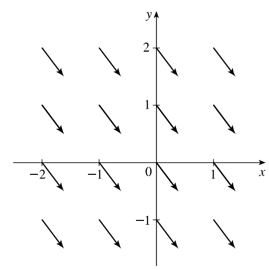
16 □ VECTOR CALCULUS

16.1 Vector Fields

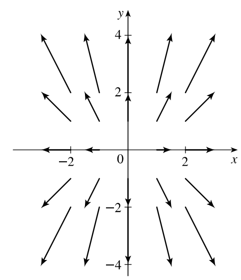
1. $\mathbf{F}(x,y) = 0.3\,\mathbf{i} - 0.4\,\mathbf{j}$

All vectors in this field are identical, with length 0.5 and parallel to $\langle 3, -4\rangle$.



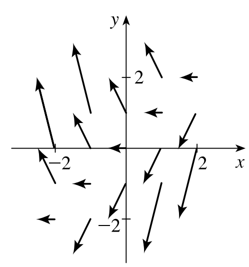
2. $\mathbf{F}(x,y) = \frac{1}{2}x\,\mathbf{i} + y\,\mathbf{j}$

The length of the vector $\frac{1}{2}x\,\mathbf{i} + y\,\mathbf{j}$ is $\sqrt{\frac{1}{4}x^2 + y^2}$. Vectors point roughly away from the origin and vectors farther from the origin are longer.



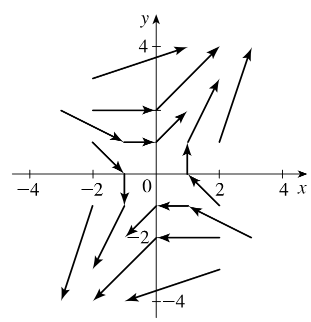
3. $\mathbf{F}(x,y) = -\frac{1}{2}\,\mathbf{i} + (y - x)\,\mathbf{j}$

The length of the vector $-\frac{1}{2}\,\mathbf{i} + (y - x)\,\mathbf{j}$ is $\sqrt{\frac{1}{4} + (y-x)^2}$. Vectors along the line $y = x$ are horizontal with length $\frac{1}{2}$.



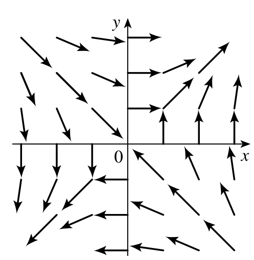
4. $\mathbf{F}(x,y) = y\,\mathbf{i} + (x + y)\,\mathbf{j}$

The length of the vector $y\,\mathbf{i} + (x + y)\,\mathbf{j}$ is $\sqrt{y^2 + (x+y)^2}$. Vectors along the x-axis are vertical, and vectors along the line $y = -x$ are horizontal with length $|y|$.



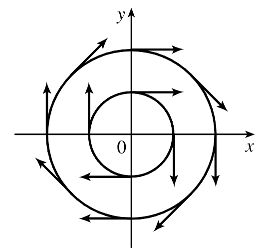
5. $\mathbf{F}(x,y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

The length of the vector $\dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$ is 1.



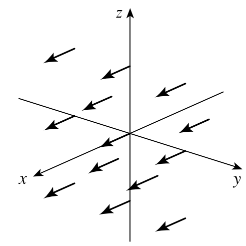
6. $\mathbf{F}(x,y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2+y^2}}$

All the vectors $\mathbf{F}(x,y)$ are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2+y^2}$.



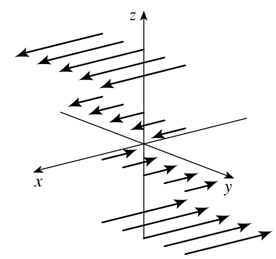
7. $\mathbf{F}(x,y) = \mathbf{i}$

All vectors in this field are identical, with length 1 and pointing in the direction of the positive x-axis.



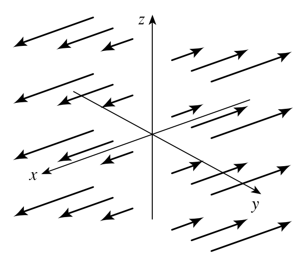
8. $\mathbf{F}(x,y,z) = z\,\mathbf{i}$

At each point (x,y,z), $\mathbf{F}(x,y,z)$ is a vector of length $|z|$. For $z > 0$, all point in the direction of the positive x-axis, while for $z < 0$, all are in the direction of the negative x-axis. In each plane $z = k$, all the vectors are identical.



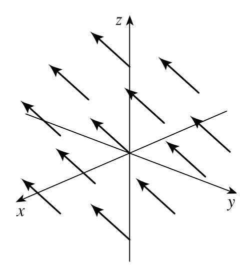
9. $\mathbf{F}(x,y,z) = -y\,\mathbf{i}$

At each point (x,y,z), $\mathbf{F}(x,y,z)$ is a vector of length $|y|$. For $y > 0$, all point in the direction of the negative x-axis, while for $y < 0$, all are in the direction of the positive x-axis. In each plane $y = k$, all the vectors are identical.



10. $\mathbf{F}(x,y,z) = \mathbf{i} + \mathbf{k}$

All vectors in this field have length $\sqrt{2}$ and point in the same direction, parallel to the xz-plane.



11. $\mathbf{F}(x,y) = \langle x, -y\rangle$ corresponds to graph IV. In the first quadrant all the vectors have positive x-components and negative y-components, in the second quadrant all vectors have negative x- and y-components, in the third quadrant all vectors have negative x-components and positive y-components, and in the fourth quadrant all vectors have positive x- and y-components. In addition, the vectors get shorter as we approach the origin.

12. $\mathbf{F}(x, y) = \langle y, x - y \rangle$ corresponds to graph III. All vectors in quadrants I and II have positive x-components while all vectors in quadrants III and IV have negative x-components. In addition, vectors along the line $y = x$ are horizontal, and vectors get shorter as we approach the origin.

13. $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$ corresponds to graph I. As in Exercise 12, all vectors in quadrants I and II have positive x-components while all vectors in quadrants III and IV have negative x-components. Vectors along the line $y = -2$ are horizontal, and the vectors are independent of x (vectors along horizontal lines are identical).

14. $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$ corresponds to graph II. All vectors in quadrants I and IV have positive y-components while all vectors in quadrants II and III have negative y-components. Also, the y-components of vectors along any vertical line remain constant while the x-component oscillates.

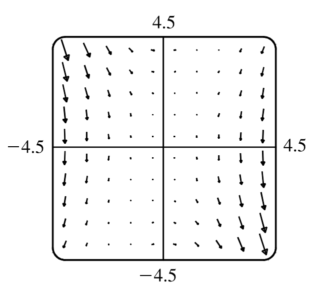
15. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph IV, since all vectors have identical length and direction.

16. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ corresponds to graph I, since the horizontal vector components remain constant, but the vectors above the xy-plane point generally upward while the vectors below the xy-plane point generally downward.

17. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph III; the projection of each vector onto the xy-plane is $x\,\mathbf{i} + y\,\mathbf{j}$, which points away from the origin, and the vectors point generally upward because their z-components are all 3.

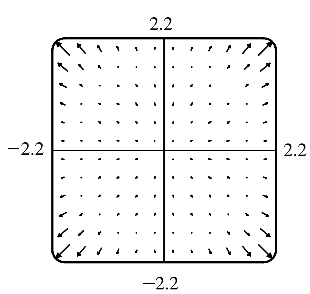
18. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ corresponds to graph II; each vector $\mathbf{F}(x, y, z)$ has the same length and direction as the position vector of the point (x, y, z), and therefore the vectors all point directly away from the origin.

19.



The vector field seems to have very short vectors near the line $y = 2x$. For $\mathbf{F}(x, y) = \langle 0, 0 \rangle$ we must have $y^2 - 2xy = 0$ and $3xy - 6x^2 = 0$. The first equation holds if $y = 0$ or $y = 2x$, and the second holds if $x = 0$ or $y = 2x$. So both equations hold [and thus $\mathbf{F}(x, y) = \mathbf{0}$] along the line $y = 2x$.

20.



From the graph, it appears that all of the vectors in the field lie on lines through the origin, and that the vectors have very small magnitudes near the circle $|\mathbf{x}| = 2$ and near the origin. Note that $\mathbf{F}(\mathbf{x}) = \mathbf{0} \quad \Leftrightarrow \quad r(r - 2) = 0 \quad \Leftrightarrow \quad r = 0$ or 2, so as we suspected, $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ for $|\mathbf{x}| = 2$ and for $|\mathbf{x}| = 0$. Note that where $r^2 - r < 0$, the vectors point towards the origin, and where $r^2 - r > 0$, they point away from the origin.

21. $f(x, y) = y \sin(xy) \quad \Rightarrow$

$$\begin{aligned}\nabla f(x, y) &= f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j} = (y \cos(xy) \cdot y)\,\mathbf{i} + [y \cdot x \cos(xy) + \sin(xy) \cdot 1]\,\mathbf{j} \\ &= y^2 \cos(xy)\,\mathbf{i} + [xy \cos(xy) + \sin(xy)]\,\mathbf{j}\end{aligned}$$

22. $f(s,t) = \sqrt{2s+3t} \quad \Rightarrow$

$$\nabla f(s,t) = f_s(s,t)\,\mathbf{i} + f_t(s,t)\,\mathbf{j} = \left[\tfrac{1}{2}(2s+3t)^{-1/2}\cdot 2\right]\mathbf{i} + \left[\tfrac{1}{2}(2s+3t)^{-1/2}\cdot 3\right]\mathbf{j} = \frac{1}{\sqrt{2s+3t}}\,\mathbf{i} + \frac{3}{2\sqrt{2s+3t}}\,\mathbf{j}$$

23. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \quad \Rightarrow$

$$\begin{aligned}\nabla f(x,y,z) &= f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}\\ &= \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)\,\mathbf{i} + \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)\,\mathbf{j} + \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)\,\mathbf{k}\\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}\end{aligned}$$

24. $f(x,y,z) = x^2ye^{y/z} \quad \Rightarrow$

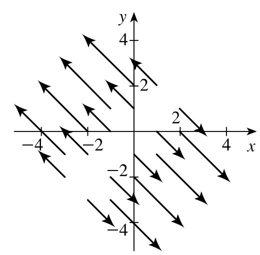
$$\begin{aligned}\nabla f(x,y,z) &= f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}\\ &= 2xye^{y/z}\,\mathbf{i} + x^2\left[y\cdot e^{yz}(1/z) + e^{y/z}\cdot 1\right]\mathbf{j} + \left[x^2ye^{y/z}(-y/z^2)\right]\mathbf{k}\\ &= 2xye^{y/z}\,\mathbf{i} + x^2e^{y/z}\left(\frac{y}{z}+1\right)\mathbf{j} - \frac{x^2y^2}{z^2}e^{y/z}\,\mathbf{k}\end{aligned}$$

25. $f(x,y) = \frac{1}{2}(x-y)^2 \quad \Rightarrow$

$\nabla f(x,y) = (x-y)(1)\,\mathbf{i} + (x-y)(-1)\,\mathbf{j} = (x-y)\,\mathbf{i} + (y-x)\,\mathbf{j}$.

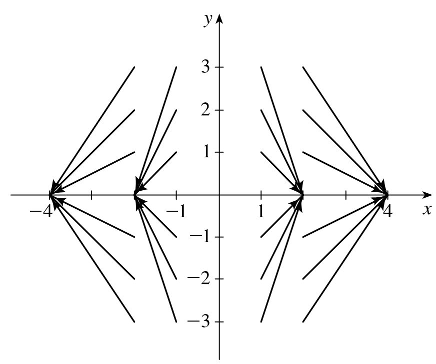
The length of $\nabla f(x,y)$ is $\sqrt{(x-y)^2+(y-x)^2} = \sqrt{2}\,|x-y|$.

The vectors are $\mathbf{0}$ along the line $y=x$. Elsewhere the vectors point away from the line $y=x$ with length that increases as the distance from the line increases.



