16.7 Surface Integrals

1. The box is a cube where each face has surface area 4. The centers of the faces are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. For each face we take the point P^*_{ij} to be the center of the face and $f(x, y, z) = \cos(x + 2y + 3z)$, so by Definition 1,

$$\begin{aligned}\iint_S f(x,y,z)\,dS &\approx [f(1,0,0)](4) + [f(-1,0,0)](4) + [f(0,1,0)](4) \\ &\quad + [f(0,-1,0)](4) + [f(0,0,1)](4) + [f(0,0,-1)](4) \\ &= 4\,[\cos 1 + \cos(-1) + \cos 2 + \cos(-2) + \cos 3 + \cos(-3)] \approx -6.93\end{aligned}$$

2. Each quarter-cylinder has surface area $\frac{1}{4}[2\pi(1)(2)] = \pi$, and the top and bottom disks have surface area $\pi(1)^2 = \pi$. We can take $(0, 0, 1)$ as a sample point in the top disk, $(0, 0, -1)$ in the bottom disk, and $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ in the four quarter-cylinders. Then $\iint_S f(x, y, z)\,dS$ can be approximated by the Riemann sum

$$\begin{aligned}&f(1,0,0)(\pi) + f(-1,0,0)(\pi) + f(0,1,0)\,(\pi) + f(0,-1,0)(\pi) + f(0,0,1)(\pi) + f(0,0,-1)(\pi) \\ &\qquad = (2 + 2 + 3 + 3 + 4 + 4)\pi = 18\pi \approx 56.5.\end{aligned}$$

3. We can use the xz- and yz-planes to divide H into four patches of equal size, each with surface area equal to $\frac{1}{8}$ the surface area of a sphere with radius $\sqrt{50}$, so $\Delta S = \frac{1}{8}(4)\pi\left(\sqrt{50}\right)^2 = 25\pi$. Then $(\pm 3, \pm 4, 5)$ are sample points in the four patches, and using a Riemann sum as in Definition 1, we have

$$\begin{aligned}\iint_H f(x,y,z)\,dS &\approx f(3,4,5)\,\Delta S + f(3,-4,5)\,\Delta S + f(-3,4,5)\,\Delta S + f(-3,-4,5)\,\Delta S \\ &= (7 + 8 + 9 + 12)(25\pi) = 900\pi \approx 2827\end{aligned}$$

4. On the surface, $f(x, y, z) = g\left(\sqrt{x^2 + y^2 + z^2}\right) = g(2) = -5$. So since the area of a sphere is $4\pi r^2$,

$\iint_S f(x,y,z)\,dS = \iint_S g(2)\,dS = -5\iint_S dS = -5[4\pi(2)^2] = -80\pi.$

5. $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (u-v)\,\mathbf{j} + (1 + 2u + v)\,\mathbf{k}$, $0 \le u \le 2$, $0 \le v \le 1$ and

$\mathbf{r}_u \times \mathbf{r}_v = (\mathbf{i} + \mathbf{j} + 2\,\mathbf{k}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k} \quad \Rightarrow \quad |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}$. Then by Formula 2,

$$\begin{aligned}\iint_S (x+y+z)\,dS &= \iint_D (u + v + u - v + 1 + 2u + v)\ |\mathbf{r}_u \times \mathbf{r}_v|\ dA = \int_0^1 \int_0^2 (4u + v + 1)\cdot\sqrt{14}\,du\,dv \\ &= \sqrt{14}\int_0^1 \left[2u^2 + uv + u\right]_{u=0}^{u=2}\,dv = \sqrt{14}\int_0^1 (2v + 10)\,dv = \sqrt{14}\left[v^2 + 10v\right]_0^1 = 11\sqrt{14}\end{aligned}$$

6. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi/2$ and

$\mathbf{r}_u \times \mathbf{r}_v = (\cos v\,\mathbf{i} + \sin v\,\mathbf{j} + \mathbf{k}) \times (-u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}) = -u\cos v\,\mathbf{i} - u\sin v\,\mathbf{j} + u\,\mathbf{k} \quad \Rightarrow$

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2u^2} = \sqrt{2}\,u$ [since $u \ge 0$]. Then by Formula 2,

$$\begin{aligned}\iint_S xyz\,dS &= \iint_D (u\cos v)(u\sin v)(u)\ |\mathbf{r}_u \times \mathbf{r}_v|\ dA = \int_0^1 \int_0^{\pi/2} (u^3 \sin v\cos v)\cdot\sqrt{2}\,u\,dv\,du \\ &= \sqrt{2}\int_0^1 u^4\,du\,\int_0^{\pi/2}\sin v\cos v\,dv = \sqrt{2}\left[\tfrac{1}{5}u^5\right]_0^1\ \left[\tfrac{1}{2}\sin^2 v\right]_0^{\pi/2} = \sqrt{2}\cdot\tfrac{1}{5}\cdot\tfrac{1}{2} = \tfrac{1}{10}\sqrt{2}\end{aligned}$$

7. $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle,\ 0 \le u \le 1,\ 0 \le v \le \pi$ and

$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle = \langle \sin v, -\cos v, u\rangle \quad \Rightarrow$

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{u^2+1}$. Then

$$\iint_S y\,dS = \iint_D (u\sin v)\ |\mathbf{r}_u \times \mathbf{r}_v|\ dA = \int_0^1 \int_0^\pi (u \sin v)\cdot\sqrt{u^2+1}\,dv\,du = \int_0^1 u\sqrt{u^2+1}\,du\ \int_0^\pi \sin v\,dv$$
$$= \left[\tfrac{1}{3}(u^2+1)^{3/2}\right]_0^1\ [-\cos v]_0^\pi = \tfrac{1}{3}(2^{3/2}-1)\cdot 2 = \tfrac{2}{3}(2\sqrt{2}-1)$$

8. $\mathbf{r}(u,v) = \langle 2uv, u^2-v^2, u^2+v^2\rangle$, $u^2+v^2 \le 1$ and

$\mathbf{r}_u \times \mathbf{r}_v = \langle 2v, 2u, 2u\rangle \times \langle 2u, -2v, 2v\rangle = \langle 8uv, 4u^2-4v^2, -4u^2-4v^2\rangle$, so

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{(8uv)^2 + (4u^2-4v^2)^2 + (-4u^2-4v^2)^2} = \sqrt{64u^2v^2 + 32u^4 + 32v^4}$$
$$= \sqrt{32(u^2+v^2)^2} = 4\sqrt{2}\,(u^2+v^2)$$

Then

$$\iint_S (x^2+y^2)\,dS = \iint_D \left[(2uv)^2 + (u^2-v^2)^2\right]\ |\mathbf{r}_u \times \mathbf{r}_v|\ dA = \iint_D (4u^2v^2 + u^4 - 2u^2v^2 + v^4)\cdot 4\sqrt{2}\,(u^2+v^2)\,dA$$
$$= 4\sqrt{2}\iint_D (u^4 + 2u^2v^2 + v^4)\,(u^2+v^2)\,dA = 4\sqrt{2}\iint_D (u^2+v^2)^3\,dA = 4\sqrt{2}\int_0^{2\pi}\int_0^1 (r^2)^3\,r\,dr\,d\theta$$
$$= 4\sqrt{2}\int_0^{2\pi} d\theta\ \int_0^1 r^7\,dr = 4\sqrt{2}\ \left[\,\theta\,\right]_0^{2\pi}\ \left[\tfrac{1}{8}r^8\right]_0^1 = 4\sqrt{2}\cdot 2\pi\cdot\tfrac{1}{8} = \sqrt{2}\,\pi$$

