10.2 Calculus with Parametric Curves

1. $x = \dfrac{t}{1+t},\; y = \sqrt{1+t} \;\Rightarrow\; \dfrac{dy}{dt} = \dfrac{1}{2}(1+t)^{-1/2} = \dfrac{1}{2\sqrt{1+t}},\; \dfrac{dx}{dt} = \dfrac{(1+t)(1)-t(1)}{(1+t)^2} = \dfrac{1}{(1+t)^2}$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1/(2\sqrt{1+t}\,)}{1/(1+t)^2} = \dfrac{(1+t)^2}{2\sqrt{1+t}} = \dfrac{1}{2}(1+t)^{3/2}$.

2. $x = te^t,\; y = t + \sin t \;\Rightarrow\; \dfrac{dy}{dt} = 1 + \cos t,\; \dfrac{dx}{dt} = te^t + e^t = e^t(t+1)$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1+\cos t}{e^t(t+1)}$.

3. $x = t^3 + 1,\; y = t^4 + t;\quad t = -1.\; \dfrac{dy}{dt} = 4t^3 + 1,\; \dfrac{dx}{dt} = 3t^2$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t^3+1}{3t^2}$. When $t = -1$, $(x,y) = (0,0)$ and $dy/dx = -3/3 = -1$, so an equation of the tangent to the curve at the point corresponding to $t = -1$ is $y - 0 = -1(x - 0)$, or $y = -x$.

4. $x = \sqrt{t},\; y = t^2 - 2t;\quad t = 4.\; \dfrac{dy}{dt} = 2t - 2,\; \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = (2t-2)2\sqrt{t} = 4(t-1)\sqrt{t}$. When $t = 4$, $(x,y) = (2,8)$ and $dy/dx = 4(3)(2) = 24$, so an equation of the tangent to the curve at the point corresponding to $t = 4$ is $y - 8 = 24(x - 2)$, or $y = 24x - 40$.

5. $x = t\cos t,\; y = t\sin t;\; t = \pi.\quad \dfrac{dy}{dt} = t\cos t + \sin t,\; \dfrac{dx}{dt} = t(-\sin t) + \cos t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t\cos t + \sin t}{-t\sin t + \cos t}$. When $t = \pi$, $(x,y) = (-\pi, 0)$ and $dy/dx = -\pi/(-1) = \pi$, so an equation of the tangent to the curve at the point corresponding to $t = \pi$ is $y - 0 = \pi[x - (-\pi)]$, or $y = \pi x + \pi^2$.

6. $x = e^t \sin \pi t,\; y = e^{2t};\quad t = 0.\; \dfrac{dy}{dt} = 2e^{2t},\; \dfrac{dx}{dt} = e^t(\pi\cos\pi t) + (\sin\pi t)e^t = e^t(\pi\cos\pi t + \sin\pi t)$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2e^{2t}}{e^t(\pi\cos\pi t + \sin\pi t)} = \dfrac{2e^t}{\pi\cos\pi t + \sin\pi t}$. When $t = 0$, $(x,y) = (0,1)$ and $dy/dx = 2/\pi$, so an equation of the tangent to the curve at the point corresponding to $t = 0$ is $y - 1 = \frac{2}{\pi}(x - 0)$, or $y = \frac{2}{\pi}x + 1$.

7. (a) $x = 1 + \ln t,\; y = t^2 + 2;\; (1,3).\quad \dfrac{dy}{dt} = 2t,\; \dfrac{dx}{dt} = \dfrac{1}{t}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{1/t} = 2t^2$. At $(1,3)$,

$x = 1 + \ln t = 1 \;\Rightarrow\; \ln t = 0 \;\Rightarrow\; t = 1$ and $\dfrac{dy}{dx} = 2$, so an equation of the tangent is $y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(b) $x = 1 + \ln t \;\Rightarrow\; \ln t = x - 1 \;\Rightarrow\; t = e^{x-1}$, so $y = t^2 + 2 = (e^{x-1})^2 + 2 = e^{2x-2} + 2$, and $y' = e^{2x-2}\cdot 2$. At $(1,3)$, $y' = e^{2(1)-2}\cdot 2 = 2$, so an equation of the tangent is $y - 3 = 2(x-1)$, or $y = 2x + 1$.

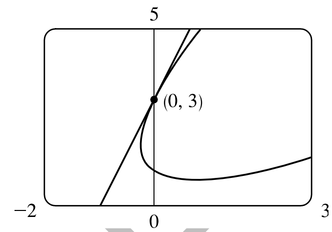
8. (a) $x = 1 + \sqrt{t},\; y = e^{t^2};\; (2,e).\quad \dfrac{dy}{dt} = e^{t^2}\cdot 2t,\; \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2te^{t^2}}{1/(2\sqrt{t}\,)} = 4t^{3/2}e^{t^2}$. At $(2,e)$,

$x = 1 + \sqrt{t} = 2 \;\Rightarrow\; \sqrt{t} = 1 \;\Rightarrow\; t = 1$ and $\dfrac{dy}{dx} = 4e$, so an equation of the tangent is $y - e = 4e(x - 2)$, or $y = 4ex - 7e$.

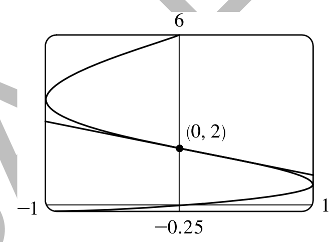
(b) $x = 1 + \sqrt{t} \;\Rightarrow\; \sqrt{t} = x - 1 \;\Rightarrow\; t = (x-1)^2$, so $y = e^{t^2} = e^{(x-1)^4}$, and $y' = e^{(x-1)^4} \cdot 4(x-1)^3$.

At $(2, e)$, $y' = e \cdot 4 = 4e$, so an equation of the tangent is $y - e = 4e(x - 2)$, or $y = 4ex - 7e$.

9. $x = t^2 - t,\; y = t^2 + t + 1;\; (0, 3)$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{2t-1}$. To find the value of t corresponding to the point $(0, 3)$, solve $x = 0 \;\Rightarrow\; t^2 - t = 0 \;\Rightarrow\; t(t-1) = 0 \;\Rightarrow\; t = 0$ or $t = 1$. Only $t = 1$ gives $y = 3$. With $t = 1$, $dy/dx = 3$, and an equation of the tangent is $y - 3 = 3(x - 0)$, or $y = 3x + 3$.



10. $x = \sin \pi t,\; y = t^2 + t;\; (0, 2)$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{\pi \cos \pi t}$. To find the value of t corresponding to the point $(0, 2)$, solve $y = 2 \;\Rightarrow\; t^2 + t - 2 = 0 \;\Rightarrow\; (t+2)(t-1) = 0 \;\Rightarrow\; t = -2$ or $t = 1$. Either value gives $dy/dx = -3/\pi$, so an equation of the tangent is $y - 2 = -\frac{3}{\pi}(x - 0)$, or $y = -\frac{3}{\pi}x + 2$.



11. $x = t^2 + 1,\; y = t^2 + t \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{2t} = 1 + \dfrac{1}{2t} \;\Rightarrow\; \dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{-1/(2t^2)}{2t} = -\dfrac{1}{4t^3}$.

