(c) Using Exercise 29, we have that $\operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{grad}\operatorname{div}\mathbf{E} - \nabla^2\mathbf{E} \quad \Rightarrow$

$$\nabla^2\mathbf{E} = \operatorname{grad}\operatorname{div}\mathbf{E} - \operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{grad} 0 + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \quad \text{[from part (a)]} \quad = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

(d) As in part (c), $\nabla^2\mathbf{H} = \operatorname{grad}\operatorname{div}\mathbf{H} - \operatorname{curl}\operatorname{curl}\mathbf{H} = \operatorname{grad} 0 + \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2}$ [using part (b)] $= \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2}$.

39. For any continuous function f on $\mathbb{R}^3$, define a vector field $\mathbf{G}(x,y,z) = \langle g(x,y,z), 0, 0\rangle$ where $g(x,y,z) = \int_0^x f(t,y,z)\,dt$. Then $\operatorname{div}\mathbf{G} = \dfrac{\partial}{\partial x}(g(x,y,z)) + \dfrac{\partial}{\partial y}(0) + \dfrac{\partial}{\partial z}(0) = \dfrac{\partial}{\partial x}\int_0^x f(t,y,z)\,dt = f(x,y,z)$ by the Fundamental Theorem of Calculus. Thus every continuous function f on $\mathbb{R}^3$ is the divergence of some vector field.

16.6 Parametric Surfaces and Their Areas

1. $P(4,-5,1)$ lies on the parametric surface $\mathbf{r}(u,v) = \langle u+v, u-2v, 3+u-v\rangle$ if and only if there are values for u and v where $u+v=4$, $u-2v=-5$, and $3+u-v=1$. From the first equation we have $u = 4-v$ and substituting into the second equation gives $4-v-2v=-5 \quad\Leftrightarrow\quad v=3$. Then $u=1$, and these values satisfy the third equation, so P does lie on the surface.

$Q(0,4,6)$ lies on $\mathbf{r}(u,v)$ if and only if $u+v=0$, $u-2v=4$, and $3+u-v=6$, but solving the first two equations simultaneoulsy gives $u=\frac{4}{3}$, $v=-\frac{4}{3}$ and these values do not satisfy the third equation, so Q does not lie on the surface.

2. $P(1,2,1)$ lies on the parametric surface $\mathbf{r}(u,v) = \left\langle 1+u-v, u+v^2, u^2-v^2\right\rangle$ if and only if there are values for u and v where $1+u-v=1$, $u+v^2=2$, and $u^2-v^2=1$. From the first equation we have $u=v$ and substituting into the third equation gives $0=1$, an impossibility, so P does not lie on the surface.

$Q(2,3,3)$ lies on $\mathbf{r}(u,v)$ if and only if $1+u-v=2$, $u+v^2=3$, and $u^2-v^2=3$. From the first equation we have $u=v+1$ and substituting into the second equation gives $v+1+v^2=3 \quad\Leftrightarrow\quad v^2+v-2=0 \quad\Leftrightarrow\quad (v+2)(v-1)=0$, so $v=-2 \quad\Rightarrow\quad u=-1$ or $v=1 \quad\Rightarrow\quad u=2$. The third equation is satisfied by $u=2$, $v=1$ so Q does lie on the surface.

3. $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (3-v)\,\mathbf{j} + (1+4u+5v)\,\mathbf{k} = \langle 0,3,1\rangle + u\,\langle 1,0,4\rangle + v\,\langle 1,-1,5\rangle$. From Example 3, we recognize this as a vector equation of a plane through the point $(0,3,1)$ and containing vectors $\mathbf{a} = \langle 1,0,4\rangle$ and $\mathbf{b} = \langle 1,-1,5\rangle$. If we wish to find a more conventional equation for the plane, a normal vector to the plane is $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 4 \\ 1 & -1 & 5 \end{vmatrix} = 4\,\mathbf{i} - \mathbf{j} - \mathbf{k}$ and an equation of the plane is $4(x-0)-(y-3)-(z-1)=0$ or $4x-y-z=-4$.

4. $\mathbf{r}(u,v) = u^2\,\mathbf{i} + u\cos v\,\mathbf{j} + u\sin v\,\mathbf{k}$, so the corresponding parametric equations for the surface are $x=u^2$, $y=u\cos v$, $z=u\sin v$. For any point (x,y,z) on the surface, we have $y^2+z^2 = u^2\cos^2 v + u^2\sin^2 v = u^2 = x$. Since no restrictions are placed on the parameters, the surface is $x = y^2+z^2$, which we recognize as a circular paraboloid whose axis is the x-axis.

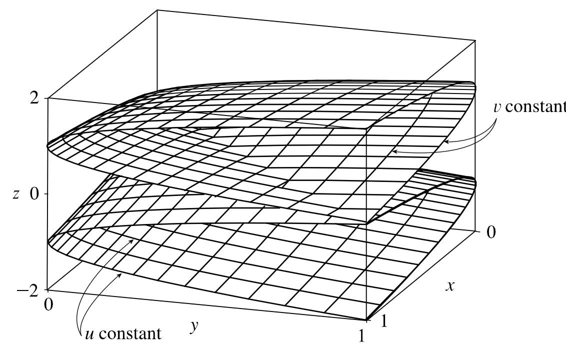
5. $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, s\rangle$, so the corresponding parametric equations for the surface are $x = s\cos t$, $y = s\sin t$, $z = s$. For any point (x,y,z) on the surface, we have $x^2 + y^2 = s^2\cos^2 t + s^2\sin^2 t = s^2 = z^2$. Since no restrictions are placed on the parameters, the surface is $z^2 = x^2 + y^2$, which we recognize as a circular cone with axis the z-axis.

6. $\mathbf{r}(s,t) = \langle 3\cos t, s, \sin t\rangle$, so the corresponding parametric equations for the surface are $x = 3\cos t$, $y = s$, $z = \sin t$. For any point (x,y,z) on the surface, we have $(x/3)^2 + z^2 = \cos^2 t + \sin^2 t = 1$, so vertical cross-sections parallel to the xz-plane are all identical ellipses. Since $y = s$ and $-1 \le s \le 1$, the surface is the portion of the elliptic cylinder $\frac{1}{9}x^2 + z^2 = 1$ corresponding to $-1 \le y \le 1$.

7. $\mathbf{r}(u,v) = \left\langle u^2, v^2, u+v\right\rangle$, $-1 \le u \le 1$, $-1 \le v \le 1$.

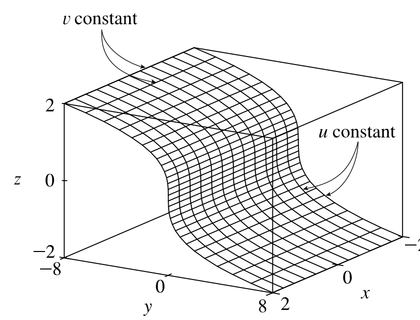
The surface has parametric equations $x = u^2$, $y = v^2$, $z = u + v$, $-1 \le u \le 1$, $-1 \le v \le 1$.

In Maple, the surface can be graphed by entering `plot3d([u^2,v^2,u+v],u=-1..1,v=-1..1);`. In Mathematica we use the `ParametricPlot3D` command. If we keep u constant at u_0, $x = u_0^2$, a constant, so the corresponding grid curves must be the curves parallel to the yz-plane. If v is constant, we have $y = v_0^2$, a constant, so these grid curves are the curves parallel to the xz-plane.



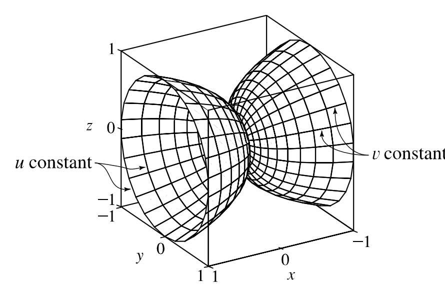
8. $\mathbf{r}(u,v) = \left\langle u, v^3, -v\right\rangle$, $-2 \le u \le 2$, $-2 \le v \le 2$.

The surface has parametric equations $x = u$, $y = v^3$, $z = -v$, $-2 \le u \le 2$, $-2 \le v \le 2$. If $u = u_0$ is constant, $x = u_0 =$ constant, so the corresponding grid curves are the curves parallel to the yz-plane. If $v = v_0$ is constant, $y = v_0^3 =$ constant, so the corresponding grid curves are the curves parallel to the xz-plane.



