2. There are six different possible orders of integration.

$$\iiint_E (xy+z^2)\,dV = \int_0^2\int_0^1\int_0^3 (xy+z^2)\,dz\,dy\,dx = \int_0^2\int_0^1 \left[xyz+\tfrac{1}{3}z^3\right]_{z=0}^{z=3} dy\,dx = \int_0^2\int_0^1 (3xy+9)\,dy\,dx$$
$$= \int_0^2 \left[\tfrac{3}{2}xy^2+9y\right]_{y=0}^{y=1} dx = \int_0^2 \left(\tfrac{3}{2}x+9\right) dx = \left[\tfrac{3}{4}x^2+9x\right]_0^2 = 21$$

$$\iiint_E (xy+z^2)\,dV = \int_0^1\int_0^2\int_0^3 (xy+z^2)\,dz\,dx\,dy = \int_0^1\int_0^2 \left[xyz+\tfrac{1}{3}z^3\right]_{z=0}^{z=3} dx\,dy = \int_0^1\int_0^2 (3xy+9)\,dx\,dy$$
$$= \int_0^1 \left[\tfrac{3}{2}x^2y+9x\right]_{x=0}^{x=2} dy = \int_0^1 (6y+18)\,dy = \left[3y^2+18y\right]_0^1 = 21$$

$$\iiint_E (xy+z^2)\,dV = \int_0^2\int_0^3\int_0^1 (xy+z^2)\,dy\,dz\,dx = \int_0^2\int_0^3 \left[\tfrac{1}{2}xy^2+yz^2\right]_{y=0}^{y=1} dz\,dx = \int_0^2\int_0^3 \left(\tfrac{1}{2}x+z^2\right) dz\,dx$$
$$= \int_0^2 \left[\tfrac{1}{2}xz+\tfrac{1}{3}z^3\right]_{z=0}^{z=3} dx = \int_0^2 \left(\tfrac{3}{2}x+9\right) dx = \left[\tfrac{3}{4}x^2+9x\right]_0^2 = 21$$

$$\iiint_E (xy+z^2)\,dV = \int_0^3\int_0^2\int_0^1 (xy+z^2)\,dy\,dx\,dz = \int_0^3\int_0^2 \left[\tfrac{1}{2}xy^2+yz^2\right]_{y=0}^{y=1} dx\,dz = \int_0^3\int_0^2 \left(\tfrac{1}{2}x+z^2\right) dx\,dz$$
$$= \int_0^3 \left[\tfrac{1}{4}x^2+xz^2\right]_{x=0}^{x=2} dz = \int_0^3 \left(1+2z^2\right) dz = \left[z+\tfrac{2}{3}z^3\right]_0^3 = 21$$

$$\iiint_E (xy+z^2)\,dV = \int_0^1\int_0^3\int_0^2 (xy+z^2)\,dx\,dz\,dy = \int_0^1\int_0^3 \left[\tfrac{1}{2}x^2y+xz^2\right]_{x=0}^{x=2} dz\,dy = \int_0^1\int_0^3 \left(2y+2z^2\right) dz\,dy$$
$$= \int_0^1 \left[2yz+\tfrac{2}{3}z^3\right]_{z=0}^{z=3} dy = \int_0^1 (6y+18)\,dy = \left[3y^2+18y\right]_0^1 = 21$$

$$\iiint_E (xy+z^2)\,dV = \int_0^3\int_0^1\int_0^2 (xy+z^2)\,dx\,dy\,dz = \int_0^3\int_0^1 \left[\tfrac{1}{2}x^2y+xz^2\right]_{x=0}^{x=2} dy\,dz = \int_0^3\int_0^1 \left(2y+2z^2\right) dy\,dz$$
$$= \int_0^3 \left[y^2+2yz^2\right]_{y=0}^{y=1} dz = \int_0^3 \left(1+2z^2\right) dz = \left[z+\tfrac{2}{3}z^3\right]_0^3 = 21$$

3. $$\int_0^2\int_0^{z^2}\int_0^{y-z}(2x-y)\,dx\,dy\,dz = \int_0^2\int_0^{z^2} \left[x^2-xy\right]_{x=0}^{x=y-z} dy\,dz = \int_0^2\int_0^{z^2} \left[(y-z)^2-(y-z)y\right] dy\,dz$$
$$= \int_0^2\int_0^{z^2} \left(z^2-yz\right) dy\,dz = \int_0^2 \left[yz^2-\tfrac{1}{2}y^2z\right]_{y=0}^{y=z^2} dz = \int_0^2 \left(z^4-\tfrac{1}{2}z^5\right) dz$$
$$= \left[\tfrac{1}{5}z^5-\tfrac{1}{12}z^6\right]_0^2 = \tfrac{32}{5}-\tfrac{64}{12} = \tfrac{16}{15}$$

4. $$\int_0^1\int_y^{2y}\int_0^{x+y} 6xy\,dz\,dx\,dy = \int_0^1\int_y^{2y} \left[6xyz\right]_{z=0}^{z=x+y} dx\,dy = \int_0^1\int_y^{2y} 6xy(x+y)\,dx\,dy = \int_0^1\int_y^{2y} (6x^2y+6xy^2)\,dx\,dy$$
$$= \int_0^1 \left[2x^3y+3x^2y^2\right]_{x=y}^{x=2y} dy = \int_0^1 23y^4\,dy = \tfrac{23}{5}y^5\Big]_0^1 = \tfrac{23}{5}$$

5. $$\int_1^2\int_0^{2z}\int_0^{\ln x} xe^{-y}\,dy\,dx\,dz = \int_1^2\int_0^{2z} \left[-xe^{-y}\right]_{y=0}^{y=\ln x} dx\,dz = \int_1^2\int_0^{2z} \left(-xe^{-\ln x}+xe^0\right) dx\,dz$$
$$= \int_1^2\int_0^{2z} (-1+x)\,dx\,dz = \int_1^2 \left[-x+\tfrac{1}{2}x^2\right]_{x=0}^{x=2z} dz$$
$$= \int_1^2 \left(-2z+2z^2\right) dz = \left[-z^2+\tfrac{2}{3}z^3\right]_1^2 = -4+\tfrac{16}{3}+1-\tfrac{2}{3} = \tfrac{5}{3}$$

6. $$\int_0^1\int_0^1\int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\,dx\,dz\,dy = \int_0^1\int_0^1 \left[\frac{z}{y+1}\cdot x\right]_{x=0}^{x=\sqrt{1-z^2}} dz\,dy = \int_0^1\int_0^1 \frac{z\sqrt{1-z^2}}{y+1}\,dz\,dy$$
$$= \int_0^1 \left[\frac{-\frac{1}{3}(1-z^2)^{3/2}}{y+1}\right]_{z=0}^{z=1} dy = \frac{1}{3}\int_0^1 \frac{1}{y+1}\,dy = \frac{1}{3}\ln(y+1)\Big]_0^1$$
$$= \tfrac{1}{3}(\ln 2-\ln 1) = \tfrac{1}{3}\ln 2$$

7. $\int_0^\pi \int_0^1 \int_0^{\sqrt{1-z^2}} z\sin x\,dy\,dz\,dx = \int_0^\pi \int_0^1 \left[yz\sin x\right]_{y=0}^{y=\sqrt{1-z^2}} dz\,dx = \int_0^\pi \int_0^1 z\sqrt{1-z^2}\,\sin x\,dz\,dx$

$= \int_0^\pi \sin x\left[-\frac{1}{3}(1-z^2)^{3/2}\right]_{z=0}^{z=1} dx = \int_0^\pi \frac{1}{3}\sin x\,dx = -\frac{1}{3}\cos x\Big]_0^\pi = -\frac{1}{3}(-1-1) = \frac{2}{3}$

8. $\int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z\,dz\,dy\,dx = \int_0^1 \int_0^1 \left[xye^z\right]_{z=0}^{z=2-x^2-y^2} dy\,dx = \int_0^1 \int_0^1 (xye^{2-x^2-y^2} - xy)\,dy\,dx$

$= \int_0^1 \left[-\frac{1}{2}xe^{2-x^2-y^2} - \frac{1}{2}xy^2\right]_{y=0}^{y=1} dx = \int_0^1 \left(-\frac{1}{2}xe^{1-x^2} - \frac{1}{2}x + \frac{1}{2}xe^{2-x^2}\right) dx$

$= \left[\frac{1}{4}e^{1-x^2} - \frac{1}{4}x^2 - \frac{1}{4}e^{2-x^2}\right]_0^1 = \frac{1}{4} - \frac{1}{4} - \frac{1}{4}e - \frac{1}{4}e + 0 + \frac{1}{4}e^2 = \frac{1}{4}e^2 - \frac{1}{2}e$

9. $\iiint_E y\,dV = \int_0^3 \int_0^x \int_{x-y}^{x+y} y\,dz\,dy\,dx = \int_0^3 \int_0^x \left[yz\right]_{z=x-y}^{z=x+y} dy\,dx = \int_0^3 \int_0^x 2y^2\,dy\,dx$

$= \int_0^3 \left[\frac{2}{3}y^3\right]_{y=0}^{y=x} dx = \int_0^3 \frac{2}{3}x^3\,dx = \frac{1}{6}x^4\Big]_0^3 = \frac{81}{6} = \frac{27}{2}$

10. $\iiint_E e^{z/y}\,dV = \int_0^1 \int_y^1 \int_0^{xy} e^{z/y}\,dz\,dx\,dy = \int_0^1 \int_y^1 \left[ye^{z/y}\right]_{z=0}^{z=xy} dx\,dy$

$= \int_0^1 \int_y^1 (ye^x - y)\,dx\,dy = \int_0^1 \left[ye^x - xy\right]_{x=y}^{x=1} dy = \int_0^1 \left(ey - y - ye^y + y^2\right) dy$

$= \left[\frac{1}{2}ey^2 - \frac{1}{2}y^2 - (y-1)e^y + \frac{1}{3}y^3\right]_0^1$ [integrate by parts]

$= \frac{1}{2}e - \frac{1}{2} + \frac{1}{3} - 1 = \frac{1}{2}e - \frac{7}{6}$

11. $\displaystyle\iiint_E \frac{z}{x^2+z^2}\,dV = \int_1^4 \int_y^4 \int_0^z \frac{z}{x^2+z^2}\,dx\,dz\,dy = \int_1^4 \int_y^4 \left[z\cdot\frac{1}{z}\tan^{-1}\frac{x}{z}\right]_{x=0}^{x=z} dz\,dy$

$= \int_1^4 \int_y^4 \left[\tan^{-1}(1) - \tan^{-1}(0)\right] dz\,dy = \int_1^4 \int_y^4 \left(\frac{\pi}{4} - 0\right) dz\,dy = \frac{\pi}{4}\int_1^4 \left[z\right]_{z=y}^{z=4} dy$

