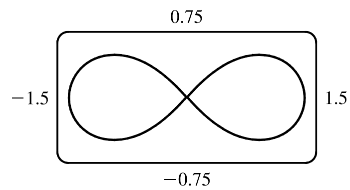
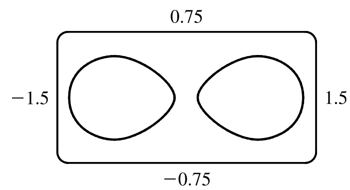
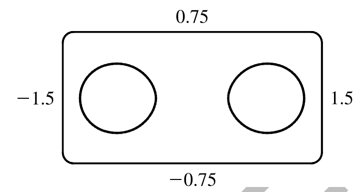
$a \approx 1.4$, the graph no longer has dimples, and has an oval shape. As $a \to \infty$, the oval becomes larger and rounder, since the c^2 and c^4 terms lose their significance. Note that the shape of the graph seems to depend only on the ratio c/a, while the size of the graph varies as c and a jointly increase.



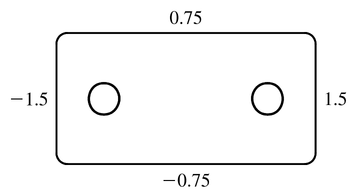
$(a, c) = (1, 1)$



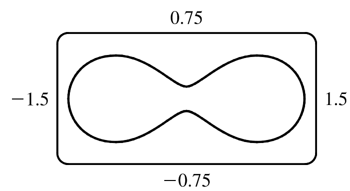
$(a, c) = (0.99, 1)$



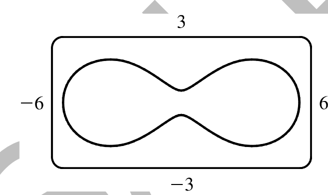
$(a, c) = (0.9, 1)$



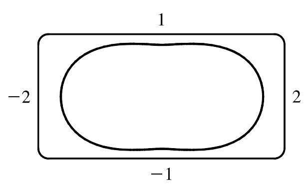
$(a, c) = (0.6, 1)$



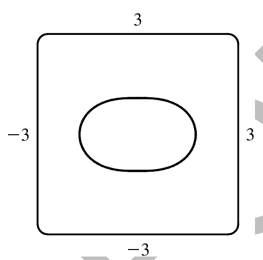
$(a, c) = (1.01, 1)$



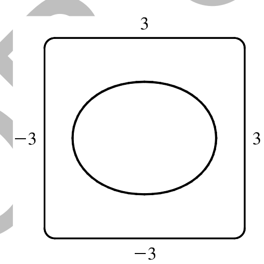
$(a, c) = (4.04, 4)$



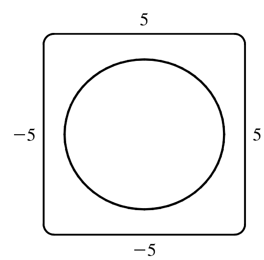
$(a, c) = (1.3, 1)$



$(a, c) = (1.5, 1)$



$(a, c) = (2, 1)$



$(a, c) = (4, 1)$

10.4 Areas and Lengths in Polar Coordinates

1. $r = e^{-\theta/4},\ \pi/2 \le \theta \le \pi$.

$$A = \int_{\pi/2}^{\pi} \tfrac{1}{2} r^2\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2}(e^{-\theta/4})^2\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2} e^{-\theta/2}\, d\theta = \tfrac{1}{2}\Big[-2e^{-\theta/2}\Big]_{\pi/2}^{\pi} = -1(e^{-\pi/2} - e^{-\pi/4}) = e^{-\pi/4} - e^{-\pi/2}$$

2. $r = \cos\theta,\ 0 \le \theta \le \pi/6$.

$$A = \int_0^{\pi/6} \tfrac{1}{2} r^2\, d\theta = \int_0^{\pi/6} \tfrac{1}{2}\cos^2\theta\, d\theta = \tfrac{1}{2}\int_0^{\pi/6} \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta = \tfrac{1}{4}\big[\theta + \tfrac{1}{2}\sin 2\theta\big]_0^{\pi/6}$$
$$= \tfrac{1}{4}\big(\tfrac{\pi}{6} + \tfrac{1}{2}\cdot\tfrac{1}{2}\sqrt{3}\big) = \tfrac{\pi}{24} + \tfrac{1}{16}\sqrt{3}$$

3. $r = \sin\theta + \cos\theta,\ 0 \le \theta \le \pi$.

$$A = \int_0^{\pi} \frac{1}{2} r^2\, d\theta = \int_0^{\pi} \frac{1}{2}(\sin\theta + \cos\theta)^2\, d\theta = \int_0^{\pi} \frac{1}{2}(\sin^2\theta + 2\sin\theta\,\cos\theta + \cos^2\theta)\, d\theta = \int_0^{\pi} \frac{1}{2}(1 + \sin 2\theta)\, d\theta$$
$$= \tfrac{1}{2}\Big[\theta - \tfrac{1}{2}\cos 2\theta\Big]_0^{\pi} = \tfrac{1}{2}\big[\big(\pi - \tfrac{1}{2}\big) - \big(0 - \tfrac{1}{2}\big)\big] = \tfrac{\pi}{2}$$

4. $r = 1/\theta,\ \pi/2 \le \theta \le 2\pi.$

$$A = \int_{\pi/2}^{2\pi} \frac{1}{2} r^2\, d\theta = \int_{\pi/2}^{2\pi} \frac{1}{2}\left(\frac{1}{\theta}\right)^2 d\theta = \int_{\pi/2}^{2\pi} \frac{1}{2}\theta^{-2}\, d\theta = \frac{1}{2}\left[-\frac{1}{\theta}\right]_{\pi/2}^{2\pi}$$
$$= \tfrac{1}{2}\left(-\tfrac{1}{2\pi} + \tfrac{2}{\pi}\right) = \tfrac{1}{2}\left(-\tfrac{1}{2\pi} + \tfrac{4}{2\pi}\right) = \tfrac{3}{4\pi}$$

5. $r^2 = \sin 2\theta,\ 0 \le \theta \le \pi/2.$

$$A = \int_0^{\pi/2} \tfrac{1}{2} r^2\, d\theta = \int_0^{\pi/2} \tfrac{1}{2}\sin 2\theta\, d\theta = \left[-\tfrac{1}{4}\cos 2\theta\right]_0^{\pi/2} = -\tfrac{1}{4}(\cos\pi - \cos 0) = -\tfrac{1}{4}(-1-1) = \tfrac{1}{2}$$

6. $r = 2 + \cos\theta,\ \pi/2 \le \theta \le \pi.$

$$A = \int_{\pi/2}^{\pi} \tfrac{1}{2} r^2\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2}(2+\cos\theta)^2\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2}(4 + 4\cos\theta + \cos^2\theta)\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2}\left[4 + 4\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right] d\theta$$
$$= \int_{\pi/2}^{\pi} \left(\tfrac{9}{4} + 2\cos\theta + \tfrac{1}{4}\cos 2\theta\right) d\theta = \left[\tfrac{9}{4}\theta + 2\sin\theta + \tfrac{1}{8}\sin 2\theta\right]_{\pi/2}^{\pi} = \left(\tfrac{9\pi}{4} + 0 + 0\right) - \left(\tfrac{9\pi}{8} + 2 + 0\right) = \tfrac{9\pi}{8} - 2$$

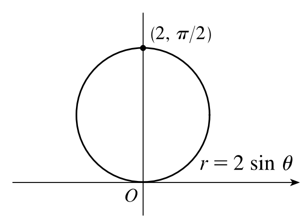
7. $r = 4 + 3\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$

$$A = \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}((4+3\sin\theta)^2\, d\theta = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} (16 + 24\sin\theta + 9\sin^2\theta)\, d\theta$$
$$= \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} (16 + 9\sin^2\theta)\, d\theta \qquad \text{[by Theorem 5.5.7(b)]}$$
$$= \tfrac{1}{2}\cdot 2\int_0^{\pi/2} \left[16 + 9\cdot\tfrac{1}{2}(1-\cos 2\theta)\right] d\theta \qquad \text{[by Theorem 5.5.7(a)]}$$
$$= \int_0^{\pi/2} \left(\tfrac{41}{2} - \tfrac{9}{2}\cos 2\theta\right) d\theta = \left[\tfrac{41}{2}\theta - \tfrac{9}{4}\sin 2\theta\right]_0^{\pi/2} = \left(\tfrac{41\pi}{4} - 0\right) - (0-0) = \tfrac{41\pi}{4}$$