9. $z = 1+2x+3y$ so $\dfrac{\partial z}{\partial x} = 2$ and $\dfrac{\partial z}{\partial y} = 3$. Then by Formula 4,

$$\iint_S x^2yz\,dS = \iint_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA = \int_0^3\int_0^2 x^2y(1+2x+3y)\,\sqrt{4+9+1}\ dy\,dx$$
$$= \sqrt{14}\int_0^3\int_0^2 (x^2y + 2x^3y + 3x^2y^2)\,dy\,dx = \sqrt{14}\int_0^3 \left[\tfrac{1}{2}x^2y^2 + x^3y^2 + x^2y^3\right]_{y=0}^{y=2}\ dx$$
$$= \sqrt{14}\int_0^3 (10x^2 + 4x^3)\,dx = \sqrt{14}\left[\tfrac{10}{3}x^3 + x^4\right]_0^3 = 171\sqrt{14}$$

10. S is the part of the plane $z = 4-2x-2y$ over the region $D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 2-x\}$. Thus

$$\iint_S\ xz\,dS = \iint_D\ x(4-2x-2y)\ \sqrt{(-2)^2+(-2)^2+1}\ dA = 3\int_0^2\int_0^{2-x} \left(4x - 2x^2 - 2xy\right)dy\,dx$$
$$= 3\int_0^2 \left[4xy - 2x^2y - xy^2\right]_{y=0}^{y=2-x}\ dx = 3\int_0^2 \left[4x(2-x) - 2x^2(2-x) - x(2-x)^2\right]dx$$
$$= 3\int_0^2 \left(x^3 - 4x^2 + 4x\right)dx = 3\left[\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 + 2x^2\right]_0^2 = 3\left(4 - \tfrac{32}{3} + 8\right) = 4$$

11. An equation of the plane through the points $(1,0,0)$, $(0,-2,0)$, and $(0,0,4)$ is $4x-2y+z=4$, so S is the region in the plane $z = 4-4x+2y$ over $D = \{(x,y) \mid 0 \le x \le 1,\ 2x-2 \le y \le 0\}$. Thus by Formula 4,

$$\iint_S\ x\,dS = \iint_D\ x\ \sqrt{(-4)^2+(2)^2+1}\,dA = \sqrt{21}\ \int_0^1\int_{2x-2}^0\ x\,dy\,dx = \sqrt{21}\ \int_0^1\ [xy]_{y=2x-2}^{y=0}\ dx$$
$$= \sqrt{21}\ \int_0^1 (-2x^2+2x)\,dx = \sqrt{21}\left[-\tfrac{2}{3}x^3 + x^2\right]_0^1 = \sqrt{21}\left(-\tfrac{2}{3}+1\right) = \tfrac{\sqrt{21}}{3}$$

12. $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ and

$$\iint_S y\,dS = \iint_D y\sqrt{\left(\sqrt{x}\,\right)^2 + \left(\sqrt{y}\,\right)^2 + 1}\ dA = \int_0^1\int_0^1 y\,\sqrt{x+y+1}\ dx\,dy$$
$$= \int_0^1 y\left[\tfrac{2}{3}(x+y+1)^{3/2}\right]_{x=0}^{x=1} dy = \int_0^1 \tfrac{2}{3}y\left[(y+2)^{3/2} - (y+1)^{3/2}\right]dy$$

[continued]

Substituting $u = y + 2$ in the first term and $t = y + 1$ in the second, we have

$$\iint_S y\,dS = \tfrac{2}{3}\int_2^3 (u-2)u^{3/2}\,du - \tfrac{2}{3}\int_1^2 (t-1)t^{3/2}\,dt = \tfrac{2}{3}\left[\tfrac{2}{7}u^{7/2} - \tfrac{4}{5}u^{5/2}\right]_2^3 - \tfrac{2}{3}\left[\tfrac{2}{7}t^{7/2} - \tfrac{2}{5}t^{5/2}\right]_1^2$$

$$= \tfrac{2}{3}\left[\tfrac{2}{7}(3^{7/2} - 2^{7/2}) - \tfrac{4}{5}(3^{5/2} - 2^{5/2}) - \tfrac{2}{7}(2^{7/2} - 1) + \tfrac{2}{5}(2^{5/2} - 1)\right]$$

$$= \tfrac{2}{3}\left(\tfrac{18}{35}\sqrt{3} + \tfrac{8}{35}\sqrt{2} - \tfrac{4}{35}\right) = \tfrac{4}{105}\left(9\sqrt{3} + 4\sqrt{2} - 2\right)$$

13. Using y and z as parameters, we have $\mathbf{r}(y,z) = (y^2+z^2)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},\ y^2+z^2 \le 1$. Then

$\mathbf{r}_y \times \mathbf{r}_z = (2y\,\mathbf{i} + \mathbf{j}) \times (2z\,\mathbf{i} + \mathbf{k}) = \mathbf{i} - 2y\,\mathbf{j} - 2z\,\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = \sqrt{1+4y^2+4z^2} = \sqrt{1+4(y^2+z^2)}$. Thus

$$\iint_S z^2\,dS = \iint_{y^2+z^2\le 1} z^2\sqrt{1+4(y^2+z^2)}\,dA = \int_0^{2\pi}\int_0^1 (r\sin\theta)^2\sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\sin^2\theta\,d\theta\int_0^1 r^3\sqrt{1+4r^2}\,dr \qquad \left[\text{let } u = 1+4r^2 \ \Rightarrow\ r^2 = \tfrac{1}{4}(u-1) \text{ and } r\,dr = \tfrac{1}{8}du\right]$$

$$= \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\int_1^5 \tfrac{1}{4}(u-1)\sqrt{u}\cdot\tfrac{1}{8}du = \pi\cdot\tfrac{1}{32}\int_1^5 (u^{3/2} - u^{1/2})\,du = \tfrac{1}{32}\pi\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^5$$

$$= \tfrac{1}{32}\pi\left[\tfrac{2}{5}(5)^{5/2} - \tfrac{2}{3}(5)^{3/2} - \tfrac{2}{5} + \tfrac{2}{3}\right] = \tfrac{1}{32}\pi\left(\tfrac{20}{3}\sqrt{5} + \tfrac{4}{15}\right) = \tfrac{1}{120}\pi\left(25\sqrt{5} + 1\right)$$

14. Using x and z as parameters, we have $\mathbf{r}(x,z) = x\,\mathbf{i} + \sqrt{x^2+z^2}\,\mathbf{j} + z\,\mathbf{k},\ x^2+z^2 \le 25$. Then

$$\mathbf{r}_x \times \mathbf{r}_z = \left(\mathbf{i} + \frac{x}{\sqrt{x^2+z^2}}\,\mathbf{j}\right) \times \left(\frac{z}{\sqrt{x^2+z^2}}\,\mathbf{j} + \mathbf{k}\right) = \frac{x}{\sqrt{x^2+z^2}}\,\mathbf{i} - \mathbf{j} + \frac{z}{\sqrt{x^2+z^2}}\,\mathbf{k} \text{ and}$$

$$|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{\frac{x^2}{x^2+z^2} + 1 + \frac{z^2}{x^2+z^2}} = \sqrt{\frac{x^2+z^2}{x^2+z^2} + 1} = \sqrt{2}. \text{ Thus}$$

$$\iint_S y^2z^2\,dS = \iint_{x^2+z^2\le 25} (x^2+z^2)z^2\sqrt{2}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^5 r^2(r\sin\theta)^2\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^5 r^5\,dr = \sqrt{2}\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{6}r^6\right]_0^5$$

$$= \sqrt{2}\,(\pi)\cdot\tfrac{1}{6}(15{,}625 - 0) = \frac{15{,}625\sqrt{2}}{6}\pi$$

