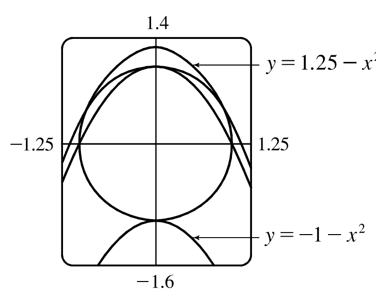
(b) $Q(x,y) = \frac{1}{2} f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2} f_{yy}(0,0)y^2$ fits the form of the polynomial function in Problem 4 with $a = \frac{1}{2} f_{xx}(0,0)$, $b = f_{xy}(0,0)$, and $c = \frac{1}{2} f_{yy}(0,0)$. Then we know Q is a paraboloid, and that Q has a local maximum, local minimum, or saddle point at $(0,0)$. Here, $D = 4ac - b^2 = 4\left(\frac{1}{2}\right) f_{xx}(0,0) \left(\frac{1}{2}\right) f_{yy}(0,0) - [f_{xy}(0,0)]^2 = f_{xx}(0,0) f_{yy}(0,0) - [f_{xy}(0,0)]^2$, and if $D > 0$ with $a = \frac{1}{2} f_{xx}(0,0) > 0 \quad \Rightarrow \quad f_{xx}(0,0) > 0$, we know from Problem 4 that Q has a local minimum at $(0,0)$. Similarly, if $D > 0$ and $a < 0 \quad \Rightarrow \quad f_{xx}(0,0) < 0$, Q has a local maximum at $(0,0)$, and if $D < 0$, Q has a saddle point at $(0,0)$.

(c) Since $f(x,y) \approx Q(x,y)$ near $(0,0)$, part (b) suggests that for $D = f_{xx}(0,0) f_{yy}(0,0) - [f_{xy}(0,0)]^2$, if $D > 0$ and $f_{xx}(0,0) > 0$, f has a local minimum at $(0,0)$. If $D > 0$ and $f_{xx}(0,0) < 0$, f has a local maximum at $(0,0)$, and if $D < 0$, f has a saddle point at $(0,0)$. Together with the conditions given in part (a), this is precisely the Second Derivatives Test from Section 14.7.

14.8 Lagrange Multipliers

1. At the extreme values of f, the level curves of f just touch the curve $g(x,y) = 8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x,y) = c$ with the largest value of c which still intersects the curve $g(x,y) = 8$ is approximately $c = 59$, and the smallest value of c corresponding to a level curve which intersects $g(x,y) = 8$ appears to be $c = 30$. Thus we estimate the maximum value of f subject to the constraint $g(x,y) = 8$ to be about 59 and the minimum to be 30.

2. (a) The values $c = \pm 1$ and $c = 1.25$ seem to give curves which are tangent to the circle. These values represent possible extreme values of the function $x^2 + y$ subject to the constraint $x^2 + y^2 = 1$.



(b) $\nabla f = \langle 2x, 1 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. So $2x = 2\lambda x \quad \Rightarrow \quad$ either $\lambda = 1$ or $x = 0$. If $\lambda = 1$, then $y = \frac{1}{2}$ and so $x = \pm \frac{\sqrt{3}}{2}$ (from the constraint). If $x = 0$, then $y = \pm 1$. Therefore f has possible extreme values at the points $(0, \pm 1)$ and $\left(\pm \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. We calculate $f\left(\pm \frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{5}{4}$ (the maximum value), $f(0,1) = 1$, and $f(0,-1) = -1$ (the minimum value). These are our answers from part (a).

3. We want to find the extreme values of $f(x,y) = x^2 - y^2$ subject to the constraint $g(x,y) = x^2 + y^2 = 1$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$, so we solve the equations $2x = 2\lambda x$, $-2y = 2\lambda y$, and $x^2 + y^2 = 1$. From the first equation we have $2x(\lambda - 1) = 0 \quad \Rightarrow \quad x = 0$ or $\lambda = 1$. If $x = 0$ then substitution into the constraint gives

$y^2 = 1 \quad \Rightarrow \quad y = \pm 1$. If $\lambda = 1$ then substitution into the second equation gives $-2y = 2y \quad \Rightarrow \quad y = 0$, and from the constraint we must have $x = \pm 1$. Thus the possible points for the extreme values of f are $(0, \pm 1)$ and $(\pm 1, 0)$. Evaluating f at these points, we see that the maximum value of f is $f(\pm 1, 0) = 1$ and the minimum is $f(0, \pm 1) = -1$.

4. $f(x, y) = 3x + y$, $g(x, y) = x^2 + y^2 = 10$, and $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 3, 1 \rangle = \langle 2\lambda x, 2\lambda y \rangle$, so $3 = 2\lambda x$, $1 = 2\lambda y$, and $x^2 + y^2 = 10$. From the first two equations we have $\dfrac{3}{2x} = \lambda = \dfrac{1}{2y} \quad \Rightarrow \quad x = 3y$ (note that the first two equations imply $x \neq 0$ and $y \neq 0$) and substitution into the third equation gives $9y^2 + y^2 = 10 \quad \Rightarrow \quad y^2 = 1 \quad \Rightarrow \quad y = \pm 1$. Then f has possible extreme values at the points $(3, 1)$ and $(-3, -1)$. We compute $f(3, 1) = 10$ and $f(-3, -1) = -10$, so the maximum value of f on $x^2 + y^2 = 10$ is $f(3, 1) = 10$ and the minimum value is $f(-3, -1) = -10$.

5. $f(x, y) = xy$, $g(x, y) = 4x^2 + y^2 = 8$, and $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle y, x \rangle = \langle 8\lambda x, 2\lambda y \rangle$, so $y = 8\lambda x$, $x = 2\lambda y$, and $4x^2 + y^2 = 8$. First note that if $x = 0$ then $y = 0$ by the first equation, and if $y = 0$ then $x = 0$ by the second equation. But this contradicts the third equation, so $x \neq 0$ and $y \neq 0$. Then from the first two equations we have
$\dfrac{y}{8x} = \lambda = \dfrac{x}{2y} \quad \Rightarrow \quad 2y^2 = 8x^2 \quad \Rightarrow \quad y^2 = 4x^2$, and substitution into the third equation gives
$4x^2 + 4x^2 = 8 \quad \Rightarrow \quad x = \pm 1$. If $x = \pm 1$ then $y^2 = 4 \quad \Rightarrow \quad y = \pm 2$, so f has possible extreme values at $(1, \pm 2)$ and $(-1, \pm 2)$. Evaluating f at these points, we see that the maximum value is $f(1, 2) = f(-1, -2) = 2$ and the minimum is $f(1, -2) = f(-1, 2) = -2$.

6. $f(x, y) = xe^y$, $g(x, y) = x^2 + y^2 = 2$, and $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle e^y, xe^y \rangle = \langle 2\lambda x, 2\lambda y \rangle$, so $e^y = 2\lambda x$, $xe^y = 2\lambda y$, and $x^2 + y^2 = 2$. First note that from the first equation $x \neq 0$. If $y = 0$, the second equation implies $x = 0$, so $y \neq 0$. Then from the first two equations we have $\dfrac{e^y}{2x} = \lambda = \dfrac{xe^y}{2y} \quad \Rightarrow \quad 2ye^y = 2x^2 e^y \quad \Rightarrow \quad y = x^2$, and substituting into the third equation gives $x^2 + (x^2)^2 = 2 \quad \Rightarrow \quad x^4 + x^2 - 2 = 0 \quad \Rightarrow \quad (x^2 + 2)(x^2 - 1) = 0 \quad \Rightarrow \quad x = \pm 1$. From $y = x^2$ we have $y = 1$, so f has possible extreme values at $(\pm 1, 1)$. Evaluating f at these points, we see that the maximum value is $f(1, 1) = e$ and the minimum is $f(-1, 1) = -e$.

7. $f(x, y, z) = 2x + 2y + z$, $g(x, y, z) = x^2 + y^2 + z^2 = 9$, and $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2, 2, 1 \rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$, so $2\lambda x = 2$, $2\lambda y = 2$, $2\lambda z = 1$, and $x^2 + y^2 + z^2 = 9$. The first three equations imply $x = \dfrac{1}{\lambda}$, $y = \dfrac{1}{\lambda}$, and $z = \dfrac{1}{2\lambda}$. But substitution into the fourth equation gives $\left(\dfrac{1}{\lambda}\right)^2 + \left(\dfrac{1}{\lambda}\right)^2 + \left(\dfrac{1}{2\lambda}\right)^2 = 9 \quad \Rightarrow \quad \dfrac{9}{4\lambda^2} = 9 \quad \Rightarrow \quad \lambda = \pm \frac{1}{2}$, so f has possible extreme values at the points $(2, 2, 1)$ and $(-2, -2, -1)$. The maximum value of f on $x^2 + y^2 + z^2 = 9$ is $f(2, 2, 1) = 9$, and the minimum is $f(-2, -2, -1) = -9$.

