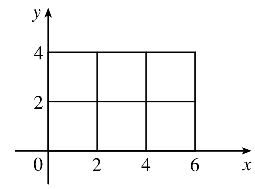
15 □ MULTIPLE INTEGRALS

15.1 Double Integrals over Rectangles

1. (a) The subrectangles are shown in the figure.



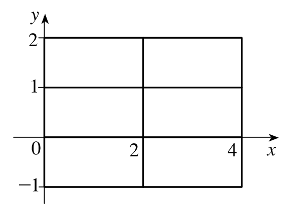
The surface is the graph of $f(x, y) = xy$ and $\Delta A = 4$, so we estimate

$$V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_i, y_j)\,\Delta A$$
$$= f(2,2)\,\Delta A + f(2,4)\,\Delta A + f(4,2)\,\Delta A + f(4,4)\,\Delta A + f(6,2)\,\Delta A + f(6,4)\,\Delta A$$
$$= 4(4) + 8(4) + 8(4) + 16(4) + 12(4) + 24(4) = 288$$

(b) $V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A = f(1,1)\,\Delta A + f(1,3)\,\Delta A + f(3,1)\,\Delta A + f(3,3)\,\Delta A + f(5,1)\,\Delta A + f(5,3)\,\Delta A$

$$= 1(4) + 3(4) + 3(4) + 9(4) + 5(4) + 15(4) = 144$$

2. (a) The subrectangles are shown in the figure.



Here $\Delta A = 2$ and we estimate

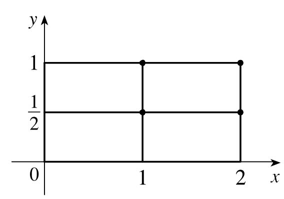
$$\iint_R \left(1 - xy^2\right) dA \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$$
$$= f(2,-1)\,\Delta A + f(2,0)\,\Delta A + f(2,1)\,\Delta A + f(4,-1)\,\Delta A + f(4,0)\,\Delta A + f(4,1)\,\Delta A$$
$$= (-1)(2) + 1(2) + (-1)(2) + (-3)(2) + 1(2) + (-3)(2) = -12$$

(b) $\iint_R \left(1 - xy^2\right) dA \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$

$$= f(0,0)\,\Delta A + f(0,1)\,\Delta A + f(0,2)\,\Delta A + f(2,0)\,\Delta A + f(2,1)\,\Delta A + f(2,2)\,\Delta A$$
$$= 1(2) + 1(2) + 1(2) + 1(2) + (-1)(2) + (-7)(2) = -8$$

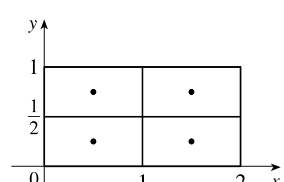
3. (a) The subrectangles are shown in the figure. Since $\Delta A = 1 \cdot \frac{1}{2} = \frac{1}{2}$, we estimate

$$\iint_R xe^{-xy}\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$$
$$= f\left(1, \tfrac{1}{2}\right)\Delta A + f(1,1)\,\Delta A + f\left(2, \tfrac{1}{2}\right)\Delta A + f(2,1)\,\Delta A$$
$$= e^{-1/2}\left(\tfrac{1}{2}\right) + e^{-1}\left(\tfrac{1}{2}\right) + 2e^{-1}\left(\tfrac{1}{2}\right) + 2e^{-2}\left(\tfrac{1}{2}\right) \approx 0.990$$

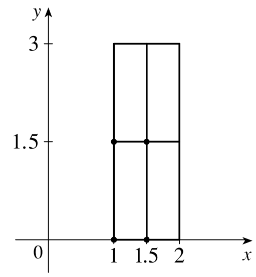


(b) $\iint_R xe^{-xy}\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A$

$$= f\left(\tfrac{1}{2}, \tfrac{1}{4}\right)\Delta A + f\left(\tfrac{1}{2}, \tfrac{3}{4}\right)\Delta A + f\left(\tfrac{3}{2}, \tfrac{1}{4}\right)\Delta A + f\left(\tfrac{3}{2}, \tfrac{3}{4}\right)\Delta A$$
$$= \tfrac{1}{2}e^{-1/8}\left(\tfrac{1}{2}\right) + \tfrac{1}{2}e^{-3/8}\left(\tfrac{1}{2}\right) + \tfrac{3}{2}e^{-3/8}\left(\tfrac{1}{2}\right) + \tfrac{3}{2}e^{-9/8}\left(\tfrac{1}{2}\right) \approx 1.151$$



4. (a) The subrectangles are shown in the figure.



The surface is the graph of $f(x, y) = 1 + x^2 + 3y$ and $\Delta A = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$, so we estimate

$$V = \iint_R (1 + x^2 + 3y)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

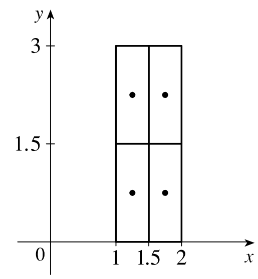
$$= f(1,0)\,\Delta A + f\left(1, \tfrac{3}{2}\right) \Delta A + f\left(\tfrac{3}{2}, 0\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{3}{2}\right) \Delta A$$

$$= 2\left(\tfrac{3}{4}\right) + \tfrac{13}{2}\left(\tfrac{3}{4}\right) + \tfrac{13}{4}\left(\tfrac{3}{4}\right) + \tfrac{31}{4}\left(\tfrac{3}{4}\right) = \tfrac{39}{2}\left(\tfrac{3}{4}\right) = \tfrac{117}{8} = 14.625$$

(b) $$V = \iint_R (1 + x^2 + 3y)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\, \Delta A$$

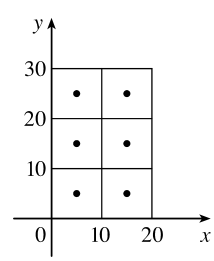
$$= f\left(\tfrac{5}{4}, \tfrac{3}{4}\right) \Delta A + f\left(\tfrac{5}{4}, \tfrac{9}{4}\right) \Delta A + f\left(\tfrac{7}{4}, \tfrac{3}{4}\right) \Delta A + f\left(\tfrac{7}{4}, \tfrac{9}{4}\right) \Delta A$$

$$= \tfrac{77}{16}\left(\tfrac{3}{4}\right) + \tfrac{149}{16}\left(\tfrac{3}{4}\right) + \tfrac{101}{16}\left(\tfrac{3}{4}\right) + \tfrac{173}{16}\left(\tfrac{3}{4}\right) = \tfrac{375}{16} = 23.4375$$



5. The values of $f(x, y) = \sqrt{52 - x^2 - y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U < V < L$. (Note that this is true no matter how R is divided into subrectangles.)

6. To approximate the volume, let R be the planar region corresponding to the surface of the water in the pool, and place R on coordinate axes so that x and y correspond to the dimensions given. Then we define $f(x, y)$ to be the depth of the water at (x, y), so the volume of water in the pool is the volume of the solid that lies above the rectangle $R = [0, 20] \times [0, 30]$ and below the graph of $f(x, y)$. We can estimate this volume using the Midpoint Rule with $m = 2$ and $n = 3$, so $\Delta A = 100$. Each subrectangle with its midpoint is shown in the figure. Then



$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A = \Delta A[f(5,5) + f(5,15) + f(5,25) + f(15,5) + f(15,15) + f(15,25)]$$

$$= 100(3 + 7 + 10 + 3 + 5 + 8) = 3600$$

Thus, we estimate that the pool contains 3600 cubic feet of water.

Alternatively, we can approximate the volume with a Riemann sum where $m = 4$, $n = 6$ and the sample points are taken to be, for example, the upper right corner of each subrectangle. Then $\Delta A = 25$ and

$$V \approx \sum_{i=1}^{4} \sum_{j=1}^{6} f(x_i, y_j)\, \Delta A$$

$$= 25[3 + 4 + 7 + 8 + 10 + 8 + 4 + 6 + 8 + 10 + 12 + 10 + 3 + 4 + 5 + 6 + 8 + 7 + 2 + 2 + 2 + 3 + 4 + 4]$$

$$= 25(140) = 3500$$

So we estimate that the pool contains 3500 ft^3 of water.

7. (a) With $m = n = 2$, we have $\Delta A = 4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

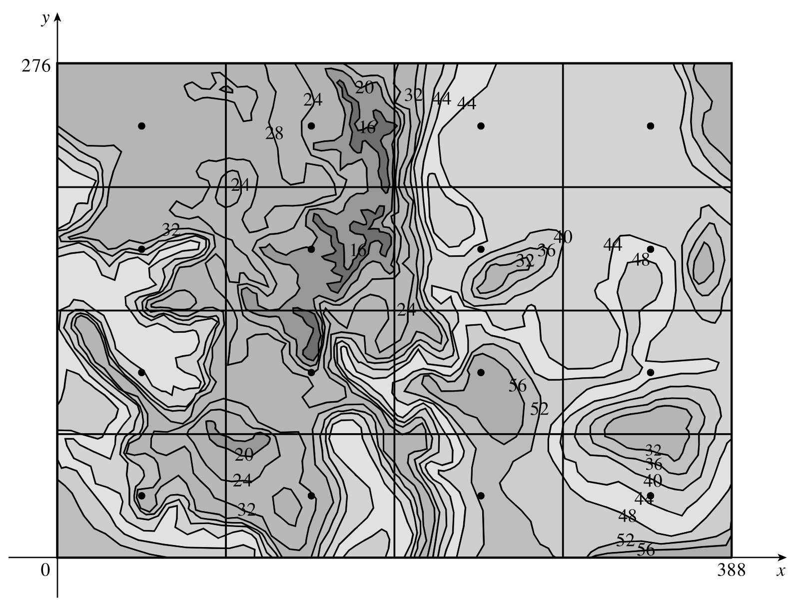
$$\iint_R f(x,y)\;dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A = \Delta A[f(1,1) + f(1,3) + f(3,1) + f(3,3)] \approx 4(27 + 4 + 14 + 17) = 248$$

(b) $f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA \approx \frac{1}{16}(248) = 15.5$

8. As in Example 9, we place the origin at the southwest corner of the state. Then $R = [0, 388] \times [0, 276]$ (in miles) is the rectangle corresponding to Colorado and we define $f(x,y)$ to be the temperature at the location (x,y). The average temperature is given by

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA = \frac{1}{388 \cdot 276} \iint_R f(x,y)\,dA$$

To use the Midpoint Rule with $m = n = 4$, we divide R into 16 regions of equal size, as shown in the figure, with the center of each subrectangle indicated.



The area of each subrectangle is $\Delta A = \frac{388}{4} \cdot \frac{276}{4} = 6693$, so using the contour map to estimate the function values at each midpoint, we have

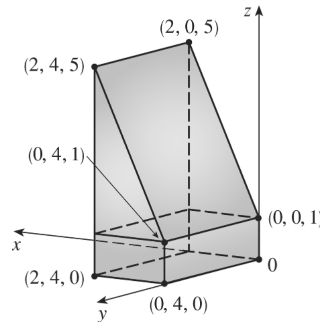
$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{4}\sum_{j=1}^{4} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A \\ &\approx \Delta A\,[31 + 28 + 52 + 43 + 43 + 25 + 57 + 46 + 36 + 20 + 42 + 45 + 30 + 23 + 43 + 41] \\ &= 6693(605)\end{aligned}$$

Therefore, $f_{\text{ave}} \approx \dfrac{6693 \cdot 605}{388 \cdot 276} \approx 37.8$, so the average temperature in Colorado at 4:00 PM on February 26, 2007, was approximately 37.8°F.

9. $z = \sqrt{2} > 0$, so we can interpret the double integral as the volume of the solid S that lies below the plane $z = \sqrt{2}$ and above the rectangle $[2, 6] \times [-1, 5]$. S is a rectangular solid, so $\iint_R \sqrt{2}\,dA = 4 \cdot 6 \cdot \sqrt{2} = 24\sqrt{2}$.

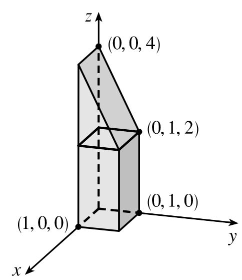
10. $z = 2x + 1 \geq 0$ for $0 \leq x \leq 2$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 2x + 1$ and above the rectangle $[0, 2] \times [0, 4]$. We can picture S as a rectangular solid (with height 1) surmounted by a triangular cylinder; thus

$$\iint_R (2x + 1)\, dA = (2)(4)(1) + \tfrac{1}{2}(2)(4)(4) = 24$$

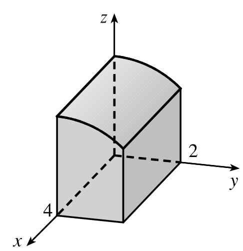


11. $z = 4 - 2y \geq 0$ for $0 \leq y \leq 1$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 4 - 2y$ and above the square $[0, 1] \times [0, 1]$. We can picture S as a rectangular solid (with height 2) surmounted by a triangular cylinder; thus

$$\iint_R (4 - 2y)\, dA = (1)(1)(2) + \tfrac{1}{2}(1)(1)(2) = 3$$



12. Here $z = \sqrt{9 - y^2}$, so $z^2 + y^2 = 9$, $z \geq 0$. Thus the integral represents the volume of the top half of the part of the circular cylinder $z^2 + y^2 = 9$ that lies above the rectangle $[0, 4] \times [0, 2]$.



