14 ☐ PARTIAL DERIVATIVES

14.1 Functions of Several Variables

1. (a) From Table 1, $f(-15, 40) = -27$, which means that if the temperature is $-15°\text{C}$ and the wind speed is 40 km/h, then the air would feel equivalent to approximately $-27°\text{C}$ without wind.

(b) The question is asking: when the temperature is $-20°\text{C}$, what wind speed gives a wind-chill index of $-30°\text{C}$? From Table 1, the speed is 20 km/h.

(c) The question is asking: when the wind speed is 20 km/h, what temperature gives a wind-chill index of $-49°\text{C}$? From Table 1, the temperature is $-35°\text{C}$.

(d) The function $W = f(-5, v)$ means that we fix T at -5 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind-chill index values for different wind speeds when the temperature is $-5°\text{C}$. From Table 1 (look at the row corresponding to $T = -5$), the function decreases and appears to approach a constant value as v increases.

(e) The function $W = f(T, 50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind-chill index values for different temperatures when the wind speed is 50 km/h . From Table 1 (look at the column corresponding to $v = 50$), the function increases almost linearly as T increases.

2. (a) From Table 3, $f(95, 70) = 124$, which means that when the actual temperature is $95°\text{F}$ and the relative humidity is 70%, the perceived air temperature is approximately $124°\text{F}$.

(b) Looking at the row corresponding to $T = 90$, we see that $f(90, h) = 100$ when $h = 60$.

(c) Looking at the column corresponding to $h = 50$, we see that $f(T, 50) = 88$ when $T = 85$.

(d) $I = f(80, h)$ means that T is fixed at 80 and h is allowed to vary, resulting in a function of h that gives the humidex values for different relative humidities when the actual temperature is $80°\text{F}$. Similarly, $I = f(100, h)$ is a function of one variable that gives the humidex values for different relative humidities when the actual temperature is $100°\text{F}$. Looking at the rows of the table corresponding to $T = 80$ and $T = 100$, we see that $f(80, h)$ increases at a relatively constant rate of approximately $1°\text{F}$ per 10% relative humidity, while $f(100, h)$ increases more quickly (at first with an average rate of change of $5°\text{F}$ per 10% relative humidity) and at an increasing rate (approximately $12°\text{F}$ per 10% relative humidity for larger values of h).

3. $P(120, 20) = 1.47(120)^{0.65}(20)^{0.35} \approx 94.2$, so when the manufacturer invests \$20 million in capital and 120,000 hours of labor are completed yearly, the monetary value of the production is about \$94.2 million.

4. If the amounts of labor and capital are both doubled, we replace L, K in the function with $2L, 2K$, giving

$$P(2L, 2K) = 1.01(2L)^{0.75}(2K)^{0.25} = 1.01(2^{0.75})(2^{0.25})L^{0.75}K^{0.25} = (2^1)1.01L^{0.75}K^{0.25} = 2P(L, K)$$

Thus, the production is doubled. It is also true for the general case $P(L, K) = bL^{\alpha}K^{1-\alpha}$:

$P(2L, 2K) = b(2L)^{\alpha}(2K)^{1-\alpha} = b(2^{\alpha})(2^{1-\alpha})L^{\alpha}K^{1-\alpha} = (2^{\alpha+1-\alpha})bL^{\alpha}K^{1-\alpha} = 2P(L, K)$.

5. (a) $f(160, 70) = 0.1091(160)^{0.425}(70)^{0.725} \approx 20.5$, which means that the surface area of a person 70 inches (5 feet 10 inches) tall who weighs 160 pounds is approximately 20.5 square feet.

(b) Answers will vary depending on the height and weight of the reader.

6. We compare the values for the wind-chill index given by Table 1 with those given by the model function:

Modeled Wind-Chill Index Values $W(T, v)$

Wind Speed (km/h)

Actual temperature (°C)

T \ V	5	10	15	20	25	30	40	50	60	70	80
5	4.08	2.66	1.74	1.07	0.52	0.05	−0.71	−1.33	−1.85	−2.30	−2.70
0	−1.59	−3.31	−4.42	−5.24	−5.91	−6.47	−7.40	−8.14	−8.77	−9.32	−9.80
−5	−7.26	−9.29	−10.58	−11.55	−12.34	−13.00	−14.08	−14.96	−15.70	−16.34	−16.91
−10	−12.93	−15.26	−16.75	−17.86	−18.76	−19.52	−20.77	−21.77	−22.62	−23.36	−24.01
−15	−18.61	−21.23	−22.91	−24.17	−25.19	−26.04	−27.45	−28.59	−29.54	−30.38	−31.11
−20	−24.28	−27.21	−29.08	−30.48	−31.61	−32.57	−34.13	−35.40	−36.47	−37.40	−38.22
−25	−29.95	−33.18	−35.24	−36.79	−38.04	−39.09	−40.82	−42.22	−43.39	−44.42	−45.32
−30	−35.62	−39.15	−41.41	−43.10	−44.46	−45.62	−47.50	−49.03	−50.32	−51.44	−52.43
−35	−41.30	−45.13	−47.57	−49.41	−50.89	−52.14	−54.19	−55.84	−57.24	−58.46	−59.53
−40	−46.97	−51.10	−53.74	−55.72	−57.31	−58.66	−60.87	−62.66	−64.17	−65.48	−66.64

The values given by the function appear to be fairly close (within 0.5) to the values in Table 1.

7. (a) According to Table 4, $f(40, 15) = 25$, which means that if a 40-knot wind has been blowing in the open sea for 15 hours, it will create waves with estimated heights of 25 feet.

(b) $h = f(30, t)$ means we fix v at 30 and allow t to vary, resulting in a function of one variable. Thus here, $h = f(30, t)$ gives the wave heights produced by 30-knot winds blowing for t hours. From the table (look at the row corresponding to $v = 30$), the function increases but at a declining rate as t increases. In fact, the function values appear to be approaching a limiting value of approximately 19, which suggests that 30-knot winds cannot produce waves higher than about 19 feet.

(c) $h = f(v, 30)$ means we fix t at 30, again giving a function of one variable. So, $h = f(v, 30)$ gives the wave heights produced by winds of speed v blowing for 30 hours. From the table (look at the column corresponding to $t = 30$), the function appears to increase at an increasing rate, with no apparent limiting value. This suggests that faster winds (lasting 30 hours) always create higher waves.

8. (a) The cost of making x small boxes, y medium boxes, and z large boxes is $C = f(x, y, z) = 8000 + 2.5x + 4y + 4.5z$ dollars.

(b) $f(3000, 5000, 4000) = 8000 + 2.5(3000) + 4(5000) + 4.5(4000) = 53{,}500$ which means that it costs \$53,500 to make 3000 small boxes, 5000 medium boxes, and 4000 large boxes.

(c) Because no partial boxes will be produced, each of x, y, and z must be a positive integer or zero.

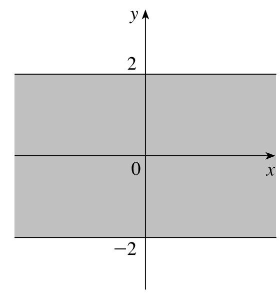
9. (a) $g(2,-1) = \cos(2+2(-1)) = \cos(0) = 1$

(b) $x+2y$ is defined for all choices of values for x and y and the cosine function is defined for all input values, so the domain of g is $\mathbb{R}^2$.

(c) The range of the cosine function is $[-1,1]$ and $x+2y$ generates all possible input values for the cosine function, so the range of $\cos(x+2y)$ is $[-1,1]$.

10. (a) $F(3,1) = 1+\sqrt{4-1^2} = 1+\sqrt{3}$

(b) $\sqrt{4-y^2}$ is defined only when $4-y^2 \geq 0$, or $y^2 \leq 4 \quad \Leftrightarrow$ $-2 \leq y \leq 2$. So the domain of F is $\{(x,y) \mid -2 \leq y \leq 2\}$.



(c) We know $0 \leq \sqrt{4-y^2} \leq 2$ so $1 \leq 1+\sqrt{4-y^2} \leq 3$. Thus the range of F is $[1,3]$.

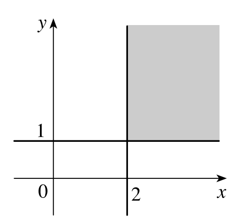
11. (a) $f(1,1,1) = \sqrt{1}+\sqrt{1}+\sqrt{1}+\ln(4-1^2-1^2-1^2) = 3+\ln 1 = 3$

(b) $\sqrt{x}$, $\sqrt{y}$, $\sqrt{z}$ are defined only when $x \geq 0$, $y \geq 0$, $z \geq 0$, and $\ln(4-x^2-y^2-z^2)$ is defined when $4-x^2-y^2-z^2 > 0 \quad \Leftrightarrow \quad x^2+y^2+z^2 < 4$, thus the domain is $\{(x,y,z) \mid x^2+y^2+z^2 < 4,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$, the portion of the interior of a sphere of radius 2, centered at the origin, that is in the first octant.

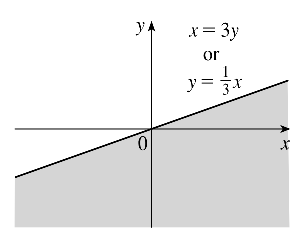
12. (a) $g(1,2,3) = 1^3 \cdot 2^2 \cdot 3\sqrt{10-1-2-3} = 12\sqrt{4} = 24$

(b) g is defined only when $10-x-y-z \geq 0 \quad \Leftrightarrow \quad z \leq 10-x-y$, so the domain is $\{(x,y,z) \mid z \leq 10-x-y\}$, the points on or below the plane $x+y+z=10$.

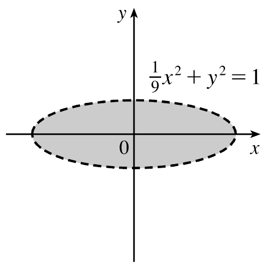
13. $\sqrt{x-2}$ is defined only when $x-2 \geq 0$, or $x \geq 2$, and $\sqrt{y-1}$ is defined only when $y-1 \geq 0$, or $y \geq 1$. So the domain of f is $\{(x,y) \mid x \geq 2,\ y \geq 1\}$.



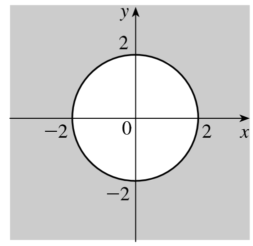
14. $\sqrt[4]{x-3y}$ is defined only when $x-3y \geq 0$, or $x \geq 3y$. So the domain of f is $\{(x,y) \mid x \geq 3y\}$ or equivalently $\{(x,y) \mid y \leq \frac{1}{3}x\}$.



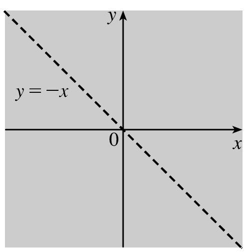
15. $\ln(9 - x^2 - 9y^2)$ is defined only when $9 - x^2 - 9y^2 > 0$, or $\frac{1}{9}x^2 + y^2 < 1$. So the domain of f is $\left\{(x, y) \mid \frac{1}{9}x^2 + y^2 < 1\right\}$, the interior of an ellipse.



