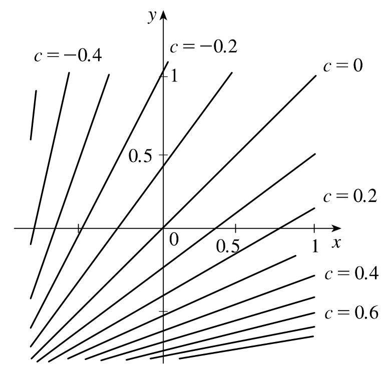
4. The level curves of $f(x,y) = \left(\dfrac{1+x}{1+y}\right)^{1/3} - 1$ are

$$\left(\frac{1+x}{1+y}\right)^{1/3} - 1 = c \quad\Rightarrow\quad \frac{1+x}{1+y} = (1+c)^3 \quad\Rightarrow$$

$$y = \frac{1+x}{(1+c)^3} - 1.$$

From the level curves, we see that increasing x (from 0) by a small amount has a similar effect on the value of f as decreasing y by a small amount. However, for larger changes, a decrease in y gives greater values of f than a similar increase in x.



14.5 The Chain Rule

1. $z = xy^3 - x^2y, \quad x = t^2+1, \quad y = t^2 - 1 \quad\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t) = 2t(y^3 - 2xy + 3xy^2 - x^2)$$

2. $z = \dfrac{x-y}{x+2y}, \quad x = e^{\pi t}, \quad y = e^{-\pi t} \quad\Rightarrow$

$$\begin{aligned}\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} &= \frac{(x+2y)(1)-(x-y)(1)}{(x+2y)^2}(\pi e^{\pi t}) + \frac{(x+2y)(-1)-(x-y)(2)}{(x+2y)^2}(-\pi e^{-\pi t}) \\ &= \frac{3y}{(x+2y)^2}\,(\pi e^{\pi t}) + \frac{-3x}{(x+2y)^2}\,(-\pi e^{-\pi t}) = \frac{3\pi}{(x+2y)^2}\,\left(ye^{\pi t} + xe^{-\pi t}\right)\end{aligned}$$

3. $z = \sin x \cos y, \quad x = \sqrt{t}, \quad y = 1/t \quad\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (\cos x\cos y)\left(\tfrac{1}{2}t^{-1/2}\right) + (-\sin x \sin y)\left(-t^{-2}\right) = \frac{1}{2\sqrt{t}}\cos x\cos y + \frac{1}{t^2}\sin x \sin y$$

4. $z = \sqrt{1+xy}, \quad x = \tan t, \quad y = \arctan t \quad\Rightarrow$

$$\begin{aligned}\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} &= \tfrac{1}{2}(1+xy)^{-1/2}(y)\cdot\sec^2 t + \tfrac{1}{2}(1+xy)^{-1/2}(x)\cdot\frac{1}{1+t^2} \\ &= \frac{1}{2\sqrt{1+xy}}\left(y\sec^2 t + \frac{x}{1+t^2}\right)\end{aligned}$$

5. $w = xe^{y/z}, \; x = t^2, \; y = 1-t, \; z = 1+2t \quad\Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = e^{y/z}\cdot 2t + xe^{y/z}\left(\frac{1}{z}\right)\cdot(-1) + xe^{y/z}\left(-\frac{y}{z^2}\right)\cdot 2 = e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$$

6. $w = \ln\sqrt{x^2+y^2+z^2} = \tfrac{1}{2}\ln(x^2+y^2+z^2), \quad x = \sin t, \quad y = \cos t, \quad z = \tan t \quad\Rightarrow$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \frac{1}{2}\cdot\frac{2x}{x^2+y^2+z^2}\cdot\cos t + \frac{1}{2}\cdot\frac{2y}{x^2+y^2+z^2}\cdot(-\sin t) + \frac{1}{2}\cdot\frac{2z}{x^2+y^2+z^2}\cdot\sec^2 t \\ &= \frac{x\cos t - y\sin t + z\sec^2 t}{x^2+y^2+z^2}\end{aligned}$$

7. $z=(x-y)^5,\quad x=s^2t,\quad y=st^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=5(x-y)^4(1)\cdot 2st+5(x-y)^4(-1)\cdot t^2=5(x-y)^4\left(2st-t^2\right)$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}=5(x-y)^4(1)\cdot s^2+5(x-y)^4(-1)\cdot 2st=5(x-y)^4\left(s^2-2st\right)$$

8. $z=\tan^{-1}(x^2+y^2),\quad x=s\ln t,\quad y=te^s \quad \Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=\frac{2x}{1+(x^2+y^2)^2}\cdot\ln t+\frac{2y}{1+(x^2+y^2)^2}\cdot te^s$$

$$=\frac{2}{1+(x^2+y^2)^2}\left(x\ln t+yte^s\right)$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}=\frac{2x}{1+(x^2+y^2)^2}\cdot\frac{s}{t}+\frac{2y}{1+(x^2+y^2)^2}\cdot e^s$$

$$=\frac{2}{1+(x^2+y^2)^2}\left(\frac{xs}{t}+ye^s\right)$$

9. $z=\ln(3x+2y),\quad x=s\sin t,\quad y=t\cos s \quad \Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=\frac{3}{3x+2y}(\sin t)+\frac{2}{3x+2y}(-t\sin s)=\frac{3\sin t-2t\sin s}{3x+2y}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}=\frac{3}{3x+2y}(s\cos t)+\frac{2}{3x+2y}(\cos s)=\frac{3s\cos t+2\cos s}{3x+2y}$$

10. $z=\sqrt{x}\,e^{xy},\quad x=1+st,\quad y=s^2-t^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=\left(\sqrt{x}\cdot e^{xy}(y)+e^{xy}\cdot\tfrac{1}{2}x^{-1/2}\right)(t)+\sqrt{x}\,e^{xy}(x)\,(2s)=\left(yt\sqrt{x}+\frac{t}{2\sqrt{x}}+2x^{3/2}s\right)e^{xy}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}=\left(\sqrt{x}\cdot e^{xy}(y)+e^{xy}\cdot\tfrac{1}{2}x^{-1/2}\right)(s)+\sqrt{x}\,e^{xy}(x)\,(-2t)=\left(ys\sqrt{x}+\frac{s}{2\sqrt{x}}-2x^{3/2}t\right)e^{xy}$$

11. $z=e^r\cos\theta,\ r=st,\ \theta=\sqrt{s^2+t^2} \quad \Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial r}\frac{\partial r}{\partial s}+\frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}=e^r\cos\theta\cdot t+e^r(-\sin\theta)\cdot\tfrac{1}{2}(s^2+t^2)^{-1/2}(2s)=te^r\cos\theta-e^r\sin\theta\cdot\frac{s}{\sqrt{s^2+t^2}}$$

$$=e^r\left(t\cos\theta-\frac{s}{\sqrt{s^2+t^2}}\sin\theta\right)$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial r}\frac{\partial r}{\partial t}+\frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}=e^r\cos\theta\cdot s+e^r(-\sin\theta)\cdot\tfrac{1}{2}(s^2+t^2)^{-1/2}(2t)=se^r\cos\theta-e^r\sin\theta\cdot\frac{t}{\sqrt{s^2+t^2}}$$

$$=e^r\left(s\cos\theta-\frac{t}{\sqrt{s^2+t^2}}\sin\theta\right)$$

12. $z=\tan(u/v),\ u=2s+3t,\ v=3s-2t \quad \Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial s}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial s}=\sec^2(u/v)(1/v)\cdot 2+\sec^2(u/v)(-uv^{-2})\cdot 3$$

$$=\frac{2}{v}\sec^2\left(\frac{u}{v}\right)-\frac{3u}{v^2}\sec^2\left(\frac{u}{v}\right)=\frac{2v-3u}{v^2}\sec^2\left(\frac{u}{v}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = \sec^2(u/v)(1/v)\cdot 3 + \sec^2(u/v)(-uv^{-2})\cdot(-2)$$
$$= \frac{3}{v}\sec^2\left(\frac{u}{v}\right) + \frac{2u}{v^2}\sec^2\left(\frac{u}{v}\right) = \frac{2u+3v}{v^2}\sec^2\left(\frac{u}{v}\right)$$

13. Let $x = g(t)$ and $y = h(t)$. Then $p(t) = f(x, y)$ and the Chain Rule (2) gives $\dfrac{dp}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$. When $t = 2$,

$x = g(2) = 4$ and $y = h(2) = 5$, so $p'(2) = f_x(4,5)\,g'(2) + f_y(4,5)\,h'(2) = (2)(-3) + (8)(6) = 42$.

14. $R(s,t) = G\left(u(s,t), v(s,t)\right) \quad\Rightarrow\quad \dfrac{\partial R}{\partial s} = \dfrac{\partial R}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial R}{\partial v}\dfrac{\partial v}{\partial s}$ and $\dfrac{\partial R}{\partial t} = \dfrac{\partial R}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial R}{\partial v}\dfrac{\partial v}{\partial t}$ by the Chain Rule (3). When $s = 1$ and $t = 2$, $u(1,2) = 5$ and $v(1,2) = 7$.

Thus $R_s(1,2) = G_u(5,7)\,u_s(1,2) + G_v(5,7)\,v_s(1,2) = (9)(4) + (-2)(2) = 32$ and

$R_t(1,2) = G_u(5,7)\,u_t(1,2) + G_v(5,7)\,v_t(1,2) = (9)(-3) + (-2)(6) = -39$.

15. $g(u,v) = f(x(u,v), y(u,v))$ where $x = e^u + \sin v$, $y = e^u + \cos v \quad\Rightarrow$

$\dfrac{\partial x}{\partial u} = e^u$, $\dfrac{\partial x}{\partial v} = \cos v$, $\dfrac{\partial y}{\partial u} = e^u$, $\dfrac{\partial y}{\partial v} = -\sin v$. By the Chain Rule (3), $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}$. Then

$g_u(0,0) = f_x(x(0,0), y(0,0))\,x_u(0,0) + f_y(x(0,0), y(0,0))\,y_u(0,0) = f_x(1,2)(e^0) + f_y(1,2)(e^0) = 2(1) + 5(1) = 7$.

Similarly, $\dfrac{\partial g}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v}$. Then

$$g_v(0,0) = f_x(x(0,0), y(0,0))\,x_v(0,0) + f_y(x(0,0), y(0,0))\,y_v(0,0) = f_x(1,2)(\cos 0) + f_y(1,2)(-\sin 0)$$
$$= 2(1) + 5(0) = 2$$

16. $g(r,s) = f(x(r,s), y(r,s))$ where $x = 2r - s$, $y = s^2 - 4r \quad\Rightarrow\quad \dfrac{\partial x}{\partial r} = 2$, $\dfrac{\partial x}{\partial s} = -1$, $\dfrac{\partial y}{\partial r} = -4$, $\dfrac{\partial y}{\partial s} = 2s$.

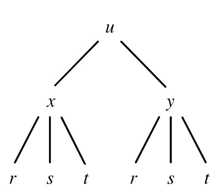
By the Chain Rule (3) $\dfrac{\partial g}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r}$. Then

$$g_r(1,2) = f_x(x(1,2), y(1,2))\,x_r(1,2) + f_y(x(1,2), y(1,2))\,y_r(1,2) = f_x(0,0)(2) + f_y(0,0)(-4)$$
$$= 4(2) + 8(-4) = -24$$

Similarly, $\dfrac{\partial g}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$. Then

$$g_s(1,2) = f_x(x(1,2), y(1,2))\,x_s(1,2) + f_y(x(1,2), y(1,2))\,y_s(1,2) = f_x(0,0)(-1) + f_y(0,0)(4)$$
$$= 4(-1) + 8(4) = 28$$

17.

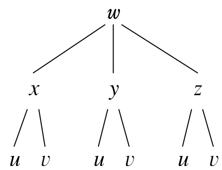


$u = f(x,y)$, $x = x(r,s,t)$, $y = y(r,s,t) \quad\Rightarrow$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

18.

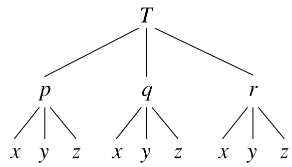


$w = f(x,y,z),\ \ x = x(u,v),\ \ y = y(u,v),\ z = z(u,v) \quad \Rightarrow$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

19.



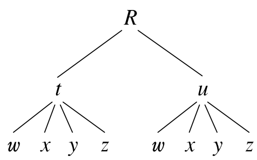
$T = F(p,q,r),\ p = p(x,y,z),\ \ q = q(x,y,z),\ r = r(x,y,z) \quad \Rightarrow$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial x},$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial y},$$

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial z}$$

20.



$R = F(t,u),\ \ t = t(w,x,y,z),\ \ u = u(w,x,y,z) \quad \Rightarrow$

$$\frac{\partial R}{\partial w} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial w} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial w},\quad \frac{\partial R}{\partial x} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial x},$$

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial y},\quad \frac{\partial R}{\partial z} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial z} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial z}$$

21. $z = x^4 + x^2 y,\ x = s + 2t - u,\ y = stu^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (4x^3 + 2xy)(1) + (x^2)(tu^2),$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (4x^3 + 2xy)(2) + (x^2)(su^2),$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (4x^3 + 2xy)(-1) + (x^2)(2stu).$$

When $s = 4$, $t = 2$, and $u = 1$ we have $x = 7$ and $y = 8$,

so $\dfrac{\partial z}{\partial s} = (1484)(1) + (49)(2) = 1582,\ \dfrac{\partial z}{\partial t} = (1484)\,(2) + (49)(4) = 3164,\ \dfrac{\partial z}{\partial u} = (1484)(-1) + (49)(16) = -700.$

22. $T = v/(2u+v) = v(2u+v)^{-1},\ u = pq\sqrt{r},\ v = p\sqrt{q}\,r \quad \Rightarrow$

$$\frac{\partial T}{\partial p} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial p} = [-v(2u+v)^{-2}(2)](q\sqrt{r}) + \frac{(2u+v)(1) - v(1)}{(2u+v)^2}\,(\sqrt{q}\,r)$$

$$= \frac{-2v}{(2u+v)^2}\,(q\sqrt{r}) + \frac{2u}{(2u+v)^2}\,(\sqrt{q}\,r),$$

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial q} = \frac{-2v}{(2u+v)^2}\,(p\sqrt{r}) + \frac{2u}{(2u+v)^2}\,\frac{pr}{2\sqrt{q}},$$

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial r} = \frac{-2v}{(2u+v)^2}\,\frac{pq}{2\sqrt{r}} + \frac{2u}{(2u+v)^2}\,(p\sqrt{q}).$$

When $p = 2$, $q = 1$, and $r = 4$ we have $u = 4$ and $v = 8$,

so $\dfrac{\partial T}{\partial p} = \left(-\frac{1}{16}\right)(2) + \left(\frac{1}{32}\right)(4) = 0,\ \dfrac{\partial T}{\partial q} = \left(-\frac{1}{16}\right)(4) + \left(\frac{1}{32}\right)(4) = -\frac{1}{8},\ \dfrac{\partial T}{\partial r} = \left(-\frac{1}{16}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{32}\right)(2) = \frac{1}{32}.$

23. $w = xy + yz + zx,\ x = r\cos\theta,\ y = r\sin\theta,\ z = r\theta \quad \Rightarrow$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (y+z)(\cos\theta) + (x+z)(\sin\theta) + (y+x)(\theta),$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta} = (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (y+x)(r).$$

When $r = 2$ and $\theta = \pi/2$ we have $x = 0$, $y = 2$, and $z = \pi$, so $\frac{\partial w}{\partial r} = (2+\pi)(0) + (0+\pi)(1) + (2+0)(\pi/2) = 2\pi$ and

$$\frac{\partial w}{\partial \theta} = (2+\pi)(-2) + (0+\pi)(0) + (2+0)(2) = -2\pi.$$

24. $P = \sqrt{u^2+v^2+w^2} = (u^2+v^2+w^2)^{1/2},\ u = xe^y,\ v = ye^x,\ w = e^{xy} \quad \Rightarrow$

$$\begin{aligned}\frac{\partial P}{\partial x} &= \frac{\partial P}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial P}{\partial w}\frac{\partial w}{\partial x} \\ &= \tfrac{1}{2}(u^2+v^2+w^2)^{-1/2}(2u)(e^y) + \tfrac{1}{2}(u^2+v^2+w^2)^{-1/2}(2v)(ye^x) + \tfrac{1}{2}(u^2+v^2+w^2)^{-1/2}(2w)(ye^{xy}) \\ &= \frac{ue^y + vye^x + wye^{xy}}{\sqrt{u^2+v^2+w^2}},\end{aligned}$$

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial P}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial P}{\partial w}\frac{\partial w}{\partial y} = \frac{u}{\sqrt{u^2+v^2+w^2}}(xe^y) + \frac{v}{\sqrt{u^2+v^2+w^2}}(e^x) + \frac{w}{\sqrt{u^2+v^2+w^2}}(xe^{xy}) \\ &= \frac{uxe^y + ve^x + wxe^{xy}}{\sqrt{u^2+v^2+w^2}}.\end{aligned}$$

When $x = 0$ and $y = 2$ we have $u = 0$, $v = 2$, and $w = 1$, so $\frac{\partial P}{\partial x} = \frac{0+4+2}{\sqrt{5}} = \frac{6}{\sqrt{5}}$ and $\frac{\partial P}{\partial y} = \frac{0+2+0}{\sqrt{5}} = \frac{2}{\sqrt{5}}$.

25. $N = \frac{p+q}{p+r},\ p = u + vw,\ q = v + uw,\ r = w + uv \quad \Rightarrow$

$$\begin{aligned}\frac{\partial N}{\partial u} &= \frac{\partial N}{\partial p}\frac{\partial p}{\partial u} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial u} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial u} \\ &= \frac{(p+r)(1)-(p+q)(1)}{(p+r)^2}(1) + \frac{(p+r)(1)-(p+q)(0)}{(p+r)^2}(w) + \frac{(p+r)(0)-(p+q)(1)}{(p+r)^2}(v) \\ &= \frac{(r-q)+(p+r)w-(p+q)v}{(p+r)^2},\end{aligned}$$

$$\frac{\partial N}{\partial v} = \frac{\partial N}{\partial p}\frac{\partial p}{\partial v} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial v} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial v} = \frac{r-q}{(p+r)^2}(w) + \frac{p+r}{(p+r)^2}(1) + \frac{-(p+q)}{(p+r)^2}(u) = \frac{(r-q)w+(p+r)-(p+q)u}{(p+r)^2},$$

$$\frac{\partial N}{\partial w} = \frac{\partial N}{\partial p}\frac{\partial p}{\partial w} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial w} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial w} = \frac{r-q}{(p+r)^2}(v) + \frac{p+r}{(p+r)^2}(u) + \frac{-(p+q)}{(p+r)^2}(1) = \frac{(r-q)v+(p+r)u-(p+q)}{(p+r)^2}.$$

When $u = 2$, $v = 3$, and $w = 4$ we have $p = 14$, $q = 11$, and $r = 10$, so $\frac{\partial N}{\partial u} = \frac{-1+(24)(4)-(25)(3)}{(24)^2} = \frac{20}{576} = \frac{5}{144}$,

$$\frac{\partial N}{\partial v} = \frac{(-1)(4)+24-(25)(2)}{(24)^2} = \frac{-30}{576} = -\frac{5}{96}, \text{ and } \frac{\partial N}{\partial w} = \frac{(-1)(3)+(24)(2)-25}{(24)^2} = \frac{20}{576} = \frac{5}{144}.$$

26. $u = xe^{ty},\ x = \alpha^2\beta,\ y = \beta^2\gamma,\ t = \gamma^2\alpha \quad \Rightarrow$

$$\frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \alpha} = e^{ty}(2\alpha\beta) + xte^{ty}(0) + xye^{ty}(\gamma^2) = e^{ty}(2\alpha\beta + xy\gamma^2),$$

$$\frac{\partial u}{\partial \beta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \beta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \beta} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \beta} = e^{ty}(\alpha^2) + xte^{ty}(2\beta\gamma) + xye^{ty}(0) = e^{ty}(\alpha^2 + 2xt\beta\gamma),$$

$$\frac{\partial u}{\partial \gamma} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \gamma} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \gamma} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \gamma} = e^{ty}(0) + xte^{ty}(\beta^2) + xye^{ty}(2\gamma\alpha) = e^{ty}(xt\beta^2 + 2xy\alpha\gamma).$$

When $\alpha = -1$, $\beta = 2$, and $\gamma = 1$ we have $x = 2$, $y = 4$, and $t = -1$, so $\dfrac{\partial u}{\partial \alpha} = e^{-4}(-4+8) = 4e^{-4}$,

$\dfrac{\partial u}{\partial \beta} = e^{-4}(1-8) = -7e^{-4}$, and $\dfrac{\partial u}{\partial \gamma} = e^{-4}(-8-16) = -24e^{-4}$.

27. $y\cos x = x^2 + y^2$, so let $F(x,y) = y\cos x - x^2 - y^2 = 0$. Then by Equation 6

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y\sin x - 2x}{\cos x - 2y} = \frac{2x + y\sin x}{\cos x - 2y}.$$

28. $\cos(xy) = 1 + \sin y$, so let $F(x,y) = \cos(xy) - 1 - \sin y = 0$. Then by Equation 6

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-\sin(xy)(y)}{-\sin(xy)(x) - \cos y} = -\frac{y\sin(xy)}{\cos y + x\sin(xy)}.$$

29. $\tan^{-1}(x^2y) = x + xy^2$, so let $F(x,y) = \tan^{-1}(x^2y) - x - xy^2 = 0$. Then

$$F_x(x,y) = \frac{1}{1+(x^2y)^2}\,(2xy) - 1 - y^2 = \frac{2xy}{1+x^4y^2} - 1 - y^2 = \frac{2xy - (1+y^2)(1+x^4y^2)}{1+x^4y^2},$$

$$F_y(x,y) = \frac{1}{1+(x^2y)^2}\,(x^2) - 2xy = \frac{x^2}{1+x^4y^2} - 2xy = \frac{x^2 - 2xy(1+x^4y^2)}{1+x^4y^2}$$

and
$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{[2xy - (1+y^2)(1+x^4y^2)]/(1+x^4y^2)}{[x^2 - 2xy(1+x^4y^2)]/(1+x^4y^2)} = \frac{(1+y^2)(1+x^4y^2) - 2xy}{x^2 - 2xy(1+x^4y^2)}$$

$$= \frac{1 + x^4y^2 + y^2 + x^4y^4 - 2xy}{x^2 - 2xy - 2x^5y^3}$$

30. $e^y\sin x = x + xy$, so let $F(x,y) = e^y\sin x - x - xy = 0$. Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{e^y\cos x - 1 - y}{e^y\sin x - x} = \dfrac{1 + y - e^y\cos x}{e^y\sin x - x}$.

31. $x^2 + 2y^2 + 3z^2 = 1$, so let $F(x,y,z) = x^2 + 2y^2 + 3z^2 - 1 = 0$. Then by Equations 7

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{6z} = -\frac{x}{3z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{6z} = -\frac{2y}{3z}.$$

32. $x^2 - y^2 + z^2 - 2z = 4$, so let $F(x,y,z) = x^2 - y^2 + z^2 - 2z - 4 = 0$. Then by Equations 7

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2z-2} = \frac{x}{1-z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-2y}{2z-2} = \frac{y}{z-1}.$$

33. $e^z = xyz$, so let $F(x, y, z) = e^z - xyz = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{-yz}{e^z - xy} = \dfrac{yz}{e^z - xy}$ and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-xz}{e^z - xy} = \frac{xz}{e^z - xy}.$$

34. $yz + x \ln y = z^2$, so let $F(x, y, z) = yz + x \ln y - z^2 = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{\ln y}{y - 2z} = \dfrac{\ln y}{2z - y}$ and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z + (x/y)}{y - 2z} = \frac{x + yz}{2yz - y^2}.$$

35. Since x and y are each functions of t, $T(x, y)$ is a function of t, so by the Chain Rule, $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt}$. After 3 seconds, $x = \sqrt{1 + t} = \sqrt{1 + 3} = 2$, $y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3$, $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{1 + t}} = \dfrac{1}{2\sqrt{1 + 3}} = \dfrac{1}{4}$, and $\dfrac{dy}{dt} = \dfrac{1}{3}$.

Then $\dfrac{dT}{dt} = T_x(2, 3)\dfrac{dx}{dt} + T_y(2, 3)\dfrac{dy}{dt} = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) = 2$. Thus the temperature is rising at a rate of $2^\circ\mathrm{C/s}$.

36. (a) Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.

(b) Since the average temperature is rising at a rate of 0.15°C/year, we know that $dT/dt = 0.15$. Since rainfall is decreasing at a rate of 0.1 cm/year, we know $dR/dt = -0.1$. Then, by the Chain Rule,

$\dfrac{dW}{dt} = \dfrac{\partial W}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial W}{\partial R}\dfrac{dR}{dt} = (-2)(0.15) + (8)(-0.1) = -1.1$. Thus we estimate that wheat production will decrease at a rate of 1.1 units/year.

37. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11T + 0.00087T^2$ and $\dfrac{\partial C}{\partial D} = 0.016$.

According to the graph, the diver is experiencing a temperature of approximately 12.5°C at $t = 20$ minutes, so

$\dfrac{\partial C}{\partial T} = 4.6 - 0.11(12.5) + 0.00087(12.5)^2 \approx 3.36$. By sketching tangent lines at $t = 20$ to the graphs given, we estimate

$\dfrac{dD}{dt} \approx \dfrac{1}{2}$ and $\dfrac{dT}{dt} \approx -\dfrac{1}{10}$. Then, by the Chain Rule, $\dfrac{dC}{dt} = \dfrac{\partial C}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial C}{\partial D}\dfrac{dD}{dt} \approx (3.36)\left(-\frac{1}{10}\right) + (0.016)\left(\frac{1}{2}\right) \approx -0.33$.

Thus the speed of sound experienced by the diver is decreasing at a rate of approximately 0.33 m/s per minute.

38. $V = \pi r^2 h/3$, so $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = \dfrac{2\pi rh}{3}1.8 + \dfrac{\pi r^2}{3}(-2.5) = 20{,}160\pi - 12{,}000\pi = 8160\pi\ \mathrm{in}^3/\mathrm{s}$.

39. (a) $V = \ell wh$, so by the Chain Rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\frac{d\ell}{dt} + \ell h\frac{dw}{dt} + \ell w\frac{dh}{dt} = 2\cdot 2\cdot 2 + 1\cdot 2\cdot 2 + 1\cdot 2\cdot(-3) = 6\ \mathrm{m}^3/\mathrm{s}.$$

(b) $S = 2(\ell w + \ell h + wh)$, so by the Chain Rule,

$$\begin{aligned}\frac{dS}{dt} &= \frac{\partial S}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} = 2(w+h)\,\frac{d\ell}{dt} + 2(\ell+h)\,\frac{dw}{dt} + 2(\ell+w)\,\frac{dh}{dt} \\ &= 2(2+2)2 + 2(1+2)2 + 2(1+2)(-3) = 10\text{ m}^2/\text{s}\end{aligned}$$

(c) $L^2 = \ell^2 + w^2 + h^2 \quad\Rightarrow\quad 2L\,\dfrac{dL}{dt} = 2\ell\,\dfrac{d\ell}{dt} + 2w\,\dfrac{dw}{dt} + 2h\,\dfrac{dh}{dt} = 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \quad\Rightarrow$

$dL/dt = 0$ m/s.

