4. By using the substitutions $x_i = \sqrt{r^2 - x_n^2 - x_{n-1}^2 - \cdots - x_{i+1}^2}\cos\theta_i$ and then applying Formulas 1 and 2 from Problem 3, we can write

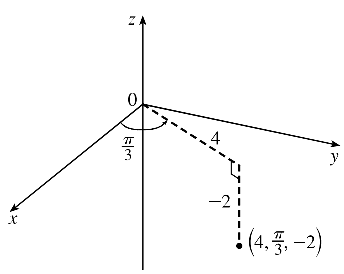
$$\begin{aligned}
V_n &= \int_{-r}^{r}\int_{-\sqrt{r^2-x_n^2}}^{\sqrt{r^2-x_n^2}}\cdots\int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}\int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}} dx_1\,dx_2\cdots dx_{n-1}\,dx_n \\
&= 2\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta_2\,d\theta_2\right]\left[\int_{-\pi/2}^{\pi/2}\cos^3\theta_3\,d\theta_3\right]\cdots\left[\int_{-\pi/2}^{\pi/2}\cos^{n-1}\theta_{n-1}\,d\theta_{n-1}\right]\left[\int_{-\pi/2}^{\pi/2}\cos^n\theta_n\,d\theta_n\right]r^n \\
&= \begin{cases} \left[2\cdot\dfrac{\pi}{2}\right]\left[\dfrac{2\cdot2}{1\cdot3}\cdot\dfrac{1\cdot3\pi}{2\cdot4}\right]\left[\dfrac{2\cdot2\cdot4}{1\cdot3\cdot5}\cdot\dfrac{1\cdot3\cdot5\pi}{2\cdot4\cdot6}\right]\cdots\left[\dfrac{2\cdot\cdots\cdot(n-2)}{1\cdot\cdots\cdot(n-1)}\cdot\dfrac{1\cdot\cdots\cdot(n-1)\,\pi}{2\cdot\cdots\cdot n}\right]r^n & n \text{ even} \\ 2\left[\dfrac{\pi}{2}\cdot\dfrac{2\cdot2}{1\cdot3}\right]\left[\dfrac{1\cdot3\pi}{2\cdot4}\cdot\dfrac{2\cdot2\cdot4}{1\cdot3\cdot5}\right]\cdots\left[\dfrac{1\cdot\cdots\cdot(n-2)\,\pi}{2\cdot\cdots\cdot(n-1)}\cdot\dfrac{2\cdot\cdots\cdot(n-1)}{1\cdot\cdots\cdot n}\right]r^n & n \text{ odd} \end{cases}
\end{aligned}$$

By canceling within each set of brackets, we find that

$$V_n = \begin{cases} \dfrac{2\pi}{2}\cdot\dfrac{2\pi}{4}\cdot\dfrac{2\pi}{6}\cdot\cdots\cdot\dfrac{2\pi}{n}r^n = \dfrac{(2\pi)^{n/2}}{2\cdot4\cdot6\cdot\cdots\cdot n}r^n = \dfrac{\pi^{n/2}}{\left(\frac{1}{2}n\right)!}r^n & n \text{ even} \\ 2\cdot\dfrac{2\pi}{3}\cdot\dfrac{2\pi}{5}\cdot\dfrac{2\pi}{7}\cdot\cdots\cdot\dfrac{2\pi}{n}r^n = \dfrac{2(2\pi)^{(n-1)/2}}{3\cdot5\cdot7\cdot\cdots\cdot n}r^n = \dfrac{2^n\left[\frac{1}{2}(n-1)\right]!\,\pi^{(n-1)/2}}{n!}r^n & n \text{ odd} \end{cases}$$

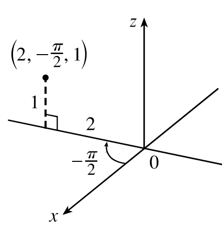
15.7 Triple Integrals in Cylindrical Coordinates

1. (a)



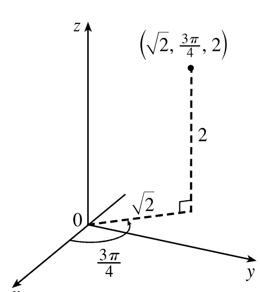
From Equations 1, $x = r\cos\theta = 4\cos\dfrac{\pi}{3} = 4\cdot\frac{1}{2} = 2$, $y = r\sin\theta = 4\sin\dfrac{\pi}{3} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$, $z = -2$, so the point is $\left(2, 2\sqrt{3}, -2\right)$ in rectangular coordinates.

(b)



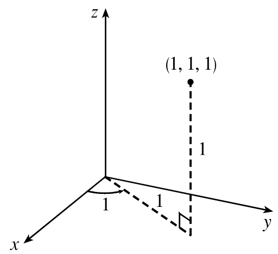
$x = 2\cos\left(-\frac{\pi}{2}\right) = 0$, $y = 2\sin\left(-\frac{\pi}{2}\right) = -2$, and $z = 1$, so the point is $(0, -2, 1)$ in rectangular coordinates.

2. (a)



$x = \sqrt{2}\cos\dfrac{3\pi}{4} = \sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = -1$, $y = \sqrt{2}\sin\dfrac{3\pi}{4} = \sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = 1$, and $z = 2$, so the point is $(-1, 1, 2)$ in rectangular coordinates.

(b)



$x = 1\cos 1 = \cos 1$, $y = 1\sin 1 = \sin 1$, and $z = 1$, so the point is $(\cos 1, \sin 1, 1) \approx (0.54, 0.84, 1)$ in rectangular coordinates.

3. (a) From Equations 2 we have $r^2 = (-1)^2 + 1^2 = 2$ so $r = \sqrt{2}$; $\tan\theta = \frac{1}{-1} = -1$ and the point $(-1, 1)$ is in the second quadrant of the xy-plane, so $\theta = \frac{3\pi}{4} + 2n\pi$; $z = 1$. Thus, one set of cylindrical coordinates is $\left(\sqrt{2}, \frac{3\pi}{4}, 1\right)$.

(b) $r^2 = (-2)^2 + \left(2\sqrt{3}\right)^2 = 16$ so $r = 4$; $\tan\theta = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$ and the point $\left(-2, 2\sqrt{3}\right)$ is in the second quadrant of the xy-plane, so $\theta = \frac{2\pi}{3} + 2n\pi$; $z = 3$. Thus, one set of cylindrical coordinates is $\left(4, \frac{2\pi}{3}, 3\right)$.

4. (a) $r^2 = \left(-\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 = 4$ so $r = 2$; $\tan\theta = \frac{\sqrt{2}}{-\sqrt{2}} = -1$ and the point $\left(-\sqrt{2}, \sqrt{2}\right)$ is in the second quadrant of the xy-plane, so $\theta = \frac{3\pi}{4} + 2n\pi$; $z = 1$. Thus, one set of cylindrical coordinates is $\left(2, \frac{3\pi}{4}, 1\right)$.

(b) $r^2 = 2^2 + 2^2 = 8$ so $r = \sqrt{8} = 2\sqrt{2}$; $\tan\theta = \frac{2}{2} = 1$ and the point $(2, 2)$ is in the first quadrant of the xy-plane, so $\theta = \frac{\pi}{4} + 2n\pi$; $z = 2$. Thus, one set of cylindrical coordinates is $\left(2\sqrt{2}, \frac{\pi}{4}, 2\right)$.

5. Since $r = 2$, the distance from any point to the z-axis is 2. Because θ and z may vary, the surface is a circular cylinder with radius 2 and the axis the z-axis. (See Figure 4.)
Also, $x^2 + y^2 = r^2 = 4$, which we recognize as an equation of this cylinder.

6. Since $\theta = \frac{\pi}{6}$ but r and z may vary, the surface is a vertical plane including the z-axis and intersecting the xy-plane in the line $y = \frac{1}{\sqrt{3}}x$. (Here we are assuming that r can be negative; if we restrict $r \geq 0$, then we get a half-plane.)

7. Since $r^2 + z^2 = 4$ and $r^2 = x^2 + y^2$, we have $x^2 + y^2 + z^2 = 4$, a sphere centered at the origin with radius 2.

8. $r = 2\sin\theta \quad\Rightarrow\quad r^2 = 2r\sin\theta \quad\Rightarrow\quad x^2 + y^2 = 2y \quad\Leftrightarrow\quad x^2 + (y-1)^2 = 1$. z doesn't appear in the equation, so any horizontal trace in $z = k$ is the circle $x^2 + (y-1)^2 = 1$, $z = k$, which has center $(0, 1, k)$ and radius 1. Thus the surface is a circular cylinder with radius 1 and axis the vertical line $x = 0$, $y = 1$.

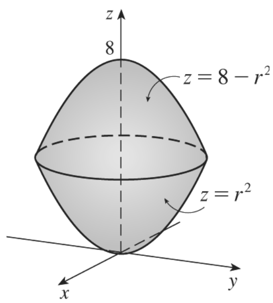
9. (a) Substituting $x^2 + y^2 = r^2$ and $x = r\cos\theta$, the equation $x^2 - x + y^2 + z^2 = 1$ becomes $r^2 - r\cos\theta + z^2 = 1$ or $z^2 = 1 + r\cos\theta - r^2$.

(b) Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation $z = x^2 - y^2$ becomes
$z = (r\cos\theta)^2 - (r\sin\theta)^2 = r^2(\cos^2\theta - \sin^2\theta)$ or $z = r^2\cos 2\theta$.

10. (a) The equation $2x^2 + 2y^2 - z^2 = 4$ can be written as $2(x^2 + y^2) - z^2 = 4$ which becomes $2r^2 - z^2 = 4$ or $z^2 = 2r^2 - 4$ in cylindrical coordinates.

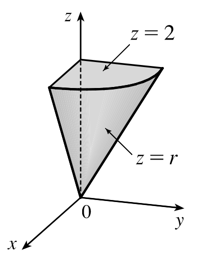
(b) Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation $2x - y + z = 1$ becomes $2r\cos\theta - r\sin\theta + z = 1$ or $z = 1 + r(\sin\theta - 2\cos\theta)$.

11.



$z = r^2 \quad \Leftrightarrow \quad z = x^2 + y^2$, a circular paraboloid opening upward with vertex the origin, and $z = 8 - r^2 \quad \Leftrightarrow \quad z = 8 - (x^2 + y^2)$, a circular paraboloid opening downward with vertex $(0, 0, 8)$. The paraboloids intersect when $r^2 = 8 - r^2 \quad \Leftrightarrow \quad r^2 = 4$. Thus $r^2 \leq z \leq 8 - r^2$ describes the solid above the paraboloid $z = x^2 + y^2$ and below the paraboloid $z = 8 - x^2 - y^2$ for $x^2 + y^2 \leq 4$.

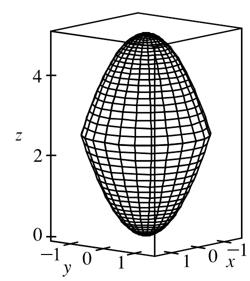
12.



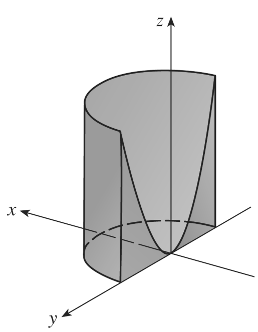
$z = r = \sqrt{x^2 + y^2}$ is a cone that opens upward. Thus $r \leq z \leq 2$ is the region above this cone and beneath the horizontal plane $z = 2$. $0 \leq \theta \leq \frac{\pi}{2}$ restricts the solid to that part of this region in the first octant.

13. We can position the cylindrical shell vertically so that its axis coincides with the z-axis and its base lies in the xy-plane. If we use centimeters as the unit of measurement, then cylindrical coordinates conveniently describe the shell as $6 \leq r \leq 7$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 20$.

14. In cylindrical coordinates, the equations are $z = r^2$ and $z = 5 - r^2$. The curve of intersection is $r^2 = 5 - r^2$ or $r = \sqrt{5/2}$. So we graph the surfaces in cylindrical coordinates, with $0 \leq r \leq \sqrt{5/2}$. In Maple, we can use the `coords=cylindrical` option in a regular `plot3d` command. In Mathematica, we can use `RevolutionPlot3D` or `ParametricPlot3D`.

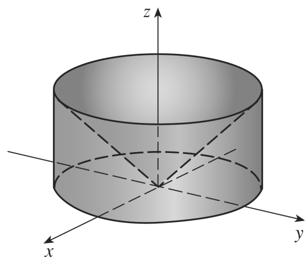


15.



The region of integration is given in cylindrical coordinates by $E = \left\{(r, \theta, z) \mid -\pi/2 \leq \theta \leq \pi/2, 0 \leq r \leq 2, 0 \leq z \leq r^2\right\}$. This represents the solid region above quadrants I and IV of the xy-plane enclosed by the circular cylinder $r = 2$, bounded above by the circular paraboloid $z = r^2$ $(z = x^2 + y^2)$, and bounded below by the xy-plane $(z = 0)$.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \int_0^2 \int_0^{r^2} r\, dz\, dr\, d\theta &= \int_{-\pi/2}^{\pi/2} \int_0^2 \left[rz\right]_{z=0}^{z=r^2} dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^2 r^3\, dr\, d\theta \\ &= \int_{-\pi/2}^{\pi/2} d\theta \int_0^2 r^3\, dr = \left[\,\theta\,\right]_{-\pi/2}^{\pi/2} \left[\tfrac{1}{4} r^4\right]_0^2 \\ &= \pi\,(4 - 0) = 4\pi\end{aligned}$$

16. 

The region of integration is given in cylindrical coordinates by $E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 2, 0 \le z \le r\}$. This represents the solid region enclosed by the circular cylinder $r = 2$, bounded above by the cone $z = r$, and bounded below by the xy-plane.

$$\int_0^2 \int_0^{2\pi} \int_0^r r\,dz\,d\theta\,dr = \int_0^2 \int_0^{2\pi} \left[rz\right]_{z=0}^{z=r} d\theta\,dr = \int_0^2 \int_0^{2\pi} r^2\,d\theta\,dr$$
$$= \int_0^2 r^2\,dr \int_0^{2\pi} d\theta = \left[\tfrac{1}{3}r^3\right]_0^2 \left[\theta\right]_0^{2\pi} = \tfrac{8}{3} \cdot 2\pi = \tfrac{16}{3}\pi$$

17. In cylindrical coordinates, E is given by $\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$. So

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi} \int_0^4 \int_{-5}^4 \sqrt{r^2}\,r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^4 r^2\,dr \int_{-5}^4 dz$$
$$= \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{3}r^3\right]_0^4 \left[z\right]_{-5}^4 = (2\pi)\left(\tfrac{64}{3}\right)(9) = 384\pi$$

18. The paraboloid $z = x^2 + y^2 = r^2$ intersects the plane $z = 4$ in the circle $x^2 + y^2 = 4$ or $r^2 = 4 \Rightarrow r = 2$, so in cylindrical coordinates, E is given by $\left\{(r, \theta, z) \,\middle|\, 0 \le \theta \le 2\pi, 0 \le r \le 2, r^2 \le z \le 4\right\}$. Thus

$$\iiint_E z\,dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 (z)\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^2 \left[\tfrac{1}{2}rz^2\right]_{z=r^2}^{z=4} dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^2 \left(8r - \tfrac{1}{2}r^5\right) dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 \left(8r - \tfrac{1}{2}r^5\right) dr = 2\pi \left[4r^2 - \tfrac{1}{12}r^6\right]_0^2$$
$$= 2\pi\left(16 - \tfrac{16}{3}\right) = \tfrac{64}{3}\pi$$

19. The paraboloid $z = 4 - x^2 - y^2 = 4 - r^2$ intersects the xy-plane in the circle $x^2 + y^2 = 4$ or $r^2 = 4 \Rightarrow r = 2$, so in cylindrical coordinates, E is given by $\left\{(r, \theta, z) \,\middle|\, 0 \le \theta \le \pi/2, 0 \le r \le 2, 0 \le z \le 4 - r^2\right\}$. Thus

$$\iiint_E (x+y+z)\,dV = \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\,r\,dz\,dr\,d\theta$$
$$= \int_0^{\pi/2} \int_0^2 \left[r^2(\cos\theta + \sin\theta)z + \tfrac{1}{2}rz^2\right]_{z=0}^{z=4-r^2} dr\,d\theta$$
$$= \int_0^{\pi/2} \int_0^2 \left[(4r^2 - r^4)(\cos\theta + \sin\theta) + \tfrac{1}{2}r(4-r^2)^2\right] dr\,d\theta$$
$$= \int_0^{\pi/2} \left[\left(\tfrac{4}{3}r^3 - \tfrac{1}{5}r^5\right)(\cos\theta + \sin\theta) - \tfrac{1}{12}(4-r^2)^3\right]_{r=0}^{r=2} d\theta$$
$$= \int_0^{\pi/2} \left[\tfrac{64}{15}(\cos\theta + \sin\theta) + \tfrac{16}{3}\right] d\theta = \left[\tfrac{64}{15}(\sin\theta - \cos\theta) + \tfrac{16}{3}\theta\right]_0^{\pi/2}$$
$$= \tfrac{64}{15}(1-0) + \tfrac{16}{3} \cdot \tfrac{\pi}{2} - \tfrac{64}{15}(0-1) - 0 = \tfrac{8}{3}\pi + \tfrac{128}{15}$$