8. $f(x,y,z)=e^{xyz}$, $g(x,y,z)=2x^2+y^2+z^2=24$, and $\nabla f=\lambda\nabla g \Rightarrow \langle yze^{xyz}, xze^{xyz}, xye^{xyz}\rangle=\langle 4\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $yze^{xyz}=4\lambda x$, $xze^{xyz}=2\lambda y$, $xye^{xyz}=2\lambda z$, and $2x^2+y^2+z^2=24$. If any of x, y, z, or λ is zero, then the first three equations imply that two of the variables x, y, z must be zero. If $x=y=z=0$ it contradicts the fourth equation, so exactly two are zero, and from the fourth equation the possibilities are $(\pm2\sqrt{3},0,0)$, $(0,\pm2\sqrt{6},0)$, $(0,0,\pm2\sqrt{6})$, all with an f-value of $e^0=1$. If none of x, y, z, λ is zero then from the first three equations we have $\dfrac{4\lambda x}{yz}=e^{xyz}=\dfrac{2\lambda y}{xz}=\dfrac{2\lambda z}{xy} \Rightarrow \dfrac{2x}{yz}=\dfrac{y}{xz}=\dfrac{z}{xy}$. This gives $2x^2z=y^2z \Rightarrow 2x^2=y^2$ and $xy^2=xz^2 \Rightarrow y^2=z^2$. Substituting into the fourth equation, we have $y^2+y^2+y^2=24 \Rightarrow y^2=8 \Rightarrow y=\pm2\sqrt{2}$, so $x^2=4 \Rightarrow x=\pm2$ and $z^2=y^2 \Rightarrow z=\pm2\sqrt{2}$, giving possible points $(\pm2,\pm2\sqrt{2},\pm2\sqrt{2})$ (all combinations). The value of f is e^{16} when all coordinates are positive or exactly two are negative, and the value is e^{-16} when all are negative or exactly one of the coordinates is negative. Thus the maximum of f subject to the constraint is e^{16} and the minimum is e^{-16}.

9. $f(x,y,z)=xy^2z$, $g(x,y,z)=x^2+y^2+z^2=4$, and $\nabla f=\lambda\nabla g \Rightarrow \langle y^2z, 2xyz, xy^2\rangle=\lambda\langle 2x,2y,2z\rangle$. Then $y^2z=2\lambda x$, $2xyz=2\lambda y$, $xy^2=2\lambda z$, and $x^2+y^2+z^2=4$.

Case 1: If $\lambda=0$, then the first equation implies that $y=0$ or $z=0$. If $y=0$, then any values of x and z satisfy the first three equations, so from the fourth equation all points $(x,0,z)$ such that $x^2+z^2=4$ are possible points. If $z=0$ then from the third equation $x=0$ or $y=0$, and from the fourth equation, the possible points are $(0,\pm2,0)$, $(\pm2,0,0)$. The f-value in all these cases is 0.

Case 2: If $\lambda\neq0$ but any one of x, y, z is zero, the first three equations imply that all three coordinates must be zero, contradicting the fourth equation. Thus if $\lambda\neq0$, none of x, y, z is zero and from the first three equations we have $\dfrac{y^2z}{2x}=\lambda=xz=\dfrac{xy^2}{2z}$. This gives $y^2z=2x^2z \Rightarrow y^2=2x^2$ and $2y^2z^2=2x^2y^2 \Rightarrow z^2=x^2$. Substituting into the fourth equation, we have $x^2+2x^2+x^2=4 \Rightarrow x^2=1 \Rightarrow x=\pm1$, so $y=\pm\sqrt{2}$ and $z=\pm1$, giving possible points $(\pm1,\pm\sqrt{2},\pm1)$ (all combinations). The value of f is 2 when x and z are the same sign and -2 when they are opposite. Thus the maximum of f subject to the constraint is $f(1,\pm\sqrt{2},1)=f(-1,\pm\sqrt{2},-1)=2$ and the minimum is $f(1,\pm\sqrt{2},-1)=f(-1,\pm\sqrt{2},1)=-2$.

10. $f(x,y,z)=\ln(x^2+1)+\ln(y^2+1)+\ln(z^2+1)$, $g(x,y,z)=x^2+y^2+z^2=12$. Then $\nabla f=\lambda\nabla g \Rightarrow$ $\left\langle \dfrac{2x}{x^2+1}, \dfrac{2y}{y^2+1}, \dfrac{2z}{z^2+1}\right\rangle=\lambda\langle 2x,2y,2z\rangle$, so $\dfrac{2x}{x^2+1}=2\lambda x$, $\dfrac{2y}{y^2+1}=2\lambda y$, $\dfrac{2z}{z^2+1}=2\lambda z$, and $x^2+y^2+z^2=12$.

First, if $\lambda=0$ then $x=y=z=0$ which contradicts the last equation, so we may assume that $\lambda\neq0$.

Case 1: If $x\neq0$, $y\neq0$, and $z\neq0$, then from the first three equations we have $\dfrac{1}{x^2+1}=\lambda=\dfrac{1}{y^2+1}=\dfrac{1}{z^2+1} \Rightarrow$

$x^2 = y^2 = z^2$, and substitution into the last equation gives $3x^2 = 12 \quad \Rightarrow \quad x = \pm 2$. Thus possible points are $(\pm 2, \pm 2, \pm 2)$ (all combinations), all of which have an f-value of $3 \ln 5$.

Case 2: If exactly one of x, y, z is zero, say $x = 0$, then from the second and third equations we have $y^2 = z^2$. Substitution into the last equation gives $2y^2 = 12 \quad \Rightarrow \quad y = \pm\sqrt{6}$. The situation is similar for $y = 0$ or $z = 0$, giving possible points $\left(0, \pm\sqrt{6}, \pm\sqrt{6}\right)$, $\left(\pm\sqrt{6}, 0, \pm\sqrt{6}\right)$, $\left(\pm\sqrt{6}, \pm\sqrt{6}, 0\right)$ (all combinations), all with an f-value of $2 \ln 7$.

Case 3: If exactly two of x, y, z are zero, then the square of the nonzero variable is 12, giving possible points $\left(0, 0, \pm 2\sqrt{3}\right)$, $\left(0, \pm 2\sqrt{3}, 0\right)$, $\left(\pm 2\sqrt{3}, 0, 0\right)$, all with an f-value of $\ln 13$.

Thus the maximum of f subject to the constraint is $3 \ln 5 \approx 4.83$ and the minimum is $\ln 13 \approx 2.56$.

11. $f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = x^4 + y^4 + z^4 = 1 \quad \Rightarrow \quad \nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \left\langle 4\lambda x^3, 4\lambda y^3, 4\lambda z^3 \right\rangle$.

Case 1: If $x \neq 0$, $y \neq 0$, and $z \neq 0$, then $\nabla f = \lambda \nabla g$ implies $\lambda = 1/(2x^2) = 1/(2y^2) = 1/(2z^2)$ or $x^2 = y^2 = z^2$ and $3x^4 = 1$ or $x = \pm \frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$ all with an f-value of $\sqrt{3}$.

Case 2: If one of the variables equals zero and the other two are not zero, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{\sqrt{2}}$ and corresponding f-value of $\sqrt{2}$.

Case 3: If exactly two of the variables are zero, then the third variable has value ± 1 with the corresponding f-value of 1.

Thus on $x^4 + y^4 + z^4 = 1$, the maximum value of f is $\sqrt{3}$ and the minimum value is 1.

12. $f(x, y, z) = x^4 + y^4 + z^4$, $g(x, y, z) = x^2 + y^2 + z^2 = 1 \quad \Rightarrow \quad \nabla f = \left\langle 4x^3, 4y^3, 4z^3 \right\rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$.

Case 1: If $x \neq 0$, $y \neq 0$, and $z \neq 0$ then $\nabla f = \lambda \nabla g$ implies $\lambda = 2x^2 = 2y^2 = 2z^2$ or $x^2 = y^2 = z^2 = \frac{1}{3}$ giving 8 points each with an f-value of $\frac{1}{3}$.

Case 2: If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f-value is $\frac{1}{2}$.

Case 3: If exactly two of the variables are 0, then the third variable has value ± 1 with corresponding f-value of 1.

Thus on $x^2 + y^2 + z^2 = 1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

13. $f(x, y, z, t) = x + y + z + t$, $g(x, y, z, t) = x^2 + y^2 + z^2 + t^2 = 1 \quad \Rightarrow \quad \langle 1, 1, 1, 1 \rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z, 2\lambda t \rangle$, so $\lambda = 1/(2x) = 1/(2y) = 1/(2z) = 1/(2t)$ and $x = y = z = t$. But $x^2 + y^2 + z^2 + t^2 = 1$, so the possible points are $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$. Thus the maximum value of f is $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2$ and the minimum value is $f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -2$.

