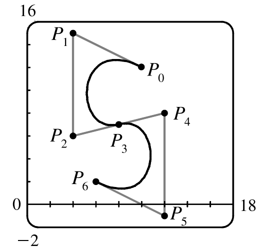
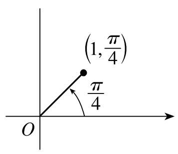
5. We use the same P_0 and P_1 as in Problem 4, and use part of our C as the top of an S. To prevent the center line from slanting up too much, we move P_2 up to $(4, 6)$ and P_3 down and to the left, to $(8, 7)$. In order to have a smooth joint between the top and bottom halves of the S (and a symmetric S), we determine points P_4, P_5, and P_6 by rotating points P_2, P_1, and P_0 about the center of the letter (point P_3). The points are therefore $P_4(12, 8)$, $P_5(12, -1)$, and $P_6(6, 2)$.



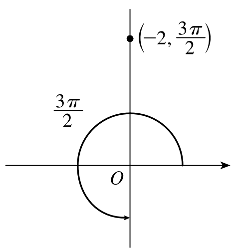
10.3 Polar Coordinates

1. (a) $\left(1, \frac{\pi}{4}\right)$



By adding 2π to $\frac{\pi}{4}$, we obtain the point $\left(1, \frac{9\pi}{4}\right)$, which satisfies the $r > 0$ requirement. The direction opposite $\frac{\pi}{4}$ is $\frac{5\pi}{4}$, so $\left(-1, \frac{5\pi}{4}\right)$ is a point that satisfies the $r < 0$ requirement.

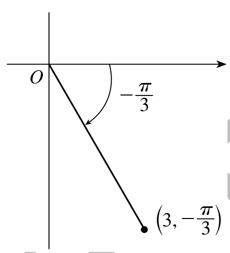
(b) $\left(-2, \frac{3\pi}{2}\right)$



$\boldsymbol{r > 0}$: $\left(-(-2), \frac{3\pi}{2} - \pi\right) = \left(2, \frac{\pi}{2}\right)$

$\boldsymbol{r < 0}$: $\left(-2, \frac{3\pi}{2} + 2\pi\right) = \left(-2, \frac{7\pi}{2}\right)$

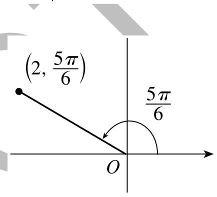
(c) $\left(3, -\frac{\pi}{3}\right)$



$\boldsymbol{r > 0}$: $\left(3, -\frac{\pi}{3} + 2\pi\right) = \left(3, \frac{5\pi}{3}\right)$

$\boldsymbol{r < 0}$: $\left(-3, -\frac{\pi}{3} + \pi\right) = \left(-3, \frac{2\pi}{3}\right)$

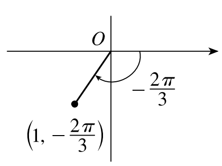
2. (a) $\left(2, \frac{5\pi}{6}\right)$



$\boldsymbol{r > 0}$: $\left(2, \frac{5\pi}{6} + 2\pi\right) = \left(2, \frac{17\pi}{6}\right)$

$\boldsymbol{r < 0}$: $\left(-2, \frac{5\pi}{6} - \pi\right) = \left(-2, -\frac{\pi}{6}\right)$

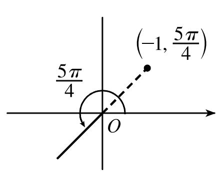
(b) $\left(1, -\frac{2\pi}{3}\right)$



$\boldsymbol{r > 0}$: $\left(1, -\frac{2\pi}{3} + 2\pi\right) = \left(1, \frac{4\pi}{3}\right)$

$\boldsymbol{r < 0}$: $\left(-1, -\frac{2\pi}{3} + \pi\right) = \left(-1, \frac{\pi}{3}\right)$

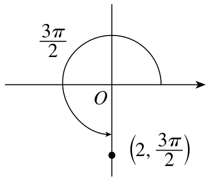
(c) $\left(-1, \frac{5\pi}{4}\right)$



$\boldsymbol{r > 0}$: $\left(-(-1), \frac{5\pi}{4} - \pi\right) = \left(1, \frac{\pi}{4}\right)$

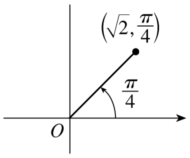
$\boldsymbol{r < 0}$: $\left(-1, \frac{5\pi}{4} - 2\pi\right) = \left(-1, -\frac{3\pi}{4}\right)$

3. (a)



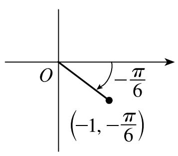
$x = 2\cos\frac{3\pi}{2} = 2(0) = 0$ and $y = 2\sin\frac{3\pi}{2} = 2(-1) = -2$ give us the Cartesian coordinates $(0, -2)$.

(b)



$x = \sqrt{2}\cos\dfrac{\pi}{4} = \sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right) = 1$ and $y = \sqrt{2}\sin\dfrac{\pi}{4} = \sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right) = 1$ give us the Cartesian coordinates $(1, 1)$.

(c)

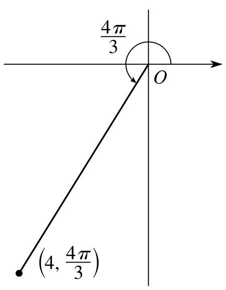


$x = -1\cos\left(-\dfrac{\pi}{6}\right) = -1\left(\dfrac{\sqrt{3}}{2}\right) = -\dfrac{\sqrt{3}}{2}$ and

$y = -1\sin\left(-\dfrac{\pi}{6}\right) = -1\left(-\dfrac{1}{2}\right) = \dfrac{1}{2}$ give us the Cartesian

coordinates $\left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$.

4. (a)

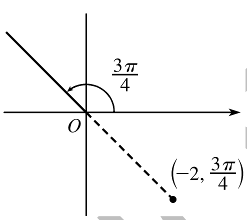


$x = 4\cos\dfrac{4\pi}{3} = 4\left(-\dfrac{1}{2}\right) = -2$ and

$y = 4\sin\dfrac{4\pi}{3} = 4\left(-\dfrac{\sqrt{3}}{2}\right) = -2\sqrt{3}$ give us the Cartesian

coordinates $\left(-2, -2\sqrt{3}\right)$.

(b)

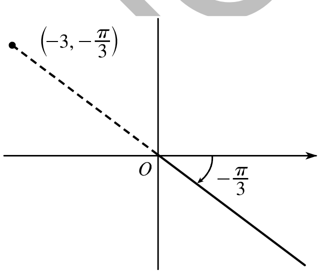


$x = -2\cos\dfrac{3\pi}{4} = -2\left(-\dfrac{\sqrt{2}}{2}\right) = \sqrt{2}$ and

$y = -2\sin\dfrac{3\pi}{4} = -2\left(\dfrac{\sqrt{2}}{2}\right) = -\sqrt{2}$ give us the Cartesian

coordinates $\left(\sqrt{2}, -\sqrt{2}\right)$.

(c)



$x = -3\cos\left(-\dfrac{\pi}{3}\right) = -3\left(\dfrac{1}{2}\right) = -\dfrac{3}{2}$ and

$y = -3\sin\left(-\dfrac{\pi}{3}\right) = -3\left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{3\sqrt{3}}{2}$ give us the Cartesian

coordinates $\left(-\dfrac{3}{2}, \dfrac{3\sqrt{3}}{2}\right)$.

5. (a) $x = -4$ and $y = 4 \quad \Rightarrow \quad r = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$ and $\tan\theta = \frac{4}{-4} = -1 \quad [\theta = -\frac{\pi}{4} + n\pi]$. Since $(-4, 4)$ is in the second quadrant, the polar coordinates are (i) $\left(4\sqrt{2}, \frac{3\pi}{4}\right)$ and (ii) $\left(-4\sqrt{2}, \frac{7\pi}{4}\right)$.

