36. (a) Here $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Then $f(\mathbf{r}) = -c/|\mathbf{r}|$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$. (See the discussion of gradient fields in Section 16.1.) Hence $\mathbf{F}$ is conservative and its line integral is independent of path. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(P_2) - f(P_1) = -\frac{c}{(x_2^2 + y_2^2 + z_2^2)^{1/2}} + \frac{c}{(x_1^2 + y_1^2 + z_1^2)^{1/2}} = c\left(\frac{1}{d_1} - \frac{1}{d_2}\right).$$

(b) In this case, $c = -(mMG) \quad \Rightarrow$

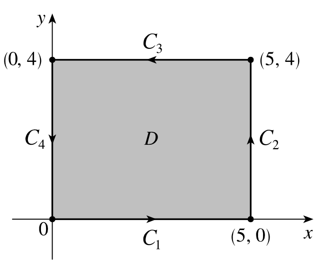
$$W = -mMG\left(\frac{1}{1.52 \times 10^{11}} - \frac{1}{1.47 \times 10^{11}}\right)$$
$$= -(5.97 \times 10^{24})(1.99 \times 10^{30})(6.67 \times 10^{-11})(-2.2377 \times 10^{-13}) \approx 1.77 \times 10^{32} \text{ J}$$

(c) In this case, $c = \epsilon q Q \quad \Rightarrow$

$$W = \epsilon q Q\left(\frac{1}{10^{-12}} - \frac{1}{5 \times 10^{-13}}\right) = \left(8.985 \times 10^9\right)(1)\left(-1.6 \times 10^{-19}\right)\left(-10^{12}\right) \approx 1400 \text{ J}.$$

16.4 Green's Theorem

1. (a)



C_1: $x = t \quad \Rightarrow \quad dx = dt, \;\; y = 0 \quad \Rightarrow \quad dy = 0\,dt, \;\; 0 \le t \le 5.$

C_2: $x = 5 \quad \Rightarrow \quad dx = 0\,dt, \;\; y = t \quad \Rightarrow \quad dy = dt, \;\; 0 \le t \le 4.$

C_3: $x = 5 - t \quad \Rightarrow \quad dx = -dt, \;\; y = 4 \quad \Rightarrow \quad dy = 0\,dt, \;\; 0 \le t \le 5.$

C_4: $x = 0 \quad \Rightarrow \quad dx = 0\,dt, \;\; y = 4 - t \quad \Rightarrow \quad dy = -dt, \;\; 0 \le t \le 4$

Thus

$$\oint_C y^2\,dx + x^2 y\,dy = \oint_{C_1 + C_2 + C_3 + C_4} y^2\,dx + x^2 y\,dy = \int_0^5 0\,dt + \int_0^4 25t\,dt + \int_0^5 (-16 + 0)\,dt + \int_0^4 0\,dt$$
$$= 0 + \left[\tfrac{25}{2}t^2\right]_0^4 + [-16t]_0^5 + 0 = 200 + (-80) = 120$$

(b) Note that C as given in part (a) is a positively oriented, piecewise-smooth, simple closed curve. Then by Green's Theorem,

$$\oint_C y^2\,dx + x^2 y\,dy = \iint_D \left[\tfrac{\partial}{\partial x}\left(x^2 y\right) - \tfrac{\partial}{\partial y}\left(y^2\right)\right] dA = \int_0^5 \int_0^4 (2xy - 2y)\,dy\,dx = \int_0^5 \left[xy^2 - y^2\right]_{y=0}^{y=4} dx$$
$$= \int_0^5 (16x - 16)\,dx = \left[8x^2 - 16x\right]_0^5 = 200 - 80 = 120$$

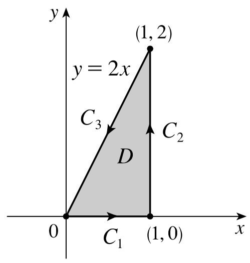
2. (a) Parametric equations for C are $x = 4\cos t, \;\; y = 4\sin t, \;\; 0 \le t \le 2\pi$. Then $dx = -4\sin t\,dt, \;\; dy = 4\cos t\,dt$ and

$$\oint_C y\,dx - x\,dy = \int_0^{2\pi} [(4\sin t)(-4\sin t) - (4\cos t)(4\cos t)]\,dt$$
$$= -16\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = -16\int_0^{2\pi} 1\,dt = -16(2\pi) = -32\pi$$

(b) Note that C as given in part (a) is a positively oriented, smooth, simple closed curve. Then by Green's Theorem,

$$\oint_C y\,dx - x\,dy = \iint_D \left[\tfrac{\partial}{\partial x}(-x) - \tfrac{\partial}{\partial y}(y)\right] dA = \iint_D (-1 - 1)\,dA = -2\iint_D dA$$
$$= -2(\text{area of } D) = -2 \cdot \pi(4)^2 = -32\pi$$

3. (a)



C_1: $x = t \Rightarrow dx = dt$, $y = 0 \Rightarrow dy = 0\,dt$, $0 \le t \le 1$.