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t < 0$.

12. $x = t^3 + 1,\; y = t^2 - t \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t-1}{3t^2} = \dfrac{2}{3t} - \dfrac{1}{3t^2} \;\Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-\frac{2}{3t^2} + \frac{2}{3t^3}}{3t^2} = \frac{\frac{2-2t}{3t^3}}{3t^2} = \frac{2(1-t)}{9t^5}.$$ The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $0 < t < 1$.

13. $x = e^t,\; y = te^{-t} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-te^{-t} + e^{-t}}{e^t} = \dfrac{e^{-t}(1-t)}{e^t} = e^{-2t}(1-t) \;\Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{e^{-2t}(-1) + (1-t)(-2e^{-2t})}{e^t} = \frac{e^{-2t}(-1-2+2t)}{e^t} = e^{-3t}(2t-3).$$ The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t > \frac{3}{2}$.

14. $x = t^2 + 1,\; y = e^t - 1 \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{e^t}{2t} \;\Rightarrow\; \dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{\frac{2te^t - e^t \cdot 2}{(2t)^2}}{2t} = \dfrac{2e^t(t-1)}{(2t)^3} = \dfrac{e^t(t-1)}{4t^3}$.

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t < 0$ or $t > 1$.

15. $x = t - \ln t,\ y = t + \ln t$ [note that $t > 0$] $\Rightarrow\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 + 1/t}{1 - 1/t} = \dfrac{t+1}{t-1} \ \Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \frac{\dfrac{(t-1)(1) - (t+1)(1)}{(t-1)^2}}{(t-1)/t} = \frac{-2t}{(t-1)^3}.$$ The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $0 < t < 1$.

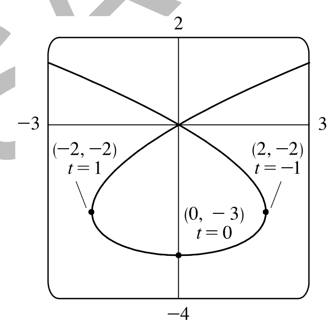
16. $x = \cos t,\ y = \sin 2t,\ 0 < t < \pi \ \Rightarrow\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2\cos 2t}{-\sin t} \ \Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \frac{\dfrac{(-\sin t)(-4\sin 2t) - (2\cos 2t)(-\cos t)}{(-\sin t)^2}}{-\sin t} = \frac{(\sin t)(8\sin t\,\cos t) + [2(1 - 2\sin^2 t)](\cos t)}{(-\sin t)\sin^2 t}$$

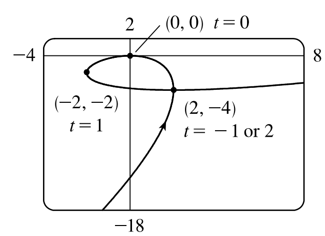
$$= \frac{(\cos t)(8\sin^2 t + 2 - 4\sin^2 t)}{(-\sin t)\sin^2 t} = -\frac{\cos t}{\sin t}\cdot\frac{4\sin^2 t + 2}{\sin^2 t} \quad [\,(-\cot t)\cdot \text{positive expression}]$$

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $-\cot t > 0 \ \Leftrightarrow\ \cot t < 0 \ \Leftrightarrow\ \frac{\pi}{2} < t < \pi$.

17. $x = t^3 - 3t,\ y = t^2 - 3.$ $\dfrac{dy}{dt} = 2t$, so $\dfrac{dy}{dt} = 0 \ \Leftrightarrow\ t = 0 \ \Leftrightarrow$ $(x, y) = (0, -3).$ $\dfrac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$, so $\dfrac{dx}{dt} = 0 \ \Leftrightarrow$ $t = -1$ or $1 \ \Leftrightarrow\ (x, y) = (2, -2)$ or $(-2, -2)$. The curve has a horizontal tangent at $(0, -3)$ and vertical tangents at $(2, -2)$ and $(-2, -2)$.

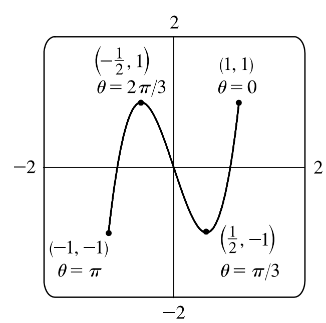


18. $x = t^3 - 3t,\ y = t^3 - 3t^2.$ $\dfrac{dy}{dt} = 3t^2 - 6t = 3t(t-2)$, so $\dfrac{dy}{dt} = 0 \ \Leftrightarrow$ $t = 0$ or $2 \ \Leftrightarrow\ (x, y) = (0, 0)$ or $(2, -4)$. $\dfrac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$, so $\dfrac{dx}{dt} = 0 \ \Leftrightarrow\ t = -1$ or $1 \ \Leftrightarrow\ (x, y) = (2, -4)$ or $(-2, -2)$. The curve has horizontal tangents at $(0, 0)$ and $(2, -4)$, and vertical tangents at $(2, -4)$ and $(-2, -2)$.



19. $x = \cos\theta,\ y = \cos 3\theta$. The whole curve is traced out for $0 \le \theta \le \pi$.

$\dfrac{dy}{d\theta} = -3\sin 3\theta$, so $\dfrac{dy}{d\theta} = 0 \ \Leftrightarrow\ \sin 3\theta = 0 \ \Leftrightarrow\ 3\theta = 0, \pi, 2\pi,$ or $3\pi \ \Leftrightarrow$ $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, or $\pi \ \Leftrightarrow\ (x, y) = (1, 1), \left(\frac{1}{2}, -1\right), \left(-\frac{1}{2}, 1\right)$, or $(-1, -1)$. $\dfrac{dx}{d\theta} = -\sin\theta$, so $\dfrac{dx}{d\theta} = 0 \ \Leftrightarrow\ \sin\theta = 0 \ \Leftrightarrow\ \theta = 0$ or $\pi \ \Leftrightarrow$ $(x, y) = (1, 1)$ or $(-1, -1)$. Both $\dfrac{dy}{d\theta}$ and $\dfrac{dx}{d\theta}$ equal 0 when $\theta = 0$ and π.



To find the slope when $\theta = 0$, we find $\displaystyle\lim_{\theta\to 0}\frac{dy}{dx} = \lim_{\theta\to 0}\frac{-3\sin 3\theta}{-\sin\theta} \overset{\text{H}}{=} \lim_{\theta\to 0}\frac{-9\cos 3\theta}{-\cos\theta} = 9$, which is the same slope when $\theta = \pi$.

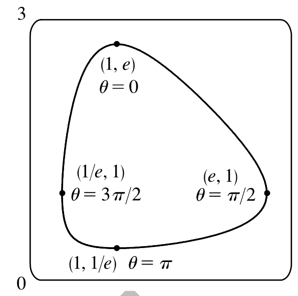
Thus, the curve has horizontal tangents at $\left(\frac{1}{2}, -1\right)$ and $\left(-\frac{1}{2}, 1\right)$, and there are no vertical tangents.

20. $x = e^{\sin\theta}$, $y = e^{\cos\theta}$. The whole curve is traced out for $0 \le \theta < 2\pi$.