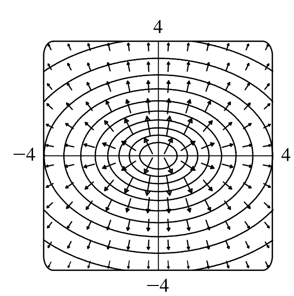
26. $f(x,y) = \frac{1}{2}(x^2-y^2) \quad \Rightarrow \quad \nabla f(x,y) = x\,\mathbf{i} - y\,\mathbf{j}$.

The length of $\nabla f(x,y)$ is $\sqrt{x^2+y^2}$. The lengths of the vectors increase as the distance from the origin increases, and the terminal point of each vector lies on the x-axis.

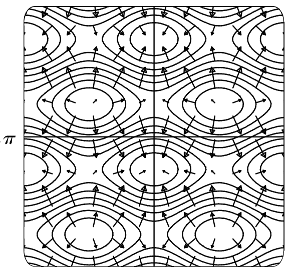


27. We graph $\nabla f(x,y) = \dfrac{2x}{1+x^2+2y^2}\,\mathbf{i} + \dfrac{4y}{1+x^2+2y^2}\,\mathbf{j}$ along with a contour map of f.

The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.



28. We graph $\nabla f(x,y) = -\sin x\,\mathbf{i} - 2\cos y\,\mathbf{j}$ along with a contour map of f.



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

29. $f(x,y) = x^2 + y^2 \quad \Rightarrow \quad \nabla f(x,y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Thus, each vector $\nabla f(x,y)$ has the same direction and twice the length of the position vector of the point (x,y), so the vectors all point directly away from the origin and their lengths increase as we move away from the origin. Hence, ∇f is graph III.

30. $f(x,y) = x(x+y) = x^2 + xy \quad \Rightarrow \quad \nabla f(x,y) = (2x+y)\,\mathbf{i} + x\,\mathbf{j}$. The y-component of each vector is x, so the vectors point upward in quadrants I and IV and downward in quadrants II and III. Also, the x-component of each vector is 0 along the line $y = -2x$ so the vectors are vertical there. Thus, ∇f is graph IV.

31. $f(x,y) = (x+y)^2 \quad \Rightarrow \quad \nabla f(x,y) = 2(x+y)\,\mathbf{i} + 2(x+y)\,\mathbf{j}$. The x- and y-components of each vector are equal, so all vectors are parallel to the line $y = x$. The vectors are $\mathbf{0}$ along the line $y = -x$ and their length increases as the distance from this line increases. Thus, ∇f is graph II.

32. $f(x,y) = \sin\sqrt{x^2+y^2} \quad \Rightarrow$

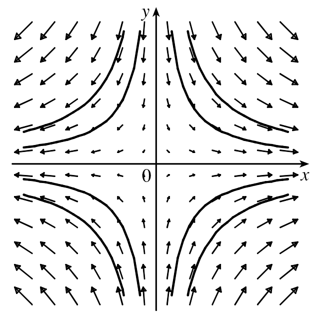
$$\begin{aligned}\nabla f(x,y) &= \left[\cos\sqrt{x^2+y^2}\cdot\tfrac{1}{2}(x^2+y^2)^{-1/2}(2x)\right]\mathbf{i} + \left[\cos\sqrt{x^2+y^2}\cdot\tfrac{1}{2}(x^2+y^2)^{-1/2}(2y)\right]\mathbf{j}\\ &= \frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,x\,\mathbf{i} + \frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,y\,\mathbf{j} \;\text{ or }\; \frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,(x\,\mathbf{i} + y\,\mathbf{j})\end{aligned}$$

Thus each vector is a scalar multiple of its position vector, so the vectors point toward or away from the origin with length that changes in a periodic fashion as we move away from the origin. ∇f is graph I.

33. At $t = 3$ the particle is at $(2,1)$ so its velocity is $\mathbf{V}(2,1) = \langle 4,3\rangle$. After 0.01 units of time, the particle's change in location should be approximately $0.01\,\mathbf{V}(2,1) = 0.01\,\langle 4,3\rangle = \langle 0.04, 0.03\rangle$, so the particle should be approximately at the point $(2.04, 1.03)$.

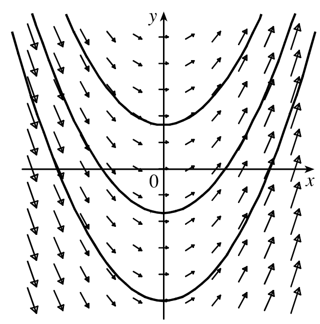
34. At $t = 1$ the particle is at $(1,3)$ so its velocity is $\mathbf{F}(1,3) = \langle 1,-1\rangle$. After 0.05 units of time, the particle's change in location should be approximately $0.05\,\mathbf{F}(1,3) = 0.05\,\langle 1,-1\rangle = \langle 0.05, -0.05\rangle$, so the particle should be approximately at the point $(1.05, 2.95)$.

35. (a) We sketch the vector field $\mathbf{F}(x,y) = x\,\mathbf{i} - y\,\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be hyperbolas with shape similar to the graph of $y = \pm 1/x$, so we might guess that the flow lines have equations $y = C/x$.



(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x,y) is $x'(t)\,\mathbf{i} + y'\,(t)\,\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = x\,\mathbf{i} - y\,\mathbf{j} \quad\Rightarrow\quad dx/dt = x,\, dy/dt = -y$. To solve these differential equations, we know $dx/dt = x \quad\Rightarrow\quad dx/x = dt \quad\Rightarrow\quad \ln|x| = t + C \quad\Rightarrow\quad x = \pm e^{t+C} = Ae^t$ for some constant A, and $dy/dt = -y \quad\Rightarrow\quad dy/y = -dt \quad\Rightarrow\quad \ln|y| = -t + K \quad\Rightarrow\quad y = \pm e^{-t+K} = Be^{-t}$ for some constant B. Therefore $xy = Ae^tBe^{-t} = AB =$ constant. If the flow line passes through $(1,1)$ then $(1)\,(1) =$ constant $= 1 \quad\Rightarrow\quad xy = 1 \quad\Rightarrow$ $y = 1/x,\, x > 0$.

36. (a) We sketch the vector field $\mathbf{F}(x,y) = \mathbf{i} + x\,\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be parabolas.



(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x,y) is $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have

$x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = \mathbf{i} + x\,\mathbf{j} \quad\Rightarrow\quad \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = x$. Thus $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{x}{1} = x$.

(c) From part (b), $dy/dx = x$. Integrating, we have $y = \frac{1}{2}x^2 + c$. Since the particle starts at the origin, we know $(0,0)$ is on the curve, so $0 = 0 + c \quad\Rightarrow\quad c = 0$ and the path the particle follows is $y = \frac{1}{2}x^2$.

16.2 Line Integrals

1. $x = t^2$ and $y = 2t$, $0 \le t \le 3$, so by Formula 3

$$\int_C y\,ds = \int_0^3 2t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^3 2t\,\sqrt{(2t)^2 + (2)^2}\,dt = \int_0^3 2t\,\sqrt{4t^2+4}\,dt$$

$$= \textstyle\int_0^3 4t\,\sqrt{t^2+1}\,dt = 2\cdot\frac{2}{3}\left(t^2+1\right)^{3/2}\Big]_0^3 = \frac{4}{3}(10^{3/2} - 1) \text{ or } \frac{4}{3}(10\sqrt{10} - 1)$$

2. $x = t^3$ and $y = t^4$, $1 \le t \le 2$, so by Formula 3

$$\begin{aligned}\int_C (x/y)\,ds &= \int_1^2 (t^3/t^4)\sqrt{(3t^2)^2 + (4t^3)^2}\,dt = \int_1^2 (1/t)\cdot t^2\sqrt{9+16t^2}\,dt = \int_1^2 t\sqrt{9+16t^2}\,dt \\ &= \tfrac{1}{32}\cdot\tfrac{2}{3}\left(9+16t^2\right)^{3/2}\Big]_1^2 = \tfrac{1}{48}(73^{3/2} - 25^{3/2}) \text{ or } \tfrac{1}{48}(73\sqrt{73} - 125)\end{aligned}$$

3. Parametric equations for C are $x = 4\cos t$, $y = 4\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then

$$\begin{aligned}\int_C xy^4\,ds &= \int_{-\pi/2}^{\pi/2}(4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt = \int_{-\pi/2}^{\pi/2} 4^5\cos t\,\sin^4 t\,\sqrt{16(\sin^2 t + \cos^2 t)}\,dt \\ &= 4^5\int_{-\pi/2}^{\pi/2}(\sin^4 t\cos t)(4)\,dt = (4)^6\left[\tfrac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = 4^6\cdot\tfrac{2}{5} = 1638.4\end{aligned}$$

4. Parametric equations for C are $x = 2+3t$, $y = 4t$, $0 \le t \le 1$. Then

$$\int_C xe^y\,ds = \int_0^1 (2+3t)\,e^{4t}\sqrt{3^2+4^2}\,dt = 5\int_0^1 (2+3t)\,e^{4t}\,dt$$

Integrating by parts with $u = 2+3t \;\Rightarrow\; du = 3\,dt$, $dv = e^{4t}\,dt \;\Rightarrow\; v = \frac{1}{4}e^{4t}$ gives

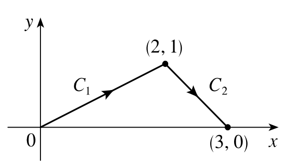
$$\int_C xe^y\,ds = 5\left[\tfrac{1}{4}(2+3t)e^{4t} - \tfrac{3}{16}e^{4t}\right]_0^1 = 5\left[\tfrac{5}{4}e^4 - \tfrac{3}{16}e^4 - \tfrac{1}{2} + \tfrac{3}{16}\right] = \tfrac{85}{16}e^4 - \tfrac{25}{16}$$

5. If we choose x as the parameter, parametric equations for C are $x = x$, $y = x^2$ for $0 \le x \le \pi$ and by Equations 7

$$\begin{aligned}\int_C \left(x^2y + \sin x\right)dy &= \int_0^\pi \left[x^2(x^2) + \sin x\right]\cdot 2x\,dx = 2\int_0^\pi \left(x^5 + x\sin x\right)dx \\ &= 2\left[\tfrac{1}{6}x^6 - x\cos x + \sin x\right]_0^\pi \quad \left[\begin{array}{c}\text{where we integrated by parts}\\ \text{in the second term}\end{array}\right] \\ &= 2\left[\tfrac{1}{6}\pi^6 + \pi + 0 - 0\right] = \tfrac{1}{3}\pi^6 + 2\pi\end{aligned}$$

6. Choosing y as the parameter, we have $x = y^3$, $y = y$, $-1 \le y \le 1$. Then

$$\int_C e^x\,dx = \int_{-1}^1 e^{y^3}\cdot 3y^2\,dy = e^{y^3}\Big]_{-1}^1 = e^1 - e^{-1} = e - \tfrac{1}{e}.$$

7. 

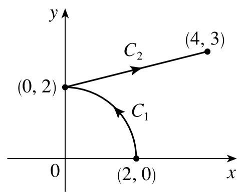
$C = C_1 + C_2$

On C_1: $x = x, y = \frac{1}{2}x \;\Rightarrow\; dy = \frac{1}{2}\,dx$, $0 \le x \le 2$.

On C_2: $x = x, y = 3 - x \;\Rightarrow\; dy = -dx$, $2 \le x \le 3$.

Then

$$\begin{aligned}\int_C (x+2y)\,dx + x^2\,dy &= \int_{C_1}(x+2y)\,dx + x^2\,dy + \int_{C_2}(x+2y)\,dx + x^2\,dy \\ &= \int_0^2 \left[x + 2\left(\tfrac{1}{2}x\right) + x^2\left(\tfrac{1}{2}\right)\right]dx + \int_2^3 \left[x + 2(3-x) + x^2(-1)\right]dx \\ &= \int_0^2 \left(2x + \tfrac{1}{2}x^2\right)dx + \int_2^3 \left(6 - x - x^2\right)dx \\ &= \left[x^2 + \tfrac{1}{6}x^3\right]_0^2 + \left[6x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_2^3 = \tfrac{16}{3} - 0 + \tfrac{9}{2} - \tfrac{22}{3} = \tfrac{5}{2}\end{aligned}$$

8. 