13. $\int_0^2 (x + 3x^2y^2)\, dx = \left[\dfrac{x^2}{2} + 3\,\dfrac{x^3}{3}\,y^2\right]_{x=0}^{x=2} = \left[\tfrac{1}{2}x^2 + x^3y^2\right]_{x=0}^{x=2} = \left[\tfrac{1}{2}(2)^2 + (2)^3y^2\right] - \left[\tfrac{1}{2}(0)^2 + (0)^3y^2\right] = 2 + 8y^2$,

$\int_0^3 (x + 3x^2y^2)\, dy = \left[xy + 3x^2\,\dfrac{y^3}{3}\right]_{y=0}^{y=3} = \left[xy + x^2y^3\right]_{y=0}^{y=3} = \left[x(3) + x^2(3)^3\right] - \left[x(0) + x^2(0)^3\right] = 3x + 27x^2$

14. $\int_0^2 y\sqrt{x+2}\, dx = \left[y \cdot \tfrac{2}{3}(x+2)^{3/2}\right]_{x=0}^{x=2} = \tfrac{2}{3}y(4)^{3/2} - \tfrac{2}{3}y(2)^{3/2} = \tfrac{16}{3}y - \tfrac{4}{3}\sqrt{2}\,y = \tfrac{4}{3}(4 - \sqrt{2}\,)\,y$,

$\int_0^3 y\sqrt{x+2}\, dy = \left[\dfrac{y^2}{2}\,\sqrt{x+2}\right]_{y=0}^{y=3} = \tfrac{1}{2}(3)^2\sqrt{x+2} - \tfrac{1}{2}(0)^2\sqrt{x+2} = \tfrac{9}{2}\sqrt{x+2}$

15. $\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx = \int_1^4 \left[3x^2y^2 - 2xy\right]_{y=0}^{y=2} dx = \int_1^4 \left[(12x^2 - 4x) - (0 - 0)\right] dx$

$$= \int_1^4 (12x^2 - 4x)\, dx = \left[4x^3 - 2x^2\right]_1^4 = (256 - 32) - (4 - 2) = 222$$

16. $\int_0^1 \int_0^1 (x+y)^2\, dx\, dy = \int_0^1 \int_0^1 (x^2 + 2xy + y^2)\, dx\, dy = \int_0^1 \left[\tfrac{1}{3}x^3 + x^2y + xy^2\right]_{x=0}^{x=1} dy$

$$= \int_0^1 \left(\tfrac{1}{3} + y + y^2\right) dy = \left[\tfrac{1}{3}y + \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3\right]_0^1 = \tfrac{1}{3} + \tfrac{1}{2} + \tfrac{1}{3} - 0 = \tfrac{7}{6}$$

17. $\int_0^1 \int_1^2 (x + e^{-y})\, dx\, dy = \int_0^1 \left[\tfrac{1}{2}x^2 + xe^{-y}\right]_{x=1}^{x=2} dy = \int_0^1 \left[(2 + 2e^{-y}) - \left(\tfrac{1}{2} + e^{-y}\right)\right] dy$

$$= \int_0^1 \left(\tfrac{3}{2} + e^{-y}\right) dy = \left[\tfrac{3}{2}y - e^{-y}\right]_0^1 = \left(\tfrac{3}{2} - e^{-1}\right) - (0 - 1) = \tfrac{5}{2} - e^{-1}$$

18. $\int_0^{\pi/6}\int_0^{\pi/2}(\sin x+\sin y)\,dy\,dx = \int_0^{\pi/6}\left[y\sin x-\cos y\right]_{y=0}^{y=\pi/2}\,dx = \int_0^{\pi/6}\left[\left(\frac{\pi}{2}\sin x-0\right)-(0-1)\right]dx$

$$= \int_0^{\pi/6}\left(\tfrac{\pi}{2}\sin x+1\right)dx = \left[-\tfrac{\pi}{2}\cos x+x\right]_0^{\pi/6}$$

$$= \left[\left(-\tfrac{\pi}{2}\cdot\tfrac{\sqrt{3}}{2}+\tfrac{\pi}{6}\right)-\left(-\tfrac{\pi}{2}+0\right)\right] = \left(\tfrac{2}{3}-\tfrac{\sqrt{3}}{4}\right)\pi$$

19. $\int_{-3}^{3}\int_0^{\pi/2}(y+y^2\cos x)\,dx\,dy = \int_{-3}^{3}\left[xy+y^2\sin x\right]_{x=0}^{x=\pi/2}\,dy = \int_{-3}^{3}\left(\frac{\pi}{2}y+y^2\right)dy$

$$= \left[\tfrac{\pi}{4}y^2+\tfrac{1}{3}y^3\right]_{-3}^{3} = \left[\left(\tfrac{9\pi}{4}+9\right)-\left(\tfrac{9\pi}{4}-9\right)\right] = 18$$

20. $\displaystyle\int_1^3\int_1^5\frac{\ln y}{xy}\,dy\,dx = \int_1^3\frac{1}{x}\,dx\int_1^5\frac{\ln y}{y}\,dy$ [by Equation 11]

$= [\ln|x|]_1^3\left[\frac{1}{2}(\ln y)^2\right]_1^5$ [substitute $u=\ln y \Rightarrow du=(1/y)\,dy$]

$= (\ln 3-0)\cdot\frac{1}{2}[(\ln 5)^2-0] = \frac{1}{2}(\ln 3)(\ln 5)^2$

21. $\displaystyle\int_1^4\int_1^2\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx = \int_1^4\left[x\ln|y|+\frac{1}{x}\cdot\frac{1}{2}y^2\right]_{y=1}^{y=2}dx = \int_1^4\left(x\ln 2+\frac{3}{2x}\right)dx = \left[\tfrac{1}{2}x^2\ln 2+\tfrac{3}{2}\ln|x|\right]_1^4$

$$= \left(8\ln 2+\tfrac{3}{2}\ln 4\right)-\left(\tfrac{1}{2}\ln 2+0\right) = \tfrac{15}{2}\ln 2+\tfrac{3}{2}\ln 4 \text{ or } \tfrac{15}{2}\ln 2+3\ln(4^{1/2}) = \tfrac{21}{2}\ln 2$$

22. $\int_0^1\int_0^2 ye^{x-y}\,dx\,dy = \int_0^1\int_0^2 ye^xe^{-y}\,dx\,dy = \int_0^2 e^x\,dx\int_0^1 ye^{-y}\,dy$ [by Equation 11]

$= [e^x]_0^2\left[(-y-1)e^{-y}\right]_0^1$ [by integrating by parts]

$= \left(e^2-e^0\right)\left[-2e^{-1}-\left(-e^0\right)\right] = (e^2-1)(1-2e^{-1})$ or $e^2-2e+2e^{-1}-1$

23. $\int_0^3\int_0^{\pi/2} t^2\sin^3\phi\,d\phi\,dt = \int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^3 t^2\,dt$ [by Equation 11] $= \int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\,d\phi\int_0^3 t^2\,dt$

$$= \left[\tfrac{1}{3}\cos^3\phi-\cos\phi\right]_0^{\pi/2}\left[\tfrac{1}{3}t^3\right]_0^3 = \left[(0-0)-\left(\tfrac{1}{3}-1\right)\right]\cdot\tfrac{1}{3}(27-0) = \tfrac{2}{3}(9) = 6$$

24. $\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx = \int_0^1 x\left[\frac{1}{3}(x^2+y^2)^{3/2}\right]_{y=0}^{y=1}dx = \frac{1}{3}\int_0^1 x[(x^2+1)^{3/2}-x^3]\,dx = \frac{1}{3}\int_0^1[x(x^2+1)^{3/2}-x^4]dx$

$$= \tfrac{1}{3}\left[\tfrac{1}{5}(x^2+1)^{5/2}-\tfrac{1}{5}x^5\right]_0^1 = \tfrac{1}{15}\left[(2^{5/2}-1)-(1-0)\right] = \tfrac{2}{15}\left(2\sqrt{2}-1\right)$$

25. $\int_0^1\int_0^1 v(u+v^2)^4\,du\,dv = \int_0^1\left[\frac{1}{5}v(u+v^2)^5\right]_{u=0}^{u=1}dv = \frac{1}{5}\int_0^1 v\left[(1+v^2)^5-(0+v^2)^5\right]dv$

$= \frac{1}{5}\int_0^1\left[v(1+v^2)^5-v^{11}\right]dv = \frac{1}{5}\left[\frac{1}{2}\cdot\frac{1}{6}(1+v^2)^6-\frac{1}{12}v^{12}\right]_0^1$

[substitute $t=1+v^2 \Rightarrow dt=2v\,dv$ in the first term]

$= \frac{1}{60}\left[(2^6-1)-(1-0)\right] = \frac{1}{60}(63-1) = \frac{31}{30}$

26. $\int_0^1\int_0^1\sqrt{s+t}\,ds\,dt = \int_0^1\left[\frac{2}{3}(s+t)^{3/2}\right]_{s=0}^{s=1}dt = \frac{2}{3}\int_0^1[(1+t)^{3/2}-t^{3/2}]\,dt = \frac{2}{3}\left[\frac{2}{5}(1+t)^{5/2}-\frac{2}{5}t^{5/2}\right]_0^1$

$$= \tfrac{4}{15}[(2^{5/2}-1)-(1-0)] = \tfrac{4}{15}\left(2^{5/2}-2\right) \text{ or } \tfrac{8}{15}\left(2\sqrt{2}-1\right)$$

27. $\iint_R x\sec^2 y\,dA = \int_0^2\int_0^{\pi/4}x\sec^2 y\,dy\,dx = \int_0^2 x\,dx\int_0^{\pi/4}\sec^2 y\,dy = \left[\frac{1}{2}x^2\right]_0^2\left[\tan y\right]_0^{\pi/4}$

$$= (2-0)\left(\tan\tfrac{\pi}{4}-\tan 0\right) = 2(1-0) = 2$$

28. $\iint_R (y + xy^{-2})\, dA = \int_1^2 \int_0^2 (y + xy^{-2})\, dx\, dy = \int_1^2 \left[xy + \frac{1}{2}x^2y^{-2}\right]_{x=0}^{x=2} dy = \int_1^2 \left(2y + 2y^{-2}\right) dy$

$= \left[y^2 - 2y^{-1}\right]_1^2 = (4 - 1) - (1 - 2) = 4$

29. $\displaystyle\iint_R \frac{xy^2}{x^2+1}\, dA = \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx = \int_0^1 \frac{x}{x^2+1}\, dx \int_{-3}^3 y^2\, dy = \left[\tfrac{1}{2}\ln(x^2+1)\right]_0^1 \left[\tfrac{1}{3}y^3\right]_{-3}^3$

$= \frac{1}{2}(\ln 2 - \ln 1) \cdot \frac{1}{3}(27 + 27) = 9 \ln 2$

30. $\displaystyle\iint_R \frac{\tan\theta}{\sqrt{1-t^2}}\, dA = \int_0^{1/2} \int_0^{\pi/3} \frac{\tan\theta}{\sqrt{1-t^2}}\, d\theta\, dt = \int_0^{1/2} \frac{1}{\sqrt{1-t^2}}\, dt \int_0^{\pi/3} \tan\theta\, d\theta = \left[\sin^{-1} t\right]_0^{1/2} \left[\ln|\sec\theta|\right]_0^{\pi/3}$

$= \left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right)\left(\ln\left|\sec\frac{\pi}{3}\right| - \ln|\sec 0|\right) = \left(\frac{\pi}{6} - 0\right)(\ln 2 - \ln 1) = \frac{\pi}{6}\ln 2$

31. $\int_0^{\pi/6} \int_0^{\pi/3} x \sin(x + y)\, dy\, dx$

$= \int_0^{\pi/6} \left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3} dx = \int_0^{\pi/6} \left[x\cos x - x\cos\left(x + \frac{\pi}{3}\right)\right] dx$

$= x\left[\sin x - \sin\left(x + \frac{\pi}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6} \left[\sin x - \sin\left(x + \frac{\pi}{3}\right)\right] dx$ [by integrating by parts separately for each term]

$= \frac{\pi}{6}\left[\frac{1}{2} - 1\right] - \left[-\cos x + \cos\left(x + \frac{\pi}{3}\right)\right]_0^{\pi/6} = -\frac{\pi}{12} - \left[-\frac{\sqrt{3}}{2} + 0 - \left(-1 + \frac{1}{2}\right)\right] = \frac{\sqrt{3}-1}{2} - \frac{\pi}{12}$

32. $\displaystyle\iint_R \frac{x}{1+xy}\, dA = \int_0^1 \int_0^1 \frac{x}{1+xy}\, dy\, dx = \int_0^1 \left[\ln(1+xy)\right]_{y=0}^{y=1} dx = \int_0^1 \left[\ln(1+x) - \ln 1\right] dx$

$= \int_0^1 \ln(1+x)\, dx = \left[(1+x)\ln(1+x) - x\right]_0^1$ [by integrating by parts]

$= (2\ln 2 - 1) - (\ln 1 - 0) = 2\ln 2 - 1$

33. $\iint_R ye^{-xy}\, dA = \int_0^3 \int_0^2 ye^{-xy}\, dx\, dy = \int_0^3 \left[-e^{-xy}\right]_{x=0}^{x=2} dy = \int_0^3 (-e^{-2y} + 1)\, dy = \left[\frac{1}{2}e^{-2y} + y\right]_0^3$

$= \frac{1}{2}e^{-6} + 3 - \left(\frac{1}{2} + 0\right) = \frac{1}{2}e^{-6} + \frac{5}{2}$

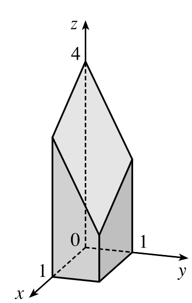
34. $\displaystyle\iint_R \frac{1}{1+x+y}\, dA = \int_1^3 \int_1^2 \frac{1}{1+x+y}\, dy\, dx = \int_1^3 \left[\ln(1+x+y)\right]_{y=1}^{y=2} dx = \int_1^3 \left[\ln(x+3) - \ln(x+2)\right] dx$

$= \left[\left((x+3)\ln(x+3) - (x+3)\right) - \left((x+2)\ln(x+2) - (x+2)\right)\right]_1^3$

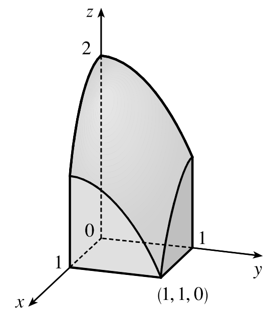
[by integrating by parts separately for each term]

$= (6\ln 6 - 6 - 5\ln 5 + 5) - (4\ln 4 - 4 - 3\ln 3 + 3) = 6\ln 6 - 5\ln 5 - 4\ln 4 + 3\ln 3$

35. $z = f(x, y) = 4 - x - 2y \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the plane $z = 4 - x - 2y$ and above $[0, 1] \times [0, 1]$.



