13 □ VECTOR FUNCTIONS

13.1 Vector Functions and Space Curves

1. The component functions $\ln(t+1)$, $\dfrac{t}{\sqrt{9-t^2}}$, and 2^t are all defined when $t+1>0 \;\Rightarrow\; t>-1$ and $9-t^2>0 \;\Rightarrow$ $-3<t<3$, so the domain of $\mathbf{r}$ is $(-1,3)$.

2. The component functions $\cos t$, $\ln t$, and $\dfrac{1}{t-2}$ are all defined when $t>0$ and $t\neq 2$, so the domain of $\mathbf{r}$ is $(0,2)\cup(2,\infty)$.

3. $\displaystyle\lim_{t\to 0} e^{-3t}=e^0=1,\ \lim_{t\to 0}\frac{t^2}{\sin^2 t}=\lim_{t\to 0}\frac{1}{\dfrac{\sin^2 t}{t^2}}=\frac{1}{\displaystyle\lim_{t\to 0}\frac{\sin^2 t}{t^2}}=\frac{1}{\left(\displaystyle\lim_{t\to 0}\frac{\sin t}{t}\right)^2}=\frac{1}{1^2}=1,$

and $\displaystyle\lim_{t\to 0}\cos 2t=\cos 0=1$. Thus

$$\lim_{t\to 0}\left(e^{-3t}\,\mathbf{i}+\frac{t^2}{\sin^2 t}\,\mathbf{j}+\cos 2t\,\mathbf{k}\right)=\left[\lim_{t\to 0} e^{-3t}\right]\mathbf{i}+\left[\lim_{t\to 0}\frac{t^2}{\sin^2 t}\right]\mathbf{j}+\left[\lim_{t\to 0}\cos 2t\right]\mathbf{k}=\mathbf{i}+\mathbf{j}+\mathbf{k}.$$

4. $\displaystyle\lim_{t\to 1}\frac{t^2-t}{t-1}=\lim_{t\to 1}\frac{t\,(t-1)}{t-1}=\lim_{t\to 1}t=1,\ \lim_{t\to 1}\sqrt{t+8}=3,\ \lim_{t\to 1}\frac{\sin \pi t}{\ln t}=\lim_{t\to 1}\frac{\pi\cos\pi t}{1/t}=-\pi$ [by l'Hospital's Rule].

Thus the given limit equals $\mathbf{i}+3\,\mathbf{j}-\pi\,\mathbf{k}$.

5. $\displaystyle\lim_{t\to\infty}\frac{1+t^2}{1-t^2}=\lim_{t\to\infty}\frac{(1/t^2)+1}{(1/t^2)-1}=\frac{0+1}{0-1}=-1,\ \lim_{t\to\infty}\tan^{-1}t=\tfrac{\pi}{2},\ \lim_{t\to\infty}\frac{1-e^{-2t}}{t}=\lim_{t\to\infty}\frac{1}{t}-\frac{1}{te^{2t}}=0-0=0.$ Thus

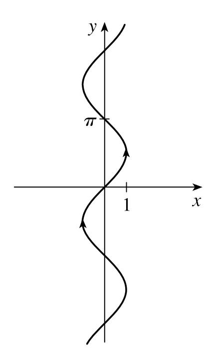
$\displaystyle\lim_{t\to\infty}\left\langle\frac{1+t^2}{1-t^2},\tan^{-1}t,\frac{1-e^{-2t}}{t}\right\rangle=\left\langle -1,\tfrac{\pi}{2},0\right\rangle.$

6. $\displaystyle\lim_{t\to\infty} te^{-t}=\lim_{t\to\infty}\frac{t}{e^t}=\lim_{t\to\infty}\frac{1}{e^t}=0$ [by l'Hospital's Rule], $\displaystyle\lim_{t\to\infty}\frac{t^3+t}{2t^3-1}=\lim_{t\to\infty}\frac{1+(1/t^2)}{2-(1/t^3)}=\frac{1+0}{2-0}=\frac{1}{2},$

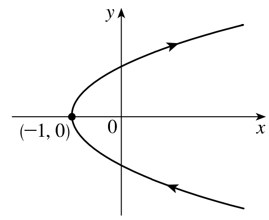
and $\displaystyle\lim_{t\to\infty}t\sin\frac{1}{t}=\lim_{t\to\infty}\frac{\sin(1/t)}{1/t}=\lim_{t\to\infty}\frac{\cos(1/t)(-1/t^2)}{-1/t^2}=\lim_{t\to\infty}\cos\frac{1}{t}=\cos 0=1$ [again by l'Hospital's Rule].

Thus $\displaystyle\lim_{t\to\infty}\left\langle te^{-t},\frac{t^3+t}{2t^3-1},t\sin\frac{1}{t}\right\rangle=\left\langle 0,\tfrac{1}{2},1\right\rangle.$

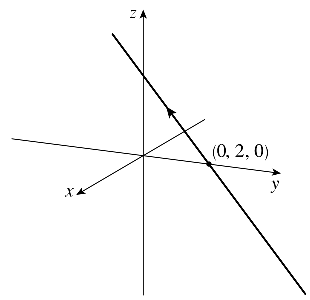
7. The corresponding parametric equations for this curve are $x=\sin t,\ y=t$. We can make a table of values, or we can eliminate the parameter: $t=y \;\Rightarrow$ $x=\sin y$, with $y\in\mathbb{R}$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



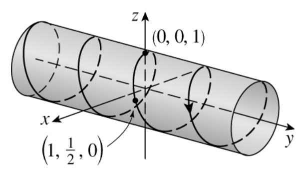
8. The corresponding parametric equations for this curve are $x = t^2 - 1,\ y = t$. We can make a table of values, or we can eliminate the parameter:
$t = y \quad \Rightarrow \quad x = y^2 - 1$, with $y \in \mathbb{R}$. Thus the curve is a parabola with vertex $(-1, 0)$ that opens to the right. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



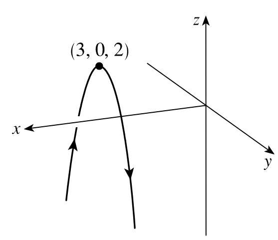
9. The corresponding parametric equations are $x = t,\ y = 2 - t,\ z = 2t$, which are parametric equations of a line through the point $(0, 2, 0)$ and with direction vector $\langle 1, -1, 2 \rangle$.



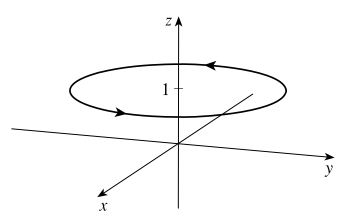
10. The corresponding parametric equations are $x = \sin \pi t,\ y = t,\ z = \cos \pi t$. Note that $x^2 + z^2 = \sin^2 \pi t + \cos^2 \pi t = 1$, so the curve lies on the circular cylinder $x^2 + z^2 = 1$. A point (x, y, z) on the curve lies directly to the left or right of the point $(x, 0, z)$ which moves clockwise (when viewed from the left) along the circle $x^2 + z^2 = 1$ in the xz-plane as t increases. Since $y = t$, the curve is a helix that spirals toward the right around the cylinder.



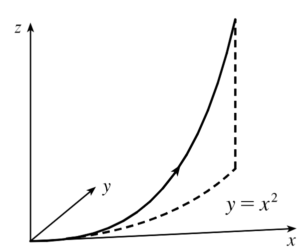
11. The corresponding parametric equations are $x = 3,\ y = t,\ z = 2 - t^2$. Eliminating the parameter in y and z gives $z = 2 - y^2$. Because $x = 3$, the curve is a parabola in the vertical plane $x = 3$ with vertex $(3, 0, 2)$.



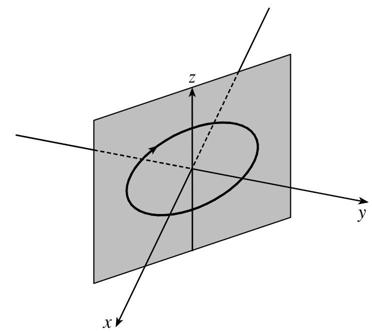
12. The corresponding parametric equations are $x = 2\cos t,\ y = 2\sin t$, $z = 1$. Eliminating the parameter in x and y gives $x^2 + y^2 = 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4$. Since $z = 1$, the curve is a circle of radius 2 centered at $(0, 0, 1)$ in the horizontal plane $z = 1$.



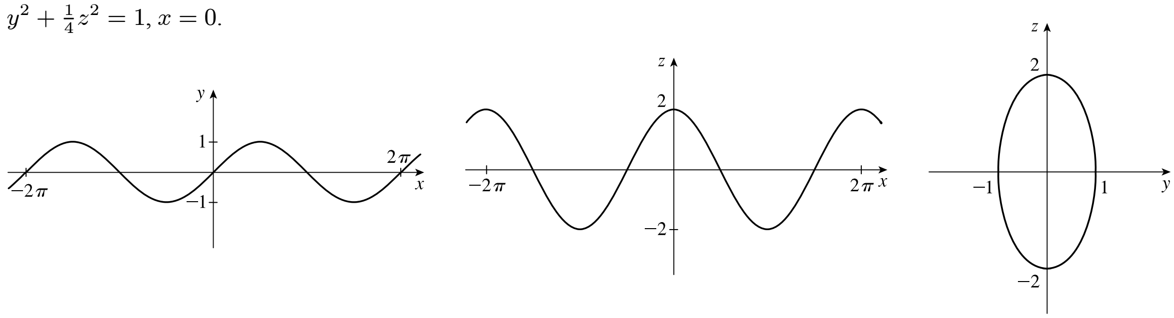
13. The parametric equations are $x = t^2$, $y = t^4$, $z = t^6$. These are positive for $t \neq 0$ and 0 when $t = 0$. So the curve lies entirely in the first octant. The projection of the graph onto the xy-plane is $y = x^2$, $y > 0$, a half parabola. The projection onto the xz-plane is $z = x^3$, $z > 0$, a half cubic, and the projection onto the yz-plane is $y^3 = z^2$.



14. If $x = \cos t,\ y = -\cos t,\ z = \sin t$, then $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$, so the curve is contained in the intersection of circular cylinders along the x- and y-axes. Furthermore, $y = -x$, so the curve is an ellipse in the plane $y = -x$, centered at the origin.

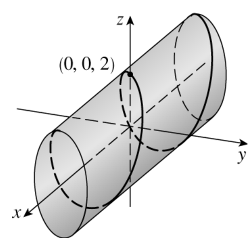


15. The projection of the curve onto the xy-plane is given by $\mathbf{r}(t) = \langle t, \sin t, 0 \rangle$ [we use 0 for the z-component] whose graph is the curve $y = \sin x$, $z = 0$. Similarly, the projection onto the xz-plane is $\mathbf{r}(t) = \langle t, 0, 2\cos t \rangle$, whose graph is the cosine wave $z = 2\cos x$, $y = 0$, and the projection onto the yz-plane is $\mathbf{r}(t) = \langle 0, \sin t, 2\cos t \rangle$ whose graph is the ellipse $y^2 + \frac{1}{4}z^2 = 1$, $x = 0$.

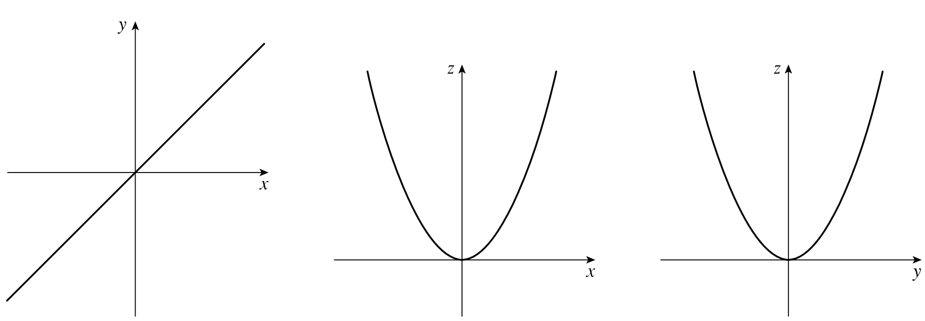


xy-plane xz-plane yz-plane

From the projection onto the yz-plane we see that the curve lies on an elliptical cylinder with axis the x-axis. The other two projections show that the curve oscillates both vertically and horizontally as we move in the x-direction, suggesting that the curve is an elliptical helix that spirals along the cylinder.



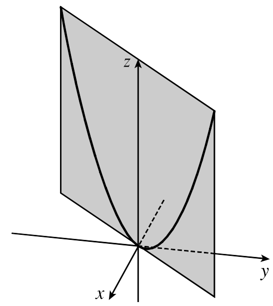
16. The projection of the curve onto the xy-plane is given by $\mathbf{r}(t) = \langle t, t, 0 \rangle$ whose graph is the line $y = x$, $z = 0$.
The projection onto the xz-plane is $\mathbf{r}(t) = \left\langle t, 0, t^2 \right\rangle$ whose graph is the parabola $z = x^2$, $y = 0$.
The projection onto the yz-plane is $\mathbf{r}(t) = \left\langle 0, t, t^2 \right\rangle$ whose graph is the parabola $z = y^2$, $x = 0$.



xy-plane xz-plane yz-plane

[continued]

From the projection onto the xy-plane we see that the curve lies on the vertical plane $y = x$. The other two projections show that the curve is a parabola contained in this plane.



17. Taking $\mathbf{r}_0 = \langle 2, 0, 0 \rangle$ and $\mathbf{r}_1 = \langle 6, 2, -2 \rangle$, we have from Equation 12.5.4

$\mathbf{r}(t) = (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)\,\langle 2, 0, 0 \rangle + t\,\langle 6, 2, -2 \rangle,\ 0 \le t \le 1$ or $\mathbf{r}(t) = \langle 2 + 4t, 2t, -2t \rangle,\ 0 \le t \le 1$.