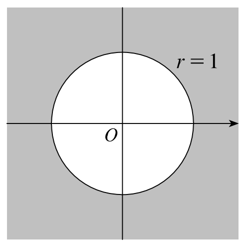
(b) $x = 3$ and $y = 3\sqrt{3} \quad \Rightarrow \quad r = \sqrt{3^2 + \left(3\sqrt{3}\right)^2} = \sqrt{9 + 27} = 6$ and $\tan\theta = \dfrac{3\sqrt{3}}{3} = \sqrt{3} \quad [\theta = \frac{\pi}{3} + n\pi]$.

Since $\left(3, 3\sqrt{3}\right)$ is in the first quadrant, the polar coordinates are (i) $\left(6, \frac{\pi}{3}\right)$ and (ii) $\left(-6, \frac{4\pi}{3}\right)$.

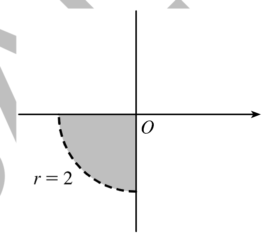
6. (a) $x = \sqrt{3}$ and $y = -1 \quad \Rightarrow \quad r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$ and $\tan\theta = \dfrac{-1}{\sqrt{3}} \quad [\theta = -\frac{\pi}{6} + n\pi]$. Since $\left(\sqrt{3}, -1\right)$ is in the fourth quadrant, the polar coordinates are (i) $\left(2, \frac{11\pi}{6}\right)$ and (ii) $\left(-2, \frac{5\pi}{6}\right)$.

(b) $x = -6$ and $y = 0 \quad \Rightarrow \quad r = \sqrt{(-6)^2 + 0^2} = 6$ and $\tan\theta = \frac{0}{-6} = 0 \quad [\theta = n\pi]$. Since $(-6, 0)$ is on the negative x-axis, the polar coordinates are (i) $(6, \pi)$ and (ii) $(-6, 0)$.

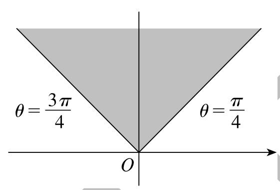
7. $r \geq 1$. The curve $r = 1$ represents a circle with center O and radius 1. So $r \geq 1$ represents the region on or outside the circle. Note that θ can take on any value.



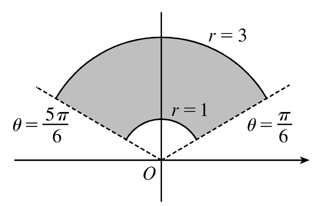
8. $0 \leq r < 2$, $\pi \leq \theta \leq 3\pi/2$. This is the region inside the circle $r = 2$ in the third quadrant.



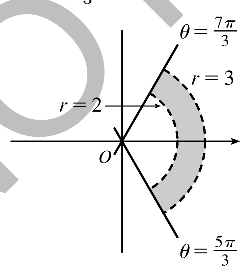
9. $r \geq 0$, $\pi/4 \leq \theta \leq 3\pi/4$.
$\theta = k$ represents a line through O.



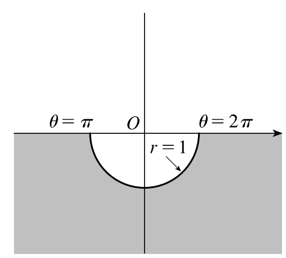
10. $1 \leq r \leq 3$, $\pi/6 < \theta < 5\pi/6$



11. $2 < r < 3$, $\frac{5\pi}{3} \leq \theta \leq \frac{7\pi}{3}$



12. $r \geq 1$, $\pi \leq \theta \leq 2\pi$



13. Converting the polar coordinates $\left(4, \frac{4\pi}{3}\right)$ and $\left(6, \frac{5\pi}{3}\right)$ to Cartesian coordinates gives us $\left(4\cos\frac{4\pi}{3}, 4\sin\frac{4\pi}{3}\right) = \left(-2, -2\sqrt{3}\right)$ and $\left(6\cos\frac{5\pi}{3}, 6\sin\frac{5\pi}{3}\right) = \left(3, -3\sqrt{3}\right)$. Now use the distance formula

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[3 - (-2)]^2 + \left[-3\sqrt{3} - \left(-2\sqrt{3}\right)\right]^2} \\ &= \sqrt{5^2 + \left(-\sqrt{3}\right)^2} = \sqrt{25 + 3} = \sqrt{28} = 2\sqrt{7} \end{aligned}$$

14. The points (r_1, θ_1) and (r_2, θ_2) in Cartesian coordinates are $(r_1 \cos\theta_1, r_1 \sin\theta_1)$ and $(r_2 \cos\theta_2, r_2 \sin\theta_2)$, respectively. The *square* of the distance between them is

$$\begin{aligned}(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2 \\ &= \left(r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\cos^2\theta_1\right) + \left(r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1\right) \\ &= r_1^2\left(\sin^2\theta_1 + \cos^2\theta_1\right) + r_2^2\left(\sin^2\theta_2 + \cos^2\theta_2\right) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ &= r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2,\end{aligned}$$

so the distance between them is $\sqrt{r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2}$.

15. $r^2 = 5 \quad\Leftrightarrow\quad x^2 + y^2 = 5$, a circle of radius $\sqrt{5}$ centered at the origin.

16. $r = 4\sec\theta \quad\Leftrightarrow\quad \dfrac{r}{\sec\theta} = 4 \quad\Leftrightarrow\quad r\cos\theta = 4 \quad\Leftrightarrow\quad x = 4$, a vertical line.

17. $r = 5\cos\theta \quad\Rightarrow\quad r^2 = 5r\cos\theta \quad\Leftrightarrow\quad x^2 + y^2 = 5x \quad\Leftrightarrow\quad x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4} \quad\Leftrightarrow\quad \left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$, a circle of radius $\frac{5}{2}$ centered at $\left(\frac{5}{2}, 0\right)$. The first two equations are actually equivalent since $r^2 = 5r\cos\theta \quad\Rightarrow\quad r(r - 5\cos\theta) = 0 \quad\Rightarrow\quad r = 0$ or $r = 5\cos\theta$. But $r = 5\cos\theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the equation $r = 5\cos\theta$ is equivalent to the compound condition ($r = 0$ or $r = 5\cos\theta$).

18. $\theta = \dfrac{\pi}{3} \quad\Rightarrow\quad \tan\theta = \tan\dfrac{\pi}{3} \quad\Rightarrow\quad \dfrac{y}{x} = \sqrt{3} \quad\Leftrightarrow\quad y = \sqrt{3}\,x$, a line through the origin.

19. $r^2\cos 2\theta = 1 \quad\Leftrightarrow\quad r^2(\cos^2\theta - \sin^2\theta) = 1 \quad\Leftrightarrow\quad (r\cos\theta)^2 - (r\sin\theta)^2 = 1 \quad\Leftrightarrow\quad x^2 - y^2 = 1$, a hyperbola centered at the origin with foci on the x-axis.

20. $r^2\sin 2\theta = 1 \quad\Leftrightarrow\quad r^2(2\sin\theta\,\cos\theta) = 1 \quad\Leftrightarrow\quad 2(r\cos\theta)(r\sin\theta) = 1 \quad\Leftrightarrow\quad 2xy = 1 \quad\Leftrightarrow\quad xy = \frac{1}{2}$, a hyperbola centered at the origin with foci on the line $y = x$.