C_2: $x = 1 \Rightarrow dx = 0\,dt$, $y = t \Rightarrow dy = dt$, $0 \le t \le 2$.

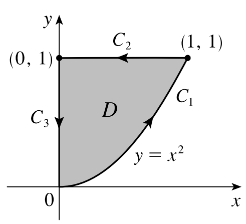
C_3: $x = 1 - t \Rightarrow dx = -dt$, $y = 2 - 2t \Rightarrow dy = -2\,dt$, $0 \le t \le 1$.

Thus

$$\begin{aligned}\oint_C xy\,dx + x^2y^3\,dy &= \oint_{C_1+C_2+C_3} xy\,dx + x^2y^3\,dy \\ &= \int_0^1 0\,dt + \int_0^2 t^3\,dt + \int_0^1 \left[-(1-t)(2-2t) - 2(1-t)^2(2-2t)^3\right] dt \\ &= 0 + \left[\tfrac{1}{4}t^4\right]_0^2 + \int_0^1 \left[-2(1-t)^2 - 16(1-t)^5\right] dt \\ &= 4 + \left[\tfrac{2}{3}(1-t)^3 + \tfrac{8}{3}(1-t)^6\right]_0^1 = 4 + 0 - \tfrac{10}{3} = \tfrac{2}{3}\end{aligned}$$

(b) $$\begin{aligned}\oint_C xy\,dx + x^2y^3\,dy &= \iint_D \left[\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx \\ &= \int_0^1 \left[\tfrac{1}{2}xy^4 - xy\right]_{y=0}^{y=2x} dx = \int_0^1 (8x^5 - 2x^2)\,dx = \tfrac{4}{3} - \tfrac{2}{3} = \tfrac{2}{3}\end{aligned}$$

4. (a)



C_1: $x = t \Rightarrow dx = dt$, $y = t^2 \Rightarrow dy = 2t\,dt$, $0 \le t \le 1$

C_2: $x = 1 - t \Rightarrow dx = -dt$, $y = 1 \Rightarrow dy = 0\,dt$, $0 \le t \le 1$

C_3: $x = 0 \Rightarrow dx = 0\,dt$, $y = 1 - t \Rightarrow dy = -dt$, $0 \le t \le 1$

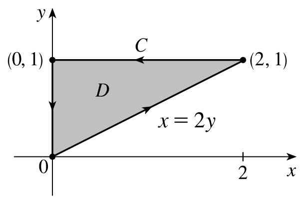
Thus

$$\begin{aligned}\oint_C x^2y^2\,dx + xy\,dy &= \oint_{C_1+C_2+C_3} x^2y^2\,dx + xy\,dy \\ &= \int_0^1 \left[t^2(t^2)^2\,dt + t(t^2)(2t\,dt)\right] + \int_0^1 \left[(1-t)^2(1)^2(-dt) + (1-t)(1)(0\,dt)\right] \\ &\qquad + \int_0^1 \left[(0)^2(1-t)^2(0\,dt) + (0)(1-t)(-dt)\right] \\ &= \int_0^1 \left(t^6 + 2t^4\right) dt + \int_0^1 \left(-1 + 2t - t^2\right) dt + \int_0^1 0\,dt \\ &= \left[\tfrac{1}{7}t^7 + \tfrac{2}{5}t^5\right]_0^1 + \left[-t + t^2 - \tfrac{1}{3}t^3\right]_0^1 + 0 = \left(\tfrac{1}{7} + \tfrac{2}{5}\right) + \left(-1 + 1 - \tfrac{1}{3}\right) = \tfrac{22}{105}\end{aligned}$$

(b) $$\begin{aligned}\oint_C x^2y^2\,dx + xy\,dy &= \iint_D \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2y^2)\right] dA = \int_0^1 \int_{x^2}^1 (y - 2x^2y)\,dy\,dx \\ &= \int_0^1 \left[\tfrac{1}{2}y^2 - x^2y^2\right]_{y=x^2}^{y=1} dx = \int_0^1 \left(\tfrac{1}{2} - x^2 - \tfrac{1}{2}x^4 + x^6\right) dx \\ &= \left[\tfrac{1}{2}x - \tfrac{1}{3}x^3 - \tfrac{1}{10}x^5 + \tfrac{1}{7}x^7\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{10} + \tfrac{1}{7} = \tfrac{22}{105}\end{aligned}$$

5. The region D enclosed by C is $[0, 3] \times [0, 4]$, so

$$\begin{aligned}\int_C ye^x\,dx + 2e^x\,dy &= \iint_D \left[\frac{\partial}{\partial x}(2e^x) - \frac{\partial}{\partial y}(ye^x)\right] dA = \int_0^3 \int_0^4 (2e^x - e^x)\,dy\,dx \\ &= \int_0^3 e^x\,dx \int_0^4 dy = \left[e^x\right]_0^3 \left[y\right]_0^4 = (e^3 - e^0)(4 - 0) = 4(e^3 - 1)\end{aligned}$$

6. 