Parametric equations are $x = 2 + 4t,\ y = 2t,\ z = -2t,\ 0 \le t \le 1$.

18. Taking $\mathbf{r}_0 = \langle -1, 2, -2 \rangle$ and $\mathbf{r}_1 = \langle -3, 5, 1 \rangle$, we have from Equation 12.5.4

$\mathbf{r}(t) = (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)\,\langle -1, 2, -2 \rangle + t\,\langle -3, 5, 1 \rangle,\ 0 \le t \le 1$ or $\mathbf{r}(t) = \langle -1 - 2t, 2 + 3t, -2 + 3t \rangle,\ 0 \le t \le 1$.

Parametric equations are $x = -1 - 2t,\ y = 2 + 3t,\ z = -2 + 3t,\ 0 \le t \le 1$.

19. Taking $\mathbf{r}_0 = \langle 0, -1, 1 \rangle$ and $\mathbf{r}_1 = \left\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \right\rangle$, we have

$\mathbf{r}(t) = (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)\,\langle 0, -1, 1 \rangle + t\left\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \right\rangle,\ 0 \le t \le 1$ or $\mathbf{r}(t) = \left\langle \frac{1}{2}t, -1 + \frac{4}{3}t, 1 - \frac{3}{4}t \right\rangle,\ 0 \le t \le 1$.

Parametric equations are $x = \frac{1}{2}t,\ y = -1 + \frac{4}{3}t,\ z = 1 - \frac{3}{4}t,\ 0 \le t \le 1$.

20. Taking $\mathbf{r}_0 = \langle a, b, c \rangle$ and $\mathbf{r}_1 = \langle u, v, w \rangle$, we have

$\mathbf{r}(t) = (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)\,\langle a, b, c \rangle + t\,\langle u, v, w \rangle,\ 0 \le t \le 1$ or $\mathbf{r}(t) = \langle a + (u-a)t, b + (v-b)t, c + (w-c)t \rangle$,

$0 \le t \le 1$. Parametric equations are $x = a + (u-a)t,\ y = b + (v-b)t,\ z = c + (w-c)t,\ 0 \le t \le 1$.

21. $x = t\cos t,\ y = t,\ z = t\sin t,\ t \ge 0$. At any point (x, y, z) on the curve, $x^2 + z^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = y^2$ so the curve lies on the circular cone $x^2 + z^2 = y^2$ with axis the y-axis. Also notice that $y \ge 0$; the graph is II.

22. $x = \cos t,\ y = \sin t,\ z = 1/(1 + t^2)$. At any point on the curve we have $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on the circular cylinder $x^2 + y^2 = 1$ with axis the z-axis. Notice that $0 < z \le 1$ and $z = 1$ only for $t = 0$. A point (x, y, z) on the curve lies directly above the point $(x, y, 0)$, which moves counterclockwise around the unit circle in the xy-plane as t increases, and $z \to 0$ as $t \to \pm\infty$. The graph must be VI.

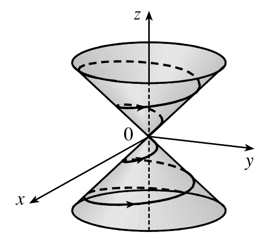
23. $x = t,\ y = 1/(1 + t^2),\ z = t^2$. At any point on the curve we have $z = x^2$, so the curve lies on a parabolic cylinder parallel to the y-axis. Notice that $0 < y \le 1$ and $z \ge 0$. Also the curve passes through $(0, 1, 0)$ when $t = 0$ and $y \to 0$, $z \to \infty$ as $t \to \pm\infty$, so the graph must be V.

24. $x = \cos t,\ y = \sin t,\ z = \cos 2t$. $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on a circular cylinder with axis the z-axis. A point (x, y, z) on the curve lies directly above or below $(x, y, 0)$, which moves around the unit circle in the xy-plane with period 2π. At the same time, the z-value of the point (x, y, z) oscillates with a period of π. So the curve repeats itself and the graph is I.

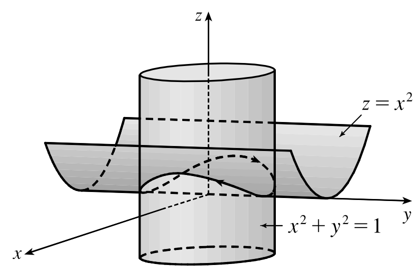
25. $x = \cos 8t,\ y = \sin 8t,\ z = e^{0.8t},\ t \geq 0.$ $x^2 + y^2 = \cos^2 8t + \sin^2 8t = 1$, so the curve lies on a circular cylinder with axis the z-axis. A point (x, y, z) on the curve lies directly above the point $(x, y, 0)$, which moves counterclockwise around the unit circle in the xy-plane as t increases. The curve starts at $(1, 0, 1)$, when $t = 0$, and $z \to \infty$ (at an increasing rate) as $t \to \infty$, so the graph is IV.

26. $x = \cos^2 t,\ y = \sin^2 t,\ z = t.$ $x + y = \cos^2 t + \sin^2 t = 1$, so the curve lies in the vertical plane $x + y = 1$. x and y are periodic, both with period π, and z increases as t increases, so the graph is III.

27. If $x = t\cos t,\ y = t\sin t,\ z = t$, then $x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = z^2$, so the curve lies on the cone $z^2 = x^2 + y^2$. Since $z = t$, the curve is a spiral on this cone.



28. Here $x^2 = \sin^2 t = z$ and $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$, so the curve is contained in the intersection of the parabolic cylinder $z = x^2$ with the circular cylinder $x^2 + y^2 = 1$. We get the complete intersection for $0 \leq t \leq 2\pi$.



29. Here $x = 2t,\ y = e^t,\ z = e^{2t}$. Then $t = x/2 \quad\Rightarrow\quad y = e^t = e^{x/2}$, so the curve lies on the cylinder $y = e^{x/2}$. Also $z = e^{2t} = e^x$, so the curve lies on the cylinder $z = e^x$. Since $z = e^{2t} = \left(e^t\right)^2 = y^2$, the curve also lies on the parabolic cylinder $z = y^2$.

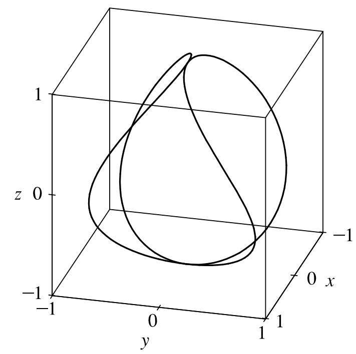
30. Here $x = t^2,\ y = \ln t,\ z = 1/t$. The domain of $\mathbf{r}$ is $(0, \infty)$, so $x = t^2 \quad\Rightarrow\quad t = \sqrt{x} \quad\Rightarrow\quad y = \ln\sqrt{x}$. Thus one surface containing the curve is the cylinder $y = \ln\sqrt{x}$ or $y = \ln x^{1/2} = \frac{1}{2}\ln x$. Also $z = 1/t = 1/\sqrt{x}$, so the curve also lies on the cylinder $z = 1/\sqrt{x}$ or $x = 1/z^2$, $z > 0$. Finally $z = 1/t \quad\Rightarrow\quad t = 1/z \quad\Rightarrow\quad y = \ln(1/z)$, so the curve also lies on the cylinder $y = \ln(1/z)$ or $y = \ln z^{-1} = -\ln z$. Note that the surface $y = \ln(xz)$ also contains the curve, since $\ln(xz) = \ln(t^2 \cdot 1/t) = \ln t = y$.

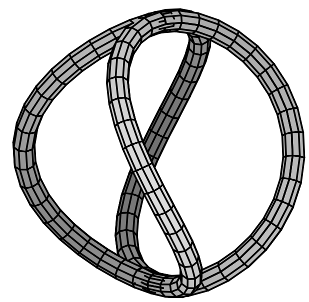
31. Parametric equations for the curve are $x = t,\ y = 0,\ z = 2t - t^2$. Substituting into the equation of the paraboloid gives $2t - t^2 = t^2 \quad\Rightarrow\quad 2t = 2t^2 \quad\Rightarrow\quad t = 0, 1$. Since $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{k}$, the points of intersection are $(0, 0, 0)$ and $(1, 0, 1)$.

32. Parametric equations for the helix are $x = \sin t$, $y = \cos t$, $z = t$. Substituting into the equation of the sphere gives $\sin^2 t + \cos^2 t + t^2 = 5 \quad\Rightarrow\quad 1 + t^2 = 5 \quad\Rightarrow\quad t = \pm 2$. Since $\mathbf{r}(2) = \langle \sin 2, \cos 2, 2 \rangle$ and $\mathbf{r}(-2) = \langle \sin(-2), \cos(-2), -2 \rangle$, the points of intersection are $(\sin 2, \cos 2, 2) \approx (0.909, -0.416, 2)$ and $(\sin(-2), \cos(-2), -2) \approx (-0.909, -0.416, -2)$.

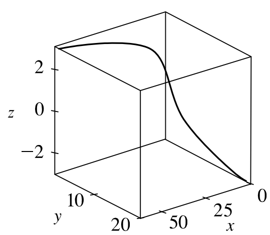
33. $\mathbf{r}(t) = \langle \cos t \, \sin 2t, \sin t \, \sin 2t, \cos 2t \rangle$.

We include both a regular plot and a plot showing a tube of radius 0.08 around the curve.

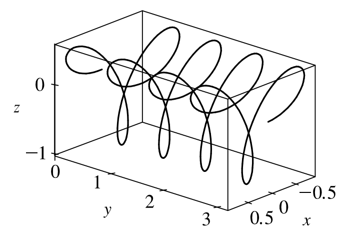




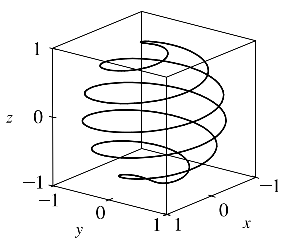
34. $\mathbf{r}(t) = \langle te^t, e^{-t}, t \rangle$



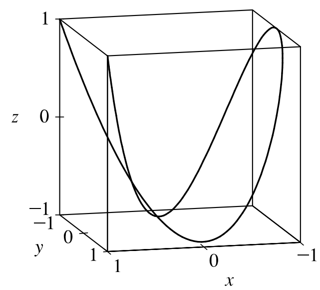
35. $\mathbf{r}(t) = \langle \sin 3t \cos t, \frac{1}{4}t, \sin 3t \sin t \rangle$



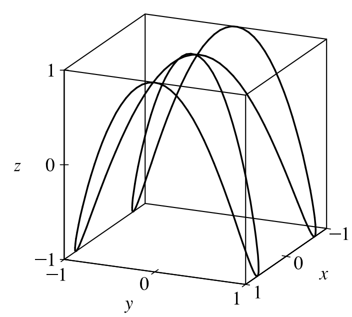
36. $\mathbf{r}(t) = \langle \cos(8 \cos t) \sin t, \sin(8 \cos t) \sin t, \cos t \rangle$



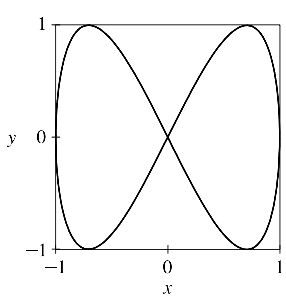
37. $\mathbf{r}(t) = \langle \cos 2t, \cos 3t, \cos 4t \rangle$



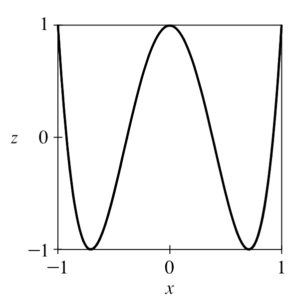
38. $x = \sin t$, $y = \sin 2t$, $z = \cos 4t$.



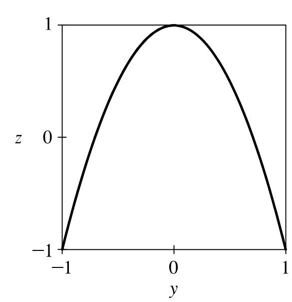
We graph the projections onto the coordinate planes.



xy-plane

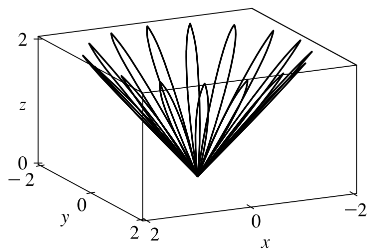


xz-plane



yz-plane

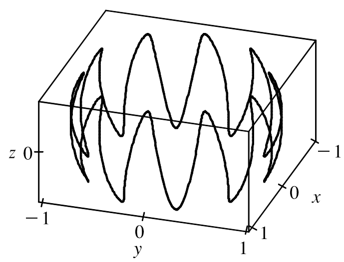
From the projection onto the xy-plane we see that from above the curve appears to be shaped like a "figure eight." The curve can be visualized as this shape wrapped around an almost parabolic cylindrical surface, the profile of which is visible in the projection onto the yz-plane.

39. 

$x = (1 + \cos 16t)\cos t$, $y = (1 + \cos 16t)\sin t$, $z = 1 + \cos 16t$. At any point on the graph,

$$x^2 + y^2 = (1 + \cos 16t)^2 \cos^2 t + (1 + \cos 16t)^2 \sin^2 t$$
$$= (1 + \cos 16t)^2 = z^2, \text{ so the graph lies on the cone } x^2 + y^2 = z^2.$$

From the graph at left, we see that this curve looks like the projection of a leaved two-dimensional curve onto a cone.

40. 

$x = \sqrt{1 - 0.25\cos^2 10t}\,\cos t$, $y = \sqrt{1 - 0.25\cos^2 10t}\,\sin t$, $z = 0.5\cos 10t$. At any point on the graph,

$$x^2 + y^2 + z^2 = (1 - 0.25\cos^2 10t)\cos^2 t + (1 - 0.25\cos^2 10t)\sin^2 t + 0.25\cos^2 t$$
$$= 1 - 0.25\cos^2 10t + 0.25\cos^2 10t = 1,$$

so the graph lies on the sphere $x^2 + y^2 + z^2 = 1$, and since $z = 0.5\cos 10t$ the graph resembles a trigonometric curve with ten peaks projected onto the sphere. We get the complete graph for $0 \le t \le 2\pi$.