$= \frac{\pi}{4}\int_1^4 (4-y)\,dy = \frac{\pi}{4}\left[4y - \frac{1}{2}y^2\right]_1^4 = \frac{\pi}{4}\left(16 - 8 - 4 + \frac{1}{2}\right) = \frac{9\pi}{8}$

12. Here $E = \{(x,y,z) \mid 0 \le x \le \pi, 0 \le y \le \pi - x, 0 \le z \le x\}$, so

$\iiint_E \sin y\,dV = \int_0^\pi \int_0^{\pi-x} \int_0^x \sin y\,dz\,dy\,dx = \int_0^\pi \int_0^{\pi-x} \left[z\sin y\right]_{z=0}^{z=x} dy\,dx = \int_0^\pi \int_0^{\pi-x} x\sin y\,dy\,dx$

$= \int_0^\pi \left[-x\cos y\right]_{y=0}^{y=\pi-x} dx = \int_0^\pi \left[-x\cos(\pi-x) + x\right] dx$

$= \left[x\sin(\pi-x) - \cos(\pi-x) + \frac{1}{2}x^2\right]_0^\pi$ [integrate by parts]

$= 0 - 1 + \frac{1}{2}\pi^2 - 0 - 1 - 0 = \frac{1}{2}\pi^2 - 2$

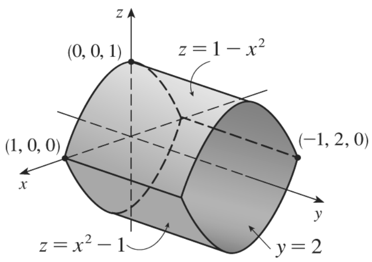
13. Here $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le 1 + x + y\}$, so

$\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} \left[6xyz\right]_{z=0}^{z=1+x+y} dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx = \int_0^1 \left[3xy^2 + 3x^2y^2 + 2xy^3\right]_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^1 (3x^2 + 3x^3 + 2x^{5/2})\,dx = \left[x^3 + \frac{3}{4}x^4 + \frac{4}{7}x^{7/2}\right]_0^1 = \frac{65}{28}$

14.

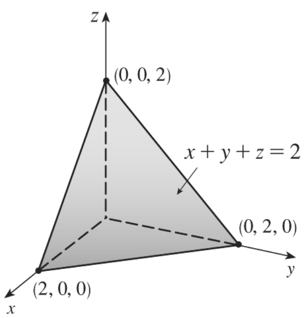


Here $E = \left\{(x, y, z) \mid -1 \leq x \leq 1,\ 0 \leq y \leq 2,\ x^2 - 1 \leq z \leq 1 - x^2\right\}$.

Thus,

$$\begin{aligned}\iiint_E (x-y)\,dV &= \int_{-1}^{1}\int_0^2\int_{x^2-1}^{1-x^2}(x-y)\,dz\,dy\,dx \\ &= \int_{-1}^{1}\int_0^2 (x-y)(1-x^2-(x^2-1))\,dy\,dx \\ &= \int_{-1}^{1}\int_0^2 (2x-2x^3-2y+2x^2y)\,dy\,dx \\ &= \int_{-1}^{1}\left[2xy-2x^3y-y^2+x^2y^2\right]_{y=0}^{y=2}\,dx \\ &= \int_{-1}^{1}(4x-4x^3-4+4x^2)\,dx \\ &= \left[2x^2-x^4-4x+\tfrac{4}{3}x^3\right]_{-1}^{1} = -\tfrac{5}{3}-\tfrac{11}{3} = -\tfrac{16}{3}\end{aligned}$$

15.

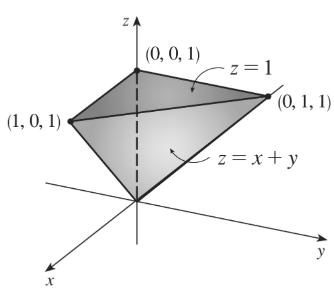


Here $T = \{(x, y, z) \mid 0 \leq x \leq 2,\ 0 \leq y \leq 2 - x,\ 0 \leq z \leq 2 - x - y\}$.

Thus,

$$\begin{aligned}\iiint_T y^2\,dV &= \int_0^2\int_0^{2-x}\int_0^{2-x-y} y^2\,dz\,dy\,dx \\ &= \int_0^2\int_0^{2-x} y^2(2-x-y)\,dy\,dx \\ &= \int_0^2\int_0^{2-x}\left[(2-x)y^2-y^3\right]dy\,dx \\ &= \int_0^2\left[(2-x)\left(\tfrac{1}{3}y^3\right)-\tfrac{1}{4}y^4\right]_{y=0}^{y=2-x}\,dx \\ &= \int_0^2\left[\tfrac{1}{3}(2-x)^4-\tfrac{1}{4}(2-x)^4\right]dx = \int_0^2 \tfrac{1}{12}(2-x)^4\,dx \\ &= \left[\tfrac{1}{12}\left(-\tfrac{1}{5}\right)(2-x)^5\right]_0^2 = -\tfrac{1}{60}(0-32) = \tfrac{8}{15}\end{aligned}$$

16.

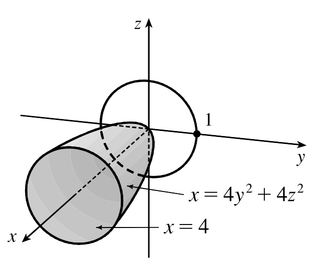


The projection of T onto the xz-plane is the triangle bounded by the lines $z = x$, $x = 0$, and $z = 1$. Then

$T = \{(x, y, z) \mid 0 \leq x \leq 1,\ x \leq z \leq 1,\ 0 \leq y \leq z - x\}$, and

$$\begin{aligned}\iiint_T xz\,dV &= \int_0^1\int_x^1\int_0^{z-x} xz\,dy\,dz\,dx = \int_0^1\int_x^1 xz(z-x)\,dz\,dx \\ &= \int_0^1\int_x^1 (xz^2-x^2z)\,dz\,dx = \int_0^1\left[\tfrac{1}{3}xz^3-\tfrac{1}{2}x^2z^2\right]_{z=x}^{z=1}\,dx \\ &= \int_0^1\left(\tfrac{1}{3}x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^4+\tfrac{1}{2}x^4\right)dx \\ &= \left[\tfrac{1}{6}x^2-\tfrac{1}{6}x^3+\tfrac{1}{30}x^5\right]_0^1 = \tfrac{1}{6}-\tfrac{1}{6}+\tfrac{1}{30} = \tfrac{1}{30}\end{aligned}$$

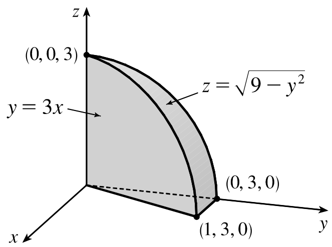
17.



The projection of E onto the yz-plane is the disk $y^2 + z^2 \leq 1$. Using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we get

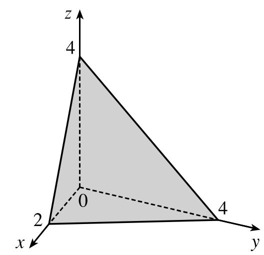
$$\begin{aligned}\iiint_E x\,dV &= \iint_D\left[\int_{4y^2+4z^2}^{4} x\,dx\right]dA = \tfrac{1}{2}\iint_D\left[4^2-(4y^2+4z^2)^2\right]dA \\ &= 8\int_0^{2\pi}\int_0^1 (1-r^4)\,r\,dr\,d\theta = 8\int_0^{2\pi}d\theta\int_0^1 (r-r^5)\,dr \\ &= 8(2\pi)\left[\tfrac{1}{2}r^2-\tfrac{1}{6}r^6\right]_0^1 = \tfrac{16\pi}{3}\end{aligned}$$

18.



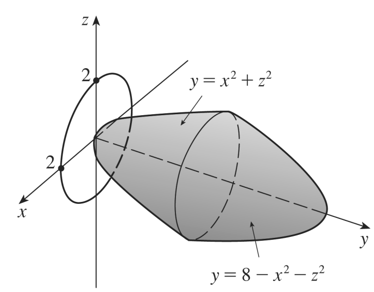
$$\begin{aligned}\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx &= \int_0^1\int_{3x}^3 \tfrac{1}{2}(9-y^2)\,dy\,dx\\ &= \int_0^1 \left[\tfrac{9}{2}y - \tfrac{1}{6}y^3\right]_{y=3x}^{y=3}\,dx\\ &= \int_0^1 \left[9 - \tfrac{27}{2}x + \tfrac{9}{2}x^3\right]dx\\ &= \left[9x - \tfrac{27}{4}x^2 + \tfrac{9}{8}x^4\right]_0^1 = \tfrac{27}{8}\end{aligned}$$

19. The plane $2x + y + z = 4$ intersects the xy-plane when $2x + y + 0 = 4 \quad\Rightarrow\quad y = 4 - 2x$, so $E = \{(x, y, z) \mid 0 \le x \le 2, 0 \le y \le 4 - 2x, 0 \le z \le 4 - 2x - y\}$ and



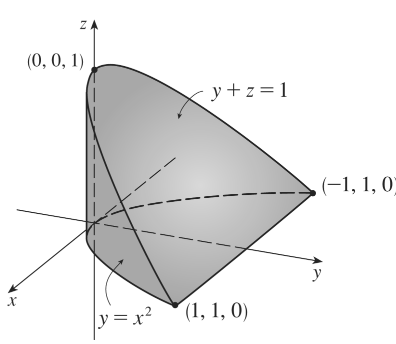
$$\begin{aligned}V &= \int_0^2\int_0^{4-2x}\int_0^{4-2x-y} dz\,dy\,dx = \int_0^2\int_0^{4-2x}(4-2x-y)\,dy\,dx\\ &= \int_0^2\left[4y - 2xy - \tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x}\,dx\\ &= \int_0^2\left[4(4-2x) - 2x(4-2x) - \tfrac{1}{2}(4-2x)^2\right]dx\\ &= \int_0^2(2x^2 - 8x + 8)\,dx = \left[\tfrac{2}{3}x^3 - 4x^2 + 8x\right]_0^2 = \tfrac{16}{3}\end{aligned}$$