8. $r = \sqrt{\ln\theta},\ 1 \le \theta \le 2\pi.$

$$A = \int_1^{2\pi} \tfrac{1}{2}\left(\sqrt{\ln\theta}\right)^2 d\theta = \int_1^{2\pi} \tfrac{1}{2}\ln\theta\, d\theta = \left[\tfrac{1}{2}\theta\ln\theta\right]_1^{2\pi} - \int_1^{2\pi} \tfrac{1}{2}\, d\theta \qquad \begin{bmatrix} u = \ln\theta, & dv = \frac{1}{2}\,d\theta \\ du = (1/\theta)\,d\theta, & v = \frac{1}{2}\theta \end{bmatrix}$$
$$= [\pi\ln(2\pi) - 0] - \left[\tfrac{1}{2}\theta\right]_1^{2\pi} = \pi\ln(2\pi) - \pi + \tfrac{1}{2}$$

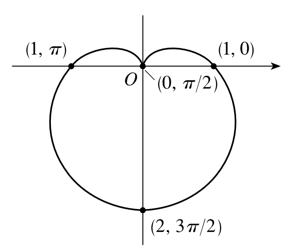
9. The area is bounded by $r = 2\sin\theta$ for $\theta = 0$ to $\theta = \pi$.



$$A = \int_0^{\pi} \tfrac{1}{2} r^2\, d\theta = \tfrac{1}{2}\int_0^{\pi} (2\sin\theta)^2\, d\theta = \tfrac{1}{2}\int_0^{\pi} 4\sin^2\theta\, d\theta$$
$$= 2\int_0^{\pi} \tfrac{1}{2}(1-\cos 2\theta) d\theta = \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi} = \pi$$

Also, note that this is a circle with radius 1, so its area is $\pi(1)^2 = \pi$.

10. $A = \int_0^{2\pi} \tfrac{1}{2} r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}(1-\sin\theta)^2\, d\theta$



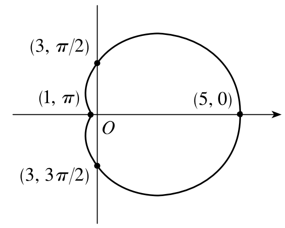
$$= \tfrac{1}{2}\int_0^{2\pi} (1 - 2\sin\theta + \sin^2\theta)\, d\theta = \tfrac{1}{2}\int_0^{2\pi} \left[1 - 2\sin\theta + \tfrac{1}{2}(1-\cos 2\theta)\right] d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi} \left(\tfrac{3}{2} - 2\sin\theta - \tfrac{1}{2}\cos 2\theta\right) d\theta = \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}$$
$$= \tfrac{1}{2}[(3\pi + 2) - (2)] = \tfrac{3\pi}{2}$$

11. $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}(3 + 2\cos\theta)^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (9 + 12\cos\theta + 4\cos^2\theta) \, d\theta$

$= \frac{1}{2}\int_0^{2\pi} \left[9 + 12\cos\theta + 4 \cdot \frac{1}{2}(1 + \cos 2\theta)\right] d\theta$

$= \frac{1}{2}\int_0^{2\pi} (11 + 12\cos\theta + 2\cos 2\theta) \, d\theta = \frac{1}{2}\big[11\theta + 12\sin\theta + \sin 2\theta\big]_0^{2\pi}$

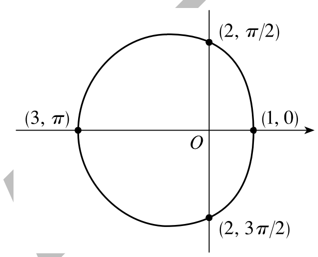
$= \frac{1}{2}(22\pi) = 11\pi$



12. $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}(2 - \cos\theta)^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}(4 - 4\cos\theta + \cos^2\theta) \, d\theta$

$= \int_0^{2\pi} \frac{1}{2}\left[4 - 4\cos\theta + \frac{1}{2}(1 + \cos 2\theta)\right] d\theta = \int_0^{2\pi} \left(\frac{9}{4} - 2\cos\theta + \frac{1}{4}\cos 2\theta\right) d\theta$

$= \left[\frac{9}{4}\theta - 2\sin\theta + \frac{1}{8}\sin 2\theta\right]_0^{2\pi} = \left(\frac{9\pi}{2} - 0 + 0\right) - (0 - 0 + 0) = \frac{9\pi}{2}$

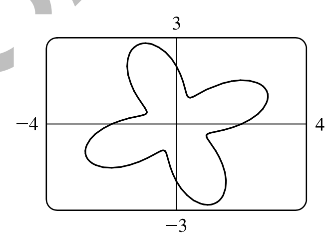


13. $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}(2 + \sin 4\theta)^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (4 + 4\sin 4\theta + \sin^2 4\theta) \, d\theta$

$= \frac{1}{2}\int_0^{2\pi} \left[4 + 4\sin 4\theta + \frac{1}{2}(1 - \cos 8\theta)\right] d\theta$

$= \frac{1}{2}\int_0^{2\pi} \left(\frac{9}{2} + 4\sin 4\theta - \frac{1}{2}\cos 8\theta\right) d\theta = \frac{1}{2}\left[\frac{9}{2}\theta - \cos 4\theta - \frac{1}{16}\sin 8\theta\right]_0^{2\pi}$

$= \frac{1}{2}[(9\pi - 1) - (-1)] = \frac{9}{2}\pi$

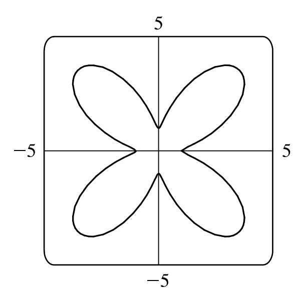


14. $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}(3 - 2\cos 4\theta)^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (9 - 12\cos 4\theta + 4\cos^2 4\theta) \, d\theta$

$= \frac{1}{2}\int_0^{2\pi} \left[9 - 12\cos 4\theta + 4 \cdot \frac{1}{2}(1 + \cos 8\theta)\right] d\theta$

$= \frac{1}{2}\int_0^{2\pi} (11 - 12\cos 4\theta + 2\cos 8\theta) \, d\theta = \frac{1}{2}\left[11\theta - 3\sin 4\theta + \frac{1}{4}\sin 8\theta\right]_0^{2\pi}$

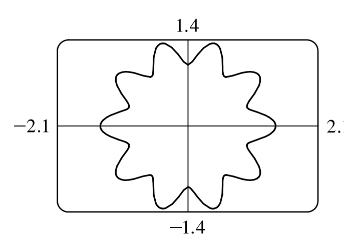
$= \frac{1}{2}(22\pi) = 11\pi$



15. $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}\left(\sqrt{1 + \cos^2 5\theta}\right)^2 d\theta$

$= \frac{1}{2}\int_0^{2\pi} (1 + \cos^2 5\theta) \, d\theta = \frac{1}{2}\int_0^{2\pi} \left[1 + \frac{1}{2}(1 + \cos 10\theta)\right] d\theta$

$= \frac{1}{2}\left[\frac{3}{2}\theta + \frac{1}{20}\sin 10\theta\right]_0^{2\pi} = \frac{1}{2}(3\pi) = \frac{3}{2}\pi$



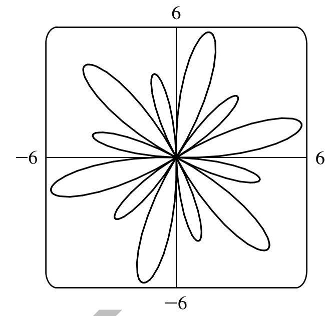
16. $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}(1 + 5\sin 6\theta)^2 \, d\theta$

$= \frac{1}{2} \int_0^{2\pi} (1 + 10\sin 6\theta + 25\sin^2 6\theta) \, d\theta$

$= \frac{1}{2} \int_0^{2\pi} \left[1 + 10\sin 6\theta + 25 \cdot \frac{1}{2}(1 - \cos 12\theta)\right] d\theta$