41. If $t = -1$, then $x = 1$, $y = 4$, $z = 0$, so the curve passes through the point $(1, 4, 0)$. If $t = 3$, then $x = 9$, $y = -8$, $z = 28$, so the curve passes through the point $(9, -8, 28)$. For the point $(4, 7, -6)$ to be on the curve, we require $y = 1 - 3t = 7 \Rightarrow t = -2$. But then $z = 1 + (-2)^3 = -7 \ne -6$, so $(4, 7, -6)$ is not on the curve.

42. The projection of the curve C of intersection onto the xy-plane is the circle $x^2 + y^2 = 4$, $z = 0$. Then we can write $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$. Since C also lies on the surface $z = xy$, we have $z = xy = (2\cos t)(2\sin t) = 4\cos t\sin t$, or $2\sin(2t)$. Then parametric equations for C are $x = 2\cos t$, $y = 2\sin t$, $z = 2\sin(2t)$, $0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\sin(2t)\,\mathbf{k}$, $0 \le t \le 2\pi$.

43. Both equations are solved for z, so we can substitute to eliminate z: $\sqrt{x^2 + y^2} = 1 + y \Rightarrow x^2 + y^2 = 1 + 2y + y^2 \Rightarrow x^2 = 1 + 2y \Rightarrow y = \frac{1}{2}(x^2 - 1)$. We can form parametric equations for the curve C of intersection by choosing a parameter $x = t$, then $y = \frac{1}{2}(t^2 - 1)$ and $z = 1 + y = 1 + \frac{1}{2}(t^2 - 1) = \frac{1}{2}(t^2 + 1)$. Thus a vector function representing C is $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2 - 1)\,\mathbf{j} + \frac{1}{2}(t^2 + 1)\,\mathbf{k}$.

44. The projection of the curve C of intersection onto the xy-plane is the parabola $y = x^2$, $z = 0$. Then we can choose the parameter $x = t \Rightarrow y = t^2$. Since C also lies on the surface $z = 4x^2 + y^2$, we have $z = 4x^2 + y^2 = 4t^2 + (t^2)^2$. Then parametric equations for C are $x = t$, $y = t^2$, $z = 4t^2 + t^4$, and the corresponding vector function is $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + (4t^2 + t^4)\,\mathbf{k}$.

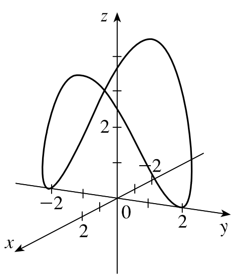
45. The projection of the curve C of intersection onto the xy-plane is the circle $x^2 + y^2 = 1$, $z = 0$, so we can write $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. Since C also lies on the surface $z = x^2 - y^2$, we have $z = x^2 - y^2 = \cos^2 t - \sin^2 t$ or $\cos 2t$. Thus parametric equations for C are $x = \cos t$, $y = \sin t$, $z = \cos 2t$, $0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k}$, $0 \le t \le 2\pi$.

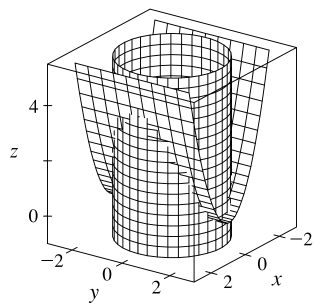
46. The projection of the curve C of intersection onto the xz-plane is the circle $x^2 + z^2 = 1$, $y = 0$, so we can write $x = \cos t$, $z = \sin t$, $0 \le t \le 2\pi$. C also lies on the surface $x^2 + y^2 + 4z^2 = 4$, and since $y \ge 0$ we can write

$$y = \sqrt{4 - x^2 - 4z^2} = \sqrt{4 - \cos^2 t - 4\sin^2 t} = \sqrt{4 - \cos^2 t - 4(1 - \cos^2 t)} = \sqrt{3\cos^2 t} = \sqrt{3}\,|\cos t|$$

Thus parametric equations for C are $x = \cos t$, $y = \sqrt{3}\,|\cos t|$, $z = \sin t$, $0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sqrt{3}\,|\cos t|\,\mathbf{j} + \sin t\,\mathbf{k}$, $0 \le t \le 2\pi$.

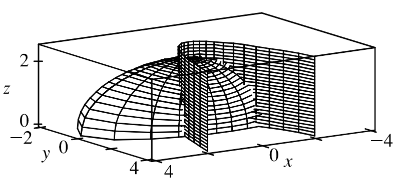
47.

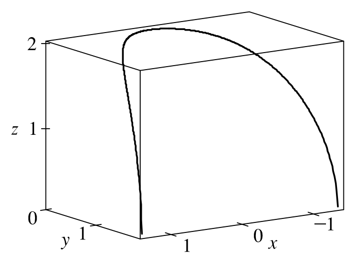




The projection of the curve C of intersection onto the xy-plane is the circle $x^2 + y^2 = 4$, $z = 0$. Then we can write $x = 2\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$. Since C also lies on the surface $z = x^2$, we have $z = x^2 = (2\cos t)^2 = 4\cos^2 t$. Then parametric equations for C are $x = 2\cos t,\ y = 2\sin t,\ z = 4\cos^2 t,\ 0 \le t \le 2\pi$.

48.





$x = t \quad\Rightarrow\quad y = t^2 \quad\Rightarrow\quad 4z^2 = 16 - x^2 - 4y^2 = 16 - t^2 - 4t^4 \quad\Rightarrow\quad z = \sqrt{4 - \left(\frac{1}{2}t\right)^2 - t^4}$.

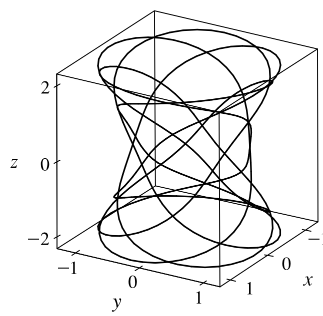
Note that z is positive because the intersection is with the top half of the ellipsoid. Hence the curve is given by $x = t,\ y = t^2,\ z = \sqrt{4 - \frac{1}{4}t^2 - t^4}$.

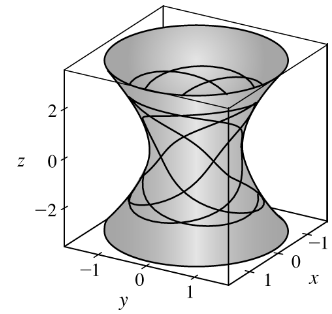
49. For the particles to collide, we require $\mathbf{r}_1(t) = \mathbf{r}_2(t) \quad\Leftrightarrow\quad \langle t^2, 7t - 12, t^2 \rangle = \langle 4t - 3, t^2, 5t - 6 \rangle$. Equating components gives $t^2 = 4t - 3$, $7t - 12 = t^2$, and $t^2 = 5t - 6$. From the first equation, $t^2 - 4t + 3 = 0 \quad\Leftrightarrow\quad (t - 3)(t - 1) = 0$ so $t = 1$ or $t = 3$. $t = 1$ does not satisfy the other two equations, but $t = 3$ does. The particles collide when $t = 3$, at the point $(9, 9, 9)$.

50. The particles collide provided $\mathbf{r}_1(t) = \mathbf{r}_2(t) \quad\Leftrightarrow\quad \langle t, t^2, t^3 \rangle = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$. Equating components gives $t = 1 + 2t$, $t^2 = 1 + 6t$, and $t^3 = 1 + 14t$. The first equation gives $t = -1$, but this does not satisfy the other equations, so the particles do not collide. For the paths to intersect, we need to find a value for t and a value for s where $\mathbf{r}_1(t) = \mathbf{r}_2(s) \quad\Leftrightarrow$

$\langle t, t^2, t^3 \rangle = \langle 1 + 2s, 1 + 6s, 1 + 14s \rangle$. Equating components, $t = 1 + 2s$, $t^2 = 1 + 6s$, and $t^3 = 1 + 14s$. Substituting the first equation into the second gives $(1 + 2s)^2 = 1 + 6s \Rightarrow 4s^2 - 2s = 0 \Rightarrow 2s(2s - 1) = 0 \Rightarrow s = 0$ or $s = \frac{1}{2}$. From the first equation, $s = 0 \Rightarrow t = 1$ and $s = \frac{1}{2} \Rightarrow t = 2$. Checking, we see that both pairs of values satisfy the third equation. Thus the paths intersect twice, at the point $(1, 1, 1)$ when $s = 0$ and $t = 1$, and at $(2, 4, 8)$ when $s = \frac{1}{2}$ and $t = 2$.

51. (a) We plot the parametric equations for $0 \leq t \leq 2\pi$ in the first figure. We get a better idea of the shape of the curve if we plot it simultaneously with the hyperboloid of one sheet from part (b), as shown in the second figure.





(b) Here $x = \frac{27}{26}\sin 8t - \frac{8}{39}\sin 18t$, $y = -\frac{27}{26}\cos 8t + \frac{8}{39}\cos 18t$, $z = \frac{144}{65}\sin 5t$.

For any point on the curve,

$$\begin{aligned} x^2 + y^2 &= \left(\tfrac{27}{26}\sin 8t - \tfrac{8}{39}\sin 18t\right)^2 + \left(-\tfrac{27}{26}\cos 8t + \tfrac{8}{39}\cos 18t\right)^2 \\ &= \tfrac{27^2}{26^2}\sin^2 8t - 2\cdot\tfrac{27\cdot 8}{26\cdot 39}\sin 8t\sin 18t + \tfrac{64}{39^2}\sin^2 18t \\ &\qquad + \tfrac{27^2}{26^2}\cos^2 8t - 2\cdot\tfrac{27\cdot 8}{26\cdot 39}\cos 8t\cos 18t + \tfrac{64}{39^2}\cos^2 18t \\ &= \tfrac{27^2}{26^2}\left(\sin^2 8t + \cos^2 8t\right) + \tfrac{64}{39^2}\left(\sin^2 18t + \cos^2 18t\right) - \tfrac{72}{169}\left(\sin 8t\sin 18t + \cos 8t\cos 18t\right) \\ &= \tfrac{27^2}{26^2} + \tfrac{64}{39^2} - \tfrac{72}{169}\cos(18t - 8t) = \tfrac{27^2}{26^2} + \tfrac{64}{39^2} - \tfrac{72}{169}\cos 10t \end{aligned}$$

using the trigonometric identities $\sin^2\theta + \cos^2\theta = 1$ and $\cos(x - y) = \cos x\cos y + \sin x\sin y$. Also $z^2 = \frac{144^2}{65^2}\sin^2 5t$, and the identity $\sin^2 x = \dfrac{1 - \cos 2x}{2}$ gives $z^2 = \frac{144^2}{65^2}\cdot\frac{1}{2}\left[1 - \cos(2\cdot 5t)\right] = \frac{144^2}{2\cdot 65^2} - \frac{144^2}{2\cdot 65^2}\cos 10t$.

Then

$$\begin{aligned} 144(x^2 + y^2) - 25z^2 &= 144\left(\tfrac{27^2}{26^2} + \tfrac{64}{39^2} - \tfrac{72}{169}\cos 10t\right) - 25\left(\tfrac{144^2}{2\cdot 65^2} - \tfrac{144^2}{2\cdot 65^2}\cos 10t\right) \\ &= 144\left(\tfrac{27^2}{26^2} + \tfrac{64}{39^2} - \tfrac{25\cdot 144}{2\cdot 65^2} - \tfrac{72}{169}\cos 10t + \tfrac{25\cdot 144}{2\cdot 65^2}\cos 10t\right) \\ &= 144\left(\tfrac{27^2}{26^2} + \tfrac{64}{39^2} - \tfrac{72}{169} - \tfrac{72}{169}\cos 10t + \tfrac{72}{169}\cos 10t\right) = 144\left(\tfrac{25}{36}\right) = 100 \end{aligned}$$

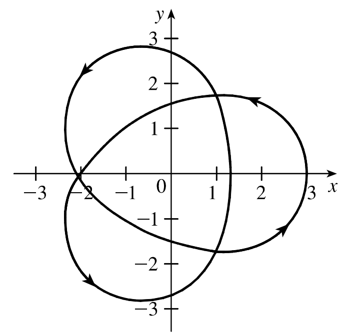
Thus the curve lies on the surface $144(x^2 + y^2) - 25z^2 = 100$ or $144x^2 + 144y^2 - 25z^2 = 100$, a hyperboloid of one sheet with axis the z-axis.

52. The projection of the curve onto the xy-plane is given by the parametric equations $x = (2 + \cos 1.5t)\cos t$, $y = (2 + \cos 1.5t)\sin t$. If we convert to polar coordinates, we have

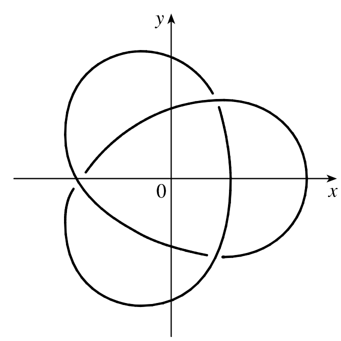
$r^2 = x^2 + y^2 = [(2 + \cos 1.5t)\cos t]^2 + [(2 + \cos 1.5t)\sin t]^2 = (2 + \cos 1.5t)^2(\cos^2 t + \sin^2 t) = (2 + \cos 1.5t)^2 \quad \Rightarrow$

$r = 2 + \cos 1.5t$. Also, $\tan\theta = \dfrac{y}{x} = \dfrac{(2 + \cos 1.5t)\sin t}{(2 + \cos 1.5t)\cos t} = \tan t \quad \Rightarrow \quad \theta = t$.