$C = C_1 + C_2$

On C_1: $x = 2\cos t \;\Rightarrow\; dx = -2\sin t\,dt,\;\; y = 2\sin t \;\Rightarrow\; dy = 2\cos t\;dt,\;\; 0 \le t \le \frac{\pi}{2}$.

On C_2: $x = 4t \;\Rightarrow\; dx = 4\,dt, y = 2 + t \;\Rightarrow\; dy = dt,\;\; 0 \le t \le 1$.

Then

$$\begin{aligned}\int_C x^2\,dx + y^2\,dy &= \int_{C_1} x^2\,dx + y^2\,dy + \int_{C_2} x^2\,dx + y^2\,dy \\ &= \int_0^{\pi/2} (2\cos t)^2(-2\sin t\,dt) + (2\sin t)^2(2\cos t\,dt) + \int_0^1 (4t)^2(4\,dt) + (2+t)^2\,dt \\ &= 8\int_0^{\pi/2}(-\cos^2 t\sin t + \sin^2 t\cos t)\,dt + \int_0^1 (65t^2 + 4t + 4)\,dt \\ &= 8\left[\tfrac{1}{3}\cos^3 t + \tfrac{1}{3}\sin^3 t\right]_0^{\pi/2} + \left[\tfrac{65}{3}t^3 + 2t^2 + 4t\right]_0^1 = 8\left(\tfrac{1}{3} - \tfrac{1}{3}\right) + \tfrac{65}{3} + 2 + 4 = \tfrac{83}{3}\end{aligned}$$

9. $x = \cos t,\;\; y = \sin t,\;\; z = t,\;\; 0 \le t \le \pi/2$. Then by Formula 9,

$$\begin{aligned}\int_C x^2 y\,ds &= \int_0^{\pi/2} (\cos t)^2(\sin t)\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 + \left(\tfrac{dz}{dt}\right)^2}\,dt \\ &= \int_0^{\pi/2} \cos^2 t\sin t\,\sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\,dt = \int_0^{\pi/2}\cos^2 t\sin t\,\sqrt{\sin^2 t + \cos^2 t + 1}\,dt \\ &= \sqrt{2}\int_0^{\pi/2}\cos^2 t\sin t\,dt = \sqrt{2}\left[-\tfrac{1}{3}\cos^3 t\right]_0^{\pi/2} = \sqrt{2}\left(0 + \tfrac{1}{3}\right) = \tfrac{\sqrt{2}}{3}\end{aligned}$$

10. Parametric equations for C are $x = 3 - 2t,\; y = 1 + t,\; z = 2 + 3t,\; 0 \le t \le 1$. Then

$$\begin{aligned}\int_C y^2 z\,ds &= \int_0^1 (1+t)^2(2+3t)\sqrt{(-2)^2 + 1^2 + 3^2}\,dt = \sqrt{14}\int_0^1 (3t^3 + 8t^2 + 7t + 2)\,dt \\ &= \sqrt{14}\left[\tfrac{3}{4}t^4 + \tfrac{8}{3}t^3 + \tfrac{7}{2}t^2 + 2t\right]_0^1 = \sqrt{14}\left(\tfrac{3}{4} + \tfrac{8}{3} + \tfrac{7}{2} + 2\right) = \tfrac{107}{12}\sqrt{14}\end{aligned}$$

11. Parametric equations for C are $x = t,\; y = 2t,\; z = 3t,\; 0 \le t \le 1$. Then

$$\int_C xe^{yz}\,ds = \int_0^1 te^{(2t)(3t)}\sqrt{1^2 + 2^2 + 3^2}\,dt = \sqrt{14}\int_0^1 te^{6t^2}\,dt = \sqrt{14}\left[\tfrac{1}{12}e^{6t^2}\right]_0^1 = \tfrac{\sqrt{14}}{12}(e^6 - 1).$$

12. $\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} = \sqrt{1^2 + (-2\sin 2t)^2 + (2\cos 2t)^2} = \sqrt{1 + 4(\sin^2 2t + \cos^2 2t)} = \sqrt{5}$. Then

$$\begin{aligned}\int_C (x^2 + y^2 + z^2)\,ds &= \int_0^{2\pi}(t^2 + \cos^2 2t + \sin^2 2t)\sqrt{5}\,dt = \sqrt{5}\int_0^{2\pi}(t^2 + 1)\,dt \\ &= \sqrt{5}\left[\tfrac{1}{3}t^3 + t\right]_0^{2\pi} = \sqrt{5}\left[\tfrac{1}{3}(8\pi^3) + 2\pi\right] = \sqrt{5}\left(\tfrac{8}{3}\pi^3 + 2\pi\right)\end{aligned}$$

13. $\int_C xye^{yz}\,dy = \int_0^1 (t)(t^2)e^{(t^2)(t^3)}\cdot 2t\,dt = \int_0^1 2t^4 e^{t^5}\,dt = \tfrac{2}{5}e^{t^5}\Big]_0^1 = \tfrac{2}{5}(e^1 - e^0) = \tfrac{2}{5}(e - 1)$

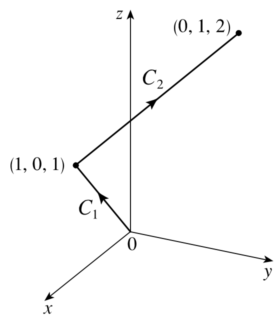
14. $\int_C y\,dx + z\,dy + x\,dz = \int_1^4 t\cdot\tfrac{1}{2}t^{-1/2}\,dt + t^2\cdot dt + \sqrt{t}\cdot 2t\,dt = \int_1^4\left(\tfrac{1}{2}t^{1/2} + t^2 + 2t^{3/2}\right)dt$

$$= \left[\tfrac{1}{3}t^{3/2} + \tfrac{1}{3}t^3 + \tfrac{4}{5}t^{5/2}\right]_1^4 = \tfrac{8}{3} + \tfrac{64}{3} + \tfrac{128}{5} - \tfrac{1}{3} - \tfrac{1}{3} - \tfrac{4}{5} = \tfrac{722}{15}$$

15. Parametric equations for C are $x = 1 + 3t,\ \ y = t,\ \ z = 2t,\ 0 \le t \le 1$. Then

$$\int_C z^2\,dx + x^2\,dy + y^2\,dz = \int_0^1 (2t)^2 \cdot 3\,dt + (1+3t)^2\,dt + t^2 \cdot 2\,dt = \int_0^1 \left(23t^2 + 6t + 1\right)dt$$
$$= \left[\tfrac{23}{3}t^3 + 3t^2 + t\right]_0^1 = \tfrac{23}{3} + 3 + 1 = \tfrac{35}{3}$$

16.



On C_1: $x = t \;\Rightarrow\; dx = dt,\ y = 0 \;\Rightarrow\; dy = 0\,dt,\ z = t \;\Rightarrow\; dz = dt,\ 0 \le t \le 1$.

On C_2: $x = 1 - t \;\Rightarrow\; dx = -dt,\ y = t \;\Rightarrow\; dy = dt,\ z = 1 + t \;\Rightarrow\; dz = dt,\ 0 \le t \le 1$.

Then

$$\int_C (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$$
$$= \int_{C_1} (y+z)\,dx + (x+z)\,dy + (x+y)\,dz + \int_{C_2} (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$$
$$= \int_0^1 (0+t)\,dt + (t+t)\cdot 0\,dt + (t+0)\,dt + \int_0^1 (t+1+t)(-dt) + (1-t+1+t)\,dt + (1-t+t)\,dt$$
$$= \int_0^1 2t\,dt + \int_0^1 (-2t+2)\,dt = \left[t^2\right]_0^1 + \left[-t^2+2t\right]_0^1 = 1 + 1 = 2$$

17. (a) Along the line $x = -3$, the vectors of $\mathbf{F}$ have positive y-components, so since the path goes upward, the integrand $\mathbf{F}\cdot\mathbf{T}$ is always positive. Therefore $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds$ is positive.

(b) All of the (nonzero) field vectors along the circle with radius 3 are pointed in the clockwise direction, that is, opposite the direction to the path. So $\mathbf{F}\cdot\mathbf{T}$ is negative, and therefore $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds$ is negative.

18. Vectors starting on C_1 point in roughly the same direction as C_1, so the tangential component $\mathbf{F}\cdot\mathbf{T}$ is positive. Then $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds$ is positive. On the other hand, no vectors starting on C_2 point in the same direction as C_2, while some vectors point in roughly the opposite direction, so we would expect $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds$ to be negative.

19. $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}$, so $\mathbf{F}(\mathbf{r}(t)) = (t^3)(t^2)^2\,\mathbf{i} - (t^3)^2\,\mathbf{j} = t^7\,\mathbf{i} - t^6\,\mathbf{j}$ and $\mathbf{r}'(t) = 3t^2\,\mathbf{i} + 2t\,\mathbf{j}$. Then

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 (t^7 \cdot 3t^2 - t^6 \cdot 2t)\,dt = \int_0^1 (3t^9 - 2t^7)\,dt = \left[\tfrac{3}{10}t^{10} - \tfrac{1}{4}t^8\right]_0^1 = \tfrac{3}{10} - \tfrac{1}{4} = \tfrac{1}{20}$.

20. $\mathbf{F}(\mathbf{r}(t)) = \left(t^2 + (t^3)^2\right)\mathbf{i} + (t^2)(-2t)\,\mathbf{j} + (t^3 - 2t)\,\mathbf{k} = (t^2 + t^6)\,\mathbf{i} - 2t^3\,\mathbf{j} + (t^3 - 2t)\,\mathbf{k},\ \mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} - 2\,\mathbf{k}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^2 (2t^3 + 2t^7 - 6t^5 - 2t^3 + 4t)\,dt = \int_0^2 (2t^7 - 6t^5 + 4t)\,dt$$
$$= \left[\tfrac{1}{4}t^8 - t^6 + 2t^2\right]_0^2 = 64 - 64 + 8 = 8$$

21. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle \sin t^3, \cos(-t^2), t^4 \right\rangle \cdot \left\langle 3t^2, -2t, 1 \right\rangle dt$

$$= \int_0^1 (3t^2 \sin t^3 - 2t\cos t^2 + t^4)\,dt = \left[-\cos t^3 - \sin t^2 + \tfrac{1}{5}t^5\right]_0^1 = \tfrac{6}{5} - \cos 1 - \sin 1$$

22. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \langle \cos t, \sin t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 1 \rangle \, dt = \int_0^\pi \sin t \cos t \, dt = \frac{1}{2} \sin^2 t \big]_0^\pi = 0$

23. $\mathbf{F}(\mathbf{r}(t)) = \sqrt{\sin^2 t + \sin t \cos t}\ \mathbf{i} + \left[(\sin t \cos t)/\sin^2 t\right] \mathbf{j} = \sqrt{\sin^2 t + \sin t \cos t}\ \mathbf{i} + \cot t \, \mathbf{j}$,

$\mathbf{r}'(t) = 2 \sin t \cos t \, \mathbf{i} + (\cos^2 t - \sin^2 t) \, \mathbf{j}$. Then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{\pi/6}^{\pi/3} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_{\pi/6}^{\pi/3} \left[2 \sin t \cos t \sqrt{\sin^2 t + \sin t \cos t} + (\cot t)(\cos^2 t - \sin^2 t) \right] dt \\ &\approx 0.5424 \end{aligned}$$

24. $$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= (\cos t \tan t) e^{\sin t} \, \mathbf{i} + (\tan t \sin t) e^{\cos t} \, \mathbf{j} + (\sin t \cos t) e^{\tan t} \, \mathbf{k} \\ &= (\sin t) e^{\sin t} \, \mathbf{i} + (\tan t \sin t) e^{\cos t} \, \mathbf{j} + (\sin t \cos t) e^{\tan t} \, \mathbf{k}, \end{aligned}$$

$\mathbf{r}'(t) = \cos t \, \mathbf{i} - \sin t \, \mathbf{j} + \sec^2 t \, \mathbf{k}$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/4} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_0^{\pi/4} \left[(\sin t \cos t) e^{\sin t} - (\tan t \sin^2 t) e^{\cos t} + (\tan t) e^{\tan t} \right] dt \approx 0.8527$$