40. $I = \dfrac{V}{R} \quad\Rightarrow$

$$\frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{V}{R^2}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{I}{R}\frac{dR}{dt} = \frac{1}{400}(-0.01) - \frac{0.08}{400}(0.03) = -0.000031\text{ A/s}$$

41. $\dfrac{dP}{dt} = 0.05$, $\dfrac{dT}{dt} = 0.15$, $V = 8.31\dfrac{T}{P}$ and $\dfrac{dV}{dt} = \dfrac{8.31}{P}\dfrac{dT}{dt} - 8.31\dfrac{T}{P^2}\dfrac{dP}{dt}$. Thus when $P = 20$ and $T = 320$,

$$\frac{dV}{dt} = 8.31\left[\frac{0.15}{20} - \frac{(0.05)(320)}{400}\right] \approx -0.27\text{ L/s}.$$

42. $P = 1.47L^{0.65}K^{0.35}$ and considering P, L, and K as functions of time t we have

$\dfrac{dP}{dt} = \dfrac{\partial P}{\partial L}\dfrac{dL}{dt} + \dfrac{\partial P}{\partial K}\dfrac{dK}{dt} = 1.47(0.65)L^{-0.35}K^{0.35}\dfrac{dL}{dt} + 1.47(0.35)L^{0.65}K^{-0.65}\dfrac{dK}{dt}$. We are given that $\dfrac{dL}{dt} = -2$ and $\dfrac{dK}{dt} = 0.5$, so when $L = 30$ and $K = 8$, the rate of change of production $\dfrac{dP}{dt}$ is $1.47(0.65)(30)^{-0.35}(8)^{0.35}(-2) + 1.47(0.35)(30)^{0.65}(8)^{-0.65}(0.5) \approx -0.596$. Thus production at that time is decreasing at a rate of about \$596,000 per year.

43. Let x be the length of the first side of the triangle and y the length of the second side. The area A of the triangle is given by $A = \frac{1}{2}xy\sin\theta$ where θ is the angle between the two sides. Thus A is a function of x, y, and θ, and x, y, and θ are each in turn functions of time t. We are given that $\dfrac{dx}{dt} = 3$, $\dfrac{dy}{dt} = -2$, and because A is constant, $\dfrac{dA}{dt} = 0$. By the Chain Rule,

$\dfrac{dA}{dt} = \dfrac{\partial A}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial A}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial A}{\partial \theta}\dfrac{d\theta}{dt} \quad\Rightarrow\quad \dfrac{dA}{dt} = \frac{1}{2}y\sin\theta\cdot\dfrac{dx}{dt} + \frac{1}{2}x\sin\theta\cdot\dfrac{dy}{dt} + \frac{1}{2}xy\cos\theta\cdot\dfrac{d\theta}{dt}$. When $x = 20$, $y = 30$, and $\theta = \pi/6$ we have

$$\begin{aligned}0 &= \tfrac{1}{2}(30)\left(\sin\tfrac{\pi}{6}\right)(3) + \tfrac{1}{2}(20)\left(\sin\tfrac{\pi}{6}\right)(-2) + \tfrac{1}{2}(20)(30)\left(\cos\tfrac{\pi}{6}\right)\frac{d\theta}{dt} \\ &= 45\cdot\tfrac{1}{2} - 20\cdot\tfrac{1}{2} + 300\cdot\frac{\sqrt{3}}{2}\cdot\frac{d\theta}{dt} = \tfrac{25}{2} + 150\sqrt{3}\,\frac{d\theta}{dt}\end{aligned}$$

Solving for $\dfrac{d\theta}{dt}$ gives $\dfrac{d\theta}{dt} = \dfrac{-25/2}{150\sqrt{3}} = -\dfrac{1}{12\sqrt{3}}$, so the angle between the sides is decreasing at a rate of $1/\left(12\sqrt{3}\right) \approx 0.048$ rad/s.

44. $f_o = \left(\dfrac{c + v_o}{c - v_s}\right) f_s = \left(\frac{332+34}{332-40}\right) 460 \approx 576.6$ Hz. v_o and v_s are functions of time t, so

$$\begin{aligned}\frac{df_o}{dt} &= \frac{\partial f_o}{\partial v_o}\frac{dv_o}{dt} + \frac{\partial f_o}{\partial v_s}\frac{dv_s}{dt} = \left(\frac{1}{c - v_s}\right) f_s \cdot \frac{dv_o}{dt} + \frac{c + v_o}{(c - v_s)^2} f_s \cdot \frac{dv_s}{dt} \\ &= \left(\tfrac{1}{332-40}\right)(460)(1.2) + \tfrac{332+34}{(332-40)^2}(460)(1.4) \approx 4.65 \text{ Hz/s}\end{aligned}$$

45. (a) By the Chain Rule, $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$, $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y} r\cos\theta$.

(b) $\left(\dfrac{\partial z}{\partial r}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 \cos^2\theta + 2\,\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2 \sin^2\theta$,

$\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\,\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y} r^2\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

46. By the Chain Rule, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x} e^s\cos t + \dfrac{\partial u}{\partial y} e^s\sin t$, $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}(-e^s\sin t) + \dfrac{\partial u}{\partial y} e^s\cos t$. Then

$\left(\dfrac{\partial u}{\partial s}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 e^{2s}\cos^2 t + 2\,\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} e^{2s}\cos t\,\sin t + \left(\dfrac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$ and

$\left(\dfrac{\partial u}{\partial t}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 e^{2s}\sin^2 t - 2\,\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} e^{2s}\cos t\,\sin t + \left(\dfrac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$. Thus

$$\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]e^{-2s} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

47. Let $u = x - y$ and $v = x + y$. Then $z = \dfrac{1}{x}\,[f(u) + g(v)]$ and

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{1}{x}\left[\frac{df}{du}\frac{\partial u}{\partial x} + \frac{dg}{dv}\frac{\partial v}{\partial x}\right] + [f(u) + g(v)]\left(-\frac{1}{x^2}\right) \\ &= \frac{1}{x}\,[f'(u)(1) + g'(v)(1)] - \frac{1}{x^2}\,[f(u) + g(v)] = \frac{1}{x}\,[f'(u) + g'(v)] - \frac{1}{x^2}\,[f(u) + g(v)]\end{aligned}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x}\left[\frac{df}{du}\frac{\partial u}{\partial y} + \frac{dg}{dv}\frac{\partial v}{\partial y}\right] = \frac{1}{x}\,[f'(u)(-1) + g'(v)(1)] = \frac{1}{x}\,[-f'(u) + g'(v)]$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{x}\left[\frac{d}{du}\,[-f'(u)]\,\frac{\partial u}{\partial y} + \frac{d}{dv}\,[g'(v)]\,\frac{\partial v}{\partial y}\right] = \frac{1}{x}\,[-f''(u)(-1) + g''(v)(1)] = \frac{1}{x}\,[f''(u) + g''(v)]$$

Thus

$$\begin{aligned}\frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right) &= \frac{\partial}{\partial x}\left(x\,[f'(u) + g'(v)] - [f(u) + g(v)]\right) \\ &= x\,[f''(u)(1) + g''(v)(1)] + [f'(u) + g'(v)]\,(1) - [f'(u)(1) + g'(v)(1)] \\ &= x\,[f''(u) + g''(v)] + f'(u) + g'(v) - f'(u) - g'(v) = x\,[f''(u) + g''(v)] \\ &= x^2 \cdot \frac{1}{x}\,[f''(u) + g''(v)] = x^2\,\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

48. Let $u = ax + y$ and $v = ax - y$. Then $z = \dfrac{1}{y}\left[f(u) + g(v)\right]$ and

$$\frac{\partial z}{\partial x} = \frac{1}{y}\left[\frac{df}{du}\frac{\partial u}{\partial x} + \frac{dg}{dv}\frac{\partial v}{\partial x}\right] = \frac{1}{y}\left[f'(u)(a) + g'(v)(a)\right] = \frac{a}{y}\left[f'(u) + g'(v)\right]$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{a}{y}\left[\frac{d}{du}\left[f'(u)\right]\frac{\partial u}{\partial x} + \frac{d}{dv}\left[g'(v)\right]\frac{\partial v}{\partial x}\right] = \frac{a}{y}\left[f''(u)(a) + g''(v)(a)\right] = \frac{a^2}{y}\left[f''(u) + g''(v)\right]$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{1}{y}\left[\frac{df}{du}\frac{\partial u}{\partial y} + \frac{dg}{dv}\frac{\partial v}{\partial y}\right] + \left[f(u) + g(v)\right]\left(-\frac{1}{y^2}\right)\\ &= \frac{1}{y}\left[f'(u)(1) + g'(v)(-1)\right] - \frac{1}{y^2}\left[f(u) + g(v)\right] = \frac{1}{y}\left[f'(u) - g'(v)\right] - \frac{1}{y^2}\left[f(u) + g(v)\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y}\left(y^2\frac{\partial z}{\partial y}\right) &= \frac{\partial}{\partial y}\left(y\left[f'(u) - g'(v)\right] - \left[f(u) + g(v)\right]\right)\\ &= y\left[f''(u)(1) - g''(v)(-1)\right] + \left[f'(u) - g'(v)\right](1) - \left[f'(u)(1) + g'(v)(-1)\right]\\ &= y\left[f''(u) + g''(v)\right] + f'(u) - g'(v) - f'(u) + g'(v) = y\left[f''(u) + g''(v)\right]\end{aligned}$$

Thus $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{a^2}{y}\left[f''(u) + g''(v)\right] = \dfrac{a^2}{y^2}\cdot y\left[f''(u) + g''(v)\right] = \dfrac{a^2}{y^2}\dfrac{\partial}{\partial y}\left(y^2\dfrac{\partial z}{\partial y}\right)$.

49. Let $u = x + at,\ v = x - at$. Then $z = f(u) + g(v)$, so $\partial z/\partial u = f'(u)$ and $\partial z/\partial v = g'(v)$.

Thus $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial t} = af'(u) - ag'(v)$ and

$$\frac{\partial^2 z}{\partial t^2} = a\frac{\partial}{\partial t}\left[f'(u) - g'(v)\right] = a\left(\frac{df'(u)}{du}\frac{\partial u}{\partial t} - \frac{dg'(v)}{dv}\frac{\partial v}{\partial t}\right) = a^2 f''(u) + a^2 g''(v).$$

Similarly, $\dfrac{\partial z}{\partial x} = f'(u) + g'(v)$ and $\dfrac{\partial^2 z}{\partial x^2} = f''(u) + g''(v)$. Thus $\dfrac{\partial^2 z}{\partial t^2} = a^2\,\dfrac{\partial^2 z}{\partial x^2}$.

50. By the Chain Rule, $\dfrac{\partial u}{\partial s} = e^s\cos t\,\dfrac{\partial u}{\partial x} + e^s\sin t\,\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial t} = -e^s\sin t\,\dfrac{\partial u}{\partial x} + e^s\cos t\,\dfrac{\partial u}{\partial y}$.

Then $\dfrac{\partial^2 u}{\partial s^2} = e^s\cos t\,\dfrac{\partial u}{\partial x} + e^s\cos t\,\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial x}\right) + e^s\sin t\,\dfrac{\partial u}{\partial y} + e^s\sin t\,\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial y}\right)$. But

$\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial x^2}\dfrac{\partial x}{\partial s} + \dfrac{\partial^2 u}{\partial y\,\partial x}\dfrac{\partial y}{\partial s} = e^s\cos t\,\dfrac{\partial^2 u}{\partial x^2} + e^s\sin t\,\dfrac{\partial^2 u}{\partial y\,\partial x}$ and

$\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial y^2}\dfrac{\partial y}{\partial s} + \dfrac{\partial^2 u}{\partial x\,\partial y}\dfrac{\partial x}{\partial s} = e^s\sin t\,\dfrac{\partial^2 u}{\partial y^2} + e^s\cos t\,\dfrac{\partial^2 u}{\partial x\,\partial y}$.

Also, by continuity of the partials, $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$. Thus

$$\begin{aligned}\frac{\partial^2 u}{\partial s^2} &= e^s\cos t\,\frac{\partial u}{\partial x} + e^s\cos t\left(e^s\cos t\,\frac{\partial^2 u}{\partial x^2} + e^s\sin t\,\frac{\partial^2 u}{\partial x\,\partial y}\right) + e^s\sin t\,\frac{\partial u}{\partial y} + e^s\sin t\left(e^s\sin t\,\frac{\partial^2 u}{\partial y^2} + e^s\cos t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &= e^s\cos t\frac{\partial u}{\partial x} + e^s\sin t\,\frac{\partial u}{\partial y} + e^{2s}\cos^2 t\,\frac{\partial^2 u}{\partial x^2} + 2e^{2s}\cos t\sin t\,\frac{\partial^2 u}{\partial x\,\partial y} + e^{2s}\sin^2 t\,\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= -e^s \cos t\,\frac{\partial u}{\partial x} - e^s \sin t\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - e^s \sin t\,\frac{\partial u}{\partial y} + e^s \cos t\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\\ &= -e^s \cos t\,\frac{\partial u}{\partial x} - e^s \sin t\left(-e^s \sin t\,\frac{\partial^2 u}{\partial x^2} + e^s \cos t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &\qquad -e^s \sin t\,\frac{\partial u}{\partial y} + e^s \cos t\left(e^s \cos t\,\frac{\partial^2 u}{\partial y^2} - e^s \sin t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &= -e^s \cos t\,\frac{\partial u}{\partial x} - e^s \sin t\,\frac{\partial u}{\partial y} + e^{2s} \sin^2 t\,\frac{\partial^2 u}{\partial x^2} - 2e^{2s} \cos\,t \sin t\,\frac{\partial^2 u}{\partial x\,\partial y} + e^{2s} \cos^2 t\,\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Thus $e^{-2s}\left(\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2}\right) = (\cos^2 t + \sin^2 t)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, as desired.

51. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\,2s + \dfrac{\partial z}{\partial y}\,2r$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r\,\partial s} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x} 2s\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y} 2r\right)\\ &= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}\,2s + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}\,2s + \frac{\partial z}{\partial x}\frac{\partial}{\partial r}\,2s + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\,2r + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}\,2r + \frac{\partial z}{\partial y}\,2\\ &= 4rs\,\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y\,\partial x}\,4s^2 + 0 + 4rs\,\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x\,\partial y}\,4r^2 + 2\,\frac{\partial z}{\partial y}\end{aligned}$$

By the continuity of the partials, $\dfrac{\partial^2 z}{\partial r \partial s} = 4rs\,\dfrac{\partial^2 z}{\partial x^2} + 4rs\,\dfrac{\partial^2 z}{\partial y^2} + (4r^2 + 4s^2)\,\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial z}{\partial y}$.

52. By the Chain Rule,

(a) $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ (b) $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x}\,r\sin\theta + \dfrac{\partial z}{\partial y}\,r\cos\theta$

(c) $$\begin{aligned}\frac{\partial^2 z}{\partial r\,\partial\theta} &= \frac{\partial^2 z}{\partial\theta\,\partial r} = \frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right) = -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\,\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\,\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right)\\ &= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial\theta} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial\theta}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\,\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial\theta} + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial\theta}\\ &= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\left(-r\sin\theta\,\frac{\partial^2 z}{\partial x^2} + r\cos\theta\,\frac{\partial^2 z}{\partial y\,\partial x}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\left(r\cos\theta\,\frac{\partial^2 z}{\partial y^2} - r\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y}\right)\\ &= -\sin\theta\,\frac{\partial z}{\partial x} - r\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x^2} + r\cos^2\theta\,\frac{\partial^2 z}{\partial y\,\partial x} + \cos\theta\,\frac{\partial z}{\partial y} + r\cos\,\theta\sin\theta\,\frac{\partial^2 z}{\partial y^2} - r\sin^2\theta\,\frac{\partial^2 z}{\partial y\,\partial x}\\ &= \cos\theta\,\frac{\partial z}{\partial y} - \sin\theta\,\frac{\partial z}{\partial x} + r\cos\,\theta\sin\theta\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2}\right) + r(\cos^2\theta - \sin^2\theta)\,\frac{\partial^2 z}{\partial y\,\partial x}\end{aligned}$$

53. $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ and $\dfrac{\partial z}{\partial\theta} = -\dfrac{\partial z}{\partial x}\,r\sin\theta + \dfrac{\partial z}{\partial y}\,r\cos\theta$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\right) + \sin\theta\left(\frac{\partial^2 z}{\partial y^2}\sin\theta + \frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\right)\\ &= \cos^2\theta\,\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + \sin^2\theta\,\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

and

$$\frac{\partial^2 z}{\partial \theta^2} = -r\cos\theta\,\frac{\partial z}{\partial x} + (-r\sin\theta)\left(\frac{\partial^2 z}{\partial x^2}\,(-r\sin\theta) + \frac{\partial^2 z}{\partial y\,\partial x}\,r\cos\theta\right)$$
$$-r\sin\theta\,\frac{\partial z}{\partial y} + r\cos\theta\left(\frac{\partial^2 z}{\partial y^2}\,r\cos\theta + \frac{\partial^2 z}{\partial x\,\partial y}\,(-r\sin\theta)\right)$$
$$= -r\cos\theta\,\frac{\partial z}{\partial x} - r\sin\theta\,\frac{\partial z}{\partial y} + r^2\sin^2\theta\,\frac{\partial^2 z}{\partial x^2} - 2r^2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + r^2\cos^2\theta\,\frac{\partial^2 z}{\partial y^2}$$

Thus

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} = (\cos^2\theta + \sin^2\theta)\,\frac{\partial^2 z}{\partial x^2} + \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 z}{\partial y^2}$$
$$-\frac{1}{r}\cos\theta\,\frac{\partial z}{\partial x} - \frac{1}{r}\sin\theta\,\frac{\partial z}{\partial y} + \frac{1}{r}\left(\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}\right)$$
$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \text{ as desired.}$$

54. (a) $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$. Then

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$
$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial y}{\partial t}\frac{\partial x}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$
$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\,\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$

(b) $\dfrac{\partial^2 z}{\partial s\,\partial t} = \dfrac{\partial}{\partial s}\left(\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}\right)$

$$= \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s\,\partial t} + \left(\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial s}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s\,\partial t}$$
$$= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x\,\partial y}\left(\frac{\partial y}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t}\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s\,\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s\,\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t}$$

55. (a) Since f is a polynomial, it has continuous second-order partial derivatives, and

$f(tx, ty) = (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 = t^3x^2y + 2t^3xy^2 + 5t^3y^3 = t^3(x^2y + 2xy^2 + 5y^3) = t^3 f(x, y)$.

Thus, f is homogeneous of degree 3.

(b) Differentiating both sides of $f(tx, ty) = t^n f(x, y)$ with respect to t using the Chain Rule, we get

$\dfrac{\partial}{\partial t}\, f(tx, ty) = \dfrac{\partial}{\partial t}\,[t^n f(x, y)] \quad \Leftrightarrow$

$\dfrac{\partial}{\partial (tx)}\, f(tx, ty)\cdot\dfrac{\partial (tx)}{\partial t} + \dfrac{\partial}{\partial (ty)}\, f(tx, ty)\cdot\dfrac{\partial (ty)}{\partial t} = x\,\dfrac{\partial}{\partial (tx)}\, f(tx, ty) + y\,\dfrac{\partial}{\partial (ty)}\, f(tx, ty) = nt^{n-1} f(x, y)$.

Setting $t = 1$: $x\,\dfrac{\partial}{\partial x}\, f(x, y) + y\,\dfrac{\partial}{\partial y}\, f(x, y) = nf(x, y)$.

56. Differentiating both sides of $f(tx,ty)=t^n f(x,y)$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial(tx)}f(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial}{\partial(ty)}f(tx,ty)\cdot\frac{\partial(ty)}{\partial t}=x\frac{\partial}{\partial(tx)}f(tx,ty)+y\frac{\partial}{\partial(ty)}f(tx,ty)=nt^{n-1}f(x,y)\text{ and}$$

differentiating again with respect to t gives

$$\begin{aligned}&x\left[\frac{\partial^2}{\partial(tx)^2}f(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial^2}{\partial(ty)\,\partial(tx)}f(tx,ty)\cdot\frac{\partial(ty)}{\partial t}\right]\\&\qquad+y\left[\frac{\partial^2}{\partial(tx)\,\partial(ty)}f(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial^2}{\partial(ty)^2}f(tx,ty)\cdot\frac{\partial(ty)}{\partial t}\right]=n(n-1)t^{n-1}f(x,y).\end{aligned}$$

Setting $t=1$ and using the fact that $f_{yx}=f_{xy}$, we have $x^2f_{xx}+2xyf_{xy}+y^2f_{yy}=n(n-1)f(x,y)$.

57. Differentiating both sides of $f(tx,ty)=t^n f(x,y)$ with respect to x using the Chain Rule, we get

$$\frac{\partial}{\partial x}f(tx,ty)=\frac{\partial}{\partial x}\left[t^nf(x,y)\right]\quad\Leftrightarrow$$

$$\frac{\partial}{\partial(tx)}f(tx,ty)\cdot\frac{\partial(tx)}{\partial x}+\frac{\partial}{\partial(ty)}f(tx,ty)\cdot\frac{\partial(ty)}{\partial x}=t^n\frac{\partial}{\partial x}f(x,y)\quad\Leftrightarrow\quad tf_x(tx,ty)=t^nf_x(x,y).$$

Thus $f_x(tx,ty)=t^{n-1}f_x(x,y)$.

58. $F(x,y,z)=0$ is assumed to define z as a function of x and y, that is, $z=f(x,y)$. So by (7), $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$ since $F_z\neq0$.

Similarly, it is assumed that $F(x,y,z)=0$ defines x as a function of y and z, that is $x=h(x,z)$. Then $F(h(y,z),y,z)=0$

and by the Chain Rule, $F_x\dfrac{\partial x}{\partial y}+F_y\dfrac{\partial y}{\partial y}+F_z\dfrac{\partial z}{\partial y}=0$. But $\dfrac{\partial z}{\partial y}=0$ and $\dfrac{\partial y}{\partial y}=1$, so $F_x\dfrac{\partial x}{\partial y}+F_y=0\quad\Rightarrow\quad\dfrac{\partial x}{\partial y}=-\dfrac{F_y}{F_x}$.

A similar calculation shows that $\dfrac{\partial y}{\partial z}=-\dfrac{F_z}{F_y}$. Thus $\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial y}\dfrac{\partial y}{\partial z}=\left(-\dfrac{F_x}{F_z}\right)\left(-\dfrac{F_y}{F_x}\right)\left(-\dfrac{F_z}{F_y}\right)=-1$.

59. Given a function defined implicitly by $F(x,y)=0$, where F is differentiable and $F_y\neq0$, we know that $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$. Let

$G(x,y)=-\dfrac{F_x}{F_y}$ so $\dfrac{dy}{dx}=G(x,y)$. Differentiating both sides with respect to x and using the Chain Rule gives

$$\frac{d^2y}{dx^2}=\frac{\partial G}{\partial x}\frac{dx}{dx}+\frac{\partial G}{\partial y}\frac{dy}{dx}\text{ where }\frac{\partial G}{\partial x}=\frac{\partial}{\partial x}\left(-\frac{F_x}{F_y}\right)=-\frac{F_yF_{xx}-F_xF_{yx}}{F_y^2},\ \frac{\partial G}{\partial y}=\frac{\partial}{\partial y}\left(-\frac{F_x}{F_y}\right)=-\frac{F_yF_{xy}-F_xF_{yy}}{F_y^2}.$$

Thus

$$\begin{aligned}\frac{d^2y}{dx^2}&=\left(-\frac{F_yF_{xx}-F_xF_{yx}}{F_y^2}\right)(1)+\left(-\frac{F_yF_{xy}-F_xF_{yy}}{F_y^2}\right)\left(-\frac{F_x}{F_y}\right)\\&=-\frac{F_{xx}F_y^2-F_{yx}F_xF_y-F_{xy}F_yF_x+F_{yy}F_x^2}{F_y^3}\end{aligned}$$

But F has continuous second derivatives, so by Clauraut's Theorem, $F_{yx}=F_{xy}$ and we have

$$\frac{d^2y}{dx^2}=-\frac{F_{xx}F_y^2-2F_{xy}F_xF_y+F_{yy}F_x^2}{F_y^3}\text{ as desired.}$$

14.6 Directional Derivatives and the Gradient Vector

1. We can approximate the directional derivative of the pressure function at K in the direction of S by the average rate of change of pressure between the points where the red line intersects the contour lines closest to K (extend the red line slightly at the left). In the direction of S, the pressure changes from 1000 millibars to 996 millibars and we estimate the distance between these two points to be approximately 50 km (using the fact that the distance from K to S is 300 km). Then the rate of change of pressure in the direction given is approximately $\frac{996-1000}{50} = -0.08$ millibar/km.

2. First we draw a line passing through Dubbo and Sydney. We approximate the directional derivative at Dubbo in the direction of Sydney by the average rate of change of temperature between the points where the line intersects the contour lines closest to Dubbo. In the direction of Sydney, the temperature changes from 30°C to 27°C. We estimate the distance between these two points to be approximately 120 km, so the rate of change of maximum temperature in the direction given is approximately $\frac{27-30}{120} = -0.025^\circ\text{C/km}$.

3. $D_\mathbf{u}\, f(-20,30) = \nabla f\,(-20,30)\cdot \mathbf{u} = f_T(-20,30)\left(\frac{1}{\sqrt{2}}\right) + f_v(-20,30)\left(\frac{1}{\sqrt{2}}\right)$.

$f_T(-20,30) = \lim\limits_{h\to 0} \dfrac{f(-20+h,30)-f(-20,30)}{h}$, so we can approximate $f_T(-20,30)$ by considering $h=\pm 5$ and using the values given in the table: $f_T(-20,30) \approx \dfrac{f(-15,30)-f(-20,30)}{5} = \dfrac{-26-(-33)}{5} = 1.4$,

$f_T(-20,30) \approx \dfrac{f(-25,30)-f(-20,30)}{-5} = \dfrac{-39-(-33)}{-5} = 1.2$. Averaging these values gives $f_T(-20,30)\approx 1.3$.

Similarly, $f_v(-20,30) = \lim\limits_{h\to 0}\dfrac{f(-20,30+h)-f(-20,30)}{h}$, so we can approximate $f_v(-20,30)$ with $h=\pm 10$:

$f_v(-20,30) \approx \dfrac{f(-20,40)-f(-20,30)}{10} = \dfrac{-34-(-33)}{10} = -0.1$,

$f_v(-20,30) \approx \dfrac{f(-20,20)-f(-20,30)}{-10} = \dfrac{-30-(-33)}{-10} = -0.3$. Averaging these values gives $f_v(-20,30)\approx -0.2$.

Then $D_\mathbf{u} f(-20,30) \approx 1.3\left(\frac{1}{\sqrt{2}}\right) + (-0.2)\left(\frac{1}{\sqrt{2}}\right) \approx 0.778$.

4. $f(x,y) = xy^3 - x^2 \quad\Rightarrow\quad f_x(x,y) = y^3 - 2x$ and $f_y(x,y) = 3xy^2$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \pi/3$, then from Equation 6, $D_\mathbf{u} f(1,2) = f_x(1,2)\cos\left(\frac{\pi}{3}\right) + f_y(1,2)\sin\left(\frac{\pi}{3}\right) = 6\cdot\frac{1}{2} + 12\cdot\frac{\sqrt{3}}{2} = 3+6\sqrt{3}$.