14. $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$, $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \quad \Rightarrow$ $\langle 1, 1, \ldots, 1 \rangle = \langle 2\lambda x_1, 2\lambda x_2, \ldots, 2\lambda x_n \rangle$, so $\lambda = 1/(2x_1) = 1/(2x_2) = \cdots = 1/(2x_n)$ and $x_1 = x_2 = \cdots = x_n$.

But $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, so $x_i = \pm 1/\sqrt{n}$ for $i = 1, \ldots, n$. Thus the maximum value of f is $f(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n}) = \sqrt{n}$ and the minimum value is $f(-1/\sqrt{n}, -1/\sqrt{n}, \ldots, -1/\sqrt{n}) = -\sqrt{n}$.

15. $f(x,y) = x^2 + y^2$, $g(x,y) = xy = 1$, and $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y \rangle = \langle \lambda y, \lambda x \rangle$, so $2x = \lambda y$, $2y = \lambda x$, and $xy = 1$. From the last equation, $x \neq 0$ and $y \neq 0$, so $2x = \lambda y \Rightarrow \lambda = 2x/y$. Substituting, we have $2y = (2x/y)\,x \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$. But $xy = 1$, so $x = y = \pm 1$ and the possible points for the extreme values of f are $(1,1)$ and $(-1,-1)$. Here there is no maximum value, since the constraint $xy = 1 \Leftrightarrow y = 1/x$ allows x or y to become arbitrarily large, and hence $f(x,y) = x^2 + y^2$ can be made arbitrarily large. The minimum value is $f(1,1) = f(-1,-1) = 2$.

16. $f(x,y,z) = x^2 + 2y^2 + 3z^2$, $g(x,y) = x + 2y + 3z = 10$, and $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 4y, 6z \rangle = \langle \lambda, 2\lambda, 3\lambda \rangle$, so $2x = \lambda$, $4y = 2\lambda$, $6z = 3\lambda$, and $x + 2y + 3z = 10$. From the first three equations we have $2x = \lambda = 2y = 2z \Rightarrow x = y = z$, and substituting into the fourth equation gives $x + 2x + 3x = 10 \Rightarrow x = \frac{5}{3} = y = z$. Thus the only possible point for an extreme value of f is $\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$. Notice here that the constraint $x + 2y + 3z = 10$ allows any of $|x|$, $|y|$, or $|z|$ to be arbitrarily large, and hence $f(x,y,z) = x^2 + 2y^2 + 3z^2$ can be made arbitrarily large. So f has no maximum value subject to the constraint. The minimum value is $f\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right) = 6\left(\frac{5}{3}\right)^2 = \frac{50}{3}$.

17. $f(x,y,z) = x + y + z$, $g(x,y,z) = x^2 + z^2 = 2$, $h(x,y,z) = x + y = 1$, and $\nabla f = \lambda \nabla g + \mu \nabla h \Rightarrow \langle 1,1,1 \rangle = \langle 2\lambda x, 0, 2\lambda z \rangle + \langle \mu, \mu, 0 \rangle$. Then $1 = 2\lambda x + \mu$, $1 = \mu$, $1 = 2\lambda z$, $x^2 + z^2 = 2$, and $x + y = 1$. Substituting $\mu = 1$ into the first equation gives $\lambda = 0$ or $x = 0$. But $\lambda = 0$ contradicts $1 = 2\lambda z$, so $x = 0$. Then $x + y = 1 \Rightarrow y = 1$ and $x^2 + z^2 = 2 \Rightarrow z = \pm\sqrt{2}$, so the possible points are $(0, 1, \pm\sqrt{2})$. The maximum value of f subject to the constraints is $f(0,1,\sqrt{2}) = 1 + \sqrt{2} \approx 2.41$ and the minimum is $f(0,1,-\sqrt{2}) = 1 - \sqrt{2} \approx -0.41$.

Note: Since $x + y = 1$ is one of the constraints, we could have solved the problem by solving $f(x,z) = 1 + z$ subject to $x^2 + z^2 = 2$.

18. $f(x,y,z) = z$, $g(x,y,z) = x^2 + y^2 - z^2 = 0$, $h(x,y,z) = x + y + z = 24$, and $\nabla f = \lambda \nabla g + \mu \nabla h \Rightarrow \langle 0,0,1 \rangle = \langle 2\lambda x, 2\lambda y, -2\lambda z \rangle + \langle \mu, \mu, \mu \rangle$. Then $0 = 2\lambda x + \mu$, $0 = 2\lambda y + \mu$, $1 = -2\lambda z + \mu$, $x^2 + y^2 - z^2 = 0$, and $x + y + z = 24$. From the first two equations we have $-2\lambda x = \mu = -2\lambda y \Rightarrow \lambda = 0$ or $x = y$. But $\lambda = 0 \Rightarrow \mu = 0$ which contradicts the third equation, so $x = y$ and substitution into the last equation gives $z = 24 - 2x$. From the fourth equation we have $x^2 + x^2 - (24 - 2x)^2 = 0 \Rightarrow -2x^2 + 96x - 576 = 0 \Rightarrow x^2 - 48x + 288 = 0 \Rightarrow$

$$x = \frac{48 \pm \sqrt{1152}}{2} = 24 \pm 12\sqrt{2} = y.$$

Now $z = 24 - 2x$, so the possible points are $\left(24 + 12\sqrt{2}, 24 + 12\sqrt{2}, -24 - 24\sqrt{2}\right)$ and $\left(24 - 12\sqrt{2}, 24 - 12\sqrt{2}, -24 + 24\sqrt{2}\right)$. The maximum of f subject to the constraints is $f\left(24 - 12\sqrt{2}, 24 - 12\sqrt{2}, -24 + 24\sqrt{2}\right) = -24 + 24\sqrt{2} \approx 9.94$ and the minimum is $f\left(24 + 12\sqrt{2}, 24 + 12\sqrt{2}, -24 - 24\sqrt{2}\right) = -24 - 24\sqrt{2} \approx -57.94$.

19. $f(x,y,z) = yz + xy$, $g(x,y,z) = xy = 1$, $h(x,y,z) = y^2 + z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle y, x+z, y\rangle$, $\lambda\nabla g = \langle \lambda y, \lambda x, 0\rangle$, $\mu\nabla h = \langle 0, 2\mu y, 2\mu z\rangle$. Then $y = \lambda y$ implies $\lambda = 1$ [$y \neq 0$ since $g(x,y,z) = 1$], $x + z = \lambda x + 2\mu y$ and $y = 2\mu z$. Thus $\mu = z/(2y) = y/(2y)$ or $y^2 = z^2$, and so $y^2 + z^2 = 1$ implies $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm\frac{1}{\sqrt{2}}$. Then $xy = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Hence the maximum of f subject to the constraints is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{3}{2}$ and the minimum is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$.

Note: Since $xy = 1$ is one of the constraints we could have solved the problem by solving $f(y,z) = yz + 1$ subject to $y^2 + z^2 = 1$.

20. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x - y = 1$, $h(x,y,z) = y^2 - z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 2x, 2y, 2z\rangle$, $\lambda\nabla g = \langle \lambda, -\lambda, 0\rangle$, and $\mu\nabla h = \langle 0, 2\mu y, -2\mu z\rangle$. Then $2x = \lambda$, $2y = -\lambda + 2\mu y$, and $2z = -2\mu z \quad \Rightarrow \quad z = 0$ or $\mu = -1$. If $z = 0$ then $y^2 - z^2 = 1$ implies $y^2 = 1 \quad \Rightarrow \quad y = \pm 1$. If $y = 1$, $x - y = 1$ implies $x = 2$, and if $y = -1$ we have $x = 0$, so possible points are $(2, 1, 0)$ and $(0, -1, 0)$. If $\mu = -1$ then $2y = -\lambda + 2\mu y$ implies $4y = -\lambda$, but $\lambda = 2x$ so $4y = -2x \quad \Rightarrow \quad x = -2y$ and $x - y = 1$ implies $-3y = 1 \quad \Rightarrow \quad y = -\frac{1}{3}$. But then $y^2 - z^2 = 1$ implies $z^2 = -\frac{8}{9}$, an impossibility. Thus the maximum value of f subject to the constraints is $f(2,1,0) = 5$ and the minimum is $f(0,-1,0) = 1$.

Note: Since $x - y = 1 \quad \Rightarrow \quad x = y + 1$ is one of the constraints we could have solved the problem by solving $f(y,z) = (y+1)^2 + y^2 + z^2$ subject to $y^2 - z^2 = 1$.