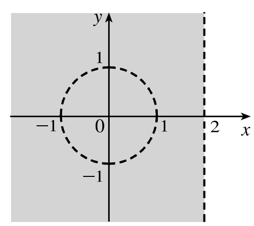
16. $\sqrt{x^2 + y^2 - 4}$ is defined only when $x^2 + y^2 - 4 \geq 0$ $\Leftrightarrow$ $x^2 + y^2 \geq 4$. So the domain of f is $\left\{(x, y) \mid x^2 + y^2 \geq 4\right\}$, the set of points on or outside a circle of radius 2 centered at the origin.



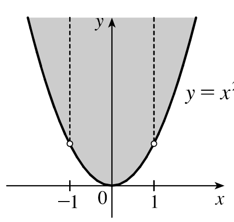
17. g is not defined if $x + y = 0$ $\Leftrightarrow$ $y = -x$ (and is defined otherwise). Thus the domain of g is $\{(x, y) \mid y \neq -x\}$, the set of all points in $\mathbb{R}^2$ that are not on the line $y = -x$.



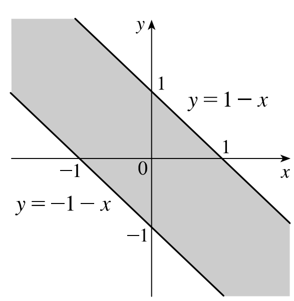
18. $\ln(2 - x)$ is defined only when $2 - x > 0$, or $x < 2$. In addition, g is not defined if $1 - x^2 - y^2 = 0$ $\Leftrightarrow$ $x^2 + y^2 = 1$. Thus the domain of g is $\left\{(x, y) \mid x < 2,\ x^2 + y^2 \neq 1\right\}$, the set of all points to the left of the line $x = 2$ and not on the unit circle.



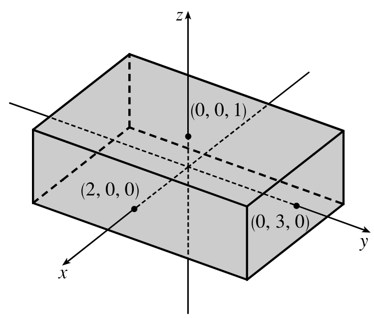
19. $\sqrt{y - x^2}$ is defined only when $y - x^2 \geq 0$, or $y \geq x^2$. In addition, f is not defined if $1 - x^2 = 0$ $\Leftrightarrow$ $x = \pm 1$. Thus the domain of f is $\left\{(x, y) \mid y \geq x^2,\ x \neq \pm 1\right\}$.



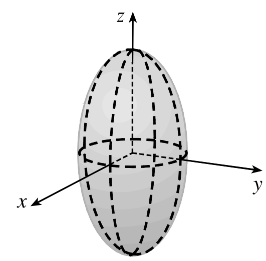
20. $\sin^{-1}(x + y)$ is defined only when $-1 \leq x + y \leq 1$ $\Leftrightarrow$ $-1 - x \leq y \leq 1 - x$. Thus the domain of f is $\{(x, y) \mid -1 - x \leq y \leq 1 - x\}$, consisting of those points on or between the parallel lines $y = -1 - x$ and $y = 1 - x$.



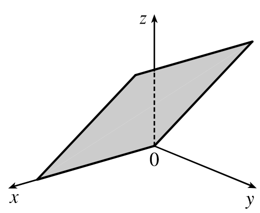
21. f is defined only when $4 - x^2 \geq 0 \quad \Leftrightarrow \quad -2 \leq x \leq 2$ and $9 - y^2 \geq 0 \quad \Leftrightarrow \quad -3 \leq y \leq 3$ and $1 - z^2 \geq 0 \quad \Leftrightarrow \quad -1 \leq z \leq 1$. Thus the domain of f is $\{(x, y, z) \mid -2 \leq x \leq 2,\ -3 \leq y \leq 3,\ -1 \leq z \leq 1\}$, a solid rectangular box with vertices $(\pm 2, \pm 3, \pm 1)$ (all combinations).



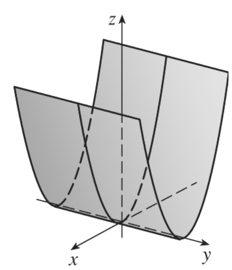
22. f is defined only when $16 - 4x^2 - 4y^2 - z^2 > 0 \quad \Rightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} < 1$. Thus, $D = \left\{ (x, y, z) \,\middle|\, \dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} < 1 \right\}$, that is, the points inside the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} = 1$.



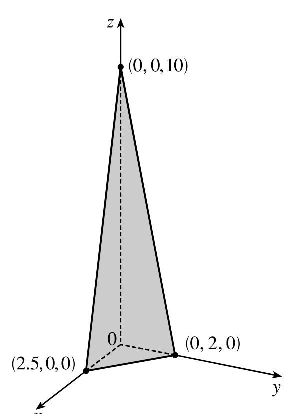
23. The graph of f has equation $z = y$, a plane which intersects the yz-plane in the line $z = y$, $x = 0$. The portion of this plane in the first octant is shown.



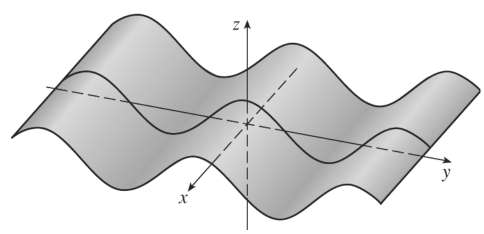
24. The graph of f has equation $z = x^2$, a parabolic cylinder.



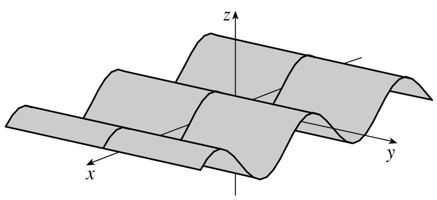
25. $z = 10 - 4x - 5y$ or $4x + 5y + z = 10$, a plane with intercepts 2.5, 2, and 10.



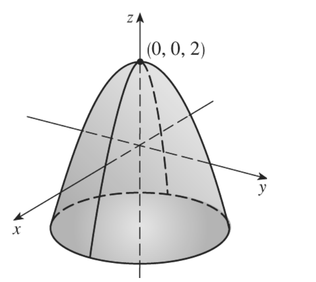
26. $z = \cos y$, a cylinder.



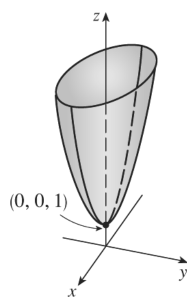
27. $z = \sin x$, a cylinder.



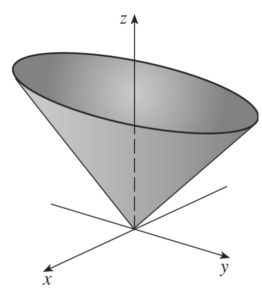
28. $z = 2 - x^2 - y^2$, a circular paraboloid opening downward with vertex at $(0, 0, 2)$.



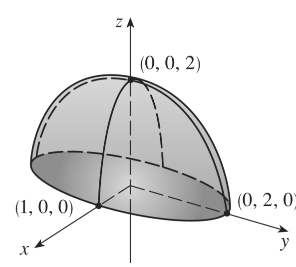
29. $z = x^2 + 4y^2 + 1$, an elliptic paraboloid opening upward with vertex at $(0, 0, 1)$.



30. $z = \sqrt{4x^2 + y^2}$ so $4x^2 + y^2 = z^2$ and $z \geq 0$, the top half of an elliptic cone.



31. $z = \sqrt{4 - 4x^2 - y^2}$ so $4x^2 + y^2 + z^2 = 4$ or $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{4} = 1$ and $z \geq 0$, the top half of an ellipsoid.



32. (a) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$. The trace in $x = 0$ is $z = \dfrac{1}{1 + y^2}$, and the trace in $y = 0$ is $z = \dfrac{1}{1 + x^2}$. The only possibility is graph III. Notice also that the level curves of f are $\dfrac{1}{1 + x^2 + y^2} = k \quad \Leftrightarrow \quad x^2 + y^2 = \dfrac{1}{k} - 1$, a family of circles for $k < 1$.

(b) $f(x, y) = \dfrac{1}{1 + x^2 y^2}$. The trace in $x = 0$ is the horizontal line $z = 1$, and the trace in $y = 0$ is also $z = 1$. Both graphs I and II have these traces; however, notice that here $z > 0$, so the graph is I.

(c) $f(x, y) = \ln(x^2 + y^2)$. The trace in $x = 0$ is $z = \ln y^2$, and the trace in $y = 0$ is $z = \ln x^2$. The level curves of f are $\ln(x^2 + y^2) = k \quad \Leftrightarrow \quad x^2 + y^2 = e^k$, a family of circles. In addition, f is large negative when $x^2 + y^2$ is small, so this is graph IV.

(d) $f(x, y) = \cos \sqrt{x^2 + y^2}$. The trace in $x = 0$ is $z = \cos \sqrt{y^2} = \cos |y| = \cos y$, and the trace in $y = 0$ is $z = \cos \sqrt{x^2} = \cos |x| = \cos x$. Notice also that the level curve $f(x, y) = 0$ is $\cos \sqrt{x^2 + y^2} = 0 \quad \Leftrightarrow$ $x^2 + y^2 = \left(\frac{\pi}{2} + n\pi\right)^2$, a family of circles, so this is graph V.

(e) $f(x, y) = |xy|$. The trace in $x = 0$ is $z = 0$, and the trace in $y = 0$ is $z = 0$, so it must be graph VI.

(f) $f(x, y) = \cos(xy)$. The trace in $x = 0$ is $z = \cos 0 = 1$, and the trace in $y = 0$ is $z = 1$. As mentioned in part (b), these traces match both graphs I and II. Here z can be negative, so the graph is II. (Also notice that the trace in $x = 1$ is $z = \cos y$, and the trace in $y = 1$ is $z = \cos x$.)

33. The point $(-3, 3)$ lies between the level curves with z-values 50 and 60. Since the point is a little closer to the level curve with $z = 60$, we estimate that $f(-3, 3) \approx 56$. The point $(3, -2)$ appears to be just about halfway between the level curves with z-values 30 and 40, so we estimate $f(3, -2) \approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

34. (a) C (Chicago) lies between level curves with pressures 1012 and 1016 mb, and since C appears to be located about one-fourth the distance from the 1012 mb isobar to the 1016 mb isobar, we estimate the pressure at Chicago to be about 1013 mb. N lies very close to a level curve with pressure 1012 mb so we estimate the pressure at Nashville to be approximately 1012 mb. S appears to be just about halfway between level curves with pressures 1008 and 1012 mb, so we estimate the pressure at San Francisco to be about 1010 mb. V lies close to a level curve with pressure 1016 mb but we can't see a level curve to its left so it is more difficult to make an accurate estimate. There are lower pressures to the right of V and V is a short distance to the left of the level curve with pressure 1016 mb, so we might estimate that the pressure at Vancouver is about 1017 mb.

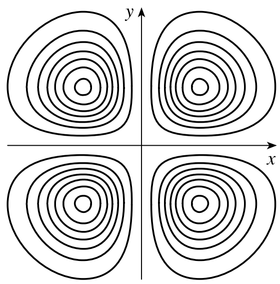
(b) Winds are stronger where the isobars are closer together (see Figure 13), and the level curves are closer near S than at the other locations, so the winds were strongest at San Francisco.