5. $f(x,y) = y\cos(xy) \quad\Rightarrow\quad f_x(x,y) = y[-\sin(xy)](y) = -y^2\sin(xy)$ and

$f_y(x,y) = y[-\sin(xy)](x) + [\cos(xy)](1) = \cos(xy) - xy\sin(xy)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \pi/4$, then from Equation 6, $D_\mathbf{u} f(0,1) = f_x(0,1)\cos\left(\frac{\pi}{4}\right) + f_y(0,1)\sin\left(\frac{\pi}{4}\right) = 0\cdot\frac{\sqrt{2}}{2} + 1\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$.

6. $f(x,y) = \sqrt{2x+3y} \quad\Rightarrow\quad f_x(x,y) = \frac{1}{2}(2x+3y)^{-1/2}(2) = 1/\sqrt{2x+3y}$ and

$f_y(x,y) = \frac{1}{2}(2x+3y)^{-1/2}(3) = 3/\left(2\sqrt{2x+3y}\right)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = -\pi/6$, then from Equation 6, $D_\mathbf{u} f(3,1) = f_x(3,1)\cos\left(-\frac{\pi}{6}\right) + f_y(3,1)\sin\left(-\frac{\pi}{6}\right) = \frac{1}{3}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\left(-\frac{1}{2}\right) = \frac{\sqrt{3}}{6} - \frac{1}{4}$.

7. $f(x,y) = x/y = xy^{-1}$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = y^{-1}\,\mathbf{i} + (-xy^{-2})\,\mathbf{j} = \dfrac{1}{y}\,\mathbf{i} - \dfrac{x}{y^2}\,\mathbf{j}$

(b) $\nabla f(2,1) = \dfrac{1}{1}\,\mathbf{i} - \dfrac{2}{1^2}\,\mathbf{j} = \mathbf{i} - 2\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\,f(2,1) = \nabla f(2,1) \cdot \mathbf{u} = (\mathbf{i} - 2\,\mathbf{j}) \cdot \left(\frac{3}{5}\,\mathbf{i} + \frac{4}{5}\,\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$

8. $f(x,y) = x^2 \ln y$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = 2x \ln y\,\mathbf{i} + (x^2/y)\,\mathbf{j}$

(b) $\nabla f(3,1) = 0\,\mathbf{i} + (9/1)\,\mathbf{j} = 9\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\,f(3,1) = \nabla f(3,1) \cdot \mathbf{u} = 9\,\mathbf{j} \cdot \left(-\frac{5}{13}\,\mathbf{i} + \frac{12}{13}\,\mathbf{j}\right) = 0 + \frac{108}{13} = \frac{108}{13}.$

9. $f(x,y,z) = x^2yz - xyz^3$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2\right\rangle$

(b) $\nabla f(2,-1,1) = \langle -4+1, 4-2, -4+6\rangle = \langle -3, 2, 2\rangle$

(c) By Equation 14, $D_{\mathbf{u}}f(2,-1,1) = \nabla f(2,-1,1) \cdot \mathbf{u} = \langle -3,2,2\rangle \cdot \left\langle 0, \frac{4}{5}, -\frac{3}{5}\right\rangle = 0 + \frac{8}{5} - \frac{6}{5} = \frac{2}{5}.$

10. $f(x,y,z) = y^2 e^{xyz}$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle y^2 e^{xyz}(yz), y^2 \cdot e^{xyz}(xz) + e^{xyz} \cdot 2y, y^2 e^{xyz}(xy)\right\rangle$

$= \left\langle y^3 z e^{xyz}, (xy^2z + 2y)e^{xyz}, xy^3 e^{xyz}\right\rangle$

(b) $\nabla f(0,1,-1) = \langle -1, 2, 0\rangle$

(c) $D_{\mathbf{u}}f(0,1,-1) = \nabla f(0,1,-1) \cdot \mathbf{u} = \langle -1,2,0\rangle \cdot \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right\rangle = -\frac{3}{13} + \frac{8}{13} + 0 = \frac{5}{13}$

11. $f(x,y) = e^x \sin y \quad\Rightarrow\quad \nabla f(x,y) = \langle e^x \sin y, e^x \cos y\rangle$, $\nabla f(0,\pi/3) = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{(-6)^2+8^2}}\,\langle -6, 8\rangle = \frac{1}{10}\,\langle -6,8\rangle = \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle$, so

$D_{\mathbf{u}}\,f(0,\pi/3) = \nabla f(0,\pi/3) \cdot \mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle = -\frac{3\sqrt{3}}{10} + \frac{4}{10} = \frac{4-3\sqrt{3}}{10}.$

12. $f(x,y) = \dfrac{x}{x^2+y^2} \quad\Rightarrow\quad \nabla f(x,y) = \left\langle \dfrac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}, \dfrac{0 - x(2y)}{(x^2+y^2)^2}\right\rangle = \left\langle \dfrac{y^2-x^2}{(x^2+y^2)^2}, -\dfrac{2xy}{(x^2+y^2)^2}\right\rangle,$

$\nabla f(1,2) = \left\langle \frac{3}{25}, -\frac{4}{25}\right\rangle$, and a unit vector in the direction of $\mathbf{v} = \langle 3,5\rangle$ is $\mathbf{u} = \frac{1}{\sqrt{9+25}}\,\langle 3,5\rangle = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right\rangle$, so

$D_{\mathbf{u}}\,f(1,2) = \nabla f(1,2) \cdot \mathbf{u} = \left\langle \frac{3}{25}, -\frac{4}{25}\right\rangle \cdot \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right\rangle = \frac{9}{25\sqrt{34}} - \frac{20}{25\sqrt{34}} = -\frac{11}{25\sqrt{34}}.$

13. $g(s,t) = s\sqrt{t} \quad\Rightarrow\quad \nabla g(s,t) = \left(\sqrt{t}\,\right)\mathbf{i} + \left(s/(2\sqrt{t}\,)\right)\mathbf{j}$, $\nabla g(2,4) = 2\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{2^2+(-1)^2}}(2\,\mathbf{i} - \mathbf{j}) = \frac{1}{\sqrt{5}}(2\,\mathbf{i} - \mathbf{j})$, so $D_{\mathbf{u}}\,g(2,4) = \nabla g(2,4) \cdot \mathbf{u} = (2\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}) \cdot \frac{1}{\sqrt{5}}(2\,\mathbf{i} - \mathbf{j}) = \frac{1}{\sqrt{5}}\left(4 - \frac{1}{2}\right) = \frac{7}{2\sqrt{5}}$ or $\frac{7\sqrt{5}}{10}$.

14. $g(u,v) = u^2 e^{-v} \Rightarrow \nabla g(u,v) = \left(2ue^{-v}\right)\mathbf{i} + \left(-u^2e^{-v}\right)\mathbf{j}$, $\nabla g(3,0) = 6\,\mathbf{i} - 9\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{3^2+4^2}}(3\,\mathbf{i} + 4\,\mathbf{j}) = \frac{1}{5}(3\,\mathbf{i} + 4\,\mathbf{j})$, so $D_{\mathbf{u}}\,g(3,0) = \nabla g(3,0)\cdot\mathbf{u} = (6\,\mathbf{i} - 9\,\mathbf{j})\cdot\frac{1}{5}(3\,\mathbf{i} + 4\,\mathbf{j}) = \frac{1}{5}\,(18 - 36) = -\frac{18}{5}$.

15. $f(x,y,z) = x^2y + y^2z \Rightarrow \nabla f(x,y,z) = \left\langle 2xy, x^2 + 2yz, y^2\right\rangle$, $\nabla f(1,2,3) = \langle 4, 13, 4\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4+1+4}}\langle 2,-1,2\rangle = \frac{1}{3}\langle 2,-1,2\rangle$, so

$D_{\mathbf{u}}\,f(1,2,3) = \nabla f(1,2,3)\cdot\mathbf{u} = \langle 4,13,4\rangle\cdot\frac{1}{3}\langle 2,-1,2\rangle = \frac{1}{3}\,(8 - 13 + 8) = \frac{3}{3} = 1$.

16. $f(x,y,z) = xy^2\tan^{-1}z \Rightarrow \nabla f(x,y,z) = \left\langle y^2\tan^{-1}z, 2xy\tan^{-1}z, \dfrac{xy^2}{1+z^2}\right\rangle$,

$\nabla f(2,1,1) = \left\langle 1\cdot\frac{\pi}{4}, 4\cdot\frac{\pi}{4}, \frac{2}{1+1}\right\rangle = \left\langle\frac{\pi}{4}, \pi, 1\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{1+1+1}}\langle 1,1,1\rangle = \frac{1}{\sqrt{3}}\,\langle 1,1,1\rangle$,

so $D_{\mathbf{u}}\,f(2,1,1) = \nabla f(2,1,1)\cdot\mathbf{u} = \left\langle\frac{\pi}{4},\pi,1\right\rangle\cdot\frac{1}{\sqrt{3}}\,\langle 1,1,1\rangle = \frac{1}{\sqrt{3}}\left(\frac{\pi}{4} + \pi + 1\right) = \frac{1}{\sqrt{3}}\left(\frac{5\pi}{4} + 1\right)$.

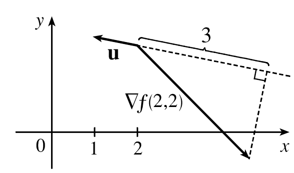
17. $h(r,s,t) = \ln(3r + 6s + 9t) \Rightarrow \nabla h(r,s,t) = \langle 3/(3r+6s+9t), 6/(3r+6s+9t), 9/(3r+6s+9t)\rangle$,

$\nabla h(1,1,1) = \left\langle\frac{1}{6},\frac{1}{3},\frac{1}{2}\right\rangle$, and a unit vector in the direction of $\mathbf{v} = 4\,\mathbf{i} + 12\,\mathbf{j} + 6\,\mathbf{k}$

is $\mathbf{u} = \frac{1}{\sqrt{16+144+36}}\,(4\,\mathbf{i} + 12\,\mathbf{j} + 6\,\mathbf{k}) = \frac{2}{7}\,\mathbf{i} + \frac{6}{7}\,\mathbf{j} + \frac{3}{7}\,\mathbf{k}$, so

$D_{\mathbf{u}}\,h(1,1,1) = \nabla h(1,1,1)\cdot\mathbf{u} = \left\langle\frac{1}{6},\frac{1}{3},\frac{1}{2}\right\rangle\cdot\left\langle\frac{2}{7},\frac{6}{7},\frac{3}{7}\right\rangle = \frac{1}{21} + \frac{2}{7} + \frac{3}{14} = \frac{23}{42}$.

18. $D_{\mathbf{u}}f(2,2) = \nabla f(2,2)\cdot\mathbf{u}$, the scalar projection of $\nabla f(2,2)$ onto $\mathbf{u}$, so we draw a perpendicular from the tip of $\nabla f(2,2)$ to the line containing $\mathbf{u}$. We can use the point $(2,2)$ to determine the scale of the axes, and we estimate the length of the projection to be approximately 3.0 units. Since the angle between $\nabla f(2,2)$ and $\mathbf{u}$ is greater than 90°, the scalar projection is negative. Thus $D_{\mathbf{u}}\,f(2,2) \approx -3$.



19. $f(x,y) = \sqrt{xy} \Rightarrow \nabla f(x,y) = \left\langle\frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x)\right\rangle = \left\langle\dfrac{y}{2\sqrt{xy}}, \dfrac{x}{2\sqrt{xy}}\right\rangle$, so $\nabla f(2,8) = \left\langle 1, \frac{1}{4}\right\rangle$.

The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 4-8\rangle = \langle 3,-4\rangle$ is $\mathbf{u} = \left\langle\frac{3}{5}, -\frac{4}{5}\right\rangle$, so

$D_{\mathbf{u}}\,f(2,8) = \nabla f(2,8)\cdot\mathbf{u} = \left\langle 1,\frac{1}{4}\right\rangle\cdot\left\langle\frac{3}{5},-\frac{4}{5}\right\rangle = \frac{2}{5}$.

20. $f(x,y,z) = xy^2z^3 \Rightarrow \nabla f(x,y,z) = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2\right\rangle$, so $\nabla f(2,1,1) = \langle 1,4,6\rangle$. The unit vector in the direction of $\overrightarrow{PQ} = \langle -2,-4,4\rangle$ is $\mathbf{u} = \frac{1}{\sqrt{4+16+16}}\langle -2,-4,4\rangle = \frac{1}{6}\,\langle -2,-4,4\rangle$, so

$D_{\mathbf{u}}\,f(2,1,1) = \nabla f(2,1,1)\cdot\mathbf{u} = \langle 1,4,6\rangle\cdot\frac{1}{6}\,\langle -2,-4,4\rangle = \frac{1}{6}\,(-2 - 16 + 24) = 1$.

21. $f(x,y) = 4y\sqrt{x} \Rightarrow \nabla f(x,y) = \left\langle 4y\cdot\frac{1}{2}x^{-1/2}, 4\sqrt{x}\right\rangle = \langle 2y/\sqrt{x}, 4\sqrt{x}\,\rangle$.

$\nabla f(4,1) = \langle 1,8\rangle$ is the direction of maximum rate of change, and the maximum rate is $|\nabla f(4,1)| = \sqrt{1+64} = \sqrt{65}$.

22. $f(s,t) = te^{st} \Rightarrow \nabla f(s,t) = \left\langle te^{st}(t), te^{st}(s) + e^{st}(1)\right\rangle = \left\langle t^2e^{st}, (st+1)e^{st}\right\rangle$.

$\nabla f(0,2) = \langle 4,1\rangle$ is the direction of maximum rate of change, and the maximum rate is $|\nabla f(0,2)| = \sqrt{16+1} = \sqrt{17}$.

23. $f(x,y) = \sin(xy) \quad \Rightarrow \quad \nabla f(x,y) = \langle y\cos(xy), x\cos(xy)\rangle$, $\nabla f(1,0) = \langle 0,1\rangle$. Thus the maximum rate of change is $|\nabla f(1,0)| = 1$ in the direction $\langle 0,1\rangle$.

24. $f(x,y,z) = x\ln(yz) \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle \ln(yz), x\cdot\dfrac{z}{yz}, x\cdot\dfrac{y}{yz}\right\rangle = \left\langle \ln(yz), \dfrac{x}{y}, \dfrac{x}{z}\right\rangle$, $\nabla f(1,2,\frac{1}{2}) = \langle 0, \frac{1}{2}, 2\rangle$. Thus the maximum rate of change is $\left|\nabla f(1,2,\frac{1}{2})\right| = \sqrt{0+\frac{1}{4}+4} = \sqrt{\frac{17}{4}} = \frac{\sqrt{17}}{2}$ in the direction $\langle 0, \frac{1}{2}, 2\rangle$ or equivalently $\langle 0,1,4\rangle$.

25. $f(x,y,z) = x/(y+z) = x(y+z)^{-1} \quad \Rightarrow$

$\nabla f(x,y,z) = \left\langle 1/(y+z), -x(y+z)^{-2}(1), -x(y+z)^{-2}(1)\right\rangle = \left\langle \dfrac{1}{y+z}, -\dfrac{x}{(y+z)^2}, -\dfrac{x}{(y+z)^2}\right\rangle$,

$\nabla f(8,1,3) = \left\langle \frac{1}{4}, -\frac{8}{4^2}, -\frac{8}{4^2}\right\rangle = \left\langle \frac{1}{4}, -\frac{1}{2}, -\frac{1}{2}\right\rangle$. Thus the maximum rate of change is

$|\nabla f(8,1,3)| = \sqrt{\frac{1}{16}+\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$ in the direction $\left\langle \frac{1}{4}, -\frac{1}{2}, -\frac{1}{2}\right\rangle$ or equivalently $\langle 1,-2,-2\rangle$.

26. $f(p,q,r) = \arctan(pqr) \quad \Rightarrow \quad \nabla f(p,q,r) = \left\langle \dfrac{qr}{1+(pqr)^2}, \dfrac{pr}{1+(pqr)^2}, \dfrac{pq}{1+(pqr)^2}\right\rangle$, $\nabla f(1,2,1) = \left\langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right\rangle$. Thus the maximum rate of change is $|\nabla f(1,2,1)| = \sqrt{\frac{4}{25}+\frac{1}{25}+\frac{4}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$ in the direction $\left\langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right\rangle$ or equivalently $\langle 2,1,2\rangle$.

27. (a) As in the proof of Theorem 15, $D_{\mathbf{u}}\,f = |\nabla f|\cos\theta$. Since the minimum value of $\cos\theta$ is -1 occurring when $\theta = \pi$, the minimum value of $D_{\mathbf{u}}\,f$ is $-|\nabla f|$ occurring when $\theta = \pi$, that is when $\mathbf{u}$ is in the opposite direction of ∇f (assuming $\nabla f \neq \mathbf{0}$).

(b) $f(x,y) = x^4y - x^2y^3 \quad \Rightarrow \quad \nabla f(x,y) = \left\langle 4x^3y - 2xy^3, x^4 - 3x^2y^2\right\rangle$, so f decreases fastest at the point $(2,-3)$ in the direction $-\nabla f(2,-3) = -\langle 12,-92\rangle = \langle -12, 92\rangle$.

28. $f(x,y) = x^2 + xy^3 \quad \Rightarrow \quad \nabla f(x,y) = \left\langle 2x+y^3, 3xy^2\right\rangle$ so $\nabla f(2,1) = \langle 5,6\rangle$. If $\mathbf{u} = \langle a,b\rangle$ is a unit vector in the desired direction then $D_{\mathbf{u}}f(2,1) = 2 \quad \Leftrightarrow \quad \langle 5,6\rangle\cdot\langle a,b\rangle = 2 \quad \Leftrightarrow \quad 5a+6b = 2 \quad \Leftrightarrow \quad b = \frac{1}{3} - \frac{5}{6}a$. But $a^2+b^2 = 1 \quad \Leftrightarrow$ $a^2 + \left(\frac{1}{3}-\frac{5}{6}a\right)^2 = 1 \quad \Leftrightarrow \quad \frac{61}{36}a^2 - \frac{5}{9}a + \frac{1}{9} = 1 \quad \Leftrightarrow \quad 61a^2 - 20a - 32 = 0$. By the quadratic formula, the solutions are

$a = \dfrac{-(-20) \pm \sqrt{(-20)^2 - 4(61)(-32)}}{2(61)} = \dfrac{20 \pm \sqrt{8208}}{122} = \dfrac{10 \pm 6\sqrt{57}}{61}$. If $a = \dfrac{10+6\sqrt{57}}{61} \approx 0.9065$ then

$b = \dfrac{1}{3} - \dfrac{5}{6}\left(\dfrac{10+6\sqrt{57}}{61}\right) \approx -0.4221$, and if $a = \dfrac{10-6\sqrt{57}}{61} \approx -0.5787$ then $b = \dfrac{1}{3} - \dfrac{5}{6}\left(\dfrac{10-6\sqrt{57}}{61}\right) \approx 0.8156$.

Thus the two directions giving a directional derivative of 2 are approximately $\langle 0.9065, -0.4221\rangle$ and $\langle -0.5787, 0.8156\rangle$.

29. The direction of fastest change is $\nabla f(x,y) = (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\mathbf{i}+\mathbf{j} \quad \Leftrightarrow \quad (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j} = k\,(\mathbf{i}+\mathbf{j}) \quad \Leftrightarrow \quad k = 2x-2$ and $k = 2y-4$. Then $2x-2 = 2y-4 \quad \Rightarrow$ $y = x+1$, so the direction of fastest change is $\mathbf{i}+\mathbf{j}$ at all points on the line $y = x+1$.

30. The fisherman is traveling in the direction $\langle -80, -60\rangle$. A unit vector in this direction is $\mathbf{u} = \frac{1}{100}\langle -80, -60\rangle = \left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle$, and if the depth of the lake is given by $f(x,y) = 200 + 0.02x^2 - 0.001y^3$, then $\nabla f(x,y) = \left\langle 0.04x, -0.003y^2\right\rangle$. $D_{\mathbf{u}}\, f(80,60) = \nabla f(80,60)\cdot\mathbf{u} = \langle 3.2, -10.8\rangle \cdot \left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle = 3.92$. Since $D_{\mathbf{u}}\, f(80,60)$ is positive, the depth of the lake is increasing near $(80,60)$ in the direction toward the buoy.

31. $T = \dfrac{k}{\sqrt{x^2+y^2+z^2}}$ and $120 = T(1,2,2) = \dfrac{k}{3}$ so $k = 360$.

(a) $\mathbf{u} = \dfrac{\langle 1,-1,1\rangle}{\sqrt{3}}$,

$$D_{\mathbf{u}}T(1,2,2) = \nabla T(1,2,2)\cdot\mathbf{u} = \left[-360\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle\right]_{(1,2,2)}\cdot\mathbf{u} = -\tfrac{40}{3}\langle 1,2,2\rangle\cdot\tfrac{1}{\sqrt{3}}\langle 1,-1,1\rangle = -\tfrac{40}{3\sqrt{3}}$$

(b) From (a), $\nabla T = -360\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle$, and since $\langle x,y,z\rangle$ is the position vector of the point (x,y,z), the vector $-\langle x,y,z\rangle$, and thus ∇T, always points toward the origin.

32. $\nabla T = -400e^{-x^2-3y^2-9z^2}\langle x, 3y, 9z\rangle$

(a) $\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1,-2,1\rangle$, $\nabla T(2,-1,2) = -400e^{-43}\langle 2,-3,18\rangle$ and

$$D_{\mathbf{u}}\,T(2,-1,2) = \left(-\frac{400e^{-43}}{\sqrt{6}}\right)(26) = -\frac{5200\sqrt{6}}{3e^{43}}\ {}^\circ\mathrm{C/m}.$$

(b) $\nabla T(2,-1,2) = 400e^{-43}\langle -2,3,-18\rangle$ or equivalently $\langle -2,3,-18\rangle$.

(c) $|\nabla T| = 400e^{-x^2-3y^2-9z^2}\sqrt{x^2+9y^2+81z^2}\ {}^\circ\mathrm{C/m}$ is the maximum rate of increase. At $(2,-1,2)$ the maximum rate of increase is $400e^{-43}\sqrt{337}\ {}^\circ\mathrm{C/m}$.

33. $\nabla V(x,y,z) = \langle 10x - 3y + yz, xz - 3x, xy\rangle$, $\nabla V(3,4,5) = \langle 38, 6, 12\rangle$

(a) $D_{\mathbf{u}}\,V(3,4,5) = \langle 38,6,12\rangle\cdot\frac{1}{\sqrt{3}}\langle 1,1,-1\rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3,4,5) = \langle 38,6,12\rangle$, or equivalently, $\langle 19,3,6\rangle$.

(c) $|\nabla V(3,4,5)| = \sqrt{38^2+6^2+12^2} = \sqrt{1624} = 2\sqrt{406}$

34. $z = f(x,y) = 1000 - 0.005x^2 - 0.01y^2 \quad\Rightarrow\quad \nabla f(x,y) = \langle -0.01x, -0.02y\rangle$ and $\nabla f(60,40) = \langle -0.6, -0.8\rangle$.

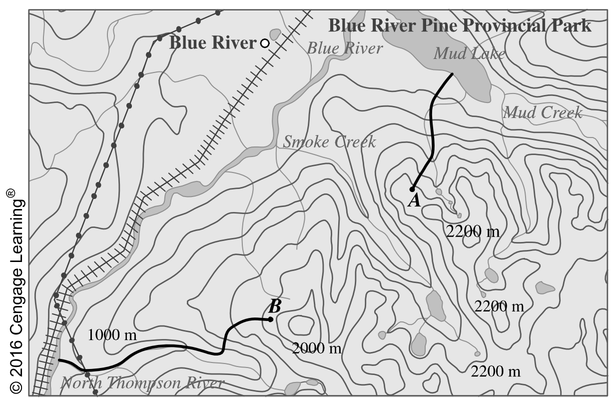
(a) Due south is in the direction of the unit vector $\mathbf{u} = -\mathbf{j}$ and $D_{\mathbf{u}}f(60,40) = \nabla f\,(60,40)\cdot\langle 0,-1\rangle = \langle -0.6,-0.8\rangle\cdot\langle 0,-1\rangle = 0.8$. Thus, if you walk due south from $(60,40,966)$ you will ascend at a rate of 0.8 vertical meters per horizontal meter.

(b) Northwest is in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}}\langle -1,1\rangle$ and $D_{\mathbf{u}}f(60,40) = \nabla f\,(60,40)\cdot\frac{1}{\sqrt{2}}\langle -1,1\rangle = \langle -0.6,-0.8\rangle\cdot\frac{1}{\sqrt{2}}\langle -1,1\rangle = -\frac{0.2}{\sqrt{2}} \approx -0.14$. Thus, if you walk northwest from $(60,40,966)$ you will descend at a rate of approximately 0.14 vertical meters per horizontal meter.

(c) $\nabla f(60, 40) = \langle -0.6, -0.8 \rangle$ is the direction of largest slope with a rate of ascent given by $|\nabla f(60, 40)| = \sqrt{(-0.6)^2 + (-0.8)^2} = 1$. The angle above the horizontal in which the path begins is given by $\tan\theta = 1 \;\Rightarrow\; \theta = 45°$.

35. A unit vector in the direction of $\overrightarrow{AB}$ is $\mathbf{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\mathbf{j}$. Thus $D_{\overrightarrow{AB}}\, f(1, 3) = f_x(1, 3) = 3$ and $D_{\overrightarrow{AC}}\, f(1, 3) = f_y(1, 3) = 26$. Therefore $\nabla f(1, 3) = \langle f_x(1, 3), f_y(1, 3) \rangle = \langle 3, 26 \rangle$, and by definition, $D_{\overrightarrow{AD}}\, f(1, 3) = \nabla f \cdot \mathbf{u}$ where $\mathbf{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$. Therefore, $D_{\overrightarrow{AD}}\, f(1, 3) = \langle 3, 26 \rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle = 3 \cdot \frac{5}{13} + 26 \cdot \frac{12}{13} = \frac{327}{13}$.

36. The curves of steepest ascent or descent are perpendicular to all of the contour lines (see Figure 12) so we sketch curves beginning at A and B that head toward lower elevations, crossing each contour line at a right angle.