36. $z = 2 - x^2 - y^2 \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the circular paraboloid $z = 2 - x^2 - y^2$ and above $[0, 1] \times [0, 1]$.



37. The solid lies under the plane $4x + 6y - 2z + 15 = 0$ or $z = 2x + 3y + \frac{15}{2}$ so

$$V = \iint_R (2x + 3y + \tfrac{15}{2})\, dA = \int_{-1}^{1} \int_{-1}^{2} (2x + 3y + \tfrac{15}{2})\, dx\, dy = \int_{-1}^{1} \left[x^2 + 3xy + \tfrac{15}{2}x\right]_{x=-1}^{x=2} dy$$
$$= \int_{-1}^{1} \left[(19 + 6y) - (-\tfrac{13}{2} - 3y)\right] dy = \int_{-1}^{1} (\tfrac{51}{2} + 9y)\, dy = \left[\tfrac{51}{2}y + \tfrac{9}{2}y^2\right]_{-1}^{1} = 30 - (-21) = 51$$

38. $$V = \iint_R (3y^2 - x^2 + 2)\, dA = \int_{-1}^{1} \int_{1}^{2} (3y^2 - x^2 + 2)\, dy\, dx = \int_{-1}^{1} \left[y^3 - x^2 y + 2y\right]_{y=1}^{y=2} dx$$
$$= \int_{-1}^{1} \left[(12 - 2x^2) - (3 - x^2)\right] dx = \int_{-1}^{1} \left(9 - x^2 dx\right) = \left[9x - \tfrac{1}{3}x^3\right]_{-1}^{1} = \tfrac{26}{3} + \tfrac{26}{3} = \tfrac{52}{3}$$

39. $$V = \int_{-2}^{2} \int_{-1}^{1} \left(1 - \tfrac{1}{4}x^2 - \tfrac{1}{9}y^2\right) dx\, dy = 4 \int_0^2 \int_0^1 \left(1 - \tfrac{1}{4}x^2 - \tfrac{1}{9}y^2\right) dx\, dy$$
$$= 4 \int_0^2 \left[x - \tfrac{1}{12}x^3 - \tfrac{1}{9}y^2 x\right]_{x=0}^{x=1} dy = 4 \int_0^2 \left(\tfrac{11}{12} - \tfrac{1}{9}y^2\right) dy = 4\left[\tfrac{11}{12}y - \tfrac{1}{27}y^3\right]_0^2 = 4 \cdot \tfrac{83}{54} = \tfrac{166}{27}$$

40. The solid lies under the surface $z = x^2 + xy^2$ and above the rectangle $R = [0, 5] \times [-2, 2]$, so its volume is

$$V = \iint_R (x^2 + xy^2)\, dA = \int_0^5 \int_{-2}^{2} (x^2 + xy^2)\, dy\, dx = \int_0^5 \left[x^2 y + \tfrac{1}{3}xy^3\right]_{y=-2}^{y=2} dx$$
$$= \int_0^5 \left[\left(2x^2 + \tfrac{8}{3}x\right) - \left(-2x^2 - \tfrac{8}{3}x\right)\right] dx = \int_0^5 (4x^2 + \tfrac{16}{3}x)\, dx$$
$$= \left[\tfrac{4}{3}x^3 + \tfrac{8}{3}x^2\right]_0^5 = \tfrac{500}{3} + \tfrac{200}{3} - 0 = \tfrac{700}{3}$$

41. The solid lies under the surface $z = 1 + x^2 y e^y$ and above the rectangle $R = [-1, 1] \times [0, 1]$, so its volume is

$$V = \iint_R (1 + x^2 y e^y)\, dA = \int_0^1 \int_{-1}^{1} (1 + x^2 y e^y)\, dx\, dy = \int_0^1 \left[x + \tfrac{1}{3}x^3 y e^y\right]_{x=-1}^{x=1} dy$$
$$= \int_0^1 (2 + \tfrac{2}{3}y e^y)\, dy = \left[2y + \tfrac{2}{3}(y - 1)\, e^y\right]_0^1 \qquad \text{[by integrating by parts in the second term]}$$
$$= (2 + 0) - \left(0 - \tfrac{2}{3}e^0\right) = 2 + \tfrac{2}{3} = \tfrac{8}{3}$$

42. The cylinder intersects the xy-plane along the line $x = 4$, so in the first octant, the solid lies below the surface $z = 16 - x^2$ and above the rectangle $R = [0, 4] \times [0, 5]$ in the xy-plane.

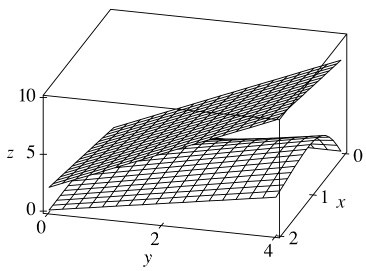
$$V = \int_0^5 \int_0^4 (16 - x^2)\, dx\, dy = \int_0^4 (16 - x^2)\, dx \int_0^5 dy$$
$$= \left[16x - \tfrac{1}{3}x^3\right]_0^4 \left[\,y\,\right]_0^5 = (64 - \tfrac{64}{3} - 0)(5 - 0) = \tfrac{640}{3}$$

43. The solid lies below the surface $z = 2 + x^2 + (y - 2)^2$ and above the plane $z = 1$ for $-1 \leq x \leq 1$, $0 \leq y \leq 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = 2 + x^2 + (y - 2)^2$ over the rectangle

$R = [-1, 1] \times [0, 4]$ and the solid that lies under $z = 1$ over R.

$$\begin{aligned} V &= \int_0^4 \int_{-1}^1 [2 + x^2 + (y-2)^2]\, dx\, dy - \int_0^4 \int_{-1}^1 (1)\, dx\, dy \\ &= \int_0^4 \left[2x + \tfrac{1}{3}x^3 + x(y-2)^2\right]_{x=-1}^{x=1} dy - \int_{-1}^1 dx \int_0^4 dy \\ &= \int_0^4 \left[(2 + \tfrac{1}{3} + (y-2)^2) - (-2 - \tfrac{1}{3} - (y-2)^2)\right] dy - [x]_{-1}^1\, [y]_0^4 \\ &= \int_0^4 \left[\tfrac{14}{3} + 2(y-2)^2\right] dy - [1 - (-1)][4 - 0] = \left[\tfrac{14}{3}y + \tfrac{2}{3}(y-2)^3\right]_0^4 - (2)(4) \\ &= \left[\left(\tfrac{56}{3} + \tfrac{16}{3}\right) - \left(0 - \tfrac{16}{3}\right)\right] - 8 = \tfrac{88}{3} - 8 = \tfrac{64}{3} \end{aligned}$$

44.



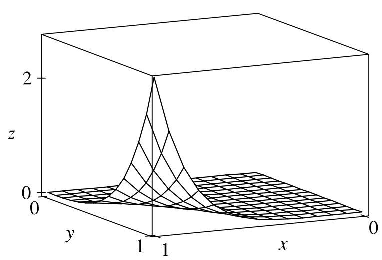
The solid lies below the plane $z = x + 2y$ and above the surface $z = \dfrac{2xy}{x^2+1}$ for $0 \le x \le 2$, $0 \le y \le 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = x + 2y$ over the rectangle $R = [0, 2] \times [0, 4]$ and the solid that lies under $z = \dfrac{2xy}{x^2+1}$ over R.

$$\begin{aligned} V &= \int_0^2 \int_0^4 (x + 2y) dy\, dx - \int_0^2 \int_0^4 \frac{2xy}{x^2+1}\, dy\, dx = \int_0^2 \left[xy + y^2\right]_{y=0}^{y=4} dx - \int_0^2 \frac{2x}{x^2+1}\, dx \int_0^4 y\, dy \\ &= \int_0^2 [(4x + 16) - (0 + 0)]\, dx - \left[\ln\left|x^2+1\right|\right]_0^2 \left[\tfrac{1}{2}y^2\right]_0^4 = \left[2x^2 + 16x\right]_0^2 - (\ln 5 - \ln 1)(8 - 0) \\ &= (8 + 32 - 0) - 8\ln 5 = 40 - 8\ln 5 \end{aligned}$$

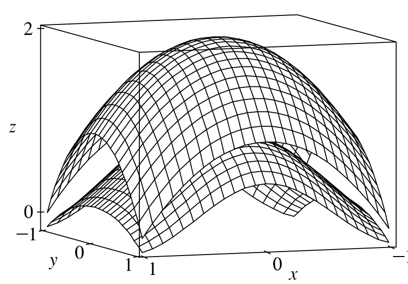
45. In Maple, we can calculate the integral by defining the integrand as `f` and then using the command `int(int(f,x=0..1),y=0..1);`. In Mathematica, we can use the command

```
Integrate[f,{x,0,1},{y,0,1}]
```

We find that $\iint_R x^5 y^3 e^{xy}\, dA = 21e - 57 \approx 0.0839$. We can use `plot3d` (in Maple) or `Plot3D` (in Mathematica) to graph the function.



46. In Maple, we can calculate the integral by defining `f:=exp(-x^2)*cos(x^2+y^2);` and `g:=2-x^2-y^2;` and then [since $2 - x^2 - y^2 > e^{-x^2}\cos(x^2 + y^2)$ for $-1 \le x \le 1$, $-1 \le y \le 1$] using the command `evalf(Int(Int(g-f,x=-1..1),y=-1..1));`. Using `Int` rather than `int` forces Maple to use purely numerical techniques in evaluating the integral.



In Mathematica, we can use the command `NIntegrate[g-f,{x,-1,1},{y,-1,1}]`. We find that $\iint_R \left[(2 - x^2 - y^2) - \left(e^{-x^2}\cos(x^2 + y^2)\right)\right] dA \approx 3.0271$. We can use the `plot3d` command (in Maple) or `Plot3D` (in Mathematica) to graph both functions on the same screen.

47. R is the rectangle $[-1,1]\times[0,5]$. Thus, $A(R)=2\cdot 5=10$ and

$$f_{\text{ave}}=\frac{1}{A(R)}\iint_R f(x,y)\,dA=\tfrac{1}{10}\int_0^5\int_{-1}^1 x^2y\,dx\,dy=\tfrac{1}{10}\int_0^5\left[\tfrac{1}{3}x^3y\right]_{x=-1}^{x=1}\,dy=\tfrac{1}{10}\int_0^5\tfrac{2}{3}y\,dy=\tfrac{1}{10}\left[\tfrac{1}{3}y^2\right]_0^5=\tfrac{5}{6}.$$

48. $A(R)=4\cdot 1=4$, so

$$\begin{aligned}f_{\text{ave}}&=\frac{1}{A(R)}\iint_R f(x,y)\,dA=\tfrac{1}{4}\int_0^4\int_0^1 e^y\sqrt{x+e^y}\,dy\,dx=\tfrac{1}{4}\int_0^4\left[\tfrac{2}{3}(x+e^y)^{3/2}\right]_{y=0}^{y=1}dx\\&=\tfrac{1}{4}\cdot\tfrac{2}{3}\int_0^4[(x+e)^{3/2}-(x+1)^{3/2}]\,dx=\tfrac{1}{6}\left[\tfrac{2}{5}(x+e)^{5/2}-\tfrac{2}{5}(x+1)^{5/2}\right]_0^4\\&=\tfrac{1}{6}\cdot\tfrac{2}{5}[(4+e)^{5/2}-5^{5/2}-e^{5/2}+1]=\tfrac{1}{15}[(4+e)^{5/2}-e^{5/2}-5^{5/2}+1]\approx 3.327\end{aligned}$$

49. $\displaystyle\iint_R\frac{xy}{1+x^4}\,dA=\int_{-1}^1\int_0^1\frac{xy}{1+x^4}\,dy\,dx=\int_{-1}^1\frac{x}{1+x^4}\,dx\int_0^1 y\,dy$ [by Equation 11] but $f(x)=\dfrac{x}{1+x^4}$ is an odd function so $\displaystyle\int_{-1}^1 f(x)\,dx=0$ (by Theorem 4.5.6 [ET 5.5.7]). Thus $\displaystyle\iint_R\frac{xy}{1+x^4}\,dA=0\cdot\int_0^1 y\,dy=0$.