35. The point $(160, 10)$, corresponding to day 160 and a depth of 10 m, lies between the isothermals with temperature values of 8 and $12^\circ\mathrm{C}$. Since the point appears to be located about three-fourths the distance from the $8^\circ\mathrm{C}$ isothermal to the $12^\circ\mathrm{C}$ isothermal, we estimate the temperature at that point to be approximately $11^\circ\mathrm{C}$. The point $(180, 5)$ lies between the 16 and $20^\circ\mathrm{C}$ isothermals, very close to the $20^\circ\mathrm{C}$ level curve, so we estimate the temperature there to be about $19.5^\circ\mathrm{C}$.

36. If we start at the origin and move along the x-axis, for example, the z-values of a cone centered at the origin increase at a constant rate, so we would expect its level curves to be equally spaced. A paraboloid with vertex the origin, on the other hand, has z-values which change slowly near the origin and more quickly as we move farther away. Thus, we would expect its level curves near the origin to be spaced more widely apart than those farther from the origin. Therefore contour map I must correspond to the paraboloid, and contour map II the cone.

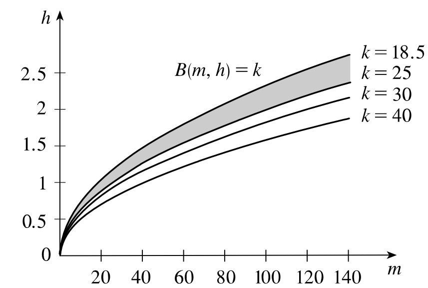
37. Near A, the level curves are very close together, indicating that the terrain is quite steep. At B, the level curves are much farther apart, so we would expect the terrain to be much less steep than near A, perhaps almost flat.

38.



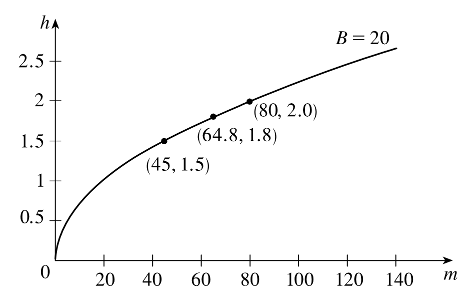
39. The level curves of $B(m,h)=\dfrac{m}{h^2}$ are $\dfrac{m}{h^2}=k \quad\Leftrightarrow\quad m=kh^2$ or equivalently $h=\sqrt{m/k}=\dfrac{1}{\sqrt{k}}\sqrt{m}$ since $m>0$, $h>0$. We draw the level curves for $k=18.5$, 25, 30, and 40.

The shaded region corresponds to BMI values between 18.5 and 25, those considered optimal. For a mass of 62 kg and a height of 152 cm (1.52 m), the BMI is $B(62,1.52)=\dfrac{62}{1.52^2}\approx 26.8$, which is outside the optimal range.

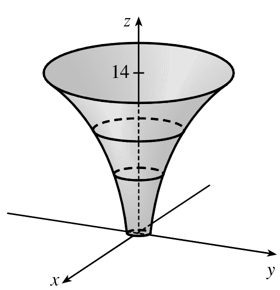


40. From Exercise 39, the body mass index function is $B(m,h)=m/h^2$. The BMI for a person 200 cm (2.0 m, about 6 ft 7 in) tall and with mass 80 kg (about 176 lb) is $B(80,2.0)=80/(2.0)^2=20$. The level curve $B(m,h)=20 \quad\Leftrightarrow\quad m=20h^2$ is shown in the graph.

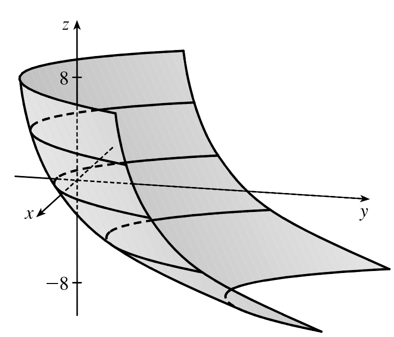
A person 1.5 m tall (about 4 ft 11 in) has a BMI on the same level curve if their mass is $m=20(1.5)^2=45$ kg (about 99 lb), and a person 1.8 m (about 5 ft 11 in) tall would have mass $m=20(1.8)^2=64.8$ kg (about 143 lb). (See the graph.)



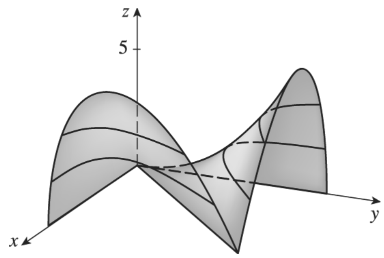
41.



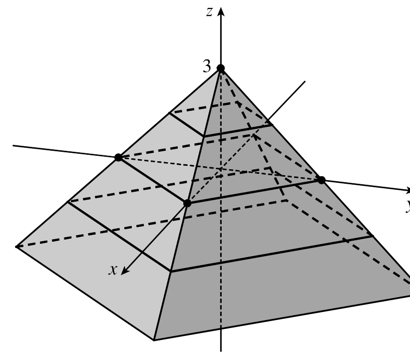
42.



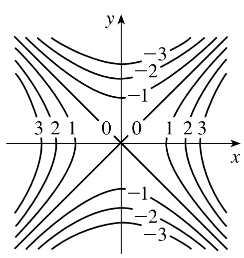
43.



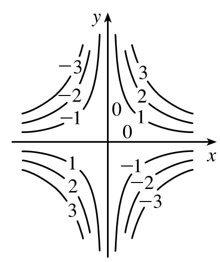
44.



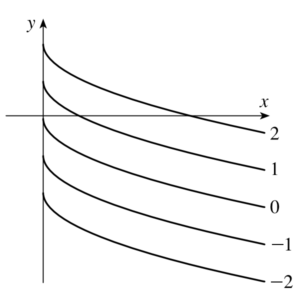
45. The level curves are $x^2 - y^2 = k$. When $k = 0$ the level curve is the pair of lines $y = \pm x$, and when $k \neq 0$ the level curves are a family of hyperbolas (oriented differently for $k > 0$ than for $k < 0$).



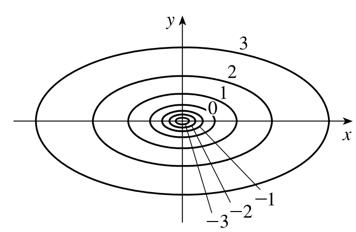
46. The level curves are $xy = k$ or $y = k/x$. When $k \neq 0$ the level curves are a family of hyperbolas. When $k = 0$ the level curve is the pair of lines $x = 0$, $y = 0$.



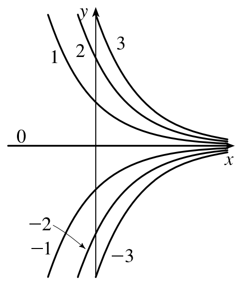
47. The level curves are $\sqrt{x} + y = k$ or $y = -\sqrt{x} + k$, a family of vertical translations of the graph of the root function $y = -\sqrt{x}$.



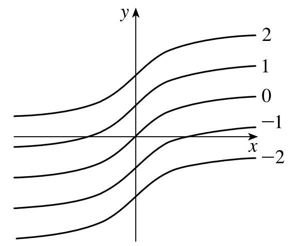
48. The level curves are $\ln(x^2 + 4y^2) = k$ or $x^2 + 4y^2 = e^k$, a family of ellipses.



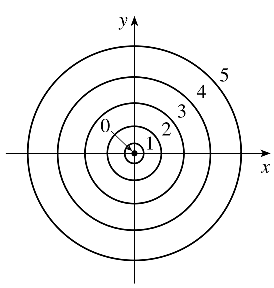
49. The level curves are $ye^x = k$ or $y = ke^{-x}$, a family of exponential curves.



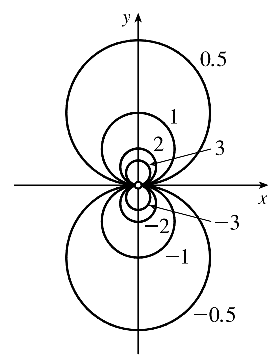
50. The level curves are $y - \arctan x = k$ or $y = (\arctan x) + k$, a family of vertical translations of the graph of the inverse tangent function $y = \arctan x$.



51. The level curves are $\sqrt[3]{x^2 + y^2} = k$ or $x^2 + y^2 = k^3$ $(k \geq 0)$, a family of circles centered at the origin with radius $k^{3/2}$.

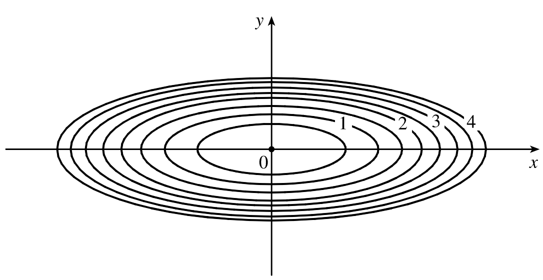


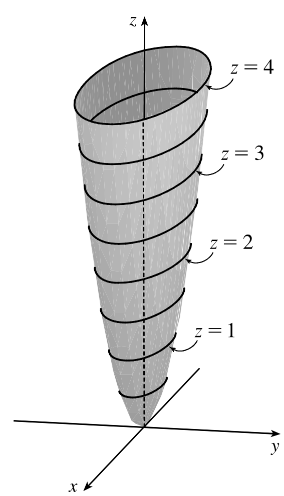
52. For $k \neq 0$ and $(x, y) \neq (0, 0)$, $k = \dfrac{y}{x^2 + y^2}$ $\Leftrightarrow$ $x^2 + y^2 - \dfrac{y}{k} = 0$ $\Leftrightarrow$ $x^2 + \left(y - \frac{1}{2k}\right)^2 = \dfrac{1}{4k^2}$, a family of circles with center $\left(0, \frac{1}{2k}\right)$ and radius $\frac{1}{2k}$ (without the origin). If $k = 0$, the level curve is the x-axis.



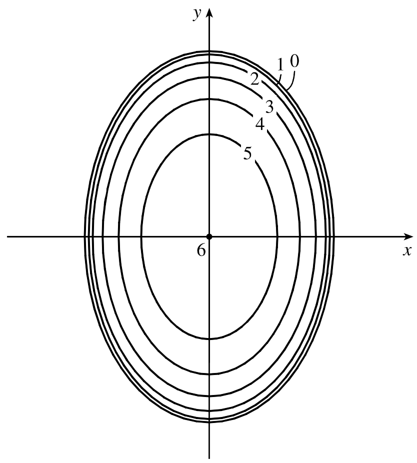
53. The contour map consists of the level curves $k = x^2 + 9y^2$, a family of ellipses with major axis the x-axis. (Or, if $k = 0$, the origin.)

The graph of $f(x, y)$ is the surface $z = x^2 + 9y^2$, an elliptic paraboloid.

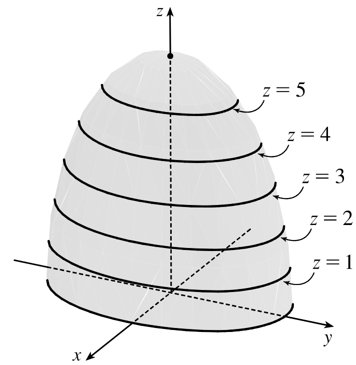




If we visualize lifting each ellipse $k = x^2 + 9y^2$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f.