21. $f(x,y) = x^2 + y^2 + 4x - 4y$. For the interior of the region, we find the critical points: $f_x = 2x + 4$, $f_y = 2y - 4$, so the only critical point is $(-2, 2)$ (which is inside the region) and $f(-2,2) = -8$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + y^2 = 9$, so $\nabla f = \lambda\nabla g \quad \Rightarrow \quad \langle 2x+4, 2y-4\rangle = \langle 2\lambda x, 2\lambda y\rangle$. Thus $2x + 4 = 2\lambda x$ and $2y - 4 = 2\lambda y$. Adding the two equations gives $2x + 2y = 2\lambda x + 2\lambda y \quad \Rightarrow \quad x + y = \lambda(x+y) \quad \Rightarrow \quad (x+y)(\lambda - 1) = 0$, so $x + y = 0 \quad \Rightarrow \quad y = -x$ or $\lambda - 1 = 0 \quad \Rightarrow \quad \lambda = 1$. But $\lambda = 1$ leads to a contradition in $2x + 4 = 2\lambda x$, so $y = -x$ and $x^2 + y^2 = 9$ implies $2y^2 = 9 \quad \Rightarrow \quad y = \pm\frac{3}{\sqrt{2}}$. We have $f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = 9 + 12\sqrt{2} \approx 25.97$ and $f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = 9 - 12\sqrt{2} \approx -7.97$, so the maximum value of f on the disk $x^2 + y^2 \leq 9$ is $f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = 9 + 12\sqrt{2}$ and the minimum is $f(-2,2) = -8$.

22. $f(x,y) = 2x^2 + 3y^2 - 4x - 5 \quad \Rightarrow \quad \nabla f = \langle 4x - 4, 6y\rangle = \langle 0, 0\rangle \quad \Rightarrow \quad x = 1, y = 0$. Thus $(1,0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x,y) = x^2 + y^2 = 16 \quad \Rightarrow \quad \lambda\nabla g = \langle 2\lambda x, 2\lambda y\rangle$, so $6y = 2\lambda y \quad \Rightarrow \quad$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \quad \Rightarrow \quad x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f\left(-2, \pm 2\sqrt{3}\right) = 47$. Thus the maximum value of $f(x,y)$ on the disk $x^2 + y^2 \leq 16$ is $f\left(-2, \pm 2\sqrt{3}\right) = 47$, and the minimum value is $f(1,0) = -7$.

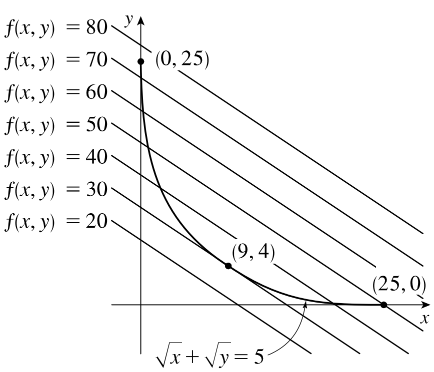
23. $f(x,y) = e^{-xy}$. For the interior of the region, we find the critical points: $f_x = -ye^{-xy}$, $f_y = -xe^{-xy}$, so the only critical point is $(0,0)$, and $f(0,0) = 1$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + 4y^2 = 1 \quad \Rightarrow$ $\lambda \nabla g = \langle 2\lambda x, 8\lambda y \rangle$, so setting $\nabla f = \lambda \nabla g$ we get $-ye^{-xy} = 2\lambda x$ and $-xe^{-xy} = 8\lambda y$. The first of these gives $e^{-xy} = -2\lambda x/y$, and then the second gives $-x(-2\lambda x/y) = 8\lambda y \quad \Rightarrow \quad x^2 = 4y^2$. Solving this last equation with the constraint $x^2 + 4y^2 = 1$ gives $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4} \approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4} \approx 0.779$. The former are the maxima on the region and the latter are the minima.

24. (a) $f(x,y) = 2x + 3y$, $g(x,y) = \sqrt{x} + \sqrt{y} = 5 \quad \Rightarrow \quad \nabla f = \langle 2, 3 \rangle = \lambda \nabla g = \lambda \left\langle \dfrac{1}{2\sqrt{x}}, \dfrac{1}{2\sqrt{y}} \right\rangle$. Then

$2 = \dfrac{\lambda}{2\sqrt{x}}$ and $3 = \dfrac{\lambda}{2\sqrt{y}}$ so $4\sqrt{x} = \lambda = 6\sqrt{y} \quad \Rightarrow \quad \sqrt{y} = \frac{2}{3}\sqrt{x}$. With $\sqrt{x} + \sqrt{y} = 5$ we have $\sqrt{x} + \frac{2}{3}\sqrt{x} = 5 \quad \Rightarrow$ $\sqrt{x} = 3 \quad \Rightarrow \quad x = 9$. Substituting into $\sqrt{y} = \frac{2}{3}\sqrt{x}$ gives $\sqrt{y} = 2$ or $y = 4$. Thus the only possible extreme value subject to the constraint is $f(9,4) = 30$. (The question remains whether this is indeed the maximum of f.)

(b) $f(25,0) = 50$ which is larger than the result of part (a).

(c)



We can see from the level curves of f that the maximum occurs at the left endpoint $(0, 25)$ of the constraint curve g. The maximum value is $f(0, 25) = 75$.

(d) Here ∇g does not exist if $x = 0$ or $y = 0$, so the method will not locate any associated points. Also, the method of Lagrange multipliers identifies points where the level curves of f share a common tangent line with the constraint curve g. This normally does not occur at an endpoint, although an absolute maximum or minimum may occur there.

(e) Here $f(9,4)$ is the absolute *minimum* of f subject to g.

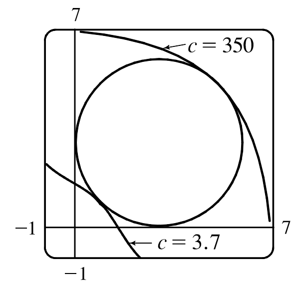
25. (a) $f(x,y) = x$, $g(x,y) = y^2 + x^4 - x^3 = 0 \quad \Rightarrow \quad \nabla f = \langle 1, 0 \rangle = \lambda \nabla g = \lambda \langle 4x^3 - 3x^2, 2y \rangle$. Then

$1 = \lambda(4x^3 - 3x^2)$ **(1)** and $0 = 2\lambda y$ **(2)**. We have $\lambda \neq 0$ from **(1)**, so **(2)** gives $y = 0$. Then, from the constraint equation, $x^4 - x^3 = 0 \quad \Rightarrow \quad x^3(x-1) = 0 \quad \Rightarrow \quad x = 0$ or $x = 1$. But $x = 0$ contradicts **(1)**, so the only possible extreme value subject to the constraint is $f(1,0) = 1$. (The question remains whether this is indeed the minimum of f.)

(b) The constraint is $y^2 + x^4 - x^3 = 0 \quad \Leftrightarrow \quad y^2 = x^3 - x^4$. The left side is non-negative, so we must have $x^3 - x^4 \geq 0$ which is true only for $0 \leq x \leq 1$. Therefore the minimum possible value for $f(x, y) = x$ is 0 which occurs for $x = y = 0$. However, $\lambda \nabla g(0, 0) = \lambda \langle 0 - 0, 0 \rangle = \langle 0, 0 \rangle$ and $\nabla f(0, 0) = \langle 1, 0 \rangle$, so $\nabla f(0, 0) \neq \lambda \nabla g(0, 0)$ for all values of λ.

(c) Here $\nabla g(0, 0) = \mathbf{0}$ but the method of Lagrange multipliers requires that $\nabla g \neq \mathbf{0}$ everywhere on the constraint curve.

26. (a) The graphs of $f(x, y) = 3.7$ and $f(x, y) = 350$ seem to be tangent to the circle, and so 3.7 and 350 are the approximate minimum and maximum values of the function $f(x, y)$ subject to the constraint $(x - 3)^2 + (y - 3)^2 = 9$.



(b) Let $g(x, y) = (x - 3)^2 + (y - 3)^2$. We calculate $f_x(x, y) = 3x^2 + 3y$, $f_y(x, y) = 3y^2 + 3x$, $g_x(x, y) = 2x - 6$, and $g_y(x, y) = 2y - 6$, and use a CAS to search for solutions to the equations $g(x, y) = (x - 3)^2 + (y - 3)^2 = 9$, $f_x = \lambda g_x$, and $f_y = \lambda g_y$. The solutions are $(x, y) = \left(3 - \frac{3}{2}\sqrt{2}, 3 - \frac{3}{2}\sqrt{2}\right) \approx (0.879, 0.879)$ and $(x, y) = \left(3 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}\right) \approx (5.121, 5.121)$. These give $f\left(3 - \frac{3}{2}\sqrt{2}, 3 - \frac{3}{2}\sqrt{2}\right) = \frac{351}{2} - \frac{243}{2}\sqrt{2} \approx 3.673$ and $f\left(3 + \frac{3}{2}\sqrt{2}, 3 + \frac{3}{2}\sqrt{2}\right) = \frac{351}{2} + \frac{243}{2}\sqrt{2} \approx 347.33$, in accordance with part (a).