$= \frac{1}{2} \int_0^{2\pi} \left[\frac{27}{2} + 10\sin 6\theta - \frac{25}{2}\cos 12\theta\right] d\theta = \frac{1}{2}\left[\frac{27}{2}\theta - \frac{5}{3}\cos 6\theta - \frac{25}{24}\sin 12\theta\right]_0^{2\pi}$

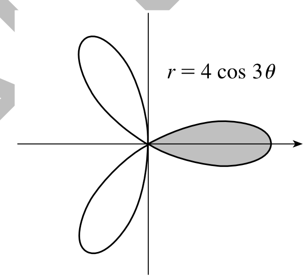
$= \frac{1}{2}\left[\left(27\pi - \frac{5}{3}\right) - \left(-\frac{5}{3}\right)\right] = \frac{27}{2}\pi$



17. The curve passes through the pole when $r = 0 \;\Rightarrow\; 4\cos 3\theta = 0 \;\Rightarrow\; \cos 3\theta = 0 \;\Rightarrow\; 3\theta = \frac{\pi}{2} + \pi n \;\Rightarrow$ $\theta = \frac{\pi}{6} + \frac{\pi}{3}n$. The part of the shaded loop above the polar axis is traced out for $\theta = 0$ to $\theta = \pi/6$, so we'll use $-\pi/6$ and $\pi/6$ as our limits of integration.

$A = \int_{-\pi/6}^{\pi/6} \frac{1}{2}(4\cos 3\theta)^2 \, d\theta = 2\int_0^{\pi/6} \frac{1}{2}(16\cos^2 3\theta) \, d\theta$

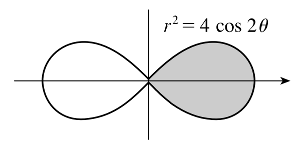
$= 16\int_0^{\pi/6} \frac{1}{2}(1 + \cos 6\theta) \, d\theta = 8\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = 8\left(\frac{\pi}{6}\right) = \frac{4}{3}\pi$



18. The curve given by $r^2 = 4\cos 2\theta$ passes through the pole when $r = 0 \;\Rightarrow\; 4\cos 2\theta = 0 \;\Rightarrow\; \cos 2\theta = 0 \;\Rightarrow$ $2\theta = \frac{\pi}{2} + \pi n \;\Rightarrow\; \theta = \frac{\pi}{4} + \frac{\pi}{2}n$. The part of the shaded loop above the polar axis is traced out for $\theta = 0$ to $\theta = \pi/4$, so we'll use $-\pi/4$ to $\pi/4$ as our limits of integration.

$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2}(4\cos 2\theta) \, d\theta = 2\int_0^{\pi/4} 2\cos 2\theta \, d\theta = 2\Big[\sin 2\theta\Big]_0^{\pi/4}$

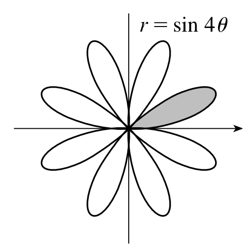
$= 2\sin\frac{\pi}{2} = 2(1) = 2$



19. $r = 0 \;\Rightarrow\; \sin 4\theta = 0 \;\Rightarrow\; 4\theta = \pi n \;\Rightarrow\; \theta = \frac{\pi}{4}n$.

$A = \int_0^{\pi/4} \frac{1}{2}(\sin 4\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi/4} \sin^2 4\theta \, d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{1}{2}(1 - \cos 8\theta) \, d\theta$

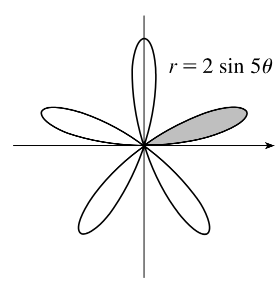
$= \frac{1}{4}\left[\theta - \frac{1}{8}\sin 8\theta\right]_0^{\pi/4} = \frac{1}{4}\left(\frac{\pi}{4}\right) = \frac{1}{16}\pi$



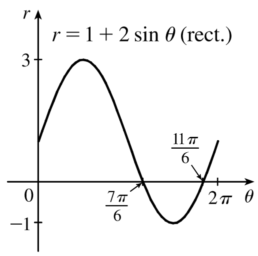
20. $r = 0 \;\Rightarrow\; 2\sin 5\theta = 0 \;\Rightarrow\; \sin 5\theta = 0 \;\Rightarrow\; 5\theta = \pi n \;\Rightarrow\; \theta = \frac{\pi}{5}n$.

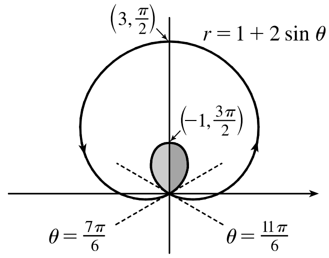
$A = \int_0^{\pi/5} \frac{1}{2}(2\sin 5\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi/5} 4\sin^2 5\theta \, d\theta$

$= 2\int_0^{\pi/5} \frac{1}{2}(1 - \cos 10\theta) \, d\theta = \left[\theta - \frac{1}{10}\sin 10\theta\right]_0^{\pi/5} = \frac{\pi}{5}$



21.





This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

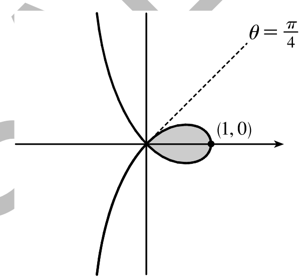
$$A = 2\int_{7\pi/6}^{3\pi/2} \tfrac{1}{2}(1 + 2\sin\theta)^2\, d\theta = \int_{7\pi/6}^{3\pi/2} \left(1 + 4\sin\theta + 4\sin^2\theta\right) d\theta = \int_{7\pi/6}^{3\pi/2} \left[1 + 4\sin\theta + 4 \cdot \tfrac{1}{2}(1 - \cos 2\theta)\right] d\theta$$

$$= \Big[\theta - 4\cos\theta + 2\theta - \sin 2\theta\Big]_{7\pi/6}^{3\pi/2} = \left(\tfrac{9\pi}{2}\right) - \left(\tfrac{7\pi}{2} + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) = \pi - \tfrac{3\sqrt{3}}{2}$$

22. To determine when the strophoid $r = 2\cos\theta - \sec\theta$ passes through the pole, we solve

$$r = 0 \quad\Rightarrow\quad 2\cos\theta - \frac{1}{\cos\theta} = 0 \quad\Rightarrow\quad 2\cos^2\theta - 1 = 0 \quad\Rightarrow\quad \cos^2\theta = \frac{1}{2} \quad\Rightarrow$$

$\cos\theta = \pm\dfrac{1}{\sqrt{2}} \quad\Rightarrow\quad \theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$ for $0 \le \theta \le \pi$ with $\theta \neq \frac{\pi}{2}$.



$$A = 2\int_0^{\pi/4} \tfrac{1}{2}(2\cos\theta - \sec\theta)^2\, d\theta = \int_0^{\pi/4} (4\cos^2\theta - 4 + \sec^2\theta)\, d\theta$$

$$= \int_0^{\pi/4} \left[4 \cdot \tfrac{1}{2}(1 + \cos 2\theta) - 4 + \sec^2\theta\right] d\theta = \int_0^{\pi/4} (-2 + 2\cos 2\theta + \sec^2\theta)\, d\theta$$

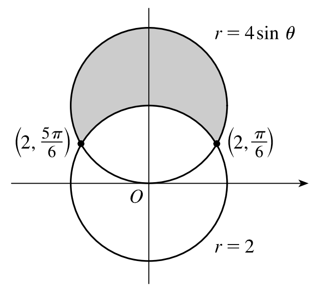
$$= \big[-2\theta + \sin 2\theta + \tan\theta\big]_0^{\pi/4} = \left(-\tfrac{\pi}{2} + 1 + 1\right) - 0 = 2 - \tfrac{\pi}{2}$$