25. $x = t^2, \;\; y = t^3, \;\; z = \sqrt{t}$ so by Formula 9,

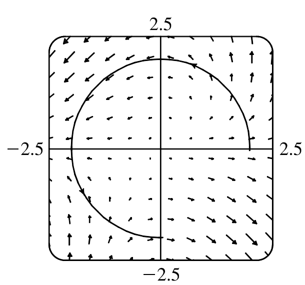
$$\begin{aligned} \int_C xy \arctan z \, ds &= \int_1^2 (t^2)(t^3) \arctan \sqrt{t} \cdot \sqrt{(2t)^2 + (3t^2)^2 + \left[1/(2\sqrt{t}\,)\right]^2} \, dt \\ &= \int_1^2 t^5 \sqrt{4t^2 + 9t^4 + 1/(4t)} \, \arctan \sqrt{t} \, dt \approx 94.8231 \end{aligned}$$

26. $x = 1 + 3t, \;\; y = 2 + t^2, \;\; z = t^4$ so by Formula 9,

$$\begin{aligned} \int_C z \ln(x + y) \, ds &= \int_{-1}^1 t^4 \ln(1 + 3t + 2 + t^2) \cdot \sqrt{(3)^2 + (2t)^2 + (4t^3)^2} \, dt \\ &= \int_{-1}^1 t^4 \sqrt{9 + 4t^2 + 16t^6} \, \ln(3 + 3t + t^2) \, dt \approx 1.7260 \end{aligned}$$

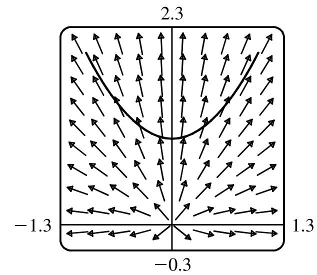
27. We graph $\mathbf{F}(x, y) = (x - y) \, \mathbf{i} + xy \, \mathbf{j}$ and the curve C. We see that most of the vectors starting on C point in roughly the same direction as C, so for these portions of C the tangential component $\mathbf{F} \cdot \mathbf{T}$ is positive. Although some vectors in the third quadrant which start on C point in roughly the opposite direction, and hence give negative tangential components, it seems reasonable that the effect of these portions of C is outweighed by the positive tangential components. Thus, we would expect $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$ to be positive.

To verify, we evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. The curve C can be represented by $\mathbf{r}(t) = 2 \cos t \, \mathbf{i} + 2 \sin t \, \mathbf{j}, \;\; 0 \le t \le \frac{3\pi}{2}$, so $\mathbf{F}(\mathbf{r}(t)) = (2 \cos t - 2 \sin t) \, \mathbf{i} + 4 \cos t \sin t \, \mathbf{j}$ and $\mathbf{r}'(t) = -2 \sin t \, \mathbf{i} + 2 \cos t \, \mathbf{j}$. Then



$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{3\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\ &= \int_0^{3\pi/2} [-2 \sin t (2 \cos t - 2 \sin t) + 2 \cos t (4 \cos t \sin t)] \, dt \\ &= 4 \int_0^{3\pi/2} (\sin^2 t - \sin t \cos t + 2 \sin t \cos^2 t) \, dt \\ &= 3\pi + \tfrac{2}{3} \qquad \text{[using a CAS]} \end{aligned}$$

28. We graph $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\,\mathbf{j}$ and the curve C. In the first quadrant, each vector starting on C points in roughly the same direction as C, so the tangential component $\mathbf{F}\cdot\mathbf{T}$ is positive. In the second quadrant, each vector starting on C points in roughly the direction opposite to C, so $\mathbf{F}\cdot\mathbf{T}$ is negative. Here, it appears that the tangential components in the first and second quadrants counteract each other, so it seems reasonable to guess that $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is zero. To verify, we evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. The curve C can be represented by



$\mathbf{r}(t) = t\,\mathbf{i} + (1+t^2)\,\mathbf{j}$, $-1 \le t \le 1$, so $\mathbf{F}(\mathbf{r}(t)) = \dfrac{t}{\sqrt{t^2+(1+t^2)^2}}\,\mathbf{i} + \dfrac{1+t^2}{\sqrt{t^2+(1+t^2)^2}}\,\mathbf{j}$ and $\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_{-1}^{1}\left(\frac{t}{\sqrt{t^2+(1+t^2)^2}} + \frac{2t(1+t^2)}{\sqrt{t^2+(1+t^2)^2}}\right)dt$$

$$= \int_{-1}^{1}\frac{t(3+2t^2)}{\sqrt{t^4+3t^2+1}}\,dt = 0 \qquad [\text{since the integrand is an odd function}]$$

29. (a) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle e^{t^2-1}, t^5\right\rangle\cdot\left\langle 2t, 3t^2\right\rangle dt = \int_0^1\left(2te^{t^2-1} + 3t^7\right)dt = \left[e^{t^2-1} + \frac{3}{8}t^8\right]_0^1 = \frac{11}{8} - 1/e$

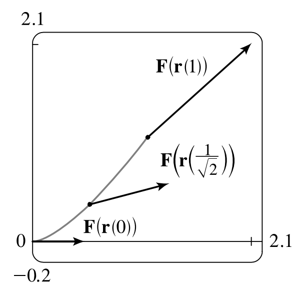
(b) $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{F}(\mathbf{r}(0)) = \left\langle e^{-1}, 0\right\rangle$;

$\mathbf{r}\left(\frac{1}{\sqrt{2}}\right) = \left\langle \frac{1}{2}, \frac{1}{2\sqrt{2}}\right\rangle$, $\mathbf{F}\left(\mathbf{r}\left(\frac{1}{\sqrt{2}}\right)\right) = \left\langle e^{-1/2}, \frac{1}{4\sqrt{2}}\right\rangle$;

$\mathbf{r}(1) = \langle 1, 1\rangle$, $\mathbf{F}(\mathbf{r}(1)) = \langle 1, 1\rangle$.

In order to generate the graph with Maple, we use the `line` command in the `plottools` package to define each of the vectors. For example,

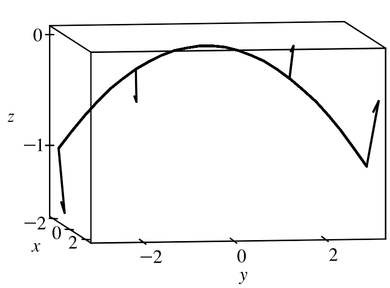
```
v1:=line([0,0],[exp(-1),0]):
```

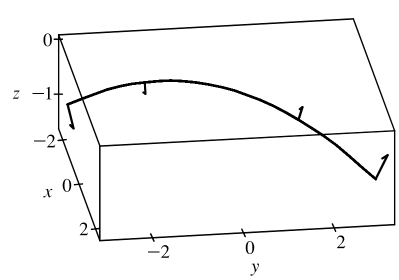


generates the vector from the vector field at the point $(0, 0)$ (but without an arrowhead) and gives it the name `v1`. To show everything on the same screen, we use the `display` command. In Mathematica, we use `ListPlot` (with the `PlotJoined -> True` option) to generate the vectors, and then `Show` to show everything on the same screen.

30. (a) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1}\left\langle 2t, t^2, 3t\right\rangle\cdot\langle 2, 3, -2t\rangle\,dt = \int_{-1}^{1}(4t + 3t^2 - 6t^2)\,dt = \left[2t^2 - t^3\right]_{-1}^{1} = -2$

(b) Now $\mathbf{F}(\mathbf{r}(t)) = \left\langle 2t, t^2, 3t\right\rangle$, so $\mathbf{F}(\mathbf{r}(-1)) = \langle -2, 1, -3\rangle$, $\mathbf{F}\left(\mathbf{r}\left(-\frac{1}{2}\right)\right) = \left\langle -1, \frac{1}{4}, -\frac{3}{2}\right\rangle$, $\mathbf{F}\left(\mathbf{r}\left(\frac{1}{2}\right)\right) = \left\langle 1, \frac{1}{4}, \frac{3}{2}\right\rangle$, and $\mathbf{F}(\mathbf{r}(1)) = \langle 2, 1, 3\rangle$.





31. $x = e^{-t}\cos 4t,\quad y = e^{-t}\sin 4t,\quad z = e^{-t},\quad 0 \le t \le 2\pi$.

Then $\dfrac{dx}{dt} = e^{-t}(-\sin 4t)(4) - e^{-t}\cos 4t = -e^{-t}(4\sin 4t + \cos 4t)$,

$\dfrac{dy}{dt} = e^{-t}(\cos 4t)(4) - e^{-t}\sin 4t = -e^{-t}(-4\cos 4t + \sin 4t)$, and $\dfrac{dz}{dt} = -e^{-t}$, so

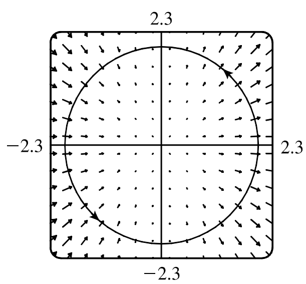
$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(-e^{-t})^2[(4\sin 4t + \cos 4t)^2 + (-4\cos 4t + \sin 4t)^2 + 1]}$$
$$= e^{-t}\sqrt{16(\sin^2 4t + \cos^2 4t) + \sin^2 4t + \cos^2 4t + 1} = 3\sqrt{2}\,e^{-t}$$

Therefore
$$\int_C x^3y^2z\,ds = \int_0^{2\pi}(e^{-t}\cos 4t)^3(e^{-t}\sin 4t)^2(e^{-t})\,(3\sqrt{2}\,e^{-t})\,dt$$
$$= \int_0^{2\pi} 3\sqrt{2}\,e^{-7t}\cos^3 4t\sin^2 4t\,dt = \tfrac{172{,}704}{5{,}632{,}705}\sqrt{2}\,(1 - e^{-14\pi})$$

32. (a) We parametrize the circle C as $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\quad 0 \le t \le 2\pi$. So $\mathbf{F}(\mathbf{r}(t)) = \langle 4\cos^2 t, 4\cos t\sin t\rangle$,

$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, and $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(-8\cos^2 t\sin t + 8\cos^2 t\sin t)\,dt = 0$.

(b)



From the graph, we see that all of the vectors in the field are perpendicular to the path. This indicates that the field does no work on the particle, since the field never pulls the particle in the direction in which it is going. In other words, at any point along C, $\mathbf{F}\cdot\mathbf{T} = 0$, and so certainly $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$.

33. We use the parametrization $x = 2\cos t$, $y = 2\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt = 2\,dt$, so $m = \int_C k\,ds = 2k\int_{-\pi/2}^{\pi/2} dt = 2k(\pi)$,

$\overline{x} = \frac{1}{2\pi k}\int_C xk\,ds = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\cos t)2\,dt = \frac{1}{2\pi}\left[4\sin t\right]_{-\pi/2}^{\pi/2} = \frac{4}{\pi}$, $\overline{y} = \frac{1}{2\pi k}\int_C yk\,ds = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\sin t)2\,dt = 0$.

Hence $(\overline{x}, \overline{y}) = \left(\frac{4}{\pi}, 0\right)$.