$\dfrac{dy}{d\theta} = -\sin\theta\, e^{\cos\theta}$, so $\dfrac{dy}{d\theta} = \theta \quad\Leftrightarrow\quad \sin\theta = 0 \quad\Leftrightarrow\quad \theta = 0 \text{ or } \pi \quad\Leftrightarrow$

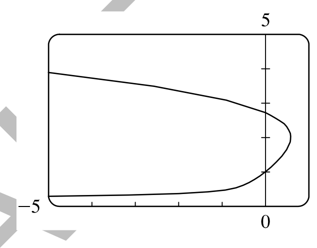
$(x, y) = (1, e)$ or $(1, 1/e)$. $\dfrac{dx}{d\theta} = \cos\theta\, e^{\sin\theta}$, so $\dfrac{dx}{d\theta} = 0 \quad\Leftrightarrow\quad \cos\theta = 0 \quad\Leftrightarrow$

$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \quad\Leftrightarrow\quad (x, y) = (e, 1)$ or $(1/e, 1)$. The curve has horizontal tangents at $(1, e)$ and $(1, 1/e)$, and vertical tangents at $(e, 1)$ and $(1/e, 1)$.



21. From the graph, it appears that the rightmost point on the curve $x = t - t^6$, $y = e^t$ is about $(0.6, 2)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0 = dx/dt = 1 - 6t^5 \quad\Leftrightarrow\quad t = 1/\sqrt[5]{6}$. Hence, the rightmost point is

$$\left(1/\sqrt[5]{6} - 1/\left(6\sqrt[5]{6}\right), e^{1/\sqrt[5]{6}}\right) = \left(5 \cdot 6^{-6/5}, e^{6^{-1/5}}\right) \approx (0.58, 2.01).$$



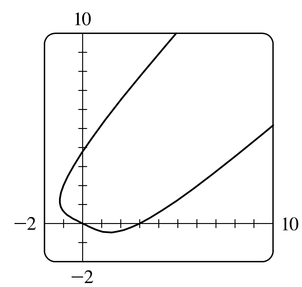
22. From the graph, it appears that the lowest point and the leftmost point on the curve $x = t^4 - 2t$, $y = t + t^4$ are $(1.5, -0.5)$ and $(-1.2, 1.2)$, respectively. To find the exact coordinates, we solve $dy/dt = 0$ (horizontal tangents) and $dx/dt = 0$ (vertical tangents).

$\dfrac{dy}{dt} = 0 \quad\Leftrightarrow\quad 1 + 4t^3 = 0 \quad\Leftrightarrow\quad t = -\dfrac{1}{\sqrt[3]{4}}$, so the lowest point is

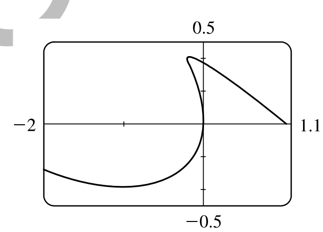
$$\left(\frac{1}{\sqrt[3]{256}} + \frac{2}{\sqrt[3]{4}}, -\frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{256}}\right) = \left(\frac{9}{\sqrt[3]{256}}, -\frac{3}{\sqrt[3]{256}}\right) \approx (1.42, -0.47).$$

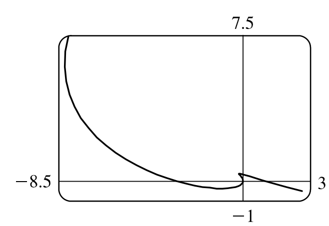
$\dfrac{dx}{dt} = 0 \quad\Leftrightarrow\quad 4t^3 - 2 = 0 \quad\Leftrightarrow\quad t = \dfrac{1}{\sqrt[3]{2}}$, so the leftmost point is

$$\left(\frac{1}{\sqrt[3]{16}} - \frac{2}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{16}}\right) = \left(-\frac{3}{\sqrt[3]{16}}, \frac{3}{\sqrt[3]{16}}\right) \approx (-1.19, 1.19).$$



23. We graph the curve $x = t^4 - 2t^3 - 2t^2$, $y = t^3 - t$ in the viewing rectangle $[-2, 1.1]$ by $[-0.5, 0.5]$. This rectangle corresponds approximately to $t \in [-1, 0.8]$.

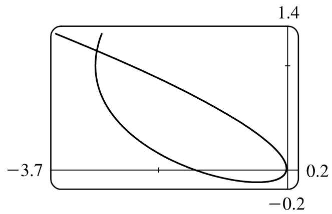


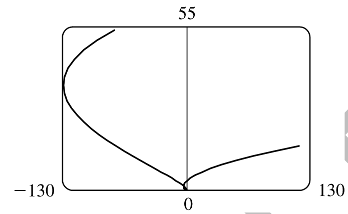


We estimate that the curve has horizontal tangents at about $(-1, -0.4)$ and $(-0.17, 0.39)$ and vertical tangents at about $(0, 0)$ and $(-0.19, 0.37)$. We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 1}{4t^3 - 6t^2 - 4t}$. The horizontal tangents occur when $dy/dt = 3t^2 - 1 = 0 \quad\Leftrightarrow\quad t = \pm\frac{1}{\sqrt{3}}$, so both horizontal tangents are shown in our graph. The vertical tangents occur when

$dx/dt = 2t(2t^2 - 3t - 2) = 0 \quad \Leftrightarrow \quad 2t(2t+1)(t-2) = 0 \quad \Leftrightarrow \quad t = 0, -\frac{1}{2}$ or 2. It seems that we have missed one vertical tangent, and indeed if we plot the curve on the t-interval $[-1.2, 2.2]$ we see that there is another vertical tangent at $(-8, 6)$.

24. We graph the curve $x = t^4 + 4t^3 - 8t^2$, $y = 2t^2 - t$ in the viewing rectangle $[-3.7, 0.2]$ by $[-0.2, 1.4]$. It appears that there is a horizontal tangent at about $(-0.4, -0.1)$, and vertical tangents at about $(-3, 1)$ and $(0, 0)$.

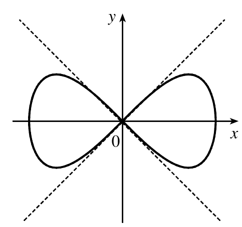




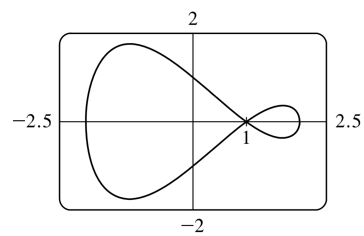
We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t-1}{4t^3 + 12t^2 - 16t}$, so there is a horizontal tangent where $dy/dt = 4t - 1 = 0 \quad \Leftrightarrow \quad t = \frac{1}{4}$.

This point (the lowest point) is shown in the first graph. There are vertical tangents where $dx/dt = 4t^3 + 12t^2 - 16t = 0 \quad \Leftrightarrow$ $4t(t^2 + 3t - 4) = 0 \quad \Leftrightarrow \quad 4t(t+4)(t-1) = 0$. We have missed one vertical tangent corresponding to $t = -4$, and if we plot the graph for $t \in [-5, 3]$, we see that the curve has another vertical tangent line at approximately $(-128, 36)$.