23. $4\sin\theta = 2 \quad\Rightarrow\quad \sin\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6} \quad\Rightarrow$

$$A = \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}[(4\sin\theta)^2 - 2^2]\, d\theta = 2\int_{\pi/6}^{\pi/2} \tfrac{1}{2}(16\sin^2\theta - 4)\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left[16 \cdot \tfrac{1}{2}(1 - \cos 2\theta) - 4\right] d\theta = \int_{\pi/6}^{\pi/2} (4 - 8\cos 2\theta)\, d\theta$$

$$= \Big[4\theta - 4\sin 2\theta\Big]_{\pi/6}^{\pi/2} = (2\pi - 0) - \left(\tfrac{2\pi}{3} - 2\sqrt{3}\right) = \tfrac{4\pi}{3} + 2\sqrt{3}$$

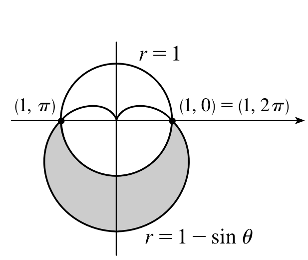


24. $1 - \sin\theta = 1 \quad\Rightarrow\quad \sin\theta = 0 \quad\Rightarrow\quad \theta = 0$ or $\pi \quad\Rightarrow$

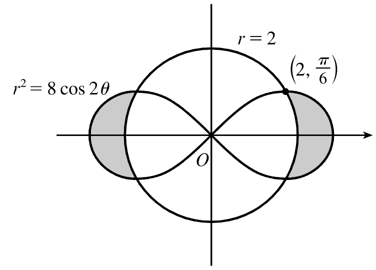
$$A = \int_{\pi}^{2\pi} \tfrac{1}{2}\left[(1 - \sin\theta)^2 - 1\right] d\theta = \tfrac{1}{2}\int_{\pi}^{2\pi} (\sin^2\theta - 2\sin\theta)\, d\theta$$

$$= \tfrac{1}{4}\int_{\pi}^{2\pi} (1 - \cos 2\theta - 4\sin\theta)\, d\theta = \tfrac{1}{4}\left[\theta - \tfrac{1}{2}\sin 2\theta + 4\cos\theta\right]_{\pi}^{2\pi}$$

$$= \tfrac{1}{4}\pi + 2$$

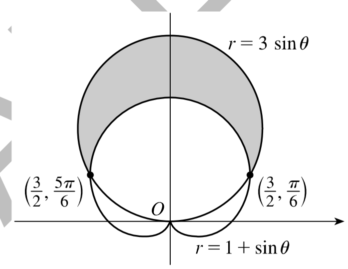


25. To find the area inside the leminiscate $r^2 = 8\cos 2\theta$ and outside the circle $r = 2$, we first note that the two curves intersect when $r^2 = 8\cos 2\theta$ and $r = 2$, that is, when $\cos 2\theta = \frac{1}{2}$. For $-\pi < \theta \le \pi$, $\cos 2\theta = \frac{1}{2} \quad\Leftrightarrow\quad 2\theta = \pm\pi/3$ or $\pm 5\pi/3 \quad\Leftrightarrow\quad \theta = \pm\pi/6$ or $\pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,



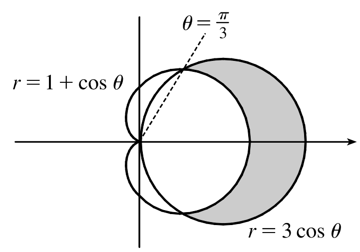
$$A = 4\int_0^{\pi/6}\left[\tfrac{1}{2}(8\cos 2\theta) - \tfrac{1}{2}(2)^2\right]d\theta = 8\int_0^{\pi/6}(2\cos 2\theta - 1)\,d\theta$$
$$= 8\left[\sin 2\theta - \theta\right]_0^{\pi/6} = 8\left(\sqrt{3}/2 - \pi/6\right) = 4\sqrt{3} - 4\pi/3$$

26. $3\sin\theta = 1 + \sin\theta \quad\Rightarrow\quad \sin\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6} \quad\Rightarrow$



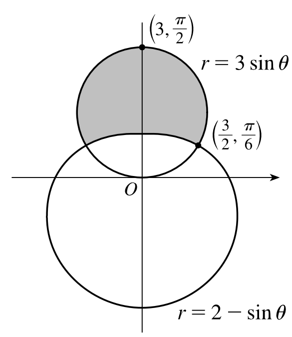
$$A = \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}\left[(3\sin\theta)^2 - (1+\sin\theta)^2\right]d\theta$$
$$= 2\int_{\pi/6}^{\pi/2} \tfrac{1}{2}\left(9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\right)d\theta$$
$$= \int_{\pi/6}^{\pi/2}\left(8\sin^2\theta - 1 - 2\sin\theta\right)d\theta$$
$$= \int_{\pi/6}^{\pi/2}\left[8\cdot\tfrac{1}{2}(1-\cos 2\theta) - 1 - 2\sin\theta\right]d\theta = \int_{\pi/6}^{\pi/2}(3 - 4\cos 2\theta - 2\sin\theta)\,d\theta$$
$$= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\pi/6}^{\pi/2} = \left(\tfrac{3\pi}{2} - 0 + 0\right) - \left(\tfrac{\pi}{2} - \sqrt{3} + \sqrt{3}\right) = \pi$$

27. $3\cos\theta = 1 + \cos\theta \quad\Leftrightarrow\quad \cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$.



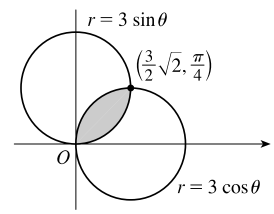
$$A = 2\int_0^{\pi/3}\tfrac{1}{2}\left[(3\cos\theta)^2 - (1+\cos\theta)^2\right]d\theta$$
$$= \int_0^{\pi/3}(8\cos^2\theta - 2\cos\theta - 1)\,d\theta = \int_0^{\pi/3}\left[4(1+\cos 2\theta) - 2\cos\theta - 1\right]d\theta$$
$$= \int_0^{\pi/3}(3 + 4\cos 2\theta - 2\cos\theta)\,d\theta = \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_0^{\pi/3}$$
$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$

28. $3\sin\theta = 2 - \sin\theta \quad\Rightarrow\quad 4\sin\theta = 2 \quad\Rightarrow\quad \sin\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.



$$A = 2\int_{\pi/6}^{\pi/2}\tfrac{1}{2}\left[(3\sin\theta)^2 - (2-\sin\theta)^2\right]d\theta$$
$$= \int_{\pi/6}^{\pi/2}\left(9\sin^2\theta - 4 + 4\sin\theta - \sin^2\theta\right]d\theta$$
$$= \int_{\pi/6}^{\pi/2}\left(8\sin^2\theta + 4\sin\theta - 4\right)d\theta$$
$$= 4\int_{\pi/6}^{\pi/2}\left[2\cdot\tfrac{1}{2}(1-\cos 2\theta) + \sin\theta - 1\right]d\theta$$
$$= 4\int_{\pi/6}^{\pi/2}(\sin\theta - \cos 2\theta)\,d\theta = 4\left[-\cos\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/2}$$
$$= 4\left[(0-0) - \left(-\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{4}\right)\right] = 4\left(\tfrac{3\sqrt{3}}{4}\right) = 3\sqrt{3}$$

29. $3\sin\theta = 3\cos\theta \quad\Rightarrow\quad \dfrac{3\sin\theta}{3\cos\theta} = 1 \quad\Rightarrow\quad \tan\theta = 1 \quad\Rightarrow\quad \theta = \frac{\pi}{4} \quad\Rightarrow$

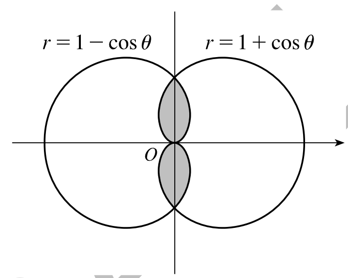


$$A = 2\int_0^{\pi/4} \tfrac{1}{2}(3\sin\theta)^2\,d\theta = \int_0^{\pi/4} 9\sin^2\theta\,d\theta = \int_0^{\pi/4} 9\cdot\tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= \int_0^{\pi/4}\left(\tfrac{9}{2} - \tfrac{9}{2}\cos 2\theta\right)d\theta = \left[\tfrac{9}{2}\theta - \tfrac{9}{4}\sin 2\theta\right]_0^{\pi/4} = \left(\tfrac{9\pi}{8} - \tfrac{9}{4}\right) - (0-0)$$

$$= \tfrac{9\pi}{8} - \tfrac{9}{4}$$

30. $A = 4\int_0^{\pi/2} \frac{1}{2}(1-\cos\theta)^2\,d\theta = 2\int_0^{\pi/2}(1 - 2\cos\theta + \cos^2\theta)\,d\theta$