20. The paraboloids intersect when $x^2 + z^2 = 8 - x^2 - z^2 \quad\Leftrightarrow\quad x^2 + z^2 = 4$, thus the intersection is the circle $x^2 + z^2 = 4$, $y = 4$. The projection of E onto the xz-plane is the disk $x^2 + z^2 \le 4$, so $E = \left\{(x, y, z) \mid x^2 + z^2 \le y \le 8 - x^2 - z^2, x^2 + z^2 \le 4\right\}$. Let $D = \left\{(x, z) \mid x^2 + z^2 \le 4\right\}$. Then using polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$, we have



$$\begin{aligned}V &= \iiint_E dV = \iint_D\left(\int_{x^2+z^2}^{8-x^2-z^2} dy\right)dA = \iint_D(8 - 2x^2 - 2z^2)\,dA\\ &= \int_0^{2\pi}\int_0^2(8-2r^2)\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^2(8r - 2r^3)\,dr\\ &= \left[\,\theta\,\right]_0^{2\pi}\left[4r^2 - \tfrac{1}{2}r^4\right]_0^2 = 2\pi(16-8) = 16\pi\end{aligned}$$

21. The plane $y + z = 1$ intersects the xy-plane in the line $y = 1$, so $E = \left\{(x, y, z) \mid -1 \le x \le 1, x^2 \le y \le 1, 0 \le z \le 1 - y\right\}$ and



$$\begin{aligned}V &= \iiint_E dV = \int_{-1}^1\int_{x^2}^1\int_0^{1-y} dz\,dy\,dx = \int_{-1}^1\int_{x^2}^1(1-y)\,dy\,dx\\ &= \int_{-1}^1\left[y - \tfrac{1}{2}y^2\right]_{y=x^2}^{y=1}\,dx = \int_{-1}^1\left(\tfrac{1}{2} - x^2 + \tfrac{1}{2}x^4\right)dx\\ &= \left[\tfrac{1}{2}x - \tfrac{1}{3}x^3 + \tfrac{1}{10}x^5\right]_{-1}^1 = \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{10} + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{10} = \tfrac{8}{15}\end{aligned}$$

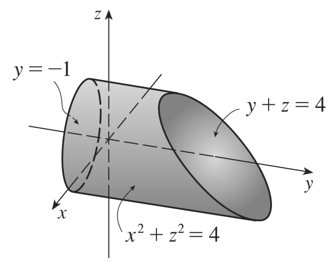
22. Here $E = \{(x,y,z) \mid -1 \le y \le 4 - z, x^2 + z^2 \le 4\}$, so

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-1}^{4-z} dy\,dz\,dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - z + 1)\,dz\,dx$$

$$= \int_{-2}^{2} \left[5z - \tfrac{1}{2}z^2\right]_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} dx = \int_{-2}^{2} 10\sqrt{4-x^2}\,dx$$

$$= 10\left[\tfrac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\tfrac{x}{2}\right)\right]_{-2}^{2} \quad \left[\begin{array}{l}\text{using trigonometric substitution or}\\ \text{Formula 30 in the Table of Integrals}\end{array}\right]$$

$$= 10\left[2\sin^{-1}(1) - 2\sin^{-1}(-1)\right] = 20\left(\tfrac{\pi}{2} - \left(-\tfrac{\pi}{2}\right)\right) = 20\pi$$



Alternatively, use polar coordinates to evaluate the double integral:

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (5 - z)\,dz\,dx = \int_{0}^{2\pi} \int_{0}^{2} (5 - r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{5}{2}r^2 - \tfrac{1}{3}r^3\sin\theta\right]_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} \left(10 - \tfrac{8}{3}\sin\theta\right) d\theta$$

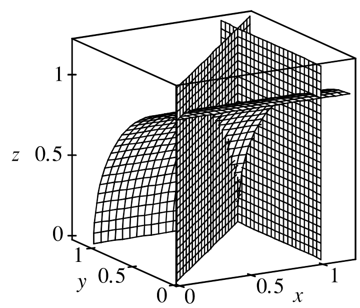
$$= 10\theta + \tfrac{8}{3}\cos\theta\Big]_0^{2\pi} = 20\pi$$

23. (a) The wedge can be described as the region

$$D = \{(x,y,z) \mid y^2 + z^2 \le 1, 0 \le x \le 1, 0 \le y \le x\}$$
$$= \left\{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le \sqrt{1-y^2}\right\}$$

So the integral expressing the volume of the wedge is

$$\iiint_D dV = \int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\,dy\,dx.$$



(b) A CAS gives $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\,dy\,dx = \frac{\pi}{4} - \frac{1}{3}$.

(Or use Formulas 30 and 87 from the Table of Integrals.)

24. (a) Divide B into 8 cubes of size $\Delta V = 8$. With $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, the Midpoint Rule gives

$$\iiint_B \sqrt{x^2+y^2+z^2}\,dV \approx \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f\left(\overline{x}_i, \overline{y}_j, \overline{z}_k\right)\Delta V$$

$$= 8[f(1,1,1) + f(1,1,3) + f(1,3,1) + f(1,3,3) + f(3,1,1)$$
$$+ f(3,1,3) + f(3,3,1) + f(3,3,3)]$$

$$\approx 239.64$$

(b) Using a CAS we have $\iiint_B \sqrt{x^2+y^2+z^2}\,dV = \int_0^4 \int_0^4 \int_0^4 \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx \approx 245.91$. This differs from the estimate in part (a) by about 2.5%.

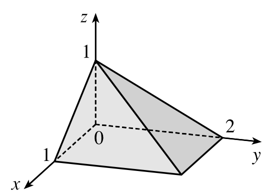
25. Here $f(x,y,z) = \cos(xyz)$ and $\Delta V = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &\approx \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(\overline{x}_i, \overline{y}_j, \overline{z}_k\right) \Delta V \\ &= \tfrac{1}{8}\left[f\left(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4}\right) + f\left(\tfrac{1}{4},\tfrac{1}{4},\tfrac{3}{4}\right) + f\left(\tfrac{1}{4},\tfrac{3}{4},\tfrac{1}{4}\right) + f\left(\tfrac{1}{4},\tfrac{3}{4},\tfrac{3}{4}\right)\right. \\ &\qquad \left. + f\left(\tfrac{3}{4},\tfrac{1}{4},\tfrac{1}{4}\right) + f\left(\tfrac{3}{4},\tfrac{1}{4},\tfrac{3}{4}\right) + f\left(\tfrac{3}{4},\tfrac{3}{4},\tfrac{1}{4}\right) + f\left(\tfrac{3}{4},\tfrac{3}{4},\tfrac{3}{4}\right)\right] \\ &= \tfrac{1}{8}\left[\cos\tfrac{1}{64} + \cos\tfrac{3}{64} + \cos\tfrac{3}{64} + \cos\tfrac{9}{64} + \cos\tfrac{3}{64} + \cos\tfrac{9}{64} + \cos\tfrac{9}{64} + \cos\tfrac{27}{64}\right] \approx 0.985\end{aligned}$$

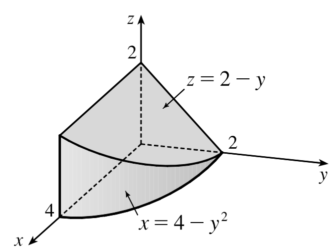
26. Here $f(x,y,z) = \sqrt{x}\, e^{xyz}$ and $\Delta V = 2 \cdot \frac{1}{2} \cdot 1 = 1$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &\approx \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(\overline{x}_i, \overline{y}_j, \overline{z}_k\right) \Delta V \\ &= 1\left[f\left(1,\tfrac{1}{4},\tfrac{1}{2}\right) + f\left(1,\tfrac{1}{4},\tfrac{3}{2}\right) + f\left(1,\tfrac{3}{4},\tfrac{1}{2}\right) + f\left(1,\tfrac{3}{4},\tfrac{3}{2}\right)\right. \\ &\qquad \left. + f\left(3,\tfrac{1}{4},\tfrac{1}{2}\right) + f\left(3,\tfrac{1}{4},\tfrac{3}{2}\right) + f\left(3,\tfrac{3}{4},\tfrac{1}{2}\right) + f\left(3,\tfrac{3}{4},\tfrac{3}{2}\right)\right] \\ &= e^{1/8} + e^{3/8} + e^{3/8} + e^{9/8} + \sqrt{3}e^{3/8} + \sqrt{3}e^{9/8} + \sqrt{3}e^{9/8} + \sqrt{3}e^{27/8} \approx 70.932\end{aligned}$$

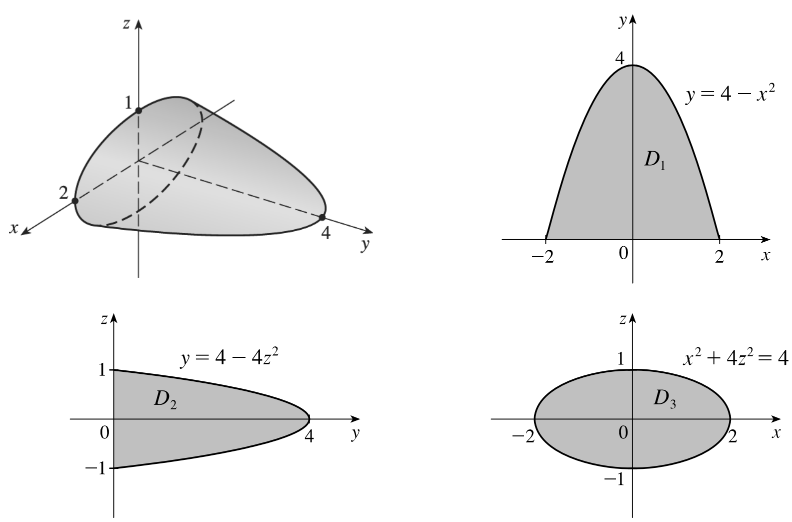
27. $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le 1 - x, 0 \le y \le 2 - 2z\}$, the solid bounded by the three coordinate planes and the planes $z = 1 - x$, $y = 2 - 2z$.



28. $E = \left\{(x,y,z) \mid 0 \le y \le 2, 0 \le z \le 2 - y, 0 \le x \le 4 - y^2\right\}$, the solid bounded by the three coordinate planes, the plane $z = 2 - y$, and the cylindrical surface $x = 4 - y^2$.



29.