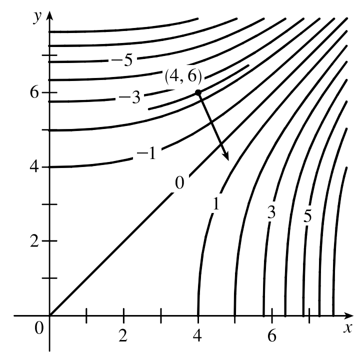
37. (a) $$\nabla(au + bv) = \left\langle \frac{\partial(au + bv)}{\partial x}, \frac{\partial(au + bv)}{\partial y} \right\rangle = \left\langle a\,\frac{\partial u}{\partial x} + b\,\frac{\partial v}{\partial x}, a\,\frac{\partial u}{\partial y} + b\,\frac{\partial v}{\partial y} \right\rangle = a\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle + b\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle = a\,\nabla u + b\,\nabla v$$

(b) $$\nabla(uv) = \left\langle v\,\frac{\partial u}{\partial x} + u\,\frac{\partial v}{\partial x}, v\,\frac{\partial u}{\partial y} + u\,\frac{\partial v}{\partial y} \right\rangle = v\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle + u\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle = v\,\nabla u + u\,\nabla v$$

(c) $$\nabla\left(\frac{u}{v}\right) = \left\langle \frac{v\,\dfrac{\partial u}{\partial x} - u\,\dfrac{\partial v}{\partial x}}{v^2}, \frac{v\,\dfrac{\partial u}{\partial y} - u\,\dfrac{\partial v}{\partial y}}{v^2} \right\rangle = \frac{v\left\langle \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y} \right\rangle - u\left\langle \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y} \right\rangle}{v^2} = \frac{v\,\nabla u - u\,\nabla v}{v^2}$$

(d) $$\nabla u^n = \left\langle \frac{\partial(u^n)}{\partial x}, \frac{\partial(u^n)}{\partial y} \right\rangle = \left\langle nu^{n-1}\,\frac{\partial u}{\partial x}, nu^{n-1}\,\frac{\partial u}{\partial y} \right\rangle = nu^{n-1}\,\nabla u$$

38. If we place the initial point of the gradient vector $\nabla f(4,6)$ at $(4,6)$, the vector is perpendicular to the level curve of f that includes $(4,6)$, so we sketch a portion of the level curve through $(4,6)$ (using the nearby level curves as a guideline) and draw a line perpendicular to the curve at $(4,6)$. The gradient vector is parallel to this line, pointing in the direction of increasing function values, and with length equal to the maximum value of the directional derivative of f at $(4,6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus the rate of change is approximately $\dfrac{-2-(-3)}{0.5} = 2$, and we sketch the gradient vector with length 2.



39. $f(x,y) = x^3 + 5x^2y + y^3 \quad \Rightarrow$

$D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot \mathbf{u} = \left\langle 3x^2+10xy, 5x^2+3y^2\right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle = \frac{9}{5}x^2 + 6xy + 4x^2 + \frac{12}{5}y^2 = \frac{29}{5}x^2 + 6xy + \frac{12}{5}y^2$. Then

$$\begin{aligned} D_{\mathbf{u}}^2 f(x,y) &= D_{\mathbf{u}}\left[D_{\mathbf{u}}f(x,y)\right] = \nabla\left[D_{\mathbf{u}}f(x,y)\right]\cdot\mathbf{u} = \left\langle \tfrac{58}{5}x + 6y, 6x + \tfrac{24}{5}y\right\rangle \cdot \left\langle \tfrac{3}{5}, \tfrac{4}{5}\right\rangle \\ &= \tfrac{174}{25}x + \tfrac{18}{5}y + \tfrac{24}{5}x + \tfrac{96}{25}y = \tfrac{294}{25}x + \tfrac{186}{25}y \end{aligned}$$

and $D_{\mathbf{u}}^2 f(2,1) = \frac{294}{25}(2) + \frac{186}{25}(1) = \frac{774}{25}$.

40. (a) From Equation 9 we have $D_{\mathbf{u}}f = \nabla f\cdot\mathbf{u} = \langle f_x, f_y\rangle\cdot\langle a,b\rangle = f_x a + f_y b$ and from Exercise 39 we have

$D_{\mathbf{u}}^2 f = D_{\mathbf{u}}\left[D_{\mathbf{u}}f\right] = \nabla\left[D_{\mathbf{u}}f\right]\cdot\mathbf{u} = \langle f_{xx}a + f_{yx}b, f_{xy}a + f_{yy}b\rangle\cdot\langle a,b\rangle = f_{xx}a^2 + f_{yx}ab + f_{xy}ab + f_{yy}b^2$.

But $f_{yx} = f_{xy}$ by Clairaut's Theorem, so $D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$.

(b) $f(x,y) = xe^{2y} \quad\Rightarrow\quad f_x = e^{2y}$, $f_y = 2xe^{2y}$, $f_{xx} = 0$, $f_{xy} = 2e^{2y}$, $f_{yy} = 4xe^{2y}$ and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4^2+6^2}}\langle 4,6\rangle = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right\rangle = \langle a,b\rangle$. Then

$$D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2 = 0\cdot\left(\tfrac{2}{\sqrt{13}}\right)^2 + 2\cdot 2e^{2y}\left(\tfrac{2}{\sqrt{13}}\right)\left(\tfrac{3}{\sqrt{13}}\right) + 4xe^{2y}\left(\tfrac{3}{\sqrt{13}}\right)^2 = \tfrac{24}{13}e^{2y} + \tfrac{36}{13}xe^{2y}.$$

41. Let $F(x,y,z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2$. Then $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$ is a level surface of F.

$F_x(x,y,z) = 4(x-2) \quad\Rightarrow\quad F_x(3,3,5) = 4$, $F_y(x,y,z) = 2(y-1) \quad\Rightarrow\quad F_y(3,3,5) = 4$, and

$F_z(x,y,z) = 2(z-3) \quad\Rightarrow\quad F_z(3,3,5) = 4$.

(a) Equation 19 gives an equation of the tangent plane at $(3,3,5)$ as $4(x-3) + 4(y-3) + 4(z-5) = 0 \quad\Leftrightarrow$ $4x + 4y + 4z = 44$ or equivalently $x + y + z = 11$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-3}{4} = \dfrac{y-3}{4} = \dfrac{z-5}{4}$ or equivalently $x - 3 = y - 3 = z - 5$. Corresponding parametric equations are $x = 3 + t$, $y = 3 + t$, $z = 5 + t$.

42. Let $F(x,y,z) = y^2 + z^2 - x$. Then $x = y^2 + z^2 + 1 \quad \Leftrightarrow \quad y^2 + z^2 - x = -1$ is a level surface of F.

$F_x(x,y,z) = -1 \quad \Rightarrow \quad F_x(3,1,-1) = -1$, $\; F_y(x,y,z) = 2y \quad \Rightarrow \quad F_y(3,1,-1) = 2$, and $F_z(x,y,z) = 2z \quad \Rightarrow$

$F_z(3,1,-1) = -2$.

(a) By Equation 19, an equation of the tangent plane at $(3,1,-1)$ is $(-1)(x-3) + 2(y-1) + (-2)[z-(-1)] = 0$ or $-x + 2y - 2z = 1$ or $x - 2y + 2z = -1$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-3}{-1} = \dfrac{y-1}{2} = \dfrac{z-(-1)}{-2}$ or equivalently

$x - 3 = \dfrac{y-1}{-2} = \dfrac{z+1}{2}$ and parametric equations $x = 3 - t$, $y = 1 + 2t$, $z = -1 - 2t$.

43. Let $F(x,y,z) = xy^2z^3$. Then $xy^2z^3 = 8$ is a level surface of F and $\nabla F(x,y,z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$.

(a) $\nabla F(2,2,1) = \langle 4,8,24\rangle$ is a normal vector for the tangent plane at $(2,2,1)$, so an equation of the tangent plane is $4(x-2) + 8(y-2) + 24(z-1) = 0$ or $4x + 8y + 24z = 48$ or equivalently $x + 2y + 6z = 12$.

(b) The normal line has direction $\nabla F(2,2,1) = \langle 4,8,24\rangle$ or equivalently $\langle 1,2,6\rangle$, so parametric equations are $x = 2 + t$, $y = 2 + 2t$, $z = 1 + 6t$, and symmetric equations are $x - 2 = \dfrac{y-2}{2} = \dfrac{z-1}{6}$.

44. Let $F(x,y,z) = xy + yz + zx$. Then $xy + yz + zx = 5$ is a level surface of F and $\nabla F(x,y,z) = \langle y+z, x+z, x+y\rangle$.

(a) $\nabla F(1,2,1) = \langle 3,2,3\rangle$ is a normal vector for the tangent plane at $(1,2,1)$, so an equation of the tangent plane is $3(x-1) + 2(y-2) + 3(z-1) = 0$ or $3x + 2y + 3z = 10$.

(b) The normal line has direction $\langle 3,2,3\rangle$, so parametric equations are $x = 1 + 3t$, $y = 2 + 2t$, $z = 1 + 3t$, and symmetric equations are $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z-1}{3}$.

45. Let $F(x,y,z) = x + y + z - e^{xyz}$. Then $x + y + z = e^{xyz}$ is the level surface $F(x,y,z) = 0$, and $\nabla F(x,y,z) = \langle 1 - yze^{xyz}, 1 - xze^{xyz}, 1 - xye^{xyz}\rangle$.

(a) $\nabla F(0,0,1) = \langle 1,1,1\rangle$ is a normal vector for the tangent plane at $(0,0,1)$, so an equation of the tangent plane is $1(x-0) + 1(y-0) + 1(z-1) = 0$ or $x + y + z = 1$.

(b) The normal line has direction $\langle 1,1,1\rangle$, so parametric equations are $x = t$, $y = t$, $z = 1 + t$, and symmetric equations are $x = y = z - 1$.

46. Let $F(x,y,z) = x^4 + y^4 + z^4 - 3x^2y^2z^2$. Then $x^4 + y^4 + z^4 = 3x^2y^2z^2$ is the level surface $F(x,y,z) = 0$, and $\nabla F(x,y,z) = \langle 4x^3 - 6xy^2z^2, 4y^3 - 6x^2yz^2, 4z^3 - 6x^2y^2z\rangle$.

(a) $\nabla F(1,1,1) = \langle -2,-2,-2\rangle$ or equivalently $\langle 1,1,1\rangle$ is a normal vector for the tangent plane at $(1,1,1)$, so an equation of the tangent plane is $1(x-1) + 1(y-1) + 1(z-1) = 0$ or $x + y + z = 3$.

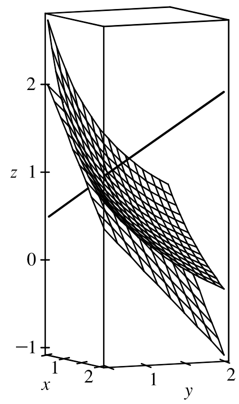
(b) The normal line has direction $\langle 1,1,1\rangle$, so parametric equations are $x = 1 + t$, $y = 1 + t$, $z = 1 + t$, and symmetric equations are $x - 1 = y - 1 = z - 1$ or equivalently $x = y = z$.

47. $F(x, y, z) = xy + yz + zx$,

$\nabla F(x, y, z) = \langle y + z, x + z, y + x\rangle$,

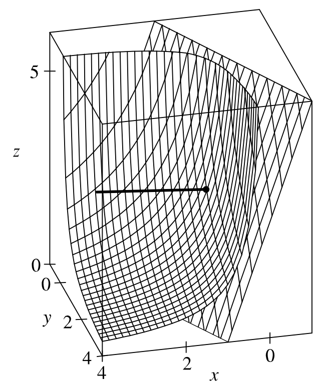
$\nabla F(1, 1, 1) = \langle 2, 2, 2\rangle$, so an equation of the tangent plane is $2x + 2y + 2z = 6$ or $x + y + z = 3$, and the normal line is given by $x - 1 = y - 1 = z - 1$ or $x = y = z$. To graph the surface we solve for z:

$z = \dfrac{3 - xy}{x + y}$.



48. $F(x, y, z) = xyz$, $\nabla F(x, y, z) = \langle yz, xz, yx\rangle$,

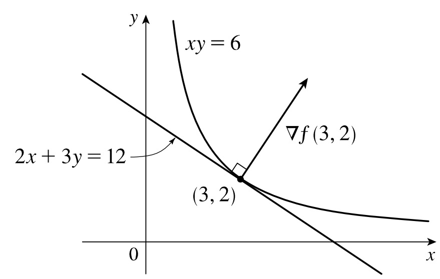
$\nabla F(1, 2, 3) = \langle 6, 3, 2\rangle$, so an equation of the tangent plane is $6x + 3y + 2z = 18$, and the normal line is given by $\dfrac{x - 1}{6} = \dfrac{y - 2}{3} = \dfrac{z - 3}{2}$ or $x = 1 + 6t$, $y = 2 + 3t$, $z = 3 + 2t$. To graph the surface we solve for z: $z = \dfrac{6}{xy}$.



49. $f(x, y) = xy \quad\Rightarrow\quad \nabla f(x, y) = \langle y, x\rangle$, $\nabla f(3, 2) = \langle 2, 3\rangle$. $\nabla f(3, 2)$ is perpendicular to the tangent line, so the tangent line has equation

$\nabla f(3, 2) \cdot \langle x - 3, y - 2\rangle = 0 \quad\Rightarrow\quad \langle 2, 3\rangle \cdot \langle x - 3, x - 2\rangle = 0 \quad\Rightarrow$

$2(x - 3) + 3(y - 2) = 0$ or $2x + 3y = 12$.

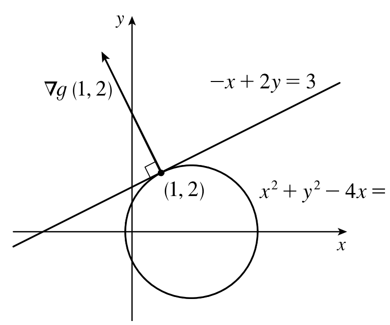


50. $g(x, y) = x^2 + y^2 - 4x \quad\Rightarrow\quad \nabla g(x, y) = \langle 2x - 4, 2y\rangle$,

$\nabla g(1, 2) = \langle -2, 4\rangle$. $\nabla g(1, 2)$ is perpendicular to the tangent line, so the tangent line has equation $\nabla g(1, 2) \cdot \langle x - 1, y - 2\rangle = 0 \quad\Rightarrow$

$\langle -2, 4\rangle \cdot \langle x - 1, y - 2\rangle = 0 \quad\Rightarrow\quad -2(x - 1) + 4(y - 2) = 0 \quad\Leftrightarrow$

$-2x + 4y = 6$ or equivalently $-x + 2y = 3$.



51. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{2z_0}{c^2} \right\rangle$. Thus an equation of the tangent plane at (x_0, y_0, z_0) is

$\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y + \dfrac{2z_0}{c^2}\,z = 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} + \dfrac{z_0^2}{c^2}\right) = 2(1) = 2$ since (x_0, y_0, z_0) is a point on the ellipsoid. Hence

$\dfrac{x_0}{a^2}\,x + \dfrac{y_0}{b^2}\,y + \dfrac{z_0}{c^2}\,z = 1$ is an equation of the tangent plane.

52. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-2z_0}{c^2} \right\rangle$, so an equation of the tangent plane at (x_0, y_0, z_0) is

$$\frac{2x_0}{a^2}\,x + \frac{2y_0}{b^2}\,y - \frac{2z_0}{c^2}\,z = 2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2}\right) = 2 \text{ or } \frac{x_0}{a^2}\,x + \frac{y_0}{b^2}\,y - \frac{z_0}{c^2}\,z = 1.$$

53. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-1}{c} \right\rangle$, so an equation of the tangent plane is $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y - \dfrac{1}{c}\,z = \dfrac{2x_0^2}{a^2} + \dfrac{2y_0^2}{b^2} - \dfrac{z_0}{c}$ or $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y = \dfrac{z}{c} + 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}\right) - \dfrac{z_0}{c}$. But $\dfrac{z_0}{c} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}$, so the equation can be written as

$$\frac{2x_0}{a^2}\,x + \frac{2y_0}{b^2}\,y = \frac{z + z_0}{c}.$$

54. Let $F(x, y, z) = x^2 + y^2 + 2z^2$; then the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is a level surface of F. $\nabla F(x, y, z) = \langle 2x, 2y, 4z\rangle$ is a normal vector to the surface at (x, y, z) and so it is a normal vector for the tangent plane there. The tangent plane is parallel to the plane $x + 2y + z = 1$ when the normal vectors of the planes are parallel, so we need a point (x_0, y_0, z_0) on the ellipsoid where $\langle 2x_0, 2y_0, 4z_0\rangle = k\,\langle 1, 2, 1\rangle$ for some $k \neq 0$. Comparing components we have $2x_0 = k \;\Rightarrow\; x_0 = k/2$, $2y_0 = 2k \;\Rightarrow\; y_0 = k$, $4z_0 = k \;\Rightarrow\; z_0 = k/4$. $(x_0, y_0, z_0) = (k/2, k, k/4)$ lies on the ellipsoid, so $(k/2)^2 + k^2 + 2(k/4)^2 = 1 \;\Rightarrow\; \frac{11}{8}k^2 = 1 \;\Rightarrow\; k^2 = \frac{8}{11} \;\Rightarrow\; k = \pm 2\sqrt{\frac{2}{11}}$. Thus the tangent planes at the points $\left(\sqrt{\frac{2}{11}}, 2\sqrt{\frac{2}{11}}, \frac{1}{2}\sqrt{\frac{2}{11}}\right)$ and $\left(-\sqrt{\frac{2}{11}}, -2\sqrt{\frac{2}{11}}, -\frac{1}{2}\sqrt{\frac{2}{11}}\right)$ are parallel to the given plane.

55. The hyperboloid $x^2 - y^2 - z^2 = 1$ is a level surface of $F(x, y, z) = x^2 - y^2 - z^2$ and $\nabla F\,(x, y, z) = \langle 2x, -2y, -2z\rangle$ is a normal vector to the surface and hence a normal vector for the tangent plane at (x, y, z). The tangent plane is parallel to the plane $z = x + y$ or $x + y - z = 0$ if and only if the corresponding normal vectors are parallel, so we need a point (x_0, y_0, z_0) on the hyperboloid where $\langle 2x_0, -2y_0, -2z_0\rangle = c\,\langle 1, 1, -1\rangle$ or equivalently $\langle x_0, -y_0, -z_0\rangle = k\,\langle 1, 1, -1\rangle$ for some $k \neq 0$. Then we must have $x_0 = k$, $y_0 = -k$, $z_0 = k$ and substituting into the equation of the hyperboloid gives $k^2 - (-k)^2 - k^2 = 1 \;\Leftrightarrow\; -k^2 = 1$, an impossibility. Thus there is no such point on the hyperboloid.

56. First note that the point $(1, 1, 2)$ is on both surfaces. The ellipsoid is a level surface of $F(x, y, z) = 3x^2 + 2y^2 + z^2$ and $\nabla F(x, y, z) = \langle 6x, 4y, 2z\rangle$. A normal vector to the surface at $(1, 1, 2)$ is $\nabla F(1, 1, 2) = \langle 6, 4, 4\rangle$ and an equation of the tangent plane there is $6(x - 1) + 4(y - 1) + 4(z - 2) = 0$ or $6x + 4y + 4z = 18$ or $3x + 2y + 2z = 9$. The sphere is a level surface of $G(x, y, z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24$ and $\nabla G(x, y, z) = \langle 2x - 8, 2y - 6, 2z - 8\rangle$. A normal vector to the sphere at $(1, 1, 2)$ is $\nabla G(1, 1, 2) = \langle -6, -4, -4\rangle$ and the tangent plane there is $-6(x - 1) - 4(y - 1) - 4(z - 2) = 0$ or $3x + 2y + 2z = 9$. Since these tangent planes are identical, the surfaces are tangent to each other at the point $(1, 1, 2)$.

57. Let (x_0, y_0, z_0) be a point on the cone [other than $(0, 0, 0)$]. The cone is a level surface of $F(x, y, z) = x^2 + y^2 - z^2$ and $\nabla F(x, y, z) = \langle 2x, 2y, -2z\rangle$, so $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, -2z_0\rangle$ is a normal vector to the cone at this point and an equation of the tangent plane there is $2x_0\,(x - x_0) + 2y_0\,(y - y_0) - 2z_0\,(z - z_0) = 0$ or $x_0x + y_0y - z_0z = x_0^2 + y_0^2 - z_0^2$. But $x_0^2 + y_0^2 = z_0^2$ so the tangent plane is given by $x_0x + y_0y - z_0z = 0$, a plane which always contains the origin.

58. Let (x_0, y_0, z_0) be a point on the sphere. Then the normal line is given by $\dfrac{x - x_0}{2x_0} = \dfrac{y - y_0}{2y_0} = \dfrac{z - z_0}{2z_0}$. For the center $(0, 0, 0)$ to be on the line, we need $-\dfrac{x_0}{2x_0} = -\dfrac{y_0}{2y_0} = -\dfrac{z_0}{2z_0}$ or equivalently $1 = 1 = 1$, which is true.

59. Let $F(x, y, z) = x^2 + y^2 - z$. Then the paraboloid is the level surface $F(x, y, z) = 0$ and $\nabla F(x, y, z) = \langle 2x, 2y, -1\rangle$, so $\nabla F(1, 1, 2) = \langle 2, 2, -1\rangle$ is a normal vector to the surface. Thus the normal line at $(1, 1, 2)$ is given by $x = 1 + 2t$, $y = 1 + 2t$, $z = 2 - t$. Substitution into the equation of the paraboloid $z = x^2 + y^2$ gives $2 - t = (1 + 2t)^2 + (1 + 2t)^2 \quad\Leftrightarrow\quad 2 - t = 2 + 8t + 8t^2 \quad\Leftrightarrow\quad 8t^2 + 9t = 0 \quad\Leftrightarrow\quad t(8t + 9) = 0$. Thus the line intersects the paraboloid when $t = 0$, corresponding to the given point $(1, 1, 2)$, or when $t = -\frac{9}{8}$, corresponding to the point $\left(-\frac{5}{4}, -\frac{5}{4}, \frac{25}{8}\right)$.

60. The ellipsoid is a level surface of $F(x, y, z) = 4x^2 + y^2 + 4z^2$ and $\nabla F(x, y, z) = \langle 8x, 2y, 8z\rangle$, so $\nabla F(1, 2, 1) = \langle 8, 4, 8\rangle$ or equivalently $\langle 2, 1, 2\rangle$ is a normal vector to the surface. Thus the normal line to the ellipsoid at $(1, 2, 1)$ is given by $x = 1 + 2t$, $y = 2 + t$, $z = 1 + 2t$. Substitution into the equation of the sphere gives $(1+2t)^2 + (2+t)^2 + (1+2t)^2 = 102 \quad\Leftrightarrow\quad 6 + 12t + 9t^2 = 102 \quad\Leftrightarrow\quad 9t^2 + 12t - 96 = 0 \quad\Leftrightarrow\quad 3(t+4)(3t-8) = 0$. Thus the line intersects the sphere when $t = -4$, corresponding to the point $(-7, -2, -7)$, and when $t = \frac{8}{3}$, corresponding to the point $\left(\frac{19}{3}, \frac{14}{3}, \frac{19}{3}\right)$.

61. Let (x_0, y_0, z_0) be a point on the surface. Then an equation of the tangent plane at the point is

$$\frac{x}{2\sqrt{x_0}} + \frac{y}{2\sqrt{y_0}} + \frac{z}{2\sqrt{z_0}} = \frac{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}{2}.$$ But $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{c}$, so the equation is $\dfrac{x}{\sqrt{x_0}} + \dfrac{y}{\sqrt{y_0}} + \dfrac{z}{\sqrt{z_0}} = \sqrt{c}$. The x-, y-, and z-intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$ and $\sqrt{cz_0}$ respectively. (The x-intercept is found by setting $y = z = 0$ and solving the resulting equation for x, and the y- and z-intercepts are found similarly.) So the sum of the intercepts is $\sqrt{c}\left(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}\right) = c$, a constant.

62. The surface $xyz = 1$ is a level surface of $F(x, y, z) = xyz$ and $\nabla F(x, y, z) = \langle yz, xz, xy\rangle$ is normal to the surface, so a normal vector for the tangent plane to the surface at (x_0, y_0, z_0) is $\langle y_0z_0, x_0z_0, x_0y_0\rangle$. An equation for the tangent plane there is $y_0z_0(x - x_0) + x_0z_0(y - y_0) + x_0y_0(z - z_0) = 0 \quad\Rightarrow\quad y_0z_0x + x_0z_0y + x_0y_0z = 3x_0y_0z_0$ or $\dfrac{x}{x_0} + \dfrac{y}{y_0} + \dfrac{z}{z_0} = 3$. If (x_0, y_0, z_0) is in the first octant, then the tangent plane cuts off a pyramid in the first octant with vertices $(0, 0, 0)$, $(3x_0, 0, 0)$, $(0, 3y_0, 0)$, $(0, 0, 3z_0)$. The base in the xy-plane is a triangle with area $\frac{1}{2}\,(3x_0)\,(3y_0)$ and the height (along the z-axis) of the pyramid is $3z_0$. The volume of the pyramid for any point (x_0, y_0, z_0) on the surface $xyz = 1$ in the first octant is $\frac{1}{3}$ (base) (height) $= \frac{1}{3}\cdot\frac{1}{2}\,(3x_0)\,(3y_0)\cdot 3z_0 = \frac{9}{2}x_0y_0z_0 = \frac{9}{2}$ since $x_0y_0z_0 = 1$.

63. If $f(x,y,z) = z - x^2 - y^2$ and $g(x,y,z) = 4x^2 + y^2 + z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1,1,2)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ will therefore be parallel to the tangent line.

We have $\nabla f(x,y,z) = \langle -2x, -2y, 1\rangle \Rightarrow \nabla f(-1,1,2) = \langle 2,-2,1\rangle$, and $\nabla g(x,y,z) = \langle 8x, 2y, 2z\rangle \Rightarrow$

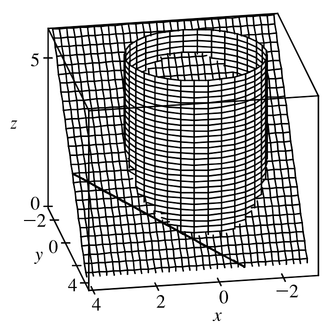
$\nabla g(-1,1,2) = \langle -8,2,4\rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = -10\,\mathbf{i} - 16\,\mathbf{j} - 12\,\mathbf{k}$.

Parametric equations are: $x = -1 - 10t,\ y = 1 - 16t,\ z = 2 - 12t$.

64. (a) Let $f(x,y,z) = y + z$ and $g(x,y,z) = x^2 + y^2$. Then the required tangent line is perpendicular to both ∇f and ∇g at $(1,2,1)$ and the vector $\mathbf{v} = \nabla f \times \nabla g$ is parallel to the tangent line. We have $\nabla f(x,y,z) = \langle 0,1,1\rangle \Rightarrow \nabla f(1,2,1) = \langle 0,1,1\rangle$, and $\nabla g(x,y,z) = \langle 2x, 2y, 0\rangle \Rightarrow \nabla g(1,2,1) = \langle 2,4,0\rangle$. Hence

$\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -4\,\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}$. So parametric equations of the desired tangent line are $x = 1 - 4t,\ y = 2 + 2t,\ z = 1 - 2t$.

(b)



65. Parametric equations for the helix are $x = \cos \pi t,\ \ y = \sin \pi t,\ \ z = t$, and substituting into the equation of the paraboloid gives $t = \cos^2 \pi t + \sin^2 \pi t \Rightarrow t = 1$. Thus the helix intersects the surface at the point $(\cos \pi, \sin \pi, 1) = (-1,0,1)$. Here $\mathbf{r}'(t) = \langle -\pi \sin \pi t, \pi \cos \pi t, 1\rangle$, so the tangent vector to the helix at that point is $\mathbf{r}'(1) = \langle -\pi \sin \pi, \pi \cos \pi, 1\rangle = \langle 0, -\pi, 1\rangle$. The paraboloid $z = x^2 + y^2 \Leftrightarrow x^2 + y^2 - z = 0$ is a level surface of $F(x,y,z) = x^2 + y^2 - z$ and $\nabla F(x,y,z) = \langle 2x, 2y, -1\rangle$, so a normal vector to the tangent plane at $(-1,0,1)$ is $\nabla F(-1,0,1) = \langle -2,0,-1\rangle$. The angle θ between $\mathbf{r}'(1)$ and $\nabla F(-1,0,1)$ is given by

$$\cos\theta = \frac{\langle 0,-\pi,1\rangle \cdot \langle -2,0,-1\rangle}{|\langle 0,-\pi,1\rangle|\,|\langle -2,0,-1\rangle|} = \frac{0+0-1}{\sqrt{0+\pi^2+1}\,\sqrt{4+0+1}} = \frac{-1}{\sqrt{5(\pi^2+1)}} \quad \Rightarrow$$

$\theta = \cos^{-1} \dfrac{-1}{\sqrt{5(\pi^2+1)}} \approx 97.8^\circ$. Because $\nabla F(-1,0,1)$ is perpendicular to the tangent plane, the angle of intersection between the helix and the paraboloid is approximately $97.8^\circ - 90^\circ = 7.8^\circ$.