20. In cylindrical coordinates E is bounded by the planes $z = 0$, $z = r\sin\theta + 4$ and the cylinders $r = 1$ and $r = 4$, so E is given by $\{(r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 1 \le r \le 4,\ 0 \le z \le r\sin\theta + 4\}$. Thus

$$\iiint_E (x-y)\,dV = \int_0^{2\pi} \int_1^4 \int_0^{r\sin\theta+4} (r\cos\theta - r\sin\theta)\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_1^4 (r^2\cos\theta - r^2\sin\theta)\left[z\right]_{z=0}^{z=r\sin\theta+4} dr\,d\theta$$
$$= \int_0^{2\pi} \int_1^4 (r^2\cos\theta - r^2\sin\theta)(r\sin\theta + 4)\,dr\,d\theta$$
$$= \int_0^{2\pi} \int_1^4 \left[r^3(\sin\theta\cos\theta - \sin^2\theta) + 4r^2(\cos\theta - \sin\theta)\right] dr\,d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{1}{4}r^4(\sin\theta\cos\theta - \sin^2\theta) + \tfrac{4}{3}r^3(\cos\theta - \sin\theta)\right]_{r=1}^{r=4} d\theta$$
$$= \int_0^{2\pi} \left[\left(64 - \tfrac{1}{4}\right)(\sin\theta\cos\theta - \sin^2\theta) + \left(\tfrac{256}{3} - \tfrac{4}{3}\right)(\cos\theta - \sin\theta)\right] d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{255}{4}(\sin\theta\cos\theta - \sin^2\theta) + 84(\cos\theta - \sin\theta)\right] d\theta$$
$$= \left[\tfrac{255}{4}\left(\tfrac{1}{2}\sin^2\theta - \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right) + 84(\sin\theta + \cos\theta)\right]_0^{2\pi} = \tfrac{255}{4}(-\pi) + 84(1) - 0 - 84(1) = -\tfrac{255}{4}\pi$$

21. In cylindrical coordinates, E is bounded by the cylinder $r = 1$, the plane $z = 0$, and the cone $z = 2r$. So $E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, 0 \le z \le 2r\}$ and

$$\iiint_E x^2\,dV = \int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left[r^3\cos^2\theta\,z\right]_{z=0}^{z=2r}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r^4\cos^2\theta\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[\tfrac{2}{5}r^5\cos^2\theta\right]_{r=0}^{r=1}\,d\theta = \tfrac{2}{5}\int_0^{2\pi}\cos^2\theta\,d\theta = \tfrac{2}{5}\int_0^{2\pi}\tfrac{1}{2}\,(1+\cos 2\theta)\,d\theta = \tfrac{1}{5}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = \tfrac{2\pi}{5}$$

22. In cylindrical coordinates E is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$, so $E = \left\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, -\sqrt{4 - r^2} \le z \le \sqrt{4 - r^2}\right\}$. Thus the volume is

$$\iiint_E\,dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta$$
$$= \int_0^{2\pi}d\theta\,\int_0^1 2r\sqrt{4-r^2}\,dr = 2\pi\left[-\tfrac{2}{3}(4-r^2)^{3/2}\right]_0^1 = \tfrac{4}{3}\pi(8 - 3^{3/2})$$

23. In cylindrical coordinates, E is bounded below by the cone $z = r$ and above by the sphere $r^2 + z^2 = 2$ or $z = \sqrt{2 - r^2}$. The cone and the sphere intersect when $2r^2 = 2 \quad\Rightarrow\quad r = 1$, so $E = \left\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, r \le z \le \sqrt{2 - r^2}\right\}$ and the volume is

$$\iiint_E\,dV = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 [rz]_{z=r}^{z=\sqrt{2-r^2}}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(r\sqrt{2-r^2} - r^2\right)dr\,d\theta$$
$$= \int_0^{2\pi}d\theta\,\int_0^1\left(r\sqrt{2-r^2} - r^2\right)dr = 2\pi\left[-\tfrac{1}{3}(2-r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^1$$
$$= 2\pi\left(-\tfrac{1}{3}\right)\left(1 + 1 - 2^{3/2}\right) = -\tfrac{2}{3}\pi\left(2 - 2\sqrt{2}\right) = \tfrac{4}{3}\pi\left(\sqrt{2} - 1\right)$$

24. In cylindrical coordinates, E is bounded below by the paraboloid $z = r^2$ and above by the sphere $r^2 + z^2 = 2$ or $z = \sqrt{2 - r^2}$. The paraboloid and the sphere intersect when $r^2 + r^4 = 2 \quad\Rightarrow\quad (r^2+2)(r^2-1) = 0 \quad\Rightarrow\quad r = 1$, so $E = \left\{(r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ r^2 \le z \le \sqrt{2 - r^2}\right\}$ and the volume is

$$\iiint_E\,dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 [rz]_{z=r^2}^{z=\sqrt{2-r^2}}\,dr\,d\theta = \int_0^{2\pi}\int_0^1\left(r\sqrt{2-r^2} - r^3\right)dr\,d\theta$$
$$= \int_0^{2\pi}d\theta\,\int_0^1\left(r\sqrt{2-r^2} - r^3\right)dr = 2\pi\left[-\tfrac{1}{3}(2-r^2)^{3/2} - \tfrac{1}{4}r^4\right]_0^1$$
$$= 2\pi(-\tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{3}\cdot 2^{3/2} - 0) = 2\pi\left(-\tfrac{7}{12} + \tfrac{2}{3}\sqrt{2}\right) = \left(-\tfrac{7}{6} + \tfrac{4}{3}\sqrt{2}\right)\pi$$

25. (a) In cylindrical coordinates, E is bounded above by the paraboloid $z = 24 - r^2$ and below by the cone $z = 2\sqrt{r^2}$ or $z = 2r$ $(r \ge 0)$. The surfaces intersect when $24 - r^2 = 2r \quad\Rightarrow\quad r^2 + 2r - 24 = 0 \quad\Rightarrow\quad (r+6)(r-4) = 0 \quad\Rightarrow\quad r = 4$, so $E = \left\{(r, \theta, z) \mid 2r \le z \le 24 - r^2, 0 \le r \le 4, 0 \le \theta \le 2\pi\right\}$ and the volume is

$$\iiint_E\,dV = \int_0^{2\pi}\int_0^4\int_{2r}^{24-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^4 r\left(24 - r^2 - 2r\right)dr\,d\theta = \int_0^{2\pi}d\theta\,\int_0^4\left(24r - r^3 - 2r^2\right)dr$$
$$= 2\pi\left[12r^2 - \tfrac{1}{4}r^4 - \tfrac{2}{3}r^3\right]_0^4 = 2\pi\left(192 - 64 - \tfrac{128}{3}\right) = \tfrac{512}{3}\pi$$

(b) For constant density K, $m = KV = \frac{512}{3}\pi K$ from part (a). Since the region is homogeneous and symmetric,

$M_{yz} = M_{xz} = 0$ and

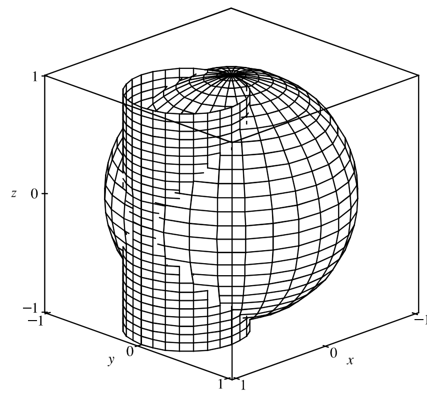
$$\begin{aligned} M_{xy} &= \int_0^{2\pi}\int_0^4\int_{2r}^{24-r^2}(zK)\,r\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^4 r\left[\tfrac12 z^2\right]_{z=2r}^{z=24-r^2}dr\,d\theta \\ &= \tfrac{K}{2}\int_0^{2\pi}\int_0^4 r[(24-r^2)^2-4r^2]\,dr\,d\theta = \tfrac{K}{2}\int_0^{2\pi}d\theta\int_0^4(576r-52r^3+r^5)\,dr \\ &= \tfrac{K}{2}(2\pi)\left[288r^2-13r^4+\tfrac16 r^6\right]_0^4 = \pi K\left(4608-3328+\tfrac{2048}{3}\right) = \tfrac{5888}{3}\pi K \end{aligned}$$

Thus $(\overline{x},\overline{y},\overline{z}) = \left(\dfrac{M_{yz}}{m},\dfrac{M_{xz}}{m},\dfrac{M_{xy}}{m}\right) = \left(0,0,\frac{5888\pi K/3}{512\pi K/3}\right) = \left(0,0,\frac{23}{2}\right)$.

26. (a) $V = \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$$\begin{aligned} &= 4\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta \\ &= 4\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta \\ &= -\tfrac43\int_0^{\pi/2}\left[(a^2-r^2)^{3/2}\right]_{r=0}^{r=a\cos\theta}d\theta \\ &= -\tfrac43\int_0^{\pi/2}\left[(a^2-a^2\cos^2\theta)^{3/2}-a^3\right]d\theta \\ &= -\tfrac43\int_0^{\pi/2}\left[(a^2\sin^2\theta)^{3/2}-a^3\right]d\theta \\ &= -\tfrac43\int_0^{\pi/2}(a^3\sin^3\theta-a^3)\,d\theta \\ &= -\frac{4a^3}{3}\int_0^{\pi/2}\left[\sin\theta\,(1-\cos^2\theta)-1\right]d\theta \\ &= -\frac{4a^3}{3}\left[-\cos\theta+\tfrac13\cos^3\theta-\theta\right]_0^{\pi/2} = -\frac{4a^3}{3}\left(-\tfrac{\pi}{2}+\tfrac23\right) = \tfrac29 a^3(3\pi-4) \end{aligned}$$

(b)



To plot the cylinder and the sphere on the same screen in Maple, we can use the sequence of commands

```
sphere:=plot3d(1,theta=0..2*Pi,phi=0..Pi,coords=spherical):
cylinder:=plot3d(cos(theta),theta=-Pi/2..Pi/2,z=-1..1,coords=cylindrical):
with(plots):
display3d({sphere,cylinder});
```

In Mathematica, we can use

```
sphere=SphericalPlot3D[1,{phi,0,Pi},{theta,0,2Pi}]
cylinder=ParametricPlot3D[{(Cos[theta])^2,Cos[theta]*Sin[theta],z},
                          {theta,-Pi/2,Pi/2},{z,-1,1}]
Show[sphere,cylinder]
```

27. The paraboloid $z = 4x^2+4y^2$ intersects the plane $z = a$ when $a = 4x^2+4y^2$ or $x^2+y^2 = \frac14 a$. So, in cylindrical coordinates, $E = \left\{(r,\theta,z) \mid 0 \le r \le \frac12\sqrt{a}, 0 \le \theta \le 2\pi, 4r^2 \le z \le a\right\}$. Thus

$$\begin{aligned} m &= \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Kr\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2}(ar-4r^3)\,dr\,d\theta \\ &= K\int_0^{2\pi}\left[\tfrac12 ar^2-r^4\right]_{r=0}^{r=\sqrt{a}/2}d\theta = K\int_0^{2\pi}\tfrac{1}{16}a^2\,d\theta = \tfrac18 a^2\pi K \end{aligned}$$

Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

$$M_{xy} = \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Krz\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2}\left(\tfrac{1}{2}a^2r - 8r^5\right)dr\,d\theta$$

$$= K\int_0^{2\pi}\left[\tfrac{1}{4}a^2r^2 - \tfrac{4}{3}r^6\right]_{r=0}^{r=\sqrt{a}/2}\,d\theta = K\int_0^{2\pi}\tfrac{1}{24}a^3\,d\theta = \tfrac{1}{12}a^3\pi K$$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\frac{2}{3}a\right)$.

28. Since density is proportional to the distance from the z-axis, we can say $\rho(x,y,z) = K\sqrt{x^2+y^2}$. Then

$$m = 2\int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} Kr^2\,dz\,dr\,d\theta = 2K\int_0^{2\pi}\int_0^a r^2\sqrt{a^2-r^2}\,dr\,d\theta$$

$$= 2K\int_0^{2\pi}\left[\tfrac{1}{8}r(2r^2-a^2)\sqrt{a^2-r^2} + \tfrac{1}{8}a^4\sin^{-1}(r/a)\right]_{r=0}^{r=a}d\theta = 2K\int_0^{2\pi}\left[\left(\tfrac{1}{8}a^4\right)\left(\tfrac{\pi}{2}\right)\right]d\theta = \tfrac{1}{4}a^4\pi^2K$$

29. The region of integration is the region above the cone $z = \sqrt{x^2+y^2}$, or $z = r$, and below the plane $z = 2$. Also, we have $-2 \le y \le 2$ with $-\sqrt{4-y^2} \le x \le \sqrt{4-y^2}$ which describes a circle of radius 2 in the xy-plane centered at $(0,0)$. Thus,

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy = \int_0^{2\pi}\int_0^2\int_r^2 (r\cos\theta)\,z\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_r^2 r^2(\cos\theta)\,z\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^2(\cos\theta)\left[\tfrac{1}{2}z^2\right]_{z=r}^{z=2}dr\,d\theta = \tfrac{1}{2}\int_0^{2\pi}\int_0^2 r^2(\cos\theta)\left(4-r^2\right)dr\,d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi}\cos\theta\,d\theta\int_0^2\left(4r^2-r^4\right)dr = \tfrac{1}{2}\left[\sin\theta\right]_0^{2\pi}\left[\tfrac{4}{3}r^3 - \tfrac{1}{5}r^5\right]_0^2 = 0$$

30. The region of integration is the region above the plane $z = 0$ and below the paraboloid $z = 9 - x^2 - y^2$. Also, we have $-3 \le x \le 3$ with $0 \le y \le \sqrt{9-x^2}$ which describes the upper half of a circle of radius 3 in the xy-plane centered at $(0,0)$. Thus,

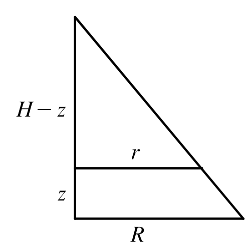
$$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx = \int_0^{\pi}\int_0^3\int_0^{9-r^2}\sqrt{r^2}\,r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^3\int_0^{9-r^2} r^2\,dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^3 r^2\left(9-r^2\right)dr\,d\theta = \int_0^{\pi}d\theta\int_0^3\left(9r^2-r^4\right)dr$$

$$= \left[\,\theta\,\right]_0^{\pi}\left[3r^3 - \tfrac{1}{5}r^5\right]_0^3 = \pi\left(81 - \tfrac{243}{5}\right) = \tfrac{162}{5}\pi$$

31. (a) The mountain comprises a solid conical region C. The work done in lifting a small volume of material ΔV with density $g(P)$ to a height $h(P)$ above sea level is $h(P)g(P)\,\Delta V$. Summing over the whole mountain we get $W = \iiint_C h(P)g(P)\,dV$.