15. Using x and z as parameters, we have $\mathbf{r}(x,z) = x\,\mathbf{i} + (x^2+4z)\,\mathbf{j} + z\,\mathbf{k},\ 0 \le x \le 1, 0 \le z \le 1$. Then

$\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 2x\,\mathbf{j}) \times (4\,\mathbf{j} + \mathbf{k}) = 2x\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+1+16} = \sqrt{4x^2+17}$. Thus

$$\iint_S x\,dS = \int_0^1\int_0^1 x\sqrt{4x^2+17}\,dz\,dx = \int_0^1 x\sqrt{4x^2+17}\,dx = \left[\tfrac{1}{8}\cdot\tfrac{2}{3}(4x^2+17)^{3/2}\right]_0^1$$

$$= \tfrac{1}{12}(21^{3/2} - 17^{3/2}) = \tfrac{1}{12}\left(21\sqrt{21} - 17\sqrt{17}\right) = \tfrac{7}{4}\sqrt{21} - \tfrac{17}{12}\sqrt{17}$$

16. The sphere intersects the cone in the circle $x^2+y^2 = \frac{1}{2}$, $z = \frac{1}{\sqrt{2}}$, so S is the portion of the sphere where $z \ge \frac{1}{\sqrt{2}}$.

Using spherical coordinates to parametrize the sphere we have $\mathbf{r}(\phi,\theta) = \sin\phi\,\cos\theta\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, and

$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$ (as in Example 1). The portion where $z \ge \frac{1}{\sqrt{2}}$ corresponds to $0 \le \phi \le \frac{\pi}{4}$, $0 \le \theta \le 2\pi$ so

$$\iint_S y^2\,dS = \int_0^{2\pi}\int_0^{\pi/4}(\sin\phi\sin\theta)^2(\sin\phi)\,d\phi\,d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta\int_0^{\pi/4}\sin^3\phi\,d\phi = \int_0^{2\pi}\sin^2\theta\,d\theta\int_0^{\pi/4}(1-\cos^2\phi)\sin\phi\,d\phi$$

$$= \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi/4} = \pi\left(\tfrac{\sqrt{2}}{12} - \tfrac{\sqrt{2}}{2} - \tfrac{1}{3} + 1\right) = \left(\tfrac{2}{3} - \tfrac{5\sqrt{2}}{12}\right)\pi$$

17. Using spherical coordinates to parametrize the sphere we have $\mathbf{r}(\phi,\theta) = 2\sin\phi\cos\theta\,\mathbf{i} + 2\sin\phi\sin\theta\,\mathbf{j} + 2\cos\phi\,\mathbf{k}$ and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 4\sin\phi$ (see Example 16.6.10). Here S is the portion of the sphere corresponding to $0 \le \phi \le \pi/2$, so

$$\begin{aligned}\iint_S (x^2z + y^2z)\,dS &= \iint_S (x^2+y^2)z\,dS = \int_0^{2\pi}\int_0^{\pi/2}(4\sin^2\phi)(2\cos\phi)(4\sin\phi)\,d\phi\,d\theta \\ &= 32\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin^3\phi\cos\phi\,d\phi = 32\,(2\pi)\,\left[\tfrac14\sin^4\phi\right]_0^{\pi/2} = 16\pi(1-0) = 16\pi\end{aligned}$$

18. S is given by $\mathbf{r}(u,v) = \cos v\,\mathbf{i} + u\,\mathbf{j} + \sin v\,\mathbf{k}$, $0 \le u \le 2$, $0 \le v \le \pi$. Then

$\mathbf{r}_u \times \mathbf{r}_v = \mathbf{j} \times (-\sin v\,\mathbf{i} + \cos v\,\mathbf{k}) = \cos v\,\mathbf{i} + \sin v\,\mathbf{k}$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\cos^2 v + \sin^2 v} = 1$, so

$$\begin{aligned}\iint_S (x+y+z)\,dS &= \int_0^\pi\int_0^2(\cos v + u + \sin v)(1)\,du\,dv = \int_0^\pi\left[u(\cos v + \sin v) + \tfrac12 u^2\right]_{u=0}^{u=2}dv \\ &= \int_0^\pi(2\cos v + 2\sin v + 2)\,dv = \left[2\sin v - 2\cos v + 2v\right]_0^\pi = 2 + 2\pi + 2 = 4 + 2\pi\end{aligned}$$

19. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x + y = 5$; and the back, S_3, in the plane $x = 0$.

On S_1: the surface is given by $\mathbf{r}(u,v) = u\,\mathbf{i} + 3\cos v\,\mathbf{j} + 3\sin v\,\mathbf{k}$, $0 \le v \le 2\pi$, and $0 \le x \le 5 - y \;\Rightarrow$ $0 \le u \le 5 - 3\cos v$. Then $\mathbf{r}_u \times \mathbf{r}_v = -3\cos v\,\mathbf{j} - 3\sin v\,\mathbf{k}$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{9\cos^2 v + 9\sin^2 v} = 3$, so

$$\begin{aligned}\iint_{S_1} xz\,dS &= \int_0^{2\pi}\int_0^{5-3\cos v} u(3\sin v)(3)\,du\,dv = 9\int_0^{2\pi}\left[\tfrac12 u^2\right]_{u=0}^{u=5-3\cos v}\sin v\,dv \\ &= \tfrac92\int_0^{2\pi}(5-3\cos v)^2\sin v\,dv = \tfrac92\left[\tfrac19(5-3\cos v)^3\right]_0^{2\pi} = 0.\end{aligned}$$

On S_2: $\mathbf{r}(y,z) = (5-y)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = |\mathbf{i} + \mathbf{j}| = \sqrt2$, where $y^2 + z^2 \le 9$ and

$$\begin{aligned}\iint_{S_2} xz\,dS &= \iint_{y^2+z^2\le 9}(5-y)z\sqrt2\,dA = \sqrt2\int_0^{2\pi}\int_0^3(5 - r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta \\ &= \sqrt2\int_0^{2\pi}\int_0^3(5r^2 - r^3\cos\theta)(\sin\theta)\,dr\,d\theta = \sqrt2\int_0^{2\pi}\left[\tfrac53 r^3 - \tfrac14 r^4\cos\theta\right]_{r=0}^{r=3}\sin\theta\,d\theta \\ &= \sqrt2\int_0^{2\pi}\left(45 - \tfrac{81}{4}\cos\theta\right)\sin\theta\,d\theta = \sqrt2\left(\tfrac{4}{81}\right)\cdot\tfrac12\left(45 - \tfrac{81}{4}\cos\theta\right)^2\Big]_0^{2\pi} = 0\end{aligned}$$

On S_3: $x = 0$ so $\iint_{S_3} xz\,dS = 0$. Hence $\iint_S xz\,dS = 0 + 0 + 0 = 0$.