The region D enclosed by C is given by $\{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le 2y\}$, so

$$\begin{aligned}\int_C (x^2+y^2)\,dx + (x^2-y^2)\,dy &= \iint_D \left[\frac{\partial}{\partial x}(x^2-y^2) - \frac{\partial}{\partial y}(x^2+y^2)\right] dA \\ &= \int_0^1 \int_0^{2y} (2x-2y)\,dx\,dy \\ &= \int_0^1 \left[x^2-2xy\right]_{x=0}^{x=2y} dy \\ &= \int_0^1 (4y^2-4y^2)\,dy = \int_0^1 0\,dy = 0\end{aligned}$$

7. $$\begin{aligned}\int_C \left(y+e^{\sqrt{x}}\right) dx + (2x+\cos y^2)\,dy &= \iint_D \left[\frac{\partial}{\partial x}(2x+\cos y^2) - \frac{\partial}{\partial y}\left(y+e^{\sqrt{x}}\right)\right] dA \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} (2-1)\,dy\,dx = \int_0^1 (\sqrt{x}-x^2)\,dx = \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}\end{aligned}$$

8. $$\begin{aligned}\int_C y^4\,dx + 2xy^3\,dy &= \iint_D \left[\frac{\partial}{\partial x}(2xy^3) - \frac{\partial}{\partial y}(y^4)\right] dA = \iint_D (2y^3-4y^3)\,dA \\ &= -2\iint_D y^3\,dA = 0\end{aligned}$$

because $f(x,y) = y^3$ is an odd function with respect to y and D is symmetric about the x-axis.

9. $$\begin{aligned}\int_C y^3\,dx - x^3\,dy &= \iint_D \left[\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3)\right] dA = \iint_D (-3x^2-3y^2)\,dA = \int_0^{2\pi}\int_0^2 (-3r^2)\,r\,dr\,d\theta \\ &= -3\int_0^{2\pi} d\theta \int_0^2 r^3\,dr = -3\big[\theta\big]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^2 = -3(2\pi)(4) = -24\pi\end{aligned}$$

10. $$\begin{aligned}\int_C (1-y^3)\,dx + (x^3+e^{y^2})\,dy &= \iint_D \left[\frac{\partial}{\partial x}(x^3+e^{y^2}) - \frac{\partial}{\partial y}(1-y^3)\right] dA = \iint_D (3x^2+3y^2)\,dA \\ &= \int_0^{2\pi}\int_2^3 (3r^2)\,r\,dr\,d\theta = 3\int_0^{2\pi} d\theta \int_2^3 r^3\,dr \\ &= 3\big[\theta\big]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_2^3 = 3(2\pi)\cdot\tfrac{1}{4}(81-16) = \tfrac{195}{2}\pi\end{aligned}$$

11. $\mathbf{F}(x,y) = \langle y\cos x - xy\sin x, xy + x\cos x\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 4-2x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C} (y\cos x - xy\sin x)\,dx + (xy + x\cos x)\,dy = -\iint_D \left[\frac{\partial}{\partial x}(xy+x\cos x) - \frac{\partial}{\partial y}(y\cos x - xy\sin x)\right] dA \\ &= -\iint_D (y - x\sin x + \cos x - \cos x + x\sin x)\,dA = -\int_0^2\int_0^{4-2x} y\,dy\,dx \\ &= -\int_0^2 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x} dx = -\int_0^2 \tfrac{1}{2}(4-2x)^2\,dx = -\int_0^2 (8-8x+2x^2)\,dx = -\left[8x-4x^2+\tfrac{2}{3}x^3\right]_0^2 \\ &= -\left(16-16+\tfrac{16}{3}-0\right) = -\tfrac{16}{3}\end{aligned}$$

12. $\mathbf{F}(x,y) = \langle e^{-x}+y^2, e^{-y}+x^2\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid -\pi/2 \le x \le \pi/2, 0 \le y \le \cos x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C} \left(e^{-x}+y^2\right) dx + \left(e^{-y}+x^2\right) dy = -\iint_D \left[\frac{\partial}{\partial x}\left(e^{-y}+x^2\right) - \frac{\partial}{\partial y}\left(e^{-x}+y^2\right)\right] dA \\ &= -\int_{-\pi/2}^{\pi/2}\int_0^{\cos x} (2x-2y)\,dy\,dx = -\int_{-\pi/2}^{\pi/2} \left[2xy-y^2\right]_{y=0}^{y=\cos x} dx \\ &= -\int_{-\pi/2}^{\pi/2} (2x\cos x - \cos^2 x)\,dx = -\int_{-\pi/2}^{\pi/2} \left[2x\cos x - \tfrac{1}{2}(1+\cos 2x)\right] dx \\ &= -\left[2x\sin x + 2\cos x - \tfrac{1}{2}\left(x + \tfrac{1}{2}\sin 2x\right)\right]_{-\pi/2}^{\pi/2} \qquad \text{[integrate by parts in the first term]} \\ &= -\left(\pi - \tfrac{1}{4}\pi - \pi - \tfrac{1}{4}\pi\right) = \tfrac{1}{2}\pi\end{aligned}$$