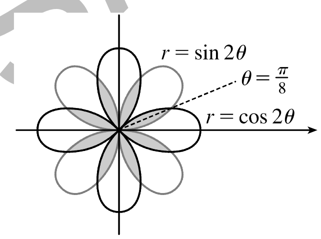
$$= 2\int_0^{\pi/2}\left[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= 2\int_0^{\pi/2}\left(\tfrac{3}{2} - 2\cos\theta + \tfrac{1}{2}\cos 2\theta\right)d\theta = \int_0^{\pi/2}(3 - 4\cos\theta + \cos 2\theta)\,d\theta$$

$$= \left[3\theta - 4\sin\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{3\pi}{2} - 4$$

31. $\sin 2\theta = \cos 2\theta \quad\Rightarrow\quad \dfrac{\sin 2\theta}{\cos 2\theta} = 1 \quad\Rightarrow\quad \tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4} \quad\Rightarrow$

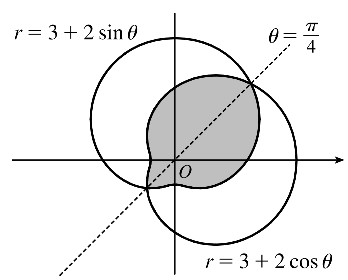
$\theta = \frac{\pi}{8} \quad\Rightarrow$



$$A = 8\cdot 2\int_0^{\pi/8} \tfrac{1}{2}\sin^2 2\theta\,d\theta = 8\int_0^{\pi/8}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= 4\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/8} = 4\left(\tfrac{\pi}{8} - \tfrac{1}{4}\cdot 1\right) = \tfrac{\pi}{2} - 1$$

32. $3 + 2\cos\theta = 3 + 2\sin\theta \quad\Rightarrow\quad \cos\theta = \sin\theta \quad\Rightarrow\quad \theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.



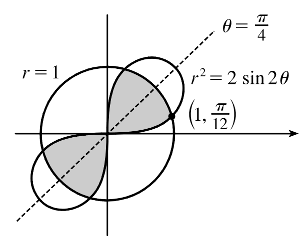
$$A = 2\int_{\pi/4}^{5\pi/4}\tfrac{1}{2}(3+2\cos\theta)^2\,d\theta = \int_{\pi/4}^{5\pi/4}(9 + 12\cos\theta + 4\cos^2\theta)\,d\theta$$

$$= \int_{\pi/4}^{5\pi/4}\left[9 + 12\cos\theta + 4\cdot\tfrac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= \int_{\pi/4}^{5\pi/4}(11 + 12\cos\theta + 2\cos 2\theta)\,d\theta = \left[11\theta + 12\sin\theta + \sin 2\theta\right]_{\pi/4}^{5\pi/4}$$

$$= \left(\tfrac{55\pi}{4} - 6\sqrt{2} + 1\right) - \left(\tfrac{11\pi}{4} + 6\sqrt{2} + 1\right) = 11\pi - 12\sqrt{2}$$

33. From the figure, we see that the shaded region is 4 times the shaded region from $\theta = 0$ to $\theta = \pi/4$. $r^2 = 2\sin 2\theta$ and $r = 1 \quad\Rightarrow$ $2\sin 2\theta = 1^2 \quad\Rightarrow\quad \sin 2\theta = \frac{1}{2} \quad\Rightarrow\quad 2\theta = \frac{\pi}{6} \quad\Rightarrow\quad \theta = \frac{\pi}{12}$.

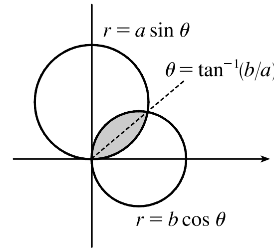


$$A = 4\int_0^{\pi/12}\tfrac{1}{2}(2\sin 2\theta)\,d\theta + 4\int_{\pi/12}^{\pi/4}\tfrac{1}{2}(1)^2\,d\theta$$

$$= \int_0^{\pi/12} 4\sin 2\theta\,d\theta + \int_{\pi/12}^{\pi/4} 2\,d\theta = \left[-2\cos 2\theta\right]_0^{\pi/12} + \left[2\theta\right]_{\pi/12}^{\pi/4}$$

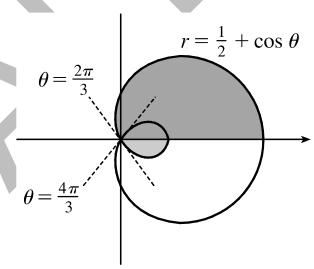
$$= \left(-\sqrt{3} + 2\right) + \left(\tfrac{\pi}{2} - \tfrac{\pi}{6}\right) = -\sqrt{3} + 2 + \tfrac{\pi}{3}$$

34. Let $\alpha = \tan^{-1}(b/a)$. Then



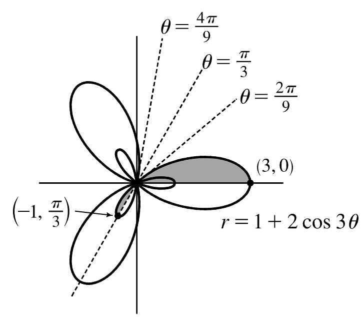
$$\begin{aligned} A &= \int_0^{\alpha} \tfrac{1}{2}(a\sin\theta)^2\,d\theta + \int_{\alpha}^{\pi/2} \tfrac{1}{2}(b\cos\theta)^2\,d\theta \\ &= \tfrac{1}{4}a^2\big[\theta - \tfrac{1}{2}\sin 2\theta\big]_0^{\alpha} + \tfrac{1}{4}b^2\big[\theta + \tfrac{1}{2}\sin 2\theta\big]_{\alpha}^{\pi/2} \\ &= \tfrac{1}{4}\alpha(a^2 - b^2) + \tfrac{1}{8}\pi b^2 - \tfrac{1}{4}(a^2+b^2)(\sin\alpha\,\cos\alpha) \\ &= \tfrac{1}{4}(a^2 - b^2)\,\tan^{-1}(b/a) + \tfrac{1}{8}\pi b^2 - \tfrac{1}{4}ab \end{aligned}$$

35. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/3$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/3$ to $\theta = \pi$), and then double that difference to obtain the desired area.



$$\begin{aligned} A &= 2\Big[\int_0^{2\pi/3} \tfrac{1}{2}\big(\tfrac{1}{2} + \cos\theta\big)^2\,d\theta - \int_{2\pi/3}^{\pi} \tfrac{1}{2}\big(\tfrac{1}{2} + \cos\theta\big)^2\,d\theta\Big] \\ &= \int_0^{2\pi/3} \big(\tfrac{1}{4} + \cos\theta + \cos^2\theta\big)\,d\theta - \int_{2\pi/3}^{\pi} \big(\tfrac{1}{4} + \cos\theta + \cos^2\theta\big)\,d\theta \\ &= \int_0^{2\pi/3} \big[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\big]\,d\theta \\ &\qquad - \int_{2\pi/3}^{\pi} \big[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\big]\,d\theta \\ &= \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi/3} - \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi} \\ &= \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right) - \left(\tfrac{\pi}{4} + \tfrac{\pi}{2}\right) + \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right) \\ &= \tfrac{\pi}{4} + \tfrac{3}{4}\sqrt{3} = \tfrac{1}{4}\big(\pi + 3\sqrt{3}\big) \end{aligned}$$

36. $r = 0 \;\Rightarrow\; 1 + 2\cos 3\theta = 0 \;\Rightarrow\; \cos 3\theta = -\frac{1}{2} \;\Rightarrow\; 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ [for $0 \le 3\theta \le 2\pi$] $\Rightarrow\; \theta = \frac{2\pi}{9}, \frac{4\pi}{9}$. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/9$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/9$ to $\theta = \pi/3$), and then double that difference to obtain the desired area.