20. Let S_1 be the lateral surface, S_2 the top disk, and S_3 the bottom disk.

On S_1: $\mathbf{r}(\theta,z) = 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 2$, $|\mathbf{r}_\theta \times \mathbf{r}_z| = 3$,

$\iint_{S_1}(x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^2(9+z^2)\,3\,dz\,d\theta = 2\pi(54+8) = 124\pi$.

On S_2: $\mathbf{r}(\theta,r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2\,\mathbf{k}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$,

$\iint_{S_2}(x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^3(r^2+4)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4} + 18\right) = \frac{153}{2}\pi$.

On S_3: $\mathbf{r}(\theta,r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$,

$\iint_{S_3}(x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^3(r^2+0)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4}\right) = \frac{81}{2}\pi$.

Hence $\iint_S\left(x^2+y^2+z^2\right)dS = 124\pi + \frac{153}{2}\pi + \frac{81}{2}\pi = 241\pi$.

21. From Exercise 5, $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (u-v)\,\mathbf{j} + (1+2u+v)\,\mathbf{k}$, $0 \le u \le 2$, $0 \le v \le 1$, and $\mathbf{r}_u \times \mathbf{r}_v = 3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$. Then

$$\begin{aligned}\mathbf{F}(\mathbf{r}(u,v)) &= (1+2u+v)e^{(u+v)(u-v)}\,\mathbf{i} - 3(1+2u+v)e^{(u+v)(u-v)}\,\mathbf{j} + (u+v)(u-v)\,\mathbf{k} \\ &= (1+2u+v)e^{u^2-v^2}\,\mathbf{i} - 3(1+2u+v)e^{u^2-v^2}\,\mathbf{j} + (u^2-v^2)\,\mathbf{k}\end{aligned}$$

Because the z-component of $\mathbf{r}_u \times \mathbf{r}_v$ is negative we use $-(\mathbf{r}_u \times \mathbf{r}_v)$ in Formula 9 for the upward orientation:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(-(\mathbf{r}_u \times \mathbf{r}_v))\,dA = \int_0^1\int_0^2 \left[-3(1+2u+v)e^{u^2-v^2} + 3(1+2u+v)e^{u^2-v^2} + 2(u^2-v^2)\right] du\,dv \\ &= \int_0^1\int_0^2 2(u^2-v^2)\,du\,dv = 2\int_0^1 \left[\tfrac13 u^3 - uv^2\right]_{u=0}^{u=2} dv = 2\int_0^1 \left(\tfrac83 - 2v^2\right) dv \\ &= 2\left[\tfrac83 v - \tfrac23 v^3\right]_0^1 = 2\left(\tfrac83 - \tfrac23\right) = 4\end{aligned}$$

22. $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$ and $\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle = \langle \sin v, -\cos v, u\rangle$. Here $\mathbf{F}(\mathbf{r}(u,v)) = v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}$ and, by Formula 9,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA = \int_0^1\int_0^\pi (v\sin v - u\sin v\cos v + u^2\cos v)\,dv\,du \\ &= \int_0^1 \left[\sin v - v\cos v - \tfrac12 u\sin^2 v + u^2\sin v\right]_{v=0}^{v=\pi} du = \int_0^1 \pi\,du = \left.\pi u\right]_0^1 = \pi\end{aligned}$$

23. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, $z = g(x,y) = 4 - x^2 - y^2$, and D is the square $[0,1]\times[0,1]$, so by Equation 10

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D [-xy(-2x) - yz(-2y) + zx]\,dA = \int_0^1\int_0^1 [2x^2y + 2y^2(4-x^2-y^2) + x(4-x^2-y^2)]\,dy\,dx \\ &= \int_0^1 \left[x^2y^2 + \tfrac83 y^3 - \tfrac23 x^2y^3 - \tfrac25 y^5 + 4xy - x^3y - \tfrac13 xy^3\right]_{y=0}^{y=1} dx \\ &= \int_0^1 \left(\tfrac13 x^2 + \tfrac{11}{3}x - x^3 + \tfrac{34}{15}\right) dx = \left[\tfrac19 x^3 + \tfrac{11}{6}x^2 - \tfrac14 x^4 + \tfrac{34}{15}x\right]_0^1 = \tfrac{713}{180}\end{aligned}$$

24. $\mathbf{F}(x,y,z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$, $z = g(x,y) = \sqrt{x^2+y^2}$, and D is the annular region $\{(x,y) \mid 1 \le x^2+y^2 \le 9\}$. Since S has downward orientation, we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= -\iint_D \left[-(-x)\left(\frac{x}{\sqrt{x^2+y^2}}\right) - (-y)\left(\frac{y}{\sqrt{x^2+y^2}}\right) + z^3\right] dA \\ &= -\iint_D \left[\frac{x^2+y^2}{\sqrt{x^2+y^2}} + \left(\sqrt{x^2+y^2}\right)^3\right] dA = -\int_0^{2\pi}\int_1^3 \left(\frac{r^2}{r} + r^3\right) r\,dr\,d\theta \\ &= -\int_0^{2\pi} d\theta \int_1^3 (r^2 + r^4)\,dr = -\big[\,\theta\,\big]_0^{2\pi}\left[\tfrac13 r^3 + \tfrac15 r^5\right]_1^3 \\ &= -2\pi\left(9 + \tfrac{243}{5} - \tfrac13 - \tfrac15\right) = -\tfrac{1712}{15}\pi\end{aligned}$$

25. $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z^2\,\mathbf{k}$, and using spherical coordinates, S is given by $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. $\mathbf{F}(\mathbf{r}(\phi,\theta)) = (\sin\phi\cos\theta)\,\mathbf{i} + (\sin\phi\sin\theta)\,\mathbf{j} + (\cos^2\phi)\,\mathbf{k}$ and, from Example 4, $\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}$. Thus

$$\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) = \sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + \sin\phi\cos^3\phi = \sin^3\phi + \sin\phi\cos^3\phi$$

and

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \left[\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\right] dA = \int_0^{2\pi}\int_0^{\pi}(\sin^3\phi+\sin\phi\cos^3\phi)\,d\phi\,d\theta \\ &= \int_0^{2\pi}d\theta\int_0^{\pi}(1-\cos^2\phi+\cos^3\phi)\sin\phi\,d\phi = (2\pi)\left[-\cos\phi+\tfrac{1}{3}\cos^3\phi-\tfrac{1}{4}\cos^4\phi\right]_0^{\pi} \\ &= 2\pi\left(1-\tfrac{1}{3}-\tfrac{1}{4}+1-\tfrac{1}{3}+\tfrac{1}{4}\right) = \tfrac{8}{3}\pi\end{aligned}$$

26. $\mathbf{F}(x,y,z) = y\,\mathbf{i} - x\,\mathbf{j} + 2z\,\mathbf{k}$, $z = g(x,y) = \sqrt{4-x^2-y^2}$ and D is the disk $\{(x,y)\mid x^2+y^2\le 4\}$. S has downward orientation, so by Equation 10,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= -\iint_D\left[-y\cdot\tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2x)-(-x)\cdot\tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2y)+2z\right]dA \\ &= -\iint_D\left(\frac{xy}{\sqrt{4-x^2-y^2}}-\frac{xy}{\sqrt{4-x^2-y^2}}+2\sqrt{4-x^2-y^2}\right)dA \\ &= -\iint_D 2\sqrt{4-x^2-y^2}\,dA = -2\int_0^{2\pi}\int_0^2\sqrt{4-r^2}\,r\,dr\,d\theta = -2\int_0^{2\pi}d\theta\int_0^2 r\sqrt{4-r^2}\,dr \\ &= -2(2\pi)\left[-\tfrac{1}{2}\cdot\tfrac{2}{3}(4-r^2)^{3/2}\right]_0^2 = -4\pi\left[0+\tfrac{1}{3}(4)^{3/2}\right] = -4\pi\cdot\tfrac{8}{3} = -\tfrac{32}{3}\pi\end{aligned}$$

27. Let S_1 be the paraboloid $y = x^2+z^2$, $0\le y\le 1$ and S_2 the disk $x^2+z^2\le 1$, $y=1$. Since S is a closed surface, we use the outward orientation.