[continued]

If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \left\{(x,y) \mid -2 \le x \le 2, 0 \le y \le 4 - x^2\right\} = \left\{(x,y) \mid 0 \le y \le 4,\ -\sqrt{4-y} \le x \le \sqrt{4-y}\,\right\}$$

$$D_2 = \left\{(y,z) \mid 0 \le y \le 4,\ -\tfrac{1}{2}\sqrt{4-y} \le z \le \tfrac{1}{2}\sqrt{4-y}\right\} = \left\{(y,z) \mid -1 \le z \le 1, 0 \le y \le 4 - 4z^2\right\}$$

$$D_3 = \left\{(x,z) \mid x^2 + 4z^2 \le 4\right\}$$

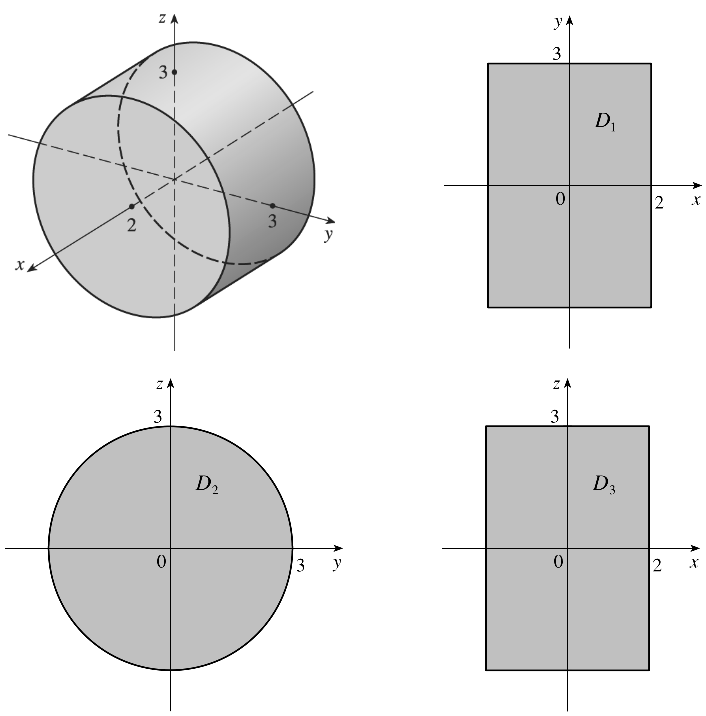
Therefore

$$\begin{aligned} E &= \left\{(x,y,z) \mid -2 \le x \le 2, 0 \le y \le 4 - x^2,\ -\tfrac{1}{2}\sqrt{4-x^2-y} \le z \le \tfrac{1}{2}\sqrt{4-x^2-y}\,\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 4,\ -\sqrt{4-y} \le x \le \sqrt{4-y},\ -\tfrac{1}{2}\sqrt{4-x^2-y} \le z \le \tfrac{1}{2}\sqrt{4-x^2-y}\,\right\} \\ &= \left\{(x,y,z) \mid -1 \le z \le 1, 0 \le y \le 4 - 4z^2,\ -\sqrt{4-y-4z^2} \le x \le \sqrt{4-y-4z^2}\,\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 4,\ -\tfrac{1}{2}\sqrt{4-y} \le z \le \tfrac{1}{2}\sqrt{4-y},\ -\sqrt{4-y-4z^2} \le x \le \sqrt{4-y-4z^2}\,\right\} \\ &= \left\{(x,y,z) \mid -2 \le x \le 2,\ -\tfrac{1}{2}\sqrt{4-x^2} \le z \le \tfrac{1}{2}\sqrt{4-x^2}, 0 \le y \le 4 - x^2 - 4z^2\right\} \\ &= \left\{(x,y,z) \mid -1 \le z \le 1,\ -\sqrt{4-4z^2} \le x \le \sqrt{4-4z^2}, 0 \le y \le 4 - x^2 - 4z^2\right\} \end{aligned}$$

Then

$$\begin{aligned} \iiint_E f(x,y,z)\,dV &= \int_{-2}^{2}\int_0^{4-x^2}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\,dz\,dy\,dx = \int_0^4\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\,dz\,dx\,dy \\ &= \int_{-1}^{1}\int_0^{4-4z^2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dy\,dz = \int_0^4\int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dz\,dy \\ &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_0^{4-x^2-4z^2} f(x,y,z)\,dy\,dz\,dx = \int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}}\int_0^{4-x^2-4z^2} f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

30.



If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x,y) \mid -2 \le x \le 2,\ -3 \le y \le 3\}$$
$$D_2 = \left\{(y,z) \mid y^2 + z^2 \le 9\right\}$$
$$D_3 = \{(x,z) \mid -2 \le x \le 2,\ -3 \le z \le 3\}$$

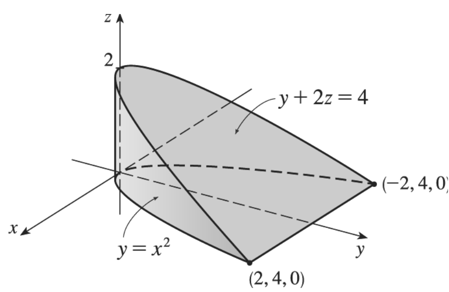
Therefore

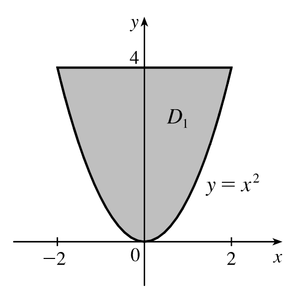
$$\begin{aligned} E &= \left\{(x,y,z) \mid -2 \le x \le 2,\ -3 \le y \le 3,\ -\sqrt{9-y^2} \le z \le \sqrt{9-y^2}\right\} \\ &= \left\{(x,y,z) \mid -3 \le y \le 3,\ -\sqrt{9-y^2} \le z \le \sqrt{9-y^2},\ -2 \le x \le 2\right\} \\ &= \left\{(x,y,z) \mid -3 \le z \le 3,\ -\sqrt{9-z^2} \le y \le \sqrt{9-z^2},\ -2 \le x \le 2\right\} \\ &= \left\{(x,y,z) \mid -2 \le x \le 2,\ -3 \le z \le 3,\ -\sqrt{9-z^2} \le y \le \sqrt{9-z^2}\right\} \end{aligned}$$

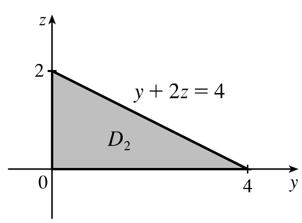
and

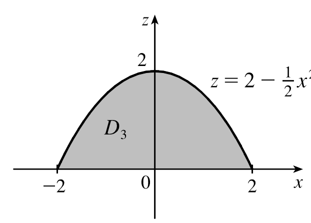
$$\begin{aligned} \iiint_E f(x,y,z)\,dV &= \int_{-2}^{2}\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)\,dz\,dy\,dx = \int_{-3}^{3}\int_{-2}^{2}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)\,dz\,dx\,dy \\ &= \int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{-2}^{2} f(x,y,z)\,dx\,dz\,dy = \int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}}\int_{-2}^{2} f(x,y,z)\,dx\,dy\,dz \\ &= \int_{-2}^{2}\int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dy\,dz\,dx = \int_{-3}^{3}\int_{-2}^{2}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

31.









If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

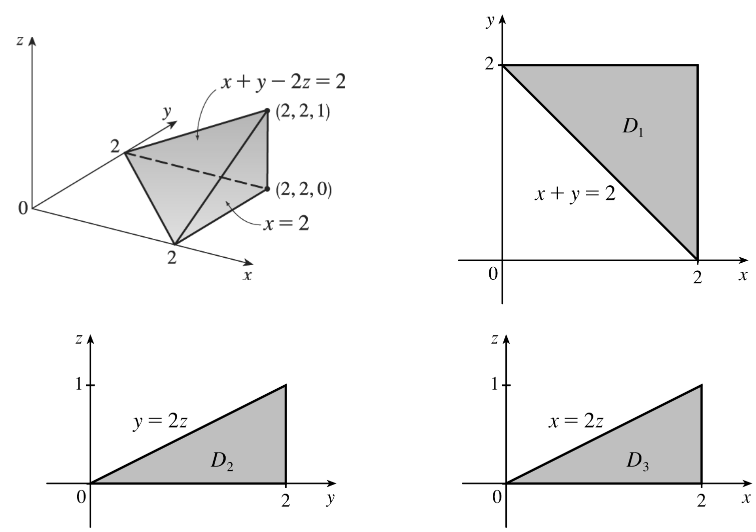
$$D_1 = \left\{(x,y) \mid -2 \le x \le 2, x^2 \le y \le 4\right\} = \left\{(x,y) \mid 0 \le y \le 4, -\sqrt{y} \le x \le \sqrt{y}\right\},$$
$$D_2 = \left\{(y,z) \mid 0 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y\right\} = \left\{(y,z) \mid 0 \le z \le 2, 0 \le y \le 4 - 2z\right\}, \text{ and}$$
$$D_3 = \left\{(x,z) \mid -2 \le x \le 2, 0 \le z \le 2 - \tfrac{1}{2}x^2\right\} = \left\{(x,z) \mid 0 \le z \le 2, -\sqrt{4-2z} \le x \le \sqrt{4-2z}\right\}$$

[continued]

Therefore
$$\begin{aligned}E &= \left\{(x,y,z) \mid -2 \le x \le 2, x^2 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y\right\}\\
&= \left\{(x,y,z) \mid 0 \le y \le 4, -\sqrt{y} \le x \le \sqrt{y}, 0 \le z \le 2 - \tfrac{1}{2}y\right\}\\
&= \left\{(x,y,z) \mid 0 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y, -\sqrt{y} \le x \le \sqrt{y}\right\}\\
&= \left\{(x,y,z) \mid 0 \le z \le 2, 0 \le y \le 4 - 2z, -\sqrt{y} \le x \le \sqrt{y}\right\}\\
&= \left\{(x,y,z) \mid -2 \le x \le 2, 0 \le z \le 2 - \tfrac{1}{2}x^2, x^2 \le y \le 4 - 2z\right\}\\
&= \left\{(x,y,z) \mid 0 \le z \le 2, -\sqrt{4-2z} \le x \le \sqrt{4-2z}, x^2 \le y \le 4 - 2z\right\}\end{aligned}$$

Then
$$\begin{aligned}\iiint_E f(x,y,z)\,dV &= \int_{-2}^{2}\int_{x^2}^{4}\int_0^{2-y/2} f(x,y,z)\,dz\,dy\,dx = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{2-y/2} f(x,y,z)\,dz\,dx\,dy\\
&= \int_0^4\int_0^{2-y/2}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dz\,dy = \int_0^2\int_0^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz\\
&= \int_{-2}^{2}\int_0^{2-x^2/2}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dz\,dx = \int_0^2\int_{-\sqrt{4-2z}}^{\sqrt{4-2z}}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dx\,dz\end{aligned}$$

32.