21. $y = 2 \quad\Leftrightarrow\quad r\sin\theta = 2 \quad\Leftrightarrow\quad r = \dfrac{2}{\sin\theta} \quad\Leftrightarrow\quad r = 2\csc\theta$

22. $y = x \quad\Rightarrow\quad \dfrac{y}{x} = 1 \;\; [x \neq 0] \quad\Rightarrow\quad \tan\theta = 1 \quad\Rightarrow\quad \theta = \tan^{-1} 1 \quad\Rightarrow\quad \theta = \dfrac{\pi}{4}$ or $\theta = \dfrac{5\pi}{4}$ [either includes the pole]

23. $y = 1 + 3x \quad\Leftrightarrow\quad r\sin\theta = 1 + 3r\cos\theta \quad\Leftrightarrow\quad r\sin\theta - 3r\cos\theta = 1 \quad\Leftrightarrow\quad r(\sin\theta - 3\cos\theta) = 1 \quad\Leftrightarrow\quad r = \dfrac{1}{\sin\theta - 3\cos\theta}$

24. $4y^2 = x \quad\Leftrightarrow\quad 4(r\sin\theta)^2 = r\cos\theta \quad\Leftrightarrow\quad 4r^2\sin^2\theta - r\cos\theta = 0 \quad\Leftrightarrow\quad r(4r\sin^2\theta - \cos\theta) = 0 \quad\Leftrightarrow\quad r = 0$ or $r = \dfrac{\cos\theta}{4\sin^2\theta} \quad\Leftrightarrow\quad r = 0$ or $r = \frac{1}{4}\cot\theta\,\csc\theta$. $r = 0$ is included in $r = \frac{1}{4}\cot\theta\,\csc\theta$ when $\theta = \frac{\pi}{2}$, so the curve is represented by the single equation $r = \frac{1}{4}\cot\theta\,\csc\theta$.

25. $x^2 + y^2 = 2cx \quad\Leftrightarrow\quad r^2 = 2cr\cos\theta \quad\Leftrightarrow\quad r^2 - 2cr\cos\theta = 0 \quad\Leftrightarrow\quad r(r - 2c\cos\theta) = 0 \quad\Leftrightarrow\quad r = 0$ or $r = 2c\cos\theta$. $r = 0$ is included in $r = 2c\cos\theta$ when $\theta = \frac{\pi}{2} + n\pi$, so the curve is represented by the single equation $r = 2c\cos\theta$.

26. $x^2 - y^2 = 4 \quad \Leftrightarrow \quad (r\cos\theta)^2 - (r\sin\theta)^2 = 4 \quad \Leftrightarrow \quad r^2\cos^2\theta - r^2\sin^2\theta = 4 \quad \Leftrightarrow \quad r^2(\cos^2\theta - \sin^2\theta) = 4 \quad \Leftrightarrow$
$r^2\cos 2\theta = 4$

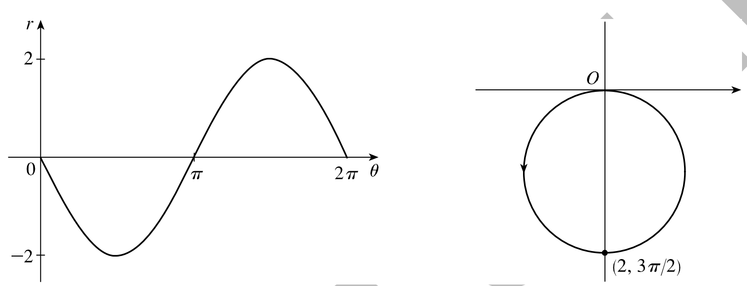
27. (a) The description leads immediately to the polar equation $\theta = \frac{\pi}{6}$, and the Cartesian equation $y = \tan\left(\frac{\pi}{6}\right)x = \frac{1}{\sqrt{3}}\,x$ is slightly more difficult to derive.

(b) The easier description here is the Cartesian equation $x = 3$.

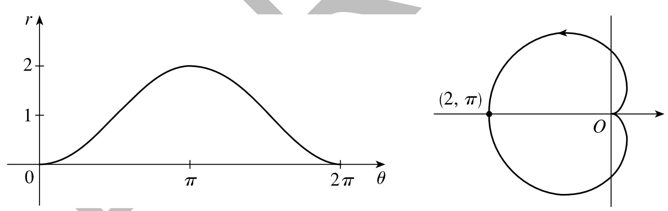
28. (a) Because its center is not at the origin, it is more easily described by its Cartesian equation, $(x-2)^2 + (y-3)^2 = 5^2$.

(b) This circle is more easily given in polar coordinates: $r = 4$. The Cartesian equation is also simple: $x^2 + y^2 = 16$.

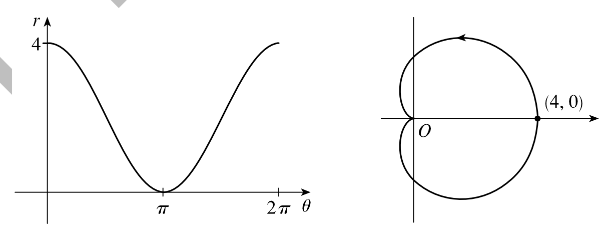
29. $r = -2\sin\theta$



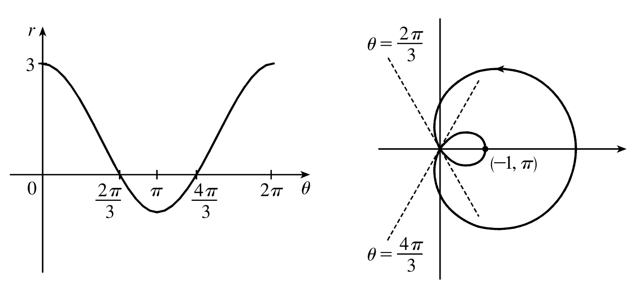
30. $r = 1 - \cos\theta$



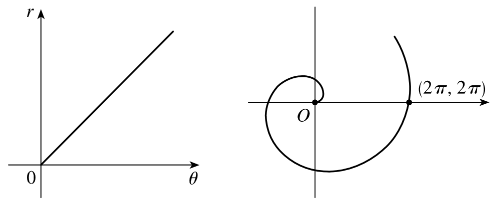
31. $r = 2(1 + \cos\theta)$



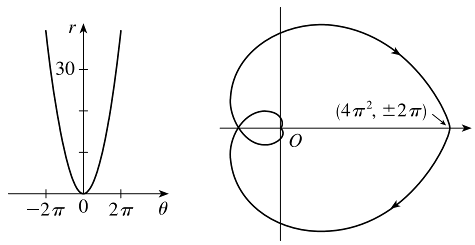
32. $r = 1 + 2\cos\theta$



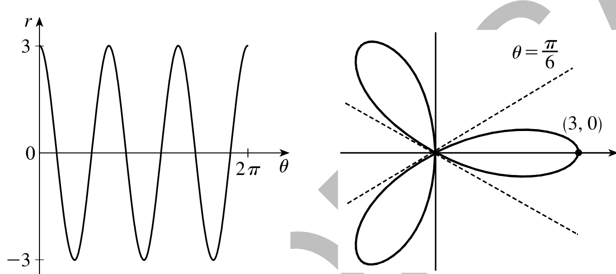
33. $r = \theta, \quad \theta \geq 0$



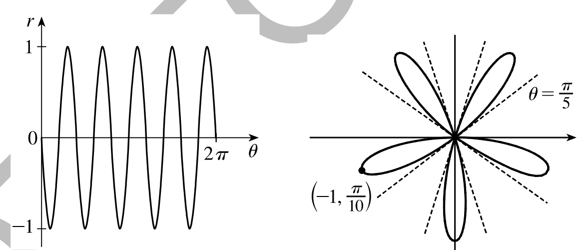
34. $r = \theta^2, \ -2\pi \leq \theta \leq 2\pi$