13. $\mathbf{F}(x,y) = \langle y - \cos y, x\sin y\rangle$ and the region D enclosed by C is the disk with radius 2 centered at $(3,-4)$.

C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{-C}(y-\cos y)\,dx + (x\sin y)\,dy = -\iint_D \left[\tfrac{\partial}{\partial x}\,(x\sin y) - \tfrac{\partial}{\partial y}\,(y-\cos y)\right] dA$$
$$= -\iint_D(\sin y - 1 - \sin y)\,dA = \iint_D\,dA = \text{area of } D = \pi(2)^2 = 4\pi$$

14. $\mathbf{F}(x,y) = \left\langle\sqrt{x^2+1}, \tan^{-1}x\right\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0\le x\le 1, x\le y\le 1\}$.

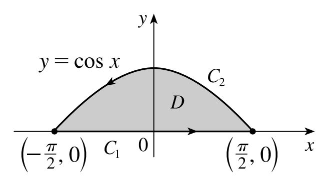
C is oriented positively, so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \sqrt{x^2+1}\,dx + \tan^{-1}x\,dy = \iint_D \left[\frac{\partial}{\partial x}\left(\tan^{-1}x\right) - \frac{\partial}{\partial y}\left(\sqrt{x^2+1}\right)\right] dA$$
$$= \int_0^1\int_x^1 \left(\frac{1}{1+x^2} - 0\right) dy\,dx = \int_0^1 \frac{1}{1+x^2}\,\left[\,y\,\right]_{y=x}^{y=1}\,dx = \int_0^1 \frac{1}{1+x^2}\,(1-x)\,dx$$
$$= \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right) dx = \left[\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

15. Here $C = C_1 + C_2$ where

C_1 can be parametrized as $x = t,\;\; y = 0,\;\; -\pi/2 \le t \le \pi/2$, and

C_2 is given by $x = -t,\;\; y = \cos t,\;\; -\pi/2\le t\le \pi/2$.



Then the line integral is

$$\oint_{C_1+C_2} x^3y^4\,dx + x^5y^4\,dy = \int_{-\pi/2}^{\pi/2}(0+0)\,dt + \int_{-\pi/2}^{\pi/2}[(-t)^3(\cos t)^4(-1) + (-t)^5(\cos t)^4(-\sin t)]\,dt$$
$$= 0 + \int_{-\pi/2}^{\pi/2}(t^3\cos^4 t + t^5\cos^4 t\sin t)\,dt = \tfrac{1}{15}\pi^4 - \tfrac{4144}{1125}\pi^2 + \tfrac{7{,}578{,}368}{253{,}125} \approx 0.0779$$

according to a CAS. The double integral is

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x}(5x^4y^4 - 4x^3y^3)\,dy\,dx = \tfrac{1}{15}\pi^4 - \tfrac{4144}{1125}\pi^2 + \tfrac{7{,}578{,}368}{253{,}125} \approx 0.0779,$$ verifying Green's

Theorem in this case.

16. We can parametrize C as $x = \cos\theta,\;\; y = 2\sin\theta,\;\; 0\le\theta\le 2\pi$. Then the line integral is

$$\oint_C P\,dx + Q\,dy = \int_0^{2\pi}\left[2\cos\theta - (\cos\theta)^3(2\sin\theta)^5\right](-\sin\theta)\,d\theta + \int_0^{2\pi}(\cos\theta)^3(2\sin\theta)^8\cdot 2\cos\theta\,d\theta$$
$$= \int_0^{2\pi}(-2\cos\theta\sin\theta + 32\cos^3\theta\sin^6\theta + 512\cos^4\theta\sin^8\theta)\,d\theta = 7\pi,$$

according to a CAS. The double integral is $\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{-1}^{1}\int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}}(3x^2y^8 + 5x^3y^4)\,dy\,dx = 7\pi.$