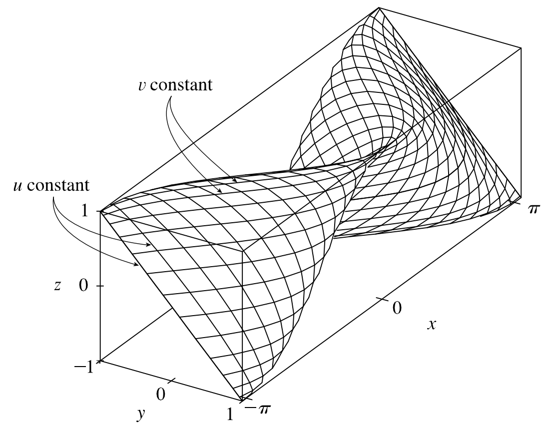
9. $\mathbf{r}(u,v) = \left\langle u^3, u\sin v, u\cos v\right\rangle$, $-1 \le u \le 1$, $0 \le v \le 2\pi$

The surface has parametric equations $x = u^3$, $y = u\sin v$, $z = u\cos v$, $-1 \le u \le 1$, $0 \le v \le 2\pi$. Note that if $u = u_0$ is constant then $x = u_0^3$ is constant and $y = u_0\sin v$, $z = u_0\cos v$ describe a circle in y, z of radius $|u_0|$, so the corresponding grid curves are circles parallel to the yz-plane. If $v = v_0$, a constant, the parametric equations become $x = u^3$, $y = u\sin v_0$, $z = u\cos v_0$. Then $y = (\tan v_0)z$, so these are the grid curves we see that lie in planes $y = kz$ that pass through the x-axis.



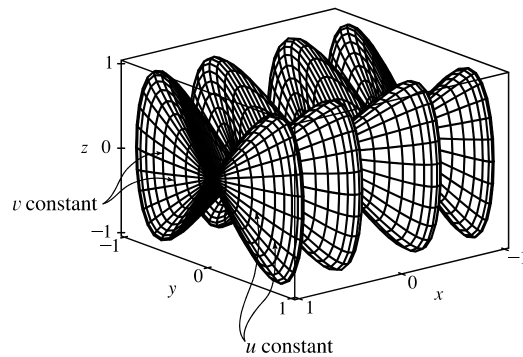
10. $\mathbf{r}(u,v) = \langle u, \sin(u+v), \sin v \rangle$, $-\pi \le u \le \pi$, $-\pi \le v \le \pi$.

The surface has parametric equations $x = u$, $y = \sin(u+v)$, $z = \sin v$, $-\pi \le u \le \pi$, $-\pi \le v \le \pi$. If $u = u_0$ is constant, $x = u_0 =$ constant, so the corresponding grid curves are the curves parallel to the yz-plane. If $v = v_0$ is constant, $z = \sin v_0 =$ constant, so the corresponding grid curves are the curves parallel to the xy-plane.

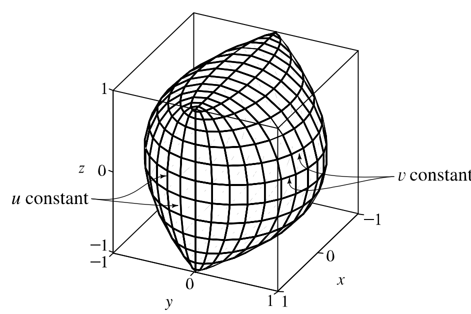


11. $x = \sin v$, $y = \cos u \sin 4v$, $z = \sin 2u \sin 4v$, $0 \le u \le 2\pi$, $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$.

Note that if $v = v_0$ is constant, then $x = \sin v_0$ is constant, so the corresponding grid curves must be parallel to the yz-plane. These are the vertically oriented grid curves we see, each shaped like a "figure-eight." When $u = u_0$ is held constant, the parametric equations become $x = \sin v$, $y = \cos u_0 \sin 4v$, $z = \sin 2u_0 \sin 4v$. Since z is a constant multiple of y, the corresponding grid curves are the curves contained in planes $z = ky$ that pass through the x-axis.



12. $x = \cos u$, $y = \sin u \, \sin v$, $z = \cos v$, $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.
If $u = u_0$ is constant, then $x = \cos u_0 =$ constant, so the corresponding grid curves are the curves parallel to the yz-plane. If $v = v_0$ is constant, then $z = \cos v_0 =$ constant, so the corresponding grid curves are the curves parallel to the xy-plane.



13. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$. The parametric equations for the surface are $x = u\cos v$, $y = u\sin v$, $z = v$. We look at the grid curves first; if we fix v, then x and y parametrize a straight line in the plane $z = v$ which intersects the z-axis. If u is held constant, the projection onto the xy-plane is circular; with $z = v$, each grid curve is a helix. The surface is a spiraling ramp, graph IV.

14. $\mathbf{r}(u,v) = uv^2\,\mathbf{i} + u^2v\,\mathbf{j} + (u^2 - v^2)\,\mathbf{k}$. The parametric equations for the surface are $x = uv^2$, $y = u^2v$, $z = u^2 - v^2$. If $u = u_0$ is held constant, then $x = u_0v^2$, $y = u_0^2v$ so $x = u_0(y/u_0^2)^2 = (1/u_0^3)y^2$, and $z = u_0^2 - v^2 = u_0^2 - (1/u_0)x$. Thus each grid curve corresponding to $u = u_0$ lies in the plane $z = u_0^2 - (1/u_0)x$ and its projection onto the xy-plane is a parabola $x = ky^2$ with axis the x-axis. Similarly, if $v = v_0$ is held constant, then $x = uv_0^2$, $y = u^2v_0$ $\Rightarrow$

$y = (x/v_0^2)^2 v_0 = (1/v_0^3)x^2$, and $z = u^2 - v_0^2 = (1/v_0)y - v_0^2$. Each grid curve lies in the plane $z = (1/v_0)y - v_0^2$ and its projection onto the xy-plane is a parabola $y = kx^2$ with axis the y-axis. The surface is graph VI.

15. $\mathbf{r}(u,v) = (u^3 - u)\,\mathbf{i} + v^2\,\mathbf{j} + u^2\,\mathbf{k}$. The parametric equations for the surface are $x = u^3 - u,\ y = v^2,\ z = u^2$. If we fix u then x and z are constant so each corresponding grid curve is contained in a line parallel to the y-axis. (Since $y = v^2 \geq 0$, the grid curves are half-lines.) If v is held constant, then $y = v^2 =$ constant, so each grid curve is contained in a plane parallel to the xz-plane. Since x and z are functions of u only, the grid curves all have the same shape. The surface is the cylinder shown in graph I.

16. $x = (1-u)(3+\cos v)\cos 4\pi u$, $y = (1-u)(3+\cos v)\sin 4\pi u$, $z = 3u + (1-u)\sin v$. These equations correspond to graph V: when $u = 0$, then $x = 3 + \cos v$, $y = 0$, and $z = \sin v$, which are equations of a circle with radius 1 in the xz-plane centered at $(3, 0, 0)$. When $u = \frac{1}{2}$, then $x = \frac{3}{2} + \frac{1}{2}\cos v$, $y = 0$, and $z = \frac{3}{2} + \frac{1}{2}\sin v$, which are equations of a circle with radius $\frac{1}{2}$ in the xz-plane centered at $\left(\frac{3}{2}, 0, \frac{3}{2}\right)$. When $u = 1$, then $x = y = 0$ and $z = 3$, giving the topmost point shown in the graph. This suggests that the grid curves with u constant are the vertically oriented circles visible on the surface. The spiralling grid curves correspond to keeping v constant.

17. $x = \cos^3 u\,\cos^3 v$, $y = \sin^3 u\,\cos^3 v$, $z = \sin^3 v$. If $v = v_0$ is held constant then $z = \sin^3 v_0$ is constant, so the corresponding grid curve lies in a horizontal plane. Several of the graphs exhibit horizontal grid curves, but the curves for this surface are neither ellipses nor straight lines, so graph III is the only possibility. (In fact, the horizontal grid curves here are members of the family $x = a\cos^3 u$, $y = a\sin^3 u$ and are called astroids.) The vertical grid curves we see on the surface correspond to $u = u_0$ held constant, as then we have $x = \cos^3 u_0\,\cos^3 v$, $y = \sin^3 u_0\,\cos^3 v$ so the corresponding grid curve lies in the vertical plane $y = (\tan^3 u_0)x$ through the z-axis.