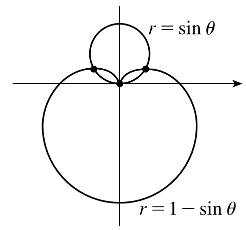
$$A = 2\Big[\int_0^{2\pi/9} \tfrac{1}{2}(1 + 2\cos 3\theta)^2\,d\theta - \int_{2\pi/9}^{\pi/3} \tfrac{1}{2}(1 + 2\cos 3\theta)^2\,d\theta\Big]$$

Now
$$\begin{aligned} r^2 &= (1 + 2\cos 3\theta)^2 = 1 + 4\cos 3\theta + 4\cos^2 3\theta = 1 + 4\cos 3\theta + 4\cdot\tfrac{1}{2}(1+\cos 6\theta) \\ &= 1 + 4\cos 3\theta + 2 + 2\cos 6\theta = 3 + 4\cos 3\theta + 2\cos 6\theta \end{aligned}$$

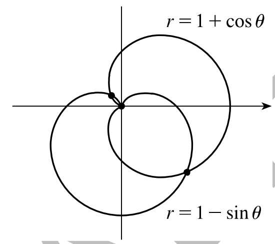
and $\int r^2\,d\theta = 3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta + C$, so

$$\begin{aligned} A &= \big[3\theta + \tfrac{4}{3}\sin 3\theta + \tfrac{1}{3}\sin 6\theta\big]_0^{2\pi/9} - \big[3\theta + \tfrac{4}{3}\sin 3\theta + \tfrac{1}{3}\sin 6\theta\big]_{2\pi/9}^{\pi/3} \\ &= \Big[\Big(\tfrac{2\pi}{3} + \tfrac{4}{3}\cdot\tfrac{\sqrt{3}}{2} + \tfrac{1}{3}\cdot\tfrac{-\sqrt{3}}{2}\Big) - 0\Big] - \Big[(\pi + 0 + 0) - \Big(\tfrac{2\pi}{3} + \tfrac{4}{3}\cdot\tfrac{\sqrt{3}}{2} + \tfrac{1}{3}\cdot\tfrac{-\sqrt{3}}{2}\Big)\Big] \\ &= \tfrac{4\pi}{3} + \tfrac{4}{3}\sqrt{3} - \tfrac{1}{3}\sqrt{3} - \pi = \tfrac{\pi}{3} + \sqrt{3} \end{aligned}$$

37. The pole is a point of intersection. $\sin\theta = 1 - \sin\theta \quad\Rightarrow\quad 2\sin\theta = 1 \quad\Rightarrow$ $\sin\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. So the other points of intersection are $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$.



38. The pole is a point of intersection. $1 + \cos\theta = 1 - \sin\theta \quad\Rightarrow$ $\cos\theta = -\sin\theta \quad\Rightarrow\quad \dfrac{\cos\theta}{\sin\theta} = -1 \quad\Rightarrow\quad \cot\theta = -1 \quad\Rightarrow\quad \theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. So the other points of intersection are $\left(1 - \frac{1}{2}\sqrt{2}, \frac{3\pi}{4}\right)$ and $\left(1 + \frac{1}{2}\sqrt{2}, \frac{7\pi}{4}\right)$.



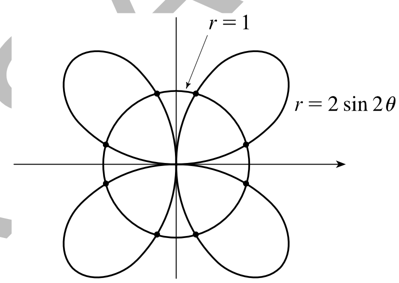
39. $2\sin 2\theta = 1 \quad\Rightarrow\quad \sin 2\theta = \frac{1}{2} \quad\Rightarrow\quad 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$.

By symmetry, the eight points of intersection are given by

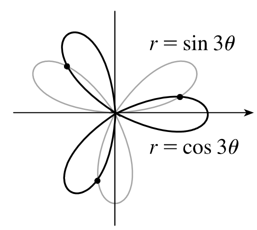
$(1, \theta)$, where $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$, and $\frac{17\pi}{12}$, and

$(-1, \theta)$, where $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}$, and $\frac{23\pi}{12}$.

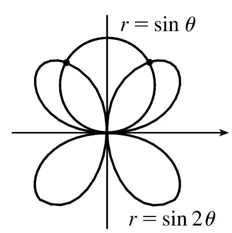
[There are many ways to describe these points.]



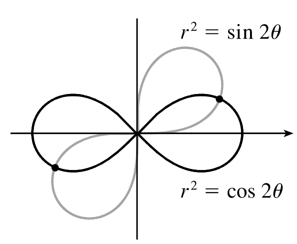
40. Clearly the pole lies on both curves. $\sin 3\theta = \cos 3\theta \quad\Rightarrow\quad \tan 3\theta = 1 \quad\Rightarrow$ $3\theta = \frac{\pi}{4} + n\pi$ [n any integer] $\quad\Rightarrow\quad \theta = \frac{\pi}{12} + \frac{\pi}{3}n \quad\Rightarrow$ $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$, or $\frac{3\pi}{4}$, so the three remaining intersection points are $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right)$, and $\left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)$.



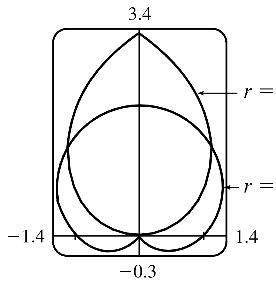
41. The pole is a point of intersection. $\sin\theta = \sin 2\theta = 2\sin\theta\,\cos\theta \quad\Leftrightarrow$ $\sin\theta\,(1 - 2\cos\theta) = 0 \quad\Leftrightarrow\quad \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \quad\Rightarrow$ $\theta = 0, \pi, \frac{\pi}{3}$, or $-\frac{\pi}{3} \quad\Rightarrow$ the other intersection points are $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$ [by symmetry].

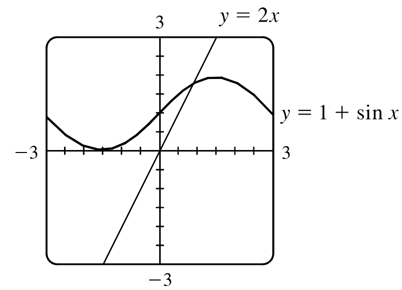


42. Clearly the pole is a point of intersection. $\sin 2\theta = \cos 2\theta \quad\Rightarrow$ $\tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4} + 2n\pi$ [since $\sin 2\theta$ and $\cos 2\theta$ must be positive in the equations] $\quad\Rightarrow\quad \theta = \frac{\pi}{8} + n\pi \quad\Rightarrow\quad \theta = \frac{\pi}{8}$ or $\frac{9\pi}{8}$. So the curves also intersect at $\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right)$ and $\left(\frac{1}{\sqrt[4]{2}}, \frac{9\pi}{8}\right)$.



43.



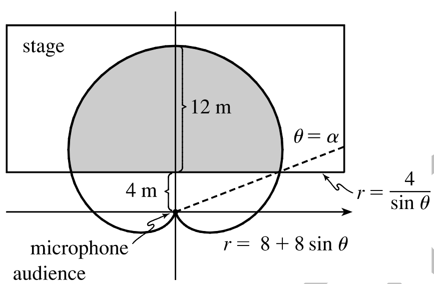


From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we find the θ-values of the intersection points to be $\alpha \approx 0.88786 \approx 0.89$ and $\pi - \alpha \approx 2.25$. (The first of these values may be more easily estimated by plotting $y = 1 + \sin x$ and $y = 2x$ in rectangular coordinates; see the second graph.) By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

$$A = 2\int_0^\alpha \tfrac{1}{2}(2\theta)^2\, d\theta + 2\int_\alpha^{\pi/2} \tfrac{1}{2}(1+\sin\theta)^2\, d\theta = \int_0^\alpha 4\theta^2\, d\theta + \int_\alpha^{\pi/2}\left[1 + 2\sin\theta + \tfrac{1}{2}(1-\cos 2\theta)\right] d\theta$$

$$= \left[\tfrac{4}{3}\theta^3\right]_0^\alpha + \left[\theta - 2\cos\theta + \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right]_\alpha^{\pi/2} = \tfrac{4}{3}\alpha^3 + \left[\left(\tfrac{\pi}{2} + \tfrac{\pi}{4}\right) - \left(\alpha - 2\cos\alpha + \tfrac{1}{2}\alpha - \tfrac{1}{4}\sin 2\alpha\right)\right] \approx 3.4645$$

44.