17. By Green's Theorem, $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C x(x+y)\,dx + xy^2\,dy = \iint_D(y^2 - x)\,dA$ where C is the path described in the question and D is the triangle bounded by C. So

$$W = \int_0^1\int_0^{1-x}(y^2-x)\,dy\,dx = \int_0^1\left[\tfrac{1}{3}y^3 - xy\right]_{y=0}^{y=1-x}dx = \int_0^1\left(\tfrac{1}{3}(1-x)^3 - x(1-x)\right)dx$$
$$= \left[-\tfrac{1}{12}(1-x)^4 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right]_0^1 = \left(-\tfrac{1}{2} + \tfrac{1}{3}\right) - \left(-\tfrac{1}{12}\right) = -\tfrac{1}{12}$$

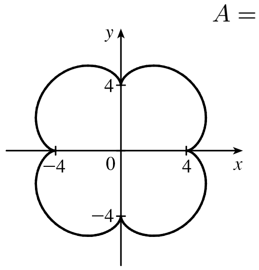
18. By Green's Theorem, $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \sin x\,dx + \left(\sin y + xy^2 + \frac{1}{3}x^3\right) dy = \iint_D (y^2 + x^2 - 0)\,dA$, where D is the region (a quarter-disk) bounded by C. Converting to polar coordinates, we have

$W = \int_0^{\pi/2} \int_0^5 r^2 \cdot r\,dr\,d\theta = \left[\,\theta\,\right]_0^{\pi/2} \left[\frac{1}{4}r^4\right]_0^5 = \frac{1}{2}\pi\left(\frac{625}{4}\right) = \frac{625}{8}\pi.$

19. Let C_1 be the arch of the cycloid from $(0,0)$ to $(2\pi,0)$, which corresponds to $0 \le t \le 2\pi$, and let C_2 be the segment from $(2\pi,0)$ to $(0,0)$, so C_2 is given by $x = 2\pi - t$, $y = 0$, $0 \le t \le 2\pi$. Then $C = C_1 \cup C_2$ is traversed clockwise, so $-C$ is oriented positively. Thus $-C$ encloses the area under one arch of the cycloid and from (5) we have

$$\begin{aligned} A = -\oint_{-C} y\,dx &= \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt + \int_0^{2\pi} 0\,(-dt) \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt + 0 = \left[t - 2\sin t + \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi \end{aligned}$$

20.



$$\begin{aligned} A = \oint_C x\,dy &= \int_0^{2\pi} (5\cos t - \cos 5t)(5\cos t - 5\cos 5t)\,dt \\ &= \int_0^{2\pi} (25\cos^2 t - 30\cos t\cos 5t + 5\cos^2 5t)\,dt \\ &= \left[25\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right) - 30\left(\tfrac{1}{8}\sin 4t + \tfrac{1}{12}\sin 6t\right) + 5\left(\tfrac{1}{2}t + \tfrac{1}{20}\sin 10t\right)\right]_0^{2\pi} \\ &\qquad [\text{Use Formula 80 in the Table of Integrals}] \\ &= 30\pi \end{aligned}$$

21. (a) Using Equation 16.2.8, we write parametric equations of the line segment as $x = (1-t)x_1 + tx_2$, $y = (1-t)y_1 + ty_2$, $0 \le t \le 1$. Then $dx = (x_2 - x_1)\,dt$ and $dy = (y_2 - y_1)\,dt$, so

$$\begin{aligned} \int_C x\,dy - y\,dx &= \int_0^1 [(1-t)x_1 + tx_2](y_2 - y_1)\,dt + [(1-t)y_1 + ty_2](x_2 - x_1)\,dt \\ &= \int_0^1 (x_1(y_2 - y_1) - y_1(x_2 - x_1) + t[(y_2 - y_1)(x_2 - x_1) - (x_2 - x_1)(y_2 - y_1)])\,dt \\ &= \int_0^1 (x_1 y_2 - x_2 y_1)\,dt = x_1 y_2 - x_2 y_1 \end{aligned}$$

(b) We apply Green's Theorem to the path $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i is the line segment that joins (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i = 1, 2, \ldots, n-1$, and C_n is the line segment that joins (x_n, y_n) to (x_1, y_1). From (5), $\frac{1}{2}\int_C x\,dy - y\,dx = \iint_D dA$, where D is the polygon bounded by C. Therefore

$$\begin{aligned} \text{area of polygon} = A(D) &= \iint_D dA = \tfrac{1}{2}\int_C x\,dy - y\,dx \\ &= \tfrac{1}{2}\left(\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \cdots + \int_{C_{n-1}} x\,dy - y\,dx + \int_{C_n} x\,dy - y\,dx\right) \end{aligned}$$

To evaluate these integrals we use the formula from (a) to get

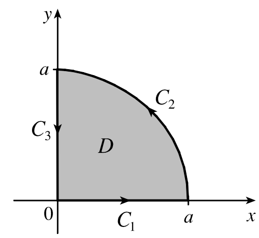
$A(D) = \frac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)].$

(c) $A = \frac{1}{2}[(0 \cdot 1 - 2 \cdot 0) + (2 \cdot 3 - 1 \cdot 1) + (1 \cdot 2 - 0 \cdot 3) + (0 \cdot 1 - (-1) \cdot 2) + (-1 \cdot 0 - 0 \cdot 1)]$

$= \frac{1}{2}(0 + 5 + 2 + 2) = \frac{9}{2}$

22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dA = \frac{1}{A}\iint_D x\,dA = \overline{x}$ and $-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_D (-2y)\,dA = \frac{1}{A}\iint_D y\,dA = \overline{y}$.