18. $x = \sin u,\ y = \cos u\,\sin v,\ z = \sin v$. If $v = v_0$ is fixed, then $z = \sin v_0$ is constant, and $x = \sin u$, $y = (\sin v_0)\cos u$ describe an ellipse that is contained in the horizontal plane $z = \sin v_0$. If $u = u_0$ is fixed, then $x = \sin u_0$ is constant, and $y = (\cos u_0)\sin v$, $z = \sin v \quad\Rightarrow\quad y = (\cos u_0)z$, so the grid curves are portions of lines through the x-axis contained in the plane $x = \sin u_0$ (parallel to the yz-plane). The surface is graph II.

19. From Example 3, parametric equations for the plane through the point $(0,0,0)$ that contains the vectors $\mathbf{a} = \langle 1, -1, 0\rangle$ and $\mathbf{b} = \langle 0, 1, -1\rangle$ are $x = 0 + u(1) + v(0) = u,\ y = 0 + u(-1) + v(1) = v - u,\ z = 0 + u(0) + v(-1) = -v$.

20. From Example 3, parametric equations for the plane through the point $(0,-1,5)$ that contains the vectors $\mathbf{a} = \langle 2, 1, 4\rangle$ and $\mathbf{b} = \langle -3, 2, 5\rangle$ are $x = 0 + u(2) + v(-3) = 2u - 3v,\ y = -1 + u(1) + v(2) = -1 + u + 2v$, $z = 5 + u(4) + v(5) = 5 + 4u + 5v$.

21. Solving the equation for x gives $x^2 = 1 + y^2 + \frac{1}{4}z^2 \quad\Rightarrow\quad x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$. (We choose the positive root since we want the part of the hyperboloid that corresponds to $x \geq 0$.) If we let y and z be the parameters, parametric equations are $y = y$, $z = z,\ x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$.

22. Solving the equation for y gives $y^2 = \frac{1}{2}(1 - x^2 - 3z^2) \quad \Rightarrow \quad y = -\sqrt{\frac{1}{2}(1 - x^2 - 3z^2)}$ (since we want the part of the ellipsoid that corresponds to $y \leq 0$). If we let x and z be the parameters, parametric equations are $x = x,\ z = z$, $y = -\sqrt{\frac{1}{2}(1 - x^2 - 3z^2)}$.

Alternate solution: The equation can be rewritten as $x^2 + \dfrac{y^2}{\left(1/\sqrt{2}\right)^2} + \dfrac{z^2}{(1/\sqrt{3})^2} = 1$, and if we let $x = u\cos v$ and $z = \dfrac{1}{\sqrt{3}}\,u\sin v$, then $y = -\sqrt{\frac{1}{2}(1 - x^2 - 3z^2)} = -\sqrt{\frac{1}{2}(1 - u^2\cos^2 v - u^2\sin^2 v)} = -\sqrt{\frac{1}{2}(1 - u^2)}$, where $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

Second alternate solution: We can adapt the formulas for converting from spherical to rectangular coordinates as follows. We let $x = \sin\phi\cos\theta,\ y = \dfrac{1}{\sqrt{2}}\sin\phi\sin\theta,\ z = \dfrac{1}{\sqrt{3}}\cos\phi$; the surface is generated for $0 \leq \phi \leq \pi,\ \pi \leq \theta \leq 2\pi$.

23. Since the cone intersects the sphere in the circle $x^2 + y^2 = 2$, $z = \sqrt{2}$ and we want the portion of the sphere above this, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $x^2 + y^2 \leq 2$.

Alternate solution: Using spherical coordinates, $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$ where $0 \leq \phi \leq \frac{\pi}{4}$ and $0 \leq \theta \leq 2\pi$.

24. We can parametrize the cylinder as $x = 3\cos\theta,\ y = y,\ z = 3\sin\theta$. To restrict the surface to that portion above the xy-plane and between the planes $y = -4$ and $y = 4$ we require $0 \leq \theta \leq \pi$, $-4 \leq y \leq 4$.

25. In spherical coordinates, parametric equations are $x = 6\sin\phi\cos\theta$, $y = 6\sin\phi\sin\theta$, $z = 6\cos\phi$. The intersection of the sphere with the plane $z = 3\sqrt{3}$ corresponds to $z = 6\cos\phi = 3\sqrt{3} \quad \Rightarrow \quad \cos\phi = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \phi = \frac{\pi}{6}$, and the plane $z = 0$ (the xy-plane) corresponds to $\phi = \frac{\pi}{2}$. Thus the surface is described by $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{2},\ 0 \leq \theta \leq 2\pi$.

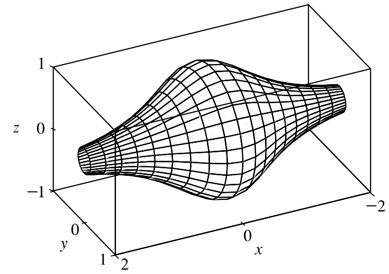
26. Using x and y as the parameters, $x = x$, $y = y$, $z = x + 3$ where $0 \leq x^2 + y^2 \leq 1$. Also, since the plane intersects the cylinder in an ellipse, the surface is a planar ellipse in the plane $z = x + 3$. Thus, parametrizing with respect to s and θ, we have $x = s\cos\theta$, $y = s\sin\theta$, $z = 3 + s\cos\theta$ where $0 \leq s \leq 1$ and $0 \leq \theta \leq 2\pi$.

27. The surface appears to be a portion of a circular cylinder of radius 3 with axis the x-axis. An equation of the cylinder is $y^2 + z^2 = 9$, and we can impose the restrictions $0 \leq x \leq 5$, $y \leq 0$ to obtain the portion shown. To graph the surface on a CAS, we can use parametric equations $x = u$, $y = 3\cos v$, $z = 3\sin v$ with the parameter domain $0 \leq u \leq 5$, $\frac{\pi}{2} \leq v \leq \frac{3\pi}{2}$. Alternatively, we can regard x and z as parameters. Then parametric equations are $x = x$, $z = z$, $y = -\sqrt{9 - z^2}$, where $0 \leq x \leq 5$ and $-3 \leq z \leq 3$.

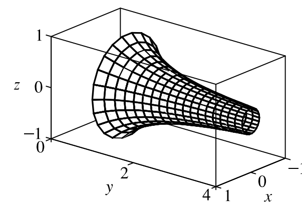
28. The surface appears to be a portion of a sphere of radius 1 centered at the origin. In spherical coordinates, the sphere has equation $\rho = 1$, and imposing the restrictions $\frac{\pi}{2} \leq \theta \leq 2\pi$, $\frac{\pi}{4} \leq \phi \leq \pi$ will give only the portion of the sphere shown. Thus, to graph the surface on a CAS we can either use spherical coordinates with the stated restrictions, or we can use parametric equations: $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$, $\frac{\pi}{2} \leq \theta \leq 2\pi$, $\frac{\pi}{4} \leq \phi \leq \pi$.

29. Using Equations 3, we have the parametrization $x = x$,

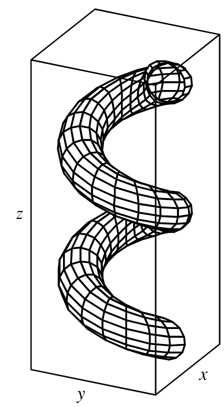
$$y = \frac{1}{1+x^2}\cos\theta, \quad z = \frac{1}{1+x^2}\sin\theta, \quad -2 \le x \le 2, \quad 0 \le \theta \le 2\pi.$$



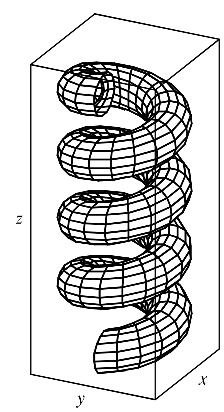
30. Letting θ be the angle of rotation about the y-axis (adapting Equations 3), we have the parametrization $x = (1/y)\cos\theta$, $y = y$, $z = (1/y)\sin\theta$, $y \ge 1$, $0 \le \theta \le 2\pi$.



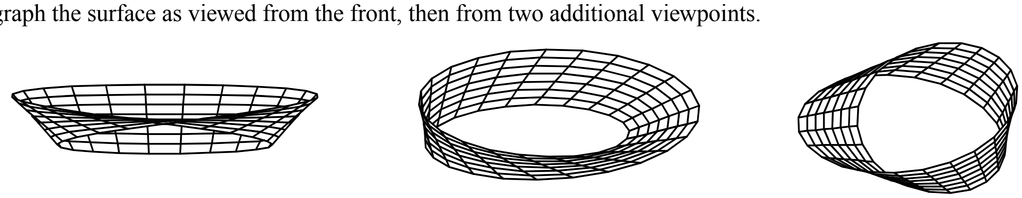
31. (a) Replacing $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$ gives parametric equations $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = u + \cos v$. From the graph, it appears that the direction of the spiral is reversed. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = 0$, draws a circle in the clockwise direction for each value of v. The original equations, on the other hand, give circular projections drawn in the counterclockwise direction. The equation for z is identical in both surfaces, so as z increases, these grid curves spiral up in opposite directions for the two surfaces.