We need to find the shaded area A in the figure. The horizontal line representing the front of the stage has equation $y = 4 \iff r\sin\theta = 4 \Rightarrow r = 4/\sin\theta$. This line intersects the curve $r = 8 + 8\sin\theta$ when $8 + 8\sin\theta = \dfrac{4}{\sin\theta} \Rightarrow$ $8\sin\theta + 8\sin^2\theta = 4 \Rightarrow 2\sin^2\theta + 2\sin\theta - 1 = 0 \Rightarrow$

$$\sin\theta = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1+\sqrt{3}}{2} \quad \text{[the other value is less than } -1\text{]} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right).$$

This angle is about 21.5° and is denoted by α in the figure.

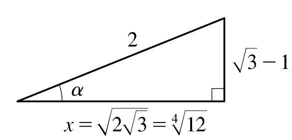
$$A = 2\int_\alpha^{\pi/2} \tfrac{1}{2}(8+8\sin\theta)^2\, d\theta - 2\int_\alpha^{\pi/2} \tfrac{1}{2}(4\csc\theta)^2\, d\theta = 64\int_\alpha^{\pi/2}(1 + 2\sin\theta + \sin^2\theta)\, d\theta - 16\int_\alpha^{\pi/2}\csc^2\theta\, d\theta$$

$$= 64\int_\alpha^{\pi/2}\left(1 + 2\sin\theta + \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta + 16\int_\alpha^{\pi/2}(-\csc^2\theta)\, d\theta = 64\left[\tfrac{3}{2}\theta - 2\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_\alpha^{\pi/2} + 16\left[\cot\theta\right]_\alpha^{\pi/2}$$

$$= 16\left[6\theta - 8\cos\theta - \sin 2\theta + \cot\theta\right]_\alpha^{\pi/a} = 16[(3\pi - 0 - 0 + 0) - (6\alpha - 8\cos\alpha - \sin 2\alpha + \cot\alpha)]$$

$$= 48\pi - 96\alpha + 128\cos\alpha + 16\sin 2\alpha - 16\cot\alpha$$

From the figure, $x^2 + \left(\sqrt{3}-1\right)^2 = 2^2 \Rightarrow x^2 = 4 - \left(3 - 2\sqrt{3} + 1\right) \Rightarrow$ $x^2 = 2\sqrt{3} = \sqrt{12}$, so $x = \sqrt{2\sqrt{3}} = \sqrt[4]{12}$. Using the trigonometric relationships for a right triangle and the identity $\sin 2\alpha = 2\sin\alpha\,\cos\alpha$, we continue:



$$A = 48\pi - 96\alpha + 128\cdot\frac{\sqrt[4]{12}}{2} + 16\cdot 2\cdot\frac{\sqrt{3}-1}{2}\cdot\frac{\sqrt[4]{12}}{2} - 16\cdot\frac{\sqrt[4]{12}}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= 48\pi - 96\alpha + 64\sqrt[4]{12} + 8\sqrt[4]{12}\left(\sqrt{3}-1\right) - 8\sqrt[4]{12}\left(\sqrt{3}+1\right) = 48\pi + 48\sqrt[4]{12} - 96\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

$$\approx 204.16\ \text{m}^2$$

45. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^\pi \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$

$= \int_0^\pi \sqrt{4(\cos^2\theta + \sin^2\theta)}\, d\theta = \int_0^\pi \sqrt{4}\, d\theta = \big[2\theta\big]_0^\pi = 2\pi$

As a check, note that the curve is a circle of radius 1, so its circumference is $2\pi(1) = 2\pi$.

46. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{(5^\theta)^2 + (5^\theta \ln 5)^2}\, d\theta = \int_0^{2\pi} \sqrt{5^{2\theta}[1 + (\ln 5)^2]}\, d\theta$

$= \sqrt{1 + (\ln 5)^2} \int_0^{2\pi} \sqrt{5^{2\theta}}\, d\theta = \sqrt{1 + (\ln 5)^2} \int_0^{2\pi} 5^\theta\, d\theta = \sqrt{1 + (\ln 5)^2} \left[\dfrac{5^\theta}{\ln 5}\right]_0^{2\pi}$

$= \sqrt{1 + (\ln 5)^2} \left(\dfrac{5^{2\pi}}{\ln 5} - \dfrac{1}{\ln 5}\right) = \dfrac{\sqrt{1 + (\ln 5)^2}}{\ln 5}(5^{2\pi} - 1)$

47. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{\theta^2(\theta^2 + 4)}\, d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta$

Now let $u = \theta^2 + 4$, so that $du = 2\theta\, d\theta$ $\left[\theta\, d\theta = \frac{1}{2}\, du\right]$ and

$$\int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta = \int_4^{4\pi^2+4} \tfrac{1}{2}\sqrt{u}\, du = \tfrac{1}{2} \cdot \tfrac{2}{3}\left[u^{3/2}\right]_4^{4(\pi^2+1)} = \tfrac{1}{3}[4^{3/2}(\pi^2 + 1)^{3/2} - 4^{3/2}] = \tfrac{8}{3}[(\pi^2 + 1)^{3/2} - 1]$$

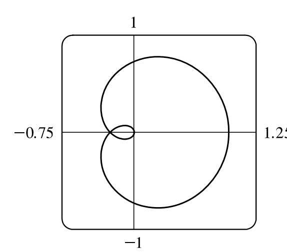
48. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{[2(1 + \cos\theta)]^2 + (-2\sin\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$

$= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta = \sqrt{8}\int_0^{2\pi} \sqrt{1 + \cos\theta}\, d\theta = \sqrt{8}\int_0^{2\pi} \sqrt{2 \cdot \tfrac{1}{2}(1 + \cos\theta)}\, d\theta$

$= \sqrt{8}\int_0^{2\pi} \sqrt{2\cos^2\dfrac{\theta}{2}}\, d\theta = \sqrt{8}\,\sqrt{2}\int_0^{2\pi} \left|\cos\dfrac{\theta}{2}\right| d\theta = 4 \cdot 2\int_0^{\pi} \cos\dfrac{\theta}{2}\, d\theta$ [by symmetry]

$= 8\left[2\sin\dfrac{\theta}{2}\right]_0^\pi = 8(2) = 16$

49. The curve $r = \cos^4(\theta/4)$ is completely traced with $0 \le \theta \le 4\pi$.



$r^2 + (dr/d\theta)^2 = [\cos^4(\theta/4)]^2 + \left[4\cos^3(\theta/4) \cdot (-\sin(\theta/4)) \cdot \tfrac{1}{4}\right]^2$

$= \cos^8(\theta/4) + \cos^6(\theta/4)\sin^2(\theta/4)$

$= \cos^6(\theta/4)[\cos^2(\theta/4) + \sin^2(\theta/4)] = \cos^6(\theta/4)$

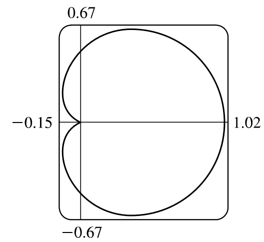
$L = \int_0^{4\pi} \sqrt{\cos^6(\theta/4)}\, d\theta = \int_0^{4\pi} \left|\cos^3(\theta/4)\right|\, d\theta$

$= 2\int_0^{2\pi} \cos^3(\theta/4)\, d\theta$ [since $\cos^3(\theta/4) \ge 0$ for $0 \le \theta \le 2\pi$] $= 8\int_0^{\pi/2} \cos^3 u\, du$ $\left[u = \frac{1}{4}\theta\right]$

$= 8\int_0^{\pi/2} (1 - \sin^2 u)\cos u\, du = 8\int_0^1 (1 - x^2)\, dx$ $\left[\begin{array}{c} x = \sin u, \\ dx = \cos u\, du \end{array}\right]$

$= 8\left[x - \tfrac{1}{3}x^3\right]_0^1 = 8\left(1 - \tfrac{1}{3}\right) = \tfrac{16}{3}$