(b) Here C is a solid right circular cone with radius $R = 62{,}000$ ft, height $H = 12{,}400$ ft, and density $g(P) = 200\ \text{lb/ft}^3$ at all points P in C. We use cylindrical coordinates:



$$\frac{r}{R} = \frac{H-z}{H} = 1 - \frac{z}{H}$$

$$W = \int_0^{2\pi}\int_0^H\int_0^{R(1-z/H)} z\cdot 200r\,dr\,dz\,d\theta = 2\pi\int_0^H 200z\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=R(1-z/H)}dz$$

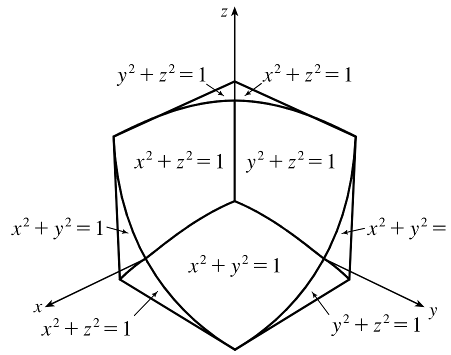
$$= 400\pi\int_0^H z\,\frac{R^2}{2}\left(1-\frac{z}{H}\right)^2dz = 200\pi R^2\int_0^H\left(z - \frac{2z^2}{H} + \frac{z^3}{H^2}\right)dz$$

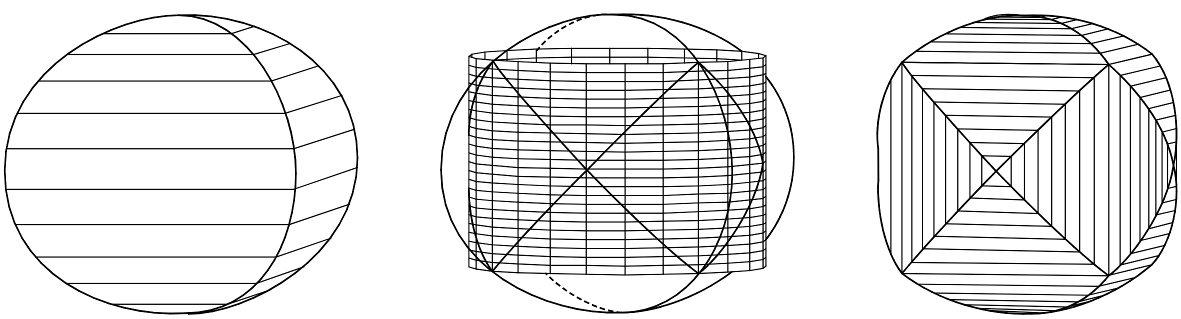
$$= 200\pi R^2\left[\frac{z^2}{2} - \frac{2z^3}{3H} + \frac{z^4}{4H^2}\right]_0^H = 200\pi R^2\left(\frac{H^2}{2} - \frac{2H^2}{3} + \frac{H^2}{4}\right)$$

$$= \tfrac{50}{3}\pi R^2H^2 = \tfrac{50}{3}\pi(62{,}000)^2(12{,}400)^2 \approx 3.1\times 10^{19}\ \text{ft-lb}$$

DISCOVERY PROJECT The Intersection of Three Cylinders

1. The three cylinders in the illustration in the text can be visualized as representing the surfaces $x^2 + y^2 = 1$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$. Then we sketch the solid of intersection with the coordinate axes and equations indicated. To be more precise, we start by finding the bounding curves of the solid (shown in the first graph below) enclosed by the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$: $x = \pm y = \pm\sqrt{1 - z^2}$ are the symmetric equations, and these can be expressed parametrically as $x = s$, $y = \pm s$, $z = \pm\sqrt{1 - s^2}$, $-1 \le s \le 1$. Now the cylinder $x^2 + y^2 = 1$ intersects these curves at the eight points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. The resulting solid has twelve curved faces bounded by "edges" which are arcs of circles, as shown in the third diagram. Each cylinder defines four of the twelve faces.



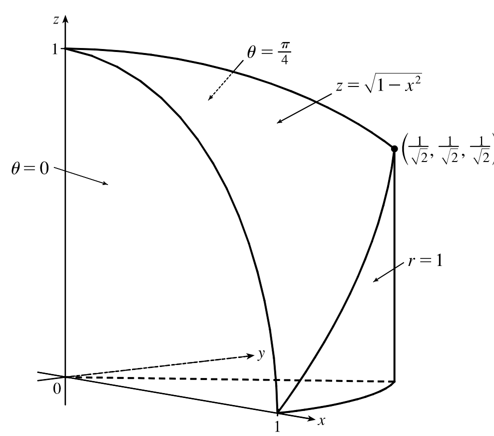


2. To find the volume, we split the solid into sixteen congruent pieces, one of which lies in the part of the first octant with $0 \le \theta \le \frac{\pi}{4}$. (Naturally, we use cylindrical coordinates!) This piece is described by

$\left\{(r, \theta, z) \mid 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{4}, 0 \le z \le \sqrt{1 - x^2}\right\}$,

and so, substituting $x = r\cos\theta$, the volume of the entire solid is

$$\begin{aligned} V &= 16 \int_0^{\pi/4} \int_0^1 \int_0^{\sqrt{1-x^2}} r\, dz\, dr\, d\theta \\ &= 16 \int_0^{\pi/4} \int_0^1 r\sqrt{1 - r^2\cos^2\theta}\, dr\, d\theta \\ &= 16 - 8\sqrt{2} \approx 4.6863 \end{aligned}$$

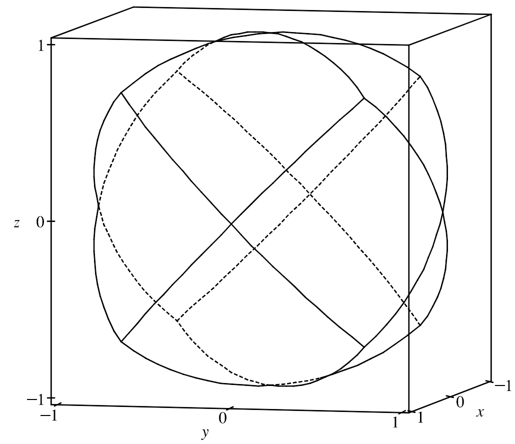


3. To graph the edges of the solid, we use parametrized curves similar to those found in Problem 1 for the intersection of two cylinders. We must restrict the parameter intervals so that each arc extends exactly to the desired vertex. One possible set of parametric equations (with all sign choices allowed) is

$x = r$, $y = \pm r$, $z = \pm\sqrt{1 - r^2}$, $-\frac{1}{\sqrt{2}} \le r \le \frac{1}{\sqrt{2}}$;

$x = \pm s$, $y = \pm\sqrt{1 - s^2}$, $z = s$, $-\frac{1}{\sqrt{2}} \le s \le \frac{1}{\sqrt{2}}$;

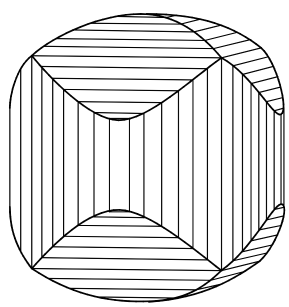
$x = \pm\sqrt{1 - t^2}$, $y = t$, $z = \pm t$, $-\frac{1}{\sqrt{2}} \le t \le \frac{1}{\sqrt{2}}$.



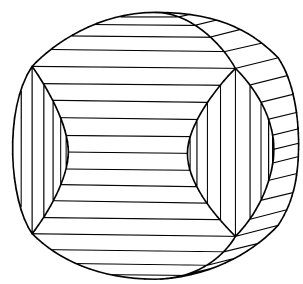
4. Let the three cylinders be $x^2 + y^2 = a^2$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$.

If $a < 1$, then the four faces defined by the cylinder $x^2 + y^2 = 1$ in Problem 1 collapse into a single face, as in the first graph. If $1 < a < \sqrt{2}$, then each pair of vertically opposed faces, defined by one of the other two cylinders, collapse into a single face, as in the second graph. If $a \ge \sqrt{2}$, then the vertical cylinder encloses the solid of intersection of the other two cylinders completely, so the solid of intersection coincides with the solid of intersection of the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$, as illustrated in Problem 1.

If we were to vary b or c instead of a, we would get solids with the same shape, but differently oriented.



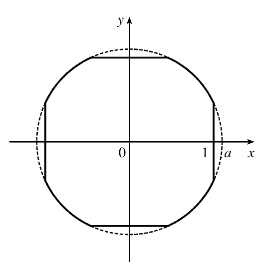
$a = 0.95, b = c = 1$

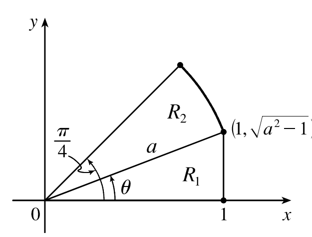


$a = 1.1, b = c = 1$

5. If $a < 1$, the solid looks similar to the first graph in Problem 4. As in Problem 2, we split the solid into sixteen congruent pieces, one of which can be described as the solid above the polar region $\{(r, \theta) \mid 0 \le r \le a, 0 \le \theta \le \frac{\pi}{4}\}$ in the xy-plane and below the surface $z = \sqrt{1 - x^2} = \sqrt{1 - r^2 \cos^2 \theta}$. Thus, the total volume is $V = 16 \int_0^{\pi/4} \int_0^a \sqrt{1 - r^2 \cos^2 \theta}\, r\, dr\, d\theta$.

If $a > 1$ and $a < \sqrt{2}$, we have a solid similar to the second graph in Problem 4. Its intersection with the xy-plane is graphed at the right. Again we split the solid into sixteen congruent pieces, one of which is the solid above the region shown in the second figure and below the surface $z = \sqrt{1 - x^2} = \sqrt{1 - r^2 \cos^2 \theta}$.





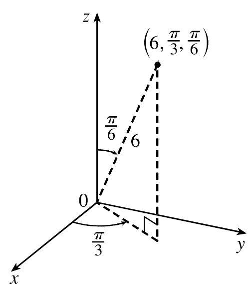
We split the region of integration where the outside boundary changes from the vertical line $x = 1$ to the circle $x^2 + y^2 = a^2$ or $r = a$. R_1 is a right triangle, so $\cos\theta = \frac{1}{a}$. Thus, the boundary between R_1 and R_2 is $\theta = \cos^{-1}\left(\frac{1}{a}\right)$ in polar coordinates, or $y = \sqrt{a^2 - 1}\,x$ in rectangular coordinates. Using rectangular coordinates for the region R_1 and polar coordinates for R_2, we find the total volume of the solid to be

$$V = 16\left[\int_0^1 \int_0^{\sqrt{a^2-1}\,x} \sqrt{1-x^2}\,dy\,dx + \int_{\cos^{-1}(1/a)}^{\pi/4} \int_0^a \sqrt{1 - r^2\cos^2\theta}\,r\,dr\,d\theta\right]$$

If $a \geq \sqrt{2}$, the cylinder $x^2 + y^2 = 1$ completely encloses the intersection of the other two cylinders, so the solid of intersection of the three cylinders coincides with the intersection of $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ as illustrated in Exercise 15.5.24. Its volume is $V = 16\int_0^1 \int_0^x \sqrt{1-x^2}\,dy\,dx$.

15.8 Triple Integrals in Spherical Coordinates

1. (a)

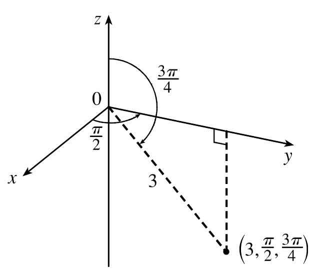


From Equations 1, $x = \rho\sin\phi\cos\theta = 6\sin\frac{\pi}{6}\cos\frac{\pi}{3} = 6\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{3}{2}$,

$y = \rho\sin\phi\sin\theta = 6\sin\frac{\pi}{6}\sin\frac{\pi}{3} = 6\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$, and

$z = \rho\cos\phi = 6\cos\frac{\pi}{6} = 6\cdot\frac{\sqrt{3}}{2} = 3\sqrt{3}$, so the point is $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 3\sqrt{3}\right)$ in rectangular coordinates.

(b)

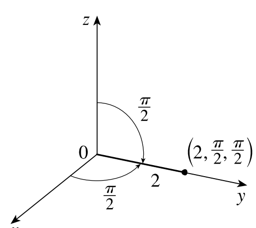


$x = 3\sin\frac{3\pi}{4}\cos\frac{\pi}{2} = 3\cdot\frac{\sqrt{2}}{2}\cdot 0 = 0$,

$y = 3\sin\frac{3\pi}{4}\sin\frac{\pi}{2} = 3\cdot\frac{\sqrt{2}}{2}\cdot 1 = \frac{3\sqrt{2}}{2}$, and

$z = 3\cos\frac{3\pi}{4} = 3\left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2}$, so the point is $\left(0, \frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$ in rectangular coordinates.

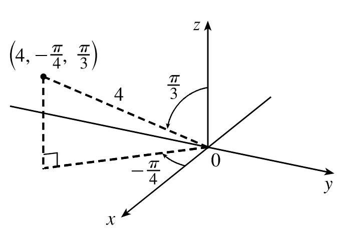
2. (a)



$x = 2\sin\frac{\pi}{2}\cos\frac{\pi}{2} = 2\cdot 1\cdot 0 = 0$, $y = 2\sin\frac{\pi}{2}\sin\frac{\pi}{2} = 2\cdot 1\cdot 1 = 2$,

$z = 2\cos\frac{\pi}{2} = 2\cdot 0 = 0$ so the point is $(0, 2, 0)$ in rectangular coordinates.

(b)



$x = 4\sin\frac{\pi}{3}\cos\left(-\frac{\pi}{4}\right) = 4\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \sqrt{6}$,

$y = 4\sin\frac{\pi}{3}\sin\left(-\frac{\pi}{4}\right) = 4\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{6}$,

$z = 4\cos\frac{\pi}{3} = 4\cdot\frac{1}{2} = 2$ so the point is $\left(\sqrt{6}, -\sqrt{6}, 2\right)$ in rectangular coordinates.

3. (a) From Equations 1 and 2, $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0^2+(-2)^2+0^2} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{0}{2} = 0 \;\Rightarrow\; \phi = \dfrac{\pi}{2}$, and

$\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{0}{2\sin(\pi/2)} = 0 \;\Rightarrow\; \theta = \dfrac{3\pi}{2}$ [since $y<0$]. Thus spherical coordinates are $\left(2, \dfrac{3\pi}{2}, \dfrac{\pi}{2}\right)$.

(b) $\rho = \sqrt{1+1+2} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{-\sqrt{2}}{2} \;\Rightarrow\; \phi = \dfrac{3\pi}{4}$, and

$\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{-1}{2\sin(3\pi/4)} = \dfrac{-1}{2\left(\sqrt{2}/2\right)} = -\dfrac{1}{\sqrt{2}} \;\Rightarrow\; \theta = \dfrac{3\pi}{4}$ [since $y>0$]. Thus spherical coordinates are $\left(2, \dfrac{3\pi}{4}, \dfrac{3\pi}{4}\right)$.

4. (a) $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+0+3} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{\sqrt{3}}{2} \;\Rightarrow\; \phi = \dfrac{\pi}{6}$, and $\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{1}{2\sin(\pi/6)} = 1 \;\Rightarrow$ $\theta = 0$. Thus spherical coordinates are $\left(2, 0, \dfrac{\pi}{6}\right)$.

(b) $\rho = \sqrt{3+1+12} = 4$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \;\Rightarrow\; \phi = \dfrac{\pi}{6}$, and $\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{\sqrt{3}}{4\sin(\pi/6)} = \dfrac{\sqrt{3}}{2} \;\Rightarrow$ $\theta = \dfrac{11\pi}{6}$ [since $y<0$]. Thus spherical coordinates are $\left(4, \dfrac{11\pi}{6}, \dfrac{\pi}{6}\right)$.

5. Since $\phi = \frac{\pi}{3}$ but ρ and θ can vary, the surface is the top half of a right circular cone with vertex at the origin and axis the positive z-axis. (See Figure 4.)

6. $\rho^2 - 3\rho + 2 = 0 \;\Rightarrow\; (\rho-1)(\rho-2) = 0 \;\Rightarrow\; \rho = 1$ or $\rho = 2$. Thus the equation represents two surfaces. In the case $\rho = 1$, the distance from any point to the origin is 1. Because θ and ϕ can vary, the surface is a sphere centered at the origin with radius 1. (See Figure 2.) Similarly, $\rho = 2$ is a sphere centered at the origin with radius 2.

Also, $\rho = 1 \;\Rightarrow\; \rho^2 = 1 \;\Rightarrow\; x^2+y^2+z^2 = 1$ which we recognize as the equation of the unit sphere, and similarly, $\rho = 2 \;\Rightarrow\; \rho^2 = 4 \;\Rightarrow\; x^2+y^2+z^2 = 4$.

7. From Equations 1 we have $z = \rho\cos\phi$, so $\rho\cos\phi = 1 \;\Leftrightarrow\; z = 1$, and the surface is the horizontal plane $z = 1$.

8. $\rho = \cos\phi \;\Rightarrow\; \rho^2 = \rho\cos\phi \;\Leftrightarrow\; x^2+y^2+z^2 = z \;\Leftrightarrow\; x^2+y^2+z^2-z+\frac{1}{4} = \frac{1}{4} \;\Leftrightarrow\; x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$. Therefore, the surface is a sphere of radius $\frac{1}{2}$ centered at $\left(0, 0, \frac{1}{2}\right)$.