On S_1: $\mathbf{F}(\mathbf{r}(x,z)) = (x^2+z^2)\,\mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_x\times\mathbf{r}_z = 2x\,\mathbf{i}-\mathbf{j}+2z\,\mathbf{k}$ (since the $\mathbf{j}$-component must be negative on S_1). Then

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \iint_{x^2+z^2\le 1}\left[-(x^2+z^2)-2z^2\right]dA = -\int_0^{2\pi}\int_0^1(r^2+2r^2\sin^2\theta)\,r\,dr\,d\theta \\ &= -\int_0^{2\pi}\int_0^1 r^3(1+2\sin^2\theta)\,dr\,d\theta = -\int_0^{2\pi}(1+1-\cos 2\theta)\,d\theta\int_0^1 r^3\,dr \\ &= -\left[2\theta-\tfrac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^1 = -4\pi\cdot\tfrac{1}{4} = -\pi\end{aligned}$$

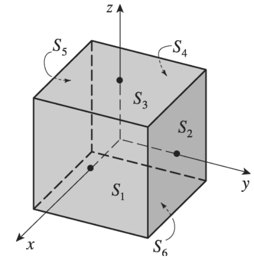
On S_2: $\mathbf{F}(\mathbf{r}(x,z)) = \mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_z\times\mathbf{r}_x = \mathbf{j}$. Then $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1}(1)\,dA = \pi$.

Hence $\iint_S \mathbf{F}\cdot d\mathbf{S} = -\pi+\pi = 0$.

28. $\mathbf{F}(x,y,z) = yz\,\mathbf{i}+zx\,\mathbf{j}+xy\,\mathbf{k}$, $z = g(x,y) = x\sin y$, and D is the rectangle $[0,2]\times[0,\pi]$, so by Equation 10

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D\left[-yz(\sin y)-zx(x\cos y)+xy\right]dA = \int_0^{\pi}\int_0^2(-xy\sin^2 y - x^3\sin y\cos y+xy)\,dx\,dy \\ &= \int_0^{\pi}\left[-\tfrac{1}{2}x^2y\sin^2 y-\tfrac{1}{4}x^4\sin y\cos y+\tfrac{1}{2}x^2y\right]_{x=0}^{x=2}dy \\ &= \int_0^{\pi}\left(-2y\sin^2 y-4\sin y\cos y+2y\right)dy \qquad \text{[integrate by parts in the first term]} \\ &= \left[\left(-\tfrac{1}{2}y^2+\tfrac{1}{2}y\sin 2y+\tfrac{1}{4}\cos 2y\right)-2\sin^2 y+y^2\right]_0^{\pi} = -\tfrac{1}{2}\pi^2+\tfrac{1}{4}+\pi^2-\tfrac{1}{4} = \tfrac{1}{2}\pi^2\end{aligned}$$

29. Here S consists of the six faces of the cube as labeled in the figure. On S_1:



$\mathbf{F} = \mathbf{i}+2y\,\mathbf{j}+3z\,\mathbf{k}$, $\mathbf{r}_y\times\mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 dy\,dz = 4$;

S_2: $\mathbf{F} = x\,\mathbf{i}+2\,\mathbf{j}+3z\,\mathbf{k}$, $\mathbf{r}_z\times\mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 2\,dx\,dz = 8$;

S_3: $\mathbf{F} = x\,\mathbf{i}+2y\,\mathbf{j}+3\,\mathbf{k}$, $\mathbf{r}_x\times\mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 3\,dx\,dy = 12$;

S_4: $\mathbf{F} = -\mathbf{i}+2y\,\mathbf{j}+3z\,\mathbf{k}$, $\mathbf{r}_z\times\mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4}\mathbf{F}\cdot d\mathbf{S} = 4$;

S_5: $\mathbf{F} = x\,\mathbf{i}-2\,\mathbf{j}+3z\,\mathbf{k}$, $\mathbf{r}_x\times\mathbf{r}_z = -\mathbf{j}$ and $\iint_{S_5}\mathbf{F}\cdot d\mathbf{S} = 8$;

S_6: $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} - 3\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_x = -\mathbf{k}$ and $\iint_{S_6} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1} \int_{-1}^{1} 3\,dx\,dy = 12$.

Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = \sum\limits_{i=1}^{6} \iint_{S_i} \mathbf{F} \cdot d\mathbf{S} = 48$.

30. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x + y = 2$; and the back, S_3, in the plane $y = 0$.

On S_1: $\mathbf{F}(\mathbf{r}(\theta, y)) = \sin\theta\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_\theta \times \mathbf{r}_y = \sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{k} \quad \Rightarrow$

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^{2-\sin\theta} (\sin^2\theta + 5\cos\theta)\,dy\,d\theta \\ &= \int_0^{2\pi} (2\sin^2\theta + 10\cos\theta - \sin^3\theta - 5\sin\theta\,\cos\theta)\,d\theta = 2\pi\end{aligned}$$

On S_2: $\mathbf{F}(\mathbf{r}(x, z)) = x\,\mathbf{i} + (2 - x)\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{i} + \mathbf{j}$.

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{x^2 + z^2 \le 1} [x + (2 - x)]\,dA = 2\pi$$

On S_3: $\mathbf{F}(\mathbf{r}(x, z)) = x\,\mathbf{i} + 5\,\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ so $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = 0$. Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = 4\pi$.

31. Here S consists of four surfaces: S_1, the top surface (a portion of the circular cylinder $y^2 + z^2 = 1$); S_2, the bottom surface (a portion of the xy-plane); S_3, the front half-disk in the plane $x = 2$, and S_4, the back half-disk in the plane $x = 0$.

On S_1: The surface is $z = \sqrt{1 - y^2}$ for $0 \le x \le 2$, $-1 \le y \le 1$ with upward orientation, so

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^2 \int_{-1}^1 \left[-x^2\,(0) - y^2\left(-\frac{y}{\sqrt{1 - y^2}}\right) + z^2\right] dy\,dx = \int_0^2 \int_{-1}^1 \left(\frac{y^3}{\sqrt{1 - y^2}} + 1 - y^2\right) dy\,dx \\ &= \int_0^2 \left[-\sqrt{1 - y^2} + \tfrac{1}{3}(1 - y^2)^{3/2} + y - \tfrac{1}{3}y^3\right]_{y=-1}^{y=1} dx = \int_0^2 \tfrac{4}{3}\,dx = \tfrac{8}{3}\end{aligned}$$

On S_2: The surface is $z = 0$ with downward orientation, so

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_{-1}^1 \left(-z^2\right) dy\,dx = \int_0^2 \int_{-1}^1 (0)\,dy\,dx = 0$$

On S_3: The surface is $x = 2$ for $-1 \le y \le 1$, $0 \le z \le \sqrt{1 - y^2}$, oriented in the positive x-direction. Regarding y and z as parameters, we have $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and

$$\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} x^2\,dz\,dy = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} 4\,dz\,dy = 4A\,(S_3) = 2\pi$$

On S_4: The surface is $x = 0$ for $-1 \le y \le 1$, $0 \le z \le \sqrt{1 - y^2}$, oriented in the negative x-direction. Regarding y and z as parameters, we use $-\left(\mathbf{r}_y \times \mathbf{r}_z\right) = -\mathbf{i}$ and

$$\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} x^2\,dz\,dy = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} (0)\;dz\,dy = 0$$

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{8}{3} + 0 + 2\pi + 0 = 2\pi + \frac{8}{3}$.