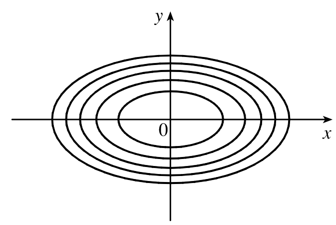
54. 

The contour map consists of the level curves $k = \sqrt{36 - 9x^2 - 4y^2} \Rightarrow 9x^2 + 4y^2 = 36 - k^2$, $k \geq 0$, a family of ellipses with major axis the y-axis. (Or, if $k = 6$, the origin.)



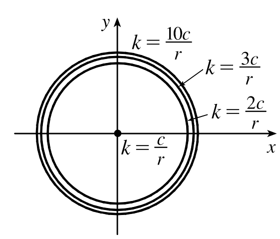
The graph of $f(x, y)$ is the surface $z = \sqrt{36 - 9x^2 - 4y^2}$, or equivalently the upper half of the ellipsoid $9x^2 + 4y^2 + z^2 = 36$. If we visualize lifting each ellipse $k = \sqrt{36 - 9x^2 - 4y^2}$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f.

55. The isothermals are given by $k = 100/(1 + x^2 + 2y^2)$ or $x^2 + 2y^2 = (100 - k)/k$ $[0 < k \leq 100]$, a family of ellipses.

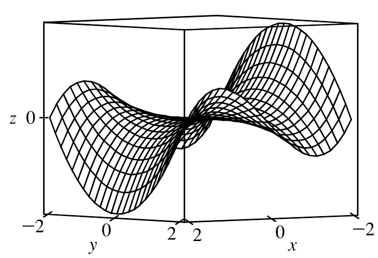


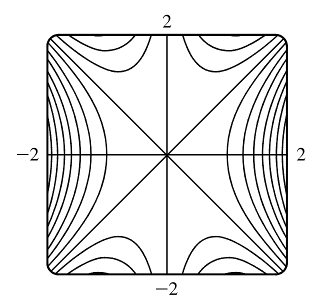
56. The equipotential curves are $k = \dfrac{c}{\sqrt{r^2 - x^2 - y^2}}$ or $x^2 + y^2 = r^2 - \left(\dfrac{c}{k}\right)^2$, a family of circles ($k \geq c/r$).

Note: As $k \to \infty$, the radius of the circle approaches r.



57. $f(x, y) = xy^2 - x^3$

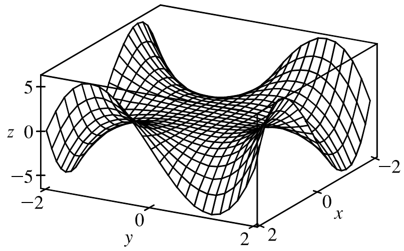


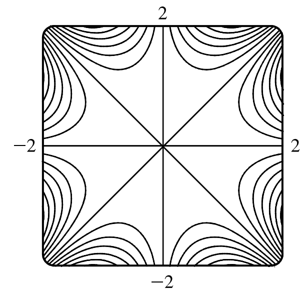


The traces parallel to the yz-plane (such as the left-front trace in the graph above) are parabolas; those parallel to the xz-plane (such as the right-front trace) are cubic curves. The surface is called a monkey saddle because a monkey sitting on the surface near the origin has places for both legs and tail to rest.

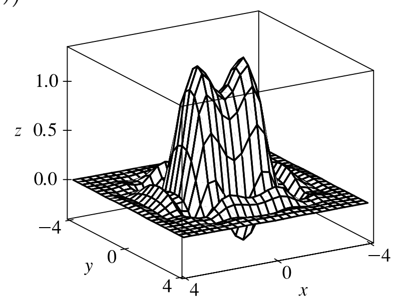
58. $f(x,y) = xy^3 - yx^3$

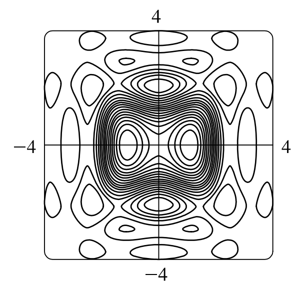
The traces parallel to either the yz-plane or the xz-plane are cubic curves.





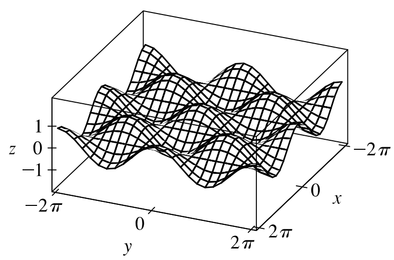
59. $f(x,y) = e^{-(x^2+y^2)/3}\left(\sin(x^2) + \cos(y^2)\right)$

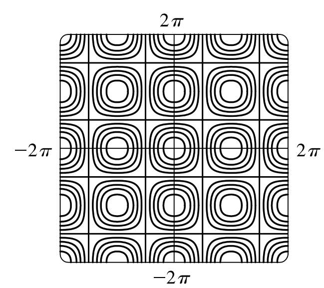




60. $f(x,y) = \cos x\, \cos y$

The traces parallel to either the yz- or xz-plane are cosine curves with amplitudes that vary from 0 to 1.





61. $z = \sin(xy)$ (a) C (b) II

Reasons: This function is periodic in both x and y, and the function is the same when x is interchanged with y, so its graph is symmetric about the plane $y = x$. In addition, the function is 0 along the x- and y-axes. These conditions are satisfied only by C and II.

62. $z = e^x \cos y$ (a) A (b) IV

Reasons: This function is periodic in y but not x, a condition satisfied only by A and IV. Also, note that traces in $x = k$ are cosine curves with amplitude that increases as x increases.

63. $z = \sin(x - y)$ (a) F (b) I

Reasons: This function is periodic in both x and y but is constant along the lines $y = x + k$, a condition satisfied only by F and I.

64. $z = \sin x - \sin y$ (a) E (b) III

Reasons: This function is periodic in both x and y, but unlike the function in Exercise 63, it is not constant along lines such as $y = x + \pi$, so the contour map is III. Also notice that traces in $y = k$ are vertically shifted copies of the sine wave $z = \sin x$, so the graph must be E.

65. $z = (1 - x^2)(1 - y^2)$ (a) B (b) VI

Reasons: This function is 0 along the lines $x = \pm 1$ and $y = \pm 1$. The only contour map in which this could occur is VI. Also note that the trace in the xz-plane is the parabola $z = 1 - x^2$ and the trace in the yz-plane is the parabola $z = 1 - y^2$, so the graph is B.

66. $z = \dfrac{x - y}{1 + x^2 + y^2}$ (a) D (b) V

Reasons: This function is not periodic, ruling out the graphs in A, C, E, and F. Also, the values of z approach 0 as we use points farther from the origin. The only graph that shows this behavior is D, which corresponds to V.

67. $k = x + 3y + 5z$ is a family of parallel planes with normal vector $\langle 1, 3, 5 \rangle$.

68. $k = x^2 + 3y^2 + 5z^2$ is a family of ellipsoids for $k > 0$ and the origin for $k = 0$.

69. Equations for the level surfaces are $k = y^2 + z^2$. For $k > 0$, we have a family of circular cylinders with axis the x-axis and radius $\sqrt{k}$. When $k = 0$ the level surface is the x-axis. (There are no level surfaces for $k < 0$.)

70. Equations for the level surfaces are $x^2 - y^2 - z^2 = k$. For $k = 0$, the equation becomes $y^2 + z^2 = x^2$ and the surface is a right circular cone with vertex the origin and axis the x-axis. For $k > 0$, we have a family of hyperboloids of two sheets with axis the x-axis, and for $k < 0$, we have a family of hyperboloids of one sheet with axis the x-axis.

71. (a) The graph of g is the graph of f shifted upward 2 units.

(b) The graph of g is the graph of f stretched vertically by a factor of 2.

(c) The graph of g is the graph of f reflected about the xy-plane.

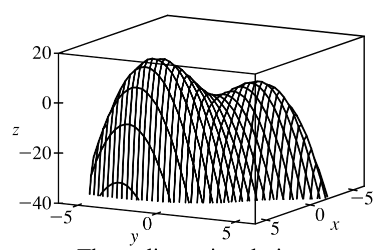
(d) The graph of $g(x, y) = -f(x, y) + 2$ is the graph of f reflected about the xy-plane and then shifted upward 2 units.

72. (a) The graph of g is the graph of f shifted 2 units in the positive x-direction.

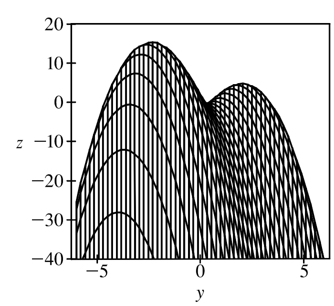
(b) The graph of g is the graph of f shifted 2 units in the negative y-direction.

(c) The graph of g is the graph of f shifted 3 units in the negative x-direction and 4 units in the positive y-direction.

73. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$



Three-dimensional view

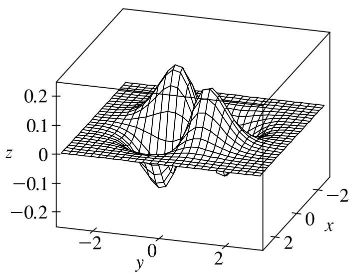


Front view

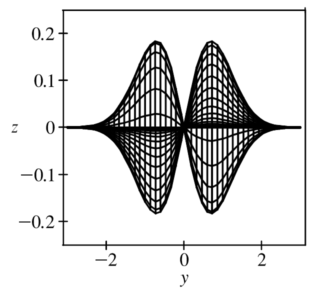
[continued]

It does appear that the function has a maximum value, at the higher of the two "hilltops." From the front view graph, the maximum value appears to be approximately 15. Both hilltops could be considered local maximum points, as the values of f there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

74. $f(x, y) = xye^{-x^2-y^2}$



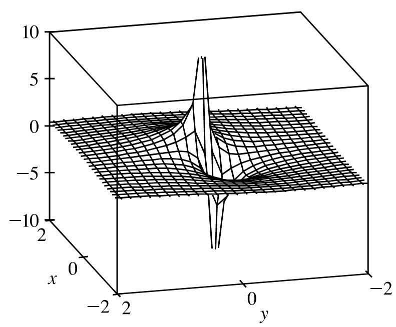
Three-dimensional view



Front view

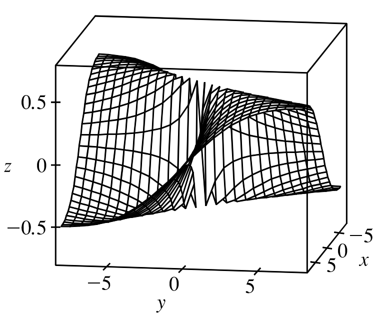
The function does have a maximum value, which it appears to achieve at two different points (the two "hilltops"). From the front view graph, we can estimate the maximum value to be approximately 0.18. These same two points can also be considered local maximum points. The two "valley bottoms" visible in the graph can be considered local minimum points, as all the neighboring points give greater values of f.

75.