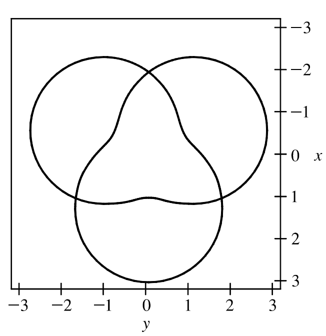
Thus the polar equation of the curve is $r = 2 + \cos 1.5\theta$. At $\theta = 0$, we have $r = 3$, and r decreases to 1 as θ increases to $\frac{2\pi}{3}$. For $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$, r increases to 3; r decreases to 1 again at $\theta = 2\pi$, increases to 3 at $\theta = \frac{8\pi}{3}$, decreases to 1 at $\theta = \frac{10\pi}{3}$, and completes the closed curve by increasing to 3 at $\theta = 4\pi$. We sketch an approximate graph as shown in the figure.



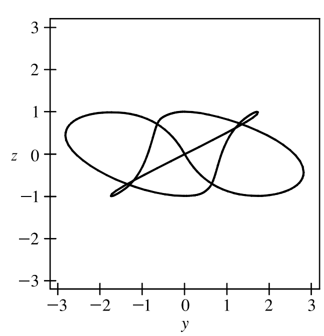
We can determine how the curve passes over itself by investigating the maximum and minimum values of z for $0 \le t \le 4\pi$. Since $z = \sin 1.5t$, z is maximized where $\sin 1.5t = 1 \quad \Rightarrow \quad 1.5t = \frac{\pi}{2}, \frac{5\pi}{2}$, or $\frac{9\pi}{2} \quad \Rightarrow$ $t = \frac{\pi}{3}, \frac{5\pi}{3}$, or 3π. z is minimized where $\sin 1.5t = -1 \quad \Rightarrow$ $1.5t = \frac{3\pi}{2}, \frac{7\pi}{2}$, or $\frac{11\pi}{2} \quad \Rightarrow \quad t = \pi, \frac{7\pi}{3}$, or $\frac{11\pi}{3}$. Note that these are precisely the values for which $\cos 1.5t = 0 \quad \Rightarrow \quad r = 2$, and on the graph of the projection, these six points appear to be at the three self-intersections we see. Comparing the maximum and minimum values of z at these intersections, we can determine where the curve passes over itself, as indicated in the figure.



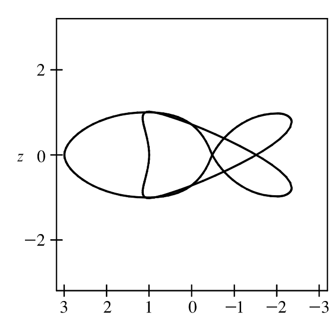
We show a computer-drawn graph of the curve from above, as well as views from the front and from the right side.



Top view

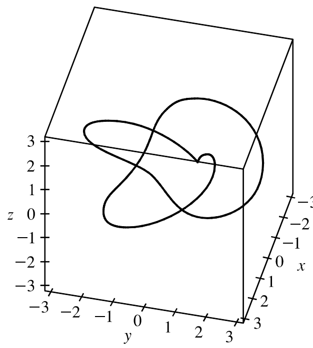
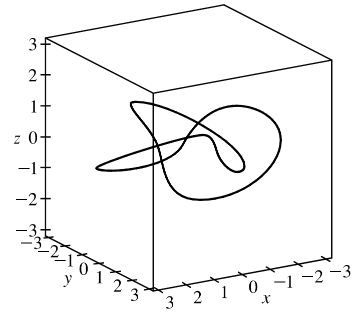
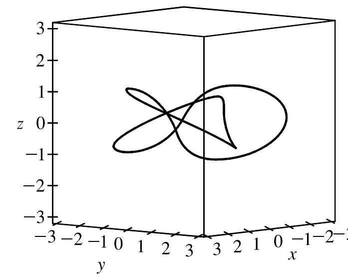


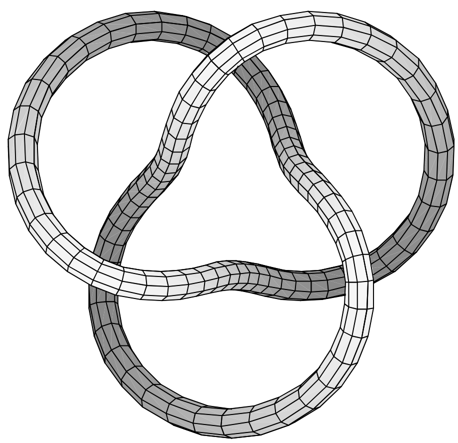
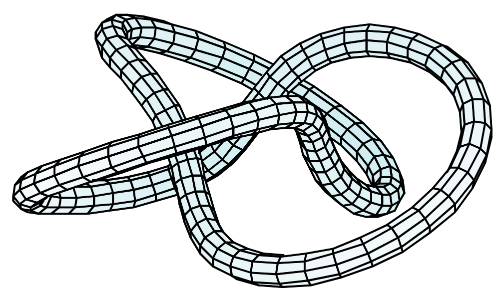
Front view



Side view

The top view graph shows a more accurate representation of the projection of the trefoil knot onto the xy-plane (the axes are rotated 90°). Notice the indentations the graph exhibits at the points corresponding to $r = 1$. Finally, we graph several additional viewpoints of the trefoil knot, along with two plots showing a tube of radius 0.2 around the curve.

53. Let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$. In each part of this problem the basic procedure is to use Equation 1 and then analyze the individual component functions using the limit properties we have already developed for real-valued functions.

(a) $\lim\limits_{t\to a} \mathbf{u}(t) + \lim\limits_{t\to a} \mathbf{v}(t) = \left\langle \lim\limits_{t\to a} u_1(t), \lim\limits_{t\to a} u_2(t), \lim\limits_{t\to a} u_3(t)\right\rangle + \left\langle \lim\limits_{t\to a} v_1(t), \lim\limits_{t\to a} v_2(t), \lim\limits_{t\to a} v_3(t)\right\rangle$ and the limits of these component functions must each exist since the vector functions both possess limits as $t \to a$. Then adding the two vectors and using the addition property of limits for real-valued functions, we have that

$$\begin{aligned}
\lim_{t\to a} \mathbf{u}(t) + \lim_{t\to a} \mathbf{v}(t) &= \left\langle \lim_{t\to a} u_1(t) + \lim_{t\to a} v_1(t), \lim_{t\to a} u_2(t) + \lim_{t\to a} v_2(t), \lim_{t\to a} u_3(t) + \lim_{t\to a} v_3(t)\right\rangle \\
&= \left\langle \lim_{t\to a} \left[u_1(t) + v_1(t)\right], \lim_{t\to a} \left[u_2(t) + v_2(t)\right], \lim_{t\to a} \left[u_3(t) + v_3(t)\right]\right\rangle \\
&= \lim_{t\to a} \langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t)\rangle \qquad \text{[using (1) backward]} \\
&= \lim_{t\to a} \left[\mathbf{u}(t) + \mathbf{v}(t)\right]
\end{aligned}$$

(b) $\lim\limits_{t\to a} c\mathbf{u}(t) = \lim\limits_{t\to a} \langle cu_1(t), cu_2(t), cu_3(t)\rangle = \left\langle \lim\limits_{t\to a} cu_1(t), \lim\limits_{t\to a} cu_2(t), \lim\limits_{t\to a} cu_3(t)\right\rangle$

$$\begin{aligned}
&= \left\langle c\lim_{t\to a} u_1(t), c\lim_{t\to a} u_2(t), c\lim_{t\to a} u_3(t)\right\rangle = c\left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle \\
&= c\lim_{t\to a} \langle u_1(t), u_2(t), u_3(t)\rangle = c\lim_{t\to a} \mathbf{u}(t)
\end{aligned}$$

(c) $\lim\limits_{t\to a}\mathbf{u}(t)\cdot\lim\limits_{t\to a}\mathbf{v}(t)=\left\langle\lim\limits_{t\to a}u_1(t),\lim\limits_{t\to a}u_2(t),\lim\limits_{t\to a}u_3(t)\right\rangle\cdot\left\langle\lim\limits_{t\to a}v_1(t),\lim\limits_{t\to a}v_2(t),\lim\limits_{t\to a}v_3(t)\right\rangle$

$$\begin{aligned}
&=\left[\lim_{t\to a}u_1(t)\right]\left[\lim_{t\to a}v_1(t)\right]+\left[\lim_{t\to a}u_2(t)\right]\left[\lim_{t\to a}v_2(t)\right]+\left[\lim_{t\to a}u_3(t)\right]\left[\lim_{t\to a}v_3(t)\right]\\
&=\lim_{t\to a}u_1(t)v_1(t)+\lim_{t\to a}u_2(t)v_2(t)+\lim_{t\to a}u_3(t)v_3(t)\\
&=\lim_{t\to a}\left[u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t)\right]=\lim_{t\to a}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right]
\end{aligned}$$

(d) $\lim\limits_{t\to a}\mathbf{u}(t)\times\lim\limits_{t\to a}\mathbf{v}(t)=\left\langle\lim\limits_{t\to a}u_1(t),\lim\limits_{t\to a}u_2(t),\lim\limits_{t\to a}u_3(t)\right\rangle\times\left\langle\lim\limits_{t\to a}v_1(t),\lim\limits_{t\to a}v_2(t),\lim\limits_{t\to a}v_3(t)\right\rangle$

$$\begin{aligned}
&=\Big\langle\left[\lim_{t\to a}u_2(t)\right]\left[\lim_{t\to a}v_3(t)\right]-\left[\lim_{t\to a}u_3(t)\right]\left[\lim_{t\to a}v_2(t)\right],\\
&\qquad\left[\lim_{t\to a}u_3(t)\right]\left[\lim_{t\to a}v_1(t)\right]-\left[\lim_{t\to a}u_1(t)\right]\left[\lim_{t\to a}v_3(t)\right],\\
&\qquad\qquad\left[\lim_{t\to a}u_1(t)\right]\left[\lim_{t\to a}v_2(t)\right]-\left[\lim_{t\to a}u_2(t)\right]\left[\lim_{t\to a}v_1(t)\right]\Big\rangle\\
&=\Big\langle\lim_{t\to a}\left[u_2(t)v_3(t)-u_3(t)v_2(t)\right],\lim_{t\to a}\left[u_3(t)v_1(t)-u_1(t)v_3(t)\right],\\
&\qquad\qquad\lim_{t\to a}\left[u_1(t)v_2(t)-u_2(t)v_1(t)\right]\Big\rangle\\
&=\lim_{t\to a}\left\langle u_2(t)v_3(t)-u_3(t)v_2(t),u_3(t)v_1(t)-u_1(t)v_3(t),u_1(t)v_2(t)-u_2(t)v_1(t)\right\rangle\\
&=\lim_{t\to a}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]
\end{aligned}$$

54. Let $\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$. If $\lim\limits_{t\to a}\mathbf{r}(t)=\mathbf{b}$, then $\lim\limits_{t\to a}\mathbf{r}(t)$ exists, so by (1),

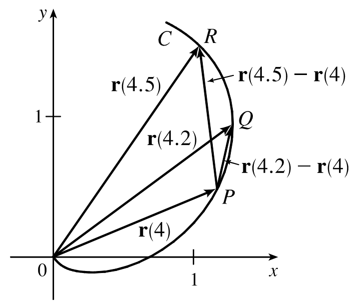
$\mathbf{b}=\lim\limits_{t\to a}\mathbf{r}(t)=\left\langle\lim\limits_{t\to a}f(t),\lim\limits_{t\to a}g(t),\lim\limits_{t\to a}h(t)\right\rangle$. By the definition of equal vectors we have $\lim\limits_{t\to a}f(t)=b_1$, $\lim\limits_{t\to a}g(t)=b_2$ and $\lim\limits_{t\to a}h(t)=b_3$. But these are limits of real-valued functions, so by the definition of limits, for every $\varepsilon>0$ there exists $\delta_1>0$, $\delta_2>0$, $\delta_3>0$ so that if $0<|t-a|<\delta_1$ then $|f(t)-b_1|<\varepsilon/3$, if $0<|t-a|<\delta_2$ then $|g(t)-b_2|<\varepsilon/3$, and if $0<|t-a|<\delta_3$ then $|h(t)-b_3|<\varepsilon/3$. Letting $\delta=$ minimum of $\{\delta_1,\delta_2,\delta_3\}$, then if $0<|t-a|<\delta$ we have $|f(t)-b_1|+|g(t)-b_2|+|h(t)-b_3|<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon$. But

$$\begin{aligned}
|\mathbf{r}(t)-\mathbf{b}|&=|\langle f(t)-b_1,g(t)-b_2,h(t)-b_3\rangle|=\sqrt{(f(t)-b_1)^2+(g(t)-b_2)^2+(h(t)-b_3)^2}\\
&\le\sqrt{[f(t)-b_1]^2}+\sqrt{[g(t)-b_2]^2}+\sqrt{[h(t)-b_3]^2}=|f(t)-b_1|+|g(t)-b_2|+|h(t)-b_3|
\end{aligned}$$

Thus for every $\varepsilon>0$ there exists $\delta>0$ such that if $0<|t-a|<\delta$ then $|\mathbf{r}(t)-\mathbf{b}|\le|f(t)-b_1|+|g(t)-b_2|+|h(t)-b_3|<\varepsilon$. Conversely, suppose for every $\varepsilon>0$, there exists $\delta>0$ such that if $0<|t-a|<\delta$ then $|\mathbf{r}(t)-\mathbf{b}|<\varepsilon\quad\Leftrightarrow\quad|\langle f(t)-b_1,g(t)-b_2,h(t)-b_3\rangle|<\varepsilon\quad\Leftrightarrow$ $\sqrt{[f(t)-b_1]^2+[g(t)-b_2]^2+[h(t)-b_3]^2}<\varepsilon\quad\Leftrightarrow\quad[f(t)-b_1]^2+[g(t)-b_2]^2+[h(t)-b_3]^2<\varepsilon^2$. But each term on the left side of the last inequality is positive, so if $0<|t-a|<\delta$, then $[f(t)-b_1]^2<\varepsilon^2$, $[g(t)-b_2]^2<\varepsilon^2$ and $[h(t)-b_3]^2<\varepsilon^2$ or, taking the square root of both sides in each of the above, $|f(t)-b_1|<\varepsilon$, $|g(t)-b_2|<\varepsilon$ and $|h(t)-b_3|<\varepsilon$. And by definition of limits of real-valued functions we have $\lim\limits_{t\to a}f(t)=b_1$, $\lim\limits_{t\to a}g(t)=b_2$ and $\lim\limits_{t\to a}h(t)=b_3$. But by (1), $\lim\limits_{t\to a}\mathbf{r}(t)=\left\langle\lim\limits_{t\to a}f(t),\lim\limits_{t\to a}g(t),\lim\limits_{t\to a}h(t)\right\rangle$, so $\lim\limits_{t\to a}\mathbf{r}(t)=\langle b_1,b_2,b_3\rangle=\mathbf{b}$.