If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x,y) \mid 0 \le x \le 2, 2 - x \le y \le 2\} = \{(x,y) \mid 0 \le y \le 2, 2 - y \le x \le 2\},$$

$$D_2 = \left\{(y,z) \mid 0 \le y \le 2, 0 \le z \le \tfrac{1}{2}y\right\} = \{(y,z) \mid 0 \le z \le 1, 2z \le y \le 2\}, \text{ and}$$

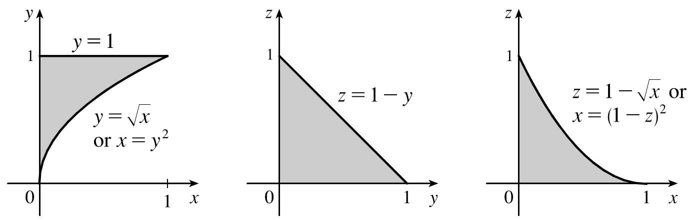
$$D_3 = \left\{(x,z) \mid 0 \le x \le 2, 0 \le z \le \tfrac{1}{2}x\right\} = \{(x,z) \mid 0 \le z \le 1, 2z \le x \le 2\}$$

Therefore
$$\begin{aligned}E &= \left\{(x,y,z) \mid 0 \le x \le 2, 2 - x \le y \le 2, 0 \le z \le \tfrac{1}{2}(x + y - 2)\right\}\\
&= \left\{(x,y,z) \mid 0 \le y \le 2, 2 - y \le x \le 2, 0 \le z \le \tfrac{1}{2}(x + y - 2)\right\}\\
&= \left\{(x,y,z) \mid 0 \le y \le 2, 0 \le z \le \tfrac{1}{2}y, 2 - y + 2z \le x \le 2\right\}\\
&= \{(x,y,z) \mid 0 \le z \le 1, 2z \le y \le 2, 2 - y + 2z \le x \le 2\}\\
&= \left\{(x,y,z) \mid 0 \le x \le 2, 0 \le z \le \tfrac{1}{2}x, 2 - x + 2z \le y \le 2\right\}\\
&= \{(x,y,z) \mid 0 \le z \le 1, 2z \le x \le 2, 2 - x + 2z \le y \le 2\}\end{aligned}$$

Then $\iiint_E f(x,y,z)\,dV = \int_0^2 \int_{2-x}^2 \int_0^{(x+y-2)/2} f(x,y,z)\,dz\,dy\,dx = \int_0^2 \int_{2-y}^2 \int_0^{(x+y-2)/2} f(x,y,z)\,dz\,dx\,dy$

$= \int_0^2 \int_0^{y/2} \int_{2-y+2z}^2 f(x,y,z)\,dx\,dz\,dy = \int_0^1 \int_{2z}^2 \int_{2-y+2z}^2 f(x,y,z)\,dx\,dy\,dz$

$= \int_0^2 \int_0^{x/2} \int_{2-x+2z}^2 f(x,y,z)\,dy\,dz\,dx = \int_0^1 \int_{2z}^2 \int_{2-x+2z}^2 f(x,y,z)\,dy\,dx\,dz$

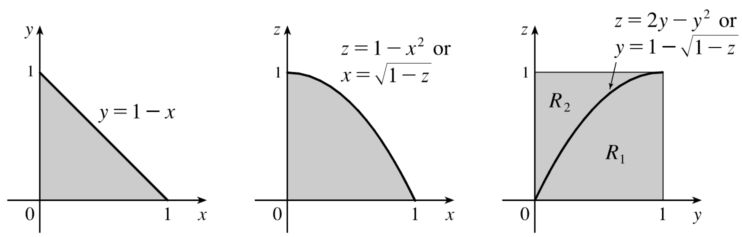
33. 

The diagrams show the projections of E onto the xy-, yz-, and xz-planes. Therefore

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\,dz\,dx\,dy = \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\,dx\,dz\,dy = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$$

34. 

The projections of E onto the xy- and xz-planes are as in the first two diagrams and so

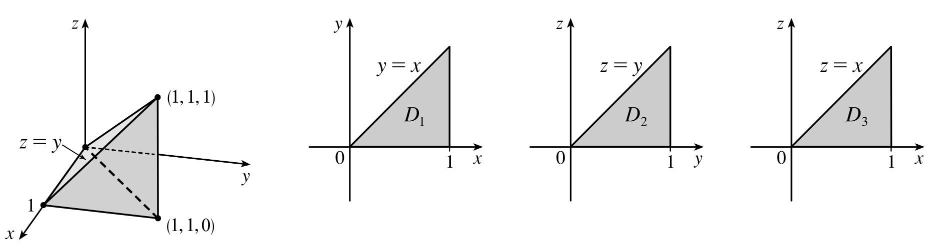
$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx = \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$

$$= \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy = \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx$$

Now the surface $z = 1 - x^2$ intersects the plane $y = 1 - x$ in a curve whose projection in the yz-plane is $z = 1 - (1-y)^2$ or $z = 2y - y^2$. So we must split up the projection of E on the yz-plane into two regions as in the third diagram. For (y,z) in R_1, $0 \le x \le 1 - y$ and for (y,z) in R_2, $0 \le x \le \sqrt{1-z}$, and so the given integral is also equal to

$$\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{2y-y^2} \int_0^{1-y} f(x,y,z)\,dx\,dz\,dy + \int_0^1 \int_{2y-y^2}^1 \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy.$$

35. 

$\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le z \le y,\ y \le x \le 1,\ 0 \le y \le 1\}$.

[continued]

If D_1, D_2, and D_3 are the projections of E onto the xy-, yz- and xz-planes then

$$D_1 = \{(x,y) \mid 0 \le y \le 1, y \le x \le 1\} = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\},$$

$$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}, \text{ and}$$

$$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le x\} = \{(x,z) \mid 0 \le z \le 1, z \le x \le 1\}.$$

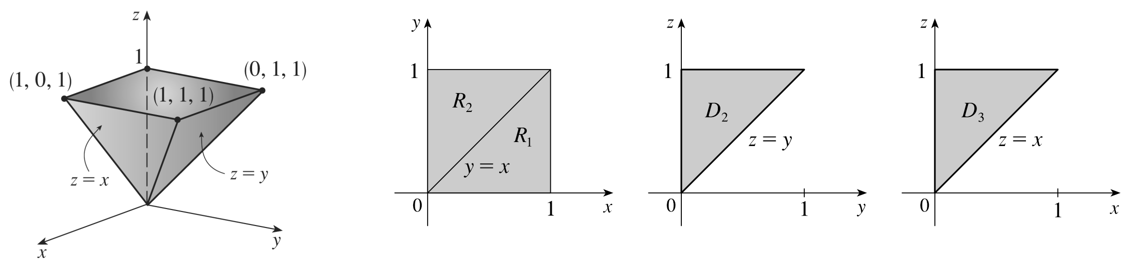
Thus we also have

$$\begin{aligned} E &= \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le y\} = \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, y \le x \le 1\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, z \le y \le 1, y \le x \le 1\} = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le x, z \le y \le x\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, z \le x \le 1, z \le y \le x\}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy &= \int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^y \int_y^1 f(x,y,z)\,dx\,dz\,dy \\ &= \int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^x \int_z^x f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1 \int_z^1 \int_z^x f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

36.



$\int_0^1 \int_y^1 \int_0^z f(x,y,z)\,dx\,dz\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le x \le z, y \le z \le 1, 0 \le y \le 1\}$.

Notice that E is bounded below by two different surfaces, so we must split the projection of E onto the xy-plane into two regions as in the second diagram. If D_1, D_2, and D_3 are the projections of E on the xy-, yz- and xz-planes then

$$\begin{aligned} D_1 = R_1 \cup R_2 &= \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\} \cup \{(x,y) \mid 0 \le x \le 1, x \le y \le 1\} \\ &= \{(x,y) \mid 0 \le y \le 1, y \le x \le 1\} \cup \{(x,y) \mid 0 \le y \le 1, 0 \le x \le y\}, \end{aligned}$$

$$D_2 = \{(y,z) \mid 0 \le y \le 1, y \le z \le 1\} = \{(y,z) \mid 0 \le z \le 1, 0 \le y \le z\}, \text{ and}$$

$$D_3 = \{(x,z) \mid 0 \le x \le 1, x \le z \le 1\} = \{(x,z) \mid 0 \le z \le 1, 0 \le x \le z\}.$$

Thus we also have

$$\begin{aligned} E &= \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, x \le z \le 1\} \cup \{(x,y,z) \mid 0 \le x \le 1, x \le y \le 1, y \le z \le 1\} \\ &= \{(x,y,z) \mid 0 \le y \le 1, y \le x \le 1, x \le z \le 1\} \cup \{(x,y,z) \mid 0 \le y \le 1, 0 \le x \le y, y \le z \le 1\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, 0 \le y \le z, 0 \le x \le z\} = \{(x,y,z) \mid 0 \le x \le 1, x \le z \le 1, 0 \le y \le z\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, 0 \le x \le z, 0 \le y \le z\}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 \int_y^1 \int_0^z f(x,y,z)\,dx\,dz\,dy &= \int_0^1 \int_0^x \int_x^1 f(x,y,z)\,dz\,dy\,dx + \int_0^1 \int_x^1 \int_y^1 f(x,y,z)\,dz\,dy\,dx \\ &= \int_0^1 \int_y^1 \int_x^1 f(x,y,z)\,dz\,dx\,dy + \int_0^1 \int_0^y \int_y^1 f(x,y,z)\,dz\,dx\,dy \\ &= \int_0^1 \int_0^z \int_0^z f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_x^1 \int_0^z f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1 \int_0^z \int_0^z f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

37. The region C is the solid bounded by a circular cylinder of radius 2 with axis the z-axis for $-2 \le z \le 2$. We can write $\iiint_C (4 + 5x^2yz^2)\,dV = \iiint_C 4\,dV + \iiint_C 5x^2yz^2\,dV$, but $f(x,y,z) = 5x^2yz^2$ is an odd function with respect to y. Since C is symmetrical about the xz-plane, we have $\iiint_C 5x^2yz^2\,dV = 0$. Thus

$\iiint_C (4 + 5x^2yz^2)\,dV = \iiint_C 4\,dV = 4 \cdot V(E) = 4 \cdot \pi(2)^2(4) = 64\pi$.

38. We can write $\iiint_B (z^3 + \sin y + 3)\,dV = \iiint_B z^3\,dV + \iiint_B \sin y\,dV + \iiint_B 3\,dV$. But z^3 is an odd function with respect to z and the region B is symmetric about the xy-plane, so $\iiint_B z^3\,dV = 0$. Similarly, $\sin y$ is an odd function with respect to y and B is symmetric about the xz-plane, so $\iiint_B \sin y\,dV = 0$. Thus

$\iiint_B (z^3 + \sin y + 3)\,dV = \iiint_B 3\,dV = 3 \cdot V(B) = 3 \cdot \frac{4}{3}\pi(1)^3 = 4\pi$.