27. $P(L, K) = bL^\alpha K^{1-\alpha}$, $g(L, K) = mL + nK = p \quad \Rightarrow \quad \nabla P = \left\langle \alpha b L^{\alpha-1} K^{1-\alpha}, (1 - \alpha) b L^\alpha K^{-\alpha} \right\rangle$, $\lambda \nabla g = \langle \lambda m, \lambda n \rangle$. Then $\alpha b (K/L)^{1-\alpha} = \lambda m$ and $(1 - \alpha) b (L/K)^\alpha = \lambda n$ and $mL + nK = p$, so $\alpha b (K/L)^{1-\alpha}/m = (1 - \alpha) b (L/K)^\alpha / n$ or $n\alpha/[m(1 - \alpha)] = (L/K)^\alpha (L/K)^{1-\alpha}$ or $L = Kn\alpha/[m(1 - \alpha)]$. Substituting into $mL + nK = p$ gives $K = (1 - \alpha)p/n$ and $L = \alpha p/m$ for the maximum production.

28. $C(L, K) = mL + nK$, $g(L, K) = bL^\alpha K^{1-\alpha} = Q \quad \Rightarrow \quad \nabla C = \langle m, n \rangle$, $\lambda \nabla g = \left\langle \lambda \alpha b L^{\alpha-1} K^{1-\alpha}, \lambda (1 - \alpha) b L^\alpha K^{-\alpha} \right\rangle$.

Then $\dfrac{m}{\alpha b} \left(\dfrac{L}{K}\right)^{1-\alpha} = \dfrac{n}{(1 - \alpha) b} \left(\dfrac{K}{L}\right)^\alpha$ and $bL^\alpha K^{1-\alpha} = Q \quad \Rightarrow \quad \dfrac{n\alpha}{m(1 - \alpha)} = \left(\dfrac{L}{K}\right)^{1-\alpha} \left(\dfrac{L}{K}\right)^\alpha \quad \Rightarrow$

$L = \dfrac{Kn\alpha}{m(1 - \alpha)}$ and so $b\left[\dfrac{Kn\alpha}{m(1 - \alpha)}\right]^\alpha K^{1-\alpha} = Q$. Hence $K = \dfrac{Q}{b\,(n\alpha/[m(1 - \alpha)])^\alpha} = \dfrac{Qm^\alpha (1 - \alpha)^\alpha}{bn^\alpha \alpha^\alpha}$

and $L = \dfrac{Qm^{\alpha-1} (1 - \alpha)^{\alpha-1}}{bn^{\alpha-1} \alpha^{\alpha-1}} = \dfrac{Qn^{1-\alpha} \alpha^{1-\alpha}}{bm^{1-\alpha} (1 - \alpha)^{1-\alpha}}$ minimizes cost.

29. Let the sides of the rectangle be x and y. Then $f(x, y) = xy$, $g(x, y) = 2x + 2y = p \quad \Rightarrow \quad \nabla f(x, y) = \langle y, x \rangle$, $\lambda \nabla g = \langle 2\lambda, 2\lambda \rangle$. Then $\lambda = \frac{1}{2}y = \frac{1}{2}x$ implies $x = y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

30. Let $f(x, y, z) = s(s - x)(s - y)(s - z)$, $g(x, y, z) = x + y + z$. Then $\nabla f = \langle -s(s - y)(s - z), -s(s - x)(s - z), -s(s - x)(s - y) \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$. Thus $(s - y)(s - z) = (s - x)(s - z)$ **(1)**, and $(s - x)(s - z) = (s - x)(s - y)$ **(2)**. **(1)** implies $x = y$ while **(2)** implies $y = z$, so $x = y = z = p/3$ and the triangle with maximum area is equilateral.

31. The distance from $(2,0,-3)$ to a point (x,y,z) on the plane is $d=\sqrt{(x-2)^2+y^2+(z+3)^2}$, so we seek to minimize $d^2=f(x,y,z)=(x-2)^2+y^2+(z+3)^2$ subject to the constraint that (x,y,z) lies on the plane $x+y+z=1$, that is, that $g(x,y,z)=x+y+z=1$. Then $\nabla f=\lambda\nabla g \Rightarrow \langle 2(x-2),2y,2(z+3)\rangle=\langle\lambda,\lambda,\lambda\rangle$, so $x=(\lambda+4)/2$, $y=\lambda/2$, $z=(\lambda-6)/2$. Substituting into the constraint equation gives $\dfrac{\lambda+4}{2}+\dfrac{\lambda}{2}+\dfrac{\lambda-6}{2}=1 \Rightarrow 3\lambda-2=2 \Rightarrow$ $\lambda=\frac{4}{3}$, so $x=\frac{8}{3}$, $y=\frac{2}{3}$, and $z=-\frac{7}{3}$. This must correspond to a minimum, so the shortest distance is

$$d=\sqrt{\left(\tfrac{8}{3}-2\right)^2+\left(\tfrac{2}{3}\right)^2+\left(-\tfrac{7}{3}+3\right)^2}=\sqrt{\tfrac{4}{3}}=\tfrac{2}{\sqrt{3}}.$$

32. The distance from $(0,1,1)$ to a point (x,y,z) on the plane is $d=\sqrt{x^2+(y-1)^2+(z-1)^2}$, so we minimize $d^2=f(x,y,z)=x^2+(y-1)^2+(z-1)^2$ subject to the constraint that (x,y,z) lies on the plane $x-2y+3z=6$, that is, $g(x,y,z)=x-2y+3z=6$. Then $\nabla f=\lambda\nabla g \Rightarrow \langle 2x,2(y-1),2(z-1)\rangle=\langle\lambda,-2\lambda,3\lambda\rangle$, so $x=\lambda/2$, $y=1-\lambda$, $z=(3\lambda+2)/2$. Substituting into the constraint equation gives $\dfrac{\lambda}{2}-2(1-\lambda)+3\cdot\dfrac{3\lambda+2}{2}=6 \Rightarrow \lambda=\frac{5}{7}$, so $x=\frac{5}{14}$, $y=\frac{2}{7}$, and $z=\frac{29}{14}$. This must correspond to a minimum, so the point on the plane closest to the point $(0,1,1)$ is $\left(\frac{5}{14},\frac{2}{7},\frac{29}{14}\right)$.

33. Let $f(x,y,z)=d^2=(x-4)^2+(y-2)^2+z^2$. Then we want to minimize f subject to the constraint $g(x,y,z)=x^2+y^2-z^2=0$. $\nabla f=\lambda\nabla g \Rightarrow \langle 2(x-4),2(y-2),2z\rangle=\langle 2\lambda x,2\lambda y,-2\lambda z\rangle$, so $x-4=\lambda x$, $y-2=\lambda y$, and $z=-\lambda z$. From the last equation we have $z+\lambda z=0 \Rightarrow z(1+\lambda)=0$, so either $z=0$ or $\lambda=-1$. But from the constraint equation we have $z=0 \Rightarrow x^2+y^2=0 \Rightarrow x=y=0$ which is not possible from the first two equations. So $\lambda=-1$ and $x-4=\lambda x \Rightarrow x=2$, $y-2=\lambda y \Rightarrow y=1$, and $x^2+y^2-z^2=0 \Rightarrow$ $4+1-z^2=0 \Rightarrow z=\pm\sqrt{5}$. This must correspond to a minimum, so the points on the cone closest to $(4,2,0)$ are $\left(2,1,\pm\sqrt{5}\right)$.

34. Let $f(x,y,z)=d^2=x^2+y^2+z^2$. Then we want to minimize f subject to the constraint $g(x,y,z)=y^2-xz=9$. $\nabla f=\lambda\nabla g \Rightarrow \langle 2x,2y,2z\rangle=\langle-\lambda z,2\lambda y,-\lambda x\rangle$, so $2x=-\lambda z$, $y=\lambda y$, and $2z=-\lambda x$. If $x=0$ then the last equation implies $z=0$, and from the constraint $y^2-xz=9$ we have $y=\pm3$. If $x\neq0$, then the first and third equations give $\lambda=-2x/z=-2z/x \Rightarrow x^2=z^2$. From the second equation we have $y=0$ or $\lambda=1$. If $y=0$ then $y^2-xz=9 \Rightarrow$ $z=-9/x$ and $x^2=z^2 \Rightarrow x^2=81/x^2 \Rightarrow x=\pm3$. Since $z=-9/x$, $x=3 \Rightarrow z=-3$ and $x=-3 \Rightarrow$ $z=3$. If $\lambda=1$, then $2x=-z$ and $2z=-x$ which implies $x=z=0$, contradicting the assumption that $x\neq0$. Thus the possible points are $(0,\pm3,0)$, $(3,0,-3)$, $(-3,0,3)$. We have $f(0,\pm3,0)=9$ and $f(3,0,-3)=f(-3,0,3)=18$, so the points on the surface that are closest to the origin are $(0,\pm3,0)$.