(b) Replacing $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$ gives parametric equations $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = u + \cos v$. From the graph, it appears that the number of coils in the surface doubles within the same parametric domain. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = 0$ (where v is constant), complete circular revolutions for $0 \le u \le \pi$ while the original surface requires $0 \le u \le 2\pi$ for a complete revolution. Thus, the new surface winds around twice as fast as the original surface, and since the equation for z is identical in both surfaces, we observe twice as many circular coils in the same z-interval.



32. First we graph the surface as viewed from the front, then from two additional viewpoints.



The surface appears as a twisted sheet, and is unusual because it has only one side. (The Möbius strip is discussed in more detail in Section 16.7.)

33. $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + 3u^2\,\mathbf{j} + (u-v)\,\mathbf{k}$.

$\mathbf{r}_u = \mathbf{i} + 6u\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_v = \mathbf{i} - \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6u\,\mathbf{i} + 2\,\mathbf{j} - 6u\,\mathbf{k}$. Since the point $(2,3,0)$ corresponds to $u=1$, $v=1$, a normal vector to the surface at $(2,3,0)$ is $-6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}$, and an equation of the tangent plane is $-6x + 2y - 6z = -6$ or $3x - y + 3z = 3$.

34. $\mathbf{r}(u,v) = (u^2+1)\,\mathbf{i} + (v^3+1)\,\mathbf{j} + (u+v)\,\mathbf{k}$.

$\mathbf{r}_u = 2u\,\mathbf{i} + \mathbf{k}$ and $\mathbf{r}_v = 3v^2\,\mathbf{j} + \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -3v^2\,\mathbf{i} - 2u\,\mathbf{j} + 6uv^2\,\mathbf{k}$. Since the point $(5,2,3)$ corresponds to $u=2$, $v=1$, a normal vector to the surface at $(5,2,3)$ is $-3\,\mathbf{i} - 4\,\mathbf{j} + 12\,\mathbf{k}$, and an equation of the tangent plane is $-3(x-5) - 4(y-2) + 12(z-3) = 0$ or $3x + 4y - 12z = -13$.

35. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}\left(1, \frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$.

$\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$ and $\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + \mathbf{k}$, so a normal vector to the surface at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ is $\mathbf{r}_u\left(1, \frac{\pi}{3}\right) \times \mathbf{r}_v\left(1, \frac{\pi}{3}\right) = \left(\frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}\right) \times \left(-\frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} + \mathbf{k}\right) = \frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{2}\,\mathbf{j} + \mathbf{k}$. Thus an equation of the tangent plane at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ is $\frac{\sqrt{3}}{2}\left(x - \frac{1}{2}\right) - \frac{1}{2}\left(y - \frac{\sqrt{3}}{2}\right) + 1\left(z - \frac{\pi}{3}\right) = 0$ or $\frac{\sqrt{3}}{2}x - \frac{1}{2}y + z = \frac{\pi}{3}$.

36. $\mathbf{r}(u,v) = \sin u\,\mathbf{i} + \cos u\,\sin v\,\mathbf{j} + \sin v\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$.

$\mathbf{r}_u = \cos u\,\mathbf{i} - \sin u\sin v\,\mathbf{j}$ and $\mathbf{r}_v = \cos u\cos v\,\mathbf{j} + \cos v\,\mathbf{k}$, so a normal vector to the surface at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$ is

$\mathbf{r}_u\left(\frac{\pi}{6}, \frac{\pi}{6}\right) \times \mathbf{r}_v\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{4}\,\mathbf{j}\right) \times \left(\frac{3}{4}\,\mathbf{j} + \frac{\sqrt{3}}{2}\,\mathbf{k}\right) = -\frac{\sqrt{3}}{8}\,\mathbf{i} - \frac{3}{4}\,\mathbf{j} + \frac{3\sqrt{3}}{8}\,\mathbf{k}$.

Thus an equation of the tangent plane at $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$ is $-\frac{\sqrt{3}}{8}\left(x - \frac{1}{2}\right) - \frac{3}{4}\left(y - \frac{\sqrt{3}}{4}\right) + \frac{3\sqrt{3}}{8}\left(z - \frac{1}{2}\right) = 0$ or

$\sqrt{3}\,x + 6y - 3\sqrt{3}\,z = \frac{\sqrt{3}}{2}$ or $2x + 4\sqrt{3}\,y - 6z = 1$.

37. $\mathbf{r}(u,v) = u^2\,\mathbf{i} + 2u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}(1,0) = (1,0,1)$.

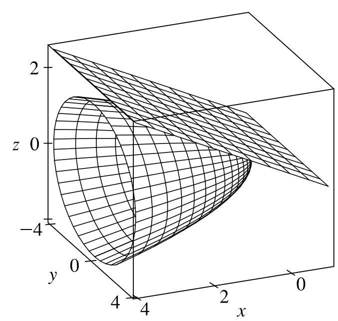
$\mathbf{r}_u = 2u\,\mathbf{i} + 2\sin v\,\mathbf{j} + \cos v\,\mathbf{k}$ and $\mathbf{r}_v = 2u\cos v\,\mathbf{j} - u\sin v\,\mathbf{k}$,

so a normal vector to the surface at the point $(1,0,1)$ is

$\mathbf{r}_u(1,0) \times \mathbf{r}_v(1,0) = (2\,\mathbf{i} + \mathbf{k}) \times (2\,\mathbf{j}) = -2\,\mathbf{i} + 4\,\mathbf{k}$.

Thus an equation of the tangent plane at $(1,0,1)$ is

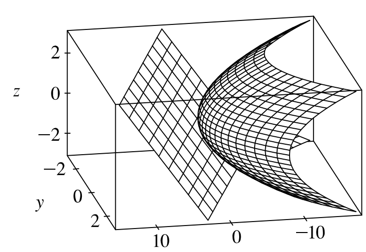
$-2(x-1) + 0(y-0) + 4(z-1) = 0$ or $-x + 2z = 1$.



38. $\mathbf{r}(u,v) = (1 - u^2 - v^2)\,\mathbf{i} - v\,\mathbf{j} - u\,\mathbf{k}$.

$\mathbf{r}_u = -2u\,\mathbf{i} - \mathbf{k}$ and $\mathbf{r}_v = -2v\,\mathbf{i} - \mathbf{j}$. Since the point $(-1,-1,-1)$ corresponds to $u=1$, $v=1$, a normal vector to the surface at $(-1,-1,-1)$ is

$\mathbf{r}_u(1,1) \times \mathbf{r}_v(1,1) = (-2\,\mathbf{i} - \mathbf{k}) \times (-2\,\mathbf{i} - \mathbf{j}) = -\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$.



Thus an equation of the tangent plane is $-1(x+1) + 2(y+1) + 2(z+1) = 0$ or $-x + 2y + 2z = -3$.