34. We use the parametrization $x = a\cos t,\quad y = a\sin t,\quad 0 \le t \le \frac{\pi}{2}$. Then

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \sqrt{(-a\sin t)^2 + (a\cos t)^2}\,dt = a\,dt$, so

$m = \int_C \rho(x,y)\,ds = \int_C kxy\,ds = \int_0^{\pi/2} k(a\cos t)(a\sin t)\,a\,dt = ka^3\int_0^{\pi/2}\cos t\sin t\,dt = ka^3\left[\frac{1}{2}\sin^2 t\right]_0^{\pi/2} = \frac{1}{2}ka^3$,

$$\overline{x} = \frac{1}{ka^3/2}\int_C x(kxy)\,ds = \frac{2}{ka^3}\int_0^{\pi/2} k(a\cos t)^2(a\sin t)a\,dt = \frac{2}{ka^3}\cdot ka^4\int_0^{\pi/2}\cos^2 t\,\sin t\,dt$$
$$= 2a\left[-\tfrac{1}{3}\cos^3 t\right]_0^{\pi/2} = 2a\left(0 + \tfrac{1}{3}\right) = \tfrac{2}{3}a, \text{ and}$$

$$\overline{y} = \frac{1}{ka^3/2}\int_C y(kxy)\,ds = \frac{2}{ka^3}\int_0^{\pi/2} k(a\cos t)(a\sin t)^2a\,dt = \frac{2}{ka^3}\cdot ka^4\int_0^{\pi/2}\sin^2 t\,\cos t\,dt$$
$$= 2a\left[\tfrac{1}{3}\sin^3 t\right]_0^{\pi/2} = 2a\left(\tfrac{1}{3} - 0\right) = \tfrac{2}{3}a.$$

Therefore the mass is $\frac{1}{2}ka^3$ and the center of mass is $(\overline{x}, \overline{y}) = \left(\frac{2}{3}a, \frac{2}{3}a\right)$.

35. (a) $\overline{x} = \dfrac{1}{m}\displaystyle\int_C x\rho(x,y,z)\,ds$, $\overline{y} = \dfrac{1}{m}\displaystyle\int_C y\rho(x,y,z)\,ds$, $\overline{z} = \dfrac{1}{m}\displaystyle\int_C z\rho(x,y,z)\,ds$ where $m = \int_C \rho(x,y,z)\,ds$.

(b) $m = \int_C k\,ds = k\int_0^{2\pi}\sqrt{4\sin^2 t + 4\cos^2 t + 9}\,dt = k\sqrt{13}\int_0^{2\pi} dt = 2\pi k\sqrt{13}$,

$$\overline{x} = \frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi} 2k\sqrt{13}\,\sin t\,dt = 0,\ \overline{y} = \frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi} 2k\sqrt{13}\,\cos t\,dt = 0,$$

$$\overline{z} = \frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi}\left(k\sqrt{13}\right)(3t)\,dt = \frac{3}{2\pi}\left(2\pi^2\right) = 3\pi.$$ Hence $(\overline{x},\overline{y},\overline{z}) = (0,0,3\pi)$.

36. $m = \int_C (x^2+y^2+z^2)\,ds = \int_0^{2\pi}(t^2+1)\sqrt{(1)^2+(-\sin t)^2+(\cos t)^2}\,dt = \int_0^{2\pi}(t^2+1)\sqrt{2}\,dt = \sqrt{2}\left(\frac{8}{3}\pi^3+2\pi\right)$,

$$\overline{x} = \frac{1}{\sqrt{2}\left(\frac{8}{3}\pi^3+2\pi\right)}\int_0^{2\pi}\sqrt{2}\,(t^3+t)\,dt = \frac{4\pi^4+2\pi^2}{\frac{8}{3}\pi^3+2\pi} = \frac{3\pi\left(2\pi^2+1\right)}{4\pi^2+3},$$

$$\overline{y} = \frac{3}{2\sqrt{2}\,\pi(4\pi^2+3)}\int_0^{2\pi}\left(\sqrt{2}\cos t\right)(t^2+1)\,dt = 0, \text{ and}$$

$$\overline{z} = \frac{3}{2\sqrt{2}\,\pi(4\pi^2+3)}\int_0^{2\pi}\left(\sqrt{2}\sin t\right)(t^2+1)\,dt = 0.$$ Hence $(\overline{x},\overline{y},\overline{z}) = \left(\dfrac{3\pi(2\pi^2+1)}{4\pi^2+3},0,0\right)$.

37. From Example 3, $\rho(x,y) = k(1-y)$, $x = \cos t$, $y = \sin t$, and $ds = dt$, $0 \le t \le \pi \quad \Rightarrow$

$$I_x = \int_C y^2\rho(x,y)\,ds = \int_0^\pi \sin^2 t\,[k(1-\sin t)]\,dt = k\int_0^\pi(\sin^2 t - \sin^3 t)\,dt$$
$$= \tfrac{1}{2}k\int_0^\pi(1-\cos 2t)\,dt - k\int_0^\pi(1-\cos^2 t)\sin t\,dt \qquad \left[\begin{array}{c}\text{Let } u=\cos t,\ du=-\sin t\,dt\\ \text{in the second integral}\end{array}\right]$$
$$= k\left[\tfrac{\pi}{2} + \int_1^{-1}(1-u^2)\,du\right] = k\left(\tfrac{\pi}{2}-\tfrac{4}{3}\right)$$

$$I_y = \int_C x^2\rho(x,y)\,ds = k\int_0^\pi\cos^2 t\,(1-\sin t)\,dt = \tfrac{k}{2}\int_0^\pi(1+\cos 2t)\,dt - k\int_0^\pi\cos^2 t\sin t\,dt$$
$= k\left(\frac{\pi}{2}-\frac{2}{3}\right)$, using the same substitution as above.

38. The wire is given as $x = 2\sin t$, $y = 2\cos t$, $z = 3t$, $0 \le t \le 2\pi$ with $\rho(x,y,z) = k$. Then

$ds = \sqrt{(2\cos t)^2 + (-2\sin t)^2 + 3^2}\,dt = \sqrt{4(\cos^2 t+\sin^2 t)+9}\,dt = \sqrt{13}\,dt$ and

$$I_x = \int_C(y^2+z^2)\rho(x,y,z)\,ds = \int_0^{2\pi}(4\cos^2 t + 9t^2)(k)\sqrt{13}\,dt = \sqrt{13}\,k\left[4\left(\tfrac{1}{2}t+\tfrac{1}{4}\sin 2t\right)+3t^3\right]_0^{2\pi}$$
$$= \sqrt{13}\,k(4\pi+24\pi^3) = 4\sqrt{13}\,\pi k(1+6\pi^2)$$

$$I_y = \int_C(x^2+z^2)\rho(x,y,z)\,ds = \int_0^{2\pi}\left(4\sin^2 t + 9t^2\right)(k)\sqrt{13}\,dt = \sqrt{13}\,k\left[4\left(\tfrac{1}{2}t-\tfrac{1}{4}\sin 2t\right)+3t^3\right]_0^{2\pi}$$
$$= \sqrt{13}\,k(4\pi+24\pi^3) = 4\sqrt{13}\,\pi k(1+6\pi^2)$$

$$I_z = \int_C(x^2+y^2)\rho(x,y,z)\,ds = \int_0^{2\pi}(4\sin^2 t + 4\cos^2 t)(k)\sqrt{13}\,dt = 4\sqrt{13}\,k\int_0^{2\pi}dt = 8\pi\sqrt{13}\,k$$

39. $$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle t-\sin t, 3-\cos t\rangle\cdot\langle 1-\cos t, \sin t\rangle\,dt$$
$$= \int_0^{2\pi}(t - t\cos t - \sin t + \sin t\cos t + 3\sin t - \sin t\cos t)\,dt$$
$$= \int_0^{2\pi}(t - t\cos t + 2\sin t)\,dt = \left[\tfrac{1}{2}t^2 - (t\sin t+\cos t) - 2\cos t\right]_0^{2\pi} \qquad \left[\begin{array}{c}\text{integrate by parts}\\ \text{in the second term}\end{array}\right]$$
$$= 2\pi^2$$

40. Choosing y as the parameter, the curve C is parametrized by $x = y^2 + 1$, $y = y$, $0 \le y \le 1$. Then

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left\langle \left(y^2+1\right)^2, ye^{y^2+1} \right\rangle \cdot \langle 2y, 1 \rangle \; dy = \int_0^1 \left[2y\left(y^2+1\right)^2 + ye^{y^2+1}\right] dy$$
$$= \left[\tfrac{1}{3}\left(y^2+1\right)^3 + \tfrac{1}{2}e^{y^2+1}\right]_0^1 = \tfrac{8}{3} + \tfrac{1}{2}e^2 - \tfrac{1}{3} - \tfrac{1}{2}e = \tfrac{1}{2}e^2 - \tfrac{1}{2}e + \tfrac{7}{3}$$

41. $\mathbf{r}(t) = \langle 2t, t, 1-t \rangle, \quad 0 \le t \le 1.$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left\langle 2t - t^2, t - (1-t)^2, 1 - t - (2t)^2 \right\rangle \cdot \langle 2, 1, -1 \rangle \, dt$$
$$= \int_0^1 \left(4t - 2t^2 + t - 1 + 2t - t^2 - 1 + t + 4t^2\right) dt = \int_0^1 \left(t^2 + 8t - 2\right) dt = \left[\tfrac{1}{3}t^3 + 4t^2 - 2t\right]_0^1 = \tfrac{7}{3}$$

42. $\mathbf{r}(t) = 2\,\mathbf{i} + t\,\mathbf{j} + 5t\,\mathbf{k}, \quad 0 \le t \le 1$. Therefore

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \frac{K\langle 2, t, 5t \rangle}{(4+26t^2)^{3/2}} \cdot \langle 0, 1, 5 \rangle \, dt = K \int_0^1 \frac{26t}{(4+26t^2)^{3/2}} \, dt = K\Big[-(4+26t^2)^{-1/2}\Big]_0^1 = K\Big(\tfrac{1}{2} - \tfrac{1}{\sqrt{30}}\Big).$$

43. (a) $\mathbf{r}(t) = at^2\,\mathbf{i} + bt^3\,\mathbf{j} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = 2at\,\mathbf{i} + 3bt^2\,\mathbf{j} \quad \Rightarrow \quad \mathbf{a}(t) = \mathbf{v}'(t) = 2a\,\mathbf{i} + 6bt\,\mathbf{j}$, and force is mass times acceleration: $\mathbf{F}(t) = m\,\mathbf{a}(t) = 2ma\,\mathbf{i} + 6mbt\,\mathbf{j}$.

(b) $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2ma\,\mathbf{i} + 6mbt\,\mathbf{j}) \cdot (2at\,\mathbf{i} + 3bt^2\,\mathbf{j}) \, dt = \int_0^1 \left(4ma^2t + 18mb^2t^3\right) dt$

$= \left[2ma^2t^2 + \tfrac{9}{2}mb^2t^4\right]_0^1 = 2ma^2 + \tfrac{9}{2}mb^2$

44. $\mathbf{r}(t) = a\sin t\,\mathbf{i} + b\cos t\,\mathbf{j} + ct\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = a\cos t\,\mathbf{i} - b\sin t\,\mathbf{j} + c\,\mathbf{k} \quad \Rightarrow \quad \mathbf{a}(t) = \mathbf{v}'(t) = -a\sin t\,\mathbf{i} - b\cos t\,\mathbf{j}$ and $\mathbf{F}(t) = m\,\mathbf{a}(t) = -ma\sin t\,\mathbf{i} - mb\cos t\,\mathbf{j}$. Thus

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (-ma\sin t\,\mathbf{i} - mb\cos t\,\mathbf{j}) \cdot (a\cos t\,\mathbf{i} - b\sin t\,\mathbf{j} + c\,\mathbf{k}) \, dt$$
$$= \int_0^{\pi/2} \left(-ma^2\sin t\cos t + mb^2\sin t\cos t\right) dt = m(b^2 - a^2)\left[\tfrac{1}{2}\sin^2 t\right]_0^{\pi/2} = \tfrac{1}{2}m(b^2 - a^2)$$

45. The combined weight of the man and the paint is 185 lb, so the force exerted (equal and opposite to that exerted by gravity) is $\mathbf{F} = 185\,\mathbf{k}$. To parametrize the staircase, let $x = 20\cos t, \;\; y = 20\sin t, \;\; z = \frac{90}{6\pi}t = \frac{15}{\pi}t, \;\; 0 \le t \le 6\pi$. Then the work done is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{6\pi} \langle 0, 0, 185 \rangle \cdot \left\langle -20\sin t, 20\cos t, \tfrac{15}{\pi} \right\rangle dt = (185)\tfrac{15}{\pi} \int_0^{6\pi} dt = (185)\left(\tfrac{15}{\pi}\right)(6\pi) \approx 1.67 \times 10^4 \text{ ft-lb}$$

