# Solutions to the First Midterm Exam – Multivariable Calculus

Math 53, October 1, 2009. Instructor: E. Frenkel

1. Consider the curve in $\mathbb{R}^2$ defined by the parametric equations

$$x = 1 - \sin^2 t, \qquad y = 2 + \cos t, \qquad 0 \leq t \leq \pi.$$

(a) Find the Cartesian equation of this curve and sketch the curve.

We have $1 - x = \sin^2 t$ and $y - 2 = \cos t$. Hence $(1-x) + (y-2)^2 = 1$, and so $x = (y-2)^2$. The set of solutions is a parabola, but since $0 \leq t \leq \pi$, we only get the part of this parabola with $1 \leq y \leq 3$.

(b) Find the points $(x, y)$ on this curve at which the tangent line to the curve has angle $\pm 45°$ with the $x$ axis.

Use the parametrization $x = (s-2)^2, y = s$. Then $dx/ds = 2(s-2), dy/ds = 1$. Hence the slope is $1/2(s-2)$. The angle with the $y$ axis is $45° = \pi/2$ if the slope is 1 or $-1$, which corresponds to the points with $s = 3/2$ and $s = 5/2$, that is $(1/4, 3/2)$ and $(1/4, 5/2)$.

You may also use the original parametrization.

2. (a) Sketch the region $D$ consisting of those points on the plane which are inside the curve $r = 1 + \sin\theta$ and outside the curve $r = 1$ (written in polar coordinates).

The picture should include the cardioid and the circle.

(b) Find the area of the region $D$.

$Area(D) = Area(D_1) - Area(D_2)$, where $D_1$ is the inside of the cardioid with $0 \leq \theta \leq \pi$, and $D_2$ is the inside of the circle with $0 \leq \theta \leq \pi$. Hence

$$\begin{aligned} Area(D) &= \int_0^\pi \frac{1}{2}[(1+\sin\theta)^2 - 1]d\theta = \int_0^\pi \sin\theta d\theta + \frac{1}{2}\int_0^\pi \sin^2\theta d\theta \\ &= 2 + \frac{\pi}{4}. \end{aligned}$$

3. (a) Find parametric equations for the line of intersection of the planes $2x + 5z + 2 = 0$ and $-x - 3y + z + 2 = 0$.

The direction vector of this line may be found as the cross-product of the normal vectors to the two planes: $\mathbf{n}_1 = \langle 2, 0, 5\rangle$ and $\mathbf{n}_2 = \langle -1, -3, 1\rangle$, which is $\langle 15, -7, -6\rangle$. A point on this line is found by solving the equations for the two planes simultaneously. For example, we can set $z = 0$, and then we have $x = -1$ from the first equation, and $y = 1$ from the second equation. Thus, we obtain the following parametric equations for the line of intersection: $x = -1 + 15t, y = 1 - 7t, z = -6t$.

(b) Find the cosine of the angle between these two planes.

$$\cos\theta = \frac{\mathbf{n}_1 \times \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{3}{\sqrt{29}\sqrt{11}}.$$

4. Consider the space curve defined by the parametric equations

$$x = t^2 - 1, \qquad y = t^3 + 6t, \qquad z = t^3 + t^2 + t.$$

(a) Find the points of intersection of this curve and the $yz$ plane.

Set $x = 0$. Then $t^2 - 1 = 0$ and we find $t = \pm 1$. The corresponding points are $(0, 7, 3)$ and $(0, -7, -1)$.

(b) Find the points on the curve at which the tangent line is orthogonal to the vector $6\mathbf{i}+\mathbf{j}-6\mathbf{k}$.

We find the components of a tangent vector by differentiating the functions: $2t\mathbf{i} + (3t^2+6)\mathbf{j} + (3t^2+2t+1)\mathbf{k}$. Hence its dot-product with the vector $6\mathbf{i}+\mathbf{j}-6\mathbf{k}$ is equal to $12t + 3t^2 + 6 - 18t^2 - 12t - 6 = -15t^2$. Hence the two vectors are orthogonal if and only if $t = 0$. The corresponding point is $(-1, 0, 0)$.

5. (a) Find an equation of the surface consisting of all points in $\mathbb{R}^3$ that are equidistant from the point $(0, -1, 0)$ and the plane $x = 1$.

The distance from a point $P = (x, y, z)$ to the point $(0, -1, 0)$ is $\sqrt{x^2 + (y+1)^2 + z^2}$, and the distance to the plane $y = 1$ is $y - 1$. Hence we obtain the equation

$$\sqrt{x^2 + (y+1)^2 + z^2} = y - 1,$$

which gives

$$x^2 + (y+1)^2 + z^2 = (y-1)^2,$$

and hence

$$y = -\frac{x^2}{4} - \frac{z^2}{4}.$$

(b) Sketch this surface. What is it called?

This is an elliptic paraboloid which does downward along the $y$ axis.

6. Consider the graph of the function $f(x, y) = x^2 + 3xy + 2y^2$.
(a) Write down the equation of the tangent plane to this graph at the point $(1, 2)$.

The equation of the tangent plane at a point $(x_0, y_0, z_0 = f(x_0, y_0))$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

which is in our case

$$z - 15 = 8(x - 1) + 11(y - 2).$$

(b) At which points of this graph is the tangent plane parallel to the plane $x + 2y - z = 10$?

The condition is that the normal vector $\langle 1, 2, -1 \rangle$ to this plane is proportional to the vector $\langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$, which is the normal vector to the tangent plane at $(x_0, y_0, z_0)$. This means that

$$2x_0 + 3y_0 = 1, \qquad 3x_0 + 4y_0 = 2,$$

from which we find the unique solution $x_0 = 2, y_0 = -1$. The corresponding point on the graph is $(2, -1, 0)$.