9. (a) From Equation 2 we have $\rho^2 = x^2+y^2+z^2$, so $x^2+y^2+z^2 = 9 \;\Leftrightarrow\; \rho^2 = 9 \;\Rightarrow\; \rho = 3$ (since $\rho \geq 0$).

(b) From Equations 1 we have $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, so the equation $x^2-y^2-z^2 = 1$ becomes $(\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2 - (\rho\cos\phi)^2 = 1 \;\Leftrightarrow\; (\rho^2\sin^2\phi)(\cos^2\theta - \sin^2\theta) - \rho^2\cos^2\phi = 1 \;\Leftrightarrow$ $\rho^2(\sin^2\phi\cos 2\theta - \cos^2\phi) = 1$.

10. (a) $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$, so the equation $z = x^2 + y^2$ becomes

$\rho\cos\phi = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$ or $\rho\cos\phi = \rho^2\sin^2\phi$. If $\rho \neq 0$, this becomes $\cos\phi = \rho\sin^2\phi$

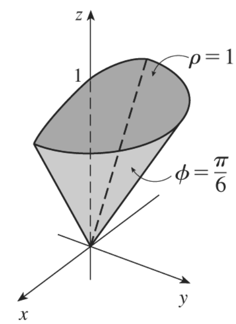
or $\rho = \cos\phi\csc^2\phi$ or $\rho = \cot\phi\csc\phi$. ($\rho = 0$ corresponds to the origin which is included in the surface.)

(b) The equation $z = x^2 - y^2$ becomes $\rho\cos\phi = (\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2$

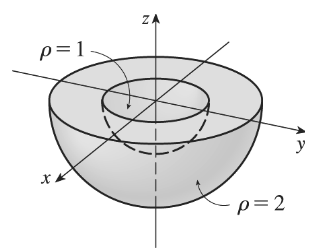
or $\rho\cos\phi = \rho^2(\sin^2\phi)(\cos^2\theta - \sin^2\theta) \quad\Leftrightarrow\quad \rho\cos\phi = \rho^2\sin^2\phi\cos 2\theta$. If $\rho \neq 0$, this becomes

$\cos\phi = \rho\sin^2\phi\cos 2\theta$. ($\rho = 0$ corresponds to the origin which is included in the surface.)

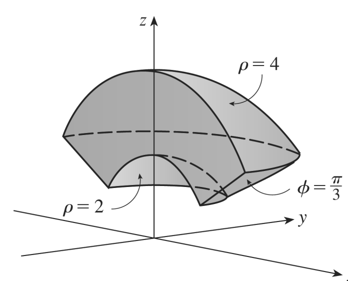
11. $\rho \leq 1$ represents the (solid) unit ball. $0 \leq \phi \leq \frac{\pi}{6}$ restricts the solid to that portion on or above the cone $\phi = \frac{\pi}{6}$, and $0 \leq \theta \leq \pi$ further restricts the solid to that portion on or to the right of the xz-plane.



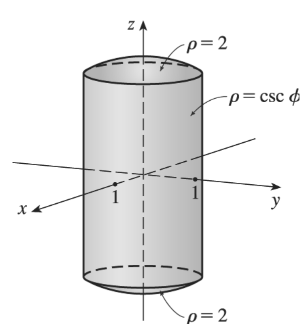
12. $1 \leq \rho \leq 2$ represents the solid region between and including the spheres of radii 1 and 2, centered at the origin. $\frac{\pi}{2} \leq \phi \leq \pi$ restricts the solid to that portion on or below the xy-plane.



13. $2 \leq \rho \leq 4$ represents the solid region between and including the spheres of radii 2 and 4, centered at the origin. $0 \leq \phi \leq \frac{\pi}{3}$ restricts the solid to that portion on or above the cone $\phi = \frac{\pi}{3}$, and $0 \leq \theta \leq \pi$ further restricts the solid to that portion on or to the right of the xz-plane.



14. $\rho \leq 2$ represents the solid sphere of radius 2 centered at the origin. Notice that $x^2 + y^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = \rho^2\sin^2\phi$. Then $\rho = \csc\phi \quad\Rightarrow\quad \rho\sin\phi = 1 \quad\Rightarrow\quad \rho^2\sin^2\phi = x^2 + y^2 = 1$, so $\rho \leq \csc\phi$ restricts the solid to that portion on or inside the circular cylinder $x^2 + y^2 = 1$.

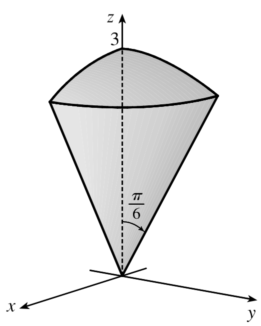


15. $z \geq \sqrt{x^2+y^2}$ because the solid lies above the cone. Squaring both sides of this inequality gives $z^2 \geq x^2+y^2 \quad\Rightarrow$ $2z^2 \geq x^2+y^2+z^2=\rho^2 \quad\Rightarrow\quad z^2=\rho^2\cos^2\phi \geq \frac{1}{2}\rho^2 \quad\Rightarrow\quad \cos^2\phi \geq \frac{1}{2}$. The cone opens upward so that the inequality is $\cos\phi \geq \frac{1}{\sqrt{2}}$, or equivalently $0 \leq \phi \leq \frac{\pi}{4}$. In spherical coordinates the sphere $z=x^2+y^2+z^2$ is $\rho\cos\phi=\rho^2 \quad\Rightarrow$ $\rho=\cos\phi$. $0 \leq \rho \leq \cos\phi$ because the solid lies below the sphere. The solid can therefore be described as the region in spherical coordinates satisfying $0 \leq \rho \leq \cos\phi$, $0 \leq \phi \leq \frac{\pi}{4}$.

16. (a) The hollow ball is a spherical shell with outer radius 15 cm and inner radius 14.5 cm. If we center the ball at the origin of the coordinate system and use centimeters as the unit of measurement, then spherical coordinates conveniently describe the hollow ball as $14.5 \leq \rho \leq 15$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$.

(b) If we position the ball as in part (a), one possibility is to take the half of the ball that is above the xy-plane which is described by $14.5 \leq \rho \leq 15$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi/2$.

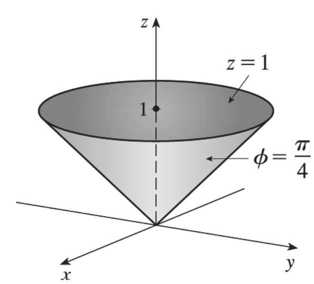
17.



The region of integration is given in spherical coordinates by $E=\{(\rho,\theta,\phi) \mid 0 \leq \rho \leq 3,\ 0 \leq \theta \leq \pi/2,\ 0 \leq \phi \leq \pi/6\}$. This represents the solid region in the first octant bounded above by the sphere $\rho=3$ and below by the cone $\phi=\pi/6$.

$$\begin{aligned}\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi &= \int_0^{\pi/6}\sin\phi\,d\phi\int_0^{\pi/2}d\theta\int_0^3\rho^2\,d\rho\\ &= \big[-\cos\phi\big]_0^{\pi/6}\,\big[\theta\big]_0^{\pi/2}\,\big[\tfrac{1}{3}\rho^3\big]_0^3\\ &= \left(1-\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)(9)=\frac{9\pi}{4}\left(2-\sqrt{3}\right)\end{aligned}$$

18. The region of integration is given in spherical coordinates by $E=\{(\rho,\theta,\phi) \mid 0 \leq \rho \leq \sec\phi,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi/4\}$. $\rho=\sec\phi \quad\Leftrightarrow\quad \rho\cos\phi=1 \quad\Leftrightarrow\quad z=1$, so E is the solid region above the cone $\phi=\pi/4$ and below the plane $z=1$.



$$\begin{aligned}\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\phi}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi &= \int_0^{\pi/4}\int_0^{2\pi}\left[\tfrac{1}{3}\rho^3\sin\phi\right]_{\rho=0}^{\rho=\sec\phi}d\theta\,d\phi\\ &= \int_0^{\pi/4}\int_0^{2\pi}\tfrac{1}{3}\sec^3\phi\sin\phi\,d\theta\,d\phi=\tfrac{1}{3}\int_0^{\pi/4}\sec^3\phi\sin\phi\,d\phi\int_0^{2\pi}d\theta\\ &= \tfrac{1}{3}\int_0^{\pi/4}\tan\phi\sec^2\phi\,d\phi\int_0^{2\pi}d\theta=\tfrac{1}{3}\left[\tfrac{1}{2}\tan^2\phi\right]_0^{\pi/4}\big[\theta\big]_0^{2\pi}\\ &= \tfrac{1}{3}\left(\tfrac{1}{2}-0\right)(2\pi)=\tfrac{\pi}{3}\end{aligned}$$

19. The solid E is most conveniently described if we use cylindrical coordinates: $E=\left\{(r,\theta,z) \mid 0 \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 3,\ 0 \leq z \leq 2\right\}$. Then $\iiint_E f(x,y,z)\,dV=\int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta,r\sin\theta,z)\,r\,dz\,dr\,d\theta$.

20. The solid E is most conveniently described if we use spherical coordinates:

$E = \left\{(\rho,\theta,\phi) \mid 1 \le \rho \le 2,\ \frac{\pi}{2} \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2}\right\}$. Then

$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/2} \int_{\pi/2}^{2\pi} \int_1^2 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$

21. In spherical coordinates, B is represented by $\{(\rho,\theta,\phi) \mid 0 \le \rho \le 5, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$. Thus

$$\begin{aligned}\iiint_B (x^2+y^2+z^2)^2\,dV &= \int_0^{\pi}\int_0^{2\pi}\int_0^5 (\rho^2)^2\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\sin\phi\,d\phi\,\int_0^{2\pi}d\theta\,\int_0^5 \rho^6\,d\rho \\ &= \big[-\cos\phi\big]_0^{\pi}\,\big[\,\theta\,\big]_0^{2\pi}\,\big[\tfrac{1}{7}\rho^7\big]_0^5 = (2)(2\pi)\big(\tfrac{78{,}125}{7}\big) \\ &= \tfrac{312{,}500}{7}\pi \approx 140{,}249.7\end{aligned}$$

22. In spherical coordinates, E is represented by $\left\{(\rho,\theta,\phi) \,\middle|\, 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{3}\right\}$. Thus

$$\begin{aligned}\iiint_E y^2z^2\,dV &= \int_0^{\pi/3}\int_0^{2\pi}\int_0^1 (\rho\sin\phi\sin\theta)^2(\rho\cos\phi)^2\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \int_0^{\pi/3}\sin^3\phi\cos^2\phi\,d\phi\,\int_0^{2\pi}\sin^2\theta\,d\theta\,\int_0^1\rho^6\,d\rho \\ &= \int_0^{\pi/3}(1-\cos^2\phi)\cos^2\phi\sin\phi\,d\phi\,\int_0^{2\pi}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta\,\int_0^1\rho^6\,d\rho \\ &= \big[\tfrac{1}{5}\cos^5\phi - \tfrac{1}{3}\cos^3\phi\big]_0^{\pi/3}\,\big[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\big]_0^{2\pi}\,\big[\tfrac{1}{7}\rho^7\big]_0^1 \\ &= \Big[\tfrac{1}{5}\big(\tfrac{1}{2}\big)^5 - \tfrac{1}{3}\big(\tfrac{1}{2}\big)^3 - \tfrac{1}{5} + \tfrac{1}{3}\Big](\pi - 0)\big(\tfrac{1}{7} - 0\big) = \tfrac{47}{480}\cdot\pi\cdot\tfrac{1}{7} = \tfrac{47}{3360}\pi\end{aligned}$$

23. In spherical coordinates, E is represented by $\{(\rho,\theta,\phi) \mid 2 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$ and

$x^2+y^2 = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = \rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right) = \rho^2\sin^2\phi$. Thus

$$\begin{aligned}\iiint_E (x^2+y^2)\,dV &= \int_0^{\pi}\int_0^{2\pi}\int_2^3 (\rho^2\sin^2\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\sin^3\phi\,d\phi\,\int_0^{2\pi}d\theta\,\int_2^3\rho^4\,d\rho \\ &= \int_0^{\pi}(1-\cos^2\phi)\,\sin\phi\,d\phi\,\big[\,\theta\,\big]_0^{2\pi}\,\big[\tfrac{1}{5}\rho^5\big]_2^3 = \big[-\cos\phi + \tfrac{1}{3}\cos^3\phi\big]_0^{\pi}\,(2\pi)\cdot\tfrac{1}{5}(243-32) \\ &= \big(1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\big)(2\pi)\big(\tfrac{211}{5}\big) = \tfrac{1688\pi}{15}\end{aligned}$$

24. In spherical coordinates, E is represented by $\{(\rho,\theta,\phi) \mid 0 \le \rho \le 3, 0 \le \theta \le \pi, 0 \le \phi \le \pi\}$. Thus

$$\begin{aligned}\iiint_E y^2\,dV &= \int_0^{\pi}\int_0^{\pi}\int_0^3 (\rho\sin\phi\sin\theta)^2\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\sin^3\phi\,d\phi\,\int_0^{\pi}\sin^2\theta\,d\theta\,\int_0^3\rho^4\,d\rho \\ &= \int_0^{\pi}(1-\cos^2\phi)\,\sin\phi\,d\phi\,\int_0^{\pi}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta\,\int_0^3\rho^4\,d\rho \\ &= \big[-\cos\phi + \tfrac{1}{3}\cos^3\phi\big]_0^{\pi}\,\big[\tfrac{1}{2}\big(\theta - \tfrac{1}{2}\sin 2\theta\big)\big]_0^{\pi}\,\big[\tfrac{1}{5}\rho^5\big]_0^3 \\ &= \big(\tfrac{2}{3}+\tfrac{2}{3}\big)\big(\tfrac{1}{2}\pi\big)\big(\tfrac{1}{5}(243)\big) = \big(\tfrac{4}{3}\big)\big(\tfrac{\pi}{2}\big)\big(\tfrac{243}{5}\big) = \tfrac{162\pi}{5}\end{aligned}$$

25. In spherical coordinates, E is represented by $\left\{(\rho,\theta,\phi) \,\middle|\, 0 \le \rho \le 1, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}\right\}$. Thus

$$\begin{aligned}\iiint_E xe^{x^2+y^2+z^2}\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\cos\theta)e^{\rho^2}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\sin^2\phi\,d\phi\,\int_0^{\pi/2}\cos\theta\,d\theta\,\int_0^1\rho^3e^{\rho^2}\,d\rho \\ &= \int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\phi)\,d\phi\,\int_0^{\pi/2}\cos\theta\,d\theta\,\left(\big[\tfrac{1}{2}\rho^2e^{\rho^2}\big]_0^1 - \int_0^1\rho e^{\rho^2}\,d\rho\right) \\ &\qquad \left[\text{integrate by parts with } u = \rho^2,\ dv = \rho e^{\rho^2}\,d\rho\right] \\ &= \big[\tfrac{1}{2}\phi - \tfrac{1}{4}\sin 2\phi\big]_0^{\pi/2}\,\big[\sin\theta\big]_0^{\pi/2}\,\big[\tfrac{1}{2}\rho^2e^{\rho^2} - \tfrac{1}{2}e^{\rho^2}\big]_0^1 = \big(\tfrac{\pi}{4} - 0\big)(1-0)\big(0+\tfrac{1}{2}\big) = \tfrac{\pi}{8}\end{aligned}$$

26. In spherical coordinates, the cone $z = \sqrt{x^2 + y^2}$ is equivalent to $\phi = \pi/4$ (as in Example 4) and E is represented by $\{(\rho, \theta, \phi) \,|\, 1 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4\}$. Also $\sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2} = \rho$, so

$$\iiint_E \sqrt{x^2 + y^2 + z^2}\, dV = \int_0^{\pi/4} \int_0^{2\pi} \int_1^2 \rho \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_0^{\pi/4} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_1^2 \rho^3 \, d\rho$$

$$= [-\cos\phi]_0^{\pi/4} \, [\,\theta\,]_0^{2\pi} \, \left[\tfrac{1}{4}\rho^4\right]_1^2 = \left(-\tfrac{\sqrt{2}}{2} + 1\right)(2\pi) \cdot \tfrac{1}{4}(16 - 1) = \tfrac{15}{2}\pi\left(1 - \tfrac{\sqrt{2}}{2}\right)$$