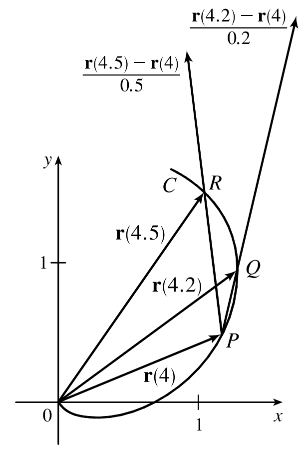
13.2 Derivatives and Integrals of Vector Functions

1. (a)



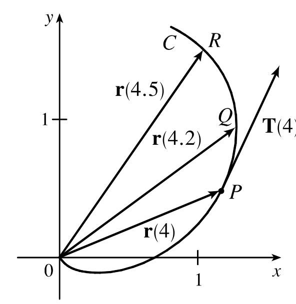
(b) $\dfrac{\mathbf{r}(4.5)-\mathbf{r}(4)}{0.5} = 2[\mathbf{r}(4.5)-\mathbf{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector $\mathbf{r}(4.5)-\mathbf{r}(4)$.

$\dfrac{\mathbf{r}(4.2)-\mathbf{r}(4)}{0.2} = 5[\mathbf{r}(4.2)-\mathbf{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\mathbf{r}(4.2)-\mathbf{r}(4)$.

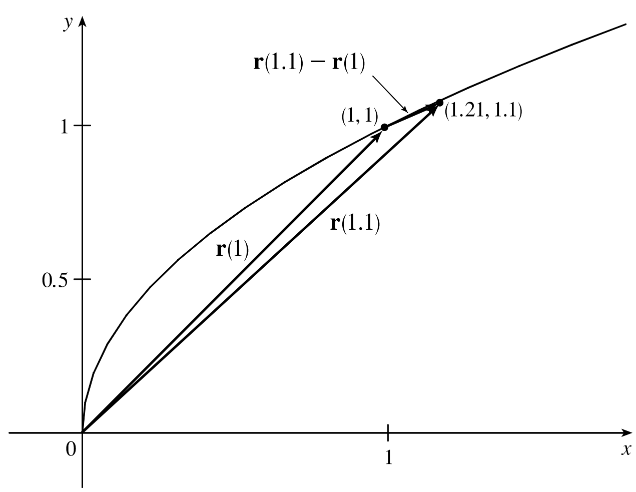


(c) By Definition 1, $\mathbf{r}'(4) = \lim\limits_{h\to 0} \dfrac{\mathbf{r}(4+h)-\mathbf{r}(4)}{h}$. $\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$.

(d) $\mathbf{T}(4)$ is a unit vector in the same direction as $\mathbf{r}'(4)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(4)$ with length 1.

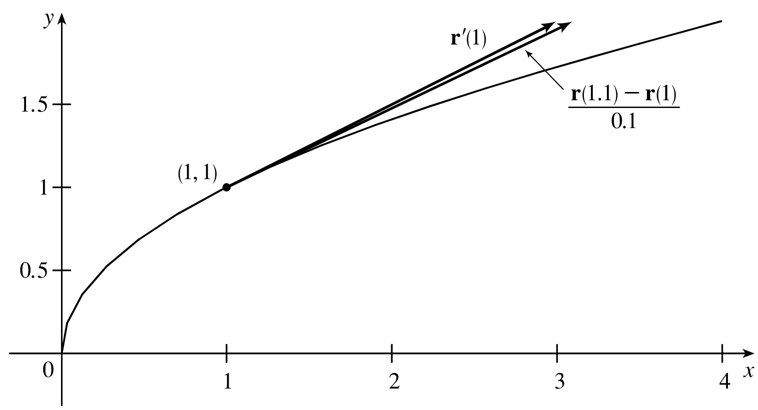


2. (a) The curve can be represented by the parametric equations $x = t^2$, $y = t$, $0 \le t \le 2$. Eliminating the parameter, we have $x = y^2$, $0 \le y \le 2$, a portion of which we graph here, along with the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1)-\mathbf{r}(1)$.



(b) Since $\mathbf{r}(t) = \langle t^2, t \rangle$, we differentiate components, giving $\mathbf{r}'(t) = \langle 2t, 1 \rangle$, so $\mathbf{r}'(1) = \langle 2, 1 \rangle$.

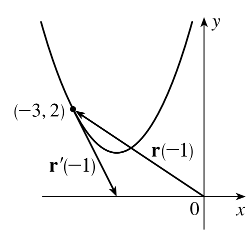
$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1} = \frac{\langle 1.21, 1.1 \rangle - \langle 1, 1 \rangle}{0.1} = 10\,\langle 0.21, 0.1 \rangle = \langle 2.1, 1 \rangle.$$



As we can see from the graph, these vectors are very close in length and direction. $\mathbf{r}'(1)$ is defined to be $\lim\limits_{h \to 0} \dfrac{\mathbf{r}(1+h) - \mathbf{r}(1)}{h}$, and we recognize $\dfrac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$ as the expression after the limit sign with $h = 0.1$. Since h is close to 0, we would expect $\dfrac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$ to be a vector close to $\mathbf{r}'(1)$.

3. $\mathbf{r}(t) = \langle t-2, t^2+1 \rangle$, $\mathbf{r}(-1) = \langle -3, 2 \rangle$. Since $(x+2)^2 = t^2 = y - 1 \;\Rightarrow\; y = (x+2)^2 + 1$, the curve is a parabola.

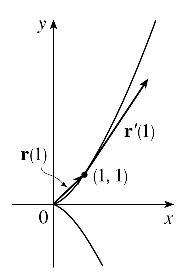
(a), (c)



(b) $\mathbf{r}'(t) = \langle 1, 2t \rangle$, $\mathbf{r}'(-1) = \langle 1, -2 \rangle$

4. $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, $\mathbf{r}(1) = \langle 1, 1 \rangle$. Since $x = t^2 = (t^3)^{2/3} = y^{2/3}$, the curve is the graph of $x = y^{2/3}$.

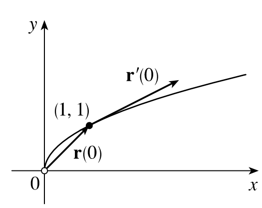
(a), (c)



(b) $\mathbf{r}'(t) = \langle 2t, 3t^2 \rangle$, $\mathbf{r}'(1) = \langle 2, 3 \rangle$

5. $\mathbf{r}(t) = e^{2t}\,\mathbf{i} + e^t\,\mathbf{j}$, $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$. Since $x = e^{2t} = (e^t)^2 = y^2$, the curve is part of a parabola. Note that here $x > 0$, $y > 0$.

(a), (c)

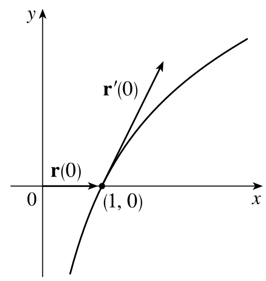


(b) $\mathbf{r}'(t) = 2e^{2t}\,\mathbf{i} + e^t\,\mathbf{j}$, $\mathbf{r}'(0) = 2\,\mathbf{i} + \mathbf{j}$

6. $\mathbf{r}(t) = e^t\,\mathbf{i} + 2t\,\mathbf{j}$, $\mathbf{r}(0) = \mathbf{i}$.

Since $x = e^t \iff t = \ln x$ and $y = 2t = 2\ln x$, the curve is the graph of $y = 2\ln x$.

(a), (c)

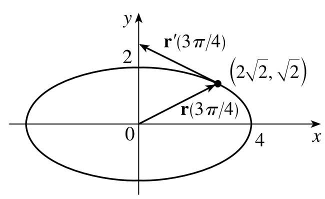


(b) $\mathbf{r}'(t) = e^t\,\mathbf{i} + 2\,\mathbf{j}$,
$\mathbf{r}'(0) = \mathbf{i} + 2\,\mathbf{j}$

7. $\mathbf{r}(t) = 4\sin t\,\mathbf{i} - 2\cos t\,\mathbf{j}$, $\mathbf{r}(3\pi/4) = 4(\sqrt{2}/2)\,\mathbf{i} - 2(-\sqrt{2}/2)\,\mathbf{j} = 2\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$.

Here $(x/4)^2 + (y/2)^2 = \sin^2 t + \cos^2 t = 1$, so the curve is the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$.

(a), (c)

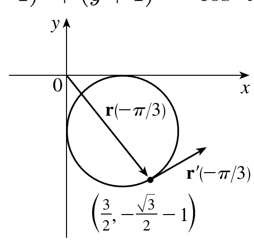


(b) $\mathbf{r}'(t) = 4\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$,
$\mathbf{r}'(3\pi/4) = -2\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$.

8. $\mathbf{r}(t) = (\cos t + 1)\,\mathbf{i} + (\sin t - 1)\,\mathbf{j}$, $\mathbf{r}(-\pi/3) = \left(\frac{1}{2} + 1\right)\mathbf{i} + \left(-\frac{\sqrt{3}}{2} - 1\right)\mathbf{j} = \frac{3}{2}\,\mathbf{i} + \left(-\frac{\sqrt{3}}{2} - 1\right)\mathbf{j} \approx 1.5\,\mathbf{i} - 1.87\,\mathbf{j}$.

Here $(x-1)^2 + (y+1)^2 = \cos^2 t + \sin^2 t = 1$, so the curve is a circle of radius 1 with center $(1, -1)$.

(a), (c)



(b) $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$,
$\mathbf{r}'(-\pi/3) = \frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} \approx 0.87\,\mathbf{i} + 0.5\,\mathbf{j}$

9. $\mathbf{r}(t) = \left\langle \sqrt{t-2}, 3, 1/t^2 \right\rangle \Rightarrow$

$$\mathbf{r}'(t) = \left\langle \frac{d}{dt}\left[\sqrt{t-2}\right], \frac{d}{dt}\,[3], \frac{d}{dt}\left[1/t^2\right] \right\rangle = \left\langle \tfrac{1}{2}(t-2)^{-1/2}, 0, -2t^{-3} \right\rangle = \left\langle \frac{1}{2\sqrt{t-2}}, 0, -\frac{2}{t^3} \right\rangle$$

10. $\mathbf{r}(t) = \left\langle e^{-t}, t - t^3, \ln t \right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle -e^{-t}, 1 - 3t^2, 1/t \right\rangle$

11. $\mathbf{r}(t) = t^2\,\mathbf{i} + \cos\left(t^2\right)\,\mathbf{j} + \sin^2 t\,\mathbf{k} \Rightarrow$

$$\mathbf{r}'(t) = 2t\,\mathbf{i} + \left[-\sin(t^2)\cdot 2t\right]\mathbf{j} + (2\sin t\cdot\cos t)\,\mathbf{k} = 2t\,\mathbf{i} - 2t\sin(t^2)\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k}$$

12. $\mathbf{r}(t) = \dfrac{1}{1+t}\,\mathbf{i} + \dfrac{t}{1+t}\,\mathbf{j} + \dfrac{t^2}{1+t}\,\mathbf{k} \Rightarrow$

$$\mathbf{r}'(t) = \frac{0 - 1(1)}{(1+t)^2}\,\mathbf{i} + \frac{(1+t)\cdot 1 - t(1)}{(1+t)^2}\,\mathbf{j} + \frac{(1+t)\cdot 2t - t^2(1)}{(1+t)^2}\,\mathbf{k} = -\frac{1}{(1+t)^2}\,\mathbf{i} + \frac{1}{(1+t)^2}\,\mathbf{j} + \frac{t^2 + 2t}{(1+t)^2}\,\mathbf{k}$$

13. $\mathbf{r}(t) = t\sin t\,\mathbf{i} + e^t\cos t\,\mathbf{j} + \sin t\cos t\,\mathbf{k} \Rightarrow$

$$\begin{aligned}\mathbf{r}'(t) &= [t\cdot\cos t + (\sin t)\cdot 1]\,\mathbf{i} + \left[e^t(-\sin t) + (\cos t)e^t\right]\mathbf{j} + [(\sin t)(-\sin t) + (\cos t)(\cos t)]\,\mathbf{k} \\ &= (t\cos t + \sin t)\,\mathbf{i} + e^t\,(\cos t - \sin t)\,\mathbf{j} + \left(\cos^2 t - \sin^2 t\right)\mathbf{k}\end{aligned}$$

14. $\mathbf{r}(t) = \sin^2 at\,\mathbf{i} + te^{bt}\,\mathbf{j} + \cos^2 ct\,\mathbf{k} \quad \Rightarrow$

$$\begin{aligned}\mathbf{r}'(t) &= [2(\sin at)\cdot(\cos at)(a)]\,\mathbf{i} + \left[t\cdot e^{bt}(b) + e^{bt}\cdot 1\right]\mathbf{j} + [2(\cos ct)\cdot(-\sin ct)(c)]\,\mathbf{k}\\ &= 2a\sin at\cos at\,\mathbf{i} + e^{bt}\,(bt+1)\,\mathbf{j} - 2c\sin ct\cos ct\,\mathbf{k}\end{aligned}$$

15. $\mathbf{r}'(t) = \mathbf{0} + \mathbf{b} + 2t\,\mathbf{c} = \mathbf{b} + 2t\,\mathbf{c}$ by Formulas 1 and 3 of Theorem 3.