46. This time m is a function of t: $m = 185 - \frac{9}{6\pi}t = 185 - \frac{3}{2\pi}t$. So let $\mathbf{F} = \left(185 - \frac{3}{2\pi}t\right)\mathbf{k}$. To parametrize the staircase, let $x = 20\cos t, \;\; y = 20\sin t, \;\; z = \frac{90}{6\pi}t = \frac{15}{\pi}t, \;\; 0 \le t \le 6\pi$. Therefore

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{6\pi} \left\langle 0, 0, 185 - \tfrac{3}{2\pi}t \right\rangle \cdot \left\langle -20\sin t, 20\cos t, \tfrac{15}{\pi} \right\rangle dt = \tfrac{15}{\pi} \int_0^{6\pi} \left(185 - \tfrac{3}{2\pi}t\right) dt$$
$$= \tfrac{15}{\pi}\left[185t - \tfrac{3}{4\pi}t^2\right]_0^{6\pi} = 90\left(185 - \tfrac{9}{2}\right) \approx 1.62 \times 10^4 \text{ ft-lb}$$

47. (a) $\mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \le t \le 2\pi$, and let $\mathbf{F} = \langle a, b \rangle$. Then

$$W = \int_C \mathbf{F} \cdot d\,\mathbf{r} = \int_0^{2\pi} \langle a, b \rangle \cdot \langle -\sin t, \cos t \rangle \; dt = \int_0^{2\pi} (-a\sin t + b\cos t) \, dt = \left[a\cos t + b\sin t\right]_0^{2\pi}$$
$$= a + 0 - a + 0 = 0$$

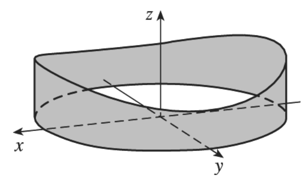
(b) Yes. $\mathbf{F}(x, y) = k\,\mathbf{x} = \langle kx, ky \rangle$ and

$$W = \int_C \mathbf{F} \cdot d\,\mathbf{r} = \int_0^{2\pi} \langle k\cos t, k\sin t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = \int_0^{2\pi} (-k\sin t\,\cos t + k\sin t\,\cos t) \, dt = \int_0^{2\pi} 0 \, dt = 0.$$

48. Consider the base of the fence in the xy-plane, centered at the origin, with the height given by $z = h(x, y)$. To graph the fence, observe that the fence is highest when $y = 0$ (where the height is 5 m) and lowest when $x = 0$ (a height of 3 m). When $y = \pm x$, the height is 4 m.

Also, the fence can be graphed using parametric equations (see Section 16.6): $x = 10\cos u,\ \ y = 10\sin u$,

$$z = v\left[4 + 0.01((10\cos u)^2 - (10\sin u)^2)\right] = v(4 + \cos^2 u - \sin^2 u)$$
$$= v(4 + \cos 2u),\ 0 \le u \le 2\pi,\ 0 \le v \le 1.$$



The area of the fence is $\int_C h(x, y)\,ds$ where C, the base of the fence, is given by $x = 10\cos t,\ \ y = 10\sin t,\ \ 0 \le t \le 2\pi$.

Then

$$\int_C h(x, y)\,ds = \int_0^{2\pi} \left[4 + 0.01((10\cos t)^2 - (10\sin t)^2)\right] \sqrt{(-10\sin t)^2 + (10\cos t)^2}\,dt$$
$$= \int_0^{2\pi} (4 + \cos 2t)\sqrt{100}\,dt = 10\left[4t + \tfrac{1}{2}\sin 2t\right]_0^{2\pi} = 10(8\pi) = 80\pi \text{ m}^2$$

If we paint both sides of the fence, the total surface area to cover is 160π m^2, and since 1 L of paint covers 100 m^2, we require $\frac{160\pi}{100} = 1.6\pi \approx 5.03$ L of paint.

49. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_a^b \langle v_1, v_2, v_3\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\,dt = \int_a^b [v_1\,x'(t) + v_2\,y'(t) + v_3\,z'(t)]\,dt$$
$$= \left[v_1\,x(t) + v_2\,y(t) + v_3\,z(t)\right]_a^b = [v_1\,x(b) + v_2\,y(b) + v_3\,z(b)] - [v_1\,x(a) + v_2\,y(a) + v_3\,z(a)]$$
$$= v_1\,[x(b) - x(a)] + v_2\,[y(b) - y(a)] + v_3\,[z(b) - z(a)]$$
$$= \langle v_1, v_2, v_3\rangle \cdot \langle x(b) - x(a), y(b) - y(a), z(b) - z(a)\rangle$$
$$= \langle v_1, v_2, v_3\rangle \cdot [\langle x(b), y(b), z(b)\rangle - \langle x(a), y(a), z(a)\rangle] = \mathbf{v} \cdot [\mathbf{r}(b) - \mathbf{r}(a)]$$

50. If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ then

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \int_a^b \langle x(t), y(t), z(t)\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\,dt = \int_a^b [x(t)\,x'(t) + y(t)\,y'(t) + z(t)\,z'(t)]\,dt$$
$$= \left[\tfrac{1}{2}[x(t)]^2 + \tfrac{1}{2}[y(t)]^2 + \tfrac{1}{2}[z(t)]^2\right]_a^b$$
$$= \tfrac{1}{2}\left\{\left([x(b)]^2 + [y(b)]^2 + [z(b)]^2\right) - \left([x(a)]^2 + [y(a)]^2 + [z(a)]^2\right)\right\}$$
$$= \tfrac{1}{2}\left[|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2\right]$$

51. The work done in moving the object is $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$. We can approximate this integral by dividing C into 7 segments of equal length $\Delta s = 2$ and approximating $\mathbf{F} \cdot \mathbf{T}$, that is, the tangential component of force, at a point (x_i^*, y_i^*) on each segment. Since C is composed of straight line segments, $\mathbf{F} \cdot \mathbf{T}$ is the scalar projection of each force vector onto C. If we choose (x_i^*, y_i^*) to be the point on the segment closest to the origin, then the work done is

$\int_C \mathbf{F}\cdot\mathbf{T}\,ds \approx \sum\limits_{i=1}^{7}\left[\mathbf{F}(x_i^*,y_i^*)\cdot\mathbf{T}(x_i^*,y_i^*)\right]\Delta s = [2+2+2+2+1+1+1](2) = 22$. Thus, we estimate the work done to be approximately 22 J.

52. Use the orientation pictured in the figure. Then since $\mathbf{B}$ is tangent to any circle that lies in the plane perpendicular to the wire, $\mathbf{B} = |\mathbf{B}|\,\mathbf{T}$ where $\mathbf{T}$ is the unit tangent to the circle C: $x = r\cos\theta,\ \ y = r\sin\theta$. Thus $\mathbf{B} = |\mathbf{B}|\,\langle -\sin\theta, \cos\theta\rangle$. Then $\int_C \mathbf{B}\cdot d\mathbf{r} = \int_0^{2\pi} |\mathbf{B}|\,\langle -\sin\theta,\cos\theta\rangle\cdot\langle -r\sin\theta, r\cos\theta\rangle\,d\theta = \int_0^{2\pi}|\mathbf{B}|\,r\,d\theta = 2\pi r\,|\mathbf{B}|$. (Note that $|\mathbf{B}|$ here is the magnitude of the field at a distance r from the wire's center.) But by Ampere's Law $\int_C \mathbf{B}\cdot d\mathbf{r} = \mu_0 I$. Hence $|\mathbf{B}| = \mu_0 I/(2\pi r)$.

16.3 The Fundamental Theorem for Line Integrals

1. C appears to be a smooth curve, and since ∇f is continuous, we know f is differentiable. Then Theorem 2 says that the value of $\int_C \nabla f\cdot d\mathbf{r}$ is simply the difference of the values of f at the terminal and initial points of C. From the graph, this is $50 - 10 = 40$.

2. C is represented by the vector function $\mathbf{r}(t) = (t^2+1)\,\mathbf{i} + (t^3+t)\,\mathbf{j}$, $0\le t\le 1$, so $\mathbf{r}'(t) = 2t\,\mathbf{i} + (3t^2+1)\,\mathbf{j}$. Since $3t^2+1\ne 0$, we have $\mathbf{r}'(t)\ne\mathbf{0}$, thus C is a smooth curve. ∇f is continuous, and hence f is differentiable, so by Theorem 2 we have $\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(2,2) - f(1,0) = 9 - 3 = 6$.

3. Let $P(x,y) = xy + y^2$ and $Q(x,y) = x^2 + 2xy$. Then $\partial P/\partial y = x + 2y$ and $\partial Q/\partial x = 2x + 2y$. Since $\partial P/\partial y \ne \partial Q/\partial x$, $\mathbf{F}(x,y) = P\,\mathbf{i} + Q\,\mathbf{j}$ is not conservative by Theorem 5.

4. $\partial(y^2 - 2x)/\partial y = 2y = \partial(2xy)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so $\mathbf{F}$ is conservative by Theorem 6. Thus, there exists a function f such that $\nabla f = \mathbf{F}$, that is, $f_x(x,y) = y^2 - 2x$ and $f_y(x,y) = 2xy$. But $f_x(x,y) = y^2 - 2x$ implies $f(x,y) = xy^2 - x^2 + g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x,y) = 2xy + g'(y)$. Thus $2xy = 2xy + g'(y)$ so $g'(y) = 0$ and $g(y) = K$ where K is a constant. Hence $f(x,y) = xy^2 - x^2 + K$ is a potential function for $\mathbf{F}$.

5. $\dfrac{\partial}{\partial y}\left(y^2e^{xy}\right) = y^2\cdot xe^{xy} + 2ye^{xy} = (xy^2+2y)e^{xy}$,

$\dfrac{\partial}{\partial x}\left[(1+xy)e^{xy}\right] = (1+xy)\cdot ye^{xy} + ye^{xy} = ye^{xy} + xy^2e^{xy} + ye^{xy} = (xy^2+2y)e^{xy}$.

Since these partial derivatives are equal and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, $\mathbf{F}$ is conservative by Theorem 6. Thus, there exists a function f such that $\nabla f = \mathbf{F}$, that is, $f_x(x,y) = y^2e^{xy}$ and $f_y(x,y) = (1+xy)e^{xy}$. But $f_x(x,y) = y^2e^{xy}$ implies $f(x,y) = ye^{xy} + g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x,y) = (1+xy)e^{xy} + g'(y)$. Thus $(1+xy)e^{xy} = (1+xy)e^{xy} + g'(y)$ so $g'(y) = 0$ and $g(y) = K$ where K is a constant. Hence $f(x,y) = ye^{xy} + K$ is a potential function for $\mathbf{F}$.

6. $\partial(ye^x)/\partial y = e^x = \partial(e^x + e^y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so $\mathbf{F}$ is conservative. Hence there exists a function f such that $\nabla f = \mathbf{F}$. Here $f_x(x,y) = ye^x$ implies $f(x,y) = ye^x + g(y)$ and then $f_y(x,y) = e^x + g'(y)$. But $f_y(x,y) = e^x + e^y$ so $g'(y) = e^y \quad \Rightarrow \quad g(y) = e^y + K$ and $f(x,y) = ye^x + e^y + K$ is a potential function for $\mathbf{F}$.

7. $\partial(ye^x + \sin y)/\partial y = e^x + \cos y = \partial(e^x + x\cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = ye^x + \sin y$ implies $f(x,y) = ye^x + x\sin y + g(y)$ and $f_y(x,y) = e^x + x\cos y + g'(y)$. But $f_y(x,y) = e^x + x\cos y$ so $g(y) = K$ and $f(x,y) = ye^x + x\sin y + K$ is a potential function for $\mathbf{F}$.

8. $\partial(2xy + y^{-2})/\partial y = 2x - 2y^{-3} = \partial(x^2 - 2xy^{-3})/\partial x$ and the domain of $\mathbf{F}$ is $\{(x,y) \mid y > 0\}$ which is open and simply-connected. Hence $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = 2xy + y^{-2}$ implies $f(x,y) = x^2y + xy^{-2} + g(y)$ and $f_y(x,y) = x^2 - 2xy^{-3} + g'(y)$. But $f_y(x,y) = x^2 - 2xy^{-3}$ so $g'(y) = 0 \quad \Rightarrow \quad g(y) = K$. Then $f(x,y) = x^2y + xy^{-2} + K$ is a potential function for $\mathbf{F}$.