39. The surface S is given by $z = f(x, y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x + 2y = 6$, so D is the triangular region given by $\left\{(x, y) \;\middle|\; 0 \le x \le 2,\ 0 \le y \le 3 - \frac{3}{2}x\right\}$. By Formula 9, the surface area of S is

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \iint_D \sqrt{1 + (-3)^2 + (-2)^2}\, dA = \sqrt{14} \iint_D dA = \sqrt{14}\, A(D) = \sqrt{14}\left(\tfrac{1}{2} \cdot 2 \cdot 3\right) = 3\sqrt{14}. \end{aligned}$$

40. $\mathbf{r}(u, v) = \langle u + v, 2 - 3u, 1 + u - v\rangle \quad \Rightarrow \quad \mathbf{r}_u = \langle 1, -3, 1\rangle$, $\mathbf{r}_v = \langle 1, 0, -1\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle 3, 2, 3\rangle$. Then by Definition 6,

$$A(S) = \iint_D |\,\mathbf{r}_u \times \mathbf{r}_v\,|\, dA = \int_0^2 \int_{-1}^1 |\,\langle 3, 2, 3\rangle\,|\, dv\, du = \sqrt{22} \int_0^2 du \int_{-1}^1 dv = \sqrt{22}\,(2)(2) = 4\sqrt{22}$$

41. Here we can write $z = f(x, y) = \frac{1}{3} - \frac{1}{3}x - \frac{2}{3}y$ and D is the disk $x^2 + y^2 \le 3$, so by Formula 9 the area of the surface is

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + \left(-\tfrac{1}{3}\right)^2 + \left(-\tfrac{2}{3}\right)^2}\, dA = \tfrac{\sqrt{14}}{3} \iint_D dA \\ &= \tfrac{\sqrt{14}}{3}\, A(D) = \tfrac{\sqrt{14}}{3} \cdot \pi\left(\sqrt{3}\right)^2 = \sqrt{14}\,\pi \end{aligned}$$

42. $z = f(x, y) = \sqrt{x^2 + y^2} \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = \dfrac{1}{2}\left(x^2 + y^2\right)^{-1/2} \cdot 2x = \dfrac{x}{\sqrt{x^2 + y^2}},\ \dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2}},$ and

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{1 + \frac{x^2 + y^2}{x^2 + y^2}} = \sqrt{2}$$

Here D is given by $\left\{(x, y) \;\middle|\; 0 \le x \le 1,\ x^2 \le y \le x\right\}$, so by Formula 9, the surface area of S is

$$A(S) = \iint_D \sqrt{2}\, dA = \int_0^1 \int_{x^2}^x \sqrt{2}\, dy\, dx = \sqrt{2} \int_0^1 \left(x - x^2\right) dx = \sqrt{2}\, \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \sqrt{2}\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \tfrac{\sqrt{2}}{6}$$

43. $z = f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x, y) \,|\, 0 \le x \le 1,\ 0 \le y \le 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + (\sqrt{x}\,)^2 + \left(\sqrt{y}\,\right)^2}\, dA = \int_0^1 \int_0^1 \sqrt{1 + x + y}\, dy\, dx \\ &= \int_0^1 \left[\tfrac{2}{3}(x + y + 1)^{3/2}\right]_{y=0}^{y=1} dx = \tfrac{2}{3} \int_0^1 \left[(x + 2)^{3/2} - (x + 1)^{3/2}\right] dx \\ &= \tfrac{2}{3}\left[\tfrac{2}{5}(x + 2)^{5/2} - \tfrac{2}{5}(x + 1)^{5/2}\right]_0^1 = \tfrac{4}{15}(3^{5/2} - 2^{5/2} - 2^{5/2} + 1) = \tfrac{4}{15}(3^{5/2} - 2^{7/2} + 1) \end{aligned}$$

44. $z = f(x, y) = 4 - 2x^2 + y$ and $D = \{(x, y) \,|\, 0 \le x \le 1,\ 0 \le y \le x\}$. Thus, by Formula 9,

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + (-4x)^2 + (1)^2}\, dA = \int_0^1 \int_0^x \sqrt{16x^2 + 2}\, dy\, dx = \int_0^1 x\sqrt{16x^2 + 2}\, dx \\ &= \tfrac{1}{32} \cdot \tfrac{2}{3}(16x^2 + 2)^{3/2}\Big]_0^1 = \tfrac{1}{48}(18^{3/2} - 2^{3/2}) = \tfrac{1}{48}(54\sqrt{2} - 2\sqrt{2}\,) = \tfrac{13}{12}\sqrt{2} \end{aligned}$$

45. $z = f(x, y) = xy$ with $x^2 + y^2 \le 1$, so $f_x = y$, $f_y = x \quad \Rightarrow$

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + y^2 + x^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\tfrac{1}{3}\left(r^2 + 1\right)^{3/2}\right]_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi} \tfrac{1}{3}\left(2\sqrt{2} - 1\right) d\theta = \tfrac{2\pi}{3}\left(2\sqrt{2} - 1\right) \end{aligned}$$

46. A parametric representation of the surface is $x = z^2 + y$, $y = y$, $z = z$ with $0 \le y \le 2$, $0 \le z \le 2$.

Hence $\mathbf{r}_y \times \mathbf{r}_z = (\mathbf{i} + \mathbf{j}) \times (2z\,\mathbf{i} + \mathbf{k}) = \mathbf{i} - \mathbf{j} - 2z\,\mathbf{k}$.

Then

$$\begin{aligned} A(S) &= \iint_D |\,\mathbf{r}_y \times \mathbf{r}_z\,|\,dA = \int_0^2\int_0^2 \sqrt{1+1+4z^2}\,dy\,dz = \int_0^2 2\sqrt{2+4z^2}\,dz \\ &= \left[2\cdot\tfrac{1}{2}\left(z\sqrt{2+4z^2} + \ln\left(2z + \sqrt{2+4z^2}\right)\right)\right]_0^2 \quad \left[\begin{array}{c}\text{Use trigonometric substitution}\\ \text{or Formula 21 in the Table of Integrals}\end{array}\right] \\ &= 6\sqrt{2} + \ln\left(4+3\sqrt{2}\right) - \ln\sqrt{2} \text{ or } 6\sqrt{2} + \ln\frac{4+3\sqrt{2}}{\sqrt{2}} = 6\sqrt{2} + \ln\left(2\sqrt{2}+3\right) \end{aligned}$$

Note: In general, if $x = f(y,z)$ then $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i} - \dfrac{\partial f}{\partial y}\mathbf{j} - \dfrac{\partial f}{\partial z}\mathbf{k}$ and $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\,dA$.

47. A parametric representation of the surface is $x = x$, $y = x^2 + z^2$, $z = z$ with $0 \le x^2 + z^2 \le 16$.

Hence $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 2x\,\mathbf{j}) \times (2z\,\mathbf{j} + \mathbf{k}) = 2x\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$.

Note: In general, if $y = f(x,z)$ then $\mathbf{r}_x \times \mathbf{r}_z = \dfrac{\partial f}{\partial x}\mathbf{i} - \mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k}$, and $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\,dA$. Then

$$\begin{aligned} A(S) &= \iint\limits_{0 \le x^2+z^2 \le 16} \sqrt{1+4x^2+4z^2}\,dA = \int_0^{2\pi}\int_0^4 \sqrt{1+4r^2}\;r\,dr\,d\theta \\ &= \int_0^{2\pi} d\theta\,\int_0^4 r\sqrt{1+4r^2}\,dr = 2\pi\left[\tfrac{1}{12}(1+4r^2)^{3/2}\right]_0^4 = \tfrac{\pi}{6}\left(65^{3/2} - 1\right) \end{aligned}$$

48. $\mathbf{r}_u = \langle \cos v, \sin v, 0\rangle$, $\mathbf{r}_v = \langle -u\sin v, u\cos v, 1\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$. Then

$$\begin{aligned} A(S) &= \int_0^\pi\int_0^1 \sqrt{1+u^2}\,du\,dv = \int_0^\pi dv\,\int_0^1 \sqrt{1+u^2}\,du \\ &= \pi\left[\tfrac{u}{2}\sqrt{u^2+1} + \tfrac{1}{2}\ln\left|u + \sqrt{u^2+1}\right|\right]_0^1 = \tfrac{\pi}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right] \end{aligned}$$

49. $\mathbf{r}_u = \langle 2u, v, 0\rangle$, $\mathbf{r}_v = \langle 0, u, v\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \left\langle v^2, -2uv, 2u^2\right\rangle$. Then

$$\begin{aligned} A(S) &= \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA = \int_0^1\int_0^2 \sqrt{v^4 + 4u^2v^2 + 4u^4}\,dv\,du = \int_0^1\int_0^2 \sqrt{(v^2+2u^2)^2}\,dv\,du \\ &= \int_0^1\int_0^2 (v^2+2u^2)\,dv\,du = \int_0^1 \left[\tfrac{1}{3}v^3 + 2u^2v\right]_{v=0}^{v=2} du = \int_0^1 \left(\tfrac{8}{3} + 4u^2\right) du = \left[\tfrac{8}{3}u + \tfrac{4}{3}u^3\right]_0^1 = 4 \end{aligned}$$

50. The cylinder encloses separate portions of the sphere in the upper and lower halves. The top half of the sphere is $z = f(x,y) = \sqrt{b^2 - x^2 - y^2}$ and D is given by $\left\{(x,y) \mid x^2 + y^2 \le a^2\right\}$. By Formula 9, the surface area of the upper enclosed portion is

$$\begin{aligned} A &= \iint_D \sqrt{1 + \left(\frac{-x}{\sqrt{b^2-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{b^2-x^2-y^2}}\right)^2}\,dA = \iint_D \sqrt{1 + \frac{x^2+y^2}{b^2-x^2-y^2}}\,dA \\ &= \iint_D \sqrt{\frac{b^2}{b^2-x^2-y^2}}\,dA = \int_0^{2\pi}\int_0^a \frac{b}{\sqrt{b^2-r^2}}\,r\,dr\,d\theta = b\int_0^{2\pi} d\theta\,\int_0^a \frac{r}{\sqrt{b^2-r^2}}\,dr \\ &= b\left[\theta\right]_0^{2\pi}\,\left[-\sqrt{b^2-r^2}\right]_0^a = 2\pi b\left(-\sqrt{b^2-a^2} + \sqrt{b^2-0}\right) = 2\pi b\left(b - \sqrt{b^2-a^2}\right) \end{aligned}$$

The lower portion of the sphere enclosed by the cylinder has identical shape, so the total area is $2A = 4\pi b\left(b - \sqrt{b^2-a^2}\right)$.