27. The solid region is given by $E = \left\{(\rho, \theta, \phi) \mid 0 \le \rho \le a, 0 \le \theta \le 2\pi, \frac{\pi}{6} \le \phi \le \frac{\pi}{3}\right\}$ and its volume is

$$V = \iiint_E dV = \int_{\pi/6}^{\pi/3} \int_0^{2\pi} \int_0^a \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_{\pi/6}^{\pi/3} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^a \rho^2 \, d\rho$$

$$= [-\cos\phi]_{\pi/6}^{\pi/3} \, [\theta]_0^{2\pi} \, \left[\tfrac{1}{3}\rho^3\right]_0^a = \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\right)(2\pi)\left(\tfrac{1}{3}a^3\right) = \tfrac{\sqrt{3}-1}{3}\pi a^3$$

28. If we center the ball at the origin, then the ball is given by $B = \{(\rho, \theta, \phi) \mid 0 \le \rho \le a, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$ and the distance from any point (x, y, z) in the ball to the center $(0, 0, 0)$ is $\sqrt{x^2 + y^2 + z^2} = \rho$. Thus the average distance is

$$\frac{1}{V(B)} \iiint_B \rho \, dV = \frac{1}{\frac{4}{3}\pi a^3} \int_0^{\pi} \int_0^{2\pi} \int_0^a \rho \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \frac{3}{4\pi a^3} \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^a \rho^3 \, d\rho$$

$$= \frac{3}{4\pi a^3} \left[-\cos\phi\right]_0^{\pi} \left[\,\theta\,\right]_0^{2\pi} \left[\tfrac{1}{4}\rho^4\right]_0^a = \frac{3}{4\pi a^3}(2)(2\pi)\left(\tfrac{1}{4}a^4\right) = \tfrac{3}{4}a$$

29. (a) Since $\rho = 4\cos\phi$ implies $\rho^2 = 4\rho\cos\phi \quad \Leftrightarrow \quad x^2 + y^2 + z^2 = 4z \quad \Leftrightarrow \quad x^2 + y^2 + (z - 2)^2 = 4$, the equation is that of a sphere of radius 2 with center at $(0, 0, 2)$. Thus

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \left[\tfrac{1}{3}\rho^3\right]_{\rho=0}^{\rho=4\cos\phi} \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \left(\tfrac{64}{3}\cos^3\phi\right) \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\tfrac{16}{3}\cos^4\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi} -\tfrac{16}{3}\left(\tfrac{1}{16} - 1\right) d\theta = 5\theta\Big]_0^{2\pi} = 10\pi$$

(b) By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$M_{xy} = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{4\cos\phi} \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \cos\phi \sin\phi \left(64\cos^4\phi\right) d\phi \, d\theta$$

$$= \int_0^{2\pi} 64 \left[-\tfrac{1}{6}\cos^6\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi} \tfrac{21}{2} \, d\theta = 21\pi$$

Hence $(\overline{x}, \overline{y}, \overline{z}) = (0, 0, 21\pi/(10\pi)) = (0, 0, 2.1)$.

30. In spherical coordinates, the sphere $x^2 + y^2 + z^2 = 4$ is equivalent to $\rho = 2$ and the cone $z = \sqrt{x^2 + y^2}$ is represented by $\phi = \frac{\pi}{4}$ (as in Example 4). Thus, the solid is given by $\left\{(\rho, \theta, \phi) \,\middle|\, 0 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ \frac{\pi}{4} \le \phi \le \frac{\pi}{2}\right\}$ and

$$V = \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_{\pi/4}^{\pi/2} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^2 \rho^2 \, d\rho$$

$$= \left[-\cos\phi\right]_{\pi/4}^{\pi/2} \left[\,\theta\,\right]_0^{2\pi} \left[\tfrac{1}{3}\rho^3\right]_0^2 = \left(\tfrac{\sqrt{2}}{2}\right)(2\pi)\left(\tfrac{8}{3}\right) = \tfrac{8\sqrt{2}\,\pi}{3}$$

31. (a) By the symmetry of the region, $M_{yz} = 0$ and $M_{xz} = 0$. Assuming constant density K,

$m = \iiint_E K\,dV = K \iiint_E dV = \frac{\pi}{8}K$ (from Example 4). Then

$$\begin{aligned} M_{xy} = \iiint_E z\,K\,dV &= K\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi}(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = K\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\cos\phi\left[\tfrac{1}{4}\rho^4\right]_{\rho=0}^{\rho=\cos\phi}d\phi\,d\theta \\ &= \tfrac{1}{4}K\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\cos\phi\left(\cos^4\phi\right)d\phi\,d\theta = \tfrac{1}{4}K\int_0^{2\pi}d\theta\int_0^{\pi/4}\cos^5\phi\sin\phi\,d\phi \\ &= \tfrac{1}{4}K\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{6}\cos^6\phi\right]_0^{\pi/4} = \tfrac{1}{4}K(2\pi)\left(-\tfrac{1}{6}\right)\left[\left(\tfrac{\sqrt{2}}{2}\right)^6 - 1\right] = -\tfrac{\pi}{12}K\left(-\tfrac{7}{8}\right) = \tfrac{7\pi}{96}K \end{aligned}$$

Thus the centroid is $(\overline{x},\overline{y},\overline{z}) = \left(\dfrac{M_{yz}}{m},\dfrac{M_{xz}}{m},\dfrac{M_{xy}}{m}\right) = \left(0,0,\dfrac{7\pi K/96}{\pi K/8}\right) = \left(0,0,\frac{7}{12}\right)$.

(b) As in Exercise 23, $x^2+y^2 = \rho^2\sin^2\phi$ and

$$\begin{aligned} I_z = \iiint_E \left(x^2+y^2\right)K\,dV &= K\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi}\left(\rho^2\sin^2\phi\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = K\int_0^{2\pi}\int_0^{\pi/4}\sin^3\phi\left[\tfrac{1}{5}\rho^5\right]_{\rho=0}^{\rho=\cos\phi}d\phi\,d\theta \\ &= \tfrac{1}{5}K\int_0^{2\pi}\int_0^{\pi/4}\sin^3\phi\cos^5\phi\,d\phi\,d\theta = \tfrac{1}{5}K\int_0^{2\pi}d\theta\int_0^{\pi/4}\cos^5\phi\left(1-\cos^2\phi\right)\sin\phi\,d\phi \\ &= \tfrac{1}{5}K\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{6}\cos^6\phi + \tfrac{1}{8}\cos^8\phi\right]_0^{\pi/4} \\ &= \tfrac{1}{5}K(2\pi)\left[-\tfrac{1}{6}\left(\tfrac{\sqrt{2}}{2}\right)^6 + \tfrac{1}{8}\left(\tfrac{\sqrt{2}}{2}\right)^8 + \tfrac{1}{6} - \tfrac{1}{8}\right] = \tfrac{2\pi}{5}K\left(\tfrac{11}{384}\right) = \tfrac{11\pi}{960}K \end{aligned}$$

32. (a) Placing the center of the base at $(0,0,0)$, $\rho(x,y,z) = K\sqrt{x^2+y^2+z^2}$ is the density function. So

$$\begin{aligned} m &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3\sin\phi\,d\rho\,d\phi\,d\theta = K\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\phi\,d\phi\int_0^a\rho^3\,d\rho \\ &= K\big[\,\theta\,\big]_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/2}\left[\tfrac{1}{4}\rho^4\right]_0^a = K(2\pi)(1)\left(\tfrac{1}{4}a^4\right) = \tfrac{1}{2}\pi K a^4 \end{aligned}$$

(b) By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$\begin{aligned} M_{xy} &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^4\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = K\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi\int_0^a\rho^4\,d\rho \\ &= K\big[\,\theta\,\big]_0^{2\pi}\left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi/2}\left[\tfrac{1}{5}\rho^5\right]_0^a = K(2\pi)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{5}a^5\right) = \tfrac{1}{5}\pi K a^5 \end{aligned}$$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\frac{2}{5}a\right)$.

(c) $I_z = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a\left(K\rho^3\sin\phi\right)\left(\rho^2\sin^2\phi\right)d\rho\,d\phi\,d\theta = K\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^a\rho^5\,d\rho$

$$= K\big[\,\theta\,\big]_0^{2\pi}\left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/2}\left[\tfrac{1}{6}\rho^6\right]_0^a = K(2\pi)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{6}a^6\right) = \tfrac{2}{9}\pi K a^6$$

33. (a) The density function is $\rho(x,y,z) = K$, a constant, and by the symmetry of the problem $M_{xz} = M_{yz} = 0$. Then

$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3\sin\phi\,\cos\phi\,d\rho\,d\phi\,d\theta = \frac{1}{2}\pi K a^4\int_0^{\pi/2}\sin\phi\,\cos\phi\,d\phi = \frac{1}{8}\pi K a^4$. But the mass is K(volume of the hemisphere) $= \frac{2}{3}\pi K a^3$, so the centroid is $\left(0,0,\frac{3}{8}a\right)$.

(b) Place the center of the base at $(0,0,0)$; the density function is $\rho(x,y,z) = K$. By symmetry, the moments of inertia about any two such diameters will be equal, so we just need to find I_x:

$$\begin{aligned} I_x &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a\left(K\rho^2\sin\phi\right)\rho^2\left(\sin^2\phi\,\sin^2\theta + \cos^2\phi\right)d\rho\,d\phi\,d\theta \\ &= K\int_0^{2\pi}\int_0^{\pi/2}\left(\sin^3\phi\,\sin^2\theta + \sin\phi\,\cos^2\phi\right)\left(\tfrac{1}{5}a^5\right)d\phi\,d\theta \\ &= \tfrac{1}{5}Ka^5\int_0^{2\pi}\left[\sin^2\theta\left(-\cos\phi + \tfrac{1}{3}\cos^3\phi\right) + \left(-\tfrac{1}{3}\cos^3\phi\right)\right]_{\phi=0}^{\phi=\pi/2}d\theta = \tfrac{1}{5}Ka^5\int_0^{2\pi}\left[\tfrac{2}{3}\sin^2\theta + \tfrac{1}{3}\right]d\theta \\ &= \tfrac{1}{5}Ka^5\left[\tfrac{2}{3}\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) + \tfrac{1}{3}\theta\right]_0^{2\pi} = \tfrac{1}{5}Ka^5\left[\tfrac{2}{3}(\pi - 0) + \tfrac{1}{3}(2\pi - 0)\right] = \tfrac{4}{15}Ka^5\pi \end{aligned}$$

34. Place the center of the base at $(0,0,0)$, then the density is $\rho(x,y,z) = Kz$, K a constant. Then

$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (K\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi K\int_0^{\pi/2}\cos\phi\sin\phi\cdot\frac{1}{4}a^4\,d\phi = \frac{1}{2}\pi Ka^4\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \frac{\pi}{4}Ka^4.$

By the symmetry of the problem $M_{xz} = M_{yz} = 0$, and

$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^4\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2}{5}\pi Ka^5\int_0^{\pi/2}\cos^2\phi\sin\phi\,d\phi = \frac{2}{5}\pi Ka^5\left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \frac{2}{15}\pi Ka^5.$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\frac{8}{15}a\right)$.

35. In spherical coordinates $z = \sqrt{x^2+y^2}$ becomes $\phi = \frac{\pi}{4}$ (as in Example 4). Then

$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}d\theta\int_0^{\pi/4}\sin\phi\,d\phi\int_0^1\rho^2\,d\rho = 2\pi\left(-\frac{\sqrt{2}}{2}+1\right)\left(\frac{1}{3}\right) = \frac{1}{3}\pi\left(2-\sqrt{2}\right),$

$M_{xy} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = 2\pi\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/4}\left(\frac{1}{4}\right) = \frac{\pi}{8}$ and by symmetry $M_{yz} = M_{xz} = 0$.

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\dfrac{3}{8\left(2-\sqrt{2}\right)}\right)$.

36. Place the center of the sphere at $(0,0,0)$, let the diameter of intersection be along the z-axis, one of the planes be the xz-plane and the other be the plane whose angle with the xz-plane is $\theta = \frac{\pi}{6}$. Then in spherical coordinates the volume is given by

$V = \int_0^{\pi/6}\int_0^{\pi}\int_0^a \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/6}d\theta\int_0^{\pi}\sin\phi\,d\phi\int_0^a\rho^2\,d\rho = \frac{\pi}{6}(2)\left(\frac{1}{3}a^3\right) = \frac{1}{9}\pi a^3.$

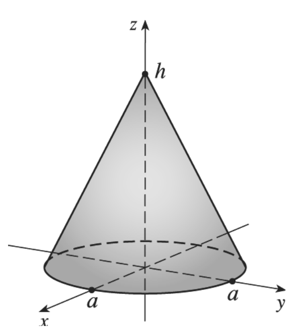
37. (a) If we orient the cylinder so that its axis is the z-axis and its base lies in the xy-plane, then the cylinder is described, in cylindrical coordinates, by $E = \{(r,\theta,z) \mid 0\le r\le a,\ 0\le\theta\le 2\pi,\ 0\le z\le h\}$. Assuming constant density K, the moment of inertia about its axis (the z-axis) is

$$\begin{aligned} I_z &= \iiint_E (x^2+y^2)\,\rho(x,y,z)\,dV = \int_0^{2\pi}\int_0^a\int_0^h K(r^2)\,r\,dz\,dr\,d\theta = K\int_0^{2\pi}d\theta\int_0^a r^3\,dr\int_0^h dz \\ &= K\left[\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^a\left[z\right]_0^h = K\,(2\pi)\left(\tfrac{1}{4}a^4\right)(h) = \tfrac{1}{2}\pi Ka^4h \end{aligned}$$

(b) By symmetry, the moments of inertia about any two diameters of the base will be equal, and one of the diameters lies on the x-axis, so we compute:

$$\begin{aligned} I_x &= \iiint_E (y^2+z^2)\,\rho(x,y,z)\,dV = \int_0^{2\pi}\int_0^a\int_0^h K(r^2\sin^2\theta+z^2)\,r\,dz\,dr\,d\theta \\ &= K\int_0^{2\pi}\int_0^a\int_0^h r^3\sin^2\theta\,dz\,dr\,d\theta + K\int_0^{2\pi}\int_0^a\int_0^h rz^2\,dz\,dr\,d\theta \\ &= K\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^a r^3\,dr\int_0^h dz + K\int_0^{2\pi}d\theta\int_0^a r\,dr\int_0^h z^2\,dz \\ &= K\left[\tfrac{1}{2}\theta-\tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^a\left[z\right]_0^h + K\left[\theta\right]_0^{2\pi}\left[\tfrac{1}{2}r^2\right]_0^a\left[\tfrac{1}{3}z^3\right]_0^h \\ &= K\,(\pi)\left(\tfrac{1}{4}a^4\right)(h) + K\,(2\pi)\left(\tfrac{1}{2}a^2\right)\left(\tfrac{1}{3}h^3\right) = \tfrac{1}{12}\pi Ka^2h(3a^2+4h^2) \end{aligned}$$

38.