23. We orient the quarter-circular region as shown in the figure.



$A = \frac{1}{4}\pi a^2$ so $\overline{x} = \dfrac{1}{\pi a^2/2}\oint_C x^2\,dy$ and $\overline{y} = -\dfrac{1}{\pi a^2/2}\oint_C y^2\,dx$.

Here $C = C_1 + C_2 + C_3$ where C_1: $x = t$, $y = 0$, $0 \le t \le a$;

C_2: $x = a\cos t$, $y = a\sin t$, $0 \le t \le \frac{\pi}{2}$; and

C_3: $x = 0$, $y = a - t$, $0 \le t \le a$. Then

$$\begin{aligned}\oint_C x^2\,dy &= \int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy = \int_0^a 0\,dt + \int_0^{\pi/2}(a\cos t)^2(a\cos t)\,dt + \int_0^a 0\,dt \\ &= \int_0^{\pi/2} a^3\cos^3 t\,dt = a^3\int_0^{\pi/2}(1-\sin^2 t)\cos t\,dt = a^3\left[\sin t - \tfrac{1}{3}\sin^3 t\right]_0^{\pi/2} = \tfrac{2}{3}a^3\end{aligned}$$

so $\overline{x} = \dfrac{1}{\pi a^2/2}\oint_C x^2\,dy = \dfrac{4a}{3\pi}$.

$$\begin{aligned}\oint_C y^2\,dx &= \int_{C_1} y^2\,dx + \int_{C_2} y^2\,dx + \int_{C_3} y^2\,dx = \int_0^a 0\,dt + \int_0^{\pi/2}(a\sin t)^2(-a\sin t)\,dt + \int_0^a 0\,dt \\ &= \int_0^{\pi/2}(-a^3\sin^3 t)\,dt = -a^3\int_0^{\pi/2}(1-\cos^2 t)\sin t\,dt = -a^3\left[\tfrac{1}{3}\cos^3 t - \cos t\right]_0^{\pi/2} = -\tfrac{2}{3}a^3,\end{aligned}$$

so $\overline{y} = -\dfrac{1}{\pi a^2/2}\oint_C y^2\,dx = \dfrac{4a}{3\pi}$. Thus $(\overline{x}, \overline{y}) = \left(\dfrac{4a}{3\pi}, \dfrac{4a}{3\pi}\right)$.

24. Here $A = \frac{1}{2}ab$ and $C = C_1 + C_2 + C_3$, where C_1: $x = x$, $y = 0$, $0 \le x \le a$;

C_2: $x = a$, $y = y$, $0 \le y \le b$; and C_3: $x = x$, $y = \frac{b}{a}x$, $x = a$ to $x = 0$. Then

$$\begin{aligned}\oint_C x^2\,dy &= \int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy = 0 + \int_0^b a^2\,dy + \int_a^0 (x^2)\left(\tfrac{b}{a}\,dx\right) \\ &= a^2 b + \tfrac{b}{a}\left[\tfrac{1}{3}x^3\right]_a^0 = a^2 b - \tfrac{1}{3}a^2 b = \tfrac{2}{3}a^2 b.\end{aligned}$$

Similarly, $\oint_C y^2\,dx = \int_{C_1} y^2\,dx + \int_{C_2} y^2\,dx + \int_{C_3} y^2\,dx = 0 + 0 + \int_a^0 \left(\frac{b}{a}x\right)^2\,dx = \frac{b^2}{a^2}\cdot\frac{1}{3}x^3\Big]_a^0 = -\frac{1}{3}ab^2$. Thus

$\overline{x} = \frac{1}{2A}\oint_C x^2\,dy = \frac{1}{ab}\cdot\frac{2}{3}a^2 b = \frac{2}{3}a$ and $\overline{y} = -\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{ab}\left(-\frac{1}{3}ab^2\right) = \frac{1}{3}b$, so $(\overline{x}, \overline{y}) = \left(\frac{2}{3}a, \frac{1}{3}b\right)$.

25. By Green's Theorem, $-\frac{1}{3}\rho\oint_C y^3\,dx = -\frac{1}{3}\rho\iint_D(-3y^2)\,dA = \iint_D y^2\rho\,dA = I_x$ and

$\frac{1}{3}\rho\oint_C x^3\,dy = \frac{1}{3}\rho\iint_D(3x^2)\,dA = \iint_D x^2\rho\,dA = I_y$.