51. From Equation 9 we have $A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA$. But if $|f_x| \le 1$ and $|f_y| \le 1$ then $0 \le (f_x)^2 \le 1$, $0 \le (f_y)^2 \le 1 \quad \Rightarrow \quad 1 \le 1 + (f_x)^2 + (f_y)^2 \le 3 \quad \Rightarrow \quad 1 \le \sqrt{1 + (f_x)^2 + (f_y)^2} \le \sqrt{3}$. By Property 15.2.11, $\iint_D 1\,dA \le \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA \le \iint_D \sqrt{3}\,dA \quad \Rightarrow \quad A(D) \le A(S) \le \sqrt{3}\,A(D) \quad \Rightarrow$ $\pi R^2 \le A(S) \le \sqrt{3}\pi R^2$.

52. $z = f(x,y) = \cos(x^2 + y^2)$ with $x^2 + y^2 \le 1$.

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + (-2x\sin(x^2+y^2))^2 + (-2y\sin(x^2+y^2))^2}\,dA \\ &= \iint_D \sqrt{1 + 4x^2\sin^2(x^2+y^2) + 4y^2\sin^2(x^2+y^2)}\,dA = \iint_D \sqrt{1 + 4(x^2+y^2)\sin^2(x^2+y^2)}\,dA \\ &= \int_0^{2\pi}\int_0^1 \sqrt{1 + 4r^2\sin^2(r^2)}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{1 + 4r^2\sin^2(r^2)}\,dr \\ &= 2\pi\int_0^1 r\sqrt{1 + 4r^2\sin^2(r^2)}\,dr \approx 4.1073 \end{aligned}$$

53. $z = f(x,y) = \ln(x^2 + y^2 + 2)$ with $x^2 + y^2 \le 1$.

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{2x}{x^2+y^2+2}\right)^2 + \left(\frac{2y}{x^2+y^2+2}\right)^2}\,dA = \iint_D \sqrt{1 + \frac{4x^2+4y^2}{(x^2+y^2+2)^2}}\,dA \\ &= \int_0^{2\pi}\int_0^1 \sqrt{1 + \frac{4r^2}{(r^2+2)^2}}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r\sqrt{\frac{(r^2+2)^2+4r^2}{(r^2+2)^2}}\,dr = 2\pi\int_0^1 \frac{r\sqrt{r^4+8r^2+4}}{r^2+2}\,dr \approx 3.5618 \end{aligned}$$

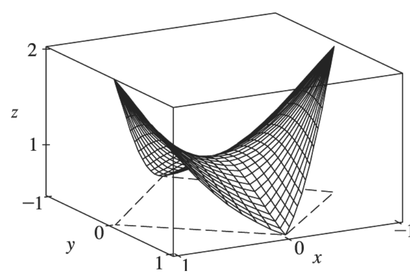
54. Let $f(x,y) = \dfrac{1+x^2}{1+y^2}$. Then $f_x = \dfrac{2x}{1+y^2}$,

$f_y = (1+x^2)\left[-\dfrac{2y}{(1+y^2)^2}\right] = -\dfrac{2y(1+x^2)}{(1+y^2)^2}$.

We use a CAS to estimate

$\int_{-1}^{1}\int_{-(1-|x|)}^{1-|x|} \sqrt{1 + f_x^2 + f_y^2}\,dy\,dx \approx 2.6959$.

In order to graph only the part of the surface above the square, we use $-(1-|x|) \le y \le 1 - |x|$ as the y-range in our plot command.



55. (a) $A(S) = \iint_D \sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}\,dA = \int_0^6\int_0^4 \sqrt{1 + \dfrac{4x^2+4y^2}{(1+x^2+y^2)^4}}\,dy\,dx$.

Using the Midpoint Rule with $f(x,y) = \sqrt{1 + \dfrac{4x^2+4y^2}{(1+x^2+y^2)^4}}$, $m = 3$, $n = 2$ we have

$$A(S) \approx \sum_{i=1}^{3}\sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A = 4\,[f(1,1) + f(1,3) + f(3,1) + f(3,3) + f(5,1) + f(5,3)] \approx 24.2055$$

(b) Using a CAS we have $A(S) = \displaystyle\int_0^6\int_0^4 \sqrt{1 + \frac{4x^2+4y^2}{(1+x^2+y^2)^4}}\,dy\,dx \approx 24.2476$. This agrees with the estimate in part (a) to the first decimal place.

56. $\mathbf{r}(u,v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$, so $\mathbf{r}_u = \langle -3\cos^2 u \sin u \cos^3 v, 3\sin^2 u \cos u \cos^3 v, 0 \rangle$,

$\mathbf{r}_v = \langle -3\cos^3 u \cos^2 v \sin v, -3\sin^3 u \cos^2 v \sin v, 3\sin^2 v \cos v \rangle$, and

$\mathbf{r}_u \times \mathbf{r}_v = \langle 9\cos u \sin^2 u \cos^4 v \sin^2 v, 9\cos^2 u \sin u \cos^4 v \sin^2 v, 9\cos^2 u \sin^2 u \cos^5 v \sin v \rangle$. Then

$$\begin{aligned} |\mathbf{r}_u \times \mathbf{r}_v| &= 9\sqrt{\cos^2 u \sin^4 u \cos^8 v \sin^4 v + \cos^4 u \sin^2 u \cos^8 v \sin^4 v + \cos^4 u \sin^4 u \cos^{10} v \sin^2 v} \\ &= 9\sqrt{\cos^2 u \sin^2 u \cos^8 v \sin^2 v\,(\sin^2 v + \cos^2 u \sin^2 u \cos^2 v)} \\ &= 9\cos^4 v\,|\cos u \sin u \sin v|\,\sqrt{\sin^2 v + \cos^2 u \sin^2 u \cos^2 v} \end{aligned}$$

Using a CAS, we have $A(S) = \int_0^\pi \int_0^{2\pi} 9\cos^4 v\,|\cos u \sin u \sin v|\,\sqrt{\sin^2 v + \cos^2 u \sin^2 u \cos^2 v}\,dv\,du \approx 4.4506$.

57. $z = 1 + 2x + 3y + 4y^2$, so

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \int_1^4 \int_0^1 \sqrt{1 + 4 + (3+8y)^2}\,dy\,dx = \int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\,dy\,dx.$$

Using a CAS, we have

$\int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\,dy\,dx = \frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left(11\sqrt{5} + 3\sqrt{14}\sqrt{5}\right) - \frac{15}{16}\ln\left(3\sqrt{5} + \sqrt{14}\sqrt{5}\right)$

or $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\frac{11\sqrt{5} + 3\sqrt{70}}{3\sqrt{5} + \sqrt{70}}$.

58. (a) $\mathbf{r}_u = a\cos v\,\mathbf{i} + b\sin v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = -au\sin v\,\mathbf{i} + bu\cos v\,\mathbf{j} + 0\,\mathbf{k}$, and

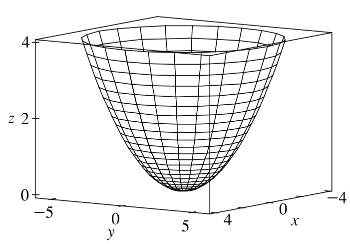
$\mathbf{r}_u \times \mathbf{r}_v = -2bu^2\cos v\,\mathbf{i} - 2au^2\sin v\,\mathbf{j} + abu\,\mathbf{k}$.