66. Parametric equations for the helix are $x = \cos(\pi t/2),\ \ y = \sin(\pi t/2),\ \ z = t$, and substituting into the equation of the sphere gives $\cos^2(\pi t/2) + \sin^2(\pi t/2) + t^2 = 2 \Rightarrow 1 + t^2 = 2 \Rightarrow t = \pm 1$. Thus the helix intersects the sphere at two points: $(\cos(\pi/2), \sin(\pi/2), 1) = (0,1,1)$, when $t = 1$, and $(\cos(-\pi/2), \sin(-\pi/2), -1) = (0,-1,-1)$, when $t = -1$. Here $\mathbf{r}'(t) = \left\langle -\frac{\pi}{2}\sin(\pi t/2), \frac{\pi}{2}\cos(\pi t/2), 1\right\rangle$, so the tangent vector to the helix at $(0,1,1)$ is $\mathbf{r}'(1) = \langle -\pi/2, 0, 1\rangle$. The sphere $x^2 + y^2 + z^2 = 2$ is a level surface of $F(x,y,z) = x^2 + y^2 + z^2$ and $\nabla F(x,y,z) = \langle 2x, 2y, 2z\rangle$, so a normal vector to the tangent plane at $(0,1,1)$ is $\nabla F(0,1,1) = \langle 0,2,2\rangle$. As in Exercise 65, the angle of intersection between the helix and the sphere is the angle between the tangent vector to the helix and the tangent plane to the sphere. The angle θ between $\mathbf{r}'(1)$ and $\nabla F(0,1,1)$ is given by

$$\cos\theta = \frac{\langle -\pi/2,0,1\rangle\cdot\langle 0,2,2\rangle}{|\langle -\pi/2,0,1\rangle|\;|\langle 0,2,2\rangle|} = \frac{2}{\sqrt{(\pi^2/4)+1}\,\sqrt{8}} = \frac{2}{\sqrt{2\pi^2+8}} \quad\Rightarrow\quad \theta = \cos^{-1}\frac{2}{\sqrt{2\pi^2+8}} \approx 67.7^\circ$$

Because $\nabla F(0,1,1)$ is perpendicular to the tangent plane, the angle between $\mathbf{r}'(1)$ and the tangent plane is approximately $90^\circ - 67.7^\circ = 22.3^\circ$.

At $(0,-1,-1)$, $\mathbf{r}'(-1) = \langle \pi/2,0,1\rangle$ and $\nabla F(0,-1,-1) = \langle 0,-2,-2\rangle$, and the angle ϕ between these vectors is given by $\cos\phi = \dfrac{\langle \pi/2,0,1\rangle\cdot\langle 0,-2,-2\rangle}{|\langle \pi/2,0,1\rangle|\;|\langle 0,-2,-2\rangle|} = \dfrac{-2}{\sqrt{2\pi^2+8}} \quad\Rightarrow\quad \phi = \cos^{-1}\dfrac{-2}{\sqrt{2\pi^2+8}} \approx 112.3^\circ$. Thus the angle between the helix and the sphere at $(0,-1,-1)$ is approximately $112.3^\circ - 90^\circ = 22.3^\circ$. (By symmetry, we would expect the angles to be identical.)

67. (a) The direction of the normal line of F is given by ∇F, and that of G by ∇G. Assuming that $\nabla F \neq 0 \neq \nabla G$, the two normal lines are perpendicular at P if $\nabla F\cdot\nabla G = 0$ at P $\Leftrightarrow$ $\langle \partial F/\partial x, \partial F/\partial y, \partial F/\partial z\rangle\cdot\langle \partial G/\partial x, \partial G/\partial y, \partial G/\partial z\rangle = 0$ at P $\Leftrightarrow$ $F_xG_x + F_yG_y + F_zG_z = 0$ at P.

(b) Here $F = x^2+y^2-z^2$ and $G = x^2+y^2+z^2-r^2$, so

$\nabla F\cdot\nabla G = \langle 2x,2y,-2z\rangle\cdot\langle 2x,2y,2z\rangle = 4x^2+4y^2-4z^2 = 4F = 0$, since the point (x,y,z) lies on the graph of $F=0$. To see that this is true without using calculus, note that $G=0$ is the equation of a sphere centered at the origin and $F=0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F=0$ and $G=0$ are everywhere orthogonal.

68. (a) The function $f(x,y) = (xy)^{1/3}$ is continuous on $\mathbb{R}^2$ since it is a composition of a polynomial and the cube root function, both of which are continuous. (See the text just after Example 14.2.8.)

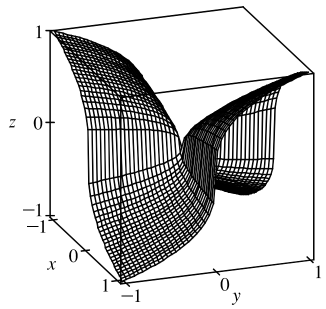
$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{(h\cdot 0)^{1/3}-0}{h} = 0,$$

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{(0\cdot h)^{1/3}-0}{h} = 0.$$

Therefore, $f_x(0,0)$ and $f_y(0,0)$ do exist and are equal to 0. Now let $\mathbf{u}$ be any unit vector other than $\mathbf{i}$ and $\mathbf{j}$ (these correspond to f_x and f_y respectively.) Then $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ where $a\neq 0$ and $b\neq 0$. Thus

$D_{\mathbf{u}}\,f(0,0) = \displaystyle\lim_{h\to 0}\frac{f(0+ha,0+hb)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt[3]{(ha)(hb)}}{h} = \lim_{h\to 0}\frac{\sqrt[3]{ab}}{h^{1/3}}$ and this limit does not exist, so $D_{\mathbf{u}}\,f(0,0)$ does not exist.

(b)



Notice that if we start at the origin and proceed in the direction of the x- or y-axis, then the graph is flat. But if we proceed in any other direction, then the graph is extremely steep.

69. Let $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$. Then we know that at the given point, $D_{\mathbf{u}}\, f = \nabla f \cdot \mathbf{u} = af_x + bf_y$ and $D_{\mathbf{v}}\, f = \nabla f \cdot \mathbf{v} = cf_x + df_y$. But these are just two linear equations in the two unknowns f_x and f_y, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y\rangle$ at the given point. In fact,

$$\nabla f = \left\langle \frac{d\,D_{\mathbf{u}}\, f - b\,D_{\mathbf{v}}\, f}{ad - bc}, \frac{a\,D_{\mathbf{v}}\, f - c\,D_{\mathbf{u}}\, f}{ad - bc} \right\rangle.$$

70. Since $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = (x_0, y_0)$, by Definition 14.4.7 we have $\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$ where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Now $\Delta z = f(\mathbf{x}) - f(\mathbf{x}_0)$, $\langle \Delta x, \Delta y\rangle = \mathbf{x} - \mathbf{x}_0$ so $(\Delta x, \Delta y) \to (0, 0)$ is equivalent to $\mathbf{x} \to \mathbf{x}_0$ and $\langle f_x(x_0, y_0)\,, f_y(x_0, y_0)\rangle = \nabla f(\mathbf{x}_0)$. Substituting into 14.4.7 gives $f(\mathbf{x}) - f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \langle \varepsilon_1, \varepsilon_2\rangle \cdot \langle \Delta x, \Delta y\rangle$ or $\langle \varepsilon_1, \varepsilon_2\rangle \cdot (\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$,

and so $\dfrac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = \dfrac{\langle \varepsilon_1, \varepsilon_2\rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|}$. But $\dfrac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}$ is a unit vector so

$\displaystyle\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\langle \varepsilon_1, \varepsilon_2\rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$ since $\varepsilon_1, \varepsilon_2 \to 0$ as $\mathbf{x} \to \mathbf{x}_0$. Hence $\displaystyle\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$.

14.7 Maximum and Minimum Values

1. (a) First we compute $D(1, 1) = f_{xx}(1, 1)\, f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1, 1) > 0$ and $f_{xx}(1, 1) > 0$, f has a local minimum at $(1, 1)$ by the Second Derivatives Test.

(b) $D(1, 1) = f_{xx}(1, 1)\, f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1, 1) < 0$, f has a saddle point at $(1, 1)$ by the Second Derivatives Test.

2. (a) $D = g_{xx}(0, 2)\, g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D < 0$, g has a saddle point at $(0, 2)$ by the Second Derivatives Test.

(b) $D = g_{xx}(0, 2)\, g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D > 0$ and $g_{xx}(0, 2) < 0$, g has a local maximum at $(0, 2)$ by the Second Derivatives Test.

(c) $D = g_{xx}(0, 2)\, g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0, 2)$.

3. In the figure, a point at approximately $(1,1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1,1)$. The level curves near $(0,0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x,y)=4+x^3+y^3-3xy \quad\Rightarrow\quad f_x(x,y)=3x^2-3y$, $f_y(x,y)=3y^2-3x$. We have critical points where these partial derivatives are equal to 0: $3x^2-3y=0$, $3y^2-3x=0$. Substituting $y=x^2$ from the first equation into the second equation gives $3(x^2)^2-3x=0 \quad\Rightarrow\quad 3x(x^3-1)=0 \quad\Rightarrow\quad x=0$ or $x=1$. Then we have two critical points, $(0,0)$ and $(1,1)$. The second partial derivatives are $f_{xx}(x,y)=6x$, $f_{xy}(x,y)=-3$, and $f_{yy}(x,y)=6y$, so $D(x,y)=f_{xx}(x,y)\,f_{yy}(x,y)-[f_{xy}(x,y)]^2=(6x)(6y)-(-3)^2=36xy-9$. Then $D(0,0)=36(0)(0)-9=-9$, and $D(1,1)=36(1)(1)-9=27$. Since $D(0,0)<0$, f has a saddle point at $(0,0)$ by the Second Derivatives Test. Since $D(1,1)>0$ and $f_{xx}(1,1)>0$, f has a local minimum at $(1,1)$.

4. In the figure, points at approximately $(-1,1)$ and $(-1,-1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1,\pm1)$. Similarly, the point $(1,0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1,0)$, $(1,1)$, and $(1,-1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

To confirm our predictions, we have $f(x,y)=3x-x^3-2y^2+y^4 \quad\Rightarrow\quad f_x(x,y)=3-3x^2,\;\; f_y(x,y)=-4y+4y^3$. Setting these partial derivatives equal to 0, we have $3-3x^2=0 \quad\Rightarrow\quad x=\pm1$ and $-4y+4y^3=0 \quad\Rightarrow$ $y\big(y^2-1\big)=0 \quad\Rightarrow\quad y=0,\pm1$. So our critical points are $(\pm1,0)$, $(\pm1,1)$, $(\pm1,-1)$.
The second partial derivatives are $f_{xx}(x,y)=-6x$, $f_{xy}(x,y)=0$, and $f_{yy}(x,y)=12y^2-4$, so
$D(x,y)=f_{xx}(x,y)\,f_{yy}(x,y)-[f_{xy}(x,y)]^2=(-6x)(12y^2-4)-(0)^2=-72xy^2+24x$.
We use the Second Derivatives Test to classify the 6 critical points:

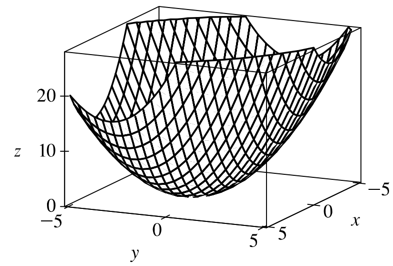
Critical Point	D	f_{xx}	Conclusion
$(1,0)$	24	-6	$D>0,\ f_{xx}<0 \quad\Rightarrow\quad f$ has a local maximum at $(1,0)$
$(1,1)$	-48		$D<0 \quad\Rightarrow\quad f$ has a saddle point at $(1,1)$
$(1,-1)$	-48		$D<0 \quad\Rightarrow\quad f$ has a saddle point at $(1,-1)$
$(-1,0)$	-24		$D<0 \quad\Rightarrow\quad f$ has a saddle point at $(-1,0)$
$(-1,1)$	48	6	$D>0,\ f_{xx}>0 \quad\Rightarrow\quad f$ has a local minimum at $(-1,1)$
$(-1,-1)$	48	6	$D>0,\ f_{xx}>0 \quad\Rightarrow\quad f$ has a local minimum at $(-1,-1)$

5. $f(x,y)=x^2+xy+y^2+y \quad\Rightarrow\quad f_x=2x+y,\;\; f_y=x+2y+1,\;\; f_{xx}=2,\;\; f_{xy}=1,\;\; f_{yy}=2$. Then $f_x=0$ implies $y=-2x$, and substitution into $f_y=x+2y+1=0$ gives $x+2\,(-2x)+1=0 \quad\Rightarrow\quad -3x=-1 \quad\Rightarrow\quad x=\frac{1}{3}$.

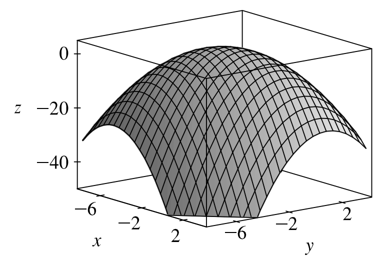
Then $y = -\frac{2}{3}$ and the only critical point is $\left(\frac{1}{3}, -\frac{2}{3}\right)$.

$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = (2)(2) - (1)^2 = 3$, and since

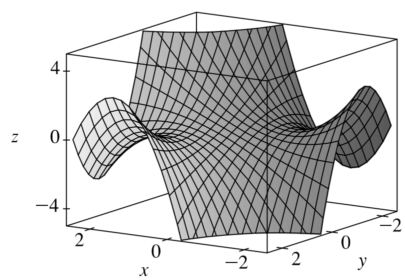
$D\left(\frac{1}{3}, -\frac{2}{3}\right) = 3 > 0$ and $f_{xx}\left(\frac{1}{3}, -\frac{2}{3}\right) = 2 > 0$, $f\left(\frac{1}{3}, -\frac{2}{3}\right) = -\frac{1}{3}$ is a local minimum by the Second Derivatives Test.



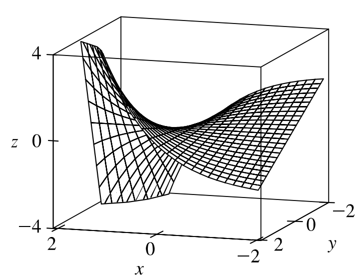
6. $f(x, y) = xy - 2x - 2y - x^2 - y^2 \quad \Rightarrow \quad f_x = y - 2 - 2x$, $f_y = x - 2 - 2y$, $f_{xx} = -2$, $f_{xy} = 1$, $f_{yy} = -2$. Then $f_x = 0$ implies $y = 2x + 2$, and substitution into $f_y = 0$ gives $x - 2 - 2(2x + 2) = 0 \quad \Rightarrow \quad -3x = 6 \quad \Rightarrow \quad x = -2$. Then $y = -2$ and the only critical point is $(-2, -2)$. $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = (-2)(-2) - 1^2 = 3$, and since $D(-2, -2) = 3 > 0$ and $f_{xx}(-2, -2) = -2 < 0$, $f(-2, -2) = 4$ is a local maximum by the Second Derivatives Test.



7. $f(x, y) = (x - y)(1 - xy) = x - y - x^2 y + xy^2 \quad \Rightarrow \quad f_x = 1 - 2xy + y^2$, $f_y = -1 - x^2 + 2xy$, $f_{xx} = -2y$, $f_{xy} = -2x + 2y$, $f_{yy} = 2x$. Then $f_x = 0$ implies $1 - 2xy + y^2 = 0$ and $f_y = 0$ implies $-1 - x^2 + 2xy = 0$. Adding the two equations gives $1 + y^2 - 1 - x^2 = 0 \quad \Rightarrow \quad y^2 = x^2 \quad \Rightarrow \quad y = \pm x$, but if $y = -x$ then $f_x = 0$ implies $1 + 2x^2 + x^2 = 0 \quad \Rightarrow \quad 3x^2 = -1$ which has no real solution. If $y = x$ then substitution into $f_x = 0$ gives $1 - 2x^2 + x^2 = 0 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$, so the critical points are $(1, 1)$ and $(-1, -1)$. Now $D(1, 1) = (-2)(2) - 0^2 = -4 < 0$ and $D(-1, -1) = (2)(-2) - 0^2 = -4 < 0$, so $(1, 1)$ and $(-1, -1)$ are saddle points.



8. $f(x, y) = y(e^x - 1) \quad \Rightarrow \quad f_x = ye^x$, $f_y = e^x - 1$, $f_{xx} = ye^x$, $f_{xy} = e^x$, $f_{yy} = 0$. Because e^x is never zero, $f_x = 0$ only when $y = 0$, and $f_y = 0$ when $e^x = 1 \quad \Rightarrow \quad x = 0$, so the only critical point is $(0, 0)$. $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = (ye^x)(0) - (e^x)^2 = -e^{2x}$, and since $D(0, 0) = -1 < 0$, $(0, 0)$ is a saddle point.



9. $f(x, y) = x^2 + y^4 + 2xy \quad \Rightarrow \quad f_x = 2x + 2y$, $f_y = 4y^3 + 2x$, $f_{xx} = 2$, $f_{xy} = 2$, $f_{yy} = 12y^2$. Then $f_x = 0$ implies $y = -x$, and substitution into $f_y = 4y^3 + 2x = 0$ gives $-4x^3 + 2x = 0 \quad \Rightarrow \quad 2x\left(1 - 2x^2\right) = 0 \quad \Rightarrow \quad x = 0$ or $x = \pm\frac{1}{\sqrt{2}}$. Thus the critical points are $(0, 0)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Now

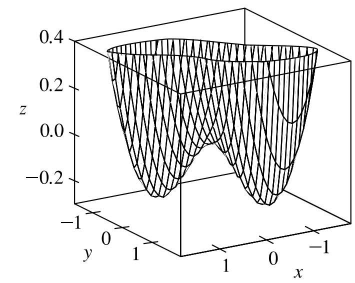
$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (2)(12y^2) - (2)^2 = 24y^2 - 4$,

so $D(0,0) = -4 < 0$ and $(0,0)$ is a saddle point.

$D\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = D\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 24\left(\frac{1}{2}\right) - 4 = 8 > 0$ and

$f_{xx}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f_{xx}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 > 0$, so $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$

and $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$ are local minima.



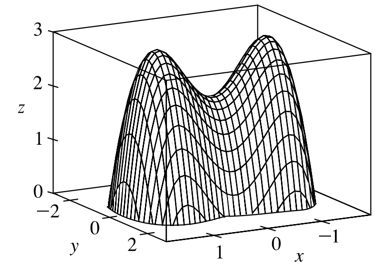
10. $f(x,y) = 2 - x^4 + 2x^2 - y^2 \;\Rightarrow\; f_x = -4x^3 + 4x,\; f_y = -2y,\; f_{xx} = -12x^2 + 4,\; f_{xy} = 0,\; f_{yy} = -2$. Then $f_x = 0$ implies $-4x(x^2 - 1) = 0$, so $x = 0$ or $x = \pm 1$, and $f_y = 0$ implies $y = 0$. Thus the critical points are $(0,0)$, $(\pm 1, 0)$.

$D(0,0) = (4)(-2) - 0^2 = -8 < 0$, so $(0,0)$ is a saddle point.

$D(1,0) = D(-1,0) = (-8)(-2) - (0)^2 = 16 > 0$, and

$f_{xx}(1,0) = f_{xx}(-1,0) = -8 < 0$, so $f(1,0) = 3$ and $f(-1,0) = 3$

are local maxima.

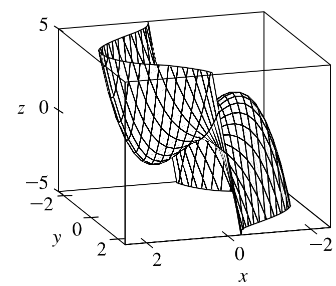


11. $f(x,y) = x^3 - 3x + 3xy^2 \;\Rightarrow\; f_x = 3x^2 - 3 + 3y^2,\; f_y = 6xy,\; f_{xx} = 6x,\; f_{xy} = 6y,\; f_{yy} = 6x$. Then $f_y = 0$ implies $x = 0$ or $y = 0$. If $x = 0$, substitution into $f_x = 0$ gives $3y^2 = 3 \;\Rightarrow\; y = \pm 1$, and if $y = 0$, substitution into $f_x = 0$ gives $x = \pm 1$. Thus the critical points are $(0, \pm 1)$ and $(\pm 1, 0)$.

$D(0, \pm 1) = 0 - 36 < 0$, so $(0, \pm 1)$ are saddle points.

$D(\pm 1, 0) = 36 - 0 > 0$, $f_{xx}(1,0) = 6 > 0$, and $f_{xx}(-1,0) = -6 < 0$,

so $f(1,0) = -2$ is a local minimum and $f(-1,0) = 2$ is a local maximum.



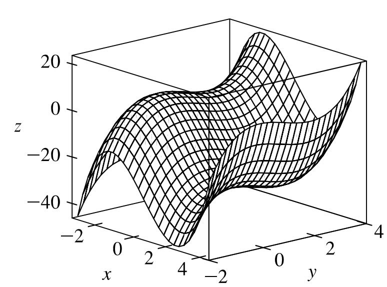
12. $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x \;\Rightarrow\; f_x = 3x^2 - 6x - 9,\; f_y = 3y^2 - 6y,\; f_{xx} = 6x - 6,\; f_{xy} = 0,\; f_{yy} = 6y - 6$. Then $f_x = 0$ implies $3(x+1)(x-3) = 0 \;\Rightarrow\; x = -1$ or $x = 3$, and $f_y = 0$ implies $3y(y-2) = 0 \;\Rightarrow\; y = 0$ or $y = 2$. Thus the critical points are $(-1,0)$, $(-1,2)$, $(3,0)$, and $(3,2)$. $D(-1,2) = (-12)(6) - (0)^2 = -72 < 0$ and

$D(3,0) = (12)(-6) - (0)^2 = -72 < 0$, so $(-1,2)$ and $(3,0)$ are saddle points. $D(-1,0) = (-12)(-6) - (0)^2 = 72 > 0$ and

$f_{xx}(-1,0) = -12 < 0$, so $f(-1,0) = 5$ is a local maximum.

$D(3,2) = (12)(6) - (0)^2 = 72 > 0$ and $f_{xx}(3,2) = 12 > 0$, so

$f(3,2) = -31$ is a local minimum.



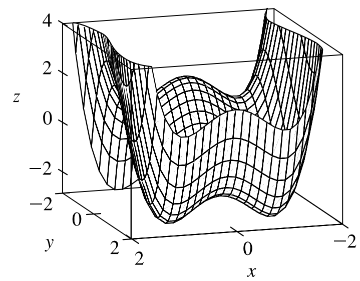
13. $f(x,y) = x^4 - 2x^2 + y^3 - 3y \quad\Rightarrow\quad f_x = 4x^3 - 4x,\ f_y = 3y^2 - 3,\ f_{xx} = 12x^2 - 4,\ f_{xy} = 0,\ f_{yy} = 6y.$

Then $f_x = 0$ implies $4x(x^2 - 1) = 0 \quad\Rightarrow\quad x = 0$ or $x = \pm 1$, and $f_y = 0$ implies $3(y^2 - 1) = 0 \quad\Rightarrow\quad y = \pm 1$.

Thus there are six critical points: $(0, \pm 1)$, $(\pm 1, 1)$, and $(\pm 1, -1)$.

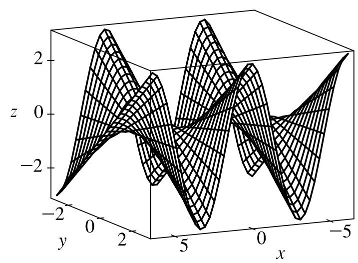
$D(0,1) = (-4)(6) - (0)^2 = -24 < 0$ and

$D(\pm 1, -1) = (8)(-6) = -48 < 0$, so $(0,1)$ and $(\pm 1, -1)$ are saddle points. $D(0,-1) = (-4)(-6) = 24 > 0$ and $f_{xx}(0,-1) = -4 < 0$, so $f(0,-1) = 2$ is a local maximum. $D(\pm 1, 1) = (8)(6) = 48 > 0$ and $f_{xx}(\pm 1, 1) = 8 > 0$, so $f(\pm 1, 1) = -3$ are local minima.



14. $f(x,y) = y\cos x \quad\Rightarrow\quad f_x = -y\sin x$, $f_y = \cos x$, $f_{xx} = -y\cos x$, $f_{xy} = -\sin x$, $f_{yy} = 0$. Then $f_y = 0$ if and only if $x = \frac{\pi}{2} + n\pi$ for n an integer. But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so $f_x = 0 \quad\Rightarrow\quad y = 0$ and the critical points are $\left(\frac{\pi}{2} + n\pi, 0\right)$, n an integer.

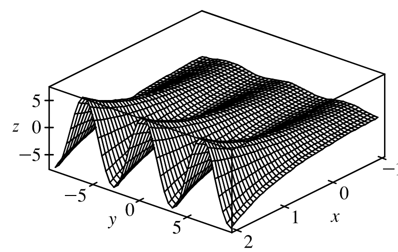
$D\left(\frac{\pi}{2} + n\pi, 0\right) = (0)(0) - (\pm 1)^2 = -1 < 0$, so each critical point is a saddle point.



15. $f(x,y) = e^x\cos y \quad\Rightarrow\quad f_x = e^x\cos y,\ f_y = -e^x\sin y.$

Now $f_x = 0$ implies $\cos y = 0$ or $y = \frac{\pi}{2} + n\pi$ for n an integer.

But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so there are no critical points.



16. $f(x,y) = xye^{-(x^2+y^2)/2} \quad\Rightarrow\quad f_x = xy\cdot e^{-(x^2+y^2)/2}(-x) + e^{-(x^2+y^2)/2}\cdot y = y(1-x^2)e^{-(x^2+y^2)/2}$,

$f_y = xy\cdot e^{-(x^2+y^2)/2}(-y) + e^{-(x^2+y^2)/2}\cdot x = x(1-y^2)e^{-(x^2+y^2)/2}$,

$f_{xx} = y\left[(1-x^2)\cdot e^{-(x^2+y^2)/2}(-x) + e^{-(x^2+y^2)/2}(-2x)\right] = xy(x^2-3)e^{-(x^2+y^2)/2}$,

$f_{xy} = (1-x^2)\left[y\cdot e^{-(x^2+y^2)/2}(-y) + e^{-(x^2+y^2)/2}(1)\right] = (1-x^2)(1-y^2)e^{-(x^2+y^2)/2}$,

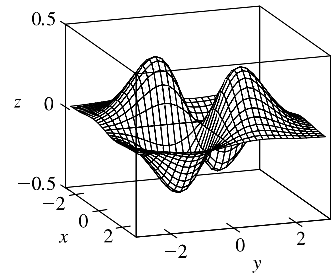
$f_{yy} = x\left[(1-y^2)\cdot e^{-(x^2+y^2)/2}(-y) + e^{-(x^2+y^2)/2}(-2y)\right] = xy(y^2-3)e^{-(x^2+y^2)/2}$.