32. Here S consists of four surfaces: S_1, the triangular face with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$; S_2, the face of the tetrahedron in the xy-plane; S_3, the face in the xz-plane; and S_4, the face in the yz-plane.

On S_1: The face is the portion of the plane $z = 1 - x - y$ for $0 \le x \le 1$, $0 \le y \le 1 - x$ with upward orientation, so

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{1-x} [-y\,(-1) - (z - y)\,(-1) + x]\,dy\,dx = \int_0^1 \int_0^{1-x} (z + x)\,dy\,dx = \int_0^1 \int_0^{1-x} (1 - y)\,dy\,dx \\ &= \int_0^1 \left[y - \tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \tfrac{1}{2} \int_0^1 \left(1 - x^2\right) dx = \tfrac{1}{2}\left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}\end{aligned}$$

On S_2: The surface is $z = 0$ with downward orientation, so

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-x} (-x)\, dy\, dx = -\int_0^1 x\,(1-x)\, dx = -\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = -\tfrac{1}{6}$$

On S_3: The surface is $y = 0$ for $0 \le x \le 1$, $0 \le z \le 1 - x$, oriented in the negative y-direction. Regarding x and z as parameters, we have $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ and

$$\begin{aligned} \iint_{S_3} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{1-x} -(z-y)\, dz\, dx = -\int_0^1 \int_0^{1-x} z\, dz\, dx = -\int_0^1 \left[\tfrac{1}{2}z^2\right]_{z=0}^{z=1-x} dx \\ &= -\tfrac{1}{2}\int_0^1 (1-x)^2\, dx = \tfrac{1}{6}\left[(1-x)^3\right]_0^1 = -\tfrac{1}{6} \end{aligned}$$

On S_4: The surface is $x = 0$ for $0 \le y \le 1$, $0 \le z \le 1 - y$, oriented in the negative x-direction. Regarding y and z as parameters, we have $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ so we use $-(\mathbf{r}_y \times \mathbf{r}_z) = -\mathbf{i}$ and

$$\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-y} (-y)\, dz\, dy = -\int_0^1 y\,(1-y)\, dy = -\left[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right]_0^1 = -\tfrac{1}{6}$$

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} = -\frac{1}{6}$.

33. $z = xe^y \quad \Rightarrow \quad \partial z/\partial x = e^y$, $\partial z/\partial y = xe^y$, so by Formula 4, a CAS gives

$\iint_S (x^2 + y^2 + z^2)\, dS = \int_0^1 \int_0^1 (x^2 + y^2 + x^2 e^{2y})\sqrt{e^{2y} + x^2 e^{2y} + 1}\, dx\, dy \approx 4.5822.$

34. $z = x^2 y^2 \quad \Rightarrow \quad \partial z/\partial x = 2xy^2,\ \partial z/\partial y = 2x^2 y$, so by Formula 4, a CAS gives

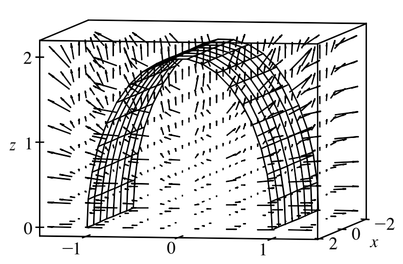
$$\begin{aligned} \iint_S xyz\, dS &= \int_0^2 \int_0^1 xy(x^2y^2)\sqrt{(2xy^2)^2 + (2x^2y)^2 + 1}\, dx\, dy \\ &= \int_0^2 \int_0^1 x^3y^3\sqrt{4x^2y^4 + 4x^4y^2 + 1}\, dx\, dy = -\tfrac{151}{33} - \tfrac{1}{220}\sqrt{3}\,\pi + \tfrac{1977}{176}\ln 7 - \tfrac{9891}{880}\ln 3 + \tfrac{3}{440}\sqrt{3}\tan^{-1}\tfrac{5}{\sqrt{3}} \end{aligned}$$

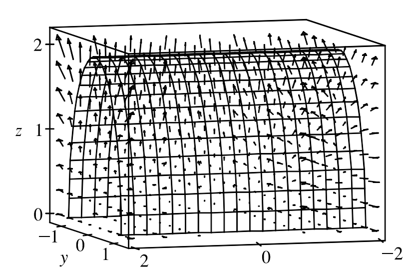
35. We use Formula 4 with $z = 3 - 2x^2 - y^2 \quad \Rightarrow \quad \partial z/\partial x = -4x$, $\partial z/\partial y = -2y$. The boundaries of the region $3 - 2x^2 - y^2 \ge 0$ are $-\sqrt{\frac{3}{2}} \le x \le \sqrt{\frac{3}{2}}$ and $-\sqrt{3 - 2x^2} \le y \le \sqrt{3 - 2x^2}$, so we use a CAS (with precision reduced to seven or fewer digits; otherwise the calculation may take a long time) to calculate

$$\iint_S x^2y^2z^2\, dS = \int_{-\sqrt{3/2}}^{\sqrt{3/2}} \int_{-\sqrt{3-2x^2}}^{\sqrt{3-2x^2}} x^2y^2(3 - 2x^2 - y^2)^2\sqrt{16x^2 + 4y^2 + 1}\, dy\, dx \approx 3.4895$$

36. The flux of $\mathbf{F}$ across S is given by $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$. Now on S, $z = g(x, y) = 2\sqrt{1 - y^2}$, so $\partial g/\partial x = 0$ and $\partial g/\partial y = -2y(1 - y^2)^{-1/2}$. Therefore, by (10),

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_{-2}^{2} \int_{-1}^{1} \left(-x^2y\left[-2y(1-y^2)^{-1/2}\right] + \left[2\sqrt{1-y^2}\right]^2 e^{x/5}\right) dy\, dx = \tfrac{1}{3}(16\pi + 80e^{2/5} - 80e^{-2/5})$$





37. If S is given by $y = h(x,z)$, then S is also the level surface $f(x,y,z) = y - h(x,z) = 0$.

$\mathbf{n} = \dfrac{\nabla f(x,y,z)}{|\nabla f(x,y,z)|} = \dfrac{-h_x\,\mathbf{i} + \mathbf{j} - h_z\,\mathbf{k}}{\sqrt{h_x^2 + 1 + h_z^2}}$, and $-\mathbf{n}$ is the unit normal that points to the left. Now we proceed as in the derivation of (10), using Formula 4 to evaluate

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\,\frac{\dfrac{\partial h}{\partial x}\,\mathbf{i} - \mathbf{j} + \dfrac{\partial h}{\partial z}\,\mathbf{k}}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2 + 1 + \left(\dfrac{\partial h}{\partial z}\right)^2}}\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + 1 + \left(\frac{\partial h}{\partial z}\right)^2}\,dA$$

where D is the projection of S onto the xz-plane. Therefore $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(P\,\frac{\partial h}{\partial x} - Q + R\,\frac{\partial h}{\partial z}\right)dA$.