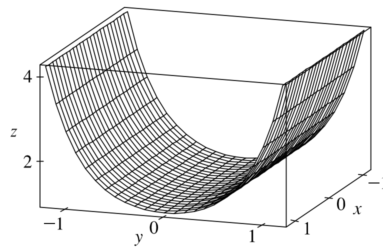
$f(x, y) = \dfrac{x + y}{x^2 + y^2}$. As both x and y become large, the function values appear to approach 0, regardless of which direction is considered. As (x, y) approaches the origin, the graph exhibits asymptotic behavior. From some directions, $f(x, y) \to \infty$, while in others $f(x, y) \to -\infty$. (These are the vertical spikes visible in the graph.) If the graph is examined carefully, however, one can see that $f(x, y)$ approaches 0 along the line $y = -x$.

76.



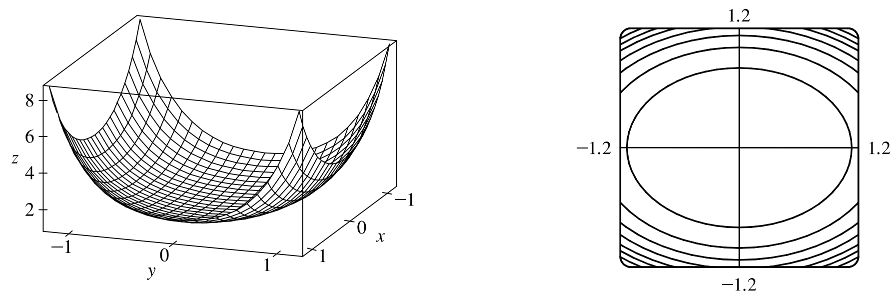
$f(x, y) = \dfrac{xy}{x^2 + y^2}$. The graph exhibits different limiting values as x and y become large or as (x, y) approaches the origin, depending on the direction being examined. For example, although f is undefined at the origin, the function values appear to be $\frac{1}{2}$ along the line $y = x$, regardless of the distance from the origin. Along the line $y = -x$, the value is always $-\frac{1}{2}$. Along the axes, $f(x, y) = 0$ for all values of (x, y) except the origin. Other directions, heading toward the origin or away from the origin, give various limiting values between $-\frac{1}{2}$ and $\frac{1}{2}$.

77. $f(x, y) = e^{cx^2+y^2}$. First, if $c = 0$, the graph is the cylindrical surface $z = e^{y^2}$ (whose level curves are parallel lines). When $c > 0$, the vertical trace above the y-axis remains fixed while the sides of the surface in the x-direction "curl" upward, giving the graph a shape resembling an elliptic paraboloid. The level curves of the surface are ellipses centered at the origin.



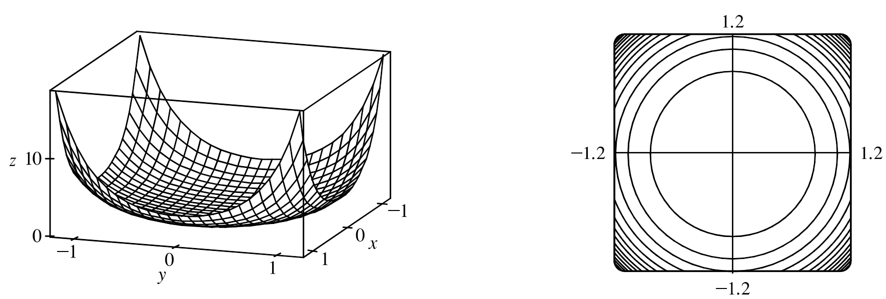
$c = 0$

For $0 < c < 1$, the ellipses have major axis the x-axis and the eccentricity increases as $c \to 0$.



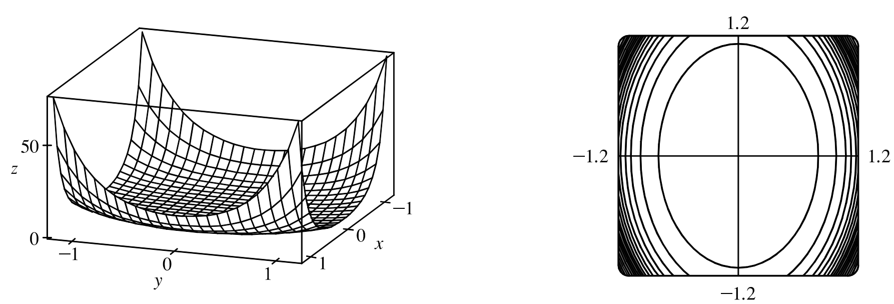
$c = 0.5$ (level curves in increments of 1)

For $c = 1$ the level curves are circles centered at the origin.



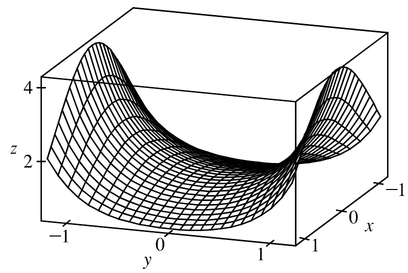
$c = 1$ (level curves in increments of 1)

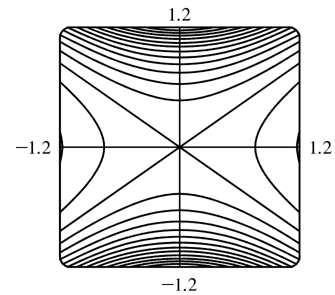
When $c > 1$, the level curves are ellipses with major axis the y-axis, and the eccentricity increases as c increases.



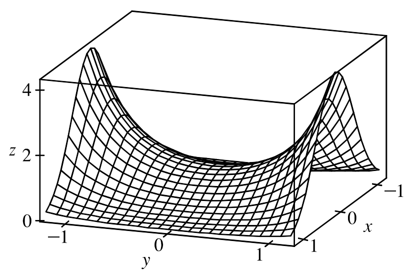
$c = 2$ (level curves in increments of 4)

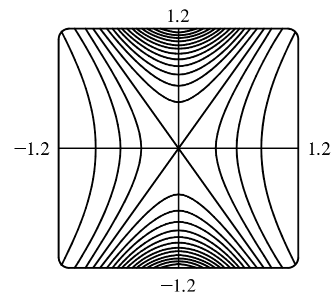
For values of $c < 0$, the sides of the surface in the x-direction curl downward and approach the xy-plane (while the vertical trace $x = 0$ remains fixed), giving a saddle-shaped appearance to the graph near the point $(0, 0, 1)$. The level curves consist of a family of hyperbolas. As c decreases, the surface becomes flatter in the x-direction and the surface's approach to the curve in the trace $x = 0$ becomes steeper, as the graphs demonstrate.





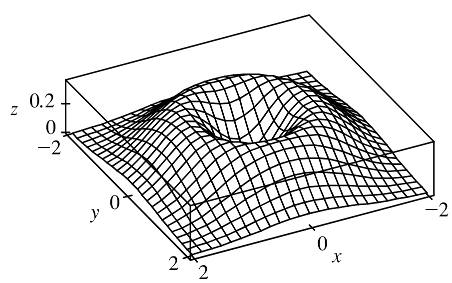
$c = -0.5$ (level curves in increments of 0.25)



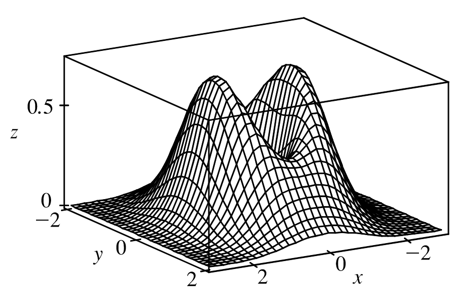


$c = -2$ (level curves in increments of 0.25)

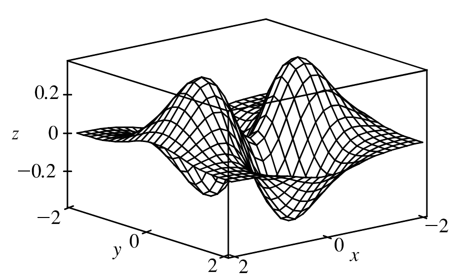
78. $z = (ax^2 + by^2)e^{-x^2-y^2}$. There are only three basic shapes which can be obtained (the fourth and fifth graphs are the reflections of the first and second ones in the xy-plane). Interchanging a and b rotates the graph by 90° about the z-axis.



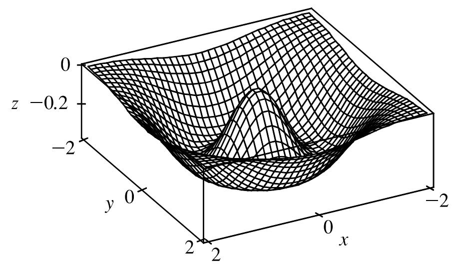
$a = 1, b = 1$



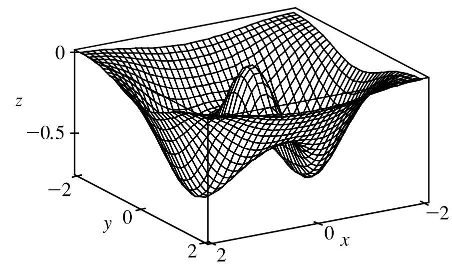
$a = 2, b = 1$



$a = 1, b = -1$



$a = -1, b = -1$

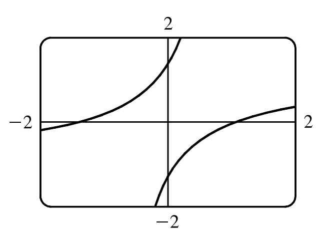


$a = -2, b = -1$

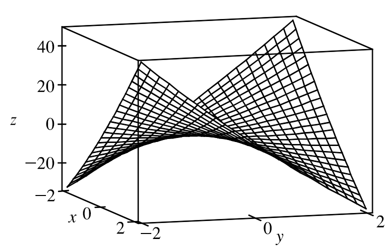
If a and b are both positive ($a \neq b$), we see that the graph has two maximum points whose height increases as a and b increase. If a and b have opposite signs, the graph has two maximum points and two minimum points, and if a and b are both negative, the graph has one maximum point and two minimum points.

79. $z = x^2 + y^2 + cxy$. When $c < -2$, the surface intersects the plane $z = k \neq 0$ in a hyperbola. (See the following graph.) It intersects the plane $x = y$ in the parabola $z = (2 + c)x^2$, and the plane $x = -y$ in the parabola $z = (2 - c)x^2$. These parabolas open in opposite directions, so the surface is a hyperbolic paraboloid.

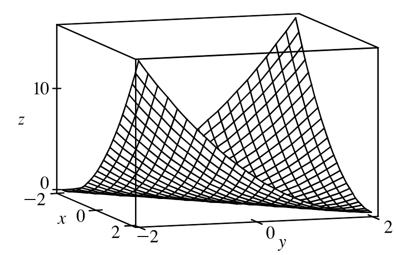
When $c = -2$ the surface is $z = x^2 + y^2 - 2xy = (x - y)^2$. So the surface is constant along each line $x - y = k$. That is, the surface is a cylinder with axis $x - y = 0$, $z = 0$. The shape of the cylinder is determined by its intersection with the plane $x + y = 0$, where $z = 4x^2$, and hence the cylinder is parabolic with minima of 0 on the line $y = x$.