Then $f_x = 0$ implies $y(1-x^2) = 0 \quad\Rightarrow\quad y = 0$ or $x = \pm 1$. Substituting $y = 0$ into $f_y = 0$ gives $xe^{-x^2/2} = 0 \quad\Rightarrow$ $x = 0$, and substituting $x = \pm 1$ into $f_y = 0$ gives $\pm(1-y^2)e^{-(1+y^2)/2} = 0 \quad\Rightarrow\quad y = \pm 1$, so the critical points are $(0,0)$, $(1, \pm 1)$, and $(-1, \pm 1)$. $D(0,0) = (0)(0) - (1)^2 = -1 < 0$, so $(0,0)$ is a saddle point.

[continued]

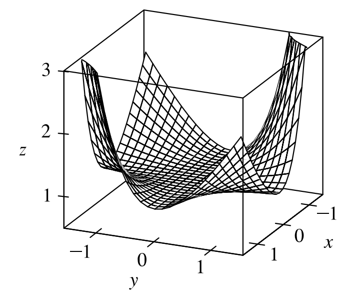
$D(1,1) = D(-1,-1) = (-2e^{-1})(-2e^{-1}) - (0)^2 = 4e^{-2} > 0$ and $f_{xx}(1,1) = f_{xx}(-1,-1) = -2e^{-1} < 0$, so $f(1,1) = f(-1,-1) = e^{-1}$ are local maxima.

$D(1,-1) = D(-1,1) = (2e^{-1})(2e^{-1}) - (0)^2 = 4e^{-2} > 0$ and $f_{xx}(1,-1) = f_{xx}(-1,1) = 2e^{-1} > 0$, so $f(1,-1) = f(-1,1) = -e^{-1}$ are local minima.



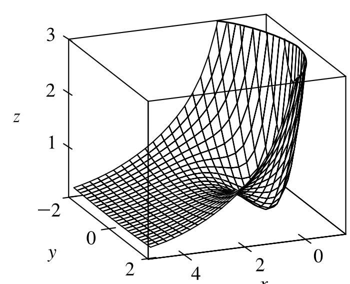
17. $f(x,y) = xy + e^{-xy} \quad\Rightarrow\quad f_x = y - ye^{-xy},\ f_y = x - xe^{-xy},\ f_{xx} = y^2e^{-xy}$, $f_{xy} = 1 - \left[y(-xe^{-xy}) + e^{-xy}(1)\right] = 1 + (xy-1)e^{-xy},\ f_{yy} = x^2e^{-xy}$. Then $f_x = 0$ implies $y(1 - e^{-xy}) = 0 \quad\Rightarrow$ $y = 0$ or $e^{-xy} = 1 \quad\Rightarrow\quad x = 0$ or $y = 0$. If $x = 0$ then $f_y = 0$ for any y-value, so all points of the form $(0, y_0)$ are critical points. If $y = 0$, then $f_y = x - xe^0 = 0$ for any x-value, so all points of the form $(x_0, 0)$ are critical points. We have $D(x_0,0) = (0)(x_0^2) - (0)^2 = 0$ and $D(0,y_0) = (y_0^2)(0) - (0)^2 = 0$, so the Second Derivatives Test gives no information.

Notice that if we let $t = xy$, then $f(x,y) = g(t) = t + e^{-t} \quad\Rightarrow$ $g'(t) = 1 - e^{-t}$. Now $g'(t) = 0$ only for $t = 0$, and $g'(t) < 0$ for $t < 0$, $g'(t) > 0$ for $t > 0$. Thus $g(0) = 1$ is a local and absolute minimum, so $f(x,y) = xy + e^{-xy} \geq 1$ for all (x,y) with equality if and only if $x = 0$ or $y = 0$. Hence all points on the x- and y-axes are local (and absolute) minima, where $f(x,y) = 1$.

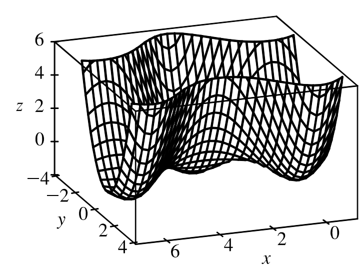


18. $f(x,y) = (x^2+y^2)e^{-x} \quad\Rightarrow\quad f_x = (x^2+y^2)(-e^{-x}) + e^{-x}(2x) = (2x - x^2 - y^2)e^{-x},\quad f_y = 2ye^{-x}$, $f_{xx} = (2x - x^2 - y^2)(-e^{-x}) + e^{-x}(2 - 2x) = (x^2 + y^2 - 4x + 2)e^{-x},\quad f_{xy} = -2ye^{-x},\quad f_{yy} = 2e^{-x}$. Then $f_y = 0$ implies $y = 0$ and substituting into $f_x = 0$ gives $(2x - x^2)e^{-x} = 0 \quad\Rightarrow$ $x(2-x) = 0 \quad\Rightarrow\quad x = 0$ or $x = 2$, so the critical points are $(0,0)$ and $(2,0)$. $D(0,0) = (2)(2) - (0)^2 = 4 > 0$ and $f_{xx}(0,0) = 2 > 0$, so $f(0,0) = 0$ is a local minimum.

$D(2,0) = (-2e^{-2})(2e^{-2}) - (0)^2 = -4e^{-4} < 0$ so $(2,0)$ is a saddle point.



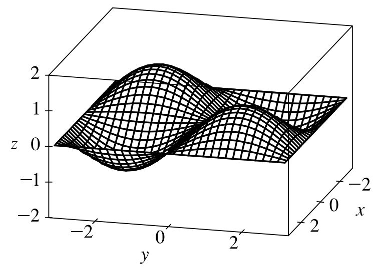
19. $f(x,y) = y^2 - 2y\cos x \quad\Rightarrow\quad f_x = 2y\sin x,\ f_y = 2y - 2\cos x$, $f_{xx} = 2y\cos x,\ f_{xy} = 2\sin x,\ f_{yy} = 2$. Then $f_x = 0$ implies $y = 0$ or $\sin x = 0 \quad\Rightarrow\quad x = 0, \pi$, or 2π for $-1 \leq x \leq 7$. Substituting $y = 0$ into $f_y = 0$ gives $\cos x = 0 \quad\Rightarrow\quad x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, substituting $x = 0$ or $x = 2\pi$ into $f_y = 0$ gives $y = 1$, and substituting $x = \pi$ into $f_y = 0$ gives $y = -1$. Thus the critical points are $(0,1)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, -1)$, $\left(\frac{3\pi}{2}, 0\right)$, and $(2\pi, 1)$.



$D\left(\frac{\pi}{2},0\right) = D\left(\frac{3\pi}{2},0\right) = -4 < 0$ so $\left(\frac{\pi}{2},0\right)$ and $\left(\frac{3\pi}{2},0\right)$ are saddle points. $D(0,1) = D(\pi,-1) = D(2\pi,1) = 4 > 0$ and $f_{xx}(0,1) = f_{xx}(\pi,-1) = f_{xx}(2\pi,1) = 2 > 0$, so $f(0,1) = f(\pi,-1) = f(2\pi,1) = -1$ are local minima.

20. $f(x,y) = \sin x\,\sin y \quad\Rightarrow\quad f_x = \cos x\,\sin y$, $f_y = \sin x\,\cos y$, $f_{xx} = -\sin x\,\sin y$, $f_{xy} = \cos x\,\cos y$, $f_{yy} = -\sin x\,\sin y$. Here we have $-\pi < x < \pi$ and $-\pi < y < \pi$, so $f_x = 0$ implies $\cos x = 0$ or $\sin y = 0$. If $\cos x = 0$ then $x = -\frac{\pi}{2}$ or $\frac{\pi}{2}$, and if $\sin y = 0$ then $y = 0$. Substituting $x = \pm\frac{\pi}{2}$ into $f_y = 0$ gives $\cos y = 0 \quad\Rightarrow\quad y = -\frac{\pi}{2}$ or $\frac{\pi}{2}$, and substituting $y = 0$ into $f_y = 0$ gives $\sin x = 0 \quad\Rightarrow\quad x = 0$. Thus the critical points are $\left(-\frac{\pi}{2},\pm\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2},\pm\frac{\pi}{2}\right)$, and $(0,0)$.

$D(0,0) = -1 < 0$ so $(0,0)$ is a saddle point.
$D\left(-\frac{\pi}{2},\pm\frac{\pi}{2}\right) = D\left(\frac{\pi}{2},\pm\frac{\pi}{2}\right) = 1 > 0$ and
$f_{xx}\left(-\frac{\pi}{2},-\frac{\pi}{2}\right) = f_{xx}\left(\frac{\pi}{2},\frac{\pi}{2}\right) = -1 < 0$ while
$f_{xx}\left(-\frac{\pi}{2},\frac{\pi}{2}\right) = f_{xx}\left(\frac{\pi}{2},-\frac{\pi}{2}\right) = 1 > 0$, so $f\left(-\frac{\pi}{2},-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2},\frac{\pi}{2}\right) = 1$
are local maxima and $f\left(-\frac{\pi}{2},\frac{\pi}{2}\right) = f\left(\frac{\pi}{2},-\frac{\pi}{2}\right) = 1$ are local minima.



21. $f(x,y) = x^2 + 4y^2 - 4xy + 2 \quad\Rightarrow\quad f_x = 2x - 4y$, $f_y = 8y - 4x$, $f_{xx} = 2$, $f_{xy} = -4$, $f_{yy} = 8$. Then $f_x = 0$ and $f_y = 0$ each implies $y = \frac{1}{2}x$, so all points of the form $\left(x_0, \frac{1}{2}x_0\right)$ are critical points and for each of these we have $D\left(x_0, \frac{1}{2}x_0\right) = (2)(8) - (-4)^2 = 0$. The Second Derivatives Test gives no information, but $f(x,y) = x^2 + 4y^2 - 4xy + 2 = (x-2y)^2 + 2 \geq 2$ with equality if and only if $y = \frac{1}{2}x$. Thus $f\left(x_0, \frac{1}{2}x_0\right) = 2$ are all local (and absolute) minima.

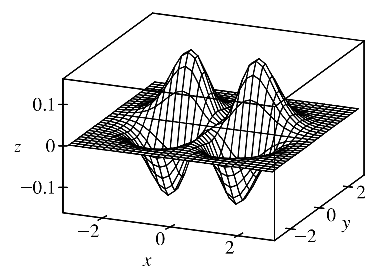
22. $f(x,y) = x^2 y e^{-x^2-y^2} \quad\Rightarrow$

$f_x = x^2 y e^{-x^2-y^2}(-2x) + 2xye^{-x^2-y^2} = 2xy(1-x^2)e^{-x^2-y^2}$,

$f_y = x^2 y e^{-x^2-y^2}(-2y) + x^2 e^{-x^2-y^2} = x^2(1-2y^2)e^{-x^2-y^2}$,

$f_{xx} = 2y(2x^4 - 5x^2 + 1)e^{-x^2-y^2}$,

$f_{xy} = 2x(1-x^2)(1-2y^2)e^{-x^2-y^2}$, $f_{yy} = 2x^2y(2y^2-3)e^{-x^2-y^2}$.



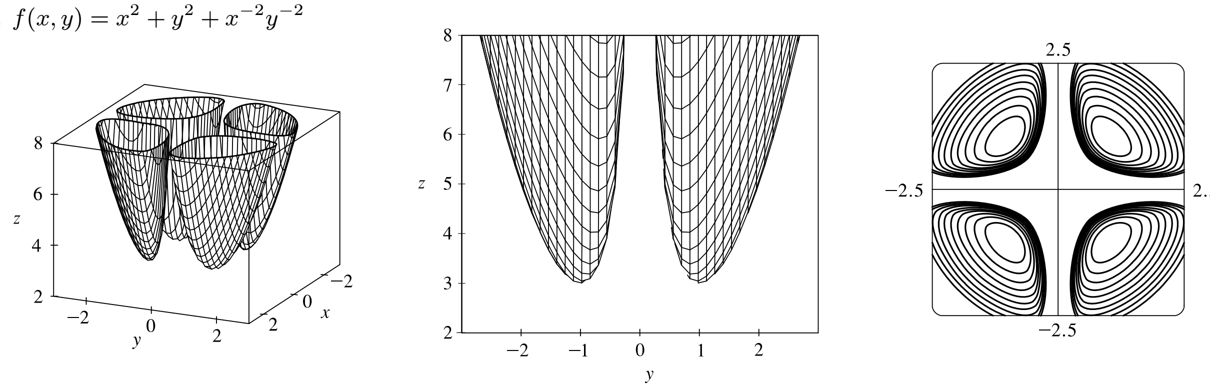
$f_x = 0$ implies $x = 0$, $y = 0$, or $x = \pm 1$. If $x = 0$ then $f_y = 0$ for any y-value, so all points of the form $(0,y)$ are critical points. If $y = 0$ then $f_y = 0 \quad\Rightarrow\quad x^2 e^{-x^2} = 0 \quad\Rightarrow\quad x = 0$, so $(0,0)$ (already included above) is a critical point. If $x = \pm 1$ then $(1-2y^2)e^{-1-y^2} = 0 \quad\Rightarrow\quad y = \pm\frac{1}{\sqrt{2}}$, so $\left(\pm 1, \frac{1}{\sqrt{2}}\right)$ and $\left(\pm 1, -\frac{1}{\sqrt{2}}\right)$ are critical points. Now

$D\left(\pm 1, \frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0$, $f_{xx}\left(\pm 1, \frac{1}{\sqrt{2}}\right) = -2\sqrt{2}\,e^{-3/2} < 0$ and $D\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0$,

$f_{xx}\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}\,e^{-3/2} > 0$, so $f\left(\pm 1, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-3/2}$ are local maximum points while

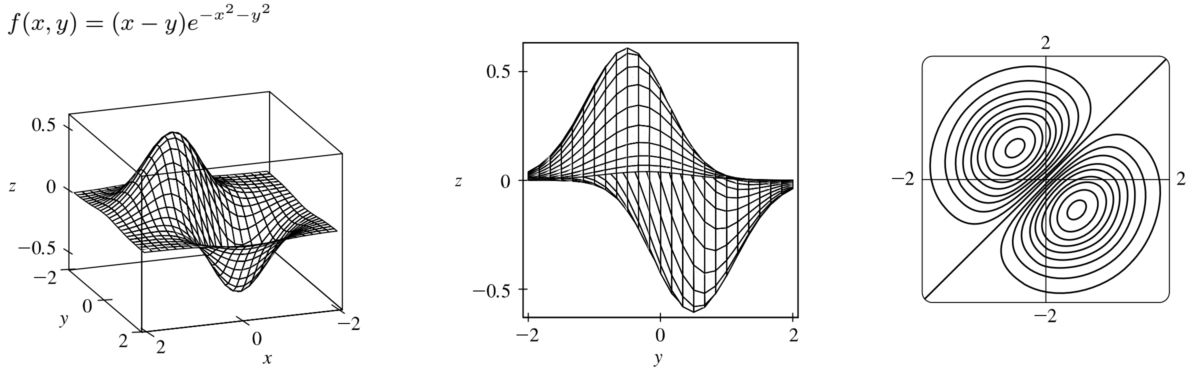
$f\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}e^{-3/2}$ are local minimum points. At all critical points $(0,y)$ we have $D(0,y) = 0$, so the Second Derivatives Test gives no information. However, if $y > 0$ then $x^2 y e^{-x^2-y^2} \geq 0$ with equality only when $x = 0$, so we have local minimum values $f(0,y) = 0$, $y > 0$. Similarly, if $y < 0$ then $x^2 y e^{-x^2-y^2} \leq 0$ with equality when $x = 0$ so $f(0,y) = 0$, $y < 0$ are local maximum values, and $(0,0)$ is a saddle point.

23. $f(x,y) = x^2 + y^2 + x^{-2}y^{-2}$



From the graphs, there appear to be local minima of about $f(1,\pm 1) = f(-1,\pm 1) \approx 3$ (and no local maxima or saddle points). $f_x = 2x - 2x^{-3}y^{-2}$, $f_y = 2y - 2x^{-2}y^{-3}$, $f_{xx} = 2 + 6x^{-4}y^{-2}$, $f_{xy} = 4x^{-3}y^{-3}$, $f_{yy} = 2 + 6x^{-2}y^{-4}$. Then $f_x = 0$ implies $2x^4y^2 - 2 = 0$ or $x^4y^2 = 1$ or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $f_y = 0$ implies $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies $2x^{-6} - 2 = 0$ or $x^6 = 1$. Thus $x = \pm 1$ and if $x = 1$, $y = \pm 1$; if $x = -1$, $y = \pm 1$. So the critical points are $(1,1)$, $(1,-1)$,$(-1,1)$ and $(-1,-1)$. Now $D(1,\pm 1) = D(-1,\pm 1) = 64 - 16 > 0$ and $f_{xx} > 0$ always, so $f(1,\pm 1) = f(-1,\pm 1) = 3$ are local minima.

24. $f(x,y) = (x-y)e^{-x^2-y^2}$



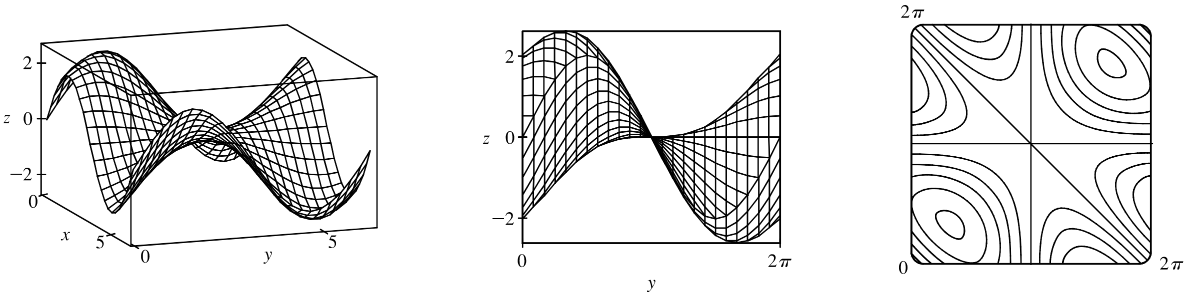
From the graphs, there appears to be a local maximum of about $f(0.5,-0.5) \approx 0.6$ and a local minimum of about $f(-0.5, 0.5) \approx -0.6$.

$f_x = (x-y)e^{-x^2-y^2}(-2x) + e^{-x^2-y^2}(1) = e^{-x^2-y^2}(1 - 2x^2 + 2xy)$,

$f_y = (x-y)e^{-x^2-y^2}(-2y) + e^{-x^2-y^2}(-1) = -e^{-x^2-y^2}(1 - 2y^2 + 2xy)$, $f_{xx} = 2e^{-x^2-y^2}(2x^3 - 3x + y - 2x^2y)$,

$f_{xy} = 2e^{-x^2-y^2}(x - y + 2x^2y - 2xy^2)$, $f_{yy} = -2e^{-x^2-y^2}(2y^3 - 3y + x - 2xy^2)$. Then $f_x = 0$ implies $1 - 2x^2 + 2xy = 0$ and $f_y = 0$ implies $1 - 2y^2 + 2xy = 0$. Subtracting these two equations gives $-2x^2 + 2y^2 = 0 \quad \Rightarrow \quad y = \pm x$. If $y = x$ then substituting into $f_x = 0$ gives $1 - 2x^2 + 2x^2 = 0$, an impossibility. Substituting $y = -x$ gives $1 - 2x^2 - 2x^2 = 0 \quad \Rightarrow \quad x^2 = \frac{1}{4} \quad \Rightarrow \quad x = \pm\frac{1}{2}$. Thus the critical points are $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Now $D\left(\frac{1}{2}, -\frac{1}{2}\right) = (-3e^{-1/2})(-3e^{-1/2}) - (e^{-1/2})^2 = 8e^{-1} > 0$ with $f_{xx}\left(\frac{1}{2}, -\frac{1}{2}\right) = -3e^{-1/2} < 0$, so $f\left(\frac{1}{2}, -\frac{1}{2}\right) = e^{-1/2} \approx 0.607$ is a local maximum, and $D\left(-\frac{1}{2}, \frac{1}{2}\right) = (3e^{-1/2})(3e^{-1/2}) - (-e^{-1/2})^2 = 8e^{-1} > 0$ with $f_{xx}\left(-\frac{1}{2}, \frac{1}{2}\right) = 3e^{-1/2} > 0$, so $f\left(-\frac{1}{2}, \frac{1}{2}\right) = -e^{-1/2} \approx -0.607$ is a local minimum.

25. $f(x,y) = \sin x + \sin y + \sin(x+y),\ 0 \le x \le 2\pi,\ 0 \le y \le 2\pi$

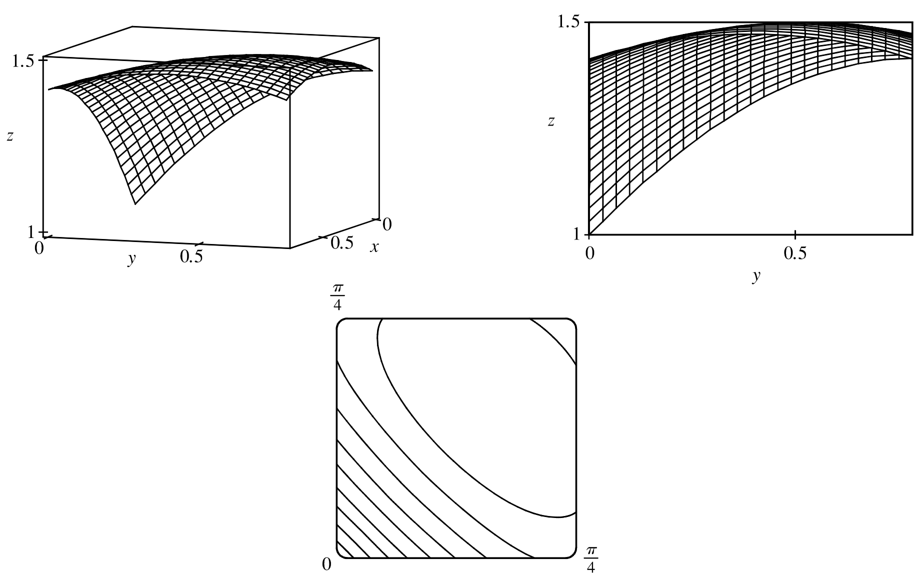


From the graphs it appears that f has a local maximum at about $(1,1)$ with value approximately 2.6, a local minimum at about $(5,5)$ with value approximately -2.6, and a saddle point at about $(3,3)$.

$f_x = \cos x + \cos(x+y),\ f_y = \cos y + \cos(x+y),\ f_{xx} = -\sin x - \sin(x+y),\ f_{yy} = -\sin y - \sin(x+y),$ $f_{xy} = -\sin(x+y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x - \cos y = 0$ or $\cos x = \cos y$. Thus $x = y$ or $x = 2\pi - y$. If $x = y$, $f_x = 0$ becomes $\cos x + \cos 2x = 0$ or $2\cos^2 x + \cos x - 1 = 0$, a quadratic in $\cos x$. Thus $\cos x = -1$ or $\frac{1}{2}$ and $x = \pi$, $\frac{\pi}{3}$, or $\frac{5\pi}{3}$, giving the critical points (π,π), $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)$. Similarly if $x = 2\pi - y$, $f_x = 0$ becomes $(\cos x) + 1 = 0$ and the resulting critical point is (π,π). Now $D(x,y) = \sin x\ \sin y + \sin x\ \sin(x+y) + \sin y\ \sin(x+y)$. So $D(\pi,\pi) = 0$ and the Second Derivatives Test doesn't apply. However, along the line $y = x$ we have $f(x,x) = 2\sin x + \sin 2x = 2\sin x + 2\sin x\ \cos x = 2\sin x(1+\cos x)$, and $f(x,x) > 0$ for $0 < x < \pi$ while $f(x,x) < 0$ for $\pi < x < 2\pi$. Thus every disk with center (π,π) contains points where f is positive as well as points where f is negative, so the graph crosses its tangent plane $(z=0)$ there and (π,π) is a saddle point.

$D\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{\pi}{3},\frac{\pi}{3}\right) < 0$ so $f\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$ is a local maximum while $D\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) > 0$, so $f\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$ is a local minimum.

26. $f(x,y) = \sin x + \sin y + \cos(x+y),\ 0 \le x \le \frac{\pi}{4},\ 0 \le y \le \frac{\pi}{4}$

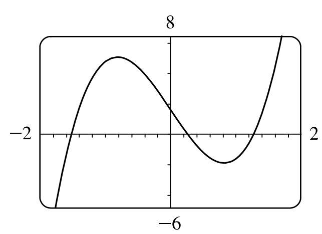


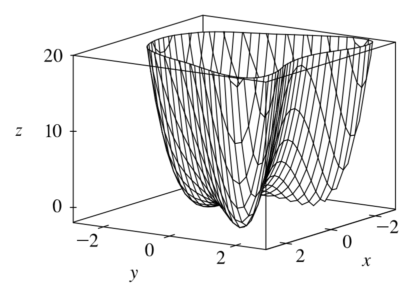
[continued]

From the graphs, it seems that f has a local maximum at about $(0.5, 0.5)$.

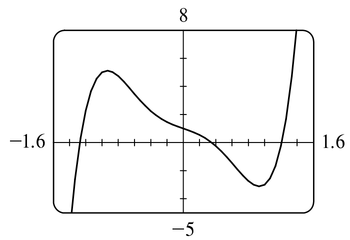
$f_x = \cos x - \sin(x+y)$, $f_y = \cos y - \sin(x+y)$, $f_{xx} = -\sin x - \cos(x+y)$, $f_{yy} = -\sin y - \cos(x+y)$, $f_{xy} = -\cos(x+y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x = \cos y$. Thus $x = y$. Substituting $x = y$ into $f_x = 0$ gives $\cos x - \sin 2x = 0$ or $\cos x(1 - 2\sin x) = 0$. But $\cos x \neq 0$ for $0 \leq x \leq \frac{\pi}{4}$ and $1 - 2\sin x = 0$ implies $x = \frac{\pi}{6}$, so the only critical point is $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. Here $f_{xx}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = -1 < 0$ and $D\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = (-1)^2 - \frac{1}{4} > 0$. Thus $f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{3}{2}$ is a local maximum.