16. To find $\mathbf{r}'(t)$, we first expand $\mathbf{r}(t) = t\,\mathbf{a}\times(\mathbf{b} + t\,\mathbf{c}) = t(\mathbf{a}\times\mathbf{b}) + t^2(\mathbf{a}\times\mathbf{c})$, so $\mathbf{r}'(t) = \mathbf{a}\times\mathbf{b} + 2t(\mathbf{a}\times\mathbf{c})$.

17. $\mathbf{r}(t) = \left\langle t^2 - 2t, 1 + 3t, \frac{1}{3}t^3 + \frac{1}{2}t^2\right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle 2t - 2, 3, t^2 + t\right\rangle \quad \Rightarrow \quad \mathbf{r}'(2) = \langle 2, 3, 6\rangle$.

So $|\mathbf{r}'(2)| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$ and $\mathbf{T}(2) = \dfrac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|} = \frac{1}{7}\langle 2, 3, 6\rangle = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right\rangle$.

18. $\mathbf{r}(t) = \left\langle \tan^{-1} t, 2e^{2t}, 8te^t\right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle 1/(1+t^2), 4e^{2t}, 8te^t + 8e^t\right\rangle \quad \Rightarrow \quad \mathbf{r}'(0) = \langle 1, 4, 8\rangle$.

So $|\mathbf{r}'(0)| = \sqrt{1^2 + 4^2 + 8^2} = \sqrt{81} = 9$ and $\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{9}\langle 1, 4, 8\rangle = \left\langle \frac{1}{9}, \frac{4}{9}, \frac{8}{9}\right\rangle$.

19. $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + 3\,\mathbf{j} + 4\cos 2t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(0) = 3\,\mathbf{j} + 4\,\mathbf{k}$. Thus

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{0^2 + 3^2 + 4^2}}\,(3\,\mathbf{j} + 4\,\mathbf{k}) = \tfrac{1}{5}(3\,\mathbf{j} + 4\,\mathbf{k}) = \tfrac{3}{5}\,\mathbf{j} + \tfrac{4}{5}\,\mathbf{k}.$$

20. $\mathbf{r}'(t) = 2\sin t\cos t\,\mathbf{i} - 2\cos t\sin t\,\mathbf{j} + 2\tan t\sec^2 t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{r}'\left(\frac{\pi}{4}\right) = 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\,\mathbf{i} - 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\,\mathbf{j} + 2\cdot 1\cdot(\sqrt{2})^2\,\mathbf{k} = \mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$ and $\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right| = \sqrt{1 + 1 + 16} = \sqrt{18} = 3\sqrt{2}$. Thus

$$\mathbf{T}\left(\tfrac{\pi}{4}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{4}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right|} = \frac{1}{3\sqrt{2}}\,(\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}) = \frac{1}{3\sqrt{2}}\,\mathbf{i} - \frac{1}{3\sqrt{2}}\,\mathbf{j} + \frac{4}{3\sqrt{2}}\,\mathbf{k}.$$

21. $\mathbf{r}(t) = \left\langle t, t^2, t^3\right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle 1, 2t, 3t^2\right\rangle$. Then $\mathbf{r}'(1) = \langle 1, 2, 3\rangle$ and $|\mathbf{r}'(1)| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, so

$\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{14}}\,\langle 1, 2, 3\rangle = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right\rangle$. $\quad \mathbf{r}''(t) = \langle 0, 2, 6t\rangle$, so

$$\begin{aligned}\mathbf{r}'(t)\times\mathbf{r}''(t) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2t & 3t^2\\ 0 & 2 & 6t\end{vmatrix} = \begin{vmatrix}2t & 3t^2\\ 2 & 6t\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & 3t^2\\ 0 & 6t\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & 2t\\ 0 & 2\end{vmatrix}\mathbf{k}\\ &= (12t^2 - 6t^2)\,\mathbf{i} - (6t - 0)\,\mathbf{j} + (2 - 0)\,\mathbf{k} = \left\langle 6t^2, -6t, 2\right\rangle\end{aligned}$$

22. $\mathbf{r}(t) = \left\langle e^{2t}, e^{-2t}, te^{2t}\right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t}\right\rangle \quad \Rightarrow \quad \mathbf{r}'(0) = \left\langle 2e^0, -2e^0, (0+1)e^0\right\rangle = \langle 2, -2, 1\rangle$

and $|\mathbf{r}'(0)| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$. Then $\mathbf{T}(0) = \dfrac{\mathbf{r}'\,(0)}{|\mathbf{r}'\,(0)|} = \frac{1}{3}\langle 2, -2, 1\rangle = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle$.

$\mathbf{r}''(t) = \left\langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t}\right\rangle \quad \Rightarrow \quad \mathbf{r}''(0) = \left\langle 4e^0, 4e^0, (0+4)e^0\right\rangle = \langle 4, 4, 4\rangle$.

$$\begin{aligned}\mathbf{r}'(t)\cdot\mathbf{r}''(t) &= \left\langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t}\right\rangle\cdot\left\langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t}\right\rangle\\ &= (2e^{2t})(4e^{2t}) + (-2e^{-2t})(4e^{-2t}) + ((2t+1)e^{2t})((4t+4)e^{2t})\\ &= 8e^{4t} - 8e^{-4t} + (8t^2 + 12t + 4)e^{4t} = (8t^2 + 12t + 12)e^{4t} - 8e^{-4t}\end{aligned}$$

23. The vector equation for the curve is $\mathbf{r}(t) = \left\langle t^2+1, 4\sqrt{t}, e^{t^2-t} \right\rangle$, so $\mathbf{r}'(t) = \left\langle 2t, 2/\sqrt{t}, (2t-1)e^{t^2-t} \right\rangle$. The point $(2,4,1)$ corresponds to $t=1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 2,2,1 \rangle$. Thus, the tangent line goes through the point $(2,4,1)$ and is parallel to the vector $\langle 2,2,1 \rangle$. Parametric equations are $x = 2+2t$, $y = 4+2t$, $z = 1+t$.

24. The vector equation for the curve is $\mathbf{r}(t) = \langle \ln(t+1), t\cos 2t, 2^t \rangle$, so $\mathbf{r}'(t) = \langle 1/(t+1), \cos 2t - 2t\sin 2t, 2^t \ln 2 \rangle$. The point $(0,0,1)$ corresponds to $t=0$, so the tangent vector there is $\mathbf{r}'(0) = \langle 1,1,\ln 2 \rangle$. Thus, the tangent line goes through the point $(0,0,1)$ and is parallel to the vector $\langle 1,1,\ln 2 \rangle$. Parametric equations are $x = 0+1\cdot t = t$, $y = 0+1\cdot t = t$, $z = 1+(\ln 2)t$.

25. The vector equation for the curve is $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t} \rangle$, so

$$\begin{aligned}\mathbf{r}'(t) &= \langle e^{-t}(-\sin t) + (\cos t)(-e^{-t}), e^{-t}\cos t + (\sin t)(-e^{-t}), (-e^{-t}) \rangle \\ &= \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), -e^{-t} \rangle\end{aligned}$$

The point $(1,0,1)$ corresponds to $t=0$, so the tangent vector there is $\mathbf{r}'(0) = \langle -e^0(\cos 0 + \sin 0), e^0(\cos 0 - \sin 0), -e^0 \rangle = \langle -1,1,-1 \rangle$. Thus, the tangent line is parallel to the vector $\langle -1,1,-1 \rangle$ and parametric equations are $x = 1+(-1)t = 1-t$, $y = 0+1\cdot t = t$, $z = 1+(-1)t = 1-t$.

26. The vector equation for the curve is $\mathbf{r}(t) = \langle \sqrt{t^2+3}, \ln(t^2+3), t \rangle$, so $\mathbf{r}'(t) = \langle t/\sqrt{t^2+3}, 2t/(t^2+3), 1 \rangle$. At $(2, \ln 4, 1)$, $t=1$ and $\mathbf{r}'(1) = \langle \frac{1}{2}, \frac{1}{2}, 1 \rangle$. Thus, parametric equations of the tangent line are $x = 2+\frac{1}{2}t$, $y = \ln 4 + \frac{1}{2}t$, $z = 1+t$.

27. First we parametrize the curve C of intersection. The projection of C onto the xy-plane is contained in the circle $x^2+y^2=25$, $z=0$, so we can write $x = 5\cos t$, $y = 5\sin t$. C also lies on the cylinder $y^2+z^2=20$, and $z \geq 0$ near the point $(3,4,2)$, so we can write $z = \sqrt{20-y^2} = \sqrt{20-25\sin^2 t}$. A vector equation then for C is $\mathbf{r}(t) = \left\langle 5\cos t, 5\sin t, \sqrt{20-25\sin^2 t} \right\rangle \;\Rightarrow\; \mathbf{r}'(t) = \left\langle -5\sin t, 5\cos t, \frac{1}{2}(20-25\sin^2 t)^{-1/2}(-50\sin t\cos t) \right\rangle$.

The point $(3,4,2)$ corresponds to $t = \cos^{-1}\left(\frac{3}{5}\right)$, so the tangent vector there is

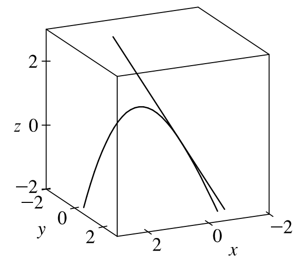
$\mathbf{r}'\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = \left\langle -5\left(\frac{4}{5}\right), 5\left(\frac{3}{5}\right), \frac{1}{2}\left(20-25\left(\frac{4}{5}\right)^2\right)^{-1/2}\left(-50\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\right) \right\rangle = \langle -4,3,-6 \rangle$.

The tangent line is parallel to this vector and passes through $(3,4,2)$, so a vector equation for the line is $\mathbf{r}(t) = (3-4t)\mathbf{i} + (4+3t)\mathbf{j} + (2-6t)\mathbf{k}$.

28. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, e^t \rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle -2\sin t, 2\cos t, e^t \rangle$. The tangent line to the curve is parallel to the plane when the curve's tangent vector is orthogonal to the plane's normal vector. Thus we require $\langle -2\sin t, 2\cos t, e^t \rangle \cdot \langle \sqrt{3}, 1, 0 \rangle = 0 \;\Rightarrow$ $-2\sqrt{3}\sin t + 2\cos t + 0 = 0 \;\Rightarrow\; \tan t = \frac{1}{\sqrt{3}} \;\Rightarrow\; t = \frac{\pi}{6}$ [since $0 \leq t \leq \pi$].

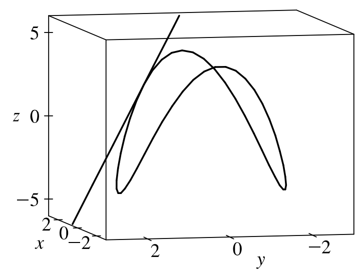
$\mathbf{r}\left(\frac{\pi}{6}\right) = \left\langle \sqrt{3}, 1, e^{\pi/6} \right\rangle$, so the point is $(\sqrt{3}, 1, e^{\pi/6})$.

29. $\mathbf{r}(t) = \langle t, e^{-t}, 2t - t^2 \rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1, -e^{-t}, 2 - 2t \rangle$. At $(0, 1, 0)$, $t = 0$ and $\mathbf{r}'(0) = \langle 1, -1, 2 \rangle$. Thus, parametric equations of the tangent line are $x = t$, $y = 1 - t$, $z = 2t$.

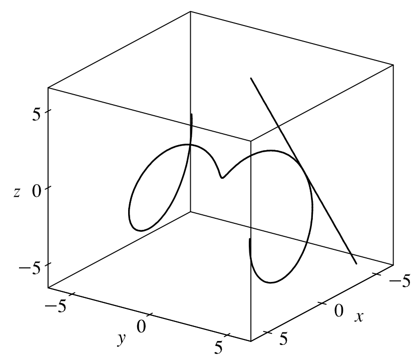


30. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\cos 2t \rangle$,

$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, -8\sin 2t \rangle$. At $(\sqrt{3}, 1, 2)$, $t = \frac{\pi}{6}$ and $\mathbf{r}'(\frac{\pi}{6}) = \langle -1, \sqrt{3}, -4\sqrt{3} \rangle$. Thus, parametric equations of the tangent line are $x = \sqrt{3} - t$, $y = 1 + \sqrt{3}\,t$, $z = 2 - 4\sqrt{3}\,t$.



31. $\mathbf{r}(t) = \langle t\cos t, t, t\sin t \rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle \cos t - t\sin t, 1, t\cos t + \sin t \rangle$. At $(-\pi, \pi, 0)$, $t = \pi$ and $\mathbf{r}'(\pi) = \langle -1, 1, -\pi \rangle$. Thus, parametric equations of the tangent line are $x = -\pi - t$, $y = \pi + t$, $z = -\pi t$.



32. (a) The tangent line at $t = 0$ is the line through the point with position vector $\mathbf{r}(0) = \langle \sin 0, 2\sin 0, \cos 0 \rangle = \langle 0, 0, 1 \rangle$, and in the direction of the tangent vector, $\mathbf{r}'(0) = \langle \pi\cos 0, 2\pi\cos 0, -\pi\sin 0 \rangle = \langle \pi, 2\pi, 0 \rangle$. So an equation of the line is $\langle x, y, z \rangle = \mathbf{r}(0) + u\,\mathbf{r}'(0) = \langle 0 + \pi u, 0 + 2\pi u, 1 \rangle = \langle \pi u, 2\pi u, 1 \rangle$.