50. $$\begin{aligned}\iint_R(1+x^2\sin y+y^2\sin x)\,dA&=\iint_R 1\,dA+\iint_R x^2\sin y\,dA+\iint_R y^2\sin x\,dA\\&=A(R)+\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}x^2\sin y\,dy\,dx+\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}y^2\sin x\,dy\,dx\\&=(2\pi)(2\pi)+\int_{-\pi}^{\pi}x^2dx\int_{-\pi}^{\pi}\sin y\,dy+\int_{-\pi}^{\pi}\sin x\,dx\int_{-\pi}^{\pi}y^2\,dy\end{aligned}$$

But $\sin x$ is an odd function, so $\int_{-\pi}^{\pi}\sin x\,dx=\int_{-\pi}^{\pi}\sin y\,dy=0$ (by Theorem 4.5.6 [ET 5.5.7]) and $\iint_R(1+x^2\sin y+y^2\sin x)\,dA=4\pi^2+0+0=4\pi^2$.

51. Let $f(x,y)=\dfrac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1\int_0^1 f(x,y)\,dy\,dx=\frac{1}{2}$ and $\int_0^1\int_0^1 f(x,y)\,dx\,dy=-\frac{1}{2}$.

To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0,0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals which diverge at their lower limits of integration.

52. (a) Loosely speaking, Fubini's Theorem says that the order of integration of a function of two variables does not affect the value of the double integral, while Clairaut's Theorem says that the order of differentiation of such a function does not affect the value of the second-order derivative. Also, both theorems require continuity (though Fubini's allows a finite number of smooth curves to contain discontinuities).

(b) To find g_{xy}, we first hold y constant and use the single-variable Fundamental Theorem of Calculus, Part 1:

$\displaystyle g_x=\frac{d}{dx}\,g(x,y)=\frac{d}{dx}\int_a^x\left(\int_c^y f(s,t)\,dt\right)ds=\int_c^y f(x,t)\,dt$. Now we use the Fundamental Theorem again:

$\displaystyle g_{xy}=\frac{d}{dy}\int_c^y f(x,t)\,dt=f(x,y).$

To find g_{yx}, we first use Fubini's Theorem to find that $\int_a^x\int_c^y f(s,t)\,dt\,ds=\int_c^y\int_a^x f(s,t)\,dt\,ds$, and then use the Fundamental Theorem twice, as above, to get $g_{yx}=f(x,y)$. So $g_{xy}=g_{yx}=f(x,y)$.

15.2 Double Integrals over General Regions

1. $\int_1^5 \int_0^x (8x - 2y)\, dy\, dx = \int_1^5 \left[8xy - y^2\right]_{y=0}^{y=x} dx = \int_1^5 [8x(x) - (x)^2 - 8x(0) + (0)^2] dx$

$= \int_1^5 7x^2\, dx = \frac{7}{3}x^3\Big]_1^5 = \frac{7}{3}(125 - 1) = \frac{868}{3}$

2. $\int_0^2 \int_0^{y^2} x^2 y\, dx\, dy = \int_0^2 \left[\frac{1}{3}x^3 y\right]_{x=0}^{x=y^2} dy = \int_0^2 \frac{1}{3}y\left[(y^2)^3 - (0)^3\right] dy$

$= \int_0^2 \frac{1}{3}y^7\, dy = \frac{1}{3}\left[\frac{1}{8}y^8\right]_0^2 = \frac{1}{3}(32 - 0) = \frac{32}{3}$

3. $\int_0^1 \int_0^y xe^{y^3}\, dx\, dy = \int_0^1 \left[\frac{1}{2}x^2 e^{y^3}\right]_{x=0}^{x=y} dy = \int_0^1 \frac{1}{2}e^{y^3}\left[(y)^2 - (0)^2\right] dy$

$= \frac{1}{2}\int_0^1 y^2 e^{y^3}\, dy = \frac{1}{2}\left[\frac{1}{3}e^{y^3}\right]_0^1 = \frac{1}{2}\cdot\frac{1}{3}\left(e^1 - e^0\right) = \frac{1}{6}(e - 1)$

4. $\int_0^{\pi/2} \int_0^x x\sin y\, dy\, dx = \int_0^{\pi/2} [x(-\cos y)]_{y=0}^{y=x} dx = \int_0^{\pi/2} (-x\cos x + x)\, dx = \int_0^{\pi/2} (x - x\cos x)\, dx$

$= \left[\frac{1}{2}x^2 - (x\sin x + \cos x)\right]_0^{\pi/2}$ (by integrating by parts in the second term)

$= \left(\frac{1}{2}\cdot\frac{\pi^2}{4} - \frac{\pi}{2} - 0\right) - (0 - 0 - 1) = \frac{\pi^2}{8} - \frac{\pi}{2} + 1$

5. $\int_0^1 \int_0^{s^2} \cos(s^3)\, dt\, ds = \int_0^1 \left[t\cos(s^3)\right]_{t=0}^{t=s^2} ds = \int_0^1 s^2\cos(s^3)\, ds = \frac{1}{3}\sin(s^3)\Big]_0^1 = \frac{1}{3}(\sin 1 - \sin 0) = \frac{1}{3}\sin 1$

6. $\int_0^1 \int_0^{e^v} \sqrt{1 + e^v}\, dw\, dv = \int_0^1 \left[w\sqrt{1 + e^v}\right]_{w=0}^{w=e^v} dv = \int_0^1 e^v\sqrt{1 + e^v}\, dv = \frac{2}{3}(1 + e^v)^{3/2}\Big]_0^1$

$= \frac{2}{3}(1 + e)^{3/2} - \frac{2}{3}(1 + 1)^{3/2} = \frac{2}{3}(1 + e)^{3/2} - \frac{4}{3}\sqrt{2}$

7. $\iint_D \frac{y}{x^2 + 1}\, dA = \int_0^4 \int_0^{\sqrt{x}} \frac{y}{x^2 + 1}\, dy\, dx = \int_0^4 \left[\frac{1}{x^2 + 1}\cdot\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^4 \frac{x}{x^2 + 1}\, dx$

$= \frac{1}{2}\left[\frac{1}{2}\ln\left|x^2 + 1\right|\right]_0^4 = \frac{1}{4}\left[\ln\left(x^2 + 1\right)\right]_0^4 = \frac{1}{4}(\ln 17 - \ln 1) = \frac{1}{4}\ln 17$

8. $\iint_D (2x + y)\, dA = \int_1^2 \int_{y-1}^1 (2x + y)\, dx\, dy = \int_1^2 \left[x^2 + xy\right]_{x=y-1}^{x=1} dy = \int_1^2 \left[1 + y - (y - 1)^2 - y(y - 1)\right] dy$

$= \int_1^2 (-2y^2 + 4y)\, dy = \left[-\frac{2}{3}y^3 + 2y^2\right]_1^2 = \left(-\frac{16}{3} + 8\right) - \left(-\frac{2}{3} + 2\right) = \frac{4}{3}$

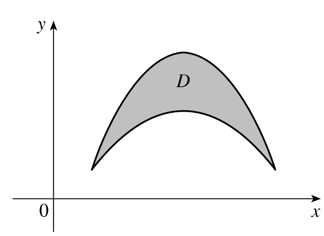
9. $\iint_D e^{-y^2}\, dA = \int_0^3 \int_0^y e^{-y^2}\, dx\, dy = \int_0^3 \left[xe^{-y^2}\right]_{x=0}^{x=y} dy = \int_0^3 \left(ye^{-y^2} - 0\right) dy = \int_0^3 ye^{-y^2}\, dy$

$= -\frac{1}{2}e^{-y^2}\Big]_0^3 = -\frac{1}{2}\left(e^{-9} - e^0\right) = \frac{1}{2}\left(1 - e^{-9}\right)$

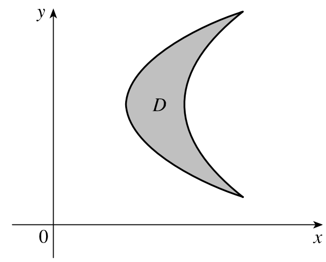
10. $\iint_D y\sqrt{x^2 - y^2}\, dA = \int_0^2 \int_0^x y\sqrt{x^2 - y^2}\, dy\, dx = \int_0^2 \left[-\frac{1}{3}(x^2 - y^2)^{3/2}\right]_{y=0}^{y=x} dx = \int_0^2 \left[0 + \frac{1}{3}(x^2)^{3/2}\right] dx$

$= \int_0^2 \frac{1}{3}x^3\, dx = \frac{1}{3}\cdot\frac{1}{4}x^4\Big]_0^2 = \frac{1}{12}(16 - 0) = \frac{4}{3}$

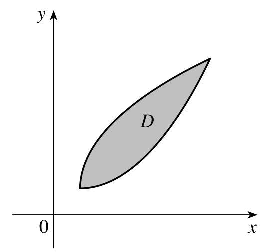
11. (a) At the right we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of x (a type I region) but not as lying between graphs of two continuous functions of y (a type II region). The regions shown in Figures 6 and 8 in the text are additional examples.



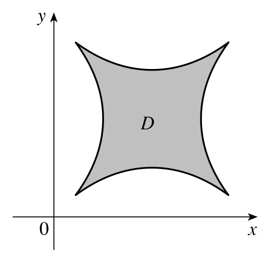
(b) Now we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of y but not as lying between graphs of two continuous functions of x. The first region shown in Figure 7 is another example.



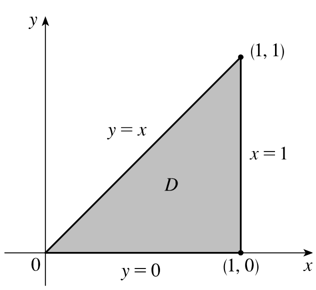
12. (a) At the right we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of x (a type I region) and also as lying between graphs of two continuous functions of y (a type II region). For additional examples see Figures 9, 11, 12, and 14–16 in the text.



(b) Now we sketch an example of a region D that can't be described as lying between the graphs of two continuous functions of x or between graphs of two continuous functions of y. The region shown in Figure 18 is another example.



13.

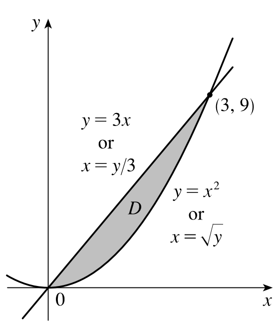


As a type I region, D lies between the lower boundary $y = 0$ and the upper boundary $y = x$ for $0 \le x \le 1$, so $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le x\}$. If we describe D as a type II region, D lies between the left boundary $x = y$ and the right boundary $x = 1$ for $0 \le y \le 1$, so $D = \{(x, y) \mid 0 \le y \le 1, y \le x \le 1\}$.

Thus $\iint_D x\, dA = \int_0^1 \int_0^x x\, dy\, dx = \int_0^1 \left[xy\right]_{y=0}^{y=x} dx = \int_0^1 x^2\, dx = \frac{1}{3}x^3\big]_0^1 = \frac{1}{3}(1-0) = \frac{1}{3}$ or

$\iint_D x\, dA = \int_0^1 \int_y^1 x\, dx\, dy = \int_0^1 \left[\frac{1}{2}x^2\right]_{x=y}^{x=1} dy = \frac{1}{2}\int_0^1 (1-y^2)\, dy = \frac{1}{2}\left[y - \frac{1}{3}y^3\right]_0^1 = \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) - 0\right] = \frac{1}{3}$.

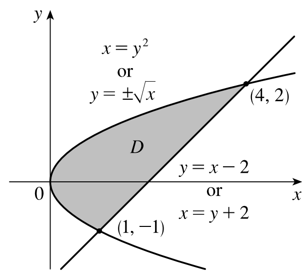
14.



The curves $y = x^2$ and $y = 3x$ intersect at points $(0, 0)$, $(3, 9)$. As a type I region, D is enclosed by the lower boundary $y = x^2$ and the upper boundary $y = 3x$ for $0 \le x \le 3$, so $D = \{(x, y) \mid 0 \le x \le 3, x^2 \le y \le 3x\}$. If we describe D as a type II region, D is enclosed by the left boundary $x = y/3$ and the right boundary $x = \sqrt{y}$ for $0 \le y \le 9$, so $D = \{(x, y) \mid 0 \le y \le 9, y/3 \le x \le \sqrt{y}\}$. Thus

$$\iint_D xy\,dA = \int_0^3\int_{x^2}^{3x} xy\,dy\,dx = \int_0^3 \left[x\cdot\tfrac{1}{2}y^2\right]_{y=x^2}^{y=3x} dx = \tfrac{1}{2}\int_0^3 x(9x^2-x^4)\,dx = \tfrac{1}{2}\int_0^3 (9x^3-x^5)\,dx$$
$$= \tfrac{1}{2}\left[9\cdot\tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_0^3 = \tfrac{1}{2}\left[\left(\tfrac{9}{4}\cdot 81 - \tfrac{1}{6}\cdot 729\right) - 0\right] = \tfrac{243}{8}$$

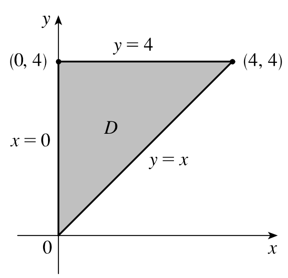
or $$\iint_D xy\,dA = \int_0^9\int_{y/3}^{\sqrt{y}} xy\,dx\,dy = \int_0^9 \left[\tfrac{1}{2}x^2y\right]_{x=y/3}^{x=\sqrt{y}} dy = \tfrac{1}{2}\int_0^9 \left(y-\tfrac{1}{9}y^2\right)y\,dy = \tfrac{1}{2}\int_0^9 \left(y^2-\tfrac{1}{9}y^3\right)dy$$
$$= \tfrac{1}{2}\left[\tfrac{1}{3}y^3 - \tfrac{1}{9}\cdot\tfrac{1}{4}y^4\right]_0^9 = \tfrac{1}{2}\left[\left(\tfrac{1}{3}\cdot 729 - \tfrac{1}{36}\cdot 6561\right) - 0\right] = \tfrac{243}{8}$$

15. 