Orient the cone so that its axis is the z-axis and its base lies in the xy-plane, as shown in the figure. (Then the z-axis is the axis of the cone and the x-axis contains a diameter of the base.) A right circular cone with axis the z-axis and vertex at the origin has equation $z^2 = c^2(x^2+y^2)$. Here we have the bottom frustum, shifted upward h units, and with $c^2 = h^2/a^2$ so that the cone includes the point $(a,0,0)$. Thus an equation of the cone in rectangular coordinates is $z = h - \frac{h}{a}\sqrt{x^2+y^2}$, $0\le z\le h$. In cylindrical coordinates, the cone is described by

$$E = \left\{(r,\theta,z) \mid 0 \le r \le a,\ 0 \le \theta \le 2\pi,\ 0 \le z \le h\left(1 - \tfrac{1}{a}r\right)\right\}$$

(a) Assuming constant density K, the moment of inertia about its axis (the z-axis) is

$$\begin{aligned} I_z &= \iiint_E (x^2+y^2)\,\rho(x,y,z)\,dV = \int_0^{2\pi}\!\int_0^a\!\int_0^{h(1-r/a)} K(r^2)\,r\,dz\,dr\,d\theta \\ &= K\int_0^{2\pi}\!\int_0^a \left[r^3 z\right]_{z=0}^{z=h(1-r/a)} dr\,d\theta = K\int_0^{2\pi}\!\int_0^a r^3 h\left(1-\tfrac{1}{a}r\right) dr\,d\theta \\ &= Kh\int_0^{2\pi} d\theta \int_0^a \left(r^3 - \tfrac{1}{a}r^4\right) dr = Kh\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac14 r^4 - \tfrac{1}{5a}r^5\right]_0^a \\ &= Kh\,(2\pi)\left(\tfrac14 a^4 - \tfrac15 a^4\right) = \tfrac{1}{10}\pi K a^4 h \end{aligned}$$

(b) By symmetry, the moments of intertia about any two diameters of the base will be equal, and one of the diameters lies on the x-axis, so we compute:

$$\begin{aligned} I_x &= \iiint_E (y^2+z^2)\,\rho(x,y,z)\,dV = \int_0^{2\pi}\!\int_0^a\!\int_0^{h(1-r/a)} K(r^2\sin^2\theta + z^2)\,r\,dz\,dr\,d\theta \\ &= K\int_0^{2\pi}\!\int_0^a \left[(r^3\sin^2\theta)z + \tfrac13 r z^3\right]_{z=0}^{z=h(1-r/a)} dr\,d\theta \\ &= K\int_0^{2\pi}\!\int_0^a \left[(r^3\sin^2\theta)\left(h\left(1-\tfrac1a r\right)\right) + \tfrac13 r\left(h\left(1-\tfrac1a r\right)\right)^3\right] dr\,d\theta \\ &= Kh\int_0^{2\pi}\!\int_0^a (r^3\sin^2\theta)\left(1-\tfrac1a r\right) dr\,d\theta + Kh^3\int_0^{2\pi}\!\int_0^a \tfrac13 r\left(1-\tfrac1a r\right)^3 dr\,d\theta \\ &= Kh\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^a\left(r^3 - \tfrac1a r^4\right)dr + \tfrac13 Kh^3\int_0^{2\pi}d\theta\int_0^a\left(r - \tfrac3a r^2 + \tfrac{3}{a^2}r^3 - \tfrac{1}{a^3}r^4\right)dr \\ &= Kh\left[\tfrac12\theta - \tfrac14\sin 2\theta\right]_0^{2\pi}\left[\tfrac14 r^4 - \tfrac{1}{5a}r^5\right]_0^a + \tfrac13 Kh^3\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac12 r^2 - \tfrac1a r^3 + \tfrac{3}{4a^2}r^4 - \tfrac{1}{5a^3}r^5\right]_0^a \\ &= Kh\,(\pi)\left(\tfrac14 a^4 - \tfrac15 a^4\right) + \tfrac13 Kh^3\,(2\pi)\left(\tfrac12 a^2 - a^2 + \tfrac34 a^2 - \tfrac15 a^2\right) \\ &= \pi Kh\left(\tfrac{1}{20}a^4\right) + \tfrac23\pi Kh^3\left(\tfrac{1}{20}a^2\right) = \pi K a^2 h\left(\tfrac{1}{20}a^2 + \tfrac{1}{30}h^2\right) \end{aligned}$$

39. In cylindrical coordinates the paraboloid is given by $z = r^2$ and the plane by $z = 2r\sin\theta$ and the projection of the intersection onto the xy-plane is the circle $r = 2\sin\theta$. Then $\iiint_E z\,dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} rz\,dz\,dr\,d\theta = \frac{5\pi}{6}$ [using a CAS].

40. (a) The region enclosed by the torus is $\{(\rho,\theta,\phi) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \pi, 0 \le \rho \le \sin\phi\}$, so its volume is

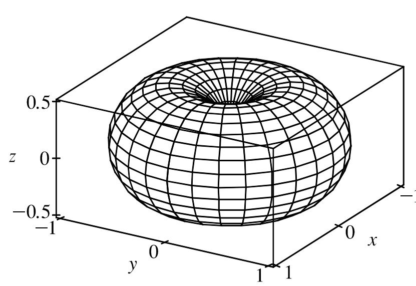
$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\pi}\frac13\sin^4\phi\,d\phi = \frac23\pi\left[\frac38\phi - \frac14\sin 2\phi + \frac{1}{16}\sin 4\phi\right]_0^{\pi} = \frac14\pi^2.$

(b) In Maple, we can plot the torus using the command

```
plot3d(sin(phi),theta=0..2*Pi,
       phi=0..Pi,coords=spherical);.
```

In Mathematica, use

```
SphericalPlot3D[Sin[phi],{phi,0,Pi},{theta,0,2Pi}].
```



41. The region E of integration is the region above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 2$ in the first octant. Because E is in the first octant we have $0 \le \theta \le \frac{\pi}{2}$. The cone has equation $\phi = \frac{\pi}{4}$ (as in Example 4), so $0 \le \phi \le \frac{\pi}{4}$, and $0 \le \rho \le \sqrt{2}$. Then the integral becomes

$$\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi/4}\sin^3\phi\,d\phi\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^{\sqrt{2}}\rho^4\,d\rho = \left(\int_0^{\pi/4}\left(1-\cos^2\phi\right)\sin\phi\,d\phi\right)\left[\tfrac12\sin^2\theta\right]_0^{\pi/2}\left[\tfrac15\rho^5\right]_0^{\sqrt{2}}$$

$$= \left[\tfrac13\cos^3\phi-\cos\phi\right]_0^{\pi/4}\cdot\tfrac12\cdot\tfrac15\left(\sqrt{2}\right)^5 = \left[\tfrac{\sqrt{2}}{12}-\tfrac{\sqrt{2}}{2}-\left(\tfrac13-1\right)\right]\cdot\tfrac{2\sqrt{2}}{5} = \tfrac{4\sqrt{2}-5}{15}$$

42. The region of integration is the solid sphere $x^2+y^2+z^2\le a^2$, so $0\le\theta\le 2\pi$, $0\le\phi\le\pi$, and $0\le\rho\le a$. Also $x^2z+y^2z+z^3=(x^2+y^2+z^2)z=\rho^2z=\rho^3\cos\phi$, so the integral becomes

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{a}\left(\rho^3\cos\phi\right)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\sin\phi\cos\phi\,d\phi\int_0^{2\pi}d\theta\int_0^{a}\rho^5\,d\rho = \left[\tfrac12\sin^2\phi\right]_0^{\pi}\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac16\rho^6\right]_0^{a}=0$$

43. The region of integration is the solid sphere $x^2+y^2+(z-2)^2\le 4$ or equivalently $\rho^2\sin^2\phi+(\rho\cos\phi-2)^2=\rho^2-4\rho\cos\phi+4\le 4 \quad\Rightarrow\quad \rho\le 4\cos\phi$, so $0\le\theta\le 2\pi$, $0\le\phi\le\frac{\pi}{2}$, and $0\le\rho\le 4\cos\phi$. Also $(x^2+y^2+z^2)^{3/2}=(\rho^2)^{3/2}=\rho^3$, so the integral becomes

$$\begin{aligned}
\int_0^{\pi/2}\int_0^{2\pi}\int_0^{4\cos\phi}\left(\rho^3\right)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi &= \int_0^{\pi/2}\int_0^{2\pi}\sin\phi\left[\tfrac16\rho^6\right]_{\rho=0}^{\rho=4\cos\phi}d\theta\,d\phi\\
&= \tfrac16\int_0^{\pi/2}\int_0^{2\pi}\sin\phi\left(4096\cos^6\phi\right)d\theta\,d\phi\\
&= \tfrac16(4096)\int_0^{\pi/2}\cos^6\phi\sin\phi\,d\phi\int_0^{2\pi}d\theta = \tfrac{2048}{3}\left[-\tfrac17\cos^7\phi\right]_0^{\pi/2}\left[\,\theta\,\right]_0^{2\pi}\\
&= \tfrac{2048}{3}\left(\tfrac17\right)(2\pi)=\tfrac{4096\pi}{21}
\end{aligned}$$

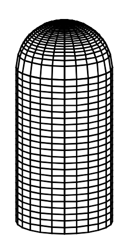
44. The solid region between the ground and an altitude of 5 km (5000 m) is given by

$$E=\left\{(\rho,\theta,\phi)\mid 6.370\times10^6\le\rho\le 6.375\times10^6, 0\le\theta\le2\pi, 0\le\phi\le\pi\right\}.$$

Then the mass of the atmosphere in this region is

$$\begin{aligned}
m=\iiint_E\delta\,dV &= \int_0^{2\pi}\int_0^{\pi}\int_{6.370\times10^6}^{6.375\times10^6}(619.09-0.000097\rho)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\
&= \int_0^{2\pi}d\theta\int_0^{\pi}\sin\phi\,d\phi\int_{6.370\times10^6}^{6.375\times10^6}\left(619.09\rho^2-0.000097\rho^3\right)d\rho\\
&= \left[\,\theta\,\right]_0^{2\pi}\left[-\cos\phi\right]_0^{\pi}\left[\tfrac{619.09}{3}\rho^3-\tfrac{0.000097}{4}\rho^4\right]_{6.370\times10^6}^{6.375\times10^6}\\
&= (2\pi)(2)\left[\tfrac{619.09}{3}\left((6.375\times10^6)^3-(6.370\times10^6)^3\right)-\tfrac{0.000097}{4}\left((6.375\times10^6)^4-(6.370\times10^6)^4\right)\right]\\
&\approx 4\pi\left(1.944\times10^{17}\right)\approx 2.44\times10^{18}\text{ kg}
\end{aligned}$$

45. In cylindrical coordinates, the equation of the cylinder is $r=3$, $0\le z\le10$.



The hemisphere is the upper part of the sphere radius 3, center $(0,0,10)$, equation $r^2+(z-10)^2=3^2$, $z\ge10$. In Maple, we can use the `coords=cylindrical` option in a regular `plot3d` command. In Mathematica, we can use `ParametricPlot3D`.

46. We begin by finding the positions of Los Angeles and Montréal in spherical coordinates, using the method described in the exercise:

Montréal	Los Angeles
$\rho = 3960$ mi	$\rho = 3960$ mi
$\theta = 360° - 73.60° = 286.40°$	$\theta = 360° - 118.25° = 241.75°$
$\phi = 90° - 45.50° = 44.50°$	$\phi = 90° - 34.06° = 55.94°$

Now we change the above to Cartesian coordinates using $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$ to get two position vectors of length 3960 mi (since both cities must lie on the surface of the earth). In particular:

Montréal: $\langle 783.67, -2662.67, 2824.47\rangle$ Los Angeles: $\langle -1552.80, -2889.91, 2217.84\rangle$

To find the angle γ between these two vectors we use the dot product:

$\langle 783.67, -2662.67, 2824.47\rangle \cdot \langle -1552.80, -2889.91, 2217.84\rangle = (3960)^2 \cos\gamma \quad\Rightarrow\quad \cos\gamma \approx 0.8126 \quad\Rightarrow$

$\gamma \approx 0.6223$ rad. The great circle distance between the cities is $s = \rho\gamma \approx 3960(0.6223) \approx 2464$ mi.

47. If E is the solid enclosed by the surface $\rho = 1 + \frac{1}{5}\sin 6\theta\,\sin 5\phi$, it can be described in spherical coordinates as

$E = \left\{(\rho,\theta,\phi) \mid 0 \le \rho \le 1 + \frac{1}{5}\sin 6\theta \sin 5\phi, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\right\}$. Its volume is given by

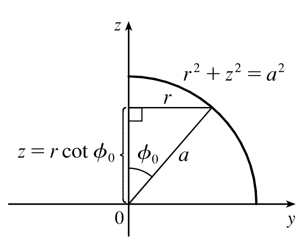
$V(E) = \iiint_E dV = \int_0^\pi \int_0^{2\pi} \int_0^{1+(\sin 6\theta\sin 5\phi)/5} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{136\pi}{99}$ [using a CAS].

48. The given integral is equal to $\lim\limits_{R\to\infty} \int_0^{2\pi}\int_0^\pi\int_0^R \rho e^{-\rho^2}\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \lim\limits_{R\to\infty} \int_0^{2\pi} d\theta \int_0^\pi \sin\phi\, d\phi \int_0^R \rho^3 e^{-\rho^2}\, d\rho$. Now use integration by parts with $u = \rho^2$, $dv = \rho e^{-\rho^2}\, d\rho$ to get

$$\begin{aligned}\lim_{R\to\infty} 2\pi(2)\left(\rho^2\left(-\tfrac{1}{2}\right)e^{-\rho^2}\Big]_0^R - \int_0^R 2\rho\left(-\tfrac{1}{2}\right)e^{-\rho^2}\, d\rho\right) &= \lim_{R\to\infty} 4\pi\left(-\tfrac{1}{2}R^2e^{-R^2} + \left[-\tfrac{1}{2}e^{-\rho^2}\right]_0^R\right)\\ &= 4\pi \lim_{R\to\infty}\left[-\tfrac{1}{2}R^2e^{-R^2} - \tfrac{1}{2}e^{-R^2} + \tfrac{1}{2}\right] = 4\pi\left(\tfrac{1}{2}\right) = 2\pi\end{aligned}$$

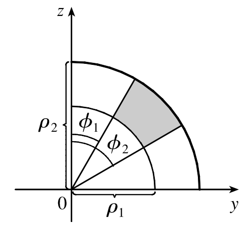
(Note that $R^2e^{-R^2} \to 0$ as $R\to\infty$ by l'Hospital's Rule.)

49. (a) From the diagram, $z = r\cot\phi_0$ to $z = \sqrt{a^2 - r^2}$, $r = 0$ to $r = a\sin\phi_0$ (or use $a^2 - r^2 = r^2\cot^2\phi_0$). Thus



$$\begin{aligned}V &= \int_0^{2\pi}\int_0^{a\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{a^2-r^2}} r\, dz\, dr\, d\theta\\ &= 2\pi\int_0^{a\sin\phi_0}\left(r\sqrt{a^2-r^2} - r^2\cot\phi_0\right) dr\\ &= \tfrac{2\pi}{3}\left[-(a^2-r^2)^{3/2} - r^3\cot\phi_0\right]_0^{a\sin\phi_0}\\ &= \tfrac{2\pi}{3}\left[-\left(a^2 - a^2\sin^2\phi_0\right)^{3/2} - a^3\sin^3\phi_0\cot\phi_0 + a^3\right]\\ &= \tfrac{2}{3}\pi a^3\left[1 - \left(\cos^3\phi_0 + \sin^2\phi_0\cos\phi_0\right)\right] = \tfrac{2}{3}\pi a^3(1-\cos\phi_0)\end{aligned}$$

(b) The wedge in question is the shaded area rotated from $\theta = \theta_1$ to $\theta = \theta_2$.



Letting

$V_{ij} =$ volume of the region bounded by the sphere of radius ρ_i and the cone with angle ϕ_j ($\theta = \theta_1$ to θ_2)

and letting V be the volume of the wedge, we have

$$
\begin{aligned}
V &= (V_{22} - V_{21}) - (V_{12} - V_{11}) \\
&= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\rho_2^3(1-\cos\phi_2) - \rho_2^3(1-\cos\phi_1) - \rho_1^3(1-\cos\phi_2) + \rho_1^3(1-\cos\phi_1)\right] \\
&= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\left(\rho_2^3 - \rho_1^3\right)(1-\cos\phi_2) - \left(\rho_2^3 - \rho_1^3\right)(1-\cos\phi_1)\right] = \tfrac{1}{3}(\theta_2 - \theta_1)\left[\left(\rho_2^3 - \rho_1^3\right)(\cos\phi_1 - \cos\phi_2)\right]
\end{aligned}
$$

Or: Show that $V = \displaystyle\int_{\theta_1}^{\theta_2} \int_{\rho_1 \sin\phi_1}^{\rho_2 \sin\phi_2} \int_{r\cot\phi_2}^{r\cot\phi_1} r\,dz\,dr\,d\theta$.

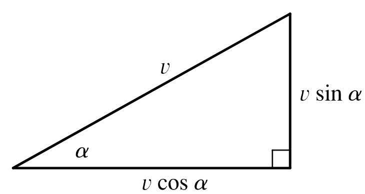
(c) By the Mean Value Theorem with $f(\rho) = \rho^3$ there exists some $\tilde{\rho}$ with $\rho_1 \le \tilde{\rho} \le \rho_2$ such that $f(\rho_2) - f(\rho_1) = f'(\tilde{\rho})(\rho_2 - \rho_1)$ or $\rho_1^3 - \rho_2^3 = 3\tilde{\rho}^2 \Delta\rho$. Similarly there exists ϕ with $\phi_1 \le \tilde{\phi} \le \phi_2$ such that $\cos\phi_2 - \cos\phi_1 = \left(-\sin\tilde{\phi}\right)\,\Delta\phi$. Substituting into the result from (b) gives

$\Delta V = (\tilde{\rho}^2\,\Delta\rho)(\theta_2 - \theta_1)(\sin\tilde{\phi})\,\Delta\phi = \tilde{\rho}^2 \sin\tilde{\phi}\,\Delta\rho\,\Delta\phi\,\Delta\theta$.