9. $\partial(y^2\cos x + \cos y)/\partial y = 2y\cos x - \sin y = \partial(2y\sin x - x\sin y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply connected. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = y^2\cos x + \cos y$ implies $f(x,y) = y^2\sin x + x\cos y + g(y)$ and $f_y(x,y) = 2y\sin x - x\sin y + g'(y)$. But $f_y(x,y) = 2y\sin x - x\sin y$ so $g'(y) = 0 \quad \Rightarrow \quad g(y) = K$ and $f(x,y) = y^2\sin x + x\cos y + K$ is a potential function for $\mathbf{F}$.

10. $\partial(\ln y + y/x)/\partial y = 1/y + 1/x = \partial(\ln x + x/y)/\partial x$ and the domain of $\mathbf{F}$ is $\{(x,y) \mid x > 0,\ y > 0\}$ which is open and simply connected. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = \ln y + y/x$ implies $f(x,y) = x\ln y + y\ln x + g(y)$ and $f_y(x,y) = x/y + \ln x + g'(y)$. But $f_y(x,y) = \ln x + x/y$ so $g'(y) = 0 \quad \Rightarrow$ $g(y) = K$ and $f(x,y) = x\ln y + y\ln x + K$ is a potential function for $\mathbf{F}$.

11. (a) $\mathbf{F}$ has continuous first-order partial derivatives and $\dfrac{\partial}{\partial y}(2xy) = 2x = \dfrac{\partial}{\partial x}(x^2)$ on $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative by Theorem 6. Then we know that the line integral of $\mathbf{F}$ is independent of path; in particular, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of C. Since all three curves have the same initial and terminal points, $\int_C \mathbf{F} \cdot d\mathbf{r}$ will have the same value for each curve.

(b) We first find a potential function f, so that $\nabla f = \mathbf{F}$. We know $f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$. Integrating $f_x(x,y)$ with respect to x, we have $f(x,y) = x^2y + g(y)$. Differentiating both sides with respect to y gives $f_y(x,y) = x^2 + g'(y)$, so we must have $x^2 + g'(y) = x^2 \quad \Rightarrow \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = K$, a constant.
Thus $f(x,y) = x^2y + K$, and we can take $K = 0$. All three curves start at $(1,2)$ and end at $(3,2)$, so by Theorem 2, $\int_C \mathbf{F} \cdot d\mathbf{r} = f(3,2) - f(1,2) = 18 - 2 = 16$ for each curve.

12. (a) If $\mathbf{F} = \nabla f$ then $f_x(x,y) = 3 + 2xy^2$ and $f_y(x,y) = 2x^2y$.

$f_x(x,y) = 3 + 2xy^2$ implies $f(x,y) = 3x + x^2y^2 + g(y)$ and $f_y(x,y) = 2x^2y + g'(y)$. But $f_y(x,y) = 2x^2y$ so $g'(y) = 0 \quad\Rightarrow\quad g(y) = K$. We can take $K = 0$, so $f(x,y) = 3x + x^2y^2$.

(b) C is a smooth curve with initial point $(1,1)$ and terminal point $\left(4, \frac{1}{4}\right)$, so by Theorem 2

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f\left(4, \frac{1}{4}\right) - f(1,1) = (12+1) - (3+1) = 9$.

13. (a) If $\mathbf{F} = \nabla f$ then $f_x(x,y) = x^2y^3$ and $f_y(x,y) = x^3y^2$.

$f_x(x,y) = x^2y^3$ implies $f(x,y) = \frac{1}{3}x^3y^3 + g(y)$ and $f_y(x,y) = x^3y^2 + g'(y)$. But $f_y(x,y) = x^3y^2$ so $g'(y) = 0 \quad\Rightarrow$ $g(y) = K$, a constant. We can take $K = 0$, so $f(x,y) = \frac{1}{3}x^3y^3$.

(b) C is a smooth curve with initial point $\mathbf{r}(0) = (0,0)$ and terminal point $\mathbf{r}(1) = (-1,3)$, so by Theorem 2

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(-1,3) - f(0,0) = -9 - 0 = -9$.

14. (a) $f_y(x,y) = x^2e^{xy}$ implies $f(x,y) = xe^{xy} + g(x) \quad\Rightarrow\quad f_x(x,y) = xye^{xy} + e^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$. But $f_x(x,y) = (1+xy)e^{xy}$ so $g'(x) = 0 \quad\Rightarrow\quad g(x) = K$. We can take $K = 0$, so $f(x,y) = xe^{xy}$.

(b) The initial point of C is $\mathbf{r}(0) = (1,0)$ and the terminal point is $\mathbf{r}(\pi/2) = (0,2)$, so

$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0,2) - f(1,0) = 0 - e^0 = -1$.

15. (a) $f_x(x,y,z) = yz$ implies $f(x,y,z) = xyz + g(y,z)$ and so $f_y(x,y,z) = xz + g_y(y,z)$. But $f_y(x,y,z) = xz$ so $g_y(y,z) = 0 \quad\Rightarrow\quad g(y,z) = h(z)$. Thus $f(x,y,z) = xyz + h(z)$ and $f_z(x,y,z) = xy + h'(z)$. But $f_z(x,y,z) = xy + 2z$, so $h'(z) = 2z \quad\Rightarrow\quad h(z) = z^2 + K$. Hence $f(x,y,z) = xyz + z^2$ (taking $K = 0$).

(b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(4,6,3) - f(1,0,-2) = 81 - 4 = 77$.

16. (a) $f_x(x,y,z) = y^2z + 2xz^2$ implies $f(x,y,z) = xy^2z + x^2z^2 + g(y,z)$ and so $f_y(x,y,z) = 2xyz + g_y(y,z)$. But $f_y(x,y,z) = 2xyz$ so $g_y(y,z) = 0 \quad\Rightarrow\quad g(y,z) = h(z)$. Thus $f(x,y,z) = xy^2z + x^2z^2 + h(z)$ and $f_z(x,y,z) = xy^2 + 2x^2z + h'(z)$. But $f_z(x,y,z) = xy^2 + 2x^2z$, so $h'(z) = 0 \quad\Rightarrow\quad h(z) = K$. Hence $f(x,y,z) = xy^2z + x^2z^2$ (taking $K = 0$).

(b) $t = 0$ corresponds to the point $(0,1,0)$ and $t = 1$ corresponds to $(1,2,1)$, so

$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,2,1) - f(0,1,0) = 5 - 0 = 5$.

17. (a) $f_x(x,y,z) = yze^{xz}$ implies $f(x,y,z) = ye^{xz} + g(y,z)$ and so $f_y(x,y,z) = e^{xz} + g_y(y,z)$. But $f_y(x,y,z) = e^{xz}$ so $g_y(y,z) = 0 \quad\Rightarrow\quad g(y,z) = h(z)$. Thus $f(x,y,z) = ye^{xz} + h(z)$ and $f_z(x,y,z) = xye^{xz} + h'(z)$. But $f_z(x,y,z) = xye^{xz}$, so $h'(z) = 0 \quad\Rightarrow\quad h(z) = K$. Hence $f(x,y,z) = ye^{xz}$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 1, -1, 0\rangle$, $\mathbf{r}(2) = \langle 5, 3, 0\rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(5,3,0) - f(1,-1,0) = 3e^0 + e^0 = 4$.

18. (a) $f_x(x, y, z) = \sin y$ implies $f(x, y, z) = x \sin y + g(y, z)$ and so $f_y(x, y, z) = x \cos y + g_y(y, z)$. But $f_y(x, y, z) = x \cos y + \cos z$ so $g_y(y, z) = \cos z \quad \Rightarrow \quad g(y, z) = y \cos z + h(z)$. Thus $f(x, y, z) = x \sin y + y \cos z + h(z)$ and $f_z(x, y, z) = -y \sin z + h'(z)$. But $f_z(x, y, z) = -y \sin z$, so $h'(z) = 0 \quad \Rightarrow$ $h(z) = K$. Hence $f(x, y, z) = x \sin y + y \cos z$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, $\mathbf{r}(\pi/2) = \langle 1, \pi/2, \pi \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, \pi/2, \pi) - f(0, 0, 0) = 1 - \frac{\pi}{2} - 0 = 1 - \frac{\pi}{2}$.

19. The functions $2xe^{-y}$ and $2y - x^2e^{-y}$ have continuous first-order derivatives on $\mathbb{R}^2$ and $\dfrac{\partial}{\partial y}\left(2xe^{-y}\right) = -2xe^{-y} = \dfrac{\partial}{\partial x}\left(2y - x^2e^{-y}\right)$, so $\mathbf{F}(x, y) = 2xe^{-y}\,\mathbf{i} + \left(2y - x^2e^{-y}\right)\mathbf{j}$ is a conservative vector field by Theorem 6 and hence the line integral is independent of path. Thus a potential function f exists, and $f_x(x, y) = 2xe^{-y}$ implies $f(x, y) = x^2e^{-y} + g(y)$ and $f_y(x, y) = -x^2e^{-y} + g'(y)$. But $f_y(x, y) = 2y - x^2e^{-y}$ so $g'(y) = 2y \quad \Rightarrow \quad g(y) = y^2 + K$. We can take $K = 0$, so $f(x, y) = x^2e^{-y} + y^2$. Then $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy = f(2, 1) - f(1, 0) = 4e^{-1} + 1 - 1 = 4/e$.

20. The functions $\sin y$ and $x \cos y - \sin y$ have continuous first-order derivatives on $\mathbb{R}^2$ and $\dfrac{\partial}{\partial y}(\sin y) = \cos y = \dfrac{\partial}{\partial x}(x \cos y - \sin y)$, so $\mathbf{F}(x, y) = \sin y\,\mathbf{i} + (x \cos y - \sin y)\,\mathbf{j}$ is a conservative vector field by Theorem 6 and hence the line integral is independent of path. Thus a potential function f exists, and $f_x(x, y) = \sin y$ implies $f(x, y) = x \sin y + g(y)$ and $f_y(x, y) = x \cos y + g'(y)$. But $f_y(x, y) = x \cos y - \sin y$ so $g'(y) = -\sin y \quad \Rightarrow \quad g(y) = \cos y + K$. We can take $K = 0$, so $f(x, y) = x \sin y + \cos y$. Then $\int_C \sin y\,dx + (x \cos y - \sin y)\,dy = f(1, \pi) - f(2, 0) = -1 - 1 = -2$.

21. If $\mathbf{F}$ is conservative, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. This means that the work done along all piecewise-smooth curves that have the described initial and terminal points is the same. Your reply: It doesn't matter which curve is chosen.

22. The curves C_1 and C_2 connect the same two points but $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Thus $\mathbf{F}$ is not independent of path, and therefore is not conservative.

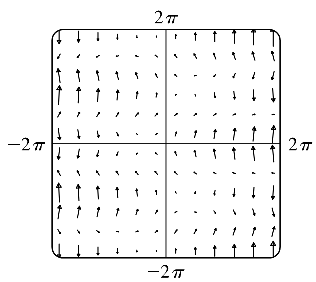
23. $\mathbf{F}(x, y) = x^3\,\mathbf{i} + y^3\,\mathbf{j}$, $W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\partial(x^3)/\partial y = 0 = \partial(y^3)/\partial x$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x(x, y) = x^3 \quad \Rightarrow \quad f(x, y) = \frac{1}{4}x^4 + g(y) \quad \Rightarrow \quad f_y(x, y) = 0 + g'(y)$. But $f_y(x, y) = y^3$ so $g'(y) = y^3 \quad \Rightarrow \quad g(y) = \frac{1}{4}y^4 + K$. We can take $K = 0 \quad \Rightarrow \quad f(x, y) = \frac{1}{4}x^4 + \frac{1}{4}y^4$. Thus $W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 2) - f(1, 0) = (4 + 4) - \left(\frac{1}{4} + 0\right) = \frac{31}{4}$.