26. By symmetry the moments of inertia about any two diameters are equal. Centering the disk at the origin, the moment of inertia about a diameter equals

$$I_y = \tfrac{1}{3}\rho\oint_C x^3\,dy = \tfrac{1}{3}\rho\int_0^{2\pi}(a^4\cos^4 t)\,dt = \tfrac{1}{3}a^4\rho\int_0^{2\pi}\left[\tfrac{3}{8} + \tfrac{1}{2}\cos 2t + \tfrac{1}{8}\cos 4t\right]dt = \tfrac{1}{3}a^4\rho\cdot\tfrac{3(2\pi)}{8} = \tfrac{1}{4}\pi a^4\rho$$

27. As in Example 5, let C' be a counterclockwise-oriented circle with center the origin and radius a, where a is chosen to be small enough so that C' lies inside C, and D the region bounded by C and C'. Here

$P = \dfrac{2xy}{(x^2+y^2)^2} \quad\Rightarrow\quad \dfrac{\partial P}{\partial y} = \dfrac{2x(x^2+y^2)^2 - 2xy\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4} = \dfrac{2x^3 - 6xy^2}{(x^2+y^2)^3}$ and

$Q = \dfrac{y^2-x^2}{(x^2+y^2)^2} \quad\Rightarrow\quad \dfrac{\partial Q}{\partial x} = \dfrac{-2x(x^2+y^2)^2 - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} = \dfrac{2x^3 - 6xy^2}{(x^2+y^2)^3}$. Thus, as in the example,

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 0\,dA = 0$$

and $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C'} \mathbf{F}\cdot d\mathbf{r}$. We parametrize C' as $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$. Then

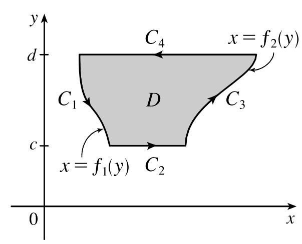
$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C'} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \frac{2\,(a\cos t)\,(a\sin t)\,\mathbf{i} + \left(a^2\sin^2 t - a^2\cos^2 t\right)\mathbf{j}}{\left(a^2\cos^2 t + a^2\sin^2 t\right)^2} \cdot \Big(-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}\Big)\,dt \\
&= \frac{1}{a}\int_0^{2\pi} \left(-\cos t\sin^2 t - \cos^3 t\right) dt = \frac{1}{a}\int_0^{2\pi} \left(-\cos t\sin^2 t - \cos t\left(1-\sin^2 t\right)\right) dt \\
&= -\frac{1}{a}\int_0^{2\pi} \cos t\,dt = -\frac{1}{a}\sin t\Big]_0^{2\pi} = 0
\end{aligned}$$

28. P and Q have continuous partial derivatives on $\mathbb{R}^2$, so by Green's Theorem we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (3-1)\,dA = 2\iint_D dA = 2\cdot A(D) = 2\cdot 6 = 12$$

29. Since C is a simple closed path which doesn't pass through or enclose the origin, there exists an open region that doesn't contain the origin but does contain D. Thus $P = -y/(x^2+y^2)$ and $Q = x/(x^2+y^2)$ have continuous partial derivatives on this open region containing D and we can apply Green's Theorem. But by Exercise 16.3.35(a), $\partial P/\partial y = \partial Q/\partial x$, so $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D 0\,dA = 0$.

30. We express D as a type II region: $D = \{(x,y) \mid f_1(y) \le x \le f_2(y),\ c \le y \le d\}$ where f_1 and f_2 are continuous functions. Then $\displaystyle\iint_D \frac{\partial Q}{\partial x}\,dA = \int_c^d \int_{f_1(y)}^{f_2(y)} \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d \left[Q(f_2(y),y) - Q(f_1(y),y)\right] dy$ by the Fundamental Theorem of Calculus. But referring to the figure, $\displaystyle\oint_C Q\,dy = \oint_{C_1+C_2+C_3+C_4} Q\,dy$.



Then $\int_{C_1} Q\,dy = \int_d^c Q(f_1(y),y)\,dy$, $\int_{C_2} Q\,dy = \int_{C_4} Q\,dy = 0$,

and $\int_{C_3} Q\,dy = \int_c^d Q(f_2(y),y)\,dy$. Hence

$\oint_C Q\,dy = \int_c^d \left[Q(f_2(y),y) - Q(f_1(y),y)\right] dy = \iint_D (\partial Q/\partial x)\,dA$.

