Department of Mathematics, University of California, Berkeley

Math 53

Alan Weinstein, Fall 2005

Final Examination, Friday, December 16th 2005

Instructions. BE SURE TO WRITE YOUR NAME AND YOUR GSI'S NAME ON YOUR BLUE BOOK. Read the problems very carefully to be sure that you understand the statements. All work should be shown in the blue book; writing should be legible and clear, and there should be enough work shown to justify your answers. **Indicate the final answers to problems by circling them.** [Point values of problems are in square brackets. There are eight problems, with a total point value is 120, for 40% of your course grade.]

PLEASE HAND IN YOUR PREPARED NOTES ALONG WITH YOUR BLUE BOOK. YOU SHOULD ***NOT*** HAND IN THIS EXAM SHEET.

1. [18 points] The vector functions $\mathbf{r}_1(t) = \langle \cos t, \sin t, t^2 \rangle$ and $\mathbf{r}_2 = \langle \cos t, -\sin t, 0 \rangle$ describe two curves which lie on the cylinder $x^2 + y^2 = 1$.
(a) The two curves go through the same point P at $t = 0$. Find that point, and find the tangent vectors of the two curves at P.
(b) For $t \neq 0$, there is a unique plane S_t through the points $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$, and P. Among all the normal vectors to the plane S_t, there is one which has the form
$\mathbf{u}(t) = \langle 1, a(t), b(t) \rangle$, where $a(t)$ and $b(t)$ are functions of t. Find $\mathbf{u}(t)$.
(c) Find the limiting position of the normal $\mathbf{u}(t)$ as t approaches zero. [Hint: use l'Hôpital's rule.]
(d) Find a normal vector for the tangent plane at P to the cylinder $x^2 + y^2 = 1$.
(e) Comment on the relation between the answers to parts (c) and (d). Do you find it surprising?

2. [12 points] The level curves of a certain function $f(x, y)$ are the lines parallel to the line $y = -x$, and the value of the function increases as one moves from lower left to upper right.
(a) Where on the ellipse $x^2 + 2y^2 = 9$ does f attain its maximum value?
(b) Where does it attain the minimum value?

3. [15 points] Find the area of the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$.

4. [15 points] (a) Describe the (solid) region of integration E for the integral

$$\int_{-5}^{5} \int_{0}^{\sqrt{25-y^2}} \int_{-\sqrt{25-x^2-y^2}}^{\sqrt{25-x^2-y^2}} \sqrt{x^2 + y^2 + z^2}\, dz\, dx\, dy$$

(b) Evaluate the integral by using spherical coordinates.

5. [15 points] In each of parts (a) and (b), evaluate as many as possible of the following five expressions. If the expression is not defined, say so. Be sure to distinguish between "zero" and "not defined". (1) The divergence of the curl; (2) the curl of the gradient; (3) the gradient of the curl; (4) the divergence of the gradient; (5) the gradient of the divergence.
There should be ten answers in all. Please present them in the order: 1a, 2a, 3a, 4a, 5a, 1b, 2b, 3b, 4b, 5b.
(a) $\mathbf{F}(x, y, z) = x^4\,\mathbf{i} + y^4\,\mathbf{k}$.
(b) $f(x, y, z) = x^4 - yz$.

PLEASE TURN OVER THE PAGE FOR THE REMAINING PROBLEMS

6. [12 points] Let E be the solid region between the sphere $x^2+y^2+z^2=1$ and the larger sphere $(x-1)^2+(y-2)^2+(z-3)^2=100$. An expanding gas is flowing into E through the small sphere, and the flux of the velocity field $\mathbf{F}$ through that sphere (with respect to the normal pointing *into* E from the hole in the middle) is equal to 20. Furthermore, the velocity field $\mathbf{F}$ has divergence everywhere equal to 3 on E. Find the flux of $\mathbf{F}$ through the larger sphere with respect to the outward normal.

7. [15 points]
(a) Find a vector field $\mathbf{F}$ whose curl is $(1+x)\,\mathbf{k}$. [Hint: guess a solution in the simplest possible form, then check, and correct if necessary.]
(b) Let S be the surface which bounds the solid region which is inside the cylinder $x^2+y^2=1$, above the xy-plane, and below the plane $z=10+x$. We may think of S as a "can with a slanted top". Use Stokes' theorem (and NOT the divergence theorem) to show that the flux of $(1+x)\,\mathbf{k}$ through the elliptical top of the can is equal to the flux of the same vector field through the circular bottom (with top and bottom both oriented by the "upward" normal). [Hint: don't forget to take the cylindrical "side" of the can into account.]
(c) Find the value of this flux.

8. [18 points] Let D be the L-shaped region in the plane whose boundary C consists of the straight line segments connecting the following points in the given order:

$$(-1,-1),\ (2,-1),\ (2,1),\ (1,1),\ (1,3),\ (-1,3),\ (-1,-1).$$

Find the line integral around C of each of the following vector fields.
(a)

$$\nabla(x^2+1-\sin(xe^y))$$

(b)

$$y\,\mathbf{i}-3x\,\mathbf{j}$$

(c)

$$\frac{y\,\mathbf{i}-x\,\mathbf{j}}{x^2+y^2}$$

[Hint: replace C by a simpler curve.]