APPLIED PROJECT Roller Derby

1. $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(m + I/r^2)v^2$, so $v^2 = \dfrac{2mgh}{m + I/r^2} = \dfrac{2gh}{1 + I^*}$.

2. The vertical component of the speed is $v\sin\alpha$, so



$$\frac{dy}{dt} = \sqrt{\frac{2gy}{1+I^*}}\sin\alpha = \sqrt{\frac{2g}{1+I^*}}\sin\alpha\,\sqrt{y}.$$

3. Solving the separable differential equation, we get $\dfrac{dy}{\sqrt{y}} = \sqrt{\dfrac{2g}{1+I^*}}\sin\alpha\,dt \quad\Rightarrow\quad 2\sqrt{y} = \sqrt{\dfrac{2g}{1+I^*}}\,(\sin\alpha)t + C$.

But $y = 0$ when $t = 0$, so $C = 0$ and we have $2\sqrt{y} = \sqrt{\dfrac{2g}{1+I^*}}\,(\sin\alpha)t$. Solving for t when $y = h$ gives

$$T = \frac{2\sqrt{h}}{\sin\alpha}\sqrt{\frac{1+I^*}{2g}} = \sqrt{\frac{2h(1+I^*)}{g\sin^2\alpha}}.$$

4. Assume that the length of each cylinder is ℓ. Then the density of the solid cylinder is $\dfrac{m}{\pi r^2 \ell}$, and from Formulas 15.6.16, its moment of inertia (using cylindrical coordinates) is

$$I_z = \iiint \frac{m}{\pi r^2 \ell}\,(x^2 + y^2)\,dV = \int_0^{\ell}\int_0^{2\pi}\int_0^{r} \frac{m}{\pi r^2 \ell}\,R^2 R\,dR\,d\theta\,dz = \frac{m}{\pi r^2 \ell}\,2\pi\ell\left[\tfrac{1}{4}R^4\right]_0^r = \frac{mr^2}{2}$$

and so $I^* = \dfrac{I_z}{mr^2} = \dfrac{1}{2}$.

[continued]

For the hollow cylinder, we consider its entire mass to lie a distance r from the axis of rotation, so $x^2 + y^2 = r^2$ is a constant. We express the density in terms of mass per unit area as $\rho = \dfrac{m}{2\pi r\ell}$, and then the moment of inertia is calculated as a double integral: $I_z = \iint (x^2 + y^2)\,\dfrac{m}{2\pi r\ell}\,dA = \dfrac{mr^2}{2\pi r\ell}\iint dA = mr^2$, so $I^* = \dfrac{I_z}{mr^2} = 1$.

5. The volume of such a ball is $\frac{4}{3}\pi(r^3 - a^3) = \frac{4}{3}\pi r(1 - b^3)$, and so its density is $\dfrac{m}{\frac{4}{3}\pi r^3(1 - b^3)}$. Using Formula 15.8.3, we get

$$\begin{aligned}
I_z &= \iiint (x^2 + y^2)\,\frac{m}{\frac{4}{3}\pi r^3(1 - b^3)}\,dV \\
&= \frac{m}{\frac{4}{3}\pi r^3(1 - b^3)}\int_a^r \int_0^{2\pi}\int_0^{\pi} (\rho^2 \sin^2\phi)(\rho^2 \sin\phi)\,d\phi\,d\theta\,d\rho \\
&= \frac{m}{\frac{4}{3}\pi r^3(1 - b^3)}\cdot 2\pi\left[-\frac{(2 + \sin^2\phi)\cos\phi}{3}\right]_0^{\pi}\left[\frac{\rho^5}{5}\right]_a^r \qquad \text{[from the Table of Integrals]} \\
&= \frac{m}{\frac{4}{3}\pi r^3(1 - b^3)}\cdot 2\pi\cdot\frac{4}{3}\cdot\frac{r^5 - a^5}{5} = \frac{2mr^5(1 - b^5)}{5r^3(1 - b^3)} = \frac{2(1 - b^5)mr^2}{5(1 - b^3)}
\end{aligned}$$

Therefore $I^* = \dfrac{2(1 - b^5)}{5(1 - b^3)}$. Since a represents the inner radius, $a \to 0$ corresponds to a solid ball, and $a \to r$ corresponds to a hollow ball.

6. For a solid ball, $a \to 0 \quad\Rightarrow\quad b \to 0$, so $I^* = \lim\limits_{b \to 0}\dfrac{2(1 - b^5)}{5(1 - b^3)} = \dfrac{2}{5}$. For a hollow ball, $a \to r \quad\Rightarrow\quad b \to 1$, so

$$I^* = \lim_{b \to 1}\frac{2(1 - b^5)}{5(1 - b^3)} = \frac{2}{5}\lim_{b \to 1}\frac{-5b^4}{-3b^2} = \frac{2}{5}\left(\frac{5}{3}\right) = \frac{2}{3} \qquad \text{[by l'Hospital's Rule]}$$

Note: We could instead have calculated $I^* = \lim\limits_{b \to 1}\dfrac{2(1 - b)(1 + b + b^2 + b^3 + b^4)}{5(1 - b)(1 + b + b^2)} = \dfrac{2 \cdot 5}{5 \cdot 3} = \dfrac{2}{3}$.

Thus the objects finish in the following order: solid ball $\left(I^* = \frac{2}{5}\right)$, solid cylinder $\left(I^* = \frac{1}{2}\right)$, hollow ball $\left(I^* = \frac{2}{3}\right)$, hollow cylinder $(I^* = 1)$.

15.9 Change of Variables in Multiple Integrals

1. $x = 2u + v,\ y = 4u - v$.

The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = (2)(-1) - (1)(4) = -6$.

2. $x = u^2 + uv,\ y = uv^2$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2u + v & u \\ v^2 & 2uv \end{vmatrix} = (2u + v)(2uv) - u(v^2) = 4u^2v + 2uv^2 - uv^2 = 4u^2v + uv^2$$

3. $x = s\cos t,\ y = s\sin t$.

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{vmatrix} = \begin{vmatrix} \cos t & -s\sin t \\ \sin t & s\cos t \end{vmatrix} = s\cos^2 t - (-s\sin^2 t) = s(\cos^2 t + \sin^2 t) = s$$

4. $x = pe^q,\ y = qe^p.$

$$\frac{\partial(x,y)}{\partial(p,q)} = \begin{vmatrix} \partial x/\partial p & \partial x/\partial q \\ \partial y/\partial p & \partial y/\partial q \end{vmatrix} = \begin{vmatrix} e^q & pe^q \\ qe^p & e^p \end{vmatrix} = e^q e^p - pe^q \cdot qe^p = e^{p+q} - pqe^{p+q} = (1-pq)e^{p+q}$$

5. $x = uv,\ y = vw,\ z = wu.$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} v & u & 0 \\ 0 & w & v \\ w & 0 & u \end{vmatrix} = v\begin{vmatrix} w & v \\ 0 & u \end{vmatrix} - u\begin{vmatrix} 0 & v \\ w & u \end{vmatrix} + 0\begin{vmatrix} 0 & w \\ w & 0 \end{vmatrix}$$

$$= v(uw - 0) - u(0 - vw) + 0 = uvw + uvw = 2uvw$$

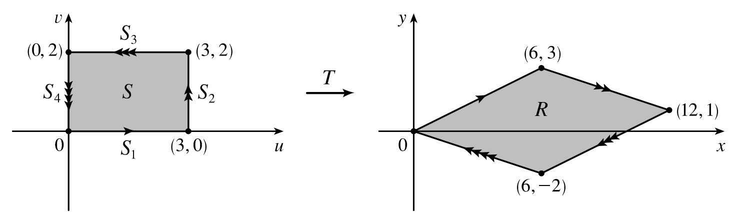
6. $x = u + vw,\ y = v + wu,\ z = w + uv.$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & w & v \\ w & 1 & u \\ v & u & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & u \\ u & 1 \end{vmatrix} - w\begin{vmatrix} w & u \\ v & 1 \end{vmatrix} + v\begin{vmatrix} w & 1 \\ v & u \end{vmatrix} = 1(1-u^2) - w\,(w - uv) + v\,(uw - v)$$

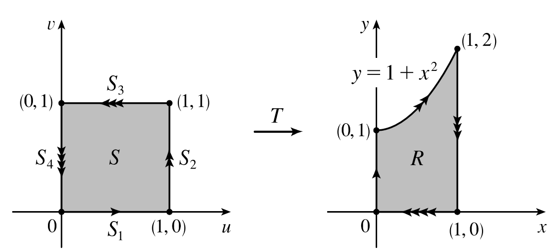
$$= 1 - u^2 - w^2 + uvw + uvw - v^2 = 1 + 2uvw - u^2 - v^2 - w^2$$

7. The transformation maps the boundary of S to the boundary of the image R, so we first look at side S_1 in the uv-plane. S_1 is described by $v = 0$, $0 \le u \le 3$, so $x = 2u + 3v = 2u$ and $y = u - v = u$. Eliminating u, we have $x = 2y$, $0 \le x \le 6$. S_2 is the line segment $u = 3$, $0 \le v \le 2$, so $x = 6 + 3v$ and $y = 3 - v$. Then $v = 3 - y \quad\Rightarrow\quad x = 6 + 3(3 - y) = 15 - 3y$, $6 \le x \le 12$. S_3 is the line segment $v = 2$, $0 \le u \le 3$, so $x = 2u + 6$ and $y = u - 2$, giving $u = y + 2 \quad\Rightarrow\quad x = 2y + 10$, $6 \le x \le 12$. Finally, S_4 is the segment $u = 0$, $0 \le v \le 2$, so $x = 3v$ and $y = -v \quad\Rightarrow\quad x = -3y$, $0 \le x \le 6$.

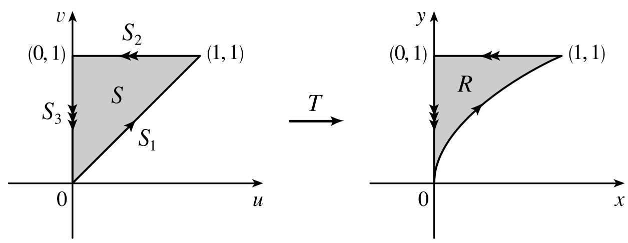
The image of set S is the region R shown in the xy-plane, a parallelogram bounded by these four segments.



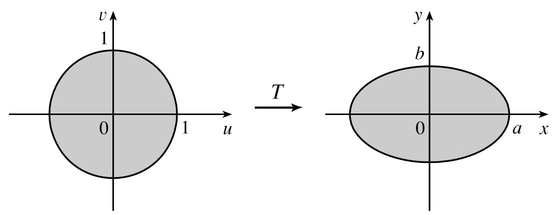
8. S_1 is the line segment $v = 0$, $0 \le u \le 1$, so $x = v = 0$ and $y = u(1 + v^2) = u$. Since $0 \le u \le 1$, the image is the line segment $x = 0$, $0 \le y \le 1$. S_2 is the segment $u = 1$, $0 \le v \le 1$, so $x = v$ and $y = u(1 + v^2) = 1 + x^2$. Thus the image is the portion of the parabola $y = 1 + x^2$ for $0 \le x \le 1$. S_3 is the segment $v = 1$, $0 \le u \le 1$, so $x = 1$ and $y = 2u$. The image is the segment $x = 1$, $0 \le y \le 2$. S_4 is described by $u = 0$, $0 \le v \le 1$, so $0 \le x = v \le 1$ and $y = u(1 + v^2) = 0$. The image is the line segment $y = 0$, $0 \le x \le 1$. Thus, the image of S is the region R bounded by the parabola $y = 1 + x^2$, the x-axis, and the lines $x = 0$, $x = 1$.



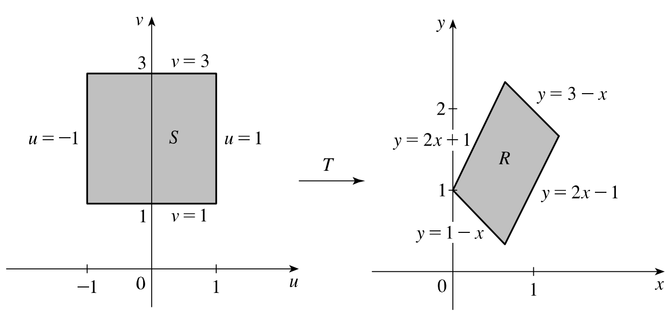
9. S_1 is the line segment $u = v$, $0 \le u \le 1$, so $y = v = u$ and $x = u^2 = y^2$. Since $0 \le u \le 1$, the image is the portion of the parabola $x = y^2$, $0 \le y \le 1$. S_2 is the segment $v = 1$, $0 \le u \le 1$, thus $y = v = 1$ and $x = u^2$, so $0 \le x \le 1$. The image is the line segment $y = 1$, $0 \le x \le 1$. S_3 is the segment $u = 0$, $0 \le v \le 1$, so $x = u^2 = 0$ and $y = v \Rightarrow 0 \le y \le 1$. The image is the segment $x = 0$, $0 \le y \le 1$. Thus, the image of S is the region R in the first quadrant bounded by the parabola $x = y^2$, the y-axis, and the line $y = 1$.



10. Substituting $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$ into $u^2 + v^2 \le 1$ gives $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$, so the image of $u^2 + v^2 \le 1$ is the elliptical region $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$.

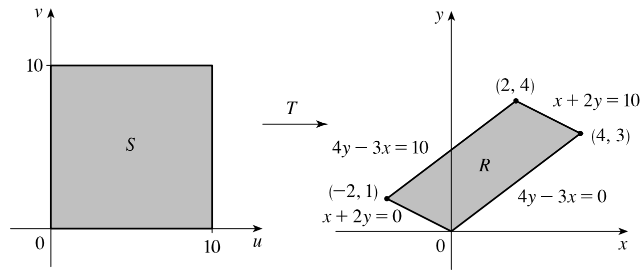


11. R is a parallelogram enclosed by the parallel lines $y = 2x - 1$, $y = 2x + 1$ and the parallel lines $y = 1 - x$, $y = 3 - x$. The first pair of equations can be written as $y - 2x = -1$, $y - 2x = 1$. If we let $u = y - 2x$ then these lines are mapped to the vertical lines $u = -1$, $u = 1$ in the uv-plane. Similarly, the second pair of equations can be written as $x + y = 1$, $x + y = 3$, and setting $v = x + y$ maps these lines to the horizontal lines $v = 1$, $v = 3$ in the uv-plane. Boundary curves are mapped to boundary curves under a transformation, so here the equations $u = y - 2x$, $v = x + y$ define a transformation T^{-1} that maps R in the xy-plane to the square S enclosed by the lines $u = -1$, $u = 1$, $v = 1$, $v = 3$ in the uv-plane. To find the transformation T that maps S to R we solve $u = y - 2x$, $v = x + y$ for x, y: Subtracting the first equation from the second gives $v - u = 3x \Rightarrow x = \frac{1}{3}(v - u)$ and adding twice the second equation to the first gives $u + 2v = 3y \Rightarrow y = \frac{1}{3}(u + 2v)$. Thus one possible transformation T (there are many) is given by $x = \frac{1}{3}(v - u)$, $y = \frac{1}{3}(u + 2v)$.

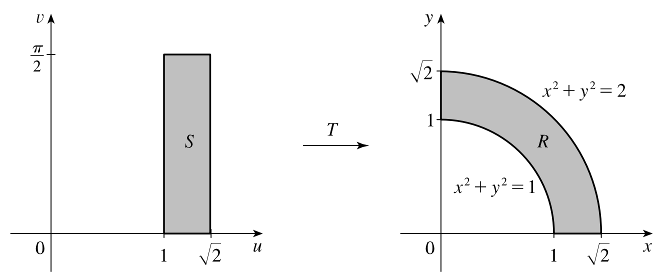


12. The boundaries of the parallelogram R are the lines $y = \frac{3}{4}x$ or $4y - 3x = 0$, $y = \frac{3}{4}x + \frac{5}{2}$ or $4y - 3x = 10$, $y = -\frac{1}{2}x$ or $x + 2y = 0$, $y = -\frac{1}{2}x + 5$ or $x + 2y = 10$. Setting $u = 4y - 3x$ and $v = x + 2y$ defines a transformation T^{-1} that maps R

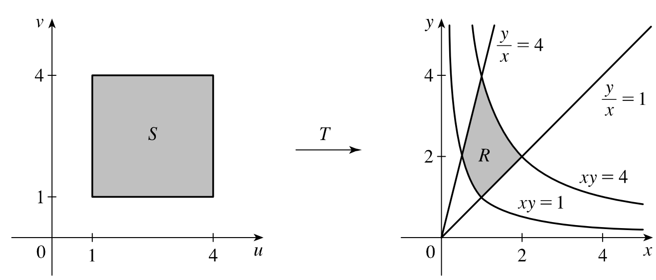
in the xy-plane to the square S enclosed by the lines $u = 0$, $u = 10$, $v = 0$, $v = 10$ in the uv-plane. Solving $u = 4y - 3x$, $v = x + 2y$ for x and y gives $2v - u = 5x \Rightarrow x = \frac{1}{5}(2v - u)$, $u + 3v = 10y \Rightarrow y = \frac{1}{10}(u + 3v)$. Thus one possible transformation T is given by $x = \frac{1}{5}(2v - u)$, $y = \frac{1}{10}(u + 3v)$.