25. $x = \cos t,\ y = \sin t \cos t. \quad dx/dt = -\sin t,$ $dy/dt = -\sin^2 t + \cos^2 t = \cos 2t.\ (x, y) = (0, 0) \quad \Leftrightarrow \quad \cos t = 0 \quad \Leftrightarrow \quad t$ is an odd multiple of $\frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $dx/dt = -1$ and $dy/dt = -1$, so $dy/dx = 1$. When $t = \frac{3\pi}{2}$, $dx/dt = 1$ and $dy/dt = -1$. So $dy/dx = -1$. Thus, $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.



26. $x = -2\cos t,\ y = \sin t + \sin 2t$. From the graph, it appears that the curve crosses itself at the point $(1, 0)$. If this is true, then $x = 1 \quad \Leftrightarrow$ $-2\cos t = 1 \quad \Leftrightarrow \quad \cos t = -\frac{1}{2} \quad \Leftrightarrow \quad t = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ for $0 \le t \le 2\pi$. Substituting either value of t into y gives $y = 0$, confirming that $(1, 0)$ is the point where the curve crosses itself. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t + 2\cos 2t}{2\sin t}$.



When $t = \dfrac{2\pi}{3}$, $\dfrac{dy}{dx} = \dfrac{-1/2 + 2(-1/2)}{2(\sqrt{3}/2)} = \dfrac{-3/2}{\sqrt{3}} = -\dfrac{\sqrt{3}}{2}$, so an equation of the tangent line is $y - 0 = -\dfrac{\sqrt{3}}{2}(x - 1)$, or $y = -\dfrac{\sqrt{3}}{2}x + \dfrac{\sqrt{3}}{2}$. Similarly, when $t = \dfrac{4\pi}{3}$, an equation of the tangent line is $y = \dfrac{\sqrt{3}}{2}x - \dfrac{\sqrt{3}}{2}$.

27. $x = r\theta - d\sin\theta,\ y = r - d\cos\theta$.

(a) $\dfrac{dx}{d\theta} = r - d\cos\theta$, $\dfrac{dy}{d\theta} = d\sin\theta$, so $\dfrac{dy}{dx} = \dfrac{d\sin\theta}{r - d\cos\theta}$.

(b) If $0 < d < r$, then $|d\cos\theta| \le d < r$, so $r - d\cos\theta \ge r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

28. $x = a\cos^3\theta,\ y = a\sin^3\theta.$

(a) $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$, so $\dfrac{dy}{dx} = -\dfrac{\sin\theta}{\cos\theta} = -\tan\theta$.

(b) The tangent is horizontal $\Leftrightarrow$ $dy/dx = 0$ $\Leftrightarrow$ $\tan\theta = 0$ $\Leftrightarrow$ $\theta = n\pi$ $\Leftrightarrow$ $(x, y) = (\pm a, 0)$.

The tangent is vertical $\Leftrightarrow$ $\cos\theta = 0$ $\Leftrightarrow$ θ is an odd multiple of $\frac{\pi}{2}$ $\Leftrightarrow$ $(x, y) = (0, \pm a)$.

(c) $dy/dx = \pm 1$ $\Leftrightarrow$ $\tan\theta = \pm 1$ $\Leftrightarrow$ θ is an odd multiple of $\frac{\pi}{4}$ $\Leftrightarrow$ $(x, y) = \left(\pm\frac{\sqrt{2}}{4}a, \pm\frac{\sqrt{2}}{4}a\right)$

[All sign choices are valid.]

29. $x = 3t^2 + 1,\ y = t^3 - 1$ $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2}{6t} = \dfrac{t}{2}$. The tangent line has slope $\dfrac{1}{2}$ when $\dfrac{t}{2} = \dfrac{1}{2}$ $\Leftrightarrow$ $t = 1$, so the point is $(4, 0)$.

30. $x = 3t^2 + 1, y = 2t^3 + 1, \dfrac{dx}{dt} = 6t, \dfrac{dy}{dt} = 6t^2$, so $\dfrac{dy}{dx} = \dfrac{6t^2}{6t} = t$ [even where $t = 0$].

So at the point corresponding to parameter value t, an equation of the tangent line is $y - (2t^3 + 1) = t[x - (3t^2 + 1)]$.

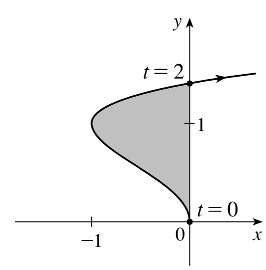
If this line is to pass through $(4, 3)$, we must have $3 - (2t^3 + 1) = t[4 - (3t^2 + 1)]$ $\Leftrightarrow$ $2t^3 - 2 = 3t^3 - 3t$ $\Leftrightarrow$ $t^3 - 3t + 2 = 0$ $\Leftrightarrow$ $(t-1)^2(t+2) = 0$ $\Leftrightarrow$ $t = 1$ or -2. Hence, the desired equations are $y - 3 = x - 4$, or $y = x - 1$, tangent to the curve at $(4, 3)$, and $y - (-15) = -2(x - 13)$, or $y = -2x + 11$, tangent to the curve at $(13, -15)$.

31. By symmetry of the ellipse about the x- and y-axes,

$$\begin{aligned} A &= 4\int_0^a y\,dx = 4\int_{\pi/2}^0 b\sin\theta\,(-a\sin\theta)\,d\theta = 4ab\int_0^{\pi/2}\sin^2\theta\,d\theta = 4ab\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta \\ &= 2ab\big[\theta - \tfrac{1}{2}\sin 2\theta\big]_0^{\pi/2} = 2ab\big(\tfrac{\pi}{2}\big) = \pi ab \end{aligned}$$

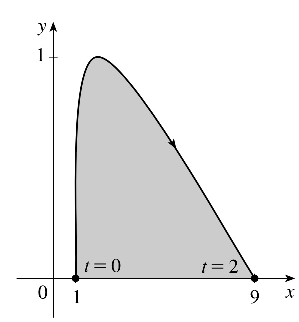
32. The curve $x = t^2 - 2t = t(t-2)$, $y = \sqrt{t}$ intersects the y-axis when $x = 0$, that is, when $t = 0$ and $t = 2$. The corresponding values of y are 0 and $\sqrt{2}$. The shaded area is given by

$$\begin{aligned} \int_{y=0}^{y=\sqrt{2}} (x_R - x_L)\,dy &= \int_{t=0}^{t=2} [0 - x(t)]\,y'(t)\,dt = -\int_0^2 (t^2 - 2t)\left(\frac{1}{2\sqrt{t}}\,dt\right) \\ &= -\int_0^2 \left(\tfrac{1}{2}t^{3/2} - t^{1/2}\right)dt = -\left[\tfrac{1}{5}t^{5/2} - \tfrac{2}{3}t^{3/2}\right]_0^2 \\ &= -\left(\tfrac{1}{5}\cdot 2^{5/2} - \tfrac{2}{3}\cdot 2^{3/2}\right) = -2^{1/2}\left(\tfrac{4}{5} - \tfrac{4}{3}\right) \\ &= -\sqrt{2}\left(-\tfrac{8}{15}\right) = \tfrac{8}{15}\sqrt{2} \end{aligned}$$