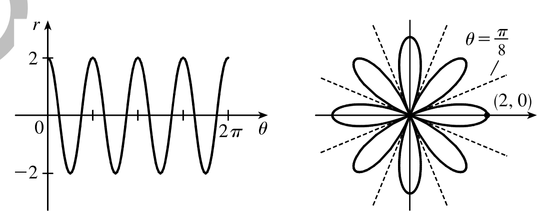
35. $r = 3\cos 3\theta$



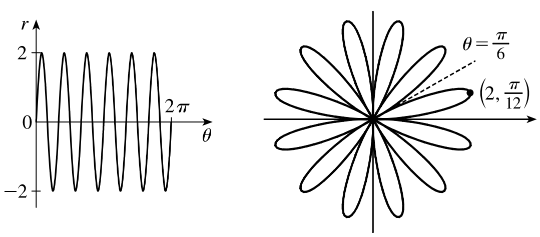
36. $r = -\sin 5\theta$



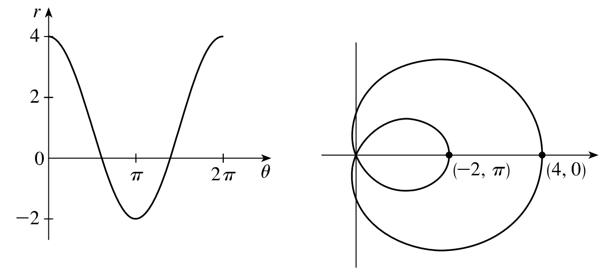
37. $r = 2\cos 4\theta$



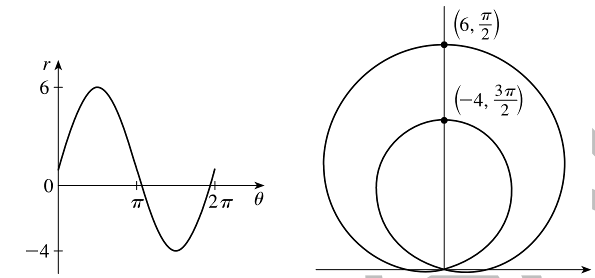
38. $r = 2\sin 6\theta$



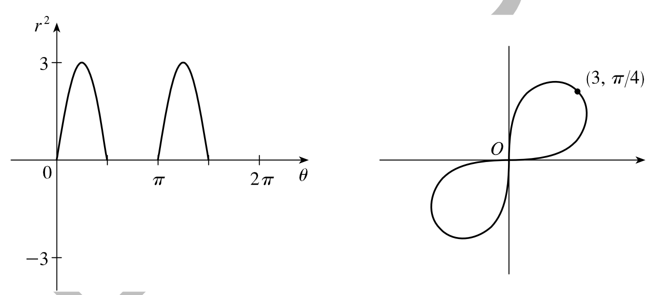
39. $r = 1 + 3\cos\theta$



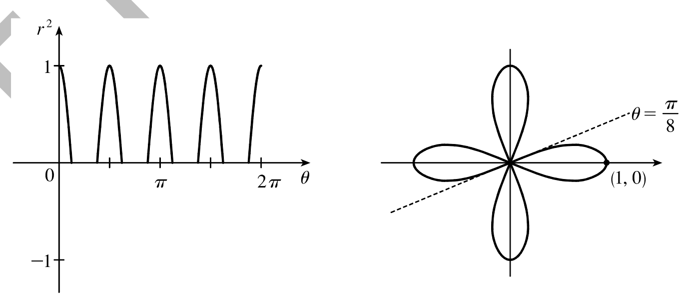
40. $r = 1 + 5\sin\theta$



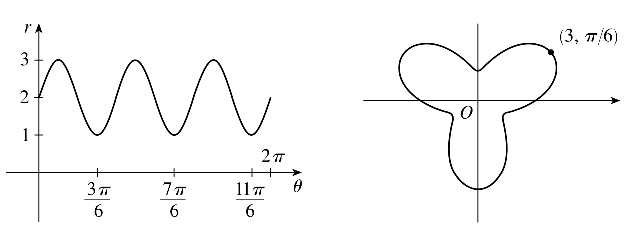
41. $r^2 = 9\sin 2\theta$



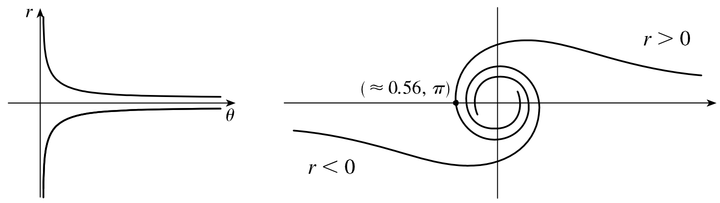
42. $r^2 = \cos 4\theta$



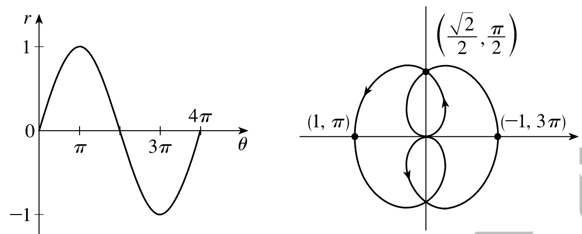
43. $r = 2 + \sin 3\theta$



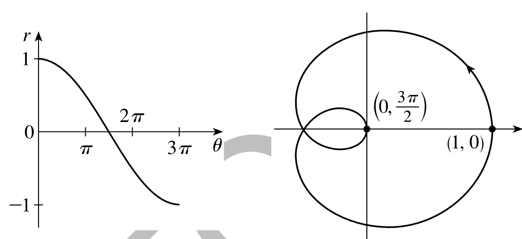
44. $r^2\theta = 1 \quad \Leftrightarrow \quad r = \pm 1/\sqrt{\theta}$ for $\theta > 0$



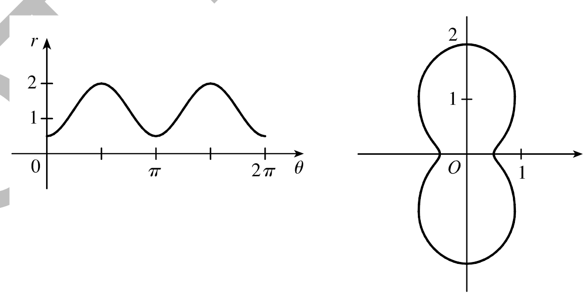
45. $r = \sin(\theta/2)$



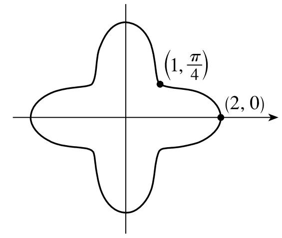
46. $r = \cos(\theta/3)$



47. For $\theta = 0$, π, and 2π, r has its minimum value of about 0.5. For $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, r attains its maximum value of 2. We see that the graph has a similar shape for $0 \le \theta \le \pi$ and $\pi \le \theta \le 2\pi$.



48. The given graph has a maximum of 2 for $\theta = 0$, a minimum of 1 for $\theta = \frac{\pi}{4}$, and then a maximum of 2 for $\theta = \frac{\pi}{2}$. This pattern is repeated 4 times for $0 \le \theta \le 2\pi$.

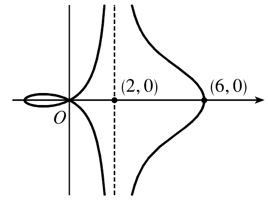


49. $x = r\cos\theta = (4 + 2\sec\theta)\cos\theta = 4\cos\theta + 2$. Now, $r \to \infty \quad\Rightarrow$

$(4 + 2\sec\theta) \to \infty \quad\Rightarrow\quad \theta \to \left(\frac{\pi}{2}\right)^-$ or $\theta \to \left(\frac{3\pi}{2}\right)^+$ [since we need only consider $0 \le \theta < 2\pi$], so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-}(4\cos\theta + 2) = 2$. Also,

$r \to -\infty \quad\Rightarrow\quad (4 + 2\sec\theta) \to -\infty \quad\Rightarrow\quad \theta \to \left(\frac{\pi}{2}\right)^+$ or $\theta \to \left(\frac{3\pi}{2}\right)^-$, so

$\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+}(4\cos\theta + 2) = 2$. Therefore, $\lim\limits_{r\to\pm\infty} x = 2 \quad\Rightarrow\quad x = 2$ is a vertical asymptote.