39. The projection of E onto the xy-plane is the disk $D = \{(x,y) \mid x^2 + y^2 \le 1\}$.

$$\begin{aligned} m &= \iiint_E \rho(x,y,z)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} 3\,dz\right] dA = \iint_D 3(1 - x^2 - y^2)\,dA \\ &= 3\int_0^1 \int_0^{2\pi} (1 - r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi} d\theta \int_0^1 (r - r^3)\,dr \\ &= 3\left[\,\theta\,\right]_0^{2\pi} \left[\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right]_0^1 = 3\,(2\pi)\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{3}{2}\pi \end{aligned}$$

$$\begin{aligned} M_{yz} &= \iiint_E x\rho(x,y,z)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} 3x\,dz\right] dA = \iint_D 3x(1 - x^2 - y^2)\,dA \\ &= 3\int_0^1 \int_0^{2\pi} (r\cos\theta)(1 - r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi} \cos\theta\,d\theta \int_0^1 (r^2 - r^4)\,dr \\ &= 3\left[\,\sin\theta\,\right]_0^{2\pi} \left[\tfrac{1}{3}r^3 - \tfrac{1}{5}r^5\right]_0^1 = 3\,(0)\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = 0 \end{aligned}$$

$$\begin{aligned} M_{xz} &= \iiint_E y\rho(x,y,z)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} 3y\,dz\right] dA = \iint_D 3y(1 - x^2 - y^2)\,dA \\ &= 3\int_0^1 \int_0^{2\pi} (r\sin\theta)(1 - r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi} \sin\theta\,d\theta \int_0^1 (r^2 - r^4)\,dr \\ &= 3\left[\,-\cos\theta\,\right]_0^{2\pi} \left[\tfrac{1}{3}r^3 - \tfrac{1}{5}r^5\right]_0^1 = 3\,(0)\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = 0 \end{aligned}$$

$$\begin{aligned} M_{xy} &= \iiint_E z\rho(x,y,z)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} 3z\,dz\right] dA = \iint_D \left[\tfrac{3}{2}z^2\right]_{z=0}^{z=1-x^2-y^2} dA \\ &= \tfrac{3}{2}\iint_D (1 - x^2 - y^2)^2\,dA = \tfrac{3}{2}\int_0^1 \int_0^{2\pi} (1 - r^2)^2\,r\,dr\,d\theta \\ &= \tfrac{3}{2}\int_0^{2\pi} d\theta \int_0^1 (r - 2r^3 + r^5)\,dr = \tfrac{3}{2}\left[\,\theta\,\right]_0^{2\pi} \left[\tfrac{1}{2}r^2 - \tfrac{1}{2}r^4 + \tfrac{1}{6}r^6\right]_0^1 \\ &= \tfrac{3}{2}\,(2\pi)\left(\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{6}\right) = \tfrac{1}{2}\pi \end{aligned}$$

Thus the mass is $\frac{3}{2}\pi$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(0, 0, \dfrac{1}{3}\right)$.

40. $m = \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4\,dx\,dz\,dy = 4\int_{-1}^1 \int_0^{1-y^2} (1 - z)\,dz\,dy = 4\int_{-1}^1 \left[z - \frac{1}{2}z^2\right]_{z=0}^{z=1-y^2} dy = 2\int_{-1}^1 (1 - y^4)\,dy = \frac{16}{5}$,

$$\begin{aligned} M_{yz} &= \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4x\,dx\,dz\,dy = 2\int_{-1}^1 \int_0^{1-y^2} (1 - z)^2\,dz\,dy = 2\int_{-1}^1 \left[-\tfrac{1}{3}(1 - z)^3\right]_{z=0}^{z=1-y^2} dy \\ &= \tfrac{2}{3}\int_{-1}^1 (1 - y^6)\,dy = \left(\tfrac{4}{3}\right)\left(\tfrac{6}{7}\right) = \tfrac{24}{21} \end{aligned}$$

[continued]

$M_{xz} = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} 4y\,dx\,dz\,dy = \int_{-1}^{1}\int_{0}^{1-y^2} 4y(1-z)\,dz\,dy$

$= \int_{-1}^{1}\left[4y(1-y^2) - 2y(1-y^2)^2\right]dy = \int_{-1}^{1}(2y - 2y^5)\,dy = 0$ [the integrand is odd]

$M_{xy} = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} 4z\,dx\,dz\,dy = \int_{-1}^{1}\int_{0}^{1-y^2}(4z-4z^2)\,dz\,dy = 2\int_{-1}^{1}\left[(1-y^2)^2 - \frac{2}{3}(1-y^2)^3\right]dy$

$= 2\int_{-1}^{1}\left[\frac{1}{3} - y^4 + \frac{2}{3}y^6\right]dy = \left[\frac{4}{3}y - \frac{4}{5}y^5 + \frac{8}{21}y^7\right]_0^1 = \frac{96}{105} = \frac{32}{35}$

Thus, $(\overline{x},\overline{y},\overline{z}) = \left(\frac{5}{14}, 0, \frac{2}{7}\right)$

41. $m = \int_0^a\int_0^a\int_0^a (x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^a\int_0^a\left[\frac{1}{3}x^3 + xy^2 + xz^2\right]_{x=0}^{x=a}dy\,dz = \int_0^a\int_0^a\left(\frac{1}{3}a^3 + ay^2 + az^2\right)dy\,dz$

$= \int_0^a\left[\frac{1}{3}a^3y + \frac{1}{3}ay^3 + ayz^2\right]_{y=0}^{y=a}dz = \int_0^a\left(\frac{2}{3}a^4 + a^2z^2\right)dz = \left[\frac{2}{3}a^4z + \frac{1}{3}a^2z^3\right]_0^a = \frac{2}{3}a^5 + \frac{1}{3}a^5 = a^5$

$M_{yz} = \int_0^a\int_0^a\int_0^a\left[x^3 + x(y^2+z^2)\right]dx\,dy\,dz = \int_0^a\int_0^a\left[\frac{1}{4}a^4 + \frac{1}{2}a^2(y^2+z^2)\right]dy\,dz$

$= \int_0^a\left(\frac{1}{4}a^5 + \frac{1}{6}a^5 + \frac{1}{2}a^3z^2\right)dz = \frac{1}{4}a^6 + \frac{1}{3}a^6 = \frac{7}{12}a^6 = M_{xz} = M_{xy}$ by symmetry of E and $\rho(x,y,z)$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(\frac{7}{12}a, \frac{7}{12}a, \frac{7}{12}a\right)$.

42. $m = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y - y^2\right]dy\,dx$

$= \int_0^1\left[\frac{1}{2}(1-x)^3 - \frac{1}{3}(1-x)^3\right]dx = \frac{1}{6}\int_0^1(1-x)^3\,dx = \frac{1}{24}$

$M_{yz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xy\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(x-x^2)y - xy^2\right]dy\,dx$

$= \int_0^1\left[\frac{1}{2}x(1-x)^3 - \frac{1}{3}x(1-x)^3\right]dx = \frac{1}{6}\int_0^1\left(x - 3x^2 + 3x^3 - x^4\right)dx = \frac{1}{6}\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right) = \frac{1}{120}$

$M_{xz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y^2\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y^2 - y^3\right]dy\,dx$

$= \int_0^1\left[\frac{1}{3}(1-x)^4 - \frac{1}{4}(1-x)^4\right]dx = \frac{1}{12}\left[-\frac{1}{5}(1-x)^5\right]_0^1 = \frac{1}{60}$

$M_{xy} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} yz\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[\frac{1}{2}y(1-x-y)^2\right]dy\,dx$

$= \frac{1}{2}\int_0^1\int_0^{1-x}\left[(1-x)^2y - 2(1-x)y^2 + y^3\right]dy\,dx = \frac{1}{2}\int_0^1\left[\frac{1}{2}(1-x)^4 - \frac{2}{3}(1-x)^4 + \frac{1}{4}(1-x)^4\right]dx$

$= \frac{1}{24}\int_0^1(1-x)^4\,dx = -\frac{1}{24}\left[\frac{1}{5}(1-x)^5\right]_0^1 = \frac{1}{120}$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)$.

43. $I_x = \int_0^L\int_0^L\int_0^L k(y^2+z^2)\,dz\,dy\,dx = k\int_0^L\int_0^L\left(Ly^2 + \frac{1}{3}L^3\right)dy\,dx = k\int_0^L \frac{2}{3}L^4\,dx = \frac{2}{3}kL^5$

By symmetry, $I_x = I_y = I_z = \frac{2}{3}kL^5$.

44. $I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} k(y^2+z^2)\,dx\,dy\,dz = ka\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}(y^2+z^2)\,dy\,dz$

$= ak\int_{-c/2}^{c/2}\left[\frac{1}{3}y^3 + z^2y\right]_{y=-b/2}^{y=b/2}dz = ak\int_{-c/2}^{c/2}\left(\frac{1}{12}b^3 + bz^2\right)dz = ak\left[\frac{1}{12}b^3z + \frac{1}{3}bz^3\right]_{-c/2}^{c/2}$

$= ak\left(\frac{1}{12}b^3c + \frac{1}{12}bc^3\right) = \frac{1}{12}kabc(b^2+c^2)$

By symmetry, $I_y = \frac{1}{12}kabc(a^2+c^2)$ and $I_z = \frac{1}{12}kabc(a^2+b^2)$.

45. $I_z = \iiint_E (x^2+y^2)\,\rho(x,y,z)\,dV = \iint\limits_{x^2+y^2\le a^2} \left[\int_0^h k(x^2+y^2)\,dz\right] dA = \iint\limits_{x^2+y^2\le a^2} k(x^2+y^2)h\,dA$

$= kh\int_0^{2\pi}\int_0^a (r^2)\,r\,dr\,d\theta = kh\int_0^{2\pi} d\theta\,\int_0^a r^3\,dr = kh(2\pi)\left[\frac{1}{4}r^4\right]_0^a = 2\pi kh\cdot\frac{1}{4}a^4 = \frac{1}{2}\pi kha^4$

46. $I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV = \iint\limits_{x^2+y^2\le h^2}\left[\int_{\sqrt{x^2+y^2}}^h k(x^2+y^2)\,dz\right] dA$

$= \iint\limits_{x^2+y^2\le h^2} k(x^2+y^2)\left(h-\sqrt{x^2+y^2}\right) dA = k\int_0^{2\pi}\int_0^h r^2(h-r)\,r\,dr\,d\theta$

$= k\int_0^{2\pi} d\theta\,\int_0^h\left(r^3h-r^4\right) dr = k(2\pi)\left[\frac{1}{4}r^4h-\frac{1}{5}r^5\right]_0^h = 2\pi k\left(\frac{1}{4}h^5-\frac{1}{5}h^5\right) = \frac{1}{10}\pi kh^5$

47. (a) $m = \int_{-1}^1\int_{x^2}^1\int_0^{1-y}\sqrt{x^2+y^2}\,dz\,dy\,dx$

(b) $(\overline{x},\overline{y},\overline{z})$ where $\overline{x} = \frac{1}{m}\int_{-1}^1\int_{x^2}^1\int_0^{1-y} x\sqrt{x^2+y^2}\,dz\,dy\,dx$, $\overline{y} = \frac{1}{m}\int_{-1}^1\int_{x^2}^1\int_0^{1-y} y\sqrt{x^2+y^2}\,dz\,dy\,dx$, and

$\overline{z} = \frac{1}{m}\int_{-1}^1\int_{x^2}^1\int_0^{1-y} z\sqrt{x^2+y^2}\,dz\,dy\,dx$.