33. The curve $x = t^3 + 1, y = 2t - t^2 = t(2-t)$ intersects the x-axis when $y = 0$, that is, when $t = 0$ and $t = 2$. The corresponding values of x are 1 and 9. The shaded area is given by

$$\begin{aligned} \int_{x=1}^{x=9} (y_T - y_B)\,dx &= \int_{t=0}^{t=2} [y(t) - 0]\,x'(t)\,dt = \int_0^2 (2t - t^2)(3t^2)\,dt \\ &= 3\int_0^2 (2t^3 - t^4)\,dt = 3\left[\tfrac{1}{2}t^4 - \tfrac{1}{5}t^5\right]_0^2 = 3\left(8 - \tfrac{32}{5}\right) = \tfrac{24}{5} \end{aligned}$$



34. By symmetry, $A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 a\sin^3\theta(-3a\cos^2\theta\,\sin\theta)\,d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\,\cos^2\theta\,d\theta$. Now

$$\begin{aligned}\int\sin^4\theta\,\cos^2\theta\,d\theta &= \int\sin^2\theta\left(\tfrac14\sin^2 2\theta\right)d\theta = \tfrac18\int(1-\cos 2\theta)\sin^2 2\theta\,d\theta\\ &= \tfrac18\int\left[\tfrac12(1-\cos 4\theta)-\sin^2 2\theta\,\cos 2\theta\right]d\theta = \tfrac{1}{16}\theta - \tfrac{1}{64}\sin 4\theta - \tfrac{1}{48}\sin^3 2\theta + C\end{aligned}$$

so $\int_0^{\pi/2}\sin^4\theta\,\cos^2\theta\,d\theta = \left[\frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta\right]_0^{\pi/2} = \frac{\pi}{32}$. Thus, $A = 12a^2\left(\frac{\pi}{32}\right) = \frac38\pi a^2$.

35. $x = r\theta - d\sin\theta,\ y = r - d\cos\theta$.

$$\begin{aligned}A &= \int_0^{2\pi r} y\,dx = \int_0^{2\pi}(r-d\cos\theta)(r-d\cos\theta)\,d\theta = \int_0^{2\pi}(r^2 - 2dr\cos\theta + d^2\cos^2\theta)\,d\theta\\ &= \left[r^2\theta - 2dr\sin\theta + \tfrac12 d^2\left(\theta + \tfrac12\sin 2\theta\right)\right]_0^{2\pi} = 2\pi r^2 + \pi d^2\end{aligned}$$

36. (a) By symmetry, the area of $\mathscr{R}$ is twice the area inside $\mathscr{R}$ above the x-axis. The top half of the loop is described by $x = t^2$, $y = t^3 - 3t$, $-\sqrt3 \le t \le 0$, so, using the Substitution Rule with $y = t^3 - 3t$ and $dx = 2t\,dt$, we find that

$$\begin{aligned}\text{area} &= 2\int_0^3 y\,dx = 2\int_0^{-\sqrt3}(t^3-3t)2t\,dt = 2\int_0^{-\sqrt3}(2t^4 - 6t^2)\,dt = 2\left[\tfrac25 t^5 - 2t^3\right]_0^{-\sqrt3}\\ &= 2\left[\tfrac25(-3^{1/2})^5 - 2(-3^{1/2})^3\right] = 2\left[\tfrac25\left(-9\sqrt3\right) - 2\left(-3\sqrt3\right)\right] = \tfrac{24}{5}\sqrt3\end{aligned}$$

(b) Here we use the formula for disks and use the Substitution Rule as in part (a):

$$\begin{aligned}\text{volume} &= \pi\int_0^3 y^2\,dx = \pi\int_0^{-\sqrt3}(t^3-3t)^2 2t\,dt = 2\pi\int_0^{-\sqrt3}(t^6 - 6t^4 + 9t^2)t\,dt = 2\pi\left[\tfrac18 t^8 - t^6 + \tfrac94 t^4\right]_0^{-\sqrt3}\\ &= 2\pi\left[\tfrac18(-3^{1/2})^8 - (-3^{1/2})^6 + \tfrac94(-3^{1/2})^4\right] = 2\pi\left[\tfrac{81}{8} - 27 + \tfrac{81}{4}\right] = \tfrac{27}{4}\pi\end{aligned}$$

(c) By symmetry, the y-coordinate of the centroid is 0. To find the x-coordinate, we note that it is the same as the x-coordinate of the centroid of the top half of $\mathscr{R}$, the area of which is $\frac12\cdot\frac{24}{5}\sqrt3 = \frac{12}{5}\sqrt3$. So, using Formula 8.3.8 with $A = \frac{12}{5}\sqrt3$, we get

$$\begin{aligned}\overline{x} &= \tfrac{5}{12\sqrt3}\int_0^3 xy\,dx = \tfrac{5}{12\sqrt3}\int_0^{-\sqrt3} t^2(t^3-3t)2t\,dt = \tfrac{5}{6\sqrt3}\left[\tfrac17 t^7 - \tfrac35 t^5\right]_0^{-\sqrt3}\\ &= \tfrac{5}{6\sqrt3}\left[\tfrac17(-3^{1/2})^7 - \tfrac35(-3^{1/2})^5\right] = \tfrac{5}{6\sqrt3}\left[-\tfrac{27}{7}\sqrt3 + \tfrac{27}{5}\sqrt3\right] = \tfrac97\end{aligned}$$

So the coordinates of the centroid of $\mathscr{R}$ are $(x, y) = \left(\frac97, 0\right)$.

37. $x = t + e^{-t},\ y = t - e^{-t},\ 0 \le t \le 2.\quad dx/dt = 1 - e^{-t}$ and $dy/dt = 1 + e^{-t}$, so
$(dx/dt)^2 + (dy/dt)^2 = (1 - e^{-t})^2 + (1 + e^{-t})^2 = 1 - 2e^{-t} + e^{-2t} + 1 + 2e^{-t} + e^{-2t} = 2 + 2e^{-2t}$.
Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^2 \sqrt{2 + 2e^{-2t}}\,dt \approx 3.1416$.

38. $x = t^2 - t,\ y = t^4,\ 1 \le t \le 4.\quad dx/dt = 2t - 1$ and $dy/dt = 4t^3$, so
$(dx/dt)^2 + (dy/dt)^2 = (2t-1)^2 + (4t^3)^2 = 4t^2 - 4t + 1 + 16t^6$.
Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_1^4 \sqrt{16t^6 + 4t^2 - 4t + 1}\,dt \approx 255.3756$.

39. $x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \le t \le 4\pi.\quad dx/dt = 1 - 2\cos t$ and $dy/dt = 2\sin t$, so
$(dx/dt)^2 + (dy/dt)^2 = (1 - 2\cos t)^2 + (2\sin t)^2 = 1 - 4\cos t + 4\cos^2 t + 4\sin^2 t = 5 - 4\cos t$.
Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^{4\pi} \sqrt{5 - 4\cos t}\,dt \approx 26.7298$.