$\mathbf{r}\left(\frac{1}{2}\right) = \left\langle \sin\frac{\pi}{2}, 2\sin\frac{\pi}{2}, \cos\frac{\pi}{2} \right\rangle = \langle 1, 2, 0 \rangle$,

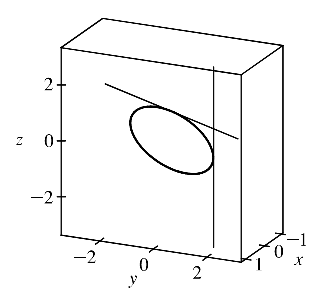
$\mathbf{r}'\left(\frac{1}{2}\right) = \left\langle \pi\cos\frac{\pi}{2}, 2\pi\cos\frac{\pi}{2}, -\pi\sin\frac{\pi}{2} \right\rangle = \langle 0, 0, -\pi \rangle$.

So the equation of the second line is

$\langle x, y, z \rangle = \langle 1, 2, 0 \rangle + v\,\langle 0, 0, -\pi \rangle = \langle 1, 2, -\pi v \rangle$.

The lines intersect where $\langle \pi u, 2\pi u, 1 \rangle = \langle 1, 2, -\pi v \rangle$, so the point of intersection is $(1, 2, 1)$.

(b)



33. The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection. Since $\mathbf{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle$ and $t = 0$ at $(0, 0, 0)$, $\mathbf{r}_1'(0) = \langle 1, 0, 0 \rangle$ is a tangent vector to $\mathbf{r}_1$ at $(0, 0, 0)$. Similarly,

$\mathbf{r}_2'(t) = \langle \cos t, 2\cos 2t, 1\rangle$ and since $\mathbf{r}_2(0) = \langle 0,0,0\rangle$, $\mathbf{r}_2'(0) = \langle 1,2,1\rangle$ is a tangent vector to $\mathbf{r}_2$ at $(0,0,0)$. If θ is the angle between these two tangent vectors, then $\cos\theta = \frac{1}{\sqrt{1}\sqrt{6}}\langle 1,0,0\rangle \cdot \langle 1,2,1\rangle = \frac{1}{\sqrt{6}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \approx 66^\circ$.

34. To find the point of intersection, we must find the values of t and s which satisfy the following three equations simultaneously: $t = 3 - s$, $1 - t = s - 2$, $3 + t^2 = s^2$. Solving the last two equations gives $t = 1$, $s = 2$ (check these in the first equation). Thus the point of intersection is $(1,0,4)$. To find the angle θ of intersection, we proceed as in Exercise 33. The tangent vectors to the respective curves at $(1,0,4)$ are $\mathbf{r}_1'(1) = \langle 1,-1,2\rangle$ and $\mathbf{r}_2'(2) = \langle -1,1,4\rangle$. So

$\cos\theta = \frac{1}{\sqrt{6}\sqrt{18}}(-1-1+8) = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$.

Note: In Exercise 33, the curves intersect when the value of both parameters is zero. However, as seen in this exercise, it is not necessary for the parameters to be of equal value at the point of intersection.

35.
$$\begin{aligned}\int_0^2 (t\,\mathbf{i} - t^3\,\mathbf{j} + 3t^5\,\mathbf{k})\,dt &= \left(\int_0^2 t\,dt\right)\mathbf{i} - \left(\int_0^2 t^3\,dt\right)\mathbf{j} + \left(\int_0^2 3t^5\,dt\right)\mathbf{k}\\ &= \left[\tfrac{1}{2}t^2\right]_0^2\,\mathbf{i} - \left[\tfrac{1}{4}t^4\right]_0^2\,\mathbf{j} + \left[\tfrac{1}{2}t^6\right]_0^2\,\mathbf{k}\\ &= \tfrac{1}{2}(4-0)\,\mathbf{i} - \tfrac{1}{4}(16-0)\,\mathbf{j} + \tfrac{1}{2}(64-0)\,\mathbf{k} = 2\,\mathbf{i} - 4\,\mathbf{j} + 32\,\mathbf{k}\end{aligned}$$

36.
$$\begin{aligned}\int_1^4 \left(2t^{3/2}\,\mathbf{i} + (t+1)\sqrt{t}\,\mathbf{k}\right)dt &= \left(\int_1^4 2t^{3/2}\,dt\right)\mathbf{i} + \left[\int_1^4 (t^{3/2} + t^{1/2})\,dt\right]\mathbf{k}\\ &= \left[\tfrac{4}{5}t^{5/2}\right]_1^4\,\mathbf{i} + \left[\tfrac{2}{5}t^{5/2} + \tfrac{2}{3}t^{3/2}\right]_1^4\,\mathbf{k}\\ &= \tfrac{4}{5}(4^{5/2}-1)\,\mathbf{i} + \left(\tfrac{2}{5}(4)^{5/2} + \tfrac{2}{3}(4)^{3/2} - \tfrac{2}{5} - \tfrac{2}{3}\right)\mathbf{k}\\ &= \tfrac{4}{5}(31)\,\mathbf{i} + \left(\tfrac{2}{5}(32) + \tfrac{2}{3}(8) - \tfrac{2}{5} - \tfrac{2}{3}\right)\mathbf{k} = \tfrac{124}{5}\,\mathbf{i} + \tfrac{256}{15}\,\mathbf{k}\end{aligned}$$

37.
$$\begin{aligned}\int_0^1 \left(\frac{1}{t+1}\,\mathbf{i} + \frac{1}{t^2+1}\,\mathbf{j} + \frac{t}{t^2+1}\,\mathbf{k}\right)dt &= \left(\int_0^1 \frac{1}{t+1}\,dt\right)\mathbf{i} + \left(\int_0^1 \frac{1}{t^2+1}\,dt\right)\mathbf{j} + \left(\int_0^1 \frac{t}{t^2+1}\,dt\right)\mathbf{k}\\ &= \left[\ln|t+1|\right]_0^1\,\mathbf{i} + \left[\tan^{-1} t\right]_0^1\,\mathbf{j} + \left[\tfrac{1}{2}\ln(t^2+1)\right]_0^1\,\mathbf{k}\\ &= (\ln 2 - \ln 1)\,\mathbf{i} + \left(\tfrac{\pi}{4} - 0\right)\mathbf{j} + \tfrac{1}{2}(\ln 2 - \ln 1)\,\mathbf{k} = \ln 2\,\mathbf{i} + \tfrac{\pi}{4}\,\mathbf{j} + \tfrac{1}{2}\ln 2\,\mathbf{k}\end{aligned}$$

38.
$$\begin{aligned}&\int_0^{\pi/4}(\sec t\tan t\,\mathbf{i} + t\cos 2t\,\mathbf{j} + \sin^2 2t\cos 2t\,\mathbf{k})\,dt\\ &= \left(\int_0^{\pi/4}\sec t\tan t\,dt\right)\mathbf{i} + \left(\int_0^{\pi/4} t\cos 2t\,dt\right)\mathbf{j} + \left(\int_0^{\pi/4}\sin^2 2t\cos 2t\,dt\right)\mathbf{k}\\ &= \left[\sec t\right]_0^{\pi/4}\,\mathbf{i} + \left(\left[\tfrac{1}{2}t\sin 2t\right]_0^{\pi/4} - \int_0^{\pi/4}\tfrac{1}{2}\sin 2t\,dt\right)\mathbf{j} + \left[\tfrac{1}{6}\sin^3 2t\right]_0^{\pi/4}\,\mathbf{k}\end{aligned}$$

[For the y-component, integrate by parts with $u = t$, $dv = \cos 2t\,dt$.]

$$\begin{aligned}&= \left(\sec\tfrac{\pi}{4} - \sec 0\right)\mathbf{i} + \left(\tfrac{\pi}{8}\sin\tfrac{\pi}{2} - 0 - \left[-\tfrac{1}{4}\cos 2t\right]_0^{\pi/4}\right)\mathbf{j} + \tfrac{1}{6}\left(\sin^3\tfrac{\pi}{2} - \sin^3 0\right)\mathbf{k}\\ &= (\sqrt{2}-1)\,\mathbf{i} + \left(\tfrac{\pi}{8} + \tfrac{1}{4}\cos\tfrac{\pi}{2} - \tfrac{1}{4}\cos 0\right)\mathbf{j} + \tfrac{1}{6}(1-0)\,\mathbf{k} = (\sqrt{2}-1)\,\mathbf{i} + \left(\tfrac{\pi}{8} - \tfrac{1}{4}\right)\mathbf{j} + \tfrac{1}{6}\,\mathbf{k}\end{aligned}$$

39.
$$\begin{aligned}\int(\sec^2 t\,\mathbf{i} + t(t^2+1)^3\,\mathbf{j} + t^2\ln t\,\mathbf{k})\,dt &= \left(\int\sec^2 t\,dt\right)\mathbf{i} + \left(\int t(t^2+1)^3\,dt\right)\mathbf{j} + \left(\int t^2\ln t\,dt\right)\mathbf{k}\\ &= \tan t\,\mathbf{i} + \tfrac{1}{8}(t^2+1)^4\,\mathbf{j} + \left(\tfrac{1}{3}t^3\ln t - \tfrac{1}{9}t^3\right)\mathbf{k} + \mathbf{C},\end{aligned}$$

where $\mathbf{C}$ is a vector constant of integration. [For the z-component, integrate by parts with $u = \ln t$, $dv = t^2\,dt$.]

40. $\displaystyle\int\left(te^{2t}\,\mathbf{i}+\frac{t}{1-t}\,\mathbf{j}+\frac{1}{\sqrt{1-t^2}}\,\mathbf{k}\right)dt=\left(\int te^{2t}\,dt\right)\mathbf{i}+\left(\int\frac{t}{1-t}\,dt\right)\mathbf{j}+\left(\int\frac{1}{\sqrt{1-t^2}}\,dt\right)\mathbf{k}$

$$=\left(\tfrac{1}{2}te^{2t}-\textstyle\int\tfrac{1}{2}e^{2t}\,dt\right)\mathbf{i}+\left[\int\left(-1+\frac{1}{1-t}\right)dt\right]\mathbf{j}+\left(\int\frac{1}{\sqrt{1-t^2}}\,dt\right)\mathbf{k}$$

$$=\left(\tfrac{1}{2}te^{2t}-\tfrac{1}{4}e^{2t}\right)\mathbf{i}+(-t-\ln|1-t|)\,\mathbf{j}+\sin^{-1}t\,\mathbf{k}+\mathbf{C}$$

41. $\mathbf{r}'(t)=2t\,\mathbf{i}+3t^2\,\mathbf{j}+\sqrt{t}\,\mathbf{k}\quad\Rightarrow\quad\mathbf{r}(t)=t^2\,\mathbf{i}+t^3\,\mathbf{j}+\frac{2}{3}t^{3/2}\,\mathbf{k}+\mathbf{C}$, where $\mathbf{C}$ is a constant vector.

But $\mathbf{i}+\mathbf{j}=\mathbf{r}(1)=\mathbf{i}+\mathbf{j}+\frac{2}{3}\mathbf{k}+\mathbf{C}$. Thus $\mathbf{C}=-\frac{2}{3}\mathbf{k}$ and $\mathbf{r}(t)=t^2\,\mathbf{i}+t^3\,\mathbf{j}+\left(\frac{2}{3}t^{3/2}-\frac{2}{3}\right)\mathbf{k}$.

42. $\mathbf{r}'(t)=t\,\mathbf{i}+e^t\,\mathbf{j}+te^t\,\mathbf{k}\quad\Rightarrow\quad\mathbf{r}(t)=\frac{1}{2}t^2\,\mathbf{i}+e^t\,\mathbf{j}+\left(te^t-e^t\right)\,\mathbf{k}+\mathbf{C}$. But $\mathbf{i}+\mathbf{j}+\mathbf{k}=\mathbf{r}(0)=\mathbf{j}-\mathbf{k}+\mathbf{C}$.

Thus $\mathbf{C}=\mathbf{i}+2\,\mathbf{k}$ and $\mathbf{r}(t)=\left(\frac{1}{2}t^2+1\right)\mathbf{i}+e^t\,\mathbf{j}+(te^t-e^t+2)\,\mathbf{k}$.

For Exercises 43–46, let $\mathbf{u}(t)=\langle u_1(t),u_2(t),u_3(t)\rangle$ and $\mathbf{v}(t)=\langle v_1(t),v_2(t),v_3(t)\rangle$. In each of these exercises, the procedure is to apply Theorem 2 so that the corresponding properties of derivatives of real-valued functions can be used.