The curves $y = x-2$ or $x = y+2$ and $x = y^2$ intersect when $y+2 = y^2 \Leftrightarrow y^2-y-2=0 \Leftrightarrow (y-2)(y+1)=0 \Leftrightarrow y=-1, y=2$, so the points of intersection are $(1,-1)$ and $(4,2)$. If we describe D as a type I region, the upper boundary curve is $y=\sqrt{x}$ but the lower boundary curve consists of two parts, $y=-\sqrt{x}$ for $0\le x\le 1$ and $y=x-2$ for $1\le x\le 4$.

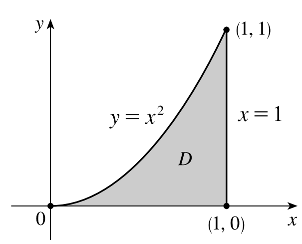
Thus $D = \{(x,y) \mid 0\le x\le 1, -\sqrt{x}\le y\le\sqrt{x}\} \cup \{(x,y)\mid 1\le x\le 4, x-2\le y\le\sqrt{x}\}$ and $\iint_D y\,dA = \int_0^1\int_{-\sqrt{x}}^{\sqrt{x}} y\,dy\,dx + \int_1^4\int_{x-2}^{\sqrt{x}} y\,dy\,dx$. If we describe D as a type II region, D is enclosed by the left boundary $x=y^2$ and the right boundary $x=y+2$ for $-1\le y\le 2$, so $D=\left\{(x,y)\mid -1\le y\le 2, y^2\le x\le y+2\right\}$ and $\iint_D y\,dA = \int_{-1}^2\int_{y^2}^{y+2} y\,dx\,dy$. In either case, the resulting iterated integrals are not difficult to evaluate but the region D is more simply described as a type II region, giving one iterated integral rather than a sum of two, so we evaluate the latter integral:

$$\iint_D y\,dA = \int_{-1}^2\int_{y^2}^{y+2} y\,dx\,dy = \int_{-1}^2 \left[xy\right]_{x=y^2}^{x=y+2} dy = \int_{-1}^2 (y+2-y^2)y\,dy = \int_{-1}^2 (y^2+2y-y^3)\,dy$$
$$= \left[\tfrac{1}{3}y^3 + y^2 - \tfrac{1}{4}y^4\right]_{-1}^2 = \left(\tfrac{8}{3}+4-4\right) - \left(-\tfrac{1}{3}+1-\tfrac{1}{4}\right) = \tfrac{9}{4}$$

16. 

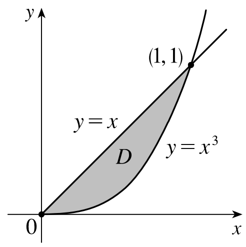
As a type I region, $D = \{(x,y)\mid 0\le x\le 4, x\le y\le 4\}$ and $\iint_D y^2e^{xy}\,dA = \int_0^4\int_x^4 y^2e^{xy}\,dy\,dx$. As a type II region, $D=\{(x,y)\mid 0\le y\le 4, 0\le x\le y\}$ and $\iint_D y^2e^{xy}\,dA = \int_0^4\int_0^y y^2e^{xy}\,dx\,dy$. Evaluating $\int y^2e^{xy}\,dy$ requires integration by parts whereas $\int y^2e^{xy}\,dx$ does not, so the iterated integral corresponding to D as a type II region appears easier to evaluate.

$$\iint_D y^2e^{xy}\,dA = \int_0^4\int_0^y y^2e^{xy}\,dx\,dy = \int_0^4 \left[ye^{xy}\right]_{x=0}^{x=y} dy = \int_0^4 \left(ye^{y^2}-y\right)dy$$
$$= \left[\tfrac{1}{2}e^{y^2} - \tfrac{1}{2}y^2\right]_0^4 = \left(\tfrac{1}{2}e^{16}-8\right) - \left(\tfrac{1}{2}-0\right) = \tfrac{1}{2}e^{16} - \tfrac{17}{2}$$

17. 

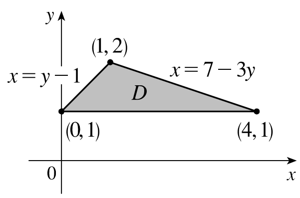
$$\int_0^1\int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 \left[x\sin y\right]_{y=0}^{y=x^2} dx = \int_0^1 x\sin x^2\,dx$$
$$= -\tfrac{1}{2}\cos x^2\Big]_0^1 = -\tfrac{1}{2}(\cos 1 - \cos 0) = \tfrac{1}{2}(1-\cos 1)$$

18.



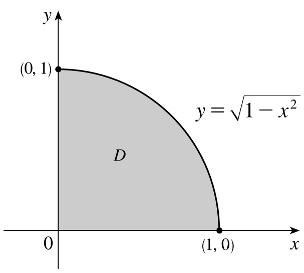
$$\iint_D (x^2+2y)\,dA = \int_0^1 \int_{x^3}^{x} (x^2+2y)\,dy\,dx = \int_0^1 \left[x^2y+y^2\right]_{y=x^3}^{y=x}\,dx$$
$$= \int_0^1 (x^3+x^2-x^5-x^6)\,dx = \left[\tfrac{1}{4}x^4+\tfrac{1}{3}x^3-\tfrac{1}{6}x^6-\tfrac{1}{7}x^7\right]_0^1$$
$$= \tfrac{1}{4}+\tfrac{1}{3}-\tfrac{1}{6}-\tfrac{1}{7} = \tfrac{23}{84}$$

19.



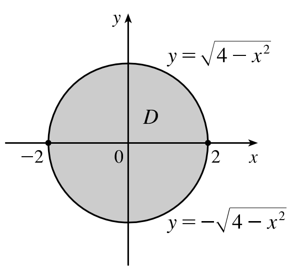
$$\iint_D y^2\,dA = \int_1^2 \int_{y-1}^{7-3y} y^2\,dx\,dy = \int_1^2 \left[xy^2\right]_{x=y-1}^{x=7-3y}\,dy$$
$$= \int_1^2 \left[(7-3y)-(y-1)\right]y^2\,dy = \int_1^2 (8y^2-4y^3)\,dy$$
$$= \left[\tfrac{8}{3}y^3-y^4\right]_1^2 = \tfrac{64}{3}-16-\tfrac{8}{3}+1 = \tfrac{11}{3}$$

20.



$$\iint_D xy\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} xy\,dy\,dx$$
$$= \int_0^1 \left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=\sqrt{1-x^2}}\,dx = \int_0^1 \tfrac{1}{2}x(1-x^2)\,dx$$
$$= \tfrac{1}{2}\int_0^1 (x-x^3)\,dx = \tfrac{1}{2}\left[\tfrac{1}{2}x^2-\tfrac{1}{4}x^4\right]_0^1$$
$$= \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{1}{4}-0\right) = \tfrac{1}{8}$$

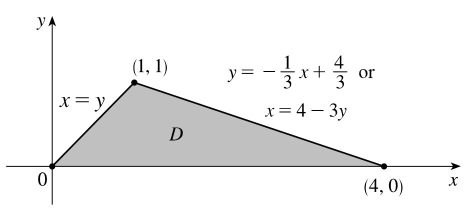
21.



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y)\,dy\,dx$$
$$= \int_{-2}^{2} \left[2xy-\tfrac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\,dx$$
$$= \int_{-2}^{2} \left[2x\sqrt{4-x^2}-\tfrac{1}{2}\left(4-x^2\right)+2x\sqrt{4-x^2}+\tfrac{1}{2}\left(4-x^2\right)\right]dx$$
$$= \int_{-2}^{2} 4x\sqrt{4-x^2}\,dx = -\tfrac{4}{3}\left(4-x^2\right)^{3/2}\Big]_{-2}^{2} = 0$$

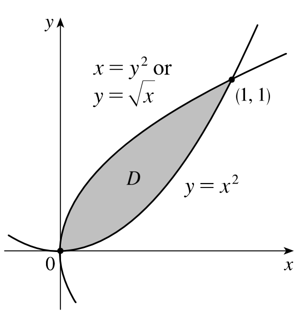
[Or, note that $4x\sqrt{4-x^2}$ is an odd function, so $\int_{-2}^{2} 4x\sqrt{4-x^2}\,dx = 0$.]

22.



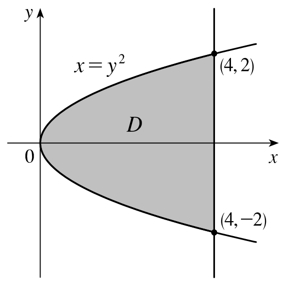
$$\iint_D y\,dA = \int_0^1 \int_y^{4-3y} y\,dx\,dy$$
$$= \int_0^1 \left[xy\right]_{x=y}^{x=4-3y}\,dy = \int_0^1 (4y-3y^2-y^2)\,dy$$
$$= \int_0^1 (4y-4y^2)\,dy = \left[2y^2-\tfrac{4}{3}y^3\right]_0^1 = 2-\tfrac{4}{3}-0 = \tfrac{2}{3}$$

23.



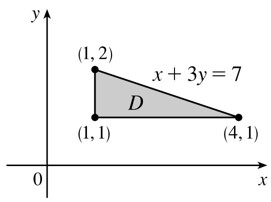
$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} (3x+2y)\,dy\,dx = \int_0^1 \left[3xy+y^2\right]_{y=x^2}^{y=\sqrt{x}}\,dx$$
$$= \int_0^1 \left[(3x\sqrt{x}+x)-(3x^3+x^4)\right]dx = \int_0^1 (3x^{3/2}+x-3x^3-x^4)\,dx$$
$$= \left[3\cdot\tfrac{2}{5}x^{5/2}+\tfrac{1}{2}x^2-\tfrac{3}{4}x^4-\tfrac{1}{5}x^5\right]_0^1 = \tfrac{6}{5}+\tfrac{1}{2}-\tfrac{3}{4}-\tfrac{1}{5}-0 = \tfrac{3}{4}$$

24.



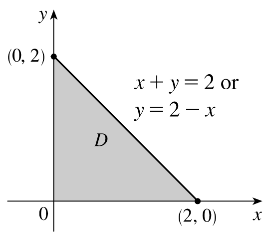
$$V = \int_{-2}^{2}\int_{y^2}^{4}(1 + x^2y^2)\,dx\,dy$$
$$= \int_{-2}^{2}\left[x + \tfrac{1}{3}x^3y^2\right]_{x=y^2}^{x=4}\,dy = \int_{-2}^{2}(4 + \tfrac{61}{3}y^2 - \tfrac{1}{3}y^8)\,dy$$
$$= \left[4y + \tfrac{61}{9}y^3 - \tfrac{1}{27}y^9\right]_{-2}^{2} = 8 + \tfrac{488}{9} - \tfrac{512}{27} + 8 + \tfrac{488}{9} - \tfrac{512}{27} = \tfrac{2336}{27}$$

25.



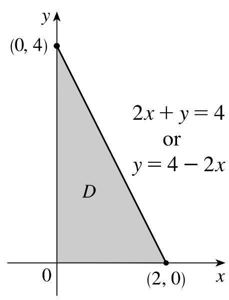
$$V = \int_{1}^{2}\int_{1}^{7-3y} xy\,dx\,dy = \int_{1}^{2}\left[\tfrac{1}{2}x^2y\right]_{x=1}^{x=7-3y}\,dy$$
$$= \tfrac{1}{2}\int_{1}^{2} y\left[(7-3y)^2 - 1\right]dy = \tfrac{1}{2}\int_{1}^{2}(48y - 42y^2 + 9y^3)\,dy$$
$$= \tfrac{1}{2}\left[24y^2 - 14y^3 + \tfrac{9}{4}y^4\right]_{1}^{2} = \tfrac{31}{8}$$

26.



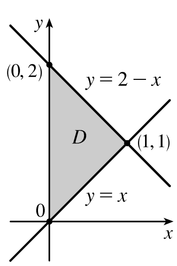
$$V = \int_{0}^{2}\int_{0}^{2-x}(x^2 + y^2 + 1)\,dy\,dx = \int_{0}^{2}\left[x^2y + \tfrac{1}{3}y^3 + y\right]_{y=0}^{y=2-x}\,dx$$
$$= \int_{0}^{2}\left[x^2(2-x) + \tfrac{1}{3}(2-x)^3 + (2-x) - 0\right]dx$$
$$= \int_{0}^{2}\left(-\tfrac{4}{3}x^3 + 4x^2 - 5x + \tfrac{14}{3}\right)dx = \left[-\tfrac{1}{3}x^4 + \tfrac{4}{3}x^3 - \tfrac{5}{2}x^2 + \tfrac{14}{3}x\right]_{0}^{2}$$
$$= -\tfrac{16}{3} + \tfrac{32}{3} - 10 + \tfrac{28}{3} - 0 = \tfrac{14}{3}$$

27.