40. $x = t + \sqrt{t},\ y = t - \sqrt{t},\ 0 \le t \le 1.$ $\dfrac{dx}{dt} = 1 + \dfrac{1}{2\sqrt{t}}$ and $\dfrac{dy}{dt} = 1 - \dfrac{1}{2\sqrt{t}}$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(1 + \frac{1}{2\sqrt{t}}\right)^2 + \left(1 - \frac{1}{2\sqrt{t}}\right)^2 = 1 + \frac{1}{\sqrt{t}} + \frac{1}{4t} + 1 - \frac{1}{\sqrt{t}} + \frac{1}{4t} = 2 + \frac{1}{2t}.$$

Thus, $L = \displaystyle\int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^1 \sqrt{2 + \frac{1}{2t}}\,dt = \lim_{t \to 0^+} \int_t^1 \sqrt{2 + \frac{1}{2t}}\,dt \approx 2.0915.$

41. $x = 1 + 3t^2,\ \ y = 4 + 2t^3,\ \ 0 \le t \le 1.$ $dx/dt = 6t$ and $dy/dt = 6t^2$, so $(dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4$.

Thus, $L = \displaystyle\int_0^1 \sqrt{36t^2 + 36t^4}\,dt = \int_0^1 6t\sqrt{1 + t^2}\,dt = 6\int_1^2 \sqrt{u}\,\left(\tfrac{1}{2}\,du\right)$ $[u = 1 + t^2,\ du = 2t\,dt]$

$$= 3\left[\tfrac{2}{3}u^{3/2}\right]_1^2 = 2(2^{3/2} - 1) = 2\left(2\sqrt{2} - 1\right)$$

42. $x = e^t - t,\ y = 4e^{t/2},\ 0 \le t \le 2.$ $dx/dt = e^t - 1$ and $dy/dt = 2e^{t/2}$, so

$(dx/dt)^2 + (dy/dt)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} - 2e^t + 1 + 4e^t = e^{2t} + 2e^t + 1 = (e^t + 1)^2$. Thus,

$$L = \int_0^2 \sqrt{(e^t + 1)^2}\,dt = \int_0^2 \left|e^t + 1\right|\,dt = \int_0^2 (e^t + 1)\,dt = \left[e^t + t\right]_0^2 = (e^2 + 2) - (1 + 0) = e^2 + 1.$$

43. $x = t\sin t,\ y = t\cos t,\ 0 \le t \le 1.$ $\dfrac{dx}{dt} = t\cos t + \sin t$ and $\dfrac{dy}{dt} = -t\sin t + \cos t$, so

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= t^2\cos^2 t + 2t\sin t\,\cos t + \sin^2 t + t^2\sin^2 t - 2t\sin t\,\cos t + \cos^2 t\\ &= t^2(\cos^2 t + \sin^2 t) + \sin^2 t + \cos^2 t = t^2 + 1.\end{aligned}$$

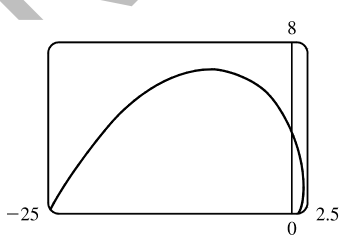
Thus, $L = \int_0^1 \sqrt{t^2 + 1}\,dt \overset{21}{=} \left[\tfrac{1}{2}t\sqrt{t^2 + 1} + \tfrac{1}{2}\ln\left(t + \sqrt{t^2 + 1}\right)\right]_0^1 = \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\ln\left(1 + \sqrt{2}\right).$

44. $x = 3\cos t - \cos 3t,\ y = 3\sin t - \sin 3t,\ 0 \le t \le \pi.$ $\dfrac{dx}{dt} = -3\sin t + 3\sin 3t$ and $\dfrac{dy}{dt} = 3\cos t - 3\cos 3t$, so

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 9\sin^2 t - 18\sin t\,\sin 3t + 9\sin^2(3t) + 9\cos^2 t - 18\cos t\,\cos 3t + 9\cos^2(3t)\\ &= 9(\cos^2 t + \sin^2 t) - 18(\cos t\,\cos 3t + \sin t\,\sin 3t) + 9[\cos^2(3t) + \sin^2(3t)]\\ &= 9(1) - 18\cos(t - 3t) + 9(1) = 18 - 18\cos(-2t) = 18(1 - \cos 2t)\\ &= 18[1 - (1 - 2\sin^2 t)] = 36\sin^2 t.\end{aligned}$$

Thus, $L = \int_0^\pi \sqrt{36\sin^2 t}\,dt = 6\int_0^\pi |\sin t|\,dt = 6\int_0^\pi \sin t\,dt = -6\left[\cos t\right]_0^\pi = -6\,(-1 - 1) = 12.$

45.



$x = e^t\cos t,\ y = e^t\sin t,\ 0 \le t \le \pi.$

$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= [e^t(\cos t - \sin t)]^2 + [e^t(\sin t + \cos t)]^2\\ &= (e^t)^2(\cos^2 t - 2\cos t\,\sin t + \sin^2 t)\\ &\qquad + (e^t)^2(\sin^2 t + 2\sin t\,\cos t + \cos^2 t\\ &= e^{2t}(2\cos^2 t + 2\sin^2 t) = 2e^{2t}\end{aligned}$$

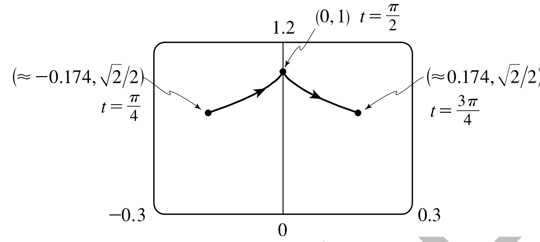
Thus, $L = \int_0^\pi \sqrt{2e^{2t}}\,dt = \int_0^\pi \sqrt{2}\,e^t\,dt = \sqrt{2}\,\left[\,e^t\,\right]_0^\pi = \sqrt{2}\,(e^\pi - 1).$

46. $x = \cos t + \ln(\tan \frac{1}{2}t), \quad y = \sin t, \quad \pi/4 \le t \le 3\pi/4.$

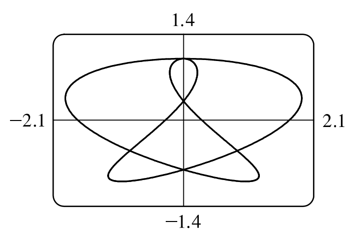
$$\frac{dx}{dt} = -\sin t + \frac{\frac{1}{2}\sec^2(t/2)}{\tan(t/2)} = -\sin t + \frac{1}{2\sin(t/2)\,\cos(t/2)} = -\sin t + \frac{1}{\sin t} \text{ and } \frac{dy}{dt} = \cos t, \text{ so}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t - 2 + \frac{1}{\sin^2 t} + \cos^2 t = 1 - 2 + \csc^2 t = \cot^2 t. \text{ Thus,}$$

$$\begin{aligned} L &= \textstyle\int_{\pi/4}^{3\pi/4} |\cot t|\, dt = 2\int_{\pi/4}^{\pi/2} \cot t\, dt \\ &= 2\Big[\ln|\sin t|\Big]_{\pi/4}^{\pi/2} = 2\left(\ln 1 - \ln \frac{1}{\sqrt{2}}\right) \\ &= 2\big(0 + \ln\sqrt{2}\big) = 2\big(\tfrac{1}{2}\ln 2\big) = \ln 2. \end{aligned}$$



47.