$c = -5, z = 2$



$c = -10$

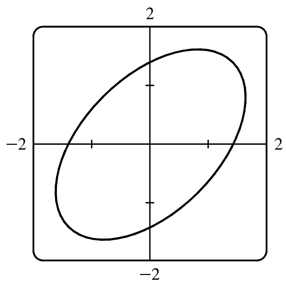


$c = -2$

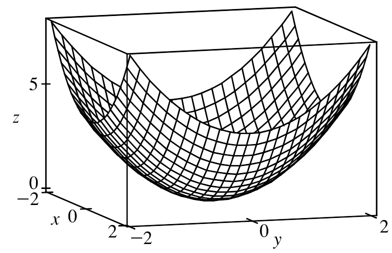
When $-2 < c \leq 0$, $z \geq 0$ for all x and y. If x and y have the same sign, then $x^2 + y^2 + cxy \geq x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. If they have opposite signs, then $cxy \geq 0$. The intersection with the surface and the plane $z = k > 0$ is an ellipse (see graph below). The intersection with the surface and the planes $x = 0$ and $y = 0$ are parabolas $z = y^2$ and $z = x^2$ respectively, so the surface is an elliptic paraboloid.

When $c > 0$ the graphs have the same shape, but are reflected in the plane $x = 0$, because $x^2 + y^2 + cxy = (-x)^2 + y^2 + (-c)(-x)y$. That is, the value of z is the same for c at (x, y) as it is for $-c$ at $(-x, y)$.

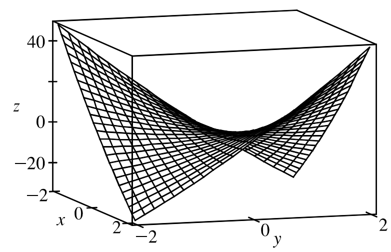
[continued]



$c = -1, z = 2$



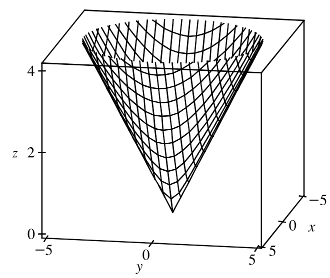
$c = 0$



$c = 10$

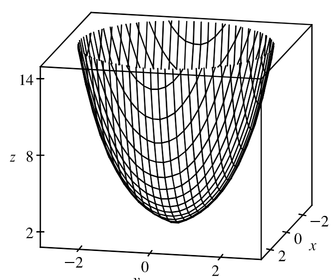
So the surface is an elliptic paraboloid for $0 < c < 2$, a parabolic cylinder for $c = 2$, and a hyperbolic paraboloid for $c > 2$.

80. First, we graph $f(x, y) = \sqrt{x^2 + y^2}$.

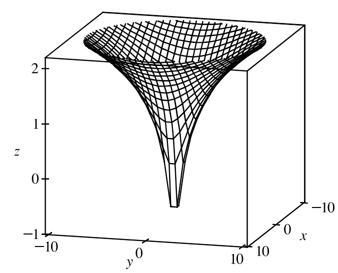


$f(x, y) = \sqrt{x^2 + y^2}$

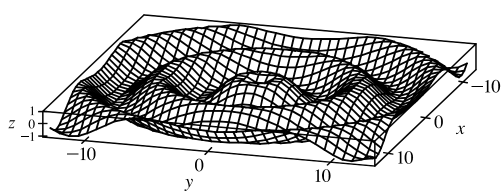
Graphs of the other four functions follow.



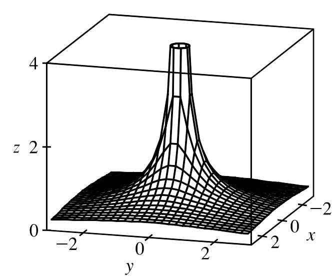
$f(x, y) = e^{\sqrt{x^2 + y^2}}$



$f(x, y) = \ln \sqrt{x^2 + y^2}$



$f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$



$f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$

Notice that each graph $f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$ exhibits radial symmetry about the z-axis and the trace in the xz-plane for

$x \geq 0$ is the graph of $z = g(x)$, $x \geq 0$. This suggests that the graph of $f(x,y) = g\left(\sqrt{x^2+y^2}\right)$ is obtained from the graph of g by graphing $z = g(x)$ in the xz-plane and rotating the curve about the z-axis.

81. (a) $P = bL^\alpha K^{1-\alpha} \quad\Rightarrow\quad \dfrac{P}{K} = bL^\alpha K^{-\alpha} \quad\Rightarrow\quad \dfrac{P}{K} = b\left(\dfrac{L}{K}\right)^\alpha \quad\Rightarrow\quad \ln\dfrac{P}{K} = \ln\left(b\left(\dfrac{L}{K}\right)^\alpha\right) \quad\Rightarrow$

$\ln\dfrac{P}{K} = \ln b + \alpha\ln\left(\dfrac{L}{K}\right)$

(b) We list the values for $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. (Historically, these values were rounded to 2 decimal places.)

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1899	0	0
1900	-0.02	-0.06
1901	-0.04	-0.02
1902	-0.04	0
1903	-0.07	-0.05
1904	-0.13	-0.12
1905	-0.18	-0.04
1906	-0.20	-0.07
1907	-0.23	-0.15
1908	-0.41	-0.38
1909	-0.33	-0.24
1910	-0.35	-0.27

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1911	-0.38	-0.34
1912	-0.38	-0.24
1913	-0.41	-0.25
1914	-0.47	-0.37
1915	-0.53	-0.34
1916	-0.49	-0.28
1917	-0.53	-0.39
1918	-0.60	-0.50
1919	-0.68	-0.57
1920	-0.74	-0.57
1921	-1.05	-0.85
1922	-0.98	-0.59

After entering the (x,y) pairs into a calculator or CAS, the resulting least squares regression line through the points is approximately $y = 0.75136x + 0.01053$, which we round to $y = 0.75x + 0.01$.

(c) Comparing the regression line from part (b) to the equation $y = \ln b + \alpha x$ with $x = \ln(L/K)$ and $y = \ln(P/K)$, we have $\alpha = 0.75$ and $\ln b = 0.01 \quad\Rightarrow\quad b = e^{0.01} \approx 1.01$. Thus, the Cobb-Douglas production function is $P = bL^\alpha K^{1-\alpha} = 1.01L^{0.75}K^{0.25}$.

14.2 Limits and Continuity

1. In general, we can't say anything about $f(3,1)$! $\displaystyle\lim_{(x,y)\to(3,1)} f(x,y) = 6$ means that the values of $f(x,y)$ approach 6 as (x,y) approaches, but is not equal to, $(3,1)$. If f is continuous, we know that $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, so $\displaystyle\lim_{(x,y)\to(3,1)} f(x,y) = f(3,1) = 6$.

2. (a) The outdoor temperature as a function of longitude, latitude, and time is continuous. Small changes in longitude, latitude, or time can produce only small changes in temperature, as the temperature doesn't jump abruptly from one value to another.

(b) Elevation is not necessarily continuous. If we think of a cliff with a sudden drop-off, a very small change in longitude or latitude can produce a comparatively large change in elevation, without all the intermediate values being attained. Elevation *can* jump from one value to another.

(c) The cost of a taxi ride is usually discontinuous. The cost normally increases in jumps, so small changes in distance traveled or time can produce a jump in cost. A graph of the function would show breaks in the surface.

3. We make a table of values of $f(x,y) = \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$ for a set of (x,y) points near the origin.

$x \backslash y$	-0.2	-0.1	-0.05	0	0.05	0.1	0.2
-0.2	-2.551	-2.525	-2.513	-2.500	-2.488	-2.475	-2.451
-0.1	-2.525	-2.513	-2.506	-2.500	-2.494	-2.488	-2.475
-0.05	-2.513	-2.506	-2.503	-2.500	-2.497	-2.494	-2.488
0	-2.500	-2.500	-2.500		-2.500	-2.500	-2.500
0.05	-2.488	-2.494	-2.497	-2.500	-2.503	-2.506	-2.513
0.1	-2.475	-2.488	-2.494	-2.500	-2.506	-2.513	-2.525
0.2	-2.451	-2.475	-2.488	-2.500	-2.513	-2.525	-2.551

As the table shows, the values of $f(x,y)$ seem to approach -2.5 as (x,y) approaches the origin from a variety of different directions. This suggests that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = -2.5$. Since f is a rational function, it is continuous on its domain. f is defined at $(0,0)$, so we can use direct substitution to establish that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \dfrac{0^20^3 + 0^30^2 - 5}{2 - 0 \cdot 0} = -\dfrac{5}{2}$, verifying our guess.

4. We make a table of values of $f(x,y) = \dfrac{2xy}{x^2 + 2y^2}$ for a set of (x,y) points near the origin.

$x \backslash y$	-0.3	-0.2	-0.1	0	0.1	0.2	0.3
-0.3	0.667	0.706	0.545	0.000	-0.545	-0.706	-0.667
-0.2	0.545	0.667	0.667	0.000	-0.667	-0.667	-0.545
-0.1	0.316	0.444	0.667	0.000	-0.667	-0.444	-0.316
0	0.000	0.000	0.000		0.000	0.000	0.000
0.1	-0.316	-0.444	-0.667	0.000	0.667	0.444	0.316
0.2	-0.545	-0.667	-0.667	0.000	0.667	0.667	0.545
0.3	-0.667	-0.706	-0.545	0.000	0.545	0.706	0.667

It appears from the table that the values of $f(x,y)$ are not approaching a single value as (x,y) approaches the origin. For verification, if we first approach $(0,0)$ along the x-axis, we have $f(x,0) = 0$, so $f(x,y) \to 0$. But if we approach $(0,0)$ along the line $y = x$, $f(x,x) = \dfrac{2x^2}{x^2 + 2x^2} = \dfrac{2}{3}$ $(x \neq 0)$, so $f(x,y) \to \frac{2}{3}$. Since f approaches different values along different paths to the origin, this limit does not exist.

5. $f(x,y) = x^2y^3 - 4y^2$ is a polynomial, and hence continuous, so we can find the limit by direct substitution:
$\lim\limits_{(x,y)\to(3,2)} f(x,y) = f(3,2) = (3)^2(2)^3 - 4(2)^2 = 56.$

6. $f(x,y) = \dfrac{x^2y + xy^2}{x^2 - y^2}$ is a rational function and hence continuous on its domain.

$(2,-1)$ is in the domain of f, so f is continuous there and $\lim\limits_{(x,y)\to(2,-1)} f(x,y) = f(2,-1) = \dfrac{(2)^2(-1) + (2)(-1)^2}{(2)^2 - (-1)^2} = -\dfrac{2}{3}$.

7. $x - y$ is a polynomial and therefore continuous. Since $\sin t$ is a continuous function, the composition $\sin(x-y)$ is also continuous. The function y is a polynomial, and hence continuous, and the product of continuous functions is continuous, so $f(x,y) = y\sin(x-y)$ is a continuous function. Then $\lim\limits_{(x,y)\to(\pi,\pi/2)} f(x,y) = f\left(\pi, \frac{\pi}{2}\right) = \frac{\pi}{2}\sin\left(\pi - \frac{\pi}{2}\right) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$.