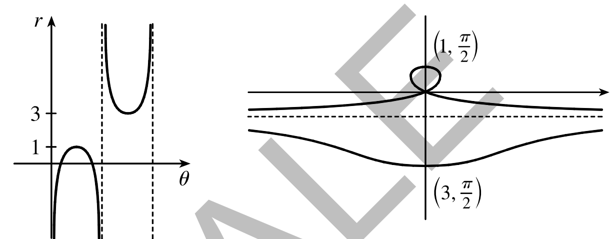


50. $y = r\sin\theta = 2\sin\theta - \csc\theta\sin\theta = 2\sin\theta - 1$.

$r \to \infty \quad\Rightarrow\quad (2 - \csc\theta) \to \infty \quad\Rightarrow$

$\csc\theta \to -\infty \quad\Rightarrow\quad \theta \to \pi^+$ [since we need only consider $0 \le \theta < 2\pi$] and so

$\lim\limits_{r\to\infty} y = \lim\limits_{\theta\to\pi^+} 2\sin\theta - 1 = -1$.



Also $r \to -\infty \quad\Rightarrow\quad (2 - \csc\theta) \to -\infty \quad\Rightarrow\quad \csc\theta \to \infty \quad\Rightarrow\quad \theta \to \pi^-$ and so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi^-} 2\sin\theta - 1 = -1$.

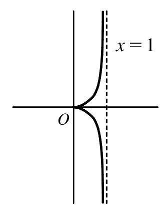
Therefore $\lim\limits_{r\to\pm\infty} y = -1 \quad\Rightarrow\quad y = -1$ is a horizontal asymptote.

51. To show that $x = 1$ is an asymptote we must prove $\lim\limits_{r\to\pm\infty} x = 1$.

$x = (r)\cos\theta = (\sin\theta\,\tan\theta)\cos\theta = \sin^2\theta$. Now, $r \to \infty \quad\Rightarrow\quad \sin\theta\,\tan\theta \to \infty \quad\Rightarrow$

$\theta \to \left(\frac{\pi}{2}\right)^-$, so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} \sin^2\theta = 1$. Also, $r \to -\infty \quad\Rightarrow\quad \sin\theta\,\tan\theta \to -\infty \quad\Rightarrow$

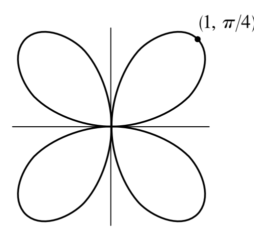
$\theta \to \left(\frac{\pi}{2}\right)^+$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} \sin^2\theta = 1$. Therefore, $\lim\limits_{r\to\pm\infty} x = 1 \quad\Rightarrow\quad x = 1$ is



a vertical asymptote. Also notice that $x = \sin^2\theta \ge 0$ for all θ, and $x = \sin^2\theta \le 1$ for all θ. And $x \ne 1$, since the curve is not defined at odd multiples of $\frac{\pi}{2}$. Therefore, the curve lies entirely within the vertical strip $0 \le x < 1$.

52. The equation is $(x^2 + y^2)^3 = 4x^2y^2$, but using polar coordinates we know that $x^2 + y^2 = r^2$ and $x = r\cos\theta$ and $y = r\sin\theta$. Substituting into the given equation: $r^6 = 4r^2\cos^2\theta\, r^2\sin^2\theta \quad\Rightarrow\quad r^2 = 4\cos^2\theta\,\sin^2\theta \quad\Rightarrow$

$r = \pm 2\cos\theta\,\sin\theta = \pm\sin 2\theta$. $r = \pm\sin 2\theta$ is sketched at right.



53. (a) We see that the curve $r = 1 + c\sin\theta$ crosses itself at the origin, where $r = 0$ (in fact the inner loop corresponds to negative r-values,) so we solve the equation of the limaçon for $r = 0 \quad\Leftrightarrow\quad c\sin\theta = -1 \quad\Leftrightarrow\quad \sin\theta = -1/c$. Now if $|c| < 1$, then this equation has no solution and hence there is no inner loop. But if $c < -1$, then on the interval $(0, 2\pi)$ the equation has the two solutions $\theta = \sin^{-1}(-1/c)$ and $\theta = \pi - \sin^{-1}(-1/c)$, and if $c > 1$, the solutions are $\theta = \pi + \sin^{-1}(1/c)$ and $\theta = 2\pi - \sin^{-1}(1/c)$. In each case, $r < 0$ for θ between the two solutions, indicating a loop.

(b) For $0 < c < 1$, the dimple (if it exists) is characterized by the fact that y has a local maximum at $\theta = \frac{3\pi}{2}$. So we determine for what c-values $\dfrac{d^2y}{d\theta^2}$ is negative at $\theta = \frac{3\pi}{2}$, since by the Second Derivative Test this indicates a maximum:

$y = r\sin\theta = \sin\theta + c\sin^2\theta \quad\Rightarrow\quad \dfrac{dy}{d\theta} = \cos\theta + 2c\sin\theta\,\cos\theta = \cos\theta + c\sin 2\theta \quad\Rightarrow\quad \dfrac{d^2y}{d\theta^2} = -\sin\theta + 2c\cos 2\theta.$

At $\theta = \frac{3\pi}{2}$, this is equal to $-(-1) + 2c(-1) = 1 - 2c$, which is negative only for $c > \frac{1}{2}$. A similar argument shows that for $-1 < c < 0$, y only has a local minimum at $\theta = \frac{\pi}{2}$ (indicating a dimple) for $c < -\frac{1}{2}$.

54. (a) $r = \ln\theta,\ 1 \le \theta \le 6\pi$. r increases as θ increases and there are almost three full revolutions. The graph must be either III or VI. As θ increases, r grows slowly in VI and quickly in III. Since $r = \ln\theta$ grows slowly, its graph must be VI.

(b) $r = \theta^2,\ 0 \le \theta \le 8\pi$. See part (a). This is graph III.

(c) The graph of $r = \cos 3\theta$ is a three-leaved rose, which is graph II.

(d) Since $-1 \le \cos 3\theta \le 1$, $1 \le 2 + \cos 3\theta \le 3$, so $r = 2 + \cos 3\theta$ is never 0; that is, the curve never intersects the pole. The graph must be I or IV. For $0 \le \theta \le 2\pi$, the graph assumes its minimum r-value of 1 three times, at $\theta = \frac{\pi}{3}$, π, and $\frac{5\pi}{3}$, so it must be graph IV.

(e) $r = \cos(\theta/2)$. For $\theta = 0$, $r = 1$, and as θ increases to π, r decreases to 0. Only graph V satisfies those values.

(f) $r = 2 + \cos(3\theta/2)$. As in part (d), this graph never intersects the pole, so it must be graph I.

55. $r = 2\cos\theta \quad\Rightarrow\quad x = r\cos\theta = 2\cos^2\theta,\ y = r\sin\theta = 2\sin\theta\,\cos\theta = \sin 2\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{2\cdot 2\cos\theta(-\sin\theta)} = \frac{\cos 2\theta}{-\sin 2\theta} = -\cot 2\theta$$

When $\theta = \dfrac{\pi}{3}$, $\dfrac{dy}{dx} = -\cot\left(2\cdot\dfrac{\pi}{3}\right) = \cot\dfrac{\pi}{3} = \dfrac{1}{\sqrt{3}}$. [*Another method:* Use Equation 3.]