38. If S is given by $x = k(y,z)$, then S is also the level surface $f(x,y,z) = x - k(y,z) = 0$.

$\mathbf{n} = \dfrac{\nabla f(x,y,z)}{|\nabla f(x,y,z)|} = \dfrac{\mathbf{i} - k_y\,\mathbf{j} - k_z\,\mathbf{k}}{\sqrt{1 + k_y^2 + k_z^2}}$, and since the x-component is positive this is the unit normal that points forward.

Now we proceed as in the derivation of (10), using Formula 4 for

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\,\frac{\mathbf{i} - \dfrac{\partial k}{\partial y}\,\mathbf{j} - \dfrac{\partial k}{\partial z}\,\mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial k}{\partial y}\right)^2 + \left(\dfrac{\partial k}{\partial z}\right)^2}}\sqrt{1 + \left(\frac{\partial k}{\partial y}\right)^2 + \left(\frac{\partial k}{\partial z}\right)^2}\,dA$$

where D is the projection of S onto the yz-plane. Therefore $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(P - Q\,\frac{\partial k}{\partial y} - R\,\frac{\partial k}{\partial z}\right)dA$.

39. $m = \iint_S K\,dS = K\cdot 4\pi\left(\frac{1}{2}a^2\right) = 2\pi a^2 K$; by symmetry $M_{xz} = M_{yz} = 0$, and

$M_{xy} = \iint_S zK\,dS = K\int_0^{2\pi}\int_0^{\pi/2}(a\cos\phi)(a^2\sin\phi)\,d\phi\,d\theta = 2\pi K a^3\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \pi K a^3$.

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\frac{1}{2}a\right)$.

40. S is given by $\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + \sqrt{x^2+y^2}\,\mathbf{k}$, $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2+y^2}{x^2+y^2}} = \sqrt{2}$ so

$$\begin{aligned} m &= \iint_S \left(10 - \sqrt{x^2+y^2}\right)dS = \iint\limits_{1\le x^2+y^2\le 16}\left(10 - \sqrt{x^2+y^2}\right)\sqrt{2}\,dA \\ &= \int_0^{2\pi}\int_1^4 \sqrt{2}\,(10-r)\,r\,dr\,d\theta = 2\pi\sqrt{2}\left[5r^2 - \tfrac{1}{3}r^3\right]_1^4 = 108\sqrt{2}\,\pi \end{aligned}$$

41. (a) $I_z = \iint_S (x^2+y^2)\rho(x,y,z)\,dS$

(b) $I_z = \iint_S (x^2+y^2)\left(10 - \sqrt{x^2+y^2}\right)dS = \iint\limits_{1\le x^2+y^2\le 16}(x^2+y^2)\left(10 - \sqrt{x^2+y^2}\right)\sqrt{2}\,dA$

$= \int_0^{2\pi}\int_1^4 \sqrt{2}\,(10r^3 - r^4)\,dr\,d\theta = 2\sqrt{2}\,\pi\left(\frac{4329}{10}\right) = \frac{4329}{5}\sqrt{2}\,\pi$

42. Using spherical coordinates to parametrize the sphere we have $\mathbf{r}(\phi,\theta) = 5\sin\phi\cos\theta\,\mathbf{i} + 5\sin\phi\sin\theta\,\mathbf{j} + 5\cos\phi\,\mathbf{k}$, and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 25\sin\phi$ (see Example 16.6.10). S is the portion of the sphere where $z \ge 4$, so $0 \le \phi \le \tan^{-1}(3/4)$ and $0 \le \theta \le 2\pi$.

(a) $m = \iint_S \rho(x,y,z)\,dS = \int_0^{2\pi} \int_0^{\tan^{-1}(3/4)} k(25\sin\phi)\,d\phi\,d\theta = 25k\int_0^{2\pi} d\theta\,\int_0^{\tan^{-1}(3/4)} \sin\phi\,d\phi$

$= 25k(2\pi)\left[-\cos\left(\tan^{-1}\frac{3}{4}\right)+1\right] = 50\pi k\left(-\frac{4}{5}+1\right) = 10\pi k.$

Because S has constant density, $\overline{x} = \overline{y} = 0$ by symmetry, and

$\overline{z} = \frac{1}{m}\iint_S z\rho(x,y,z)\,dS = \frac{1}{10\pi k}\int_0^{2\pi}\int_0^{\tan^{-1}(3/4)} k(5\cos\phi)(25\sin\phi)\,d\phi\,d\theta$

$= \frac{1}{10\pi k}(125k)\int_0^{2\pi} d\theta\,\int_0^{\tan^{-1}(3/4)} \sin\phi\cos\phi\,d\phi = \frac{1}{10\pi k}(125k)(2\pi)\left[\frac{1}{2}\sin^2\phi\right]_0^{\tan^{-1}(3/4)} = 25\cdot\frac{1}{2}\left(\frac{3}{5}\right)^2 = \frac{9}{2},$

so the center of mass is $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\frac{9}{2}\right)$.

(b) $I_z = \iint_S (x^2+y^2)\rho(x,y,z)\,dS = \int_0^{2\pi}\int_0^{\tan^{-1}(3/4)} k(25\sin^2\phi)(25\sin\phi)\,d\phi\,d\theta$

$= 625k\int_0^{2\pi} d\theta\,\int_0^{\tan^{-1}(3/4)} \sin^3\phi\,d\phi = 625k(2\pi)\left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\tan^{-1}(3/4)}$

$= 1250\pi k\left[\frac{1}{3}\left(\frac{4}{5}\right)^3 - \frac{4}{5} - \frac{1}{3} + 1\right] = 1250\pi k\left(\frac{14}{375}\right) = \frac{140}{3}\pi k$

43. The rate of flow through the cylinder is the flux $\iint_S \rho\mathbf{v}\cdot\mathbf{n}\,dS = \iint_S \rho\mathbf{v}\cdot d\mathbf{S}$. We use the parametric representation $\mathbf{r}(u,v) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j} + v\,\mathbf{k}$ for S, where $0\le u\le 2\pi$, $0\le v\le 1$, so $\mathbf{r}_u = -2\sin u\,\mathbf{i} + 2\cos u\,\mathbf{j}$, $\mathbf{r}_v = \mathbf{k}$, and the outward orientation is given by $\mathbf{r}_u\times\mathbf{r}_v = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j}$. Then

$$\begin{aligned}\iint_S \rho\mathbf{v}\cdot d\mathbf{S} &= \rho\int_0^{2\pi}\int_0^1 \left(v\,\mathbf{i} + 4\sin^2 u\,\mathbf{j} + 4\cos^2 u\,\mathbf{k}\right)\cdot(2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j})\,dv\,du \\ &= \rho\int_0^{2\pi}\int_0^1 \left(2v\cos u + 8\sin^3 u\right)dv\,du = \rho\int_0^{2\pi}\left(\cos u + 8\sin^3 u\right)du \\ &= \rho\left[\sin u + 8\left(-\tfrac{1}{3}\right)(2+\sin^2 u)\cos u\right]_0^{2\pi} = 0\text{ kg/s}\end{aligned}$$