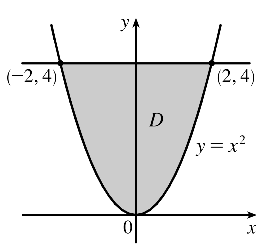
$$V = \int_{0}^{2}\int_{0}^{4-2x}(4 - 2x - y)\,dy\,dx = \int_{0}^{2}\left[4y - 2xy - \tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x}\,dx$$
$$= \int_{0}^{2}\left[4(4-2x) - 2x(4-2x) - \tfrac{1}{2}(4-2x)^2 - 0\right]dx$$
$$= \int_{0}^{2}\left(2x^2 - 8x + 8\right)dx = \left[\tfrac{2}{3}x^3 - 4x^2 + 8x\right]_{0}^{2} = \tfrac{16}{3} - 16 + 16 - 0 = \tfrac{16}{3}$$

28.



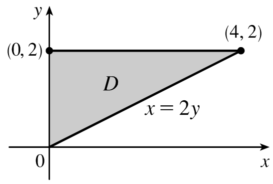
$$V = \int_{0}^{1}\int_{x}^{2-x} x\,dy\,dx$$
$$= \int_{0}^{1}\left[xy\right]_{y=x}^{y=2-x}\,dx = \int_{0}^{1}(2x - 2x^2)\,dx$$
$$= \left[x^2 - \tfrac{2}{3}x^3\right]_{0}^{1} = \tfrac{1}{3}$$

29.



$$V = \int_{-2}^{2}\int_{x^2}^{4} x^2\,dy\,dx$$
$$= \int_{-2}^{2}\left[x^2y\right]_{y=x^2}^{y=4}\,dx = \int_{-2}^{2}(4x^2 - x^4)\,dx$$
$$= \left[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right]_{-2}^{2} = \tfrac{32}{3} - \tfrac{32}{5} + \tfrac{32}{3} - \tfrac{32}{5} = \tfrac{128}{15}$$

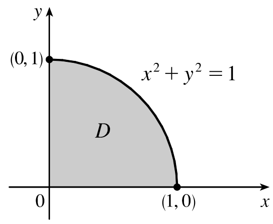
30.



$$V = \int_0^2 \int_0^{2y} \sqrt{4-y^2}\,dx\,dy = \int_0^2 \left[x\sqrt{4-y^2}\right]_{x=0}^{x=2y} dy$$

$$= \int_0^2 2y\sqrt{4-y^2}\,dy = \left[-\tfrac{2}{3}\left(4-y^2\right)^{3/2}\right]_0^2 = 0 + \tfrac{16}{3} = \tfrac{16}{3}$$

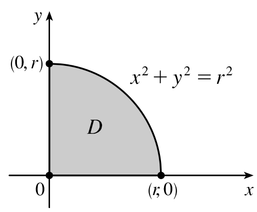
31.



$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1-x^2}{2}\,dx = \tfrac{1}{2}\left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

32.



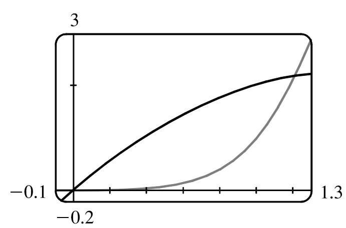
By symmetry, the desired volume V is 8 times the volume V_1 in the first octant. Now

$$V_1 = \int_0^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2}\,dx\,dy = \int_0^r \left[x\sqrt{r^2-y^2}\right]_{x=0}^{x=\sqrt{r^2-y^2}} dy$$

$$= \int_0^r (r^2-y^2)\,dy = \left[r^2y - \tfrac{1}{3}y^3\right]_0^r = \tfrac{2}{3}r^3$$

Thus $V = \frac{16}{3}r^3$.

33.



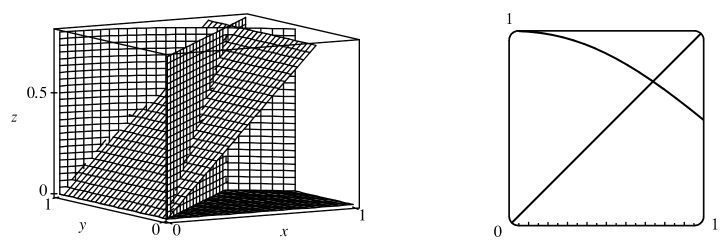
From the graph, it appears that the two curves intersect at $x = 0$ and at $x \approx 1.213$. Thus the desired integral is

$$\iint_D x\,dA \approx \int_0^{1.213}\int_{x^4}^{3x-x^2} x\,dy\,dx = \int_0^{1.213} \left[xy\right]_{y=x^4}^{y=3x-x^2} dx$$

$$= \int_0^{1.213} (3x^2 - x^3 - x^5)\,dx = \left[x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_0^{1.213}$$

$$\approx 0.713$$

34.



The desired solid is shown in the first graph. From the second graph, we estimate that $y = \cos x$ intersects $y = x$ at $x \approx 0.7391$. Therefore the volume of the solid is

$$V \approx \int_0^{0.7391}\int_x^{\cos x} x\,dy\,dx = \int_0^{0.7391} \left[xy\right]_{y=x}^{y=\cos x} dx$$

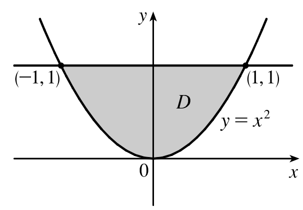
$$= \int_0^{0.7391} (x\cos x - x^2)\,dx = \left[\cos x + x\sin x - \tfrac{1}{3}x^3\right]_0^{0.7391} \approx 0.1024$$

Note: There is a different solid which can also be construed to satisfy the conditions stated in the exercise. This is the solid bounded by all of the given surfaces, as well as the plane $y = 0$. In case you calculated the volume of this solid and want to check your work, its volume is $V \approx \int_0^{0.7391}\int_0^x x\,dy\,dx + \int_{0.7391}^{\pi/2}\int_0^{\cos x} x\,dy\,dx \approx 0.4684$.

35. The region of integration is bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$ which intersect at $(\pm 1, 0)$ with $1 - x^2 \geq x^2 - 1$ on $[-1, 1]$. Within this region, the plane $z = 2x + 2y + 10$ is above the plane $z = 2 - x - y$, so

$$\begin{aligned}
V &= \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2x + 2y + 10)\, dy\, dx - \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2 - x - y)\, dy\, dx \\
&= \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2x + 2y + 10 - (2 - x - y))\, dy\, dx \\
&= \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (3x + 3y + 8)\, dy\, dx = \int_{-1}^{1} \left[3xy + \tfrac{3}{2}y^2 + 8y\right]_{y=x^2-1}^{y=1-x^2} dx \\
&= \int_{-1}^{1} \left[3x(1 - x^2) + \tfrac{3}{2}(1 - x^2)^2 + 8(1 - x^2) - 3x(x^2 - 1) - \tfrac{3}{2}(x^2 - 1)^2 - 8(x^2 - 1)\right] dx \\
&= \int_{-1}^{1} (-6x^3 - 16x^2 + 6x + 16)\, dx = \left[-\tfrac{3}{2}x^4 - \tfrac{16}{3}x^3 + 3x^2 + 16x\right]_{-1}^{1} \\
&= -\tfrac{3}{2} - \tfrac{16}{3} + 3 + 16 + \tfrac{3}{2} - \tfrac{16}{3} - 3 + 16 = \tfrac{64}{3}
\end{aligned}$$

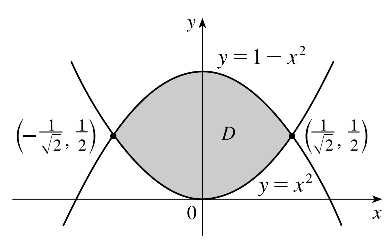
36. The two planes intersect in the line $y = 1$, $z = 3$, so the region of integration is the plane region enclosed by the parabola $y = x^2$ and the line $y = 1$. We have $2 + y \geq 3y$ for $0 \leq y \leq 1$, so the solid region is bounded above by $z = 2 + y$ and bounded below by $z = 3y$.



$$\begin{aligned}
V &= \int_{-1}^{1} \int_{x^2}^{1} (2 + y)\, dy\, dx - \int_{-1}^{1} \int_{x^2}^{1} (3y)\, dy\, dx = \int_{-1}^{1} \int_{x^2}^{1} (2 + y - 3y)\, dy\, dx \\
&= \int_{-1}^{1} \int_{x^2}^{1} (2 - 2y)\, dy\, dx = \int_{-1}^{1} \left[2y - y^2\right]_{y=x^2}^{y=1} dx \\
&= \int_{-1}^{1} (1 - 2x^2 + x^4)\, dx = x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\big]_{-1}^{1} = \tfrac{16}{15}
\end{aligned}$$

37. The region of integration is bounded by the curves $y = x^2$ and $y = 1 - x^2$ which intersect at $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

The solid lies under the graph of $z = 3$ and above the graph of $z = y$, so its volume is

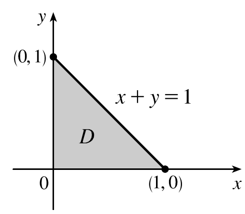


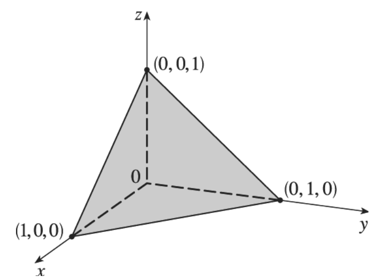
$$\begin{aligned}
V &= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2}^{1-x^2} 3\, dy\, dx - \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2}^{1-x^2} y\, dy\, dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2}^{1-x^2} (3 - y)\, dy\, dx \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[3y - \tfrac{1}{2}y^2\right]_{y=x^2}^{y=1-x^2} dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[\left(3(1 - x^2) - \tfrac{1}{2}(1 - x^2)^2\right) - \left(3x^2 - \tfrac{1}{2}(x^2)^2\right)\right] dx \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(\tfrac{5}{2} - 5x^2\right) dx = \left[\tfrac{5}{2}x - \tfrac{5}{3}x^3\right]_{-1/\sqrt{2}}^{1/\sqrt{2}} = \left(\tfrac{5}{2\sqrt{2}} - \tfrac{5}{6\sqrt{2}}\right) - \left(-\tfrac{5}{2\sqrt{2}} + \tfrac{5}{6\sqrt{2}}\right) \\
&= \tfrac{10}{3\sqrt{2}} \text{ or } \tfrac{5\sqrt{2}}{3}
\end{aligned}$$

38. The region of integration is the portion of the first quadrant bounded by the axes and the curve $y = \sqrt{4 - x^2}$. The solid lies under the graph of $z = x + y$ and above the graph of $z = xy$, so its volume is

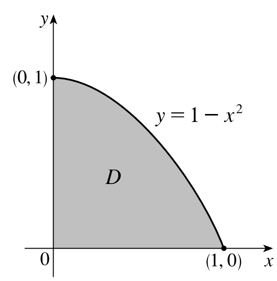
$$\begin{aligned} V &= \int_0^2 \int_0^{\sqrt{4-x^2}} (x + y)\, dy\, dx - \int_0^2 \int_0^{\sqrt{4-x^2}} xy\, dy\, dx = \int_0^2 \int_0^{\sqrt{4-x^2}} (x + y - xy)\, dy\, dx \\ &= \int_0^2 \left[xy + \tfrac{1}{2}y^2 - \tfrac{1}{2}xy^2 \right]_{y=0}^{y=\sqrt{4-x^2}} dx = \int_0^2 \left[x\sqrt{4 - x^2} + \tfrac{1}{2}(4 - x^2) - \tfrac{1}{2}x(4 - x^2) - 0 \right] dx \\ &= \int_0^2 \left(x\sqrt{4 - x^2} + 2 - \tfrac{1}{2}x^2 - 2x + \tfrac{1}{2}x^3 \right) dx = \left[-\tfrac{1}{3}(4 - x^2)^{3/2} + 2x - \tfrac{1}{6}x^3 - x^2 + \tfrac{1}{8}x^4 \right]_0^2 \\ &= \left(4 - \tfrac{4}{3} - 4 + 2 \right) - \left(-\tfrac{1}{3} \cdot 4^{3/2} \right) = \tfrac{2}{3} + \tfrac{8}{3} = \tfrac{10}{3} \end{aligned}$$

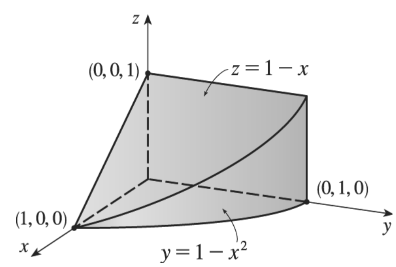
39. The solid lies below the plane $z = 1 - x - y$ or $x + y + z = 1$ and above the region $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$ in the xy-plane. The solid is a tetrahedron.





40. The solid lies below the plane $z = 1 - x$ and above the region $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1 - x^2\}$ in the xy-plane.