56. $r = 2 + \sin 3\theta \quad\Rightarrow\quad x = r\cos\theta = (2 + \sin 3\theta)\cos\theta,\ y = r\sin\theta = (2 + \sin 3\theta)\sin\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(2+\sin 3\theta)\cos\theta + \sin\theta(3\cos 3\theta)}{(2+\sin 3\theta)(-\sin\theta) + \cos\theta(3\cos 3\theta)}$$

When $\theta = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = \dfrac{\left(2+\sin\frac{3\pi}{4}\right)\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\left(3\cos\frac{3\pi}{4}\right)}{\left(2+\sin\frac{3\pi}{4}\right)\left(-\sin\frac{\pi}{4}\right) + \cos\frac{\pi}{4}\left(3\cos\frac{3\pi}{4}\right)} = \dfrac{\left(2+\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot 3\left(-\frac{\sqrt{2}}{2}\right)}{\left(2+\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}\cdot 3\left(-\frac{\sqrt{2}}{2}\right)}$

$$= \frac{\sqrt{2} + \frac{1}{2} - \frac{3}{2}}{-\sqrt{2} - \frac{1}{2} - \frac{3}{2}} = \frac{\sqrt{2} - 1}{-\sqrt{2} - 2}, \text{ or, equivalently, } 2 - \frac{3}{2}\sqrt{2}.$$

57. $r = 1/\theta \quad\Rightarrow\quad x = r\cos\theta = (\cos\theta)/\theta,\ y = r\sin\theta = (\sin\theta)/\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-1/\theta^2) + (1/\theta)\cos\theta}{\cos\theta(-1/\theta^2) - (1/\theta)\sin\theta}\cdot\frac{\theta^2}{\theta^2} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}$$

When $\theta = \pi$, $\dfrac{dy}{dx} = \dfrac{-0 + \pi(-1)}{-(-1) - \pi(0)} = \dfrac{-\pi}{1} = -\pi$.

58. $r = \cos(\theta/3) \quad\Rightarrow\quad x = r\cos\theta = \cos(\theta/3)\cos\theta,\ y = r\sin\theta = \cos(\theta/3)\sin\theta \quad\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos(\theta/3)\cos\theta + \sin\theta\left(-\frac{1}{3}\sin(\theta/3)\right)}{\cos(\theta/3)(-\sin\theta) + \cos\theta\left(-\frac{1}{3}\sin(\theta/3)\right)}$$

When $\theta = \pi$, $\dfrac{dy}{dx} = \dfrac{\frac{1}{2}(-1) + (0)\left(-\sqrt{3}/6\right)}{\frac{1}{2}(0) + (-1)\left(-\sqrt{3}/6\right)} = \dfrac{-1/2}{\sqrt{3}/6} = -\dfrac{3}{\sqrt{3}} = -\sqrt{3}$.

59. $r = \cos 2\theta \quad \Rightarrow \quad x = r\cos\theta = \cos 2\theta\,\cos\theta,\ y = r\sin\theta = \cos 2\theta\,\sin\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos 2\theta\,\cos\theta + \sin\theta\,(-2\sin 2\theta)}{\cos 2\theta\,(-\sin\theta) + \cos\theta\,(-2\sin 2\theta)}$$

When $\theta = \frac{\pi}{4}$, $\dfrac{dy}{dx} = \dfrac{0\left(\sqrt{2}/2\right) + \left(\sqrt{2}/2\right)(-2)}{0\left(-\sqrt{2}/2\right) + \left(\sqrt{2}/2\right)(-2)} = \dfrac{-\sqrt{2}}{-\sqrt{2}} = 1.$

60. $r = 1 + 2\cos\theta \quad \Rightarrow \quad x = r\cos\theta = (1 + 2\cos\theta)\cos\theta,\ y = r\sin\theta = (1 + 2\cos\theta)\sin\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(1 + 2\cos\theta)\cos\theta + \sin\theta\,(-2\sin\theta)}{(1 + 2\cos\theta)(-\sin\theta) + \cos\theta\,(-2\sin\theta)}$$

When $\theta = \frac{\pi}{3}$, $\dfrac{dy}{dx} = \dfrac{2\left(\frac{1}{2}\right) + \left(\sqrt{3}/2\right)\left(-\sqrt{3}\right)}{2\left(-\sqrt{3}/2\right) + \frac{1}{2}\left(-\sqrt{3}\right)} \cdot \dfrac{2}{2} = \dfrac{2 - 3}{-2\sqrt{3} - \sqrt{3}} = \dfrac{-1}{-3\sqrt{3}} = \dfrac{\sqrt{3}}{9}.$

61. $r = 3\cos\theta \quad \Rightarrow \quad x = r\cos\theta = 3\cos\theta\,\cos\theta,\ \ y = r\sin\theta = 3\cos\theta\,\sin\theta \quad \Rightarrow$

$\frac{dy}{d\theta} = -3\sin^2\theta + 3\cos^2\theta = 3\cos 2\theta = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \quad \Leftrightarrow \quad \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

So the tangent is horizontal at $\left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right)$ and $\left(-\frac{3}{\sqrt{2}}, \frac{3\pi}{4}\right)$ $\left[\text{same as } \left(\frac{3}{\sqrt{2}}, -\frac{\pi}{4}\right)\right]$.

$\frac{dx}{d\theta} = -6\sin\theta\cos\theta = -3\sin 2\theta = 0 \quad \Rightarrow \quad 2\theta = 0$ or $\pi \quad \Leftrightarrow \quad \theta = 0$ or $\frac{\pi}{2}$. So the tangent is vertical at $(3, 0)$ and $\left(0, \frac{\pi}{2}\right)$.

62. $r = 1 - \sin\theta \quad \Rightarrow \quad x = r\cos\theta = \cos\theta\,(1 - \sin\theta),\ \ y = r\sin\theta = \sin\theta\,(1 - \sin\theta) \quad \Rightarrow$

$\frac{dy}{d\theta} = \sin\theta\,(-\cos\theta) + (1 - \sin\theta)\,\cos\theta = \cos\theta\,(1 - 2\sin\theta) = 0 \quad \Rightarrow \quad \cos\theta = 0$ or $\sin\theta = \frac{1}{2} \quad \Rightarrow$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, or $\frac{3\pi}{2} \quad \Rightarrow \quad$ horizontal tangent at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$, and $\left(2, \frac{3\pi}{2}\right)$.

$$\begin{aligned}\frac{dx}{d\theta} &= \cos\theta\,(-\cos\theta) + (1 - \sin\theta)(-\sin\theta) = -\cos^2\theta - \sin\theta + \sin^2\theta = 2\sin^2\theta - \sin\theta - 1 \\ &= (2\sin\theta + 1)(\sin\theta - 1) = 0 \quad \Rightarrow\end{aligned}$$

$\sin\theta = -\frac{1}{2}$ or $1 \quad \Rightarrow \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$, or $\frac{\pi}{2} \quad \Rightarrow \quad$ vertical tangent at $\left(\frac{3}{2}, \frac{7\pi}{6}\right)$, $\left(\frac{3}{2}, \frac{11\pi}{6}\right)$, and $\left(0, \frac{\pi}{2}\right)$.

Note that the tangent is vertical, not horizontal, when $\theta = \frac{\pi}{2}$, since

$$\lim_{\theta \to (\pi/2)^-} \frac{dy/d\theta}{dx/d\theta} = \lim_{\theta \to (\pi/2)^-} \frac{\cos\theta\,(1 - 2\sin\theta)}{(2\sin\theta + 1)(\sin\theta - 1)} = \infty \text{ and } \lim_{\theta \to (\pi/2)^+} \frac{dy/d\theta}{dx/d\theta} = -\infty.$$

63. $r = 1 + \cos\theta \quad \Rightarrow \quad x = r\cos\theta = \cos\theta\,(1 + \cos\theta),\ \ y = r\sin\theta = \sin\theta\,(1 + \cos\theta) \quad \Rightarrow$

$\frac{dy}{d\theta} = (1 + \cos\theta)\,\cos\theta - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0 \quad \Rightarrow \quad \cos\theta = \frac{1}{2}$ or $-1 \quad \Rightarrow$

$\theta = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3} \quad \Rightarrow \quad$ horizontal tangent at $\left(\frac{3}{2}, \frac{\pi}{3}\right)$, $(0, \pi)$, and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$.