$A(S) = \int_0^{2\pi}\int_0^2 |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \int_0^{2\pi}\int_0^2 \sqrt{4b^2u^4\cos^2 v + 4a^2u^4\sin^2 v + a^2b^2u^2}\,du\,dv$

(b) $x^2 = a^2u^2\cos^2 v$, $y^2 = b^2u^2\sin^2 v$, $z = u^2 \quad\Rightarrow\quad x^2/a^2 + y^2/b^2 = u^2 = z$ which is an elliptic paraboloid. To find D, notice that $0 \le u \le 2 \quad\Rightarrow\quad 0 \le z \le 4 \quad\Rightarrow\quad 0 \le x^2/a^2 + y^2/b^2 \le 4$. Therefore, using Formula 9, we have

$$A(S) = \int_{-2a}^{2a} \int_{-b\sqrt{4-(x^2/a^2)}}^{b\sqrt{4-(x^2/a^2)}} \sqrt{1 + (2x/a^2)^2 + (2y/b^2)^2}\,dy\,dx.$$

(c)



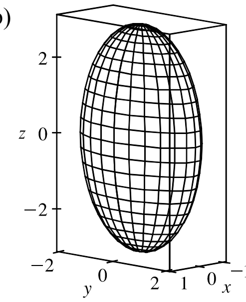
(d) We substitute $a = 2$, $b = 3$ in the integral in part (a) to get $A(S) = \int_0^{2\pi}\int_0^2 2u\sqrt{9u^2\cos^2 v + 4u^2\sin^2 v + 9}\,du\,dv$. We use a CAS to estimate the integral accurate to four decimal places. To speed up the calculation, we can set `Digits:=7;` (in Maple) or use the approximation command `N` (in Mathematica). We find that $A(S) \approx 115.6596$.

59. (a) $x = a\sin u\cos v$, $y = b\sin u\sin v$, $z = c\cos u \quad\Rightarrow$

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= (\sin u\cos v)^2 + (\sin u\sin v)^2 + (\cos u)^2 \\ &= \sin^2 u + \cos^2 u = 1 \end{aligned}$$

and since the ranges of u and v are sufficient to generate the entire graph, the parametric equations represent an ellipsoid.

(b)



(c) From the parametric equations (with $a = 1$, $b = 2$, and $c = 3$),

we calculate $\mathbf{r}_u = \cos u \cos v\,\mathbf{i} + 2\cos u \sin v\,\mathbf{j} - 3\sin u\,\mathbf{k}$ and

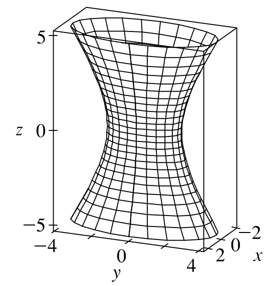
$\mathbf{r}_v = -\sin u \sin v\,\mathbf{i} + 2\sin u \cos v\,\mathbf{j}$. So $\mathbf{r}_u \times \mathbf{r}_v = 6\sin^2 u \cos v\,\mathbf{i} + 3\sin^2 u \sin v\,\mathbf{j} + 2\sin u \cos u\,\mathbf{k}$, and the surface area is given by $A(S) = \int_0^{2\pi}\int_0^{\pi} |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \int_0^{2\pi}\int_0^{\pi} \sqrt{36\sin^4 u \cos^2 v + 9\sin^4 u \sin^2 v + 4\cos^2 u \sin^2 u}\,du\,dv$

60. (a) $x = a\cosh u\,\cos v,\quad y = b\cosh u\,\sin v,\quad z = c\sinh u \quad\Rightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \cosh^2 u\,\cos^2 v + \cosh^2 u\,\sin^2 v - \sinh^2 u$$
$$= \cosh^2 u - \sinh^2 u = 1$$

and the parametric equations represent a hyperboloid of one sheet.

(b)



(c) $\mathbf{r}_u = \sinh u\,\cos v\,\mathbf{i} + 2\sinh u\,\sin v\,\mathbf{j} + 3\cosh u\,\mathbf{k}$ and

$\mathbf{r}_v = -\cosh u\,\sin v\,\mathbf{i} + 2\cosh u\,\cos v\,\mathbf{j}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6\cosh^2 u\,\cos v\,\mathbf{i} - 3\cosh^2 u\,\sin v\,\mathbf{j} + 2\cosh u\,\sinh u\,\mathbf{k}$.

We integrate between $u = \sinh^{-1}(-1) = -\ln\left(1 + \sqrt{2}\right)$ and $u = \sinh^{-1} 1 = \ln\left(1 + \sqrt{2}\right)$, since then z varies between -3 and 3, as desired. So the surface area is

$$A(S) = \int_0^{2\pi}\int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})} |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$$
$$= \int_0^{2\pi}\int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})} \sqrt{36\cosh^4 u\,\cos^2 v + 9\cosh^4 u\,\sin^2 v + 4\cosh^2 u\,\sinh^2 u}\,du\,dv$$

61. To find the region D: $z = x^2 + y^2$ implies $z + z^2 = 4z$ or $z^2 - 3z = 0$. Thus $z = 0$ or $z = 3$ are the planes where the surfaces intersect. But $x^2 + y^2 + z^2 = 4z$ implies $x^2 + y^2 + (z-2)^2 = 4$, so $z = 3$ intersects the upper hemisphere. Thus $(z-2)^2 = 4 - x^2 - y^2$ or $z = 2 + \sqrt{4 - x^2 - y^2}$. Therefore D is the region inside the circle $x^2 + y^2 + (3-2)^2 = 4$, that is, $D = \left\{(x, y) \mid x^2 + y^2 \le 3\right\}$.

$$A(S) = \iint_D \sqrt{1 + [(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2}\,dA$$
$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{1 + \frac{r^2}{4 - r^2}}\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2r\,dr}{\sqrt{4 - r^2}}\,d\theta = \int_0^{2\pi}\left[-2(4 - r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta$$
$$= \int_0^{2\pi}(-2 + 4)\,d\theta = 2\theta\Big]_0^{2\pi} = 4\pi$$

62. We first find the area of the face of the surface that intersects the positive y-axis. A parametric representation of the surface is $x = x,\ \ y = \sqrt{1 - z^2},\ \ z = z$ with $x^2 + z^2 \le 1$. Then $\mathbf{r}(x, z) = \left\langle x, \sqrt{1 - z^2}, z\right\rangle \quad\Rightarrow\quad \mathbf{r}_x = \langle 1, 0, 0\rangle$,

$\mathbf{r}_z = \left\langle 0, -z/\sqrt{1 - z^2}, 1\right\rangle$ and $\mathbf{r}_x \times \mathbf{r}_z = \left\langle 0, -1, -z/\sqrt{1 - z^2}\right\rangle \quad\Rightarrow\quad |\,\mathbf{r}_x \times \mathbf{r}_z\,| = \sqrt{1 + \dfrac{z^2}{1 - z^2}} = \dfrac{1}{\sqrt{1 - z^2}}$.

$$A(S) = \iint_{x^2 + z^2 \le 1} |\,\mathbf{r}_x \times \mathbf{r}_z\,|\,dA = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1 - z^2}}\,dx\,dz = 4\int_0^1\int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1 - z^2}}\,dx\,dz \quad \left[\begin{array}{c}\text{by the symmetry}\\ \text{of the surface}\end{array}\right]$$

This integral is improper [when $z = 1$], so

$$A(S) = \lim_{t\to 1^-} 4\int_0^t \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\,dx\,dz = \lim_{t\to 1^-} 4\int_0^t \frac{\sqrt{1-z^2}}{\sqrt{1-z^2}}\,dz = \lim_{t\to 1^-} 4\int_0^t dz = \lim_{t\to 1^-} 4t = 4$$

Since the complete surface consists of four congruent faces, the total surface area is $4(4) = 16$.

Alternate solution: The face of the surface that intersects the positive y-axis can also be parametrized as $\mathbf{r}(x,\theta) = \langle x, \cos\theta, \sin\theta\rangle$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $x^2 + z^2 \le 1 \quad\Leftrightarrow\quad x^2 + \sin^2\theta \le 1 \quad\Leftrightarrow$ $-\sqrt{1-\sin^2\theta} \le x \le \sqrt{1-\sin^2\theta} \quad\Leftrightarrow\quad -\cos\theta \le x \le \cos\theta$. Then $\mathbf{r}_x = \langle 1,0,0\rangle$, $\mathbf{r}_\theta = \langle 0, -\sin\theta, \cos\theta\rangle$ and $\mathbf{r}_x \times \mathbf{r}_\theta = \langle 0, -\cos\theta, -\sin\theta\rangle \quad\Rightarrow\quad |\,\mathbf{r}_x \times \mathbf{r}_\theta\,| = 1$, so $A(S) = \int_{-\pi/2}^{\pi/2}\int_{-\cos\theta}^{\cos\theta} 1\,dx\,d\theta = \int_{-\pi/2}^{\pi/2} 2\cos\theta\,d\theta = \left.2\sin\theta\right]_{-\pi/2}^{\pi/2} = 4$. Again, the area of the complete surface is $4(4) = 16$.