24. $\mathbf{F}(x, y) = (2x + y)\,\mathbf{i} + x\,\mathbf{j}$, $W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\partial(2x + y)/\partial y = 1 = \partial(x)/\partial x$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x(x, y) = 2x + y \quad \Rightarrow \quad f(x, y) = x^2 + xy + g(y) \quad \Rightarrow \quad f_y(x, y) = x + g'(y)$. But $f_y(x, y) = x$ so $g'(y) = 0 \quad \Rightarrow \quad g(y) = K$. We can take $K = 0 \quad \Rightarrow \quad f(x, y) = x^2 + xy$. Thus $W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 3) - f(1, 1) = (16 + 12) - (1 + 1) = 26$.

25. We know that if the vector field (call it $\mathbf{F}$) is conservative, then around any closed path C, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. But take C to be a circle centered at the origin, oriented counterclockwise. All of the field vectors that start on C are roughly in the direction of motion along C, so the integral around C will be positive. Therefore the field is not conservative.

26. If a vector field $\mathbf{F}$ is conservative, then around any closed path C, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. For any closed path we draw in the field, it appears that some vectors on the curve point in approximately the same direction as the curve and a similar number point in roughly the opposite direction. (Some appear perpendicular to the curve as well.) Therefore it is plausible that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C which means $\mathbf{F}$ is conservative.

27.



From the graph, it appears that $\mathbf{F}$ is conservative, since around all closed paths, the number and size of the field vectors pointing in directions similar to that of the path seem to be roughly the same as the number and size of the vectors pointing in the opposite direction. To check, we calculate $\dfrac{\partial}{\partial y}(\sin y) = \cos y = \dfrac{\partial}{\partial x}(1 + x\cos y)$. Thus $\mathbf{F}$ is conservative, by Theorem 6.

28. $\nabla f(x,y) = \cos(x - 2y)\,\mathbf{i} - 2\cos(x - 2y)\,\mathbf{j}$

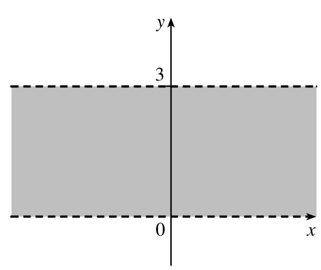
(a) We use Theorem 2: $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C_1 starts at $t = a$ and ends at $t = b$. So because $f(0,0) = \sin 0 = 0$ and $f(\pi,\pi) = \sin(\pi - 2\pi) = 0$, one possible curve C_1 is the straight line from $(0,0)$ to (π,π); that is, $\mathbf{r}(t) = \pi t\,\mathbf{i} + \pi t\,\mathbf{j},\ 0 \le t \le 1$.

(b) From (a), $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So because $f(0,0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}, 0\right) = 1$, one possible curve C_2 is $\mathbf{r}(t) = \frac{\pi}{2} t\,\mathbf{i}$, $0 \le t \le 1$, the straight line from $(0,0)$ to $\left(\frac{\pi}{2}, 0\right)$.

29. Since $\mathbf{F}$ is conservative, there exists a function f such that $\mathbf{F} = \nabla f$, that is, $P = f_x$, $Q = f_y$, and $R = f_z$. Since P, Q, and R have continuous first order partial derivatives, Clairaut's Theorem says that $\partial P/\partial y = f_{xy} = f_{yx} = \partial Q/\partial x$, $\partial P/\partial z = f_{xz} = f_{zx} = \partial R/\partial x$, and $\partial Q/\partial z = f_{yz} = f_{zy} = \partial R/\partial y$.

30. Here $\mathbf{F}(x,y,z) = y\,\mathbf{i} + x\,\mathbf{j} + xyz\,\mathbf{k}$. Then using the notation of Exercise 29, $\partial P/\partial z = 0$ while $\partial R/\partial x = yz$. Since these aren't equal, $\mathbf{F}$ is not conservative. Thus by Theorem 4, the line integral of $\mathbf{F}$ is not independent of path.

31. $D = \{(x,y) \mid 0 < y < 3\}$ consists of those points between, but not on, the horizontal lines $y = 0$ and $y = 3$.

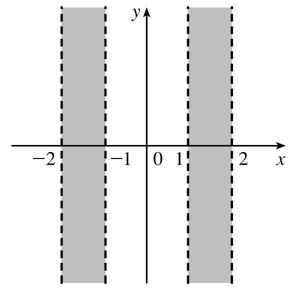


(a) Since D does not include any of its boundary points, it is open. More formally, at any point in D there is a disk centered at that point that lies entirely in D.

(b) Any two points chosen in D can always be joined by a path that lies entirely in D, so D is connected. (D consists of just one "piece.")

(c) D is connected and it has no holes, so it's simply-connected. (Every simple closed curve in D encloses only points that are in D.)

32. $D = \{(x, y) \mid 1 < |x| < 2\}$ consists of those points between, but not on, the vertical lines $x = 1$ and $x = 2$, together with the points between the vertical lines $x = -1$ and $x = -2$.

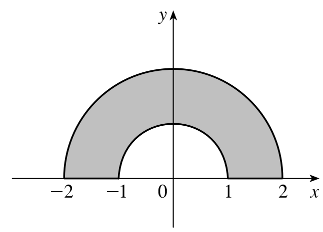


(a) The region does not include any of its boundary points, so it is open.

(b) D consists of two separate pieces, so it is not connected. [For instance, both the points $(-1.5, 0)$ and $(1.5, 0)$ lie in D but they cannot be joined by a path that lies entirely in D.]

(c) Because D is not connected, it's not simply-connected.

33. $D = \left\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4,\ y \geq 0\right\}$ is the semiannular region in the upper half-plane between circles centered at the origin of radii 1 and 2 (including all boundary points).

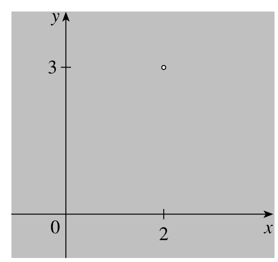


(a) D includes boundary points, so it is not open. [Note that at any boundary point, $(1, 0)$ for instance, any disk centered there cannot lie entirely in D.]

(b) The region consists of one piece, so it's connected.

(c) D is connected and has no holes, so it's simply-connected.

34. $D = \{(x, y) \mid (x, y) \neq (2, 3)\}$ consists of all points in the xy-plane except for $(2, 3)$.



(a) D has only one boundary point, namely $(2, 3)$, which is not included, so the region is open.

(b) D is connected, as it consists of only one piece.

(c) D is not simply-connected, as it has a hole at $(2, 3)$. Thus any simple closed curve that encloses $(2, 3)$ lies in D but includes a point that is not in D.

35. (a) $P = -\dfrac{y}{x^2 + y^2}$, $\dfrac{\partial P}{\partial y} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$ and $Q = \dfrac{x}{x^2 + y^2}$, $\dfrac{\partial Q}{\partial x} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$. Thus $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) C_1: $x = \cos t$, $y = \sin t$, $0 \leq t \leq \pi$, C_2: $x = \cos t$, $y = \sin t$, $t = 2\pi$ to $t = \pi$. Then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \frac{(-\sin t)(-\sin t) + (\cos t)(\cos t)}{\cos^2 t + \sin^2 t}\, dt = \int_0^\pi dt = \pi \text{ and } \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^\pi dt = -\pi$$

Since these aren't equal, the line integral of $\mathbf{F}$ isn't independent of path. (Or notice that $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi$ where C_3 is the circle $x^2 + y^2 = 1$, and apply the contrapositive of Theorem 3.) This doesn't contradict Theorem 6, since the domain of $\mathbf{F}$, which is $\mathbb{R}^2$ except the origin, isn't simply-connected.

36. (a) Here $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Then $f(\mathbf{r}) = -c/|\mathbf{r}|$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$. (See the discussion of gradient fields in Section 16.1.) Hence $\mathbf{F}$ is conservative and its line integral is independent of path. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$.

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = f(P_2) - f(P_1) = -\frac{c}{(x_2^2 + y_2^2 + z_2^2)^{1/2}} + \frac{c}{(x_1^2 + y_1^2 + z_1^2)^{1/2}} = c\left(\frac{1}{d_1} - \frac{1}{d_2}\right).$$

(b) In this case, $c = -(mMG) \quad\Rightarrow$

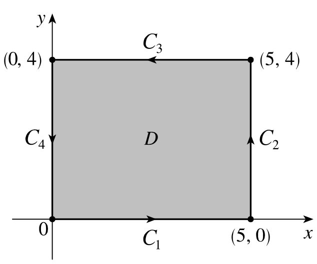
$$\begin{aligned} W &= -mMG\left(\frac{1}{1.52\times 10^{11}} - \frac{1}{1.47\times 10^{11}}\right) \\ &= -(5.97\times 10^{24})(1.99\times 10^{30})(6.67\times 10^{-11})(-2.2377\times 10^{-13}) \approx 1.77\times 10^{32}\text{ J} \end{aligned}$$

(c) In this case, $c = \epsilon q Q \quad\Rightarrow$

$$W = \epsilon qQ\left(\frac{1}{10^{-12}} - \frac{1}{5\times 10^{-13}}\right) = \left(8.985\times 10^9\right)(1)\left(-1.6\times 10^{-19}\right)\left(-10^{12}\right) \approx 1400\text{ J}.$$

16.4 Green's Theorem

1. (a)



C_1: $x = t \;\Rightarrow\; dx = dt,\; y = 0 \;\Rightarrow\; dy = 0\,dt,\; 0 \le t \le 5.$

C_2: $x = 5 \;\Rightarrow\; dx = 0\,dt,\; y = t \;\Rightarrow\; dy = dt,\; 0 \le t \le 4.$

C_3: $x = 5 - t \;\Rightarrow\; dx = -dt,\; y = 4 \;\Rightarrow\; dy = 0\,dt,\; 0 \le t \le 5.$

C_4: $x = 0 \;\Rightarrow\; dx = 0\,dt,\; y = 4 - t \;\Rightarrow\; dy = -dt,\; 0 \le t \le 4$

Thus
$$\begin{aligned} \oint_C y^2\,dx + x^2y\,dy &= \oint_{C_1 + C_2 + C_3 + C_4} y^2\,dx + x^2y\,dy = \int_0^5 0\,dt + \int_0^4 25t\,dt + \int_0^5 (-16 + 0)\,dt + \int_0^4 0\,dt \\ &= 0 + \left[\tfrac{25}{2}t^2\right]_0^4 + [-16t]_0^5 + 0 = 200 + (-80) = 120 \end{aligned}$$

(b) Note that C as given in part (a) is a positively oriented, piecewise-smooth, simple closed curve. Then by Green's Theorem,

$$\begin{aligned} \oint_C y^2\,dx + x^2y\,dy &= \iint_D \left[\tfrac{\partial}{\partial x}(x^2y) - \tfrac{\partial}{\partial y}(y^2)\right] dA = \int_0^5\int_0^4 (2xy - 2y)\,dy\,dx = \int_0^5 \left[xy^2 - y^2\right]_{y=0}^{y=4} dx \\ &= \int_0^5 (16x - 16)\,dx = \left[8x^2 - 16x\right]_0^5 = 200 - 80 = 120 \end{aligned}$$

2. (a) Parametric equations for C are $x = 4\cos t,\;\; y = 4\sin t,\;\; 0 \le t \le 2\pi$. Then $dx = -4\sin t\,dt,\; dy = 4\cos t\,dt$ and

$$\begin{aligned} \oint_C y\,dx - x\,dy &= \int_0^{2\pi} [(4\sin t)(-4\sin t) - (4\cos t)(4\cos t)]\,dt \\ &= -16\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = -16\int_0^{2\pi} 1\,dt = -16(2\pi) = -32\pi \end{aligned}$$

(b) Note that C as given in part (a) is a positively oriented, smooth, simple closed curve. Then by Green's Theorem,

$$\begin{aligned} \oint_C y\,dx - x\,dy &= \iint_D \left[\tfrac{\partial}{\partial x}(-x) - \tfrac{\partial}{\partial y}(y)\right] dA = \iint_D (-1 - 1)\,dA = -2\iint_D dA \\ &= -2(\text{area of } D) = -2\cdot\pi(4)^2 = -32\pi \end{aligned}$$