13. R is a portion of an annular region (see the figure) that is easily described in polar coordinates as $R = \left\{(r, \theta) \mid 1 \le r \le \sqrt{2},\ 0 \le \theta \le \pi/2\right\}$. If we converted a double integral over R to polar coordinates the resulting region of integration is a rectangle (in the $r\theta$-plane), so we can create a transformation T here by letting u play the role of r and v the role of θ. Thus T is defined by $x = u \cos v$, $y = u \sin v$ and T maps the rectangle $S = \left\{(u, v) \mid 1 \le u \le \sqrt{2},\ 0 \le v \le \pi/2\right\}$ in the uv-plane to R in the xy-plane.



14. The boundaries of the region R are the curves $y = 1/x$ or $xy = 1$, $y = 4/x$ or $xy = 4$, $y = x$ or $y/x = 1$, $y = 4x$ or $y/x = 4$. Setting $u = xy$ and $v = y/x$ defines a transformation T^{-1} that maps R in the xy-plane to the square S enclosed by the lines $u = 1$, $u = 4$, $v = 1$, $v = 4$ in the uv-plane. Solving $u = xy$, $v = y/x$ for x and y gives $x^2 = u/v \Rightarrow$ $x = \sqrt{u/v}$ [since x, y, u, v are all positive], $y^2 = uv \Rightarrow y = \sqrt{uv}$. Thus one possible transformation T is given by $x = \sqrt{u/v}$, $y = \sqrt{uv}$.



15. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ and $x - 3y = (2u+v) - 3(u+2v) = -u - 5v$. To find the region S in the uv-plane that corresponds to R we first find the corresponding boundary under the given transformation. The line through $(0,0)$ and $(2,1)$ is $y = \frac{1}{2}x$ which is the image of $u + 2v = \frac{1}{2}(2u+v) \;\Rightarrow\; v = 0$; the line through $(2,1)$ and $(1,2)$ is $x + y = 3$ which is the image of $(2u+v) + (u+2v) = 3 \;\Rightarrow\; u + v = 1$; the line through $(0,0)$ and $(1,2)$ is $y = 2x$ which is the image of $u + 2v = 2(2u+v) \;\Rightarrow\; u = 0$. Thus S is the triangle $0 \le v \le 1 - u$, $0 \le u \le 1$ in the uv-plane and

$$\iint_R (x-3y)\,dA = \int_0^1 \int_0^{1-u} (-u-5v)\,|3|\,dv\,du = -3\int_0^1 \left[uv + \tfrac{5}{2}v^2\right]_{v=0}^{v=1-u} du$$
$$= -3\int_0^1 \left(u - u^2 + \tfrac{5}{2}(1-u)^2\right) du = -3\left[\tfrac{1}{2}u^2 - \tfrac{1}{3}u^3 - \tfrac{5}{6}(1-u)^3\right]_0^1 = -3\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{5}{6}\right) = -3$$

16. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = \dfrac{1}{4}$, $4x + 8y = 4 \cdot \frac{1}{4}(u+v) + 8 \cdot \frac{1}{4}(v - 3u) = 3v - 5u$. R is a parallelogram bounded by the lines $x - y = -4$, $x - y = 4$, $3x + y = 0$, $3x + y = 8$. Since $u = x - y$ and $v = 3x + y$, R is the image of the rectangle enclosed by the lines $u = -4$, $u = 4$, $v = 0$, and $v = 8$. Thus

$$\iint_R (4x+8y)\,dA = \int_{-4}^4 \int_0^8 (3v - 5u)\left|\tfrac{1}{4}\right| dv\,du = \tfrac{1}{4}\int_{-4}^4 \left[\tfrac{3}{2}v^2 - 5uv\right]_{v=0}^{v=8} du$$
$$= \tfrac{1}{4}\int_{-4}^4 (96 - 40u)\,du = \tfrac{1}{4}\left[96u - 20u^2\right]_{-4}^4 = 192$$

17. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$, $x^2 = 4u^2$ and the planar ellipse $9x^2 + 4y^2 \le 36$ is the image of the disk $u^2 + v^2 \le 1$. Thus

$$\iint_R x^2\,dA = \iint\limits_{u^2+v^2\le 1} (4u^2)(6)\,du\,dv = \int_0^{2\pi}\int_0^1 (24r^2\cos^2\theta)\,r\,dr\,d\theta = 24\int_0^{2\pi}\cos^2\theta\,d\theta \int_0^1 r^3\,dr$$
$$= 24\left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^1 = 24(\pi)\left(\tfrac{1}{4}\right) = 6\pi$$

18. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \dfrac{4}{\sqrt{3}}$, $x^2 - xy + y^2 = 2u^2 + 2v^2$ and the planar ellipse $x^2 - xy + y^2 \le 2$ is the image of the disk $u^2 + v^2 \le 1$. Thus

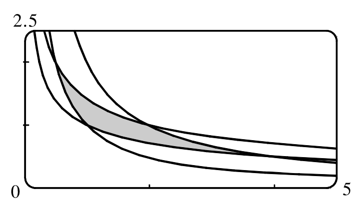
$$\iint_R (x^2 - xy + y^2)\,dA = \iint\limits_{u^2+v^2\le 1} (2u^2 + 2v^2)\left(\tfrac{4}{\sqrt{3}}\,du\,dv\right) = \int_0^{2\pi}\int_0^1 \tfrac{8}{\sqrt{3}}r^3\,dr\,d\theta = \tfrac{4\pi}{\sqrt{3}}$$

19. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$, $xy = u$, $y = x$ is the image of the parabola $v^2 = u$, $y = 3x$ is the image of the parabola $v^2 = 3u$, and the hyperbolas $xy = 1$, $xy = 3$ are the images of the lines $u = 1$ and $u = 3$ respectively. Thus

$$\iint_R xy\,dA = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u\left(\frac{1}{v}\right) dv\,du = \int_1^3 u\left(\ln\sqrt{3u} - \ln\sqrt{u}\right) du = \int_1^3 u\ln\sqrt{3}\,du = 4\ln\sqrt{3} = 2\ln 3.$$

20. Here $y = \dfrac{v}{u}$, $x = \dfrac{u^2}{v}$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{vmatrix} = \dfrac{1}{v}$ and R is the image of the square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$. So

$$\iint_R y^2\,dA = \int_1^2\int_1^2 \frac{v^2}{u^2}\left(\frac{1}{v}\right) du\,dv = \int_1^2 \frac{v}{2}\,dv = \frac{3}{4}$$



21. (a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and since $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$, $w = \dfrac{z}{c}$ the solid enclosed by the ellipsoid is the image of the ball $u^2 + v^2 + w^2 \le 1$. So

$$\iiint_E dV = \iiint\limits_{u^2+v^2+w^2 \le 1} abc\,du\,dv\,dw = (abc)(\text{volume of the ball}) = \tfrac{4}{3}\pi abc$$

(b) If we approximate the surface of the earth by the ellipsoid $\dfrac{x^2}{6378^2} + \dfrac{y^2}{6378^2} + \dfrac{z^2}{6356^2} = 1$, then we can estimate the volume of the earth by finding the volume of the solid E enclosed by the ellipsoid. From part (a), this is $\iiint_E dV = \frac{4}{3}\pi(6378)(6378)(6356) \approx 1.083 \times 10^{12}$ km^3.

(c) The moment of intertia about the z-axis is $I_z = \iiint_E \left(x^2 + y^2\right) \rho(x,y,z)\,dV$, where E is the solid enclosed by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. As in part (a), we use the transformation $x = au$, $y = bv$, $z = cw$, so $\left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| = abc$ and

$$\begin{aligned}
I_z &= \iiint_E \left(x^2+y^2\right) k\,dV = \iiint\limits_{u^2+v^2+w^2 \le 1} k(a^2u^2 + b^2v^2)(abc)\,du\,dv\,dw \\
&= abck \int_0^\pi \int_0^{2\pi} \int_0^1 (a^2\rho^2 \sin^2\phi \cos^2\theta + b^2\rho^2 \sin^2\phi \sin^2\theta)\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi \\
&= abck \left[a^2 \int_0^\pi \int_0^{2\pi} \int_0^1 (\rho^2 \sin^2\phi \cos^2\theta)\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi + b^2 \int_0^\pi \int_0^{2\pi} \int_0^1 (\rho^2 \sin^2\phi \sin^2\theta)\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi\right] \\
&= a^3bck \int_0^\pi \sin^3\phi\,d\phi \int_0^{2\pi} \cos^2\theta\,d\theta \int_0^1 \rho^4\,d\rho + ab^3ck \int_0^\pi \sin^3\phi\,d\phi \int_0^{2\pi} \sin^2\theta\,d\theta \int_0^1 \rho^4\,d\rho \\
&= a^3bck \left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^\pi \left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{1}{5}\rho^5\right]_0^1 + ab^3ck \left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^\pi \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{1}{5}\rho^5\right]_0^1 \\
&= a^3bck\left(\tfrac{4}{3}\right)(\pi)\left(\tfrac{1}{5}\right) + ab^3ck\left(\tfrac{4}{3}\right)(\pi)\left(\tfrac{1}{5}\right) = \tfrac{4}{15}\pi(a^2+b^2)abck
\end{aligned}$$

22. R is the region enclosed by the curves $xy = a$, $xy = b$, $xy^{1.4} = c$, and $xy^{1.4} = d$, so if we let $u = xy$ and $v = xy^{1.4}$ then R is the image of the rectangle enclosed by the lines $u = a$, $u = b$ ($a < b$) and $v = c$, $v = d$ ($c < d$). Now
$x = u/y \quad\Rightarrow\quad v = (u/y)y^{1.4} = uy^{0.4} \quad\Rightarrow\quad y^{0.4} = u^{-1}v \quad\Rightarrow\quad y = (u^{-1}v)^{1/0.4} = u^{-2.5}v^{2.5}$ and
$x = uy^{-1} = u(u^{-2.5}v^{2.5})^{-1} = u^{3.5}v^{-2.5}$, so
$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3.5u^{2.5}v^{-2.5} & -2.5u^{3.5}v^{-3.5} \\ -2.5u^{-3.5}v^{2.5} & 2.5u^{-2.5}v^{1.5} \end{vmatrix} = 8.75v^{-1} - 6.25v^{-1} = 2.5v^{-1}$. Thus the area of R, and the work done by the engine, is

$$\iint_R dA = \int_a^b \int_c^d \left|2.5v^{-1}\right| dv\,du = 2.5 \int_a^b du \int_c^d (1/v)\,dv = 2.5\big[u\big]_a^b \big[\ln|v|\big]_c^d = 2.5(b-a)(\ln d - \ln c) = 2.5(b-a)\ln\frac{d}{c}.$$

23. Letting $u = x - 2y$ and $v = 3x - y$, we have $x = \frac{1}{5}(2v - u)$ and $y = \frac{1}{5}(v - 3u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{vmatrix} = \dfrac{1}{5}$ and R is the image of the rectangle enclosed by the lines $u = 0$, $u = 4$, $v = 1$, and $v = 8$. Thus

$$\iint_R \frac{x-2y}{3x-y}\,dA = \int_0^4 \int_1^8 \frac{u}{v}\left|\frac{1}{5}\right| dv\,du = \frac{1}{5}\int_0^4 u\,du \int_1^8 \frac{1}{v}\,dv = \tfrac{1}{5}\left[\tfrac{1}{2}u^2\right]_0^4 \big[\ln|v|\big]_1^8 = \tfrac{8}{5}\ln 8.$$

24. Letting $u = x + y$ and $v = x - y$, we have $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\dfrac{1}{2}$ and R is the image of the rectangle enclosed by the lines $u = 0$, $u = 3$, $v = 0$, and $v = 2$. Thus

$$\begin{aligned} \iint_R (x+y)\,e^{x^2-y^2}\,dA &= \int_0^3 \int_0^2 ue^{uv}\left|-\tfrac{1}{2}\right| dv\,du = \tfrac{1}{2}\int_0^3 \left[e^{uv}\right]_{v=0}^{v=2} du = \tfrac{1}{2}\int_0^3 (e^{2u}-1)\,du \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}e^{2u} - u\right]_0^3 = \tfrac{1}{2}\left(\tfrac{1}{2}e^6 - 3 - \tfrac{1}{2}\right) = \tfrac{1}{4}(e^6 - 7) \end{aligned}$$

25. Letting $u = y - x$, $v = y + x$, we have $y = \frac{1}{2}(u + v)$, $x = \frac{1}{2}(v - u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\dfrac{1}{2}$ and R is the image of the trapezoidal region with vertices $(-1, 1)$, $(-2, 2)$, $(2, 2)$, and $(1, 1)$. Thus

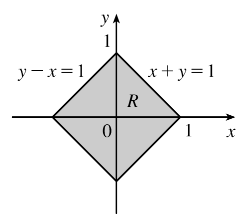
$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \int_1^2 \int_{-v}^{v} \cos\frac{u}{v}\left|-\frac{1}{2}\right| du\,dv = \frac{1}{2}\int_1^2 \left[v\sin\frac{u}{v}\right]_{u=-v}^{u=v} dv = \frac{1}{2}\int_1^2 2v\sin(1)\,dv = \tfrac{3}{2}\sin 1$$

26. Letting $u = 3x$, $v = 2y$, we have $9x^2 + 4y^2 = u^2 + v^2$, $x = \frac{1}{3}u$, and $y = \frac{1}{2}v$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{6}$ and R is the image of the quarter-disk D given by $u^2 + v^2 \leq 1$, $u \geq 0$, $v \geq 0$. Thus

$$\iint_R \sin(9x^2 + 4y^2)\,dA = \iint_D \tfrac{1}{6}\sin(u^2 + v^2)\,du\,dv = \int_0^{\pi/2}\int_0^1 \tfrac{1}{6}\sin(r^2)\,r\,dr\,d\theta = \tfrac{\pi}{12}\left[-\tfrac{1}{2}\cos r^2\right]_0^1 = \tfrac{\pi}{24}(1 - \cos 1)$$

27. Let $u = x + y$ and $v = -x + y$. Then $u + v = 2y \Rightarrow y = \frac{1}{2}(u + v)$ and $u - v = 2x \Rightarrow x = \frac{1}{2}(u - v)$.

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$. Now $|u| = |x + y| \leq |x| + |y| \leq 1 \Rightarrow -1 \leq u \leq 1$,



and $|v| = |-x + y| \leq |x| + |y| \leq 1 \Rightarrow -1 \leq v \leq 1$. R is the image of the square region with vertices $(1, 1)$, $(1, -1)$, $(-1, -1)$, and $(-1, 1)$.

So $\iint_R e^{x+y}\,dA = \frac{1}{2}\int_{-1}^1\int_{-1}^1 e^u\,du\,dv = \frac{1}{2}\left[e^u\right]_{-1}^1\left[v\right]_{-1}^1 = e - e^{-1}$.

28. Let $u = x + y$ and $v = y$, then $x = u - v$, $y = v$, $\dfrac{\partial(x,y)}{\partial(u,v)} = 1$ and R is the image under T of the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$. Thus

$$\iint_R f(x+y)\,dA = \int_0^1\int_0^u (1)\,f(u)\,dv\,du = \int_0^1 f(u)\left[\,v\,\right]_{v=0}^{v=u} du = \int_0^1 uf(u)\,du \quad \text{as desired.}$$

15 Review

TRUE-FALSE QUIZ

1. This is true by Fubini's Theorem.

2. False. $\int_0^1\int_0^x \sqrt{x + y^2}\,dy\,dx$ describes the region of integration as a Type I region. To reverse the order of integration, we must consider the region as a Type II region: $\int_0^1\int_y^1 \sqrt{x + y^2}\,dx\,dy$.

3. True by Equation 15.1.11.