41. The two bounding curves $y = x^3 - x$ and $y = x^2 + x$ intersect at the origin and at $x = 2$, with $x^2 + x > x^3 - x$ on $(0, 2)$. Using a CAS, we find that the volume of the solid is

$$V = \int_0^2 \int_{x^3 - x}^{x^2 + x} (x^3y^4 + xy^2)\, dy\, dx = \frac{13{,}984{,}735{,}616}{14{,}549{,}535}$$

42. For $|x| \le 1$ and $|y| \le 1$, $2x^2 + y^2 < 8 - x^2 - 2y^2$. Also, the cylinder is described by the inequalities $-1 \le x \le 1$, $-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$. So the volume is given by

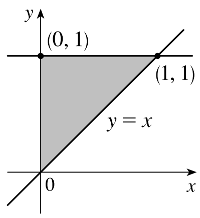
$$V = \int_{-1}^1 \int_{-\sqrt{1 - x^2}}^{\sqrt{1 - x^2}} \left[(8 - x^2 - 2y^2) - (2x^2 + y^2) \right] dy\, dx = \frac{13\pi}{2} \qquad \text{[using a CAS]}$$

43. The two surfaces intersect in the circle $x^2 + y^2 = 1$, $z = 0$ and the region of integration is the disk D: $x^2 + y^2 \le 1$.

Using a CAS, the volume is $\iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx = \dfrac{\pi}{2}$.

44. The projection onto the xy-plane of the intersection of the two surfaces is the circle $x^2 + y^2 = 2y \quad \Rightarrow$ $x^2 + y^2 - 2y = 0 \quad \Rightarrow \quad x^2 + (y-1)^2 = 1$, so the region of integration is given by $-1 \le x \le 1$, $1 - \sqrt{1-x^2} \le y \le 1 + \sqrt{1-x^2}$. In this region, $2y \ge x^2 + y^2$ so, using a CAS, the volume is

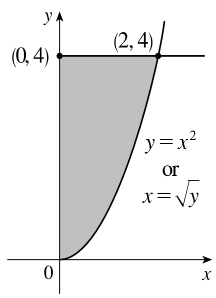
$$V = \int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} [2y - (x^2 + y^2)]\, dy\, dx = \frac{\pi}{2}$$

45. 

Because the region of integration is

$D = \{(x,y) \mid 0 \le x \le y, 0 \le y \le 1\} = \{(x,y) \mid x \le y \le 1, 0 \le x \le 1\}$

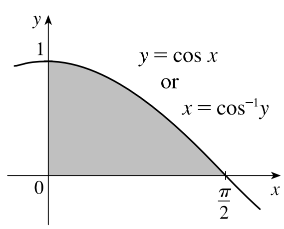
we have $\int_0^1 \int_0^y f(x,y)\, dx\, dy = \iint_D f(x,y)\, dA = \int_0^1 \int_x^1 f(x,y)\, dy\, dx$.

46. 

Because the region of integration is

$$\begin{aligned} D &= \left\{(x,y) \mid x^2 \le y \le 4, 0 \le x \le 2\right\} \\ &= \left\{(x,y) \mid 0 \le x \le \sqrt{y}, 0 \le y \le 4\right\} \end{aligned}$$

we have $\int_0^2 \int_{x^2}^4 f(x,y)\, dy\, dx = \iint_D f(x,y)\, dA = \int_0^4 \int_0^{\sqrt{y}} f(x,y)\, dx\, dy$.

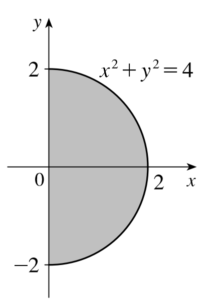
47. 

Because the region of integration is

$$\begin{aligned} D &= \{(x,y) \mid 0 \le y \le \cos x, 0 \le x \le \pi/2\} \\ &= \left\{(x,y) \mid 0 \le x \le \cos^{-1} y, 0 \le y \le 1\right\} \end{aligned}$$

we have

$\int_0^{\pi/2} \int_0^{\cos x} f(x,y)\, dy\, dx = \iint_D f(x,y)\, dA = \int_0^1 \int_0^{\cos^{-1} y} f(x,y)\, dx\, dy$.

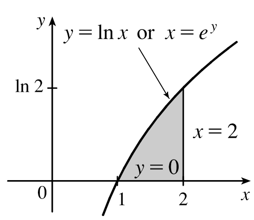
48. 

Because the region of integration is

$$\begin{aligned} D &= \left\{(x,y) \mid 0 \le x \le \sqrt{4-y^2}, -2 \le y \le 2\right\} \\ &= \left\{(x,y) \mid -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, 0 \le x \le 2\right\} \end{aligned}$$

we have

$\int_{-2}^2 \int_0^{\sqrt{4-y^2}} f(x,y)\, dx\, dy = \iint_D f(x,y)\, dA = \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\, dy\, dx$.

49. 

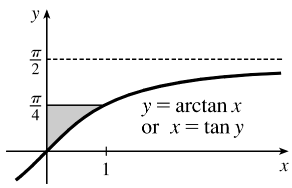
Because the region of integration is

$D = \{(x,y) \mid 0 \le y \le \ln x, 1 \le x \le 2\} = \{(x,y) \mid e^y \le x \le 2, 0 \le y \le \ln 2\}$

we have

$$\int_1^2 \int_0^{\ln x} f(x,y)\, dy\, dx = \iint_D f(x,y)\, dA = \int_0^{\ln 2} \int_{e^y}^2 f(x,y)\, dx\, dy$$

50.



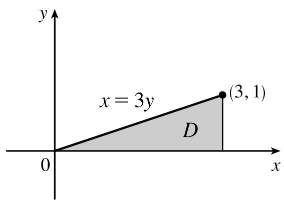
Because the region of integration is

$$D = \left\{(x,y) \mid \arctan x \le y \le \tfrac{\pi}{4}, 0 \le x \le 1\right\}$$
$$= \left\{(x,y) \mid 0 \le x \le \tan y, 0 \le y \le \tfrac{\pi}{4}\right\}$$

we have

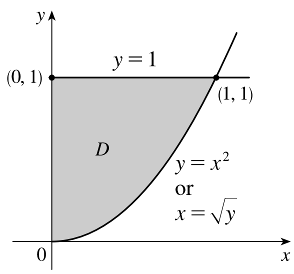
$$\int_0^1 \int_{\arctan x}^{\pi/4} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA = \int_0^{\pi/4} \int_0^{\tan y} f(x,y)\,dx\,dy$$

51.



$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 \left[e^{x^2} y\right]_{y=0}^{y=x/3}\,dx$$
$$= \int_0^3 \left(\frac{x}{3}\right) e^{x^2}\,dx = \tfrac{1}{6}\, e^{x^2}\Big]_0^3 = \frac{e^9 - 1}{6}$$

52.

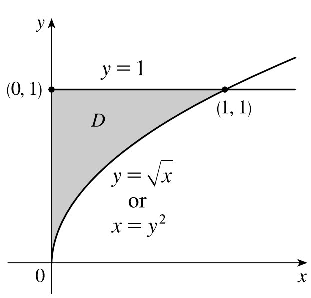


$$\int_0^1 \int_{x^2}^1 \sqrt{y}\,\sin y\,dy\,dx = \int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\,\sin y\,dx\,dy = \int_0^1 \sqrt{y}\,\sin y\,[x]_{x=0}^{x=\sqrt{y}}\,dy$$
$$= \int_0^1 (\sqrt{y}\,\sin y)\,(\sqrt{y} - 0)\,dy = \int_0^1 y\,\sin y\,dy$$
$$= -y\cos y\big]_0^1 + \textstyle\int_0^1 \cos y\,dy$$

[by integrating by parts with $u = y$, $dv = \sin y\,dy$]

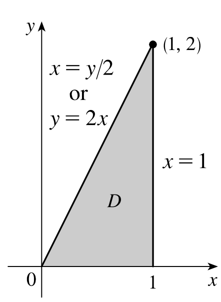
$$= [-y\cos y + \sin y]_0^1 = -\cos 1 + \sin 1 - 0 = \sin 1 - \cos 1$$

53.



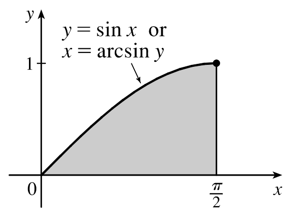
$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}\,dy\,dx = \int_0^1 \int_0^{y^2} \sqrt{y^3+1}\,dx\,dy = \int_0^1 \sqrt{y^3+1}\,[x]_{x=0}^{x=y^2}\,dy$$
$$= \int_0^1 y^2\sqrt{y^3+1}\,dy = \tfrac{2}{9}\left(y^3+1\right)^{3/2}\Big]_0^1$$
$$= \tfrac{2}{9}\left(2^{3/2} - 1^{3/2}\right) = \tfrac{2}{9}\left(2\sqrt{2} - 1\right)$$

54.



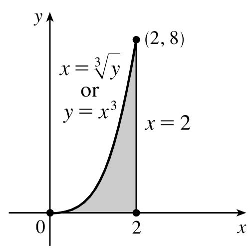
$$\int_0^2 \int_{y/2}^1 y\cos(x^3 - 1)\,dx\,dy = \int_0^1 \int_0^{2x} y\cos(x^3 - 1)\,dy\,dx$$
$$= \int_0^1 \cos(x^3 - 1)\,\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=2x}\,dx$$
$$= \int_0^1 2x^2\cos(x^3 - 1)\,dx = \tfrac{2}{3}\sin(x^3 - 1)\Big]_0^1$$
$$= \tfrac{2}{3}\,[0 - \sin(-1)] = -\tfrac{2}{3}\sin(-1) = \tfrac{2}{3}\sin 1$$

55.



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,dx\,dy$$
$$= \int_0^{\pi/2}\int_0^{\sin x} \cos x\,\sqrt{1+\cos^2 x}\,dy\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\big[\,y\,\big]_{y=0}^{y=\sin x}\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\sin x\,dx \qquad \begin{bmatrix}\text{Let } u=\cos x,\, du=-\sin x\,dx,\\ dx=du/(-\sin x)\end{bmatrix}$$
$$= \int_1^0 -u\,\sqrt{1+u^2}\,du = -\tfrac{1}{3}\left(1+u^2\right)^{3/2}\Big]_1^0$$
$$= \tfrac{1}{3}\left(\sqrt{8}-1\right) = \tfrac{1}{3}\left(2\sqrt{2}-1\right)$$

56.



$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$
$$= \int_0^2 e^{x^4}\Big[\,y\,\Big]_{y=0}^{y=x^3}\,dx = \int_0^2 x^3 e^{x^4}\,dx$$
$$= \tfrac{1}{4}e^{x^4}\Big]_0^2 = \tfrac{1}{4}(e^{16}-1)$$

57. $D = \{(x,y) \mid 0 \le x \le 1,\ -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0,\ x+1 \le y \le 1\}$
$\cup\, \{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le -x-1\}$, all type I.

$$\iint_D x^2\,dA = \int_0^1 \int_{1-x}^1 x^2\,dy\,dx + \int_{-1}^0 \int_{x+1}^1 x^2\,dy\,dx + \int_0^1 \int_{-1}^{x-1} x^2\,dy\,dx + \int_{-1}^0 \int_{-1}^{-x-1} x^2\,dy\,dx$$
$$= 4\int_0^1 \int_{1-x}^1 x^2\,dy\,dx \qquad \text{[by symmetry of the regions and because } f(x,y)=x^2 \ge 0\text{]}$$
$$= 4\int_0^1 x^3\,dx = 4\left[\tfrac{1}{4}x^4\right]_0^1 = 1$$

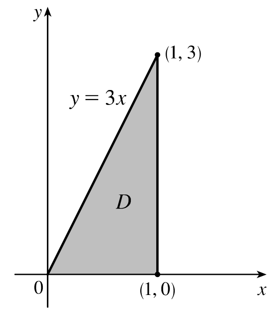
58. $D = \left\{(x,y) \mid -1 \le y \le 0,\ -1 \le x \le y - y^3\right\} \cup \left\{(x,y) \mid 0 \le y \le 1,\ \sqrt{y}-1 \le x \le y-y^3\right\}$, both type II.

$$\iint_D y\,dA = \int_{-1}^0 \int_{-1}^{y-y^3} y\,dx\,dy + \int_0^1 \int_{\sqrt{y}-1}^{y-y^3} y\,dx\,dy = \int_{-1}^0 \big[xy\big]_{x=-1}^{x=y-y^3}\,dy + \int_0^1 \big[xy\big]_{x=\sqrt{y}-1}^{x=y-y^3}\,dy$$
$$= \int_{-1}^0 (y^2 - y^4 + y)\,dy + \int_0^1 (y^2 - y^4 - y^{3/2} + y)\,dy$$
$$= \left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5 + \tfrac{1}{2}y^2\right]_{-1}^0 + \left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5 - \tfrac{2}{5}y^{5/2} + \tfrac{1}{2}y^2\right]_0^1$$
$$= \left(0 - \tfrac{11}{30}\right) + \left(\tfrac{7}{30} - 0\right) = -\tfrac{2}{15}$$

59. Since $x^2 + y^2 \le 1$ on S, we must have $0 \le x^2 \le 1$ and $0 \le y^2 \le 1$, so $0 \le x^2y^2 \le 1 \quad\Rightarrow\quad 3 \le 4 - x^2y^2 \le 4 \quad\Rightarrow$
$\sqrt{3} \le \sqrt{4 - x^2y^2} \le 2$. Here we have $A(S) = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$, so by Property 11,
$\sqrt{3}A(S) \le \iint_S \sqrt{4 - x^2y^2}\,dA \le 2A(S) \quad\Rightarrow\quad \frac{\sqrt{3}}{2}\pi \le \iint_S \sqrt{4 - x^2y^2}\,dA \le \pi$ or we can say
$2.720 < \iint_S \sqrt{4 - x^2y^2}\,dA < 3.142$. (We have rounded the lower bound down and the upper bound up to preserve the inequalities.)