63. Let $A(S_1)$ be the surface area of that portion of the surface which lies above the plane $z = 0$. Then $A(S) = 2A(S_1)$. Following Example 10, a parametric representation of S_1 is $x = a\sin\phi\cos\theta$, $y = a\sin\phi\sin\theta$, $z = a\cos\phi$ and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2\sin\phi$. For D, $0 \le \phi \le \frac{\pi}{2}$ and for each fixed ϕ, $\left(x - \frac{1}{2}a\right)^2 + y^2 \le \left(\frac{1}{2}a\right)^2$ or $\left[a\sin\phi\cos\theta - \frac{1}{2}a\right]^2 + a^2\sin^2\phi\sin^2\theta \le (a/2)^2$ implies $a^2\sin^2\phi - a^2\sin\phi\cos\theta \le 0$ or $\sin\phi\,(\sin\phi - \cos\theta) \le 0$. But $0 \le \phi \le \frac{\pi}{2}$, so $\cos\theta \ge \sin\phi$ or $\sin\left(\frac{\pi}{2} + \theta\right) \ge \sin\phi$ or $\phi - \frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \phi$. Hence $D = \left\{(\phi,\theta) \mid 0 \le \phi \le \frac{\pi}{2}, \phi - \frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \phi\right\}$. Then

$$\begin{aligned} A(S_1) &= \int_0^{\pi/2}\int_{\phi-(\pi/2)}^{(\pi/2)-\phi} a^2\sin\phi\,d\theta\,d\phi = a^2\int_0^{\pi/2}(\pi - 2\phi)\sin\phi\,d\phi \\ &= a^2\left[(-\pi\cos\phi) - 2(-\phi\cos\phi + \sin\phi)\right]_0^{\pi/2} = a^2(\pi - 2) \end{aligned}$$

Thus $A(S) = 2a^2(\pi - 2)$.

Alternate solution: Working on S_1 we could parametrize the portion of the sphere by $x = x$, $y = y$, $z = \sqrt{a^2 - x^2 - y^2}$.

Then $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2}{a^2 - x^2 - y^2} + \dfrac{y^2}{a^2 - x^2 - y^2}} = \dfrac{a}{\sqrt{a^2 - x^2 - y^2}}$ and

$$\begin{aligned} A(S_1) &= \iint\limits_{0 \le (x-(a/2))^2 + y^2 \le (a/2)^2} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \frac{a}{\sqrt{a^2 - r^2}}\,r\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} \left.-a(a^2 - r^2)^{1/2}\right]_{r=0}^{r=a\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} a^2[1 - (1 - \cos^2\theta)^{1/2}]\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} a^2(1 - |\sin\theta|)\,d\theta = 2a^2\int_0^{\pi/2}(1 - \sin\theta)\,d\theta = 2a^2\left(\tfrac{\pi}{2} - 1\right) \end{aligned}$$

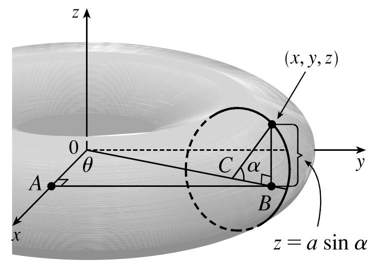
Thus $A(S) = 4a^2\left(\frac{\pi}{2} - 1\right) = 2a^2(\pi - 2)$.

Notes:

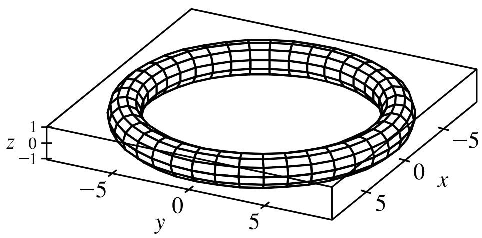
(1) Perhaps working in spherical coordinates is the most obvious approach here. However, you must be careful in setting up D.

(2) In the alternate solution, you can avoid having to use $|\sin\theta|$ by working in the first octant and then multiplying by 4. However, if you set up S_1 as above and arrived at $A(S_1) = a^2\pi$, you now see your error.

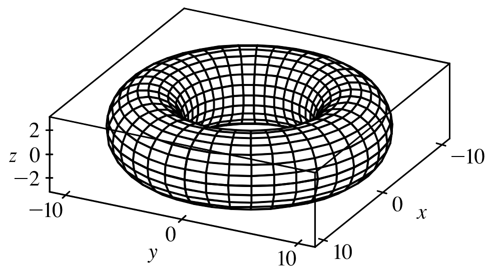
64. (a) Here $z = a\sin\alpha$, $y = |AB|$, and $x = |OA|$. But $|OB| = |OC| + |CB| = b + a\cos\alpha$ and $\sin\theta = \dfrac{|AB|}{|OB|}$ so that $y = |OB|\sin\theta = (b + a\cos\alpha)\sin\theta$. Similarly $\cos\theta = \dfrac{|OA|}{|OB|}$ so $x = (b + a\cos\alpha)\cos\theta$. Hence a parametric representation for the torus is $x = b\cos\theta + a\cos\alpha\,\cos\theta$, $y = b\sin\theta + a\cos\alpha\,\sin\theta$, $z = a\sin\alpha$, where $0 \le \alpha \le 2\pi$, $0 \le \theta \le 2\pi$.



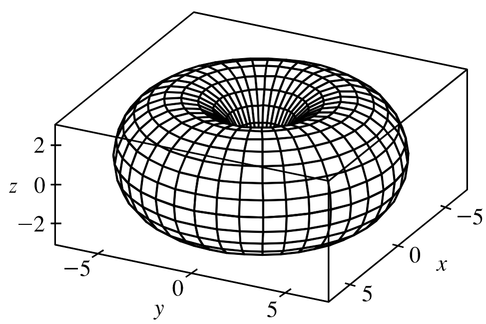
(b)



$a = 1, b = 8$



$a = 3, b = 8$



$a = 3, b = 4$

(c) $x = b\cos\theta + a\cos\alpha\,\cos\theta$, $y = b\sin\theta + a\cos\alpha\,\sin\theta$, $z = a\sin\alpha$, so $\mathbf{r}_\alpha = \langle -a\sin\alpha\,\cos\theta, -a\sin\alpha\,\sin\theta, a\cos\alpha\rangle$, $\mathbf{r}_\theta = \langle -(b + a\cos\alpha)\sin\theta, (b + a\cos\alpha)\cos\theta, 0\rangle$ and

$$\begin{aligned}\mathbf{r}_\alpha \times \mathbf{r}_\theta &= \left(-ab\cos\alpha\,\cos\theta - a^2\cos\alpha\,\cos^2\theta\right)\mathbf{i} + \left(-ab\sin\alpha\,\cos\theta - a^2\sin\alpha\,\cos^2\theta\right)\mathbf{j} \\ &\quad + \left(-ab\cos^2\alpha\,\sin\theta - a^2\cos^2\alpha\,\sin\theta\,\cos\theta - ab\sin^2\alpha\,\sin\theta - a^2\sin^2\alpha\,\sin\theta\,\cos\theta\right)\mathbf{k} \\ &= -a\,(b + a\cos\alpha)\,[(\cos\theta\,\cos\alpha)\,\mathbf{i} + (\sin\theta\,\cos\alpha)\,\mathbf{j} + (\sin\alpha)\,\mathbf{k}]\end{aligned}$$

Then $|\mathbf{r}_\alpha \times \mathbf{r}_\theta| = a(b + a\cos\alpha)\sqrt{\cos^2\theta\,\cos^2\alpha + \sin^2\theta\,\cos^2\alpha + \sin^2\alpha} = a(b + a\cos\alpha)$.

Note: $b > a$, $-1 \le \cos\alpha \le 1$ so $|b + a\cos\alpha| = b + a\cos\alpha$. Hence

$A\,(S) = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)\,d\alpha\,d\theta = 2\pi\left[ab\alpha + a^2\sin\alpha\right]_0^{2\pi} = 4\pi^2 ab$.