(c) $I_z = \int_{-1}^1\int_{x^2}^1\int_0^{1-y}(x^2+y^2)\sqrt{x^2+y^2}\,dz\,dy\,dx = \int_{-1}^1\int_{x^2}^1\int_0^{1-y}(x^2+y^2)^{3/2}\,dz\,dy\,dx$

48. (a) $m = \int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$

(b) $(\overline{x},\overline{y},\overline{z})$ where $\overline{x} = m^{-1}\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} x\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$,

$\overline{y} = m^{-1}\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$,

$\overline{z} = m^{-1}\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$

(c) $I_z = \int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}}(x^2+y^2)(1+x+y+z)\,dz\,dx\,dy$

49. (a) $m = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y(1+x+y+z)\,dz\,dy\,dx = \frac{3\pi}{32}+\frac{11}{24}$

(b) $(\overline{x},\overline{y},\overline{z}) = \Big(m^{-1}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y x(1+x+y+z)\,dz\,dy\,dx,$

$m^{-1}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y y(1+x+y+z)\,dz\,dy\,dx,$

$m^{-1}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y z(1+x+y+z)\,dz\,dy\,dx\Big)$

$$= \left(\frac{28}{9\pi+44}, \frac{30\pi+128}{45\pi+220}, \frac{45\pi+208}{135\pi+660}\right)$$

(c) $$I_z = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y(x^2+y^2)(1+x+y+z)\,dz\,dy\,dx = \frac{68+15\pi}{240}$$

50. (a) $m = \int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}}(x^2+y^2)\,dz\,dy\,dx = \frac{56}{5} = 11.2$

(b) $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} x(x^2+y^2)\, dz\, dy\, dx \approx 0.375$,

$\overline{y} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} y(x^2+y^2)\, dz\, dy\, dx = \frac{45\pi}{64} \approx 2.209$,

$\overline{z} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z(x^2+y^2)\, dz\, dy\, dx = \frac{15}{16} = 0.9375$.

(c) $I_z = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2+y^2)^2\, dz\, dy\, dx = \frac{10{,}464}{175} \approx 59.79$

51. (a) $f(x,y,z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x,y,z)\, dV = 1$. Here we have

$$\begin{aligned}\iiint_{\mathbb{R}^3} f(x,y,z)\, dV &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\, dz\, dy\, dx = \int_0^2\int_0^2\int_0^2 Cxyz\, dz\, dy\, dx \\ &= C\int_0^2 x\, dx \int_0^2 y\, dy \int_0^2 z\, dz = C\left[\tfrac{1}{2}x^2\right]_0^2 \left[\tfrac{1}{2}y^2\right]_0^2 \left[\tfrac{1}{2}z^2\right]_0^2 = 8C\end{aligned}$$

Then we must have $8C = 1 \quad \Rightarrow \quad C = \frac{1}{8}$.

(b) $$\begin{aligned}P(X \le 1, Y \le 1, Z \le 1) &= \int_{-\infty}^1\int_{-\infty}^1\int_{-\infty}^1 f(x,y,z)\, dz\, dy\, dx = \int_0^1\int_0^1\int_0^1 \tfrac{1}{8}xyz\, dz\, dy\, dx \\ &= \tfrac{1}{8}\int_0^1 x\, dx \int_0^1 y\, dy \int_0^1 z\, dz = \tfrac{1}{8}\left[\tfrac{1}{2}x^2\right]_0^1 \left[\tfrac{1}{2}y^2\right]_0^1 \left[\tfrac{1}{2}z^2\right]_0^1 = \tfrac{1}{8}\left(\tfrac{1}{2}\right)^3 = \tfrac{1}{64}\end{aligned}$$

(c) $P(X+Y+Z \le 1) = P((X,Y,Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x+y+z=1$. The plane $x+y+z=1$ meets the xy-plane in the line $x+y=1$, so we have

$$\begin{aligned}P(X+Y+Z \le 1) &= \iiint_E f(x,y,z)\, dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} \tfrac{1}{8}xyz\, dz\, dy\, dx \\ &= \tfrac{1}{8}\int_0^1\int_0^{1-x} xy\left[\tfrac{1}{2}z^2\right]_{z=0}^{z=1-x-y} dy\, dx = \tfrac{1}{16}\int_0^1\int_0^{1-x} xy(1-x-y)^2\, dy\, dx \\ &= \tfrac{1}{16}\int_0^1\int_0^{1-x} [(x^3-2x^2+x)y + (2x^2-2x)y^2 + xy^3]\, dy\, dx \\ &= \tfrac{1}{16}\int_0^1 \left[(x^3-2x^2+x)\tfrac{1}{2}y^2 + (2x^2-2x)\tfrac{1}{3}y^3 + x\left(\tfrac{1}{4}y^4\right)\right]_{y=0}^{y=1-x} dx \\ &= \tfrac{1}{192}\int_0^1 (x - 4x^2 + 6x^3 - 4x^4 + x^5)\, dx = \tfrac{1}{192}\left(\tfrac{1}{30}\right) = \tfrac{1}{5760}\end{aligned}$$

52. (a) $f(x,y,z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x,y,z)\, dV = 1$. Here we have

$$\begin{aligned}\iiint_{\mathbb{R}^3} f(x,y,z)\, dV &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\, dz\, dy\, dx = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} Ce^{-(0.5x+0.2y+0.1z)}\, dz\, dy\, dx \\ &= C\int_0^{\infty} e^{-0.5x}\, dx \int_0^{\infty} e^{-0.2y}\, dy \int_0^{\infty} e^{-0.1z}\, dz \\ &= C\lim_{t\to\infty}\int_0^t e^{-0.5x}\, dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\, dy \lim_{t\to\infty}\int_0^t e^{-0.1z}\, dz \\ &= C\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t \lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \\ &= C\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right] \lim_{t\to\infty}\left[-10(e^{-0.1t}-1)\right] \\ &= C\cdot(-2)(0-1)\cdot(-5)(0-1)\cdot(-10)(0-1) = 100C\end{aligned}$$

So we must have $100C = 1 \quad \Rightarrow \quad C = \frac{1}{100}$.

(b) We have no restriction on Z, so

$$\begin{aligned}P(X \le 1, Y \le 1) &= \int_{-\infty}^1\int_{-\infty}^1\int_{-\infty}^{\infty} f(x,y,z)\, dz\, dy\, dx = \int_0^1\int_0^1\int_0^{\infty} \tfrac{1}{100}e^{-(0.5x+0.2y+0.1z)}\, dz\, dy\, dx \\ &= \tfrac{1}{100}\int_0^1 e^{-0.5x}\, dx \int_0^1 e^{-0.2y}\, dy \int_0^{\infty} e^{-0.1z}\, dz \\ &= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1 \left[-5e^{-0.2y}\right]_0^1 \lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \qquad \text{[by part (a)]} \\ &= \tfrac{1}{100}(2-2e^{-0.5})(5-5e^{-0.2})(10) = (1-e^{-0.5})(1-e^{-0.2}) \approx 0.07132\end{aligned}$$

(c) $P(X \le 1, Y \le 1, Z \le 1) = \int_{-\infty}^{1} \int_{-\infty}^{1} \int_{-\infty}^{1} f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^1 \int_0^1 \frac{1}{100} e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx$

$$= \tfrac{1}{100} \int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^1 e^{-0.1z}\,dz$$
$$= \tfrac{1}{100} \left[-2e^{-0.5x}\right]_0^1 \left[-5e^{-0.2y}\right]_0^1 \left[-10e^{-0.1z}\right]_0^1$$
$$= (1-e^{-0.5})(1-e^{-0.2})(1-e^{-0.1}) \approx 0.006787$$

53. $V(E) = L^3 \quad \Rightarrow \quad f_{\text{ave}} = \dfrac{1}{L^3} \displaystyle\int_0^L \int_0^L \int_0^L xyz\,dx\,dy\,dz = \dfrac{1}{L^3} \int_0^L x\,dx \int_0^L y\,dy \int_0^L z\,dz$

$$= \frac{1}{L^3} \left[\frac{x^2}{2}\right]_0^L \left[\frac{y^2}{2}\right]_0^L \left[\frac{z^2}{2}\right]_0^L = \frac{1}{L^3} \frac{L^2}{2} \frac{L^2}{2} \frac{L^2}{2} = \frac{L^3}{8}$$

54. The height of each point is given by its z-coordinate, so the average height of the points in $E = \left\{(x,y,z) \mid x^2+y^2+z^2 \le 1,\ z \ge 0\right\}$ is

$$\frac{1}{V(E)} \iiint_E z\,dV$$

Here $V(E) = \frac{1}{2} \cdot \frac{4}{3}\pi(1)^3 = \frac{2}{3}\pi$ [half the volume of a sphere], so

$$\tfrac{1}{V(E)} \iiint_E z\,dV = \tfrac{1}{2\pi/3} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} z\,dz\,dy\,dx = \tfrac{3}{2\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\tfrac{1}{2}z^2\right]_{z=0}^{z=\sqrt{1-x^2-y^2}} dy\,dx$$
$$= \tfrac{3}{2\pi} \cdot \tfrac{1}{2} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx = \tfrac{3}{4\pi} \int_0^{2\pi} \int_0^1 (1-r^2)\,r\,dr\,d\theta$$
$$= \tfrac{3}{4\pi} \int_0^{2\pi} d\theta \int_0^1 (r-r^3)\,dr = \tfrac{3}{4\pi}(2\pi)\left[\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right]_0^1 = \tfrac{3}{2}\left(\tfrac{1}{4}\right) = \tfrac{3}{8}$$

55. (a) The triple integral will attain its maximum when the integrand $1-x^2-2y^2-3z^2$ is positive in the region E and negative everywhere else. For if E contains some region F where the integrand is negative, the integral could be increased by excluding F from E, and if E fails to contain some part G of the region where the integrand is positive, the integral could be increased by including G in E. So we require that $x^2+2y^2+3z^2 \le 1$. This describes the region bounded by the ellipsoid $x^2+2y^2+3z^2=1$.