50. The curve $r = \cos^2(\theta/2)$ is completely traced with $0 \le \theta \le 2\pi$.



$$\begin{aligned} r^2 + (dr/d\theta)^2 &= [\cos^2(\theta/2)]^2 + \left[2\cos(\theta/2)\cdot(-\sin(\theta/2))\cdot\tfrac{1}{2}\right]^2 \\ &= \cos^4(\theta/2) + \cos^2(\theta/2)\sin^2(\theta/2) \\ &= \cos^2(\theta/2)[\cos^2(\theta/2) + \sin^2(\theta/2)] \\ &= \cos^2(\theta/2) \end{aligned}$$

$L = \int_0^{2\pi} \sqrt{\cos^2(\theta/2)}\,d\theta = \int_0^{2\pi} |\cos(\theta/2)|\,d\theta = 2\int_0^{\pi} \cos(\theta/2)\,d\theta$ [since $\cos(\theta/2) \ge 0$ for $0 \le \theta \le \pi$]

$= 4\int_0^{\pi/2} \cos u\,du$ $\left[u = \frac{1}{2}\theta\right]$ $= 4\big[\sin u\big]_0^{\pi/2} = 4(1-0) = 4$

51. One loop of the curve $r = \cos 2\theta$ is traced with $-\pi/4 \le \theta \le \pi/4$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos^2 2\theta + (-2\sin 2\theta)^2 = \cos^2 2\theta + 4\sin^2 2\theta = 1 + 3\sin^2 2\theta \quad\Rightarrow$$

$$L = \int_{-\pi/4}^{\pi/4} \sqrt{1 + 3\sin^2 2\theta}\,d\theta \approx 2.4221.$$

52. $r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = \tan^2\theta + (\sec^2\theta)^2 \quad\Rightarrow\quad L = \displaystyle\int_{\pi/6}^{\pi/3} \sqrt{\tan^2\theta + \sec^4\theta}\,d\theta \approx 1.2789$

53. The curve $r = \sin(6\sin\theta)$ is completely traced with $0 \le \theta \le \pi$. $r = \sin(6\sin\theta) \Rightarrow$

$\dfrac{dr}{d\theta} = \cos(6\sin\theta)\cdot 6\cos\theta$, so $r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = \sin^2(6\sin\theta) + 36\cos^2\theta\cos^2(6\sin\theta) \quad\Rightarrow$

$$L = \int_0^{\pi} \sqrt{\sin^2(6\sin\theta) + 36\cos^2\theta\,\cos^2(6\sin\theta)}\,d\theta \approx 8.0091.$$

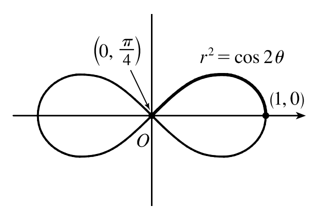
54. The curve $r = \sin(\theta/4)$ is completely traced with $0 \le \theta \le 8\pi$. $r = \sin(\theta/4) \Rightarrow \dfrac{dr}{d\theta} = \frac{1}{4}\cos(\theta/4)$, so

$r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = \sin^2(\theta/4) + \frac{1}{16}\cos^2(\theta/4) \quad\Rightarrow\quad L = \displaystyle\int_0^{8\pi} \sqrt{\sin^2(\theta/4) + \tfrac{1}{16}\cos^2(\theta/4)}\,d\theta \approx 17.1568.$

55. (a) From (10.2.6),

$$\begin{aligned} S &= \textstyle\int_a^b 2\pi y\sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta \\ &= \textstyle\int_a^b 2\pi y\sqrt{r^2 + (dr/d\theta)^2}\,d\theta \qquad \text{[from the derivation of Equation 10.4.5]} \\ &= \textstyle\int_a^b 2\pi r\sin\theta\sqrt{r^2 + (dr/d\theta)^2}\,d\theta \end{aligned}$$

(b) The curve $r^2 = \cos 2\theta$ goes through the pole when $\cos 2\theta = 0 \Rightarrow$ $2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$. We'll rotate the curve from $\theta = 0$ to $\theta = \frac{\pi}{4}$ and double this value to obtain the total surface area generated.



$r^2 = \cos 2\theta \Rightarrow 2r\dfrac{dr}{d\theta} = -2\sin 2\theta \Rightarrow \left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{\sin^2 2\theta}{r^2} = \dfrac{\sin^2 2\theta}{\cos 2\theta}.$

$$S = 2\int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\sin\theta\sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta}\,d\theta = 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\sin\theta\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}\,d\theta$$

$$= 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\sin\theta\frac{1}{\sqrt{\cos 2\theta}}\,d\theta = 4\pi\int_0^{\pi/4}\sin\theta\,d\theta = 4\pi\big[-\cos\theta\big]_0^{\pi/4} = -4\pi\left(\tfrac{\sqrt{2}}{2} - 1\right) = 2\pi\left(2 - \sqrt{2}\right)$$

56. (a) Rotation around $\theta = \frac{\pi}{2}$ is the same as rotation around the y-axis, that is, $S = \int_a^b 2\pi x\, ds$ where $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$ for a parametric equation, and for the special case of a polar equation, $x = r\cos\theta$ and $ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta = \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$ [see the derivation of Equation 10.4.5]. Therefore, for a polar equation rotated around $\theta = \frac{\pi}{2}$, $S = \int_a^b 2\pi r\cos\theta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$.

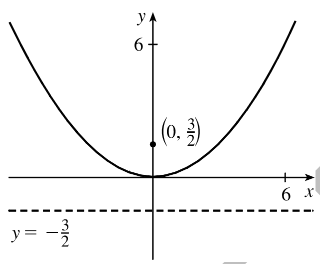
(b) As in the solution for Exercise 55(b), we can double the surface area generated by rotating the curve from $\theta = 0$ to $\theta = \frac{\pi}{4}$ to obtain the total surface area.

$$S = 2\int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\cos\theta\sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta}\, d\theta = 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\cos\theta\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}\, d\theta$$
$$= 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\cos\theta\,\frac{1}{\sqrt{\cos 2\theta}}\, d\theta = 4\pi\int_0^{\pi/4}\cos\theta\, d\theta = 4\pi\big[\sin\theta\big]_0^{\pi/4} = 4\pi\left(\frac{\sqrt{2}}{2} - 0\right) = 2\sqrt{2}\,\pi$$

10.5 Conic Sections

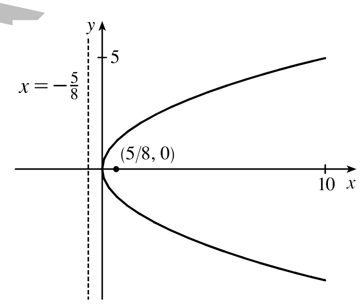
1. $x^2 = 6y$ and $x^2 = 4py \;\Rightarrow\; 4p = 6 \;\Rightarrow\; p = \frac{3}{2}$.

The vertex is $(0,0)$, the focus is $\left(0, \frac{3}{2}\right)$, and the directrix is $y = -\frac{3}{2}$.



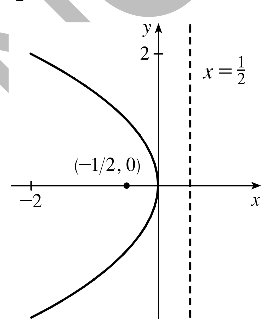
2. $2y^2 = 5x \;\Rightarrow\; y^2 = \frac{5}{2}x$. $4p = \frac{5}{2} \;\Rightarrow\; p = \frac{5}{8}$.

The vertex is $(0,0)$, the focus is $\left(\frac{5}{8}, 0\right)$, and the directrix is $x = -\frac{5}{8}$.



3. $2x = -y^2 \;\Rightarrow\; y^2 = -2x$. $4p = -2 \;\Rightarrow\; p = -\frac{1}{2}$.

The vertex is $(0,0)$, the focus is $\left(-\frac{1}{2}, 0\right)$, and the directrix is $x = \frac{1}{2}$.



4. $3x^2 + 8y = 0 \;\Rightarrow\; 3x^2 = -8y \;\Rightarrow\; x^2 = -\frac{8}{3}y$.

$4p = -\frac{8}{3} \;\Rightarrow\; p = -\frac{2}{3}$. The vertex is $(0,0)$, the focus is $\left(0, -\frac{2}{3}\right)$, and the directrix is $y = \frac{2}{3}$.