43. $\displaystyle\frac{d}{dt}\left[\mathbf{u}(t)+\mathbf{v}(t)\right]=\frac{d}{dt}\left\langle u_1(t)+v_1(t),u_2(t)+v_2(t),u_3(t)+v_3(t)\right\rangle$

$$=\left\langle\frac{d}{dt}\left[u_1(t)+v_1(t)\right],\frac{d}{dt}\left[u_2(t)+v_2(t)\right],\frac{d}{dt}\left[u_3(t)+v_3(t)\right]\right\rangle$$

$$=\left\langle u_1'(t)+v_1'(t),u_2'(t)+v_2'(t),u_3'(t)+v_3'(t)\right\rangle$$

$$=\left\langle u_1'(t),u_2'(t),u_3'(t)\right\rangle+\left\langle v_1'(t),v_2'(t),v_3'(t)\right\rangle=\mathbf{u}'(t)+\mathbf{v}'(t)$$

44. $\displaystyle\frac{d}{dt}\left[f(t)\,\mathbf{u}(t)\right]=\frac{d}{dt}\left\langle f(t)u_1(t),f(t)u_2(t),f(t)u_3(t)\right\rangle$

$$=\left\langle\frac{d}{dt}\left[f(t)u_1(t)\right],\frac{d}{dt}\left[f(t)u_2(t)\right],\frac{d}{dt}\left[f(t)u_3(t)\right]\right\rangle$$

$$=\left\langle f'(t)u_1(t)+f(t)u_1'(t),f'(t)u_2(t)+f(t)u_2'(t),f'(t)u_3(t)+f(t)u_3'(t)\right\rangle$$

$$=f'(t)\left\langle u_1(t),u_2(t),u_3(t)\right\rangle+f(t)\left\langle u_1'(t),u_2'(t),u_3'(t)\right\rangle=f'(t)\,\mathbf{u}(t)+f(t)\,\mathbf{u}'(t)$$

45. $\displaystyle\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]=\frac{d}{dt}\left\langle u_2(t)v_3(t)-u_3(t)v_2(t),u_3(t)v_1(t)-u_1(t)v_3(t),u_1(t)v_2(t)-u_2(t)v_1(t)\right\rangle$

$$\begin{aligned}&=\langle u_2'v_3(t)+u_2(t)v_3'(t)-u_3'(t)v_2(t)-u_3(t)v_2'(t),\\&\qquad u_3'(t)v_1(t)+u_3(t)v_1'(t)-u_1'(t)v_3(t)-u_1(t)v_3'(t),\\&\qquad\quad u_1'(t)v_2(t)+u_1(t)v_2'(t)-u_2'(t)v_1(t)-u_2(t)v_1'(t)\rangle\\&=\langle u_2'(t)v_3(t)-u_3'(t)v_2(t),u_3'(t)v_1(t)-u_1'(t)v_3(t),u_1'(t)v_2(t)-u_2'(t)v_1(t)\rangle\\&\qquad+\langle u_2(t)v_3'(t)-u_3(t)v_2'(t),u_3(t)v_1'(t)-u_1(t)v_3'(t),u_1(t)v_2'(t)-u_2(t)v_1'(t)\rangle\\&=\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)\end{aligned}$$

Alternate solution: Let $\mathbf{r}(t)=\mathbf{u}(t)\times\mathbf{v}(t)$. Then

$$\begin{aligned}\mathbf{r}(t+h)-\mathbf{r}(t)&=\left[\mathbf{u}(t+h)\times\mathbf{v}(t+h)\right]-\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]\\&=\left[\mathbf{u}(t+h)\times\mathbf{v}(t+h)\right]-\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]+\left[\mathbf{u}(t+h)\times\mathbf{v}(t)\right]-\left[\mathbf{u}(t+h)\times\mathbf{v}(t)\right]\\&=\mathbf{u}(t+h)\times\left[\mathbf{v}(t+h)-\mathbf{v}(t)\right]+\left[\mathbf{u}(t+h)-\mathbf{u}(t)\right]\times\mathbf{v}(t)\end{aligned}$$

(Be careful of the order of the cross product.) Dividing through by h and taking the limit as $h \to 0$ we have

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{u}(t+h) \times [\mathbf{v}(t+h) - \mathbf{v}(t)]}{h} + \lim_{h\to 0} \frac{[\mathbf{u}(t+h) - \mathbf{u}(t)] \times \mathbf{v}(t)}{h} = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$$

by Exercise 13.1.53(a) and Definition 1.

46. $\dfrac{d}{dt}\,[\mathbf{u}(f(t))] = \dfrac{d}{dt}\,\langle u_1(f(t)), u_2(f(t)), u_3(f(t))\rangle = \left\langle \dfrac{d}{dt}\,[u_1(f(t))]\,, \dfrac{d}{dt}\,[u_2(f(t))]\,, \dfrac{d}{dt}\,[u_3(f(t))]\right\rangle$

$$= \langle f'(t)u_1'(f(t)), f'(t)u_2'(f(t)), f'(t)u_3'(f(t))\rangle = f'(t)\,\mathbf{u}'(t)$$

47. $\dfrac{d}{dt}\,[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$ [by Formula 4 of Theorem 3]

$$\begin{aligned} &= \langle \cos t, -\sin t, 1\rangle \cdot \langle t, \cos t, \sin t\rangle + \langle \sin t, \cos t, t\rangle \cdot \langle 1, -\sin t, \cos t\rangle \\ &= t\cos t - \cos t\,\sin t + \sin t + \sin t - \cos t\,\sin t + t\cos t \\ &= 2t\cos t + 2\sin t - 2\cos t\,\sin t \end{aligned}$$

48. $\dfrac{d}{dt}\,[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$ [by Formula 5 of Theorem 3]

$$\begin{aligned} &= \langle \cos t, -\sin t, 1\rangle \times \langle t, \cos t, \sin t\rangle + \langle \sin t, \cos t, t\rangle \times \langle 1, -\sin t, \cos t\rangle \\ &= \left\langle -\sin^2 t - \cos t, t - \cos t\,\sin t, \cos^2 t + t\sin t\right\rangle \\ &\qquad + \left\langle \cos^2 t + t\sin t, t - \cos t\,\sin t, -\sin^2 t - \cos t\right\rangle \\ &= \left\langle \cos^2 t - \sin^2 t - \cos t + t\sin t, 2t - 2\cos t\,\sin t, \cos^2 t - \sin^2 t - \cos t + t\sin t\right\rangle \end{aligned}$$

49. By Formula 4 of Theorem 3, $f'(t) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$, and $\mathbf{v}'(t) = \left\langle 1, 2t, 3t^2\right\rangle$, so

$f'(2) = \mathbf{u}'(2)\cdot\mathbf{v}(2) + \mathbf{u}(2)\cdot\mathbf{v}'(2) = \langle 3, 0, 4\rangle\cdot\langle 2, 4, 8\rangle + \langle 1, 2, -1\rangle\cdot\langle 1, 4, 12\rangle = 6 + 0 + 32 + 1 + 8 - 12 = 35.$

50. By Formula 5 of Theorem 3, $\mathbf{r}'(t) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$, so

$$\begin{aligned} \mathbf{r}'(2) &= \mathbf{u}'(2)\times\mathbf{v}(2) + \mathbf{u}(2)\times\mathbf{v}'(2) = \langle 3, 0, 4\rangle\times\langle 2, 4, 8\rangle + \langle 1, 2, -1\rangle\times\langle 1, 4, 12\rangle \\ &= \langle -16, -16, 12\rangle + \langle 28, -13, 2\rangle = \langle 12, -29, 14\rangle \end{aligned}$$

51. $\mathbf{r}(t) = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t \quad\Rightarrow\quad \mathbf{r}'(t) = -\mathbf{a}\,\omega\sin\omega t + \mathbf{b}\,\omega\cos\omega t$ by Formulas 1 and 3 of Theorem 3. Then

$$\begin{aligned} \mathbf{r}(t)\times\mathbf{r}'(t) &= (\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t)\times(-\mathbf{a}\,\omega\sin\omega t + \mathbf{b}\,\omega\cos\omega t) \\ &= (\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t)\times(-\mathbf{a}\,\omega\sin\omega t) + (\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t)\times(\mathbf{b}\,\omega\cos\omega t) \\ &\qquad\qquad \text{[by Property 3 of Theorem 12.4.11]} \\ &= \mathbf{a}\cos\omega t\times(-\mathbf{a}\,\omega\sin\omega t) + \mathbf{b}\sin\omega t\times(-\mathbf{a}\,\omega\sin\omega t) + \mathbf{a}\cos\omega t\times\mathbf{b}\,\omega\cos\omega t + \mathbf{b}\sin\omega t\times\mathbf{b}\,\omega\cos\omega t \\ &\qquad\qquad \text{[by Property 4]} \\ &= (\cos\omega t)\,(-\omega\sin\omega t)\,(\mathbf{a}\times\mathbf{a}) + (\sin\omega t)\,(-\omega\sin\omega t)\,(\mathbf{b}\times\mathbf{a}) + (\cos\omega t)\,(\omega\cos\omega t)\,(\mathbf{a}\times\mathbf{b}) \\ &\qquad\qquad + (\sin\omega t)\,(\omega\cos\omega t)\,(\mathbf{b}\times\mathbf{b}) \qquad \text{[by Property 2]} \\ &= \mathbf{0} + \left(\omega\sin^2\omega t\right)(\mathbf{a}\times\mathbf{b}) + \left(\omega\cos^2\omega t\right)(\mathbf{a}\times\mathbf{b}) + \mathbf{0} \qquad \text{[by Property 1 and Example 12.4.2]} \\ &= \omega\left(\sin^2\omega t + \cos^2\omega t\right)(\mathbf{a}\times\mathbf{b}) = \omega\,(\mathbf{a}\times\mathbf{b}) = \omega\,\mathbf{a}\times\mathbf{b} \qquad \text{[by Property 2]} \end{aligned}$$

52. From Exercise 51, $\mathbf{r}'(t) = -\mathbf{a}\,\omega\sin\omega t + \mathbf{b}\,\omega\cos\omega t \quad\Rightarrow\quad \mathbf{r}''(t) = -\mathbf{a}\,\omega^2\cos\omega t - \mathbf{b}\,\omega^2\sin\omega t$. Then

$$\begin{aligned}\mathbf{r}''(t) + \omega^2\mathbf{r}(t) &= \left(-\mathbf{a}\,\omega^2\cos\omega t - \mathbf{b}\,\omega^2\sin\omega t\right) + \omega^2\left(\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t\right)\\ &= -\mathbf{a}\,\omega^2\cos\omega t - \mathbf{b}\,\omega^2\sin\omega t + \mathbf{a}\,\omega^2\cos\omega t + \mathbf{b}\,\omega^2\sin\omega t = \mathbf{0}\end{aligned}$$

53. $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right] = \mathbf{r}'(t)\times\mathbf{r}'(t) + \mathbf{r}(t)\times\mathbf{r}''(t)$ by Formula 5 of Theorem 3. But $\mathbf{r}'(t)\times\mathbf{r}'(t) = \mathbf{0}$ (by Example 12.4.2).

Thus, $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right] = \mathbf{r}(t)\times\mathbf{r}''(t)$.

54.
$$\begin{aligned}\frac{d}{dt}\left(\mathbf{u}(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right]\right) &= \mathbf{u}'(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right] + \mathbf{u}(t)\cdot\frac{d}{dt}\left[\mathbf{v}(t)\times\mathbf{w}(t)\right]\\ &= \mathbf{u}'(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right] + \mathbf{u}(t)\cdot\left[\mathbf{v}'(t)\times\mathbf{w}(t) + \mathbf{v}(t)\times\mathbf{w}'(t)\right]\\ &= \mathbf{u}'(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right] + \mathbf{u}(t)\cdot\left[\mathbf{v}'(t)\times\mathbf{w}(t)\right] + \mathbf{u}(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}'(t)\right]\\ &= \mathbf{u}'(t)\cdot\left[\mathbf{v}(t)\times\mathbf{w}(t)\right] - \mathbf{v}'(t)\cdot\left[\mathbf{u}(t)\times\mathbf{w}(t)\right] + \mathbf{w}'(t)\cdot\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]\end{aligned}$$

55. $\dfrac{d}{dt}\left|\mathbf{r}(t)\right| = \dfrac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]^{1/2} = \tfrac{1}{2}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]^{-1/2}\left[2\mathbf{r}(t)\cdot\mathbf{r}'(t)\right] = \dfrac{1}{\left|\mathbf{r}(t)\right|}\,\mathbf{r}(t)\cdot\mathbf{r}'(t)$

56. Since $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, we have $0 = 2\mathbf{r}(t)\cdot\mathbf{r}'(t) = \dfrac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right] = \dfrac{d}{dt}\left|\mathbf{r}(t)\right|^2$. Thus $\left|\mathbf{r}(t)\right|^2$, and so $\left|\mathbf{r}(t)\right|$, is a constant, and hence the curve lies on a sphere with center the origin.

57. Since $\mathbf{u}(t) = \mathbf{r}(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right]$,

$$\begin{aligned}\mathbf{u}'(t) &= \mathbf{r}'(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right] + \mathbf{r}(t)\cdot\frac{d}{dt}\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right]\\ &= 0 + \mathbf{r}(t)\cdot\left[\mathbf{r}''(t)\times\mathbf{r}''(t) + \mathbf{r}'(t)\times\mathbf{r}'''(t)\right] && \left[\text{since } \mathbf{r}'(t)\perp\mathbf{r}'(t)\times\mathbf{r}''(t)\right]\\ &= \mathbf{r}(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}'''(t)\right] && \left[\text{since } \mathbf{r}''(t)\times\mathbf{r}''(t) = \mathbf{0}\right]\end{aligned}$$

58. The tangent vector $\mathbf{r}'(t)$ is defined as $\lim\limits_{h\to 0}\dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$. Here we assume that this limit exists and $\mathbf{r}'(t)\neq\mathbf{0}$; then we know that this vector lies on the tangent line to the curve. As in Figure 1, let points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$. The vector $\mathbf{r}(t+h)-\mathbf{r}(t)$ points from P to Q, so $\mathbf{r}(t+h)-\mathbf{r}(t) = \overrightarrow{PQ}$. If $h>0$ then $t<t+h$, so Q lies "ahead" of P on the curve. If h is sufficiently small (we can take h to be as small as we like since $h\to 0$) then $\overrightarrow{PQ}$ approximates the curve from P to Q and hence points approximately in the direction of the curve as t increases. Since h is positive, $\dfrac{1}{h}\,\overrightarrow{PQ} = \dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ points in the same direction. If $h<0$, then $t>t+h$ so Q lies "behind" P on the curve. For h sufficiently small, $\overrightarrow{PQ}$ approximates the curve but points in the direction of decreasing t. However, h is negative, so $\dfrac{1}{h}\,\overrightarrow{PQ} = \dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ points in the opposite direction, that is, in the direction of increasing t. In both cases, the difference quotient $\dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ points in the direction of increasing t. The tangent vector $\mathbf{r}'(t)$ is the limit of this difference quotient, so it must also point in the direction of increasing t.