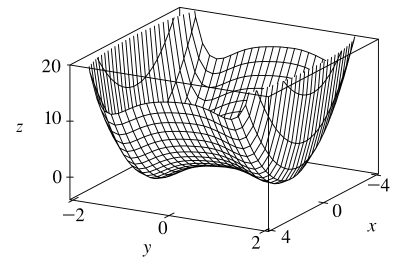
27. $f(x,y) = x^4 + y^4 - 4x^2y + 2y \quad\Rightarrow\quad f_x(x,y) = 4x^3 - 8xy$ and $f_y(x,y) = 4y^3 - 4x^2 + 2$. $f_x = 0 \quad\Rightarrow$ $4x(x^2 - 2y) = 0$, so $x = 0$ or $x^2 = 2y$. If $x = 0$ then substitution into $f_y = 0$ gives $4y^3 = -2 \quad\Rightarrow\quad y = -\frac{1}{\sqrt[3]{2}}$, so $\left(0, -\frac{1}{\sqrt[3]{2}}\right)$ is a critical point. Substituting $x^2 = 2y$ into $f_y = 0$ gives $4y^3 - 8y + 2 = 0$. Using a graph, solutions are approximately $y = -1.526$, 0.259, and 1.267. (Alternatively, we could have used a calculator or a CAS to find these roots.) We have $x^2 = 2y \quad\Rightarrow\quad x = \pm\sqrt{2y}$, so $y = -1.526$ gives no real-valued solution for x, but $y = 0.259 \quad\Rightarrow\quad x \approx \pm 0.720$ and $y = 1.267 \quad\Rightarrow\quad x \approx \pm 1.592$. Thus to three decimal places, the critical points are $\left(0, -\frac{1}{\sqrt[3]{2}}\right) \approx (0, -0.794)$, $(\pm 0.720, 0.259)$, and $(\pm 1.592, 1.267)$. Now since $f_{xx} = 12x^2 - 8y$, $f_{xy} = -8x$, $f_{yy} = 12y^2$, and $D = (12x^2 - 8y)(12y^2) - 64x^2$, we have $D(0, -0.794) > 0$, $f_{xx}(0, -0.794) > 0$, $D(\pm 0.720, 0.259) < 0$, $D(\pm 1.592, 1.267) > 0$, and $f_{xx}(\pm 1.592, 1.267) > 0$. Therefore $f(0, -0.794) \approx -1.191$ and $f(\pm 1.592, 1.267) \approx -1.310$ are local minima, and $(\pm 0.720, 0.259)$ are saddle points. There is no highest point on the graph, but the lowest points are approximately $(\pm 1.592, 1.267, -1.310)$.



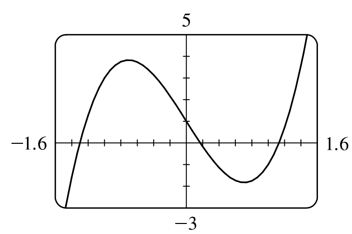


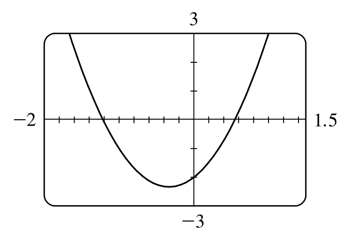
28. $f(x,y) = y^6 - 2y^4 + x^2 - y^2 + y \quad\Rightarrow\quad f_x(x,y) = 2x$ and $f_y(x,y) = 6y^5 - 8y^3 - 2y + 1$. $f_x = 0$ implies $x = 0$, and the graph of f_y shows that the roots of $f_y = 0$ are approximately $y = -1.273$, 0.347, and 1.211. (Alternatively, we could have found the roots of $f_y = 0$ directly, using a calculator or CAS.) So to three decimal places, the critical points are $(0, -1.273)$, $(0, 0.347)$, and $(0, 1.211)$. Now since $f_{xx} = 2$, $f_{xy} = 0$, $f_{yy} = 30y^4 - 24y^2 - 2$, and $D = 60y^4 - 48y^2 - 4$, we have $D(0, -1.273) > 0$, $f_{xx}(0, -1.273) > 0$, $D(0, 0.347) < 0$, $D(0, 1.211) > 0$, and $f_{xx}(0, 1.211) > 0$, so $f(0, -1.273) \approx -3.890$ and $f(0, 1.211) \approx -1.403$ are local minima, and $(0, 0.347)$ is a saddle point. The lowest point on the graph is approximately $(0, -1.273, -3.890)$.

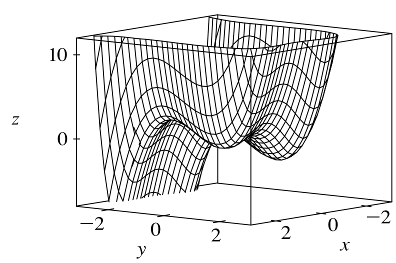




29. $f(x,y) = x^4 + y^3 - 3x^2 + y^2 + x - 2y + 1 \quad \Rightarrow \quad f_x(x,y) = 4x^3 - 6x + 1$ and $f_y(x,y) = 3y^2 + 2y - 2$. From the graphs, we see that to three decimal places, $f_x = 0$ when $x \approx -1.301$, 0.170, or 1.131, and $f_y = 0$ when $y \approx -1.215$ or x 0.549. (Alternatively, we could have used a calculator or a CAS to find these roots. We could also use the quadratic formula to find the solutions of $f_y = 0$.) So, to three decimal places, f has critical points at $(-1.301, -1.215)$, $(-1.301, 0.549)$, $(0.170, -1.215)$, $(0.170, 0.549)$, $(1.131, -1.215)$, and $(1.131, 0.549)$. Now since $f_{xx} = 12x^2 - 6$, $f_{xy} = 0$, $f_{yy} = 6y + 2$, and $D = (12x^2 - 6)(6y + 2)$, we have $D(-1.301, -1.215) < 0$, $D(-1.301, 0.549) > 0$, $f_{xx}(-1.301, 0.549) > 0$, $D(0.170, -1.215) > 0$, $f_{xx}(0.170, -1.215) < 0$, $D(0.170, 0.549) < 0$, $D(1.131, -1.215) < 0$, $D(1.131, 0.549) > 0$, and $f_{xx}(1.131, 0.549) > 0$. Therefore, to three decimal places, $f(-1.301, 0.549) \approx -3.145$ and $f(1.131, 0.549) \approx -0.701$ are local minima, $f(0.170, -1.215) \approx 3.197$ is a local maximum, and $(-1.301, -1.215)$, $(0.170, 0.549)$, and $(1.131, -1.215)$ are saddle points. There is no highest or lowest point on the graph.







30. $f(x,y) = 20e^{-x^2-y^2} \sin 3x \cos 3y \quad \Rightarrow$

$$\begin{aligned} f_x(x,y) &= 20 \cos 3y \left[e^{-x^2-y^2} (3 \cos 3x) + (\sin 3x) e^{-x^2-y^2} (-2x) \right] \\ &= 20e^{-x^2-y^2} \cos 3y \, (3 \cos 3x - 2x \sin 3x) \\ f_y(x,y) &= 20 \sin 3x \left[e^{-x^2-y^2} (-3 \sin 3y) + (\cos 3y) e^{-x^2-y^2} (-2y) \right] \\ &= -20e^{-x^2-y^2} \sin 3x \, (3 \sin 3y + 2y \cos 3y) \end{aligned}$$

Now $f_x = 0$ implies $\cos 3y = 0$ or $3 \cos 3x - 2x \sin 3x = 0$. For $|y| \le 1$, the solutions to $\cos 3y = 0$ are $y = \pm\frac{\pi}{6} \approx \pm 0.524$. Using a graph (or a calculator or CAS), we estimate the roots of $3 \cos 3x - 2x \sin 3x$ for $|x| \le 1$ to be

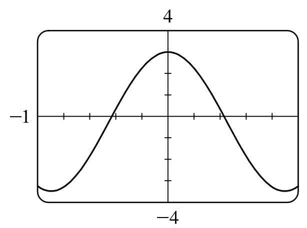
$x \approx \pm 0.430$. $f_y = 0$ implies $\sin 3x = 0$, so $x = 0$, or $3\sin 3y + 2y\cos 3y = 0$. From a graph (or calculator or CAS), the roots of $3\sin 3y + 2y\cos 3y$ between -1 and 1 are approximately 0 and ± 0.872. So to three decimal places, f has critical points at $(\pm 0.430, 0)$, $(0.430, \pm 0.872)$, $(-0.430, \pm 0.872)$, and $(0, \pm 0.524)$. Now

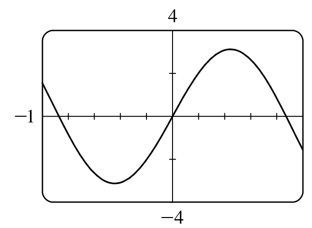
$$f_{xx} = 20e^{-x^2-y^2}\cos 3y[(4x^2 - 11)\sin 3x - 12x\cos 3x]$$

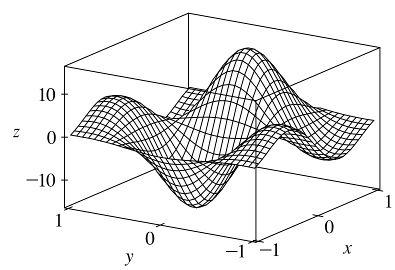
$$f_{xy} = -20e^{-x^2-y^2}(3\cos 3x - 2x\sin 3x)(3\sin 3y + 2y\cos 3y)$$

$$f_{yy} = 20e^{-x^2-y^2}\sin 3x[(4y^2 - 11)\cos 3y - 12y\sin 3y]$$

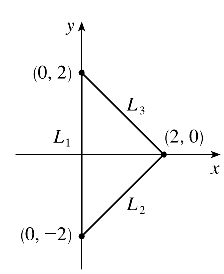
and $D = f_{xx}f_{yy} - f_{xy}^2$. Then $D(\pm 0.430, 0) > 0$, $f_{xx}(0.430, 0) < 0$, $f_{xx}(-0.430, 0) > 0$, $D(0.430, \pm 0.872) > 0$, $f_{xx}(0.430, \pm 0.872) > 0$, $D(-0.430, \pm 0.872) > 0$, $f_{xx}(-0.430, \pm 0.872) < 0$, and $D(0, \pm 0.524) < 0$, so $f(0.430, 0) \approx 15.973$ and $f(-0.430, \pm 0.872) \approx 6.459$ are local maxima, $f(-0.430, 0) \approx -15.973$ and $f(0.430, \pm 0.872) \approx -6.459$ are local minima, and $(0, \pm 0.524)$ are saddle points. The highest point on the graph is approximately $(0.430, 0, 15.973)$ and the lowest point is approximately $(-0.430, 0, -15.973)$.





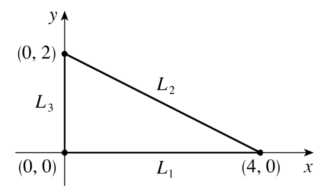


31. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. Here $f_x = 2x - 2$, $f_y = 2y$, and setting $f_x = f_y = 0$ gives $(1, 0)$ as the only critical point (which is inside D), where $f(1, 0) = -1$. Along L_1: $x = 0$ and $f(0, y) = y^2$ for $-2 \le y \le 2$, a quadratic function which attains its minimum at $y = 0$, where $f(0, 0) = 0$, and its maximum at $y = \pm 2$, where $f(0, \pm 2) = 4$. Along L_2: $y = x - 2$ for $0 \le x \le 2$, and $f(x, x - 2) = 2x^2 - 6x + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2}$, a quadratic which attains its minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, -\frac{1}{2}\right) = -\frac{1}{2}$, and its maximum at $x = 0$, where $f(0, -2) = 4$. Along L_3: $y = 2 - x$ for $0 \le x \le 2$, and $f(x, 2 - x) = 2x^2 - 6x + 4 = 2\left(x - \frac{3}{2}\right)^2 - \frac{1}{2}$, a quadratic which attains its minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, \frac{1}{2}\right) = -\frac{1}{2}$, and its maximum at $x = 0$, where $f(0, 2) = 4$. Thus the absolute maximum of f on D is $f(0, \pm 2) = 4$ and the absolute minimum is $f(1, 0) = -1$.



32. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. $f_x = 1 - y$, $f_y = 1 - x$, and setting $f_x = f_y = 0$ gives $(1, 1)$ as the only critical point (which is inside D), where $f(1, 1) = 1$. Along L_1: $y = 0$ and $f(x, 0) = x$ for $0 \le x \le 4$, an increasing function in x, so the maximum value is $f(4, 0) = 4$ and the minimum value is $f(0, 0) = 0$. Along L_2: $y = 2 - \frac{1}{2}x$ and $f\left(x, 2 - \frac{1}{2}x\right) = \frac{1}{2}x^2 - \frac{3}{2}x + 2 = \frac{1}{2}\left(x - \frac{3}{2}\right)^2 + \frac{7}{8}$ for $0 \le x \le 4$, a quadratic function which has a minimum at $x = \frac{3}{2}$, where $f\left(\frac{3}{2}, \frac{5}{4}\right) = \frac{7}{8}$, and a maximum at $x = 4$, where $f(4, 0) = 4$.

Along L_3: $x = 0$ and $f(0, y) = y$ for $0 \le y \le 2$, an increasing function in y, so the maximum value is $f(0, 2) = 2$ and the minimum value is $f(0, 0) = 0$. Thus the absolute maximum of f on D is $f(4, 0) = 4$ and the absolute minimum is $f(0, 0) = 0$.



33. $f_x(x, y) = 2x + 2xy$, $f_y(x, y) = 2y + x^2$, and setting $f_x = f_y = 0$ gives $(0, 0)$ as the only critical point in D, with $f(0, 0) = 4$.

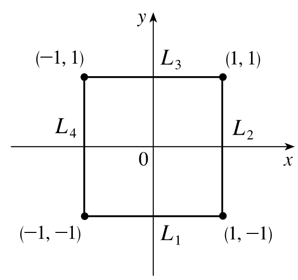
On L_1: $y = -1$, $f(x, -1) = 5$, a constant.

On L_2: $x = 1$, $f(1, y) = y^2 + y + 5$, a quadratic in y which attains its maximum at $(1, 1)$, $f(1, 1) = 7$ and its minimum at $\left(1, -\frac{1}{2}\right)$, $f\left(1, -\frac{1}{2}\right) = \frac{19}{4}$.

On L_3: $f(x, 1) = 2x^2 + 5$ which attains its maximum at $(-1, 1)$ and $(1, 1)$ with $f(\pm 1, 1) = 7$ and its minimum at $(0, 1)$, $f(0, 1) = 5$.

On L_4: $f(-1, y) = y^2 + y + 5$ with maximum at $(-1, 1)$, $f(-1, 1) = 7$ and minimum at $\left(-1, -\frac{1}{2}\right)$, $f\left(-1, -\frac{1}{2}\right) = \frac{19}{4}$.

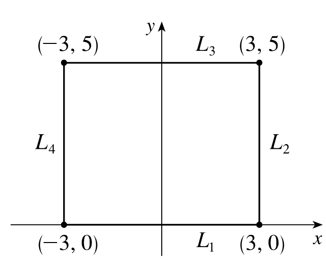
Thus the absolute maximum is attained at both $(\pm 1, 1)$ with $f(\pm 1, 1) = 7$ and the absolute minimum on D is attained at $(0, 0)$ with $f(0, 0) = 4$.



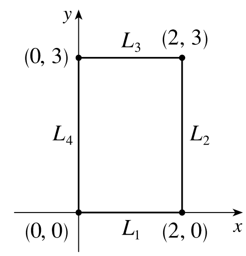
34. $f(x, y) = x^2 + xy + y^2 - 6y \quad \Rightarrow \quad f_x = 2x + y$, $\ f_y = x + 2y - 6$. Then $f_x = 0$ implies $y = -2x$, and substituting into $f_y = 0$ gives $x - 4x - 6 = 0 \quad \Rightarrow \quad x = -2$, so the only critical point is $(-2, 4)$ (which is in D) where $f(-2, 4) = -12$.

Along L_1: $y = 0$, so $f(x, 0) = x^2$, $-3 \le x \le 3$, which has a maximum value at $x = \pm 3$ where $f(\pm 3, 0) = 9$ and a minimum value at $x = 0$, where $f(0, 0) = 0$. Along L_2: $x = 3$, so $f(3, y) = 9 - 3y + y^2 = \left(y - \frac{3}{2}\right)^2 + \frac{27}{4}$, $0 \le y \le 5$, which has a maximum value at $y = 5$ where $f(3, 5) = 19$ and a minimum value at $y = \frac{3}{2}$ where $f(3, \frac{3}{2}) = \frac{27}{4}$.

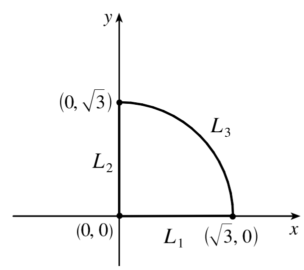
Along L_3: $y = 5$, so $f(x, 5) = x^2 + 5x - 5 = \left(x + \frac{5}{2}\right)^2 - \frac{45}{4}$, $-3 \le x \le 3$, which has a maximum value at $x = 3$ where $f(3, 5) = 19$ and a minimum value at $x = -\frac{5}{2}$, where $f\left(-\frac{5}{2}, 5\right) = -\frac{45}{4}$. Along L_4: $x = -3$, so $f(-3, y) = 9 - 9y + y^2 = \left(y - \frac{9}{2}\right)^2 - \frac{45}{4}$, $0 \le y \le 5$, which has a maximum value at $y = 0$ where $f(-3, 0) = 9$ and a minimum value at $y = \frac{9}{2}$ where $f\left(-3, \frac{9}{2}\right) = -\frac{45}{4}$. Thus the absolute maximum of f on D is $f(3, 5) = 19$ and the absolute minimum is $f(-2, 4) = -12$.



35. $f(x,y) = x^2 + 2y^2 - 2x - 4y + 1 \quad \Rightarrow \quad f_x = 2x - 2,\ f_y = 4y - 4$. Setting $f_x = 0$ and $f_y = 0$ gives $(1,1)$ as the only critical point (which is inside D), where $f(1,1) = -2$. Along L_1: $y = 0$, so $f(x,0) = x^2 - 2x + 1 = (x-1)^2$, $0 \le x \le 2$, which has a maximum value both at $x = 0$ and $x = 2$ where $f(0,0) = f(2,0) = 1$ and a minimum value at $x = 1$, where $f(1,0) = 0$. Along L_2: $x = 2$, so $f(2,y) = 2y^2 - 4y + 1 = 2(y-1)^2 - 1$, $0 \le y \le 3$, which has a maximum value at $y = 3$ where $f(2,3) = 7$ and a minimum value at $y = 1$ where $f(2,1) = -1$. Along L_3: $y = 3$, so $f(x,3) = x^2 - 2x + 7 = (x-1)^2 + 6$, $0 \le x \le 2$, which has a maximum value both at $x = 0$ and $x = 2$ where $f(0,3) = f(2,3) = 7$ and a minimum value at $x = 1$, where $f(1,3) = 6$. Along L_4: $x = 0$, so $f(0,y) = 2y^2 - 4y + 1 = 2(y-1)^2 - 1$, $0 \le y \le 3$, which has a maximum value at $y = 3$ where $f(0,3) = 7$ and a minimum value at $y = 1$ where $f(0,1) = -1$. Thus the absolute maximum is attained at both $(0,3)$ and $(2,3)$, where $f(0,3) = f(2,3) = 7$, and the absolute minimum is $f(1,1) = -2$.



36. $f_x = y^2$ and $f_y = 2xy$, and since $f_x = 0 \quad \Leftrightarrow \quad y = 0$, there are no critical points in the interior of D. Along L_1: $y = 0$ and $f(x,0) = 0$. Along L_2: $x = 0$ and $f(0,y) = 0$. Along L_3: $y = \sqrt{3 - x^2}$, so let $g(x) = f\left(x, \sqrt{3 - x^2}\right) = 3x - x^3$ for $0 \le x \le \sqrt{3}$. Then $g'(x) = 3 - 3x^2 = 0 \quad \Leftrightarrow \quad x = 1$. The maximum value is $f\left(1, \sqrt{2}\right) = 2$ and the minimum occurs both at $x = 0$ and $x = \sqrt{3}$ where $f\left(0, \sqrt{3}\right) = f\left(\sqrt{3}, 0\right) = 0$. Thus the absolute maximum of f on D is $f\left(1, \sqrt{2}\right) = 2$, and the absolute minimum is 0 which occurs at all points along L_1 and L_2.

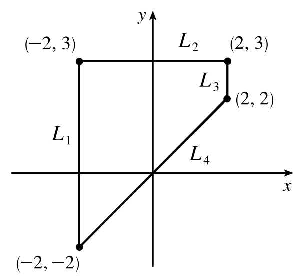


37. $f_x(x,y) = 6x^2$ and $f_y(x,y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0,0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ so let $g(x) = f(x,y) = 2x^3 + (1 - x^2)^2 = x^4 + 2x^3 - 2x^2 + 1$, $-1 \le x \le 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \quad \Rightarrow \quad x = 0$, -2, or $\frac{1}{2}$. $f(0, \pm 1) = g(0) = 1$, $f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$, and $(-2, -3)$ is not in D. Checking the endpoints, we get $f(-1,0) = g(-1) = -2$ and $f(1,0) = g(1) = 2$. Thus the absolute maximum and minimum of f on D are $f(1,0) = 2$ and $f(-1,0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos\theta$, $y = \sin\theta$, so $f(\cos\theta, \sin\theta) = 2\cos^3\theta + \sin^4\theta$, $0 \le \theta \le 2\pi$.

38. $f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -3y^2 + 12$ and the critical points are $(1,2)$, $(1,-2)$, $(-1,2)$, and $(-1,-2)$. But only $(1,2)$ and $(-1,2)$ are in D and $f(1,2) = 14$, $f(-1,2) = 18$. Along L_1: $x = -2$ and $f(-2,y) = -2 - y^3 + 12y$,

$-2 \le y \le 3$, which has a maximum at $y = 2$ where $f(-2, 2) = 14$ and a minimum at $y = -2$ where $f(-2, -2) = -18$. Along L_2: $x = 2$ and $f(2, y) = 2 - y^3 + 12y$, $2 \le y \le 3$, which has a maximum at $y = 2$ where $f(2, 2) = 18$ and a minimum at $y = 3$ where $f(2, 3) = 11$. Along L_3: $y = 3$ and $f(x, 3) = x^3 - 3x + 9$, $-2 \le x \le 2$, which has a maximum at $x = -1$ and $x = 2$ where $f(-1, 3) = f(2, 3) = 11$ and a minimum at $x = 1$ and $x = -2$ where $f(1, 3) = f(-2, 3) = 7$. Along L_4: $y = x$ and $f(x, x) = 9x$, $-2 \le x \le 2$, which has a maximum at $x = 2$ where $f(2, 2) = 18$ and a minimum at $x = -2$ where $f(-2, -2) = -18$. So the absolute maximum value of f on D is $f(2, 2) = 18$ and the minimum is $f(-2, -2) = -18$.



39. $f(x, y) = -(x^2 - 1)^2 - (x^2 y - x - 1)^2 \quad \Rightarrow \quad f_x(x, y) = -2(x^2 - 1)(2x) - 2(x^2 y - x - 1)(2xy - 1)$ and $f_y(x, y) = -2(x^2 y - x - 1)x^2$. Setting $f_y(x, y) = 0$ gives either $x = 0$ or $x^2 y - x - 1 = 0$.

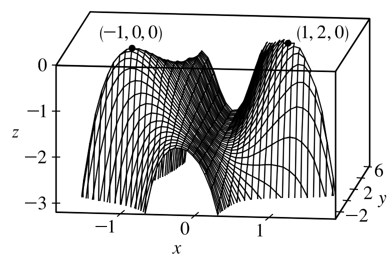
There are no critical points for $x = 0$, since $f_x(0, y) = -2$, so we set $x^2 y - x - 1 = 0 \quad \Leftrightarrow \quad y = \dfrac{x + 1}{x^2} \quad [x \ne 0]$,

so $f_x\left(x, \dfrac{x+1}{x^2}\right) = -2(x^2 - 1)(2x) - 2\left(x^2 \dfrac{x+1}{x^2} - x - 1\right)\left(2x \dfrac{x+1}{x^2} - 1\right) = -4x(x^2 - 1)$. Therefore

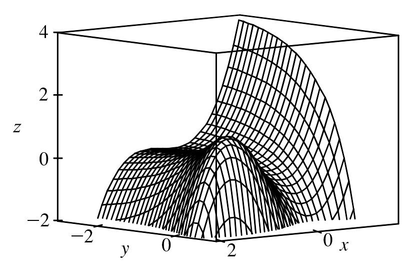
$f_x(x, y) = f_y(x, y) = 0$ at the points $(1, 2)$ and $(-1, 0)$. To classify these critical points, we calculate $f_{xx}(x, y) = -12x^2 - 12x^2 y^2 + 12xy + 4y + 2$, $f_{yy}(x, y) = -2x^4$, and $f_{xy}(x, y) = -8x^3 y + 6x^2 + 4x$. In order to use the Second Derivatives Test we calculate

$D(-1, 0) = f_{xx}(-1, 0)\, f_{yy}(-1, 0) - [f_{xy}(-1, 0)]^2 = 16 > 0$,

$f_{xx}(-1, 0) = -10 < 0$, $D(1, 2) = 16 > 0$, and $f_{xx}(1, 2) = -26 < 0$, so both $(-1, 0)$ and $(1, 2)$ give local maxima.



40. $f(x, y) = 3xe^y - x^3 - e^{3y}$ is differentiable everywhere, so the requirement for critical points is that $f_x = 3e^y - 3x^2 = 0$ **(1)** and $f_y = 3xe^y - 3e^{3y} = 0$ **(2)**. From **(1)** we obtain $e^y = x^2$, and then **(2)** gives $3x^3 - 3x^6 = 0 \quad \Rightarrow \quad x = 1$ or 0, but only $x = 1$ is valid, since $x = 0$ makes **(1)** impossible. So substituting $x = 1$ into **(1)** gives $y = 0$, and the only critical point is $(1, 0)$.



The Second Derivatives Test shows that this gives a local maximum, since

$D(1, 0) = \left[-6x(3xe^y - 9e^{3y}) - (3e^y)^2\right]_{(1,0)} = 27 > 0$ and $f_{xx}(1, 0) = [-6x]_{(1,0)} = -6 < 0$. But $f(1, 0) = 1$ is not an absolute maximum because, for instance, $f(-3, 0) = 17$. This can also be seen from the graph.

41. Let d be the distance from $(2, 0, -3)$ to any point (x, y, z) on the plane $x + y + z = 1$, so $d = \sqrt{(x-2)^2 + y^2 + (z+3)^2}$ where $z = 1 - x - y$, and we minimize $d^2 = f(x, y) = (x-2)^2 + y^2 + (4 - x - y)^2$. Then

$f_x(x,y) = 2(x-2) + 2(4-x-y)(-1) = 4x + 2y - 12$, $f_y(x,y) = 2y + 2(4-x-y)(-1) = 2x + 4y - 8$. Solving $4x + 2y - 12 = 0$ and $2x + 4y - 8 = 0$ simultaneously gives $x = \frac{8}{3}$, $y = \frac{2}{3}$, so the only critical point is $\left(\frac{8}{3}, \frac{2}{3}\right)$. An absolute minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for $x = \frac{8}{3}$, $y = \frac{2}{3}$ for which $d = \sqrt{\left(\frac{8}{3}-2\right)^2 + \left(\frac{2}{3}\right)^2 + \left(4 - \frac{8}{3} - \frac{2}{3}\right)^2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$.

42. Here the distance d from a point on the plane to the point $(0,1,1)$ is $d = \sqrt{x^2 + (y-1)^2 + (z-1)^2}$, where $z = 2 - \frac{1}{3}x + \frac{2}{3}y$. We can minimize $d^2 = f(x,y) = x^2 + (y-1)^2 + \left(1 - \frac{1}{3}x + \frac{2}{3}y\right)^2$, so

$f_x(x,y) = 2x + 2\left(1 - \frac{1}{3}x + \frac{2}{3}y\right)\left(-\frac{1}{3}\right) = \frac{20}{9}x - \frac{4}{9}y - \frac{2}{3}$ and

$f_y(x,y) = 2(y-1) + 2\left(1 - \frac{1}{3}x + \frac{2}{3}y\right)\left(\frac{2}{3}\right) = -\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3}$. Solving $\frac{20}{9}x - \frac{4}{9}y - \frac{2}{3} = 0$ and $-\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3} = 0$ simultaneously gives $x = \frac{5}{14}$ and $y = \frac{2}{7}$, so the only critical point is $\left(\frac{5}{14}, \frac{2}{7}\right)$.