35. $f(x,y,z)=xyz$, $g(x,y,z)=x+y+z=100 \Rightarrow \nabla f=\langle yz,xz,xy\rangle=\lambda\nabla g=\langle\lambda,\lambda,\lambda\rangle$. Then $\lambda=yz=xz=xy$ implies $x=y=z=\frac{100}{3}$.

36. Minimize $f(x,y,z)=x^2+y^2+z^2$ subject to $g(x,y,z)=x+y+z=12$ with $x>0$, $y>0$, $z>0$. Then $\nabla f=\lambda\nabla g \Rightarrow \langle 2x,2y,2z\rangle=\lambda\langle 1,1,1\rangle \Rightarrow 2x=\lambda, 2y=\lambda, 2z=\lambda \Rightarrow x=y=z$, so $x+y+z=12 \Rightarrow 3x=12 \Rightarrow x=4=y=z$. By comparing nearby values we can confirm that this gives a minimum and not a maximum. Thus the three numbers are 4, 4, and 4.

37. If the dimensions are $2x$, $2y$, and $2z$, then maximize $f(x,y,z)=(2x)(2y)(2z)=8xyz$ subject to $g(x,y,z)=x^2+y^2+z^2=r^2$ ($x>0$, $y>0$, $z>0$). Then $\nabla f=\lambda\nabla g \Rightarrow \langle 8yz,8xz,8xy\rangle=\lambda\langle 2x,2y,2z\rangle \Rightarrow$ $8yz=2\lambda x$, $8xz=2\lambda y$, and $8xy=2\lambda z$, so $\lambda=\dfrac{4yz}{x}=\dfrac{4xz}{y}=\dfrac{4xy}{z}$. This gives $x^2z=y^2z \Rightarrow x^2=y^2$ (since $z\neq 0$) and $xy^2=xz^2 \Rightarrow z^2=y^2$, so $x^2=y^2=z^2 \Rightarrow x=y=z$, and substituting into the constraint equation gives $3x^2=r^2 \Rightarrow x=r/\sqrt{3}=y=z$. Thus the largest volume of such a box is

$$f\left(\tfrac{r}{\sqrt{3}},\tfrac{r}{\sqrt{3}},\tfrac{r}{\sqrt{3}}\right)=8\left(\tfrac{r}{\sqrt{3}}\right)\left(\tfrac{r}{\sqrt{3}}\right)\left(\tfrac{r}{\sqrt{3}}\right)=\frac{8}{3\sqrt{3}}r^3.$$

38. If the dimensions of the box are x, y, and z then minimize $f(x,y,z)=2xy+2xz+2yz$ subject to $g(x,y,z)=xyz=1000$ ($x>0$, $y>0$, $z>0$). Then $\nabla f=\lambda\nabla g \Rightarrow \langle 2y+2z,2x+2z,2x+2y\rangle=\lambda\langle yz,xz,xy\rangle \Rightarrow 2y+2z=\lambda yz$, $2x+2z=\lambda xz$, $2x+2y=\lambda xy$. Solving for λ in each equation gives $\lambda=\dfrac{2}{z}+\dfrac{2}{y}=\dfrac{2}{z}+\dfrac{2}{x}=\dfrac{2}{y}+\dfrac{2}{x} \Rightarrow x=y=z$. From $xyz=1000$ we have $x^3=1000 \Rightarrow x=10$ and the dimensions of the box are $x=y=z=10$ cm.

39. $f(x,y,z)=xyz$, $g(x,y,z)=x+2y+3z=6 \Rightarrow \nabla f=\langle yz,xz,xy\rangle=\lambda\nabla g=\langle\lambda,2\lambda,3\lambda\rangle$. Then $\lambda=yz=\frac{1}{2}xz=\frac{1}{3}xy$ implies $x=2y$, $z=\frac{2}{3}y$. But $2y+2y+2y=6$ so $y=1$, $x=2$, $z=\frac{2}{3}$ and the volume is $V=\frac{4}{3}$.

40. $f(x,y,z)=xyz$, $g(x,y,z)=xy+yz+xz=32 \Rightarrow \nabla f=\langle yz,xz,xy\rangle=\lambda\nabla g=\langle\lambda(y+z),\lambda(x+z),\lambda(x+y)\rangle$. Then $\lambda(y+z)=yz$ **(1)**, $\lambda(x+z)=xz$ **(2)**, and $\lambda(x+y)=xy$ **(3)**. And **(1)** minus **(2)** implies $\lambda(y-x)=z(y-x)$ so $x=y$ or $\lambda=z$. If $\lambda=z$, then **(1)** implies $z(y+z)=yz$ or $z=0$ which is false. Thus $x=y$. Similarly **(2)** minus **(3)** implies $\lambda(z-y)=x(z-y)$ so $y=z$ or $\lambda=x$. As above, $\lambda\neq x$, so $x=y=z$ and $3x^2=32$ or $x=y=z=\frac{8}{\sqrt{6}}$ cm.

41. $f(x,y,z)=xyz$, $g(x,y,z)=4(x+y+z)=c \Rightarrow \nabla f=\langle yz,xz,xy\rangle$, $\lambda\nabla g=\langle 4\lambda,4\lambda,4\lambda\rangle$. Thus $4\lambda=yz=xz=xy$ or $x=y=z=\frac{1}{12}c$ are the dimensions giving the maximum volume.

42. $C(x,y,z)=5xy+2xz+2yz$, $g(x,y,z)=xyz=V \Rightarrow$ $\nabla C=\langle 5y+2z,5x+2z,2x+2y\rangle=\lambda\nabla g=\langle\lambda yz,\lambda xz,\lambda xy\rangle$. Then $\lambda yz=5y+2z$ **(1)**, $\lambda xz=5x+2z$ **(2)**, $\lambda xy=2(x+y)$ **(3)**, and $xyz=V$ **(4)**. Now **(1)** − **(2)** implies $\lambda z(y-x)=5(y-x)$, so $x=y$ or $\lambda=5/z$, but z can't be 0, so $x=y$. Then twice **(2)** minus five times **(3)** together with $x=y$ implies $\lambda y(2x-5y)=2(2z-5y)$ which gives

$z = \frac{5}{2}y$ [again $\lambda \neq 2/y$ or else **(3)** implies $y = 0$]. Hence $\frac{5}{2}y^3 = V$ and the dimensions which minimize cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$ units.

43. If the dimensions of the box are given by x, y, and z, then we need to find the maximum value of $f(x, y, z) = xyz$ $[x, y, z > 0]$ subject to the constraint $L = \sqrt{x^2 + y^2 + z^2}$ or $g(x, y, z) = x^2 + y^2 + z^2 = L^2$. $\nabla f = \lambda \nabla g \Rightarrow$ $\langle yz, xz, xy\rangle = \lambda\langle 2x, 2y, 2z\rangle$, so $yz = 2\lambda x \Rightarrow \lambda = \dfrac{yz}{2x}$, $xz = 2\lambda y \Rightarrow \lambda = \dfrac{xz}{2y}$, and $xy = 2\lambda z \Rightarrow \lambda = \dfrac{xy}{2z}$. Thus $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{2y} \Rightarrow x^2 = y^2$ [since $z \neq 0$] $\Rightarrow x = y$ and $\lambda = \dfrac{yz}{2x} = \dfrac{xy}{2z} \Rightarrow x = z$ [since $y \neq 0$]. Substituting into the constraint equation gives $x^2 + x^2 + x^2 = L^2 \Rightarrow x^2 = L^2/3 \Rightarrow x = L/\sqrt{3} = y = z$ and the maximum volume is $\left(L/\sqrt{3}\right)^3 = L^3/\left(3\sqrt{3}\right)$.