44. A parametric representation for the hemisphere S is $\mathbf{r}(\phi,\theta) = 3\sin\phi\,\cos\theta\,\mathbf{i} + 3\sin\phi\,\sin\theta\,\mathbf{j} + 3\cos\phi\,\mathbf{k}$, $0\le\phi\le\pi/2$, $0\le\theta\le 2\pi$. Then $\mathbf{r}_\phi = 3\cos\phi\,\cos\theta\,\mathbf{i} + 3\cos\phi\,\sin\theta\,\mathbf{j} - 3\sin\phi\,\mathbf{k}$, $\mathbf{r}_\theta = -3\sin\phi\sin\theta\,\mathbf{i} + 3\sin\phi\cos\theta\,\mathbf{j}$, and the outward orientation is given by $\mathbf{r}_\phi\times\mathbf{r}_\theta = 9\sin^2\phi\,\cos\theta\,\mathbf{i} + 9\sin^2\phi\,\sin\theta\,\mathbf{j} + 9\sin\phi\,\cos\phi\,\mathbf{k}$. The rate of flow through S is

$$\begin{aligned}\iint_S \rho\mathbf{v}\cdot d\mathbf{S} &= \rho\int_0^{\pi/2}\int_0^{2\pi}(3\sin\phi\,\sin\theta\,\mathbf{i} + 3\sin\phi\,\cos\theta\,\mathbf{j})\cdot\left(9\sin^2\phi\,\cos\theta\,\mathbf{i} + 9\sin^2\phi\sin\theta\,\mathbf{j} + 9\sin\phi\,\cos\phi\,\mathbf{k}\right)d\theta\,d\phi \\ &= 27\rho\int_0^{\pi/2}\int_0^{2\pi}\left(\sin^3\phi\,\sin\theta\,\cos\theta + \sin^3\phi\,\sin\theta\,\cos\theta\right)d\theta\,d\phi = 54\rho\int_0^{\pi/2}\sin^3\phi\,d\phi\,\int_0^{2\pi}\sin\theta\,\cos\theta\,d\theta \\ &= 54\rho\left[-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi/2}\,\left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi} = 0\text{ kg/s}\end{aligned}$$

45. S consists of the hemisphere S_1 given by $z = \sqrt{a^2-x^2-y^2}$ and the disk S_2 given by $0\le x^2+y^2\le a^2$, $z=0$.

On S_1: $\mathbf{E} = a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + 2a\cos\phi\,\mathbf{k}$,

$\mathbf{T}_\phi\times\mathbf{T}_\theta = a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}$. Thus

$$\begin{aligned}\iint_{S_1}\mathbf{E}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2}(a^3\sin^3\phi + 2a^3\sin\phi\,\cos^2\phi)\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2}(a^3\sin\phi + a^3\sin\phi\,\cos^2\phi)\,d\phi\,d\theta = (2\pi)a^3\left(1+\tfrac{1}{3}\right) = \tfrac{8}{3}\pi a^3\end{aligned}$$

On S_2: $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\mathbf{r}_y\times\mathbf{r}_x = -\mathbf{k}$ so $\iint_{S_2}\mathbf{E}\cdot d\mathbf{S} = 0$. Hence the total charge is $q = \varepsilon_0\iint_S\mathbf{E}\cdot d\mathbf{S} = \frac{8}{3}\pi a^3\varepsilon_0$.

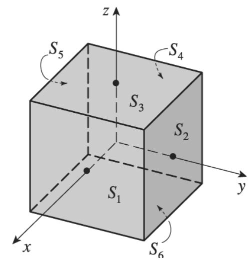
46. Referring to the figure, on

S_1: $\mathbf{E} = \mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^{1} \int_{-1}^{1} dy\,dz = 4$;

S_2: $\mathbf{E} = x\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^{1} \int_{-1}^{1} dx\,dz = 4$;

S_3: $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^{1} \int_{-1}^{1} dx\,dy = 4$;

S_4: $\mathbf{E} = -\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4} \mathbf{E} \cdot d\mathbf{S} = 4$.



Similarly $\iint_{S_5} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_6} \mathbf{E} \cdot d\mathbf{S} = 4$. Hence $q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S} = \varepsilon_0 \sum\limits_{i=1}^{6} \iint_{S_i} \mathbf{E} \cdot d\mathbf{S} = 24\varepsilon_0$.

47. $K\nabla u = 6.5(4y\,\mathbf{j} + 4z\,\mathbf{k})$. S is given by $\mathbf{r}(x,\theta) = x\,\mathbf{i} + \sqrt{6}\,\cos\theta\,\mathbf{j} + \sqrt{6}\,\sin\theta\,\mathbf{k}$ and since we want the inward heat flow, we use $\mathbf{r}_x \times \mathbf{r}_\theta = -\sqrt{6}\,\cos\theta\,\mathbf{j} - \sqrt{6}\,\sin\theta\,\mathbf{k}$. Then the rate of heat flow inward is given by

$\iint_S (-K\,\nabla u) \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^4 -(6.5)(-24)\,dx\,d\theta = (2\pi)(156)(4) = 1248\pi$.

48. $u(x,y,z) = c/\sqrt{x^2+y^2+z^2}$,

$$\begin{aligned} \mathbf{F} = -K\,\nabla u &= -K\left[-\frac{cx}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{i} - \frac{cy}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{j} - \frac{cz}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{k}\right] \\ &= \frac{cK}{(x^2+y^2+z^2)^{3/2}}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \end{aligned}$$

and the outward unit normal is $\mathbf{n} = \dfrac{1}{a}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$.

Thus $\mathbf{F} \cdot \mathbf{n} = \dfrac{cK}{a(x^2+y^2+z^2)^{3/2}}\,(x^2+y^2+z^2)$, but on S, $x^2+y^2+z^2 = a^2$ so $\mathbf{F} \cdot \mathbf{n} = \dfrac{cK}{a^2}$. Hence the rate of heat flow across S is $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{cK}{a^2} \iint_S dS = \frac{cK}{a^2}\,(4\pi a^2) = 4\pi Kc$.

49. Let S be a sphere of radius a centered at the origin. Then $|\mathbf{r}| = a$ and $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3 = \left(c/a^3\right)(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$. A parametric representation for S is $\mathbf{r}(\phi,\theta) = a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi = a\cos\phi\,\cos\theta\,\mathbf{i} + a\cos\phi\,\sin\theta\,\mathbf{j} - a\sin\phi\,\mathbf{k}$, $\mathbf{r}_\theta = -a\sin\phi\,\sin\theta\,\mathbf{i} + a\sin\phi\,\cos\theta\,\mathbf{j}$, and the outward orientation is given by $\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}$. The flux of $\mathbf{F}$ across S is

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^\pi \int_0^{2\pi} \frac{c}{a^3}\,(a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}) \\ &\qquad \cdot \left(a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}\right) d\theta\,d\phi \\ &= \frac{c}{a^3} \int_0^\pi \int_0^{2\pi} a^3\left(\sin^3\phi + \sin\phi\,\cos^2\phi\right) d\theta\,d\phi = c\int_0^\pi \int_0^{2\pi} \sin\phi\,d\theta\,d\phi = 4\pi c \end{aligned}$$

Thus the flux does not depend on the radius a.