Math 53 - Chapter 16 Conclusion

1 Types of Integrals

1.1 ... of a function f

1.1.1 ... on a curve C (any dimension)

$$\int\limits_C f \; \mathrm{d}s$$

How to compute:

$$\vec{r}(t) \rightsquigarrow \mathrm{d}s = |r'(t)| \, \mathrm{d}t$$

1.1.2 ... on a surface S (only in $\mathbb{R}^3$)

$$\iint\limits_S f \; \mathrm{d}S$$

How to compute:

$$\vec{r}(u,v) \rightsquigarrow \mathrm{d}S = |\vec{r}_u \times \vec{r}_v| \, \mathrm{d}u \; \mathrm{d}v$$

1.2 ... of a vecor field $\vec{F}$

Note: In $\mathbb{R}^n$, we can only compute the **work** along an **1-dimensional** manifold (curve), or the **flux** through a $(\mathbf{n-1})$**-dimensional** manifold.

1.2.1 ... of the work along a curve C (any dimension)

$$\int\limits_C \vec{F} \cdot \; \mathrm{d}\vec{r} = \int\limits_C \vec{F} \cdot \hat{\mathrm{T}} \; \mathrm{d}s = \int P \; \mathrm{d}x + Q \; \mathrm{d}y \; (+R \; \mathrm{d}z)$$

How to compute:

$$\int P \; \mathrm{d}x + Q \; \mathrm{d}y \; (+R \; \mathrm{d}z)$$

$$\begin{aligned} \vec{r}(t) \rightsquigarrow \mathrm{d}x &= x'(t) \; \mathrm{d}t \\ \mathrm{d}y &= y'(t) \; \mathrm{d}t \\ (\mathrm{d}z &= z'(t) \; \mathrm{d}t) \end{aligned}$$

1.2.2 ... of the flux through a curve C (only in $\mathbb{R}^2$)

$$\int\limits_C \vec{F} \cdot \hat{\mathrm{n}} \; \mathrm{d}S = \int\limits_C -Q \; \mathrm{d}x + P \; \mathrm{d}y$$

Convention: $\hat{\mathrm{n}}$ is obtained by rotating $\hat{\mathrm{T}}$ $90°$ counter-clockwise (s.t. $\hat{\mathrm{n}}$ of a closed, counter-clockwise curve will be pointing outwards).

How to compute:

$$\begin{aligned} \vec{r}(t) \rightsquigarrow \mathrm{d}x &= x'(t) \; \mathrm{d}t \\ \mathrm{d}y &= y'(t) \; \mathrm{d}t \end{aligned}$$

1.2.3 ... of the flux through a surface S (only in $\mathbb{R}^3$)

$$\iint\limits_S \vec{F}\cdot\ \mathrm{d}\vec{S} = \iint\limits_S \vec{F}\cdot \hat{\mathrm{n}}\ \mathrm{d}S$$

Note: Need to specify where $\hat{\mathrm{n}}$ points.

How to compute:

$$\vec{r}(u,v) \rightsquigarrow \mathrm{d}\vec{S} = \pm(\vec{r}_u \times \vec{r}_v)\ \mathrm{d}u\ \mathrm{d}v$$

Good to know:

$$\begin{aligned}&\text{for } \vec{r}(x,y) = \langle x, y, f(x,y)\rangle\\ &\quad \vec{r}_u \times \vec{r}_v = \langle -f_x, -f_y, 1\rangle\end{aligned}$$

$$\begin{aligned}&\text{for a sphere of radius } a\\ &\quad \hat{\mathrm{n}} = \frac{\langle x, y, z\rangle}{a}\\ &\quad \mathrm{d}S = a^2 \sin\phi\ \mathrm{d}\phi\ \mathrm{d}\theta\end{aligned}$$

$$\begin{aligned}&\text{for a cylinder } x^2+y^2=a^2\\ &\quad \hat{\mathrm{n}} = \frac{\langle x, y, 0\rangle}{a}\\ &\quad \mathrm{d}S = a\ \mathrm{d}z\ \mathrm{d}\theta\end{aligned}$$

2 Theorems of Vector Fields

2.1 $\vec{F}$ is the gradient of a function f (only in $\mathbb{R}^3$)

$$\begin{aligned}&\vec{F} = \nabla f \ (\vec{F} \text{ is conservative})\\ \Leftrightarrow\ &\oint\limits_C \vec{F}\cdot\ \mathrm{d}\vec{r} = 0\ \forall \text{ closed } C \text{ (Fundamental Theorem of Line Integral)}\\ \Leftrightarrow\ &\int\limits_C \vec{F}\cdot\ \mathrm{d}\vec{r} \text{ depends only on endpoints of } C\\ \Rightarrow\ &\text{curl } \vec{F} = 0/\vec{0}\\ \Leftarrow\ &\text{curl } \vec{F} = 0/\vec{0} \text{ if } \vec{F} \text{ is defined on a simple connected domain}\\ &(\text{counterexample}: \vec{F} = \frac{\langle -y, x\rangle}{x^2+y^2})\end{aligned}$$

2.2 $\vec{F}$ is the curl of another vector field $\vec{G}$ (only in $\mathbb{R}^3$)

$$\begin{aligned}
&\vec{F} = \text{curl } \vec{G} \\
\Leftrightarrow &\oiint_S \vec{F} \cdot \, \mathrm{d}\vec{S} = 0 \;\forall \text{ closed } S \\
\Leftrightarrow &\iint_S \vec{F} \cdot \, \mathrm{d}\vec{S} \text{ depends only on the boundary curve of } S \\
\Rightarrow &\text{div } \vec{F} = 0 \\
\Leftarrow &\text{div } \vec{F} = 0 \text{ under some additional assumptions (e.g. all of } \mathbb{R}^3) \\
&(\text{counterexample}: \vec{F} = \nabla f \text{ where } f = 1/\rho)
\end{aligned}$$

3 Theorems of Integrals

3.1 Fundamental Theorem of Calculus (only in $\mathbb{R}$)

$$\int_a^b F'(t)\ \mathrm{d}t = F(b) - F(a)$$

3.2 Fundamental Theorem of Line Integrals (any dimension)

$$\int_C \nabla f \cdot \, \mathrm{d}\vec{r} = f(\vec{r}_1) - f(\vec{r}_0)$$

3.3 Green's Theorem for Work (only in $\mathbb{R}^2$)

$$\iint_D \text{curl } \vec{F} \cdot \hat{\mathrm{k}}\ \mathrm{d}A = \oint_C \vec{F} \cdot \, \mathrm{d}\vec{r} \text{ (C is oriented counter-clockwise)}$$

3.4 Green's Theorem for Flux (only in $\mathbb{R}^2$)

$$\iint_D \text{div } \vec{F}\ \mathrm{d}A = \oint_C \vec{F} \cdot \hat{\mathrm{n}}\ \mathrm{d}s \text{ (C is oriented counter-clockwise)}$$

3.5 Stoke's Theorem (only in $\mathbb{R}^3$)

$$\iint_S \text{curl } \vec{F} \cdot \, \mathrm{d}\vec{S} = \oint_C \vec{F} \cdot \, \mathrm{d}\vec{r} \text{ (C is compatibly counter-clockwise)}$$

3.6 Divergence Theorem (only in $\mathbb{R}^3$)

$$\iiint_D \text{div } \vec{F}\ \mathrm{d}V = \oiint_S \vec{F} \cdot \, \mathrm{d}\vec{S} \text{ ($\mathrm{d}\vec{S}$ points outwards)}$$