44. Let the dimensions of the box be x, y, and z, so its volume is $f(x, y, z) = xyz$, its surface area is $2xy + 2yz + 2xz = 1500$ and its total edge length is $4x + 4y + 4z = 200$. We find the extreme values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = xy + yz + xz = 750$ and $h(x, y, z) = x + y + z = 50$. Then $\nabla f = \langle yz, xz, xy\rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda(y + z), \lambda(x + z), \lambda(x + y)\rangle + \langle \mu, \mu, \mu\rangle$. So $yz = \lambda(y + z) + \mu$ **(1)**, $xz = \lambda(x + z) + \mu$ **(2)**, and $xy = \lambda(x + y) + \mu$ **(3)**. Notice that the box can't be a cube or else $x = y = z = \frac{50}{3}$ but then $xy + yz + xz = \frac{2500}{3} \neq 750$. Assume x is the distinct side, that is, $x \neq y$, $x \neq z$. Then **(1)** minus **(2)** implies $z\,(y - x) = \lambda(y - x)$ or $\lambda = z$, and **(1)** minus **(3)** implies $y(z - x) = \lambda(z - x)$ or $\lambda = y$. So $y = z = \lambda$ and $x + y + z = 50$ implies $x = 50 - 2\lambda$; also $xy + yz + xz = 750$ implies $x(2\lambda) + \lambda^2 = 750$. Hence $50 - 2\lambda = \dfrac{750 - \lambda^2}{2\lambda}$ or $3\lambda^2 - 100\lambda + 750 = 0$ and $\lambda = \dfrac{50 \pm 5\sqrt{10}}{3}$, giving the points $\left(\frac{1}{3}\left(50 \mp 10\sqrt{10}\right), \frac{1}{3}\left(50 \pm 5\sqrt{10}\right), \frac{1}{3}\left(50 \pm 5\sqrt{10}\right)\right)$. Thus the minimum of f is $f\left(\frac{1}{3}\left(50 - 10\sqrt{3}\right), \frac{1}{3}\left(50 + 5\sqrt{10}\right), \frac{1}{3}\left(50 + 5\sqrt{10}\right)\right) = \frac{1}{27}\left(87{,}500 - 2500\sqrt{10}\right)$, and its maximum is $f\left(\frac{1}{3}\left(50 + 10\sqrt{10}\right), \frac{1}{3}\left(50 - 5\sqrt{10}\right), \frac{1}{3}\left(50 - 5\sqrt{10}\right)\right) = \frac{1}{27}\left(87{,}500 + 2500\sqrt{10}\right)$.

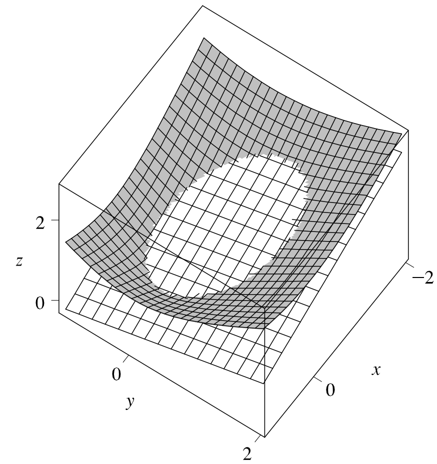
Note: If either y or z is the distinct side, then symmetry gives the same result.

45. We need to find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints $g(x, y, z) = x + y + 2z = 2$ and $h(x, y, z) = x^2 + y^2 - z = 0$. $\nabla f = \langle 2x, 2y, 2z\rangle$, $\lambda \nabla g = \langle \lambda, \lambda, 2\lambda\rangle$ and $\mu \nabla h = \langle 2\mu x, 2\mu y, -\mu\rangle$. Thus we need $2x = \lambda + 2\mu x$ **(1)**, $2y = \lambda + 2\mu y$ **(2)**, $2z = 2\lambda - \mu$ **(3)**, $x + y + 2z = 2$ **(4)**, and $x^2 + y^2 - z = 0$ **(5)**. From **(1)** and **(2)**, $2(x - y) = 2\mu(x - y)$, so if $x \neq y$, $\mu = 1$. Putting this in **(3)** gives $2z = 2\lambda - 1$ or $\lambda = z + \frac{1}{2}$, but putting $\mu = 1$ into **(1)** says $\lambda = 0$. Hence $z + \frac{1}{2} = 0$ or $z = -\frac{1}{2}$. Then **(4)** and **(5)** become $x + y - 3 = 0$ and $x^2 + y^2 + \frac{1}{2} = 0$. The last equation cannot be true, so this case gives no solution. So we must have $x = y$. Then **(4)** and **(5)** become $2x + 2z = 2$ and $2x^2 - z = 0$ which imply $z = 1 - x$ and $z = 2x^2$. Thus $2x^2 = 1 - x$ or $2x^2 + x - 1 = (2x - 1)(x + 1) = 0$ so $x = \frac{1}{2}$ or

$x = -1$. The two points to check are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$: $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ and $f(-1, -1, 2) = 6$. Thus $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the point on the ellipse nearest the origin and $(-1, -1, 2)$ is the one farthest from the origin.

46. (a) After plotting $z = \sqrt{x^2 + y^2}$, the top half of the cone, and the plane $z = (5 - 4x + 3y)/8$ we see the ellipse formed by the intersection of the surfaces. The ellipse can be plotted explicitly using cylindrical coordinates (see Section 15.7): The cone is given by $z = r$, and the plane is $4r\cos\theta - 3r\sin\theta + 8z = 5$. Substituting $z = r$ into the plane equation gives $4r\cos\theta - 3r\sin\theta + 8r = 5 \quad\Rightarrow\quad r = \dfrac{5}{4\cos\theta - 3\sin\theta + 8}$.

Since $z = r$ on the ellipse, parametric equations (in cylindrical coordinates) are $\theta = t,\ r = z = \dfrac{5}{4\cos t - 3\sin t + 8},\ 0 \le t \le 2\pi$.



(b) We need to find the extreme values of $f(x, y, z) = z$ subject to the two constraints $g(x, y, z) = 4x - 3y + 8z = 5$ and $h(x, y, z) = x^2 + y^2 - z^2 = 0$.

$\nabla f = \lambda \nabla g + \mu \nabla h \quad\Rightarrow\quad \langle 0, 0, 1\rangle = \lambda\langle 4, -3, 8\rangle + \mu\langle 2x, 2y, -2z\rangle$, so we need $4\lambda + 2\mu x = 0 \quad\Rightarrow\quad x = -\frac{2\lambda}{\mu}$ **(1)**, $-3\lambda + 2\mu y = 0 \quad\Rightarrow\quad y = \frac{3\lambda}{2\mu}$ **(2)**, $8\lambda - 2\mu z = 1 \quad\Rightarrow\quad z = \frac{8\lambda - 1}{2\mu}$ **(3)**, $4x - 3y + 8z = 5$ **(4)**, and $x^2 + y^2 = z^2$ **(5)**. [Note that $\mu \neq 0$, else $\lambda = 0$ from **(1)**, but substitution into **(3)** gives a contradiction.]

Substituting **(1)**, **(2)**, and **(3)** into **(4)** gives $4\left(-\frac{2\lambda}{\mu}\right) - 3\left(\frac{3\lambda}{2\mu}\right) + 8\left(\frac{8\lambda - 1}{2\mu}\right) = 5 \quad\Rightarrow\quad \mu = \frac{39\lambda - 8}{10}$ and into **(5)** gives $\left(-\frac{2\lambda}{\mu}\right)^2 + \left(\frac{3\lambda}{2\mu}\right)^2 = \left(\frac{8\lambda - 1}{2\mu}\right)^2 \quad\Rightarrow\quad 16\lambda^2 + 9\lambda^2 = (8\lambda - 1)^2 \quad\Rightarrow\quad 39\lambda^2 - 16\lambda + 1 = 0 \quad\Rightarrow\quad \lambda = \frac{1}{13}$ or $\lambda = \frac{1}{3}$.

If $\lambda = \frac{1}{13}$ then $\mu = -\frac{1}{2}$ and $x = \frac{4}{13}, y = -\frac{3}{13}, z = \frac{5}{13}$. If $\lambda = \frac{1}{3}$ then $\mu = \frac{1}{2}$ and $x = -\frac{4}{3}, y = 1, z = \frac{5}{3}$. Thus the highest point on the ellipse is $\left(-\frac{4}{3}, 1, \frac{5}{3}\right)$ and the lowest point is $\left(\frac{4}{13}, -\frac{3}{13}, \frac{5}{13}\right)$.

47. $f(x, y, z) = ye^{x-z},\ g(x, y, z) = 9x^2 + 4y^2 + 36z^2 = 36,\ h(x, y, z) = xy + yz = 1. \quad \nabla f = \lambda\nabla g + \mu\nabla h \quad\Rightarrow$ $\left\langle ye^{x-z}, e^{x-z}, -ye^{x-z}\right\rangle = \lambda\langle 18x, 8y, 72z\rangle + \mu\langle y, x + z, y\rangle$, so $ye^{x-z} = 18\lambda x + \mu y$, $e^{x-z} = 8\lambda y + \mu(x + z)$, $-ye^{x-z} = 72\lambda z + \mu y$, $9x^2 + 4y^2 + 36z^2 = 36$, $xy + yz = 1$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ, and μ (in Maple, use the `allvalues` command), we get 4 real-valued solutions:

$$\begin{array}{lllll} x \approx 0.222444, & y \approx -2.157012, & z \approx -0.686049, & \lambda \approx -0.200401, & \mu \approx 2.108584 \\ x \approx -1.951921, & y \approx -0.545867, & z \approx 0.119973, & \lambda \approx 0.003141, & \mu \approx -0.076238 \\ x \approx 0.155142, & y \approx 0.904622, & z \approx 0.950293, & \lambda \approx -0.012447, & \mu \approx 0.489938 \\ x \approx 1.138731, & y \approx 1.768057, & z \approx -0.573138, & \lambda \approx 0.317141, & \mu \approx 1.862675 \end{array}$$

Substituting these values into f gives $f(0.222444, -2.157012, -0.686049) \approx -5.3506$,

$f(-1.951921, -0.545867, 0.119973) \approx -0.0688$, $f(0.155142, 0.904622, 0.950293) \approx 0.4084$, $f(1.138731, 1.768057, -0.573138) \approx 9.7938$. Thus the maximum is approximately 9.7938, and the minimum is approximately -5.3506.