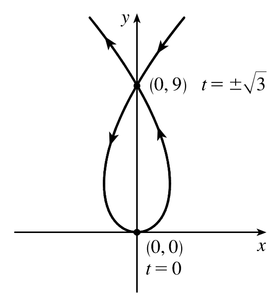
The figure shows the curve $x = \sin t + \sin 1.5t$, $y = \cos t$ for $0 \le t \le 4\pi$.
$dx/dt = \cos t + 1.5\cos 1.5t$ and $dy/dt = -\sin t$, so
$(dx/dt)^2 + (dy/dt)^2 = \cos^2 t + 3\cos t\,\cos 1.5t + 2.25\cos^2 1.5t + \sin^2 t$.
Thus, $L = \int_0^{4\pi} \sqrt{1 + 3\cos t\,\cos 1.5t + 2.25\cos^2 1.5t}\, dt \approx 16.7102$.

48. $x = 3t - t^3$, $y = 3t^2$. $\quad dx/dt = 3 - 3t^2$ and $dy/dt = 6t$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = (3 + 3t^2)^2$$

and the length of the loop is given by

$$\begin{aligned} L &= \int_{-\sqrt{3}}^{\sqrt{3}} (3 + 3t^2)\, dt = 2\int_0^{\sqrt{3}} (3 + 3t^2)\, dt = 2\left[3t + t^3\right]_0^{\sqrt{3}} \\ &= 2\big(3\sqrt{3} + 3\sqrt{3}\big) = 12\sqrt{3} \end{aligned}$$



49. $x = t - e^t,\ y = t + e^t,\ -6 \le t \le 6.$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 - e^t)^2 + (1 + e^t)^2 = (1 - 2e^t + e^{2t}) + (1 + 2e^t + e^{2t}) = 2 + 2e^{2t}$, so $L = \int_{-6}^{6} \sqrt{2 + 2e^{2t}}\, dt$.

Set $f(t) = \sqrt{2 + 2e^{2t}}$. Then by Simpson's Rule with $n = 6$ and $\Delta t = \frac{6 - (-6)}{6} = 2$, we get

$L \approx \frac{2}{3}[f(-6) + 4f(-4) + 2f(-2) + 4f(0) + 2f(2) + 4f(4) + f(6)] \approx 612.3053$.

50. $x = 2a\cot\theta \quad\Rightarrow\quad dx/dt = -2a\csc^2\theta$ and $y = 2a\sin^2\theta \quad\Rightarrow\quad dy/dt = 4a\sin\theta\,\cos\theta = 2a\sin 2\theta$.

So $L = \int_{\pi/4}^{\pi/2} \sqrt{4a^2\csc^4\theta + 4a^2\sin^2 2\theta}\, d\theta = 2a\int_{\pi/4}^{\pi/2} \sqrt{\csc^4\theta + \sin^2 2\theta}\, d\theta$. Using Simpson's Rule with

$n = 4$, $\Delta\theta = \frac{\pi/2 - \pi/4}{4} = \frac{\pi}{16}$, and $f(\theta) = \sqrt{\csc^4\theta + \sin^2 2\theta}$, we get

$L \approx 2a \cdot S_4 = (2a)\,\frac{\pi}{16\cdot 3}\left[f\left(\frac{\pi}{4}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{3\pi}{8}\right) + 4f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 2.2605a$.

51. $x = \sin^2 t,\ y = \cos^2 t,\ 0 \le t \le 3\pi.$

$(dx/dt)^2 + (dy/dt)^2 = (2\sin t\,\cos t)^2 + (-2\cos t\sin t)^2 = 8\sin^2 t\cos^2 t = 2\sin^2 2t \quad\Rightarrow$

Distance $= \int_0^{3\pi} \sqrt{2}\,|\sin 2t|\ dt = 6\sqrt{2}\int_0^{\pi/2} \sin 2t\,dt$ [by symmetry] $= -3\sqrt{2}\left[\cos 2t\right]_0^{\pi/2} = -3\sqrt{2}\,(-1-1) = 6\sqrt{2}$.

The full curve is traversed as t goes from 0 to $\frac{\pi}{2}$, because the curve is the segment of $x+y=1$ that lies in the first quadrant (since $x, y \geq 0$), and this segment is completely traversed as t goes from 0 to $\frac{\pi}{2}$. Thus, $L = \int_0^{\pi/2} \sin 2t\,dt = \sqrt{2}$, as above.

52. $x = \cos^2 t,\ y = \cos t,\ 0 \leq t \leq 4\pi$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-2\cos t\sin t)^2 + (-\sin t)^2 = \sin^2 t\,(4\cos^2 t + 1)$

$$\begin{aligned}\text{Distance} &= \int_0^{4\pi} |\sin t|\,\sqrt{4\cos^2 t + 1}\,dt = 4\int_0^{\pi} \sin t\,\sqrt{4\cos^2 t+1}\,dt \\ &= -4\int_1^{-1} \sqrt{4u^2+1}\,du \quad [u=\cos t,\, du = -\sin t\,dt] \ = 4\int_{-1}^{1} \sqrt{4u^2+1}\,du \\ &= 8\int_0^1 \sqrt{4u^2+1}\,du = 8\int_0^{\tan^{-1}2} \sec\theta\cdot\tfrac{1}{2}\sec^2\theta\,d\theta \qquad [2u = \tan\theta,\ 2\,du = \sec^2\theta\,d\theta] \\ &= 4\int_0^{\tan^{-1}2} \sec^3\theta\,d\theta \overset{71}{=} \Big[2\sec\theta\tan\theta + 2\ln|\sec\theta+\tan\theta|\Big]_0^{\tan^{-1}2} = 4\sqrt{5} + 2\ln\!\left(\sqrt{5}+2\right)\end{aligned}$$

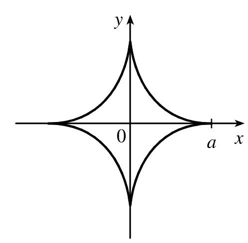
Thus, $L = \int_0^{\pi} |\sin t|\,\sqrt{4\cos^2 t+1}\,dt = \sqrt{5} + \frac{1}{2}\ln\!\left(\sqrt{5}+2\right)$.

53. $x = a\sin\theta,\ y = b\cos\theta,\ 0 \leq \theta \leq 2\pi$.

$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= (a\cos\theta)^2 + (-b\sin\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2(1-\sin^2\theta) + b^2\sin^2\theta \\ &= a^2 - (a^2-b^2)\sin^2\theta = a^2 - c^2\sin^2\theta = a^2\left(1 - \frac{c^2}{a^2}\sin^2\theta\right) = a^2(1-e^2\sin^2\theta)\end{aligned}$$

So $L = 4\int_0^{\pi/2} \sqrt{a^2\left(1-e^2\sin^2\theta\right)}\,d\theta$ [by symmetry] $= 4a\int_0^{\pi/2} \sqrt{1-e^2\sin^2\theta}\,d\theta$.