(b) The maximum value of $\iiint_E (1-x^2-2y^2-3z^2)\,dV$ occurs when E is the solid region bounded by the ellipsoid $x^2+2y^2+3z^2=1$. The projection of E on the xy-plane is the planar region bounded by the ellipse $x^2+2y^2=1$, so

$$E = \left\{(x,y,z) \mid -1 \le x \le 1, -\sqrt{\tfrac{1}{2}(1-x^2)} \le y \le \sqrt{\tfrac{1}{2}(1-x^2)}, -\sqrt{\tfrac{1}{3}(1-x^2-2y^2)} \le z \le \sqrt{\tfrac{1}{3}(1-x^2-2y^2)}\right\}$$

and

$$\iiint_E (1-x^2-2y^2-3z^2)\,dV = \int_{-1}^{1} \int_{-\sqrt{\frac{1}{2}(1-x^2)}}^{\sqrt{\frac{1}{2}(1-x^2)}} \int_{-\sqrt{\frac{1}{3}(1-x^2-2y^2)}}^{\sqrt{\frac{1}{3}(1-x^2-2y^2)}} (1-x^2-2y^2-3z^2)\,dz\,dy\,dx = \frac{4\sqrt{6}}{45}\pi$$

using a CAS.

DISCOVERY PROJECT Volumes of Hyperspheres

In this project we use V_n to denote the n-dimensional volume of an n-dimensional hypersphere.

1. The interior of the circle is the set of points $\left\{(x,y) \mid -r \le y \le r,\ -\sqrt{r^2-y^2} \le x \le \sqrt{r^2-y^2}\right\}$. So, substituting $y = r\sin\theta$ and then using Formula 64 to evaluate the integral, we get

$$V_2 = \int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx\,dy = \int_{-r}^{r} 2\sqrt{r^2-y^2}\,dy = \int_{-\pi/2}^{\pi/2} 2r\sqrt{1-\sin^2\theta}\,(r\cos\theta\,d\theta)$$

$$= 2r^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = 2r^2\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2r^2\left(\tfrac{\pi}{2}\right) = \pi r^2$$

2. The region of integration is
$\left\{(x,y,z) \mid -r \le z \le r, -\sqrt{r^2-z^2} \le y \le \sqrt{r^2-z^2}, -\sqrt{r^2-z^2-y^2} \le x \le \sqrt{r^2-z^2-y^2}\right\}$. Substituting $y = \sqrt{r^2-z^2}\sin\theta$ and using Formula 64 to integrate $\cos^2\theta$, we get

$$V_3 = \int_{-r}^{r}\int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}}\int_{-\sqrt{r^2-z^2-y^2}}^{\sqrt{r^2-z^2-y^2}} dx\,dy\,dz = \int_{-r}^{r}\int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}} 2\sqrt{r^2-z^2-y^2}\,dy\,dz$$

$$= \int_{-r}^{r}\int_{-\pi/2}^{\pi/2} 2\sqrt{r^2-z^2}\sqrt{1-\sin^2\theta}\left(\sqrt{r^2-z^2}\cos\theta\,d\theta\right)dz$$

$$= 2\left[\int_{-r}^{r}(r^2-z^2)\,dz\right]\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta\right] = 2\left(\frac{4r^3}{3}\right)\left(\frac{\pi}{2}\right) = \frac{4\pi r^3}{3}$$

3. Here we substitute $y = \sqrt{r^2-w^2-z^2}\sin\theta$ and, later, $w = r\sin\phi$. Because $\int_{-\pi/2}^{\pi/2}\cos^p\theta\,d\theta$ seems to occur frequently in these calculations, it is useful to find a general formula for that integral. From Exercises 7.1.49-50, we have

$$\int_0^{\pi/2}\sin^{2k}x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2k}\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\pi/2}\sin^{2k+1}x\,dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot 2k}{1\cdot 3\cdot 5\cdot\cdots\cdot(2k+1)}$$

and from the symmetry of the sine and cosine functions, we can conclude that

$$\int_{-\pi/2}^{\pi/2}\cos^{2k}x\,dx = 2\int_0^{\pi/2}\sin^{2k}x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)\pi}{2\cdot 4\cdot 6\cdot\cdots\cdot 2k} \tag{1}$$

$$\int_{-\pi/2}^{\pi/2}\cos^{2k+1}x\,dx = 2\int_0^{\pi/2}\sin^{2k+1}x\,dx = \frac{2\cdot 2\cdot 4\cdot 6\cdot\cdots\cdot 2k}{1\cdot 3\cdot 5\cdot\cdots\cdot(2k+1)} \tag{2}$$

Thus $$V_4 = \int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}}\int_{-\sqrt{r^2-w^2-z^2-y^2}}^{\sqrt{r^2-w^2-z^2-y^2}} dx\,dy\,dz\,dw$$

$$= 2\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}}\sqrt{r^2-w^2-z^2-y^2}\,dy\,dz\,dw$$

$$= 2\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\pi/2}^{\pi/2}(r^2-w^2-z^2)\cos^2\theta\,d\theta\,dz\,dw$$

$$= 2\left[\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}(r^2-w^2-z^2)\,dz\,dw\right]\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta\right]$$

$$= 2\left(\tfrac{\pi}{2}\right)\left[\int_{-r}^{r}\tfrac{4}{3}(r^2-w^2)^{3/2}\,dw\right] = \pi\left(\tfrac{4}{3}\right)\int_{-\pi/2}^{\pi/2} r^4\cos^4\phi\,d\phi = \frac{4\pi}{3}r^4\cdot\frac{1\cdot 3\cdot\pi}{2\cdot 4} = \frac{\pi^2 r^4}{2}$$

4. By using the substitutions $x_i = \sqrt{r^2 - x_n^2 - x_{n-1}^2 - \cdots - x_{i+1}^2}\cos\theta_i$ and then applying Formulas 1 and 2 from Problem 3, we can write

$$V_n = \int_{-r}^{r}\int_{-\sqrt{r^2-x_n^2}}^{\sqrt{r^2-x_n^2}}\cdots\int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}\int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}} dx_1\,dx_2\cdots dx_{n-1}\,dx_n$$

$$= 2\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta_2\,d\theta_2\right]\left[\int_{-\pi/2}^{\pi/2}\cos^3\theta_3\,d\theta_3\right]\cdots\left[\int_{-\pi/2}^{\pi/2}\cos^{n-1}\theta_{n-1}\,d\theta_{n-1}\right]\left[\int_{-\pi/2}^{\pi/2}\cos^{n}\theta_{n}\,d\theta_{n}\right]r^n$$

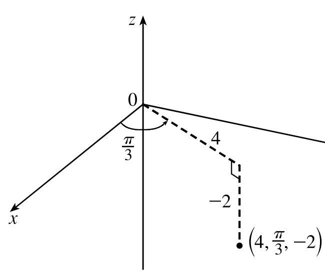
$$= \begin{cases} \left[2\cdot\dfrac{\pi}{2}\right]\left[\dfrac{2\cdot 2}{1\cdot 3}\cdot\dfrac{1\cdot 3\pi}{2\cdot 4}\right]\left[\dfrac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\cdot\dfrac{1\cdot 3\cdot 5\pi}{2\cdot 4\cdot 6}\right]\cdots\left[\dfrac{2\cdot\cdots\cdot(n-2)}{1\cdot\cdots\cdot(n-1)}\cdot\dfrac{1\cdot\cdots\cdot(n-1)\,\pi}{2\cdot\cdots\cdot n}\right]r^n & n\text{ even} \\ 2\left[\dfrac{\pi}{2}\cdot\dfrac{2\cdot 2}{1\cdot 3}\right]\left[\dfrac{1\cdot 3\pi}{2\cdot 4}\cdot\dfrac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\right]\cdots\left[\dfrac{1\cdot\cdots\cdot(n-2)\,\pi}{2\cdot\cdots\cdot(n-1)}\cdot\dfrac{2\cdot\cdots\cdot(n-1)}{1\cdot\cdots\cdot n}\right]r^n & n\text{ odd}\end{cases}$$

By canceling within each set of brackets, we find that

$$V_n = \begin{cases} \dfrac{2\pi}{2}\cdot\dfrac{2\pi}{4}\cdot\dfrac{2\pi}{6}\cdot\cdots\cdot\dfrac{2\pi}{n}r^n = \dfrac{(2\pi)^{n/2}}{2\cdot 4\cdot 6\cdot\cdots\cdot n}r^n = \dfrac{\pi^{n/2}}{\left(\frac{1}{2}n\right)!}r^n & n\text{ even} \\ 2\cdot\dfrac{2\pi}{3}\cdot\dfrac{2\pi}{5}\cdot\dfrac{2\pi}{7}\cdot\cdots\cdot\dfrac{2\pi}{n}r^n = \dfrac{2(2\pi)^{(n-1)/2}}{3\cdot 5\cdot 7\cdot\cdots\cdot n}r^n = \dfrac{2^n\left[\frac{1}{2}(n-1)\right]!\,\pi^{(n-1)/2}}{n!}r^n & n\text{ odd}\end{cases}$$

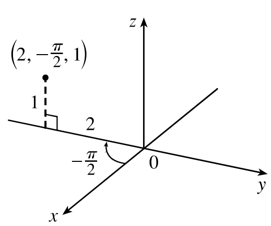
15.7 Triple Integrals in Cylindrical Coordinates

1. (a)



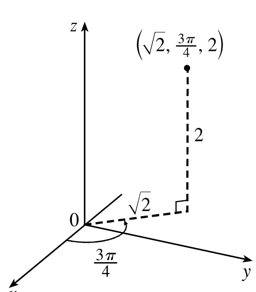
From Equations 1, $x = r\cos\theta = 4\cos\dfrac{\pi}{3} = 4\cdot\frac{1}{2} = 2$, $y = r\sin\theta = 4\sin\dfrac{\pi}{3} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$, $z = -2$, so the point is $\left(2, 2\sqrt{3}, -2\right)$ in rectangular coordinates.

(b)



$x = 2\cos\left(-\frac{\pi}{2}\right) = 0$, $y = 2\sin\left(-\frac{\pi}{2}\right) = -2$, and $z = 1$, so the point is $(0, -2, 1)$ in rectangular coordinates.

2. (a)



$x = \sqrt{2}\cos\dfrac{3\pi}{4} = \sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = -1$,

$y = \sqrt{2}\sin\dfrac{3\pi}{4} = \sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = 1$, and $z = 2$,

so the point is $(-1, 1, 2)$ in rectangular coordinates.