31. Using the first part of (5), we have that $\iint_R dx\,dy = A(R) = \int_{\partial R} x\,dy$. But $x = g(u,v)$, and $dy = \dfrac{\partial h}{\partial u}\,du + \dfrac{\partial h}{\partial v}\,dv$, and we orient ∂S by taking the positive direction to be that which corresponds, under the mapping, to the positive direction along ∂R, so

$$\begin{aligned}
\int_{\partial R} x\,dy = \int_{\partial S} g(u,v)\left(\frac{\partial h}{\partial u}\,du + \frac{\partial h}{\partial v}\,dv\right) &= \int_{\partial S} g(u,v)\,\frac{\partial h}{\partial u}\,du + g(u,v)\,\frac{\partial h}{\partial v}\,dv \\
&= \pm\iint_S \left[\tfrac{\partial}{\partial u}\left(g(u,v)\,\tfrac{\partial h}{\partial v}\right) - \tfrac{\partial}{\partial v}\left(g(u,v)\,\tfrac{\partial h}{\partial u}\right)\right] dA \qquad \text{[using Green's Theorem in the } uv\text{-plane]} \\
&= \pm\iint_S \left(\tfrac{\partial g}{\partial u}\tfrac{\partial h}{\partial v} + g(u,v)\,\tfrac{\partial^2 h}{\partial u\,\partial v} - \tfrac{\partial g}{\partial v}\tfrac{\partial h}{\partial u} - g(u,v)\,\tfrac{\partial^2 h}{\partial v\,\partial u}\right) dA \qquad \text{[using the Chain Rule]} \\
&= \pm\iint_S \left(\tfrac{\partial x}{\partial u}\tfrac{\partial y}{\partial v} - \tfrac{\partial x}{\partial v}\tfrac{\partial y}{\partial u}\right) dA \quad \text{[by the equality of mixed partials]} \quad = \pm\iint_S \tfrac{\partial(x,y)}{\partial(u,v)}\,du\,dv
\end{aligned}$$

The sign is chosen to be positive if the orientation that we gave to ∂S corresponds to the usual positive orientation, and it is negative otherwise. In either case, since $A(R)$ is positive, the sign chosen must be the same as the sign of $\dfrac{\partial\,(x,y)}{\partial(u,v)}$.

Therefore $A(R) = \iint_R dx\,dy = \iint_S \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$.

16.5 Curl and Divergence

1. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(x^2y^2z) - \frac{\partial}{\partial z}(x^2yz^2)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(x^2y^2z) - \frac{\partial}{\partial z}(xy^2z^2)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(x^2yz^2) - \frac{\partial}{\partial y}(xy^2z^2)\right]\mathbf{k}$$

$$= (2x^2yz - 2x^2yz)\,\mathbf{i} - (2xy^2z - 2xy^2z)\,\mathbf{j} + (2xyz^2 - 2xyz^2)\,\mathbf{k} = \mathbf{0}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}\,(xy^2z^2) + \dfrac{\partial}{\partial y}\,(x^2yz^2) + \dfrac{\partial}{\partial z}\,(x^2y^2z) = y^2z^2 + x^2z^2 + x^2y^2$

2. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x^3yz^2 & y^4z^3 \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(y^4z^3) - \frac{\partial}{\partial z}(x^3yz^2)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(y^4z^3) - \frac{\partial}{\partial z}(0)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(x^3yz^2) - \frac{\partial}{\partial y}(0)\right]\mathbf{k}$$

$$= (4y^3z^3 - 2x^3yz)\,\mathbf{i} - (0-0)\,\mathbf{j} + (3x^2yz^2 - 0)\,\mathbf{k} = (4y^3z^3 - 2x^3yz)\,\mathbf{i} + 3x^2yz^2\,\mathbf{k}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}\,(0) + \dfrac{\partial}{\partial y}\,(x^3yz^2) + \dfrac{\partial}{\partial z}\,(y^4z^3) = 0 + x^3z^2 + 3y^4z^2 = x^3z^2 + 3y^4z^2$

3. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xye^z & 0 & yze^x \end{vmatrix} = (ze^x - 0)\,\mathbf{i} - (yze^x - xye^z)\,\mathbf{j} + (0 - xe^z)\,\mathbf{k}$

$$= ze^x\,\mathbf{i} + (xye^z - yze^x)\,\mathbf{j} - xe^z\,\mathbf{k}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}\,(xye^z) + \dfrac{\partial}{\partial y}\,(0) + \dfrac{\partial}{\partial z}\,(yze^x) = ye^z + 0 + ye^x = y(e^z + e^x)$

4. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin yz & \sin zx & \sin xy \end{vmatrix}$

$$= (x\cos xy - x\cos zx)\,\mathbf{i} - (y\cos xy - y\cos yz)\,\mathbf{j} + (z\cos zx - z\cos yz)\,\mathbf{k}$$

$$= x(\cos xy - \cos zx)\,\mathbf{i} + y(\cos yz - \cos xy)\,\mathbf{j} + z(\cos zx - \cos yz)\,\mathbf{k}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}\,(\sin yz) + \dfrac{\partial}{\partial y}\,(\sin zx) + \dfrac{\partial}{\partial z}\,(\sin xy) = 0 + 0 + 0 = 0$