8. $2x - y$ is a polynomial and therefore continuous. Since $\sqrt{t}$ is continuous for $t \geq 0$, the composition $\sqrt{2x-y}$ is continuous where $2x - y \geq 0$. The function e^u is continuous everywhere, so the composition $f(x,y) = e^{\sqrt{2x-y}}$ is a continuous function for $2x - y \geq 0$. If $x = 3$ and $y = 2$ then $2x - y \geq 0$, so $\lim\limits_{(x,y)\to(3,2)} f(x,y) = f(3,2) = e^{\sqrt{2(3)-2}} = e^2$.

9. $f(x,y) = (x^4 - 4y^2)/(x^2 + 2y^2)$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = x^4/x^2 = x^2$ for $x \neq 0$, so $f(x,y) \to 0$. Now approach $(0,0)$ along the y-axis. For $y \neq 0$, $f(0,y) = -4y^2/2y^2 = -2$, so $f(x,y) \to -2$. Since f has two different limits along two different lines, the limit does not exist.

10. $f(x,y) = (5y^4\cos^2 x)/(x^4 + y^4)$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = 0/x^4 = 0$ for $x \neq 0$, so $f(x,y) \to 0$. Next approach $(0,0)$ along the y-axis. For $y \neq 0$, $f(0,y) = 5y^4/y^4 = 5$, so $f(x,y) \to 5$. Since f has two different limits along two different lines, the limit does not exist.

11. $f(x,y) = (y^2\sin^2 x)/(x^4 + y^4)$. On the x-axis, $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the line $y = x$, $f(x,x) = \dfrac{x^2\sin^2 x}{x^4 + x^4} = \dfrac{\sin^2 x}{2x^2} = \dfrac{1}{2}\left(\dfrac{\sin x}{x}\right)^2$ for $x \neq 0$ and $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$, so $f(x,y) \to \frac{1}{2}$. Since f has two different limits along two different lines, the limit does not exist.

12. $f(x,y) = \dfrac{xy - y}{(x-1)^2 + y^2}$. On the x-axis, $f(x,0) = 0/(x-1)^2 = 0$ for $x \neq 1$, so $f(x,y) \to 0$ as $(x,y) \to (1,0)$ along the x-axis. Approaching $(1,0)$ along the line $y = x - 1$, $f(x,x-1) = \dfrac{x(x-1) - (x-1)}{(x-1)^2 + (x-1)^2} = \dfrac{(x-1)^2}{2(x-1)^2} = \dfrac{1}{2}$ for $x \neq 1$, so $f(x,y) \to \frac{1}{2}$ along this line. Thus the limit does not exist.

13. $f(x,y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$. We can see that the limit along any line through $(0,0)$ is 0, as well as along other paths through $(0,0)$ such as $x = y^2$ and $y = x^2$. So we suspect that the limit exists and equals 0; we use the Squeeze Theorem to prove our assertion. Since $|y| \leq \sqrt{x^2 + y^2}$, we have $\dfrac{|y|}{\sqrt{x^2+y^2}} \leq 1$ and so $0 \leq \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| \leq |x|$. Now $|x| \to 0$ as $(x,y) \to (0,0)$, so $\left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| \to 0$ and hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

14. $f(x,y)=\dfrac{x^3-y^3}{x^2+xy+y^2}=\dfrac{(x-y)(x^2+xy+y^2)}{x^2+xy+y^2}=x-y$ for $(x,y)\neq(0,0)$. [Note that $x^2+xy+y^2=0$ only when $(x,y)=(0,0)$.] Thus $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}(x-y)=0-0=0$.

15. Let $f(x,y)=\dfrac{xy^2\cos y}{x^2+y^4}$. Then $f(x,0)=0$ for $x\neq 0$, so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x-axis. Approaching $(0,0)$ along the y-axis or the line $y=x$ also gives a limit of 0. But $f\left(y^2,y\right)=\dfrac{y^2y^2\cos y}{(y^2)^2+y^4}=\dfrac{y^4\cos y}{2y^4}=\dfrac{\cos y}{2}$ for $y\neq 0$, so $f(x,y)\to\frac{1}{2}\cos 0=\frac{1}{2}$ as $(x,y)\to(0,0)$ along the parabola $x=y^2$. Thus the limit doesn't exist.

16. We can use the Squeeze Theorem to show that $\displaystyle\lim_{(x,y)\to(0,0)}\frac{xy^4}{x^4+y^4}=0$:

$0\leq\dfrac{|x|\,y^4}{x^4+y^4}\leq|x|$ since $0\leq\dfrac{y^4}{x^4+y^4}\leq 1$, and $|x|\to 0$ as $(x,y)\to(0,0)$, so $\dfrac{|x|\,y^4}{x^4+y^4}\to 0\quad\Rightarrow\quad\dfrac{xy^4}{x^4+y^4}\to 0$ as $(x,y)\to(0,0)$.

17.
$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}&=\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\cdot\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}\\&=\lim_{(x,y)\to(0,0)}\frac{\left(x^2+y^2\right)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2}=\lim_{(x,y)\to(0,0)}\left(\sqrt{x^2+y^2+1}+1\right)=2\end{aligned}$$

18. $f(x,y)=xy^4/(x^2+y^8)$. On the x-axis, $f(x,0)=0$ for $x\neq 0$, so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x-axis. Approaching $(0,0)$ along the curve $x=y^4$ gives $f(y^4,y)=y^8/2y^8=\frac{1}{2}$ for $y\neq 0$, so along this path $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$. Thus the limit does not exist.

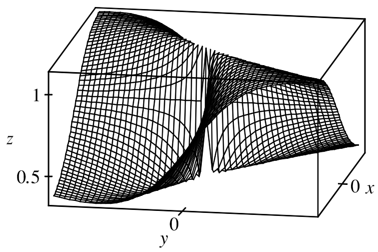
19. e^{y^2} is a composition of continuous functions and hence continuous. xz is a continuous function and $\tan t$ is continuous for $t\neq\frac{\pi}{2}+n\pi$ (n an integer), so the composition $\tan(xz)$ is continuous for $xz\neq\frac{\pi}{2}+n\pi$. Thus the product $f(x,y,z)=e^{y^2}\tan(xz)$ is a continuous function for $xz\neq\frac{\pi}{2}+n\pi$. If $x=\pi$ and $z=\frac{1}{3}$ then $xz\neq\frac{\pi}{2}+n\pi$, so $\displaystyle\lim_{(x,y,z)\to(\pi,0,1/3)} f(x,y,z)=f(\pi,0,1/3)=e^{0^2}\tan(\pi\cdot 1/3)=1\cdot\tan(\pi/3)=\sqrt{3}$.

20. $f(x,y,z)=\dfrac{xy+yz}{x^2+y^2+z^2}$. Then $f(x,0,0)=0/x^2=0$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the x-axis, $f(x,y,z)\to 0$. But $f(x,x,0)=x^2/(2x^2)=\frac{1}{2}$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the line $y=x$, $z=0$, $f(x,y,z)\to\frac{1}{2}$. Thus the limit doesn't exist.

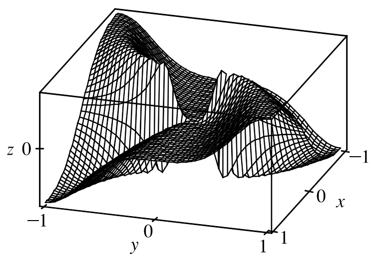
21. $f(x,y,z)=\dfrac{xy+yz^2+xz^2}{x^2+y^2+z^4}$. Then $f(x,0,0)=0/x^2=0$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the x-axis, $f(x,y,z)\to 0$. But $f(x,x,0)=x^2/(2x^2)=\frac{1}{2}$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the line $y=x$, $z=0$, $f(x,y,z)\to\frac{1}{2}$. Thus the limit doesn't exist.

22. We can use the Squeeze Theorem to show that $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2+y^2+z^2} = 0$:

$0 \le \dfrac{x^2y^2z^2}{x^2+y^2+z^2} \le x^2y^2$ since $0 \le \dfrac{z^2}{x^2+y^2+z^2} \le 1$, and $x^2y^2 \to 0$ as $(x,y,z) \to (0,0,0)$, so $\dfrac{x^2y^2z^2}{x^2+y^2+z^2} \to 0$ as $(x,y,z) \to (0,0,0)$.

23. 

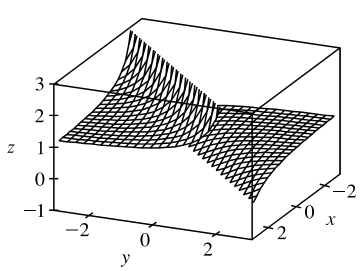
From the ridges on the graph, we see that as $(x,y) \to (0,0)$ along the lines under the two ridges, $f(x,y)$ approaches different values. So the limit does not exist.

24. 

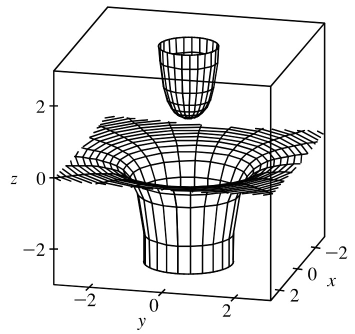
From the graph, it appears that as we approach the origin along the lines $x = 0$ or $y = 0$, the function is everywhere 0, whereas if we approach the origin along a certain curve it has a constant value of about $\frac{1}{2}$. [In fact, $f(y^3, y) = y^6/(2y^6) = \frac{1}{2}$ for $y \ne 0$, so $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along the curve $x = y^3$.] Since the function approaches different values depending on the path of approach, the limit does not exist.

25. $h(x,y) = g(f(x,y)) = (2x+3y-6)^2 + \sqrt{2x+3y-6}$. Since f is a polynomial, it is continuous on $\mathbb{R}^2$ and g is continuous on its domain $\{t \mid t \ge 0\}$. Thus h is continuous on its domain $\{(x,y) \mid 2x+3y-6 \ge 0\} = \left\{(x,y) \mid y \ge -\frac{2}{3}x + 2\right\}$, which consists of all points on or above the line $y = -\frac{2}{3}x + 2$.

26. $h(x,y) = g(f(x,y)) = \dfrac{1-xy}{1+x^2y^2} + \ln\left(\dfrac{1-xy}{1+x^2y^2}\right)$. f is a rational function, so it is continuous on its domain. Because $1 + x^2y^2 > 0$, the domain of f is $\mathbb{R}^2$, so f is continuous everywhere. g is continuous on its domain $\{t \mid t > 0\}$. Thus h is continuous on its domain $\left\{(x,y) \,\middle|\, \dfrac{1-xy}{1+x^2y^2} > 0\right\} = \{(x,y) \mid xy < 1\}$ which consists of all points between (but not on) the two branches of the hyperbola $y = 1/x$.

27. 

From the graph, it appears that f is discontinuous along the line $y = x$. If we consider $f(x,y) = e^{1/(x-y)}$ as a composition of functions, $g(x,y) = 1/(x-y)$ is a rational function and therefore continuous except where $x - y = 0 \quad \Leftrightarrow \quad y = x$. Since the function $h(t) = e^t$ is continuous everywhere, the composition $h(g(x,y)) = e^{1/(x-y)} = f(x,y)$ is continuous except along the line $y = x$, as we suspected.

28. 