$\frac{dx}{d\theta} = -(1 + \cos\theta)\sin\theta - \cos\theta\sin\theta = -\sin\theta\,(1 + 2\cos\theta) = 0 \quad \Rightarrow \quad \sin\theta = 0$ or $\cos\theta = -\frac{1}{2} \quad \Rightarrow$

$\theta = 0, \pi, \frac{2\pi}{3}$, or $\frac{4\pi}{3} \quad \Rightarrow \quad$ vertical tangent at $(2, 0)$, $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$, and $\left(\frac{1}{2}, \frac{4\pi}{3}\right)$.

Note that the tangent is horizontal, not vertical when $\theta = \pi$, since $\displaystyle\lim_{\theta \to \pi} \frac{dy/d\theta}{dx/d\theta} = 0$.

64. $r = e^\theta \quad \Rightarrow \quad x = r\cos\theta = e^\theta\cos\theta,\ \ y = r\sin\theta = e^\theta\sin\theta \quad \Rightarrow$

$\frac{dy}{d\theta} = e^\theta\sin\theta + e^\theta\cos\theta = e^\theta(\sin\theta + \cos\theta) = 0 \quad \Rightarrow \quad \sin\theta = -\cos\theta \quad \Rightarrow \quad \tan\theta = -1 \quad \Rightarrow$

$\theta = -\frac{1}{4}\pi + n\pi$ [n any integer] $\Rightarrow$ horizontal tangents at $\left(e^{\pi(n-1/4)}, \pi\left(n - \frac{1}{4}\right)\right)$.

$\frac{dx}{d\theta} = e^{\theta}\cos\theta - e^{\theta}\sin\theta = e^{\theta}(\cos\theta - \sin\theta) = 0 \quad\Rightarrow\quad \sin\theta = \cos\theta \quad\Rightarrow\quad \tan\theta = 1 \quad\Rightarrow$

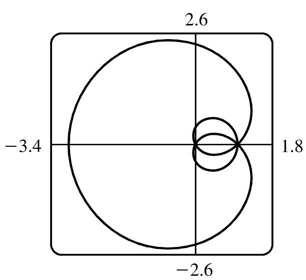
$\theta = \frac{1}{4}\pi + n\pi$ [n any integer] $\Rightarrow$ vertical tangents at $\left(e^{\pi(n+1/4)}, \pi\left(n + \frac{1}{4}\right)\right)$.

65. $r = a\sin\theta + b\cos\theta \quad\Rightarrow\quad r^2 = ar\sin\theta + br\cos\theta \quad\Rightarrow\quad x^2 + y^2 = ay + bx \quad\Rightarrow$

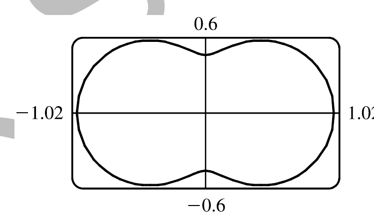
$x^2 - bx + \left(\frac{1}{2}b\right)^2 + y^2 - ay + \left(\frac{1}{2}a\right)^2 = \left(\frac{1}{2}b\right)^2 + \left(\frac{1}{2}a\right)^2 \quad\Rightarrow\quad \left(x - \frac{1}{2}b\right)^2 + \left(y - \frac{1}{2}a\right)^2 = \frac{1}{4}(a^2 + b^2)$, and this is a circle with center $\left(\frac{1}{2}b, \frac{1}{2}a\right)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$.

66. These curves are circles which intersect at the origin and at $\left(\frac{1}{\sqrt{2}}\,a, \frac{\pi}{4}\right)$. At the origin, the first circle has a horizontal tangent and the second a vertical one, so the tangents are perpendicular here. For the first circle [$r = a\sin\theta$], $dy/d\theta = a\cos\theta\,\sin\theta + a\sin\theta\,\cos\theta = a\sin 2\theta = a$ at $\theta = \frac{\pi}{4}$ and $dx/d\theta = a\cos^2\theta - a\sin^2\theta = a\cos 2\theta = 0$ at $\theta = \frac{\pi}{4}$, so the tangent here is vertical. Similarly, for the second circle [$r = a\cos\theta$], $dy/d\theta = a\cos 2\theta = 0$ and $dx/d\theta = -a\sin 2\theta = -a$ at $\theta = \frac{\pi}{4}$, so the tangent is horizontal, and again the tangents are perpendicular.

67. $r = 1 + 2\sin(\theta/2)$. The parameter interval is $[0, 4\pi]$.

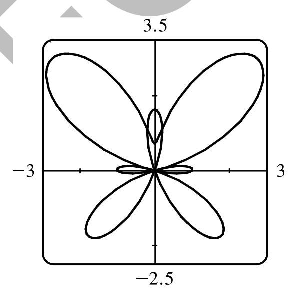


68. $r = \sqrt{1 - 0.8\sin^2\theta}$. The parameter interval is $[0, 2\pi]$.



69. $r = e^{\sin\theta} - 2\cos(4\theta)$.

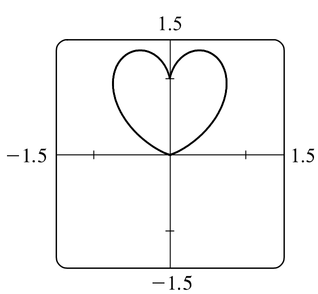
The parameter interval is $[0, 2\pi]$.

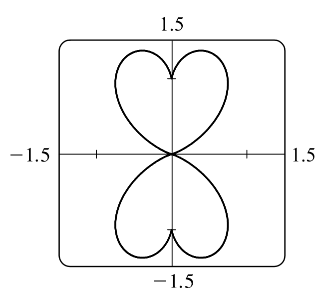


70. $r = |\tan\theta|^{|\cot\theta|}$.

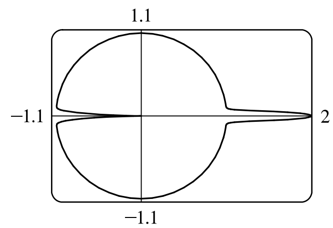
The parameter interval $[0, \pi]$ produces the heart-shaped valentine curve shown in the first window.

The complete curve, including the reflected heart, is produced by the parameter interval $[0, 2\pi]$, but perhaps you'll agree that the first curve is more appropriate.

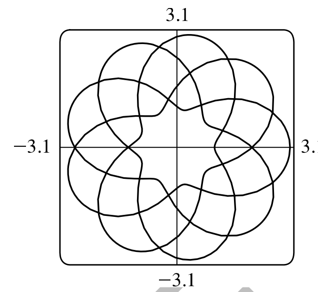




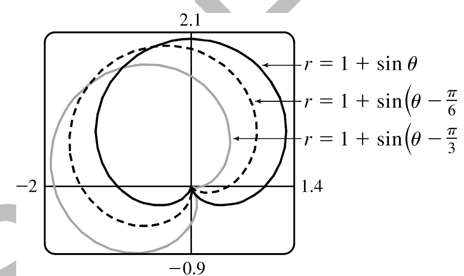
71. $r = 1 + \cos^{999}\theta$. The parameter interval is $[0, 2\pi]$.



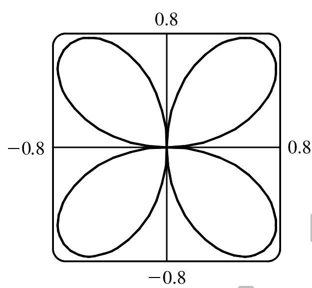
72. $r = 2 + \cos(9\theta/4)$. The parameter interval is $[0, 8\pi]$.