This point must correspond to the minimum distance, so the point on the plane closest to $(0,1,1)$ is $\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)$.

43. Let d be the distance from the point $(4,2,0)$ to any point (x,y,z) on the cone, so $d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$ where $z^2 = x^2 + y^2$, and we minimize $d^2 = (x-4)^2 + (y-2)^2 + x^2 + y^2 = f(x,y)$. Then $f_x(x,y) = 2\,(x-4) + 2x = 4x - 8$, $f_y(x,y) = 2\,(y-2) + 2y = 4y - 4$, and the critical points occur when $f_x = 0 \;\Rightarrow\; x = 2$, $f_y = 0 \;\Rightarrow\; y = 1$. Thus the only critical point is $(2,1)$. An absolute minimum exists (since there is a minimum distance from the cone to the point) which must occur at a critical point, so the points on the cone closest to $(4,2,0)$ are $\left(2, 1, \pm\sqrt{5}\right)$.

44. The distance from the origin to a point (x,y,z) on the surface is $d = \sqrt{x^2 + y^2 + z^2}$ where $y^2 = 9 + xz$, so we minimize $d^2 = x^2 + 9 + xz + z^2 = f(x,z)$. Then $f_x = 2x + z$, $f_z = x + 2z$, and $f_x = 0$, $f_z = 0 \;\Rightarrow\; x = 0, z = 0$, so the only critical point is $(0,0)$. $D(0,0) = (2)(2) - 1 = 3 > 0$ with $f_{xx}(0,0) = 2 > 0$, so this is a minimum. Thus $y^2 = 9 + 0 \;\Rightarrow\; y = \pm 3$ and the points on the surface closest to the origin are $(0, \pm 3, 0)$.

45. Let x, y, z be the positive numbers. Then $x + y + z = 100 \;\Rightarrow\; z = 100 - x - y$, and we want to maximize $xyz = xy(100 - x - y) = 100xy - x^2y - xy^2 = f(x,y)$ for $0 < x, y, z < 100$. $f_x = 100y - 2xy - y^2$, $f_y = 100x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 100 - 2x - 2y$. Then $f_x = 0$ implies $y(100 - 2x - y) = 0 \;\Rightarrow\; y = 100 - 2x$ (since $y > 0$). Substituting into $f_y = 0$ gives $x[100 - x - 2(100 - 2x)] = 0 \;\Rightarrow\; 3x - 100 = 0$ (since $x > 0$) $\;\Rightarrow\; x = \frac{100}{3}$. Then $y = 100 - 2\left(\frac{100}{3}\right) = \frac{100}{3}$, and the only critical point is $\left(\frac{100}{3}, \frac{100}{3}\right)$. $D\left(\frac{100}{3}, \frac{100}{3}\right) = \left(-\frac{200}{3}\right)\left(-\frac{200}{3}\right) - \left(-\frac{100}{3}\right)^2 = \frac{10{,}000}{3} > 0$ and $f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3} < 0$. Thus $f\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local maximum. It is also the absolute maximum (compare to the values of f as x, y, or $z \to 0$ or 100), so the numbers are $x = y = z = \frac{100}{3}$.

46. Let x, y, z, be the positive numbers. Then $x + y + z = 12$ and we want to minimize $x^2 + y^2 + z^2 = x^2 + y^2 + (12 - x - y)^2 = f(x,y)$ for $0 < x, y < 12$. $f_x = 2x + 2(12 - x - y)(-1) = 4x + 2y - 24$,

$f_y = 2y + 2(12 - x - y)(-1) = 2x + 4y - 24$, $f_{xx} = 4$, $f_{xy} = 2$, $f_{yy} = 4$. Then $f_x = 0$ implies $4x + 2y = 24$ or $y = 12 - 2x$ and substituting into $f_y = 0$ gives $2x + 4(12 - 2x) = 24 \quad \Rightarrow \quad 6x = 24 \quad \Rightarrow \quad x = 4$ and then $y = 4$, so the only critical point is $(4, 4)$. $D(4, 4) = 16 - 4 > 0$ and $f_{xx}(4, 4) = 4 > 0$, so $f(4, 4)$ is a local minimum. $f(4, 4)$ is also the absolute minimum [compare to the values of f as $x, y \to 0$ or 12] so the numbers are $x = y = z = 4$.

47. Center the sphere at the origin so that its equation is $x^2 + y^2 + z^2 = r^2$, and orient the inscribed rectangular box so that its edges are parallel to the coordinate axes. Any vertex of the box satisfies $x^2 + y^2 + z^2 = r^2$, so take (x, y, z) to be the vertex in the first octant. Then the box has length $2x$, width $2y$, and height $2z = 2\sqrt{r^2 - x^2 - y^2}$ with volume given by

$V(x, y) = (2x)(2y)\left(2\sqrt{r^2 - x^2 - y^2}\right) = 8xy\sqrt{r^2 - x^2 - y^2}$ for $0 < x < r$, $0 < y < r$. Then

$V_x = (8xy) \cdot \frac{1}{2}(r^2 - x^2 - y^2)^{-1/2}(-2x) + \sqrt{r^2 - x^2 - y^2} \cdot 8y = \dfrac{8y(r^2 - 2x^2 - y^2)}{\sqrt{r^2 - x^2 - y^2}}$ and $V_y = \dfrac{8x(r^2 - x^2 - 2y^2)}{\sqrt{r^2 - x^2 - y^2}}$.

Setting $V_x = 0$ gives $y = 0$ or $2x^2 + y^2 = r^2$, but $y > 0$ so only the latter solution applies. Similarly, $V_y = 0$ with $x > 0$ implies $x^2 + 2y^2 = r^2$. Substituting, we have $2x^2 + y^2 = x^2 + 2y^2 \quad \Rightarrow \quad x^2 = y^2 \quad \Rightarrow \quad y = x$. Then $x^2 + 2y^2 = r^2 \quad \Rightarrow \quad 3x^2 = r^2 \quad \Rightarrow \quad x = \sqrt{r^2/3} = r/\sqrt{3} = y$. Thus the only critical point is $\left(r/\sqrt{3}, r/\sqrt{3}\right)$. There must be a maximum volume and here it must occur at a critical point, so the maximum volume occurs when $x = y = r/\sqrt{3}$ and the maximum volume is $V\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right) = 8\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)\sqrt{r^2 - \left(\frac{r}{\sqrt{3}}\right)^2 - \left(\frac{r}{\sqrt{3}}\right)^2} = \dfrac{8}{3\sqrt{3}}\, r^3$.

48. Let x, y, and z be the dimensions of the box. We wish to minimize surface area $= 2xy + 2xz + 2yz$, but we have volume $= xyz = 1000 \quad \Rightarrow \quad z = \dfrac{1000}{xy}$ so we minimize

$f(x, y) = 2xy + 2x\left(\dfrac{1000}{xy}\right) + 2y\left(\dfrac{1000}{xy}\right) = 2xy + \dfrac{2000}{y} + \dfrac{2000}{x}$. Then $f_x = 2y - \dfrac{2000}{x^2}$ and $f_y = 2x - \dfrac{2000}{y^2}$. Setting $f_x = 0$ implies $y = \dfrac{1000}{x^2}$ and substituting into $f_y = 0$ gives $x - \dfrac{x^4}{1000} = 0 \quad \Rightarrow \quad x^3 = 1000$ [since $x \neq 0$] $\Rightarrow \quad x = 10$. The surface area has a minimum but no maximum and it must occur at a critical point, so the minimal surface area occurs for a box with dimensions $x = 10$ cm, $y = 1000/10^2 = 10$ cm, $z = 1000/10^2 = 10$ cm.

49. Maximize $f(x, y) = \dfrac{xy}{3}(6 - x - 2y)$, then the maximum volume is $V = xyz$.

$f_x = \frac{1}{3}(6y - 2xy - y^2) = \frac{1}{3}y(6 - 2x - 2y)$ and $f_y = \frac{1}{3}x(6 - x - 4y)$. Setting $f_x = 0$ and $f_y = 0$ gives the critical point $(2, 1)$ which geometrically must give a maximum. Thus the volume of the largest such box is $V = (2)(1)\left(\frac{2}{3}\right) = \frac{4}{3}$.

50. Surface area $= 2(xy + xz + yz) = 64\ \text{cm}^2$, so $xy + xz + yz = 32$ or $z = \dfrac{32 - xy}{x + y}$. Maximize the volume

$f(x, y) = xy\,\dfrac{32 - xy}{x + y}$. Then $f_x = \dfrac{32y^2 - 2xy^3 - x^2y^2}{(x + y)^2} = y^2\,\dfrac{32 - 2xy - x^2}{(x + y)^2}$ and $f_y = x^2\,\dfrac{32 - 2xy - y^2}{(x + y)^2}$. Setting

$f_x = 0$ implies $y = \dfrac{32 - x^2}{2x}$ and substituting into $f_y = 0$ gives $32(4x^2) - (32 - x^2)(4x^2) - (32 - x^2)^2 = 0$ or

$3x^4 + 64x^2 - (32)^2 = 0$. Thus $x^2 = \frac{64}{6}$ or $x = \frac{8}{\sqrt{6}}$, $y = \frac{64/3}{16/\sqrt{6}} = \frac{8}{\sqrt{6}}$ and $z = \frac{8}{\sqrt{6}}$. Thus the box is a cube with edge length $\frac{8}{\sqrt{6}}$ cm.

51. Let the dimensions be x, y, and z; then $4x + 4y + 4z = c$ and the volume is

$V = xyz = xy\left(\frac{1}{4}c - x - y\right) = \frac{1}{4}cxy - x^2y - xy^2$, $x > 0$, $y > 0$. Then $V_x = \frac{1}{4}cy - 2xy - y^2$ and $V_y = \frac{1}{4}cx - x^2 - 2xy$, so $V_x = 0 = V_y$ when $2x + y = \frac{1}{4}c$ and $x + 2y = \frac{1}{4}c$. Solving, we get $x = \frac{1}{12}c$, $y = \frac{1}{12}c$ and $z = \frac{1}{4}c - x - y = \frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus the box is a cube with edge length $\frac{1}{12}c$.

52. The cost equals $5xy + 2(xz + yz)$ and $xyz = V$, so $C(x, y) = 5xy + 2V(x + y)/(xy) = 5xy + 2V(x^{-1} + y^{-1})$. Then $C_x = 5y - 2Vx^{-2}$, $C_y = 5x - 2\,Vy^{-2}$, $C_x = 0$ implies $y = 2\,V/(5x^2)$, $C_y = 0$ implies $x = \sqrt[3]{\frac{2}{5}V} = y$. Thus the dimensions of the aquarium which minimize the cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

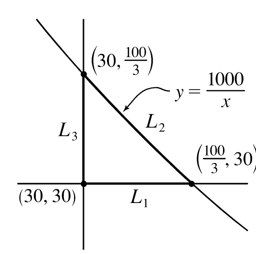
53. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000\text{ cm}^3$. Then

$f(x, y) = xy + [64{,}000(x + y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$.

And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now $D(x, y) = [(2)(64{,}000)]^2x^{-3}y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum. Thus the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.

54. Let x be the length of the north and south walls, y the length of the east and west walls, and z the height of the building. The heat loss is given by $h = 10(2yz) + 8(2xz) + 1(xy) + 5(xy) = 6xy + 16xz + 20yz$. The volume is 4000 m^3, so $xyz = 4000$, and we substitute $z = \frac{4000}{xy}$ to obtain the heat loss function $h(x, y) = 6xy + 80{,}000/x + 64{,}000/y$.

(a) Since $z = \frac{4000}{xy} \geq 4$, $xy \leq 1000 \quad \Rightarrow \quad y \leq 1000/x$. Also $x \geq 30$ and $y \geq 30$, so the domain of h is $D = \{(x, y) \mid x \geq 30, 30 \leq y \leq 1000/x\}$.



(b) $h(x, y) = 6xy + 80{,}000x^{-1} + 64{,}000y^{-1} \quad \Rightarrow$

$h_x = 6y - 80{,}000x^{-2}$, $h_y = 6x - 64{,}000y^{-2}$.

$h_x = 0$ implies $6x^2y = 80{,}000 \quad \Rightarrow \quad y = \dfrac{80{,}000}{6x^2}$ and substituting into

$h_y = 0$ gives $6x = 64{,}000\left(\dfrac{6x^2}{80{,}000}\right)^2 \quad \Rightarrow \quad x^3 = \dfrac{80{,}000^2}{6 \cdot 64{,}000} = \dfrac{50{,}000}{3}$, so

$x = \sqrt[3]{\dfrac{50{,}000}{3}} = 10\sqrt[3]{\dfrac{50}{3}} \quad \Rightarrow \quad y = \dfrac{80}{\sqrt[3]{60}}$, and the only critical point of h is $\left(10\sqrt[3]{\dfrac{50}{3}}, \dfrac{80}{\sqrt[3]{60}}\right) \approx (25.54, 20.43)$

which is not in D. Next we check the boundary of D.

On L_1: $y = 30$, $h(x, 30) = 180x + 80{,}000/x + 6400/3$, $30 \le x \le \frac{100}{3}$. Since $h'(x, 30) = 180 - 80{,}000/x^2 > 0$ for $30 \le x \le \frac{100}{3}$, $h(x, 30)$ is an increasing function with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$.

On L_2: $y = 1000/x$, $h(x, 1000/x) = 6000 + 64x + 80{,}000/x$, $30 \le x \le \frac{100}{3}$.

Since $h'(x, 1000/x) = 64 - 80{,}000/x^2 < 0$ for $30 \le x \le \frac{100}{3}$, $h(x, 1000/x)$ is a decreasing function with minimum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$.

On L_3: $x = 30$, $h(30, y) = 180y + 64{,}000/y + 8000/3$, $30 \le y \le \frac{100}{3}$. $h'(30, y) = 180 - 64{,}000/y^2 > 0$ for $30 \le y \le \frac{100}{3}$, so $h(30, y)$ is an increasing function of y with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$.

Thus the absolute minimum of h is $h(30, 30) = 10{,}200$, and the dimensions of the building that minimize heat loss are walls 30 m in length and height $\frac{4000}{30^2} = \frac{40}{9} \approx 4.44$ m.

(c) From part (b), the only critical point of h, which gives a local (and absolute) minimum, is approximately $h(25.54, 20.43) \approx 9396$. So a building of volume 4000 m^2 with dimensions $x \approx 25.54$ m, $y \approx 20.43$ m, $z \approx \frac{4000}{(25.54)(20.43)} \approx 7.67$ m has the least amount of heat loss.

55. Let x, y, z be the dimensions of the rectangular box. Then the volume of the box is xyz and

$L = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow \quad L^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad z = \sqrt{L^2 - x^2 - y^2}$.

Substituting, we have volume $V(x, y) = xy\sqrt{L^2 - x^2 - y^2}$ $(x, y > 0)$.

$$V_x = xy \cdot \tfrac{1}{2}(L^2 - x^2 - y^2)^{-1/2}(-2x) + y\sqrt{L^2 - x^2 - y^2} = y\sqrt{L^2 - x^2 - y^2} - \frac{x^2 y}{\sqrt{L^2 - x^2 - y^2}},$$

$V_y = x\sqrt{L^2 - x^2 - y^2} - \dfrac{xy^2}{\sqrt{L^2 - x^2 - y^2}}$. $V_x = 0$ implies $y(L^2 - x^2 - y^2) = x^2 y \quad \Rightarrow \quad y(L^2 - 2x^2 - y^2) = 0 \quad \Rightarrow$ $2x^2 + y^2 = L^2$ (since $y > 0$), and $V_y = 0$ implies $x(L^2 - x^2 - y^2) = xy^2 \quad \Rightarrow \quad x(L^2 - x^2 - 2y^2) = 0 \quad \Rightarrow$ $x^2 + 2y^2 = L^2$ (since $x > 0$). Substituting $y^2 = L^2 - 2x^2$ into $x^2 + 2y^2 = L^2$ gives $x^2 + 2L^2 - 4x^2 = L^2 \quad \Rightarrow$ $3x^2 = L^2 \quad \Rightarrow \quad x = L/\sqrt{3}$ (since $x > 0$) and then $y = \sqrt{L^2 - 2\left(L/\sqrt{3}\right)^2} = L/\sqrt{3}$.

So the only critical point is $\left(L/\sqrt{3}, L/\sqrt{3}\right)$ which, from the geometrical nature of the problem, must give an absolute maximum. Thus the maximum volume is $V\left(L/\sqrt{3}, L/\sqrt{3}\right) = \left(L/\sqrt{3}\right)^2 \sqrt{L^2 - \left(L/\sqrt{3}\right)^2 - \left(L/\sqrt{3}\right)^2} = L^3/\left(3\sqrt{3}\right)$ cubic units.

56. $Y(N, P) = kNPe^{-N-P} \quad \Rightarrow \quad Y_N = kP\left[N\left(-e^{-N-P}\right) + e^{-N-P}(1)\right] = kP(1 - N)e^{-N-P}$,

$Y_P = kN\left[P\left(-e^{-N-P}\right) + e^{-N-P}(1)\right] = kN(1 - P)e^{-N-P}$. Here $N \ge 0$ and $P \ge 0$, but if either $N = 0$ or $P = 0$ then the yield is zero. Assuming that $N > 0$ and $P > 0$, $Y_N = 0$ implies $N = 1$ and $Y_P = 0$ implies $P = 1$, so the only critical point in $\{(N, P) \mid N > 0, P > 0\}$ is $(1, 1)$ where $Y(1, 1) = ke^{-2}$.

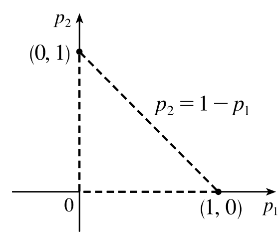
$D(N, P) = Y_{NN}Y_{PP} - (Y_{NP})^2 = \left[kP(N-2)e^{-N-P}\right]\left[kN(P-2)e^{-N-P}\right] - \left[k(1-N)(1-P)e^{-N-P}\right]^2 \quad \Rightarrow$

$D(1, 1) = \left(-ke^{-2}\right)\left(-ke^{-2}\right) - (0)^2 = k^2e^{-4} > 0$ and $Y_{NN}(1, 1) = -ke^{-2} < 0$, so $Y(1, 1) = ke^{-2}$ is a local maximum. $Y(1, 1)$ is also the absolute maximum (we have only one critical point, and $Y \to 0$ as $N \to 0$ or $P \to 0$ and $Y \to 0$ as N or P grow large) so the best yield is achieved when both the nitrogen and phosphorus levels are 1 (measured in appropriate units).

57. (a) We are given that $p_1 + p_2 + p_3 = 1 \quad \Rightarrow \quad p_3 = 1 - p_1 - p_2$, so

$H = -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 = -p_1 \ln p_1 - p_2 \ln p_2 - (1 - p_1 - p_2) \ln (1 - p_1 - p_2)$.

(b) Because p_i is a proportion we have $0 \le p_i \le 1$, but H is undefined unless $p_1 > 0$, $p_2 > 0$, and $1 - p_1 - p_2 > 0 \quad \Leftrightarrow \quad p_1 + p_2 < 1$. This last restriction forces $p_1 < 1$ and $p_2 < 1$, so the domain of H is $\{(p_1, p_2) \mid 0 < p_1 < 1,\ p_2 < 1 - p_1\}$. It is the interior of the triangle drawn in the figure.



(c) $$H_{p_1} = -\left[p_1 \cdot (1/p_1) + (\ln p_1) \cdot 1\right] - \left[(1 - p_1 - p_2) \cdot (-1)/(1 - p_1 - p_2) + \ln (1 - p_1 - p_2) \cdot (-1)\right]$$
$$= -1 - \ln p_1 + 1 + \ln (1 - p_1 - p_2) = \ln (1 - p_1 - p_2) - \ln p_1$$

Similarly $H_{p_2} = \ln (1 - p_1 - p_2) - \ln p_2$. Then $H_{p_1} = 0$ implies $\ln (1 - p_1 - p_2) = \ln p_1 \quad \Rightarrow \quad 1 - p_1 - p_2 = p_1 \quad \Rightarrow \quad p_2 = 1 - 2p_1$, and $H_{p_2} = 0$ implies $\ln (1 - p_1 - p_2) = \ln p_2 \quad \Rightarrow \quad p_1 = 1 - 2p_2$. Substituting, we have $p_1 = 1 - 2(1 - 2p_1) \quad \Rightarrow$ $3p_1 = 1 \quad \Rightarrow \quad p_1 = \frac{1}{3}$, and then $p_2 = 1 - 2\left(\frac{1}{3}\right) = \frac{1}{3}$. Thus the only critical point is $\left(\frac{1}{3}, \frac{1}{3}\right)$.

$D(p_1, p_2) = H_{p_1p_1}H_{p_2p_2} - (H_{p_1p_2})^2 = \left(\dfrac{-1}{1 - p_1 - p_2} - \dfrac{1}{p_1}\right)\left(\dfrac{-1}{1 - p_1 - p_2} - \dfrac{1}{p_2}\right) - \left(\dfrac{-1}{1 - p_1 - p_2}\right)^2$, so

$D\left(\frac{1}{3}, \frac{1}{3}\right) = (-6)(-6) - (-3)^2 = 27 > 0$ and $H_{p_1p_1}\left(\frac{1}{3}, \frac{1}{3}\right) = -6 < 0$. Thus $H\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{1}{3} \ln \frac{1}{3} - \frac{1}{3} \ln \frac{1}{3} - \frac{1}{3} \ln \frac{1}{3} = -\ln \frac{1}{3} = \ln 3$ is a local maximum. Here it is also the absolute maximum, so the maximum value of H is $\ln 3$, which occurs for $p_1 = p_2 = p_3 = \frac{1}{3}$ (all three species have equal proportion in the ecosystem).

58. Since $p + q + r = 1$ we can substitute $p = 1 - r - q$ into P giving $P = P(q, r) = 2(1 - r - q)q + 2(1 - r - q)r + 2rq = 2q - 2q^2 + 2r - 2r^2 - 2rq$. Since p, q and r represent proportions and $p + q + r = 1$, we know $q \ge 0$, $r \ge 0$, and $q + r \le 1$. Thus, we want to find the absolute maximum of the continuous function $P(q, r)$ on the closed set D enclosed by the lines $q = 0$, $r = 0$, and $q + r = 1$. To find any critical points, we set the partial derivatives equal to zero: $P_q(q,r) = 2 - 4q - 2r = 0$ and $P_r(q,r) = 2 - 4r - 2q = 0$. The first equation gives $r = 1 - 2q$, and substituting into the second equation we have $2 - 4(1-2q) - 2q = 0 \quad \Rightarrow \quad q = \frac{1}{3}$. Then we have one critical point, $\left(\frac{1}{3}, \frac{1}{3}\right)$, where $P\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{3}$. Next we find the maximum values of P on the boundary of D which consists of three line segments. For the segment given by $r = 0$, $0 \le q \le 1$, $P(q,r) = P(q,0) = 2q - 2q^2$, $0 \le q \le 1$. This represents a parabola with maximum value $P\left(\frac{1}{2}, 0\right) = \frac{1}{2}$. On the segment $q = 0$, $0 \le r \le 1$ we have $P(0,r) = 2r - 2r^2$, $0 \le r \le 1$. This represents a parabola with maximum value $P\left(0, \frac{1}{2}\right) = \frac{1}{2}$. Finally, on the segment $q + r = 1$, $0 \le q \le 1$, $P(q,r) = P(q, 1-q) = 2q - 2q^2$, $0 \le q \le 1$ which has a maximum value of $P\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$. Comparing these values with the value of P at the critical point, we see that the absolute maximum value of $P(q,r)$ on D is $\frac{2}{3}$.

59. Note that here the variables are m and b, and $f(m,b) = \sum\limits_{i=1}^{n} [y_i - (mx_i + b)]^2$. Then $f_m = \sum\limits_{i=1}^{n} -2x_i[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} \left(x_i y_i - m x_i^2 - b x_i\right) = 0$ or $\sum\limits_{i=1}^{n} x_i y_i = m \sum\limits_{i=1}^{n} x_i^2 + b \sum\limits_{i=1}^{n} x_i$ and $f_b = \sum\limits_{i=1}^{n} -2[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} y_i = m \sum\limits_{i=1}^{n} x_i + \sum\limits_{i=1}^{n} b = m\left(\sum\limits_{i=1}^{n} x_i\right) + nb$. Thus we have the two desired equations.

Now $f_{mm} = \sum\limits_{i=1}^{n} 2x_i^2$, $f_{bb} = \sum\limits_{i=1}^{n} 2 = 2n$ and $f_{mb} = \sum\limits_{i=1}^{n} 2x_i$. And $f_{mm}(m,b) > 0$ always and

$D(m,b) = 4n\left(\sum\limits_{i=1}^{n} x_i^2\right) - 4\left(\sum\limits_{i=1}^{n} x_i\right)^2 = 4\left[n\left(\sum\limits_{i=1}^{n} x_i^2\right) - \left(\sum\limits_{i=1}^{n} x_i\right)^2\right] > 0$ always so the solutions of these two equations do indeed minimize $\sum\limits_{i=1}^{n} d_i^2$.

60. Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron that passes through the point $(1,2,3)$. Writing the equation of the plane as $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$, the volume of the tetrahedron is given by $V = \dfrac{abc}{6}$. But $(1,2,3)$ must lie on the plane, so we need $\dfrac{1}{a} + \dfrac{2}{b} + \dfrac{3}{c} = 1$ $(\star)$ and thus can think of c as a function of a and b. Then $V_a = \dfrac{b}{6}\left(c + a\dfrac{\partial c}{\partial a}\right)$ and $V_b = \dfrac{a}{6}\left(c + b\dfrac{\partial c}{\partial b}\right)$. Differentiating $(\star)$ with respect to a we get $-a^{-2} - 3c^{-2}\dfrac{\partial c}{\partial a} = 0 \quad \Rightarrow$ $\dfrac{\partial c}{\partial a} = \dfrac{-c^2}{3a^2}$, and differentiating $(\star)$ with respect to b gives $-2b^{-2} - 3c^{-2}\dfrac{\partial c}{\partial b} = 0 \quad \Rightarrow \quad \dfrac{\partial c}{\partial b} = \dfrac{-2c^2}{3b^2}$. Then $V_a = \dfrac{b}{6}\left(c + a\dfrac{-c^2}{3a^2}\right) = 0 \quad \Rightarrow \quad c = 3a$, and $V_b = \dfrac{a}{6}\left(c + b\dfrac{-2c^2}{3b^2}\right) = 0 \quad \Rightarrow \quad c = \frac{3}{2}b$. Thus $3a = \frac{3}{2}b$ or $b = 2a$. Putting these into $(\star)$ gives $\frac{3}{a} = 1$ or $a = 3$ and then $b = 6$, $c = 9$. Thus the equation of the required plane is $\dfrac{x}{3} + \dfrac{y}{6} + \dfrac{z}{9} = 1$ or $6x + 3y + 2z = 18$.