3, 4)$ is

$$\begin{aligned} f(x, y, z) &\approx f(2, 3, 4) + f_x(2, 3, 4)(x - 2) + f_y(2, 3, 4)(y - 3) + f_z(2, 3, 4)(z - 4) \\ &= 40 + 60(x - 2) + \tfrac{24}{5}(y - 3) + \tfrac{32}{5}(z - 4) = 60x + \tfrac{24}{5}y + \tfrac{32}{5}z - 120 \end{aligned}$$

Then $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2} = f(1.98, 3.01, 3.97) \approx 60(1.98) + \frac{24}{5}(3.01) + \frac{32}{5}(3.97) - 120 = 38.656$.

34. (a) $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus the maximum error in the calculated area is about $dA = 6(0.002) + \frac{5}{2}(0.002) = 0.017\ \text{m}^2$ or $170\ \text{cm}^2$.

(b) $z = \sqrt{x^2 + y^2}$, $dz = \dfrac{x}{\sqrt{x^2 + y^2}}\,dx + \dfrac{y}{\sqrt{x^2 + y^2}}\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus the maximum error in the calculated hypotenuse length is about $dz = \frac{5}{13}(0.002) + \frac{12}{13}(0.002) = \frac{0.17}{65} \approx 0.0026$ m or 0.26 cm.

35. $\dfrac{du}{dp} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dp} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dp} + \dfrac{\partial u}{\partial z}\dfrac{dz}{dp} = 2xy^3(1 + 6p) + 3x^2y^2(pe^p + e^p) + 4z^3(p\cos p + \sin p)$

36. $\dfrac{\partial v}{\partial s} = \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial s} = \left(2x\sin y + y^2e^{xy}\right)(1) + \left(x^2\cos y + xye^{xy} + e^{xy}\right)(t).$

$s = 0, t = 1 \quad\Rightarrow\quad x = 2, y = 0$, so $\dfrac{\partial v}{\partial s} = 0 + (4 + 1)(1) = 5.$

$\dfrac{\partial v}{\partial t} = \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial t} = \left(2x\sin y + y^2e^{xy}\right)(2) + \left(x^2\cos y + xye^{xy} + e^{xy}\right)(s) = 0 + 0 = 0.$