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Spring 2001, Math 53M

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Bechtel Auditorium

Final Examination

8:00-11:00 AM

1. (60 points, 6 points apiece) Find the following. If an expression is undefined, say so.

(a) The area of the region of the plane described in polar coordinates by the conditions

$$0 \leq \theta \leq 1, \quad 0 \leq r \leq 1 + e^\theta.$$

(b) A unit vector perpendicular to both $\langle 1, 2, 3 \rangle$ and $\langle 4, 5, 6 \rangle$.

(c) The length of the curve given by $x = t^2$, $y = (t^3/3) - t$, where $-1 \leq t \leq 1$.

(d) $\frac{d}{dt} f(g(t^2), g(t^3))$, where f is a differentiable function of two variables and g is a differentiable function of one variable. The answer should be expressed in terms of f , g , and their derivatives and/or partial derivatives.

(e) $\int_0^1 \int_{1-x}^{1+x} xy \, dy \, dx.$

(f) An expression for $\iiint_E f(x, y, z) \, dV$, as an iterated integral, where $E = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 100\}$, and f is a continuous function. (Do not change coordinates.)

(g) $\int_C \mathbf{F} \cdot d\mathbf{r}$, where \mathbf{F} is the vector field $\langle 1, 2, y \rangle$ and C is the curve given by $\mathbf{r}(t) = \langle t^2, t^3, t^5 \rangle$ for $0 \leq t \leq 10$.

(h) An expression for $\iint_{D_1} f(x, y) \, dx \, dy$ as an integral over D_2 , where D_1 is a region of the x - y -plane, and D_2 is a region of the u - v -plane which is mapped in a one-to-one fashion onto D_1 by the transformation $x = uv$, $y = u^2/v$; and where f is a continuous function on D_1 .

(i) $\int_C xy^{-1} \, dx$, where C is the segment of the curve $y = x^3$ between $x = -1$ and $x = 1$.

(j) $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where \mathbf{F} is the constant vector field $\langle 2, 1, -1 \rangle$, and S is the parallelogram with vertices $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$ and $(1, -2, 3)$, and upward orientation.

2. (12 points; 4 points each) In each part, give the definition asked for. Note that you are *not* asked to give examples or other related information.

(a) Define the partial derivative $f_1(a, b)$ of a function f at a point (a, b) of its domain. (There are many other symbols for this, e.g., $(\partial f / \partial x)(a, b)$, but we are asking for a definition, not alternative notation. A brief general statement of how one would find this partial derivative would be an acceptable answer.)

(b) What does it mean to say that a function f of two variables is *differentiable*?

(c) If f is a function of two variables, and \mathbf{u} a unit vector in the plane, what is meant by the *directional derivative* $D_{\mathbf{u}} f$?

3. (12 points) (a) (6 points) Find constants a and b such that the vector field $\langle 3x^2y \sin y - 2x^2 \cos y, ax^3y \cos y + bx^3 \sin y \rangle$ is the gradient of a function $f(x, y)$. (You do not have to find the function f .)

(b) (6 points) Suppose a and b are as in part (a), so that \mathbf{F} is the gradient of a function. Prove that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every curve C beginning at $(0, 0)$ and ending at $(0, 1)$. (Suggestion: first show this fact for one particular curve. You do not have to have done part (a) to do part (b).) 另一种方法是?

4. (8 points) Let S_1 be the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and S_2 the hemisphere $z = -\sqrt{1 - x^2 - y^2}$, both with upward orientation, and let E be the solid ball $\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$. If \mathbf{F} is any vector field whose component functions have continuous partial derivatives on an open set containing E , write an equation expressing the triple integral over E of the divergence of \mathbf{F} in terms of surface integrals over S_1 and S_2 . 表达式是?

5. (8 points) Let \mathbf{F} be the vector field $\langle xy, yz^2, zx^3 \rangle + \nabla e^{x + \sin y}$, and let S be the part of the surface $z = xy^2(1 - x - y)^3$ lying above the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, with upward orientation. Compute $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$. (Suggestion: Treat the two summands of \mathbf{F} separately.)