73. It appears that the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{6}\right)$ is the same shape as the graph of $r = 1 + \sin\theta$, but rotated counterclockwise about the origin by $\frac{\pi}{6}$. Similarly, the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$ is rotated by $\frac{\pi}{3}$. In general, the graph of $r = f(\theta - \alpha)$ is the same shape as that of $r = f(\theta)$, but rotated counterclockwise through α about the origin. That is, for any point (r_0, θ_0) on the curve $r = f(\theta)$, the point $(r_0, \theta_0 + \alpha)$ is on the curve $r = f(\theta - \alpha)$, since $r_0 = f(\theta_0) = f((\theta_0 + \alpha) - \alpha)$.



74.



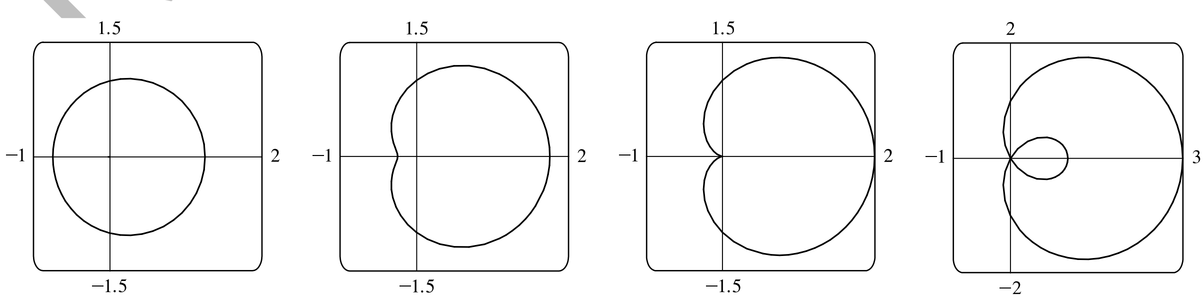
From the graph, the highest points seem to have $y \approx 0.77$. To find the exact value, we solve $dy/d\theta = 0$. $y = r\sin\theta = \sin\theta\,\sin 2\theta \quad \Rightarrow$

$$\begin{aligned} dy/d\theta &= 2\sin\theta\,\cos 2\theta + \cos\theta\,\sin 2\theta \\ &= 2\sin\theta\,(2\cos^2\theta - 1) + \cos\theta\,(2\sin\theta\,\cos\theta) \\ &= 2\sin\theta\,(3\cos^2\theta - 1) \end{aligned}$$

In the first quadrant, this is 0 when $\cos\theta = \frac{1}{\sqrt{3}} \quad \Leftrightarrow \quad \sin\theta = \sqrt{\frac{2}{3}} \quad \Leftrightarrow$

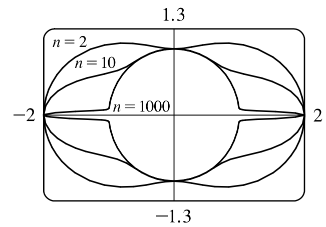
$y = 2\sin^2\theta\cos\theta = 2 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} = \frac{4}{9}\sqrt{3} \approx 0.77$.

75. Consider curves with polar equation $r = 1 + c\cos\theta$, where c is a real number. If $c = 0$, we get a circle of radius 1 centered at the pole. For $0 < c \le 0.5$, the curve gets slightly larger, moves right, and flattens out a bit on the left side. For $0.5 < c < 1$, the left side has a dimple shape. For $c = 1$, the dimple becomes a cusp. For $c > 1$, there is an internal loop. For $c \ge 0$, the rightmost point on the curve is $(1 + c, 0)$. For $c < 0$, the curves are reflections through the vertical axis of the curves with $c > 0$.

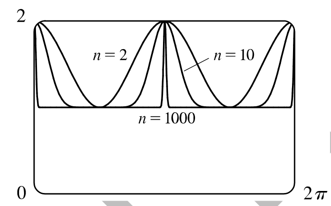


$c = 0.25$ $\quad$ $c = 0.75$ $\quad$ $c = 1$ $\quad$ $c = 2$

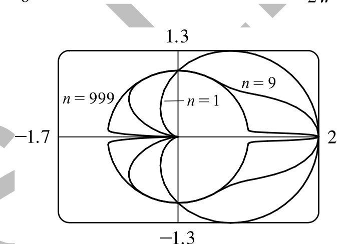
76. Consider the polar curves $r = 1 + \cos^n \theta$, where n is a positive integer. First, let $\boldsymbol{n}$ **be an even positive integer**. The first figure shows that the curve has a peanut shape for $n = 2$, but as n increases, the ends are squeezed. As n becomes large, the curves look more and more like the unit circle, but with spikes to the points $(2, 0)$ and $(2, \pi)$.



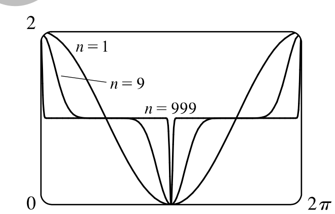
The second figure shows r as a function of θ in Cartesian coordinates for the same values of n. We can see that for large n, the graph is similar to the graph of $y = 1$, but with spikes to $y = 2$ for $x = 0$, π, and 2π. (Note that when $0 < \cos\theta < 1$, $\cos^{1000}\theta$ is very small.)



Next, let $\boldsymbol{n}$ **be an odd positive integer**. The third figure shows that the curve is a cardioid for $n = 1$, but as n increases, the heart shape becomes more pronounced. As n becomes large, the curves again look more like the unit circle, but with an outward spike to $(2, 0)$ and an inward spike to $(0, \pi)$.



The fourth figure shows r as a function of θ in Cartesian coordinates for the same values of n. We can see that for large n, the graph is similar to the graph of $y = 1$, but spikes to $y = 2$ for $x = 0$ and π, and to $y = 0$ for $x = \pi$.

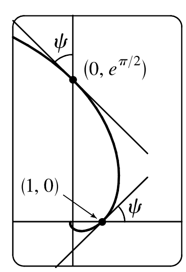


77. $$\tan\psi = \tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\,\tan\theta} = \frac{\dfrac{dy}{dx} - \tan\theta}{1 + \dfrac{dy}{dx}\tan\theta} = \frac{\dfrac{dy/d\theta}{dx/d\theta} - \tan\theta}{1 + \dfrac{dy/d\theta}{dx/d\theta}\tan\theta}$$

$$= \frac{\dfrac{dy}{d\theta} - \dfrac{dx}{d\theta}\tan\theta}{\dfrac{dx}{d\theta} + \dfrac{dy}{d\theta}\tan\theta} = \frac{\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right) - \tan\theta\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right)}{\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right) + \tan\theta\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right)} = \frac{r\cos\theta + r\cdot\dfrac{\sin^2\theta}{\cos\theta}}{\dfrac{dr}{d\theta}\cos\theta + \dfrac{dr}{d\theta}\cdot\dfrac{\sin^2\theta}{\cos\theta}}$$

$$= \frac{r\cos^2\theta + r\sin^2\theta}{\dfrac{dr}{d\theta}\cos^2\theta + \dfrac{dr}{d\theta}\sin^2\theta} = \frac{r}{dr/d\theta}$$

78. (a) $r = e^\theta \;\Rightarrow\; dr/d\theta = e^\theta$, so by Exercise 77, $\tan\psi = r/e^\theta = 1 \;\Rightarrow\; \psi = \arctan 1 = \frac{\pi}{4}$.



(b) The Cartesian equation of the tangent line at $(1, 0)$ is $y = x - 1$, and that of the tangent line at $(0, e^{\pi/2})$ is $y = e^{\pi/2} - x$.

(c) Let a be the tangent of the angle between the tangent and radial lines, that is, $a = \tan\psi$. Then, by Exercise 77, $a = \dfrac{r}{dr/d\theta} \;\Rightarrow\; \dfrac{dr}{d\theta} = \dfrac{1}{a}r \;\Rightarrow\; r = Ce^{\theta/a}$ (by Theorem 9.4.2).