We can see a circular break in the graph, corresponding approximately to the unit circle, where f is discontinuous. Since $f(x,y)=\dfrac{1}{1-x^2-y^2}$ is a rational function, it is continuous except where $1-x^2-y^2=0 \quad \Leftrightarrow$ $x^2+y^2=1$, confirming our observation that f is discontinuous on the circle $x^2+y^2=1$.

29. The functions xy and $1+e^{x-y}$ are continuous everywhere, and $1+e^{x-y}$ is never zero, so $F(x,y)=\dfrac{xy}{1+e^{x-y}}$ is continuous on its domain $\mathbb{R}^2$.

30. $F(x,y)=\cos\sqrt{1+x-y}=g(f(x,y))$ where $f(x,y)=\sqrt{1+x-y}$, continuous on its domain $\{(x,y)\mid 1+x-y\geq 0\}=\{(x,y)\mid y\leq x+1\}$, and $g(t)=\cos t$ is continuous everywhere. Thus F is continuous on its domain $\{(x,y)\mid y\leq x+1\}$.

31. $F(x,y)=\dfrac{1+x^2+y^2}{1-x^2-y^2}$ is a rational function and thus is continuous on its domain $\{(x,y)\mid 1-x^2-y^2\neq 0\}=\{(x,y)\mid x^2+y^2\neq 1\}$.

32. The functions e^x+e^y and $e^{xy}-1$ are continuous everywhere, so $H(x,y)=\dfrac{e^x+e^y}{e^{xy}-1}$ is continuous except where $e^{xy}-1=0 \quad\Rightarrow\quad xy=0 \quad\Rightarrow\quad x=0$ or $y=0$. Thus H is continuous on its domain $\{(x,y)\mid x\neq 0,\ y\neq 0\}$.

33. $\sqrt{x}$ is continuous on its domain $\{(x,y)\mid x\geq 0\}$ and $\sqrt{1-x^2-y^2}$ is continuous on its domain $\{(x,y)\mid 1-x^2-y^2\geq 0\}=\{(x,y)\mid x^2+y^2\leq 1\}$, so the sum $G(x,y)=\sqrt{x}+\sqrt{1-x^2-y^2}$ is continuous for $x\geq 0$ and $x^2+y^2\leq 1$, that is, $\{(x,y)\mid x^2+y^2\leq 1,\ x\geq 0\}$. This is the right half of the unit disk.

34. $G(x,y)=\ln(1+x-y)=g(f(x,y))$ where $f(x,y)=1+x-y$, a polynomial and hence continuous on $\mathbb{R}^2$, and $g(t)=\ln t$, continuous on its domain $\{t\mid t>0\}$. Thus G is continuous on its domain $\{(x,y)\mid 1+x-y>0\}=\{(x,y)\mid y<x+1\}$, the region in $\mathbb{R}^2$ below the line $y=x+1$.

35. $f(x,y,z)=h(g(x,y,z))$ where $g(x,y,z)=x^2+y^2+z^2$, a polynomial that is continuous everywhere, and $h(t)=\arcsin t$, continuous on $[-1,1]$. Thus f is continuous on its domain $\{(x,y,z)\mid -1\leq x^2+y^2+z^2\leq 1\}=\{(x,y,z)\mid x^2+y^2+z^2\leq 1\}$, so f is continuous on the unit ball.

36. $\sqrt{y-x^2}$ is continuous on its domain $\{(x,y)\mid y-x^2\geq 0\}=\{(x,y)\mid y\geq x^2\}$ and $\ln z$ is continuous on its domain $\{z\mid z>0\}$, so the product $f(x,y,z)=\sqrt{y-x^2}\ln z$ is continuous for $y\geq x^2$ and $z>0$, that is, $\{(x,y,z)\mid y\geq x^2,\ z>0\}$.

37. $f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. Since $x^2 \leq 2x^2 + y^2$, we have $\left|x^2y^3/(2x^2+y^2)\right| \leq \left|y^3\right|$. We know that $\left|y^3\right| \to 0$ as $(x,y) \to (0,0)$. So, by the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^3}{2x^2+y^2} = 0$. But $f(0,0) = 1$, so f is discontinuous at $(0,0)$. Therefore, f is continuous on the set $\{(x,y) \mid (x,y) \neq (0,0)\}$.

38. $f(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But $f(x,x) = x^2/(3x^2) = \frac{1}{3}$ for $x \neq 0$, so $f(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$ along the line $y = x$. Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is not continuous at $(0,0)$ and the largest set on which f is continuous is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

39. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2} = \lim\limits_{r\to0^+} \dfrac{(r\cos\theta)^3 + (r\sin\theta)^3}{r^2} = \lim\limits_{r\to0^+} (r\cos^3\theta + r\sin^3\theta) = 0$

40. $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim\limits_{r\to0^+} r^2\ln r^2 = \lim\limits_{r\to0^+} \dfrac{\ln r^2}{1/r^2} = \lim\limits_{r\to0^+} \dfrac{(1/r^2)(2r)}{-2/r^3}$ [using l'Hospital's Rule]

$$= \lim_{r\to0^+} (-r^2) = 0$$

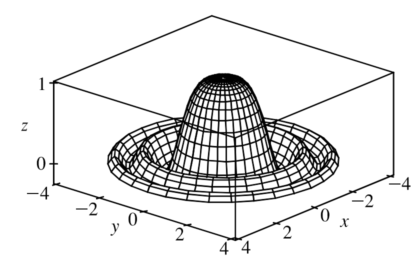
41. $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{-x^2-y^2}-1}{x^2+y^2} = \lim\limits_{r\to0^+} \dfrac{e^{-r^2}-1}{r^2} = \lim\limits_{r\to0^+} \dfrac{e^{-r^2}(-2r)}{2r}$ [using l'Hospital's Rule]

$$= \lim_{r\to0^+} -e^{-r^2} = -e^0 = -1$$

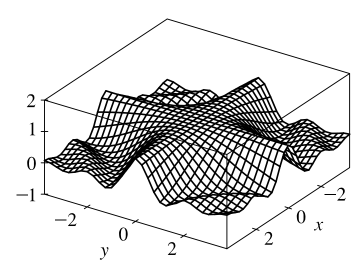
42. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = \lim\limits_{r\to0^+} \dfrac{\sin(r^2)}{r^2}$, which is an indeterminate form of type $0/0$. Using l'Hospital's Rule, we get

$\lim\limits_{r\to0^+} \dfrac{\sin(r^2)}{r^2} \overset{\text{H}}{=} \lim\limits_{r\to0^+} \dfrac{2r\cos(r^2)}{2r} = \lim\limits_{r\to0^+} \cos(r^2) = 1.$

Or: Use the fact that $\lim\limits_{\theta\to0} \dfrac{\sin\theta}{\theta} = 1$.



43. $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{xy} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$



From the graph, it appears that f is continuous everywhere. We know xy is continuous on $\mathbb{R}^2$ and $\sin t$ is continuous everywhere, so $\sin(xy)$ is continuous on $\mathbb{R}^2$ and $\dfrac{\sin(xy)}{xy}$ is continuous on $\mathbb{R}^2$

[continued]

except possibly where $xy = 0$. To show that f is continuous at those points, consider any point (a, b) in $\mathbb{R}^2$ where $ab = 0$. Because xy is continuous, $xy \to ab = 0$ as $(x, y) \to (a, b)$. If we let $t = xy$, then $t \to 0$ as $(x, y) \to (a, b)$ and $\lim\limits_{(x,y)\to(a,b)} \dfrac{\sin(xy)}{xy} = \lim\limits_{t\to 0} \dfrac{\sin(t)}{t} = 1$ by Equation 2.4.2 [ET 3.3.2]. Thus $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$ and f is continuous on $\mathbb{R}^2$.

44. (a) $f(x, y) = \begin{cases} 0 & \text{if } y \le 0 \quad \text{or} \quad y \ge x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$ Consider the path $y = mx^a$, $0 < a < 4$. [The path does not pass through $(0, 0)$ if $a \le 0$ except for the trivial case where $m = 0$.] If $mx^a \le 0$ then $f(x, mx^a) = 0$. If $mx^a > 0$ then $mx^a = |mx^a| = |m|\ |x^a|$ and $mx^a \ge x^4 \quad \Leftrightarrow \quad |m|\ |x^a| \ge x^4 \quad \Leftrightarrow \quad \dfrac{x^4}{|x^a|} \le |m| \quad \Leftrightarrow \quad |x|^{4-a} \le |m|$ whenever x^a is defined. Then $mx^a \ge x^4 \quad \Leftrightarrow \quad |x| \le |m|^{1/(4-a)}$ so $f(x, mx^a) = 0$ for $|x| \le |m|^{1/(4-a)}$ and $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along this path.

(b) If we approach $(0, 0)$ along the path $y = x^5$, $x > 0$ then we have $f(x, x^5) = 1$ for $0 < x < 1$ because $0 < x^5 < x^4$ there. Thus $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along this path, but in part (a) we found a limit of 0 along other paths, so $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ doesn't exist and f is discontinuous at $(0, 0)$.

(c) First we show that f is discontinuous at any point $(a, 0)$ on the x-axis. If we approach $(a, 0)$ along the path $x = a$, $y > 0$ then $f(a, y) = 1$ for $0 < y < a^4$, so $f(x, y) \to 1$ as $(x, y) \to (a, 0)$ along this path. If we approach $(a, 0)$ along the path $x = a$, $y < 0$ then $f(a, y) = 0$ since $y < 0$ and $f(x, y) \to 0$ as $(x, y) \to (a, 0)$. Thus the limit does not exist and f is discontinuous on the line $y = 0$. f is also discontinuous on the curve $y = x^4$: For any point (a, a^4) on this curve, approaching the point along the path $x = a$, $y > a^4$ gives $f(a, y) = 0$ since $y > a^4$, so $f(x, y) \to 0$ as $(x, y) \to (a, a^4)$. But approaching the point along the path $x = a$, $y < a^4$ gives $f(a, y) = 1$ for $y > 0$, so $f(x, y) \to 1$ as $(x, y) \to (a, a^4)$ and the limit does not exist there.

45. Since $|\mathbf{x} - \mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}| \cos\theta \ge |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}| = (|\mathbf{x}| - |\mathbf{a}|)^2$, we have $\big||\mathbf{x}| - |\mathbf{a}|\big| \le |\mathbf{x} - \mathbf{a}|$. Let $\epsilon > 0$ be given and set $\delta = \epsilon$. Then if $0 < |\mathbf{x} - \mathbf{a}| < \delta$, $\big||\mathbf{x}| - |\mathbf{a}|\big| \le |\mathbf{x} - \mathbf{a}| < \delta = \epsilon$. Hence $\lim_{\mathbf{x}\to\mathbf{a}} |\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$.

46. Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that if $0 < |\mathbf{x} - \mathbf{a}| < \delta$ then $|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| < \epsilon$. But $|\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| = |\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})|$ and $|\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})| \le |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}|$ by Exercise 12.3.61 (the Cauchy-Schwartz Inequality). Set $\delta = \epsilon / |\mathbf{c}|$. Then if $0 < |\mathbf{x} - \mathbf{a}| < \delta$, $|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| \le |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}| < |\mathbf{c}|\,\delta = |\mathbf{c}|\,(\epsilon / |\mathbf{c}|) = \epsilon$. So f is continuous on $\mathbb{R}^n$.