48. $f(x,y,z) = x + y + z,\ g(x,y,z) = x^2 - y^2 - z = 0,\ h(x,y,z) = x^2 + z^2 = 4$.

$\nabla f = \lambda \nabla g + \mu \nabla h \quad \Rightarrow \quad \langle 1,1,1 \rangle = \lambda \langle 2x, -2y, -1 \rangle + \mu \langle 2x, 0, 2z \rangle$, so $1 = 2\lambda x + 2\mu x$, $1 = -2\lambda y$, $1 = -\lambda + 2\mu z$, $x^2 - y^2 = z$, $x^2 + z^2 = 4$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ, and μ, we get 4 real-valued solutions:

$$\begin{array}{lllll}
x \approx -1.652878, & y \approx -1.964194, & z \approx -1.126052, & \lambda \approx 0.254557, & \mu \approx -0.557060 \\
x \approx -1.502800, & y \approx 0.968872, & z \approx 1.319694, & \lambda \approx -0.516064, & \mu \approx 0.183352 \\
x \approx -0.992513, & y \approx 1.649677, & z \approx -1.736352, & \lambda \approx -0.303090, & \mu \approx -0.200682 \\
x \approx 1.895178, & y \approx 1.718347, & z \approx 0.638984, & \lambda \approx -0.290977, & \mu \approx 0.554805
\end{array}$$

Substituting these values into f gives $f(-1.652878, -1.964194, -1.126052) \approx -4.7431$, $f(-1.502800, 0.968872, 1.319694) \approx 0.7858$, $f(-0.992513, 1.649677, -1.736352) \approx -1.0792$, $f(1.895178, 1.718347, 0.638984) \approx 4.2525$. Thus the maximum is approximately 4.2525, and the minimum is approximately -4.7431.

49. (a) We wish to maximize $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$ subject to

$g(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n = c$ and $x_i > 0$.

$\nabla f = \left\langle \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n), \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n), \ldots, \frac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) \right\rangle$

and $\lambda \nabla g = \langle \lambda, \lambda, \ldots, \lambda \rangle$, so we need to solve the system of equations

$$\begin{aligned}
\tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n) = \lambda \quad &\Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_1 \\
\tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n) = \lambda \quad &\Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_2 \\
&\vdots \\
\tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) = \lambda \quad &\Rightarrow \quad x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = n\lambda x_n
\end{aligned}$$

This implies $n\lambda x_1 = n\lambda x_2 = \cdots = n\lambda x_n$. Note $\lambda \neq 0$, otherwise we can't have all $x_i > 0$. Thus $x_1 = x_2 = \cdots = x_n$. But $x_1 + x_2 + \cdots + x_n = c \quad \Rightarrow \quad nx_1 = c \quad \Rightarrow \quad x_1 = \dfrac{c}{n} = x_2 = x_3 = \cdots = x_n$. Then the only point where f can have an extreme value is $\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right)$. Since we can choose values for $(x_1, x_2, \ldots, x_n)$ that make f as close to zero (but not equal) as we like, f has no minimum value. Thus the maximum value is

$f\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right) = \sqrt[n]{\dfrac{c}{n} \cdot \dfrac{c}{n} \cdot \cdots \cdot \dfrac{c}{n}} = \dfrac{c}{n}$.

(b) From part (a), $\frac{c}{n}$ is the maximum value of f. Thus $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{c}{n}$. But $x_1 + x_2 + \cdots + x_n = c$, so $\sqrt[n]{x_1 x_2 \cdots x_n} \leq \dfrac{x_1 + x_2 + \cdots + x_n}{n}$. These two means are equal when f attains its maximum value $\frac{c}{n}$, but this can occur only at the point $\left(\frac{c}{n}, \frac{c}{n}, \ldots, \frac{c}{n}\right)$ we found in part (a). So the means are equal only when $x_1 = x_2 = x_3 = \cdots = x_n = \frac{c}{n}$.

50. (a) Let $f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum\limits_{i=1}^{n} x_i y_i$, $g(x_1, \ldots, x_n) = \sum\limits_{i=1}^{n} x_i^2$, and $h(x_1, \ldots, x_n) = \sum\limits_{i=1}^{n} y_i^2$. Then $\nabla f = \nabla \sum\limits_{i=1}^{n} x_i y_i = \langle y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_n \rangle$, $\nabla g = \nabla \sum\limits_{i=1}^{n} x_i^2 = \langle 2x_1, 2x_2, \ldots, 2x_n, 0, 0, \ldots, 0 \rangle$ and $\nabla h = \nabla \sum\limits_{i=1}^{n} y_i^2 = \langle 0, 0, \ldots, 0, 2y_1, 2y_2, \ldots, 2y_n \rangle$. So $\nabla f = \lambda \nabla g + \mu \nabla h \quad \Leftrightarrow \quad y_i = 2\lambda x_i$ and $x_i = 2\mu y_i$, $1 \leq i \leq n$. Then $1 = \sum\limits_{i=1}^{n} y_i^2 = \sum\limits_{i=1}^{n} 4\lambda^2 x_i^2 = 4\lambda^2 \sum\limits_{i=1}^{n} x_i^2 = 4\lambda^2 \quad \Rightarrow \quad \lambda = \pm\frac{1}{2}$. If $\lambda = \frac{1}{2}$ then $y_i = 2\left(\frac{1}{2}\right)x_i = x_i$, $1 \leq i \leq n$. Thus $\sum\limits_{i=1}^{n} x_i y_i = \sum\limits_{i=1}^{n} x_i^2 = 1$. Similarly if $\lambda = -\frac{1}{2}$ we get $y_i = -x_i$ and $\sum\limits_{i=1}^{n} x_i y_i = -1$. Similarly we get $\mu = \pm\frac{1}{2}$ giving $y_i = \pm x_i$, $1 \leq i \leq n$, and $\sum\limits_{i=1}^{n} x_i y_i = \pm 1$. Thus the maximum value of $\sum\limits_{i=1}^{n} x_i y_i$ is 1.

(b) Here we assume $\sum\limits_{i=1}^{n} a_i^2 \neq 0$ and $\sum\limits_{i=1}^{n} b_i^2 \neq 0$. (If $\sum\limits_{i=1}^{n} a_i^2 = 0$, then each $a_i = 0$ and so the inequality is trivially true.)

$x_i = \dfrac{a_i}{\sqrt{\sum a_j^2}} \quad \Rightarrow \quad \sum x_i^2 = \dfrac{\sum a_i^2}{\sum a_j^2} = 1$, and $y_i = \dfrac{b_i}{\sqrt{\sum b_j^2}} \quad \Rightarrow \quad \sum y_i^2 = \dfrac{\sum b_i^2}{\sum b_j^2} = 1$. Therefore, from part (a), $\sum x_i y_i = \sum \dfrac{a_i b_i}{\sqrt{\sum a_j^2}\sqrt{\sum b_j^2}} \leq 1 \quad \Leftrightarrow \quad \sum a_i b_i \leq \sqrt{\sum a_j^2}\sqrt{\sum b_j^2}$.

APPLIED PROJECT Rocket Science

1. Initially the rocket engine has mass $M_r = M_1$ and payload mass $P = M_2 + M_3 + A$. Then the change in velocity resulting from the first stage is $\Delta V_1 = -c\ln\left(1 - \dfrac{(1-S)M_1}{M_2 + M_3 + A + M_1}\right)$. After the first stage is jettisoned we can consider the rocket engine to have mass $M_r = M_2$ and the payload to have mass $P = M_3 + A$. The resulting change in velocity from the second stage is $\Delta V_2 = -c\ln\left(1 - \dfrac{(1-S)M_2}{M_3 + A + M_2}\right)$. When only the third stage remains, we have $M_r = M_3$ and $P = A$, so the resulting change in velocity is $\Delta V_3 = -c\ln\left(1 - \dfrac{(1-S)M_3}{A + M_3}\right)$. Since the rocket started from rest, the final velocity