60. T is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ so $A(T) = \frac{1}{2}(1)(2) = 1$. We have $0 \le \sin^4(x+y) \le 1$ for all x, y, and Property 11 gives $0 \cdot A(T) \le \iint_T \sin^4(x+y)\,dA \le 1 \cdot A(T) \quad \Rightarrow \quad 0 \le \iint_T \sin^4(x+y)\,dA \le 1$.

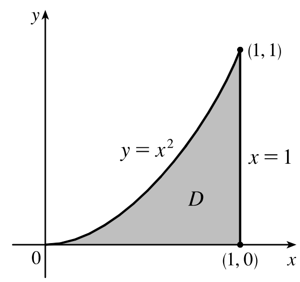
61. The average value of a function f of two variables defined on a rectangle R was defined in Section 15.1 as $f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$. Extending this definition to general regions D, we have $f_{\text{ave}} = \frac{1}{A(D)} \iint_D f(x,y)\,dA$.



Here $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 3x\}$, so $A(D) = \frac{1}{2}(1)(3) = \frac{3}{2}$ and

$$f_{\text{ave}} = \frac{1}{A(D)} \iint_D f(x,y)\,dA = \frac{1}{3/2} \int_0^1 \int_0^{3x} xy\,dy\,dx$$
$$= \frac{2}{3} \int_0^1 \left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=3x} dx = \frac{1}{3} \int_0^1 9x^3\,dx = \tfrac{3}{4}x^4\Big]_0^1 = \frac{3}{4}$$

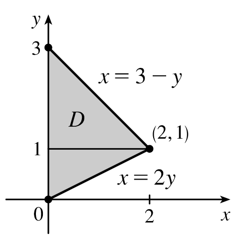
62. Here $D = \left\{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^2\right\}$, so $A(D) = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3}$ and



$$f_{\text{ave}} = \frac{1}{A(D)} \iint_D f(x,y)\,dA = \frac{1}{1/3} \int_0^1 \int_0^{x^2} x \sin y\,dy\,dx$$
$$= 3 \int_0^1 \left[-x \cos y\right]_{y=0}^{y=x^2} dx$$
$$= 3 \int_0^1 \left[x - x\cos(x^2)\right] dx = 3\left[\tfrac{1}{2}x^2 - \tfrac{1}{2}\sin(x^2)\right]_0^1$$
$$= 3\left(\tfrac{1}{2} - \tfrac{1}{2}\sin 1 - 0\right) = \tfrac{3}{2}(1 - \sin 1)$$

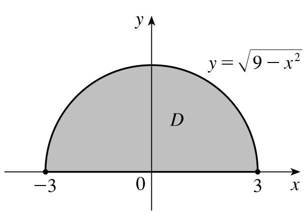
63. Since $m \le f(x,y) \le M$, $\iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA$ by (8) $\Rightarrow$ $m \iint_D 1\,dA \le \iint_D f(x,y)\,dA \le M \iint_D 1\,dA$ by (7) $\Rightarrow$ $mA(D) \le \iint_D f(x,y)\,dA \le MA(D)$ by (10).

64.



$$\iint_D f(x,y)\,dA = \int_0^1 \int_0^{2y} f(x,y)\,dx\,dy + \int_1^3 \int_0^{3-y} f(x,y)\,dx\,dy$$
$$= \int_0^2 \int_{x/2}^{3-x} f(x,y)\,dy\,dx$$

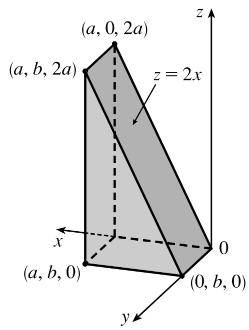
65.

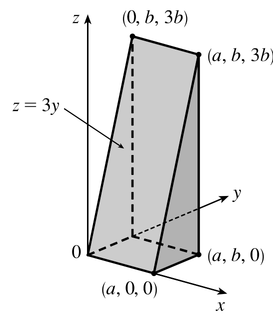


First we can write $\iint_D (x+2)\,dA = \iint_D x\,dA + \iint_D 2\,dA$. But $f(x,y) = x$ is an odd function with respect to x [that is, $f(-x,y) = -f(x,y)$] and D is symmetric with respect to x. Consequently, the volume above D and below the graph of f is the same as the volume below D and above the graph of f, so $\iint_D x\,dA = 0$. Also, $\iint_D 2\,dA = 2 \cdot A(D) = 2 \cdot \frac{1}{2}\pi(3)^2 = 9\pi$ since D is a half disk of radius 3. Thus $\iint_D (x+2)\,dA = 0 + 9\pi = 9\pi$.

66. The graph of $f(x,y) = \sqrt{R^2 - x^2 - y^2}$ is the top half of the sphere $x^2 + y^2 + z^2 = R^2$, centered at the origin with radius R, and D is the disk in the xy-plane also centered at the origin with radius R. Thus $\iint_D \sqrt{R^2 - x^2 - y^2}\,dA$ represents the volume of a half ball of radius R which is $\frac{1}{2} \cdot \frac{4}{3}\pi R^3 = \frac{2}{3}\pi R^3$.

67. We can write $\iint_D (2x+3y)\,dA = \iint_D 2x\,dA + \iint_D 3y\,dA$. $\iint_D 2x\,dA$ represents the volume of the solid lying under the plane $z = 2x$ and above the rectangle D. This solid region is a triangular cylinder with length b and whose cross-section is a triangle with width a and height $2a$. (See the first figure.)

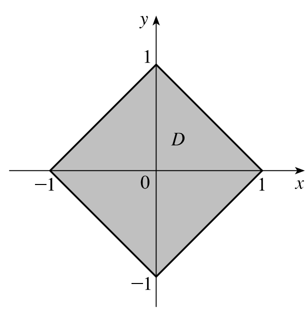




Thus its volume is $\frac{1}{2} \cdot a \cdot 2a \cdot b = a^2 b$. Similarly, $\iint_D 3y\,dA$ represents the volume of a triangular cylinder with length a, triangular cross-section with width b and height $3b$, and volume $\frac{1}{2} \cdot b \cdot 3b \cdot a = \frac{3}{2}ab^2$. (See the second figure.) Thus

$$\iint_D (2x+3y)\,dA = a^2 b + \tfrac{3}{2}ab^2$$

68.

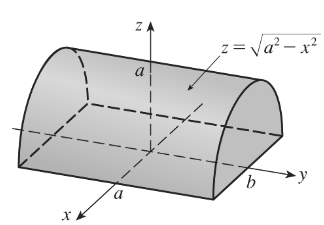


In the first quadrant, x and y are positive and the boundary of D is $x + y = 1$. But D is symmetric with respect to both axes because of the absolute values, so the region of integration is the square shown at the left. To evaluate the double integral, we first write $\iint_D (2 + x^2y^3 - y^2 \sin x)\,dA = \iint_D 2\,dA + \iint_D x^2y^3\,dA - \iint_D y^2 \sin x\,dA$. Now $f(x,y) = x^2y^3$ is odd with respect to y [that is, $f(x,-y) = -f(x,y)$] and D is symmetric with respect to y, so $\iint_D x^2y^3\,dA = 0$.

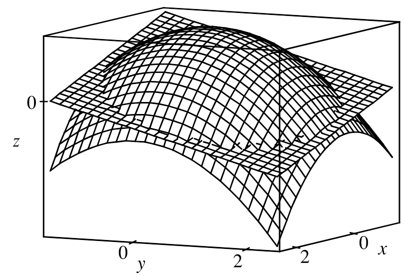
Similarly, $g(x,y) = y^2 \sin x$ is odd with respect to x [since $g(-x,y) = -g(x,y)$] and D is symmetric with respect to x, so $\iint_D y^2 \sin x\,dA = 0$. D is a square with side length $\sqrt{2}$, so $\iint_D 2\,dA = 2 \cdot A(D) = 2\left(\sqrt{2}\right)^2 = 4$, and $\iint_D (2 + x^2y^3 - y^2 \sin x)\,dA = 4 + 0 + 0 = 4$.

69. $\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right) dA = \iint_D ax^3\,dA + \iint_D by^3\,dA + \iint_D \sqrt{a^2 - x^2}\,dA$. Now ax^3 is odd with respect to x and by^3 is odd with respect to y, and the region of integration is symmetric with respect to both x and y, so $\iint_D ax^3\,dA = \iint_D by^3\,dA = 0$.

$\iint_D \sqrt{a^2 - x^2}\,dA$ represents the volume of the solid region under the graph of $z = \sqrt{a^2 - x^2}$ and above the rectangle D, namely a half circular cylinder with radius a and length $2b$ (see the figure) whose volume is $\frac{1}{2} \cdot \pi r^2 h = \frac{1}{2}\pi a^2 (2b) = \pi a^2 b$. Thus $\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right) dA = 0 + 0 + \pi a^2 b = \pi a^2 b$.



70. To find the equations of the boundary curves, we require that the z-values of the two surfaces be the same. In Maple, we use the command `solve(4-x^2-y^2=1-x-y,y);` and in Mathematica, we use `Solve[4-x^2-y^2==1-x-y,y]`. We find that the curves have equations $y = \dfrac{1 \pm \sqrt{13 + 4x - 4x^2}}{2}$. To find the two points of intersection of these curves, we use the CAS to solve $13 + 4x - 4x^2 = 0$, finding that $x = \frac{1 \pm \sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is



$$V = \int_{(1-\sqrt{14})/2}^{(1+\sqrt{14})/2} \int_{(1-\sqrt{13+4x-4x^2})/2}^{(1+\sqrt{13+4x-4x^2})/2} [(4 - x^2 - y^2) - (1 - x - y)]\, dy\, dx = \frac{49\pi}{8}$$

15.3 Double Integrals in Polar Coordinates

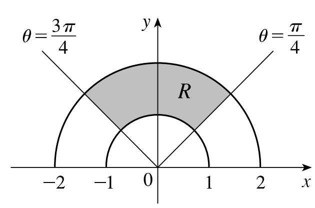
1. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 5, 0 \le \theta \le 2\pi\}$.
Thus $\iint_R f(x,y)\, dA = \int_0^{2\pi} \int_2^5 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

2. The region R is more easily described by rectangular coordinates: $R = \{(x,y) \mid -1 \le x \le 1,\ -x \le y \le 1\}$.
Thus $\iint_R f(x,y)\, dA = \int_{-1}^{1} \int_{-x}^{1} f(x,y)\, dy\, dx$.

3. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 0 \le r \le 1, \pi \le \theta \le 2\pi\}$.
Thus $\iint_R f(x,y)\, dA = \int_{\pi}^{2\pi} \int_0^1 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

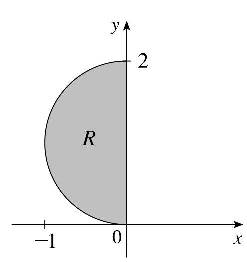
4. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 0 \le r \le 3,\ -\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\right\}$.
Thus $\iint_R f(x,y)\, dA = \int_{-\pi/4}^{3\pi/4} \int_0^3 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

5. The integral $\int_{\pi/4}^{3\pi/4} \int_1^2 r\, dr\, d\theta$ represents the area of the region $R = \{(r,\theta) \mid 1 \le r \le 2, \pi/4 \le \theta \le 3\pi/4\}$, the top quarter portion of a ring (annulus).



$$\begin{aligned}\int_{\pi/4}^{3\pi/4} \int_1^2 r\, dr\, d\theta &= \left(\int_{\pi/4}^{3\pi/4} d\theta\right)\left(\int_1^2 r\, dr\right) \\ &= \left[\theta\right]_{\pi/4}^{3\pi/4} \left[\tfrac{1}{2}r^2\right]_1^2 = \left(\tfrac{3\pi}{4} - \tfrac{\pi}{4}\right) \cdot \tfrac{1}{2}(4 - 1) = \tfrac{\pi}{2} \cdot \tfrac{3}{2} = \tfrac{3\pi}{4}\end{aligned}$$

6. The integral $\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\, dr\, d\theta$ represents the area of the region $R = \{(r,\theta) \mid 0 \le r \le 2\sin\theta, \pi/2 \le \theta \le \pi\}$. Since $r = 2\sin\theta \;\Rightarrow\; r^2 = 2r\sin\theta \;\Leftrightarrow\; x^2 + y^2 = 2y \;\Leftrightarrow\; x^2 + (y-1)^2 = 1$, R is the portion in the second quadrant of a disk of radius 1 with center $(0, 1)$.



$$\begin{aligned}\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\, dr\, d\theta &= \int_{\pi/2}^{\pi} \left[\tfrac{1}{2}r^2\right]_{r=0}^{r=2\sin\theta} d\theta = \int_{\pi/2}^{\pi} 2\sin^2\theta\, d\theta \\ &= \int_{\pi/2}^{\pi} 2 \cdot \tfrac{1}{2}(1 - \cos 2\theta)\, d\theta = \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/2}^{\pi} \\ &= \pi - 0 - \tfrac{\pi}{2} + 0 = \tfrac{\pi}{2}\end{aligned}$$