54. $x = a\cos^3\theta,\ y = a\sin^3\theta$.

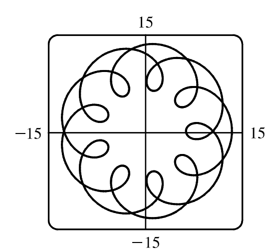


$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= (-3a\cos^2\theta\,\sin\theta)^2 + (3a\sin^2\theta\,\cos\theta)^2 \\ &= 9a^2\cos^4\theta\,\sin^2\theta + 9a^2\sin^4\theta\,\cos^2\theta \\ &= 9a^2\sin^2\theta\,\cos^2\theta(\cos^2\theta+\sin^2\theta) = 9a^2\sin^2\theta\,\cos^2\theta.\end{aligned}$$

The graph has four-fold symmetry and the curve in the first quadrant corresponds to $0 \leq \theta \leq \pi/2$. Thus,

$$\begin{aligned}L &= 4\int_0^{\pi/2} 3a\sin\theta\,\cos\theta\,d\theta \qquad [\text{since } a > 0 \text{ and } \sin\theta \text{ and } \cos\theta \text{ are positive for } 0 \leq \theta \leq \pi/2] \\ &= 12a\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} = 12a\left(\tfrac{1}{2} - 0\right) = 6a\end{aligned}$$

55. (a) $x = 11\cos t - 4\cos(11t/2),\ y = 11\sin t - 4\sin(11t/2)$.

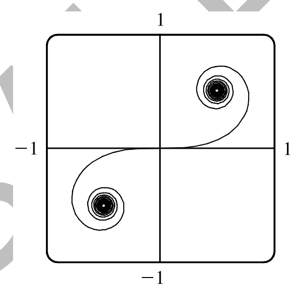


Notice that $0 \leq t \leq 2\pi$ does not give the complete curve because $x(0) \neq x(2\pi)$. In fact, we must take $t \in [0, 4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{2\pi}{11/2} = \frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π.

(b) We use the CAS to find the derivatives dx/dt and dy/dt, and then use Theorem 5 to find the arc length. Recent versions of Maple express the integral $\int_0^{4\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$ as $88E\left(2\sqrt{2}\, i\right)$, where $E(x)$ is the elliptic integral $\int_0^1 \dfrac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}\, dt$ and i is the imaginary number $\sqrt{-1}$.

Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi));` to estimate the length, and find that the arc length is approximately 294.03. Derive's `Para_arc_length` function in the utility file `Int_apps` simplifies the integral to $11\int_0^{4\pi} \sqrt{-4\cos t\, \cos\left(\frac{11t}{2}\right) - 4\sin t\, \sin\left(\frac{11t}{2}\right) + 5}\, dt$.

56. (a) It appears that as $t \to \infty$, $(x, y) \to \left(\frac{1}{2}, \frac{1}{2}\right)$, and as $t \to -\infty$, $(x, y) \to \left(-\frac{1}{2}, -\frac{1}{2}\right)$.



(b) By the Fundamental Theorem of Calculus, $dx/dt = \cos\left(\frac{\pi}{2}t^2\right)$ and $dy/dt = \sin\left(\frac{\pi}{2}t^2\right)$, so by Theorem 5, the length of the curve from the origin to the point with parameter value t is

$$L = \int_0^t \sqrt{\left(\tfrac{dx}{du}\right)^2 + \left(\tfrac{dy}{du}\right)^2}\, du = \int_0^t \sqrt{\cos^2\left(\tfrac{\pi}{2}u^2\right) + \sin^2\left(\tfrac{\pi}{2}u^2\right)}\, du$$
$$= \int_0^t 1\, du = t \qquad [\text{or } -t \text{ if } t < 0]$$

We have used u as the dummy variable so as not to confuse it with the upper limit of integration.

57. $x = t\sin t,\ y = t\cos t,\ 0 \le t \le \pi/2.$ $dx/dt = t\cos t + \sin t$ and $dy/dt = -t\sin t + \cos t$, so

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= t^2\cos^2 t + 2t\sin t\, \cos t + \sin^2 t + t^2\sin^2 t - 2t\sin t\, \cos t + \cos^2 t \\ &= t^2(\cos^2 t + \sin^2 t) + \sin^2 t + \cos^2 t = t^2 + 1\end{aligned}$$

$S = \int 2\pi y\, ds = \int_0^{\pi/2} 2\pi t\cos t\sqrt{t^2+1}\, dt \approx 4.7394.$

58. $x = \sin t,\ y = \sin 2t,\ 0 \le t \le \pi/2.$ $dx/dt = \cos t$ and $dy/dt = 2\cos 2t$, so $(dx/dt)^2 + (dy/dt)^2 = \cos^2 t + 4\cos^2 2t$.

$S = \int 2\pi y\, ds = \int_0^{\pi/2} 2\pi\sin 2t\sqrt{\cos^2 t + 4\cos^2 2t}\, dt \approx 8.0285.$

59. $x = t + e^t,\ y = e^{-t},\ 0 \le t \le 1.$

$dx/dt = 1 + e^t$ and $dy/dt = -e^{-t}$, so $(dx/dt)^2 + (dy/dt)^2 = (1+e^t)^2 + (-e^{-t})^2 = 1 + 2e^t + e^{2t} + e^{-2t}$.

$S = \int 2\pi y\, ds = \int_0^1 2\pi e^{-t}\sqrt{1 + 2e^t + e^{2t} + e^{-2t}}\, dt \approx 10.6705.$

60. $x = t^2 - t^3,\ y = t + t^4,\ 0 \le t \le 1.$

$(dx/dt)^2 + (dy/dt)^2 = (2t - 3t^2)^2 + (1 + 4t^3)^2 = 4t^2 - 12t^3 + 9t^4 + 1 + 8t^3 + 16t^6$, so

$S = \int 2\pi y\, ds = \int_0^1 2\pi(t + t^4)\sqrt{16t^6 + 9t^4 - 4t^3 + 4t^2 + 1}\, dt \approx 12.7176.$

61. $x = t^3,\ y = t^2,\ 0 \le t \le 1.\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(3t^2\right)^2 + (2t)^2 = 9t^4 + 4t^2.$

$$S = \int_0^1 2\pi y \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt = \int_0^1 2\pi t^2 \sqrt{9t^4 + 4t^2}\,dt = 2\pi \int_0^1 t^2 \sqrt{t^2(9t^2+4)}\,dt$$

$$= 2\pi \int_4^{13} \left(\frac{u-4}{9}\right)\sqrt{u}\left(\tfrac{1}{18}\,du\right) \quad \begin{bmatrix} u = 9t^2+4,\ t^2 = (u-4)/9, \\ du = 18t\,dt, \text{so } t\,dt = \frac{1}{18}\,du \end{bmatrix} \quad = \frac{2\pi}{9\cdot 18}\int_4^{13}\left(u^{3/2} - 4u^{1/2}\right)du$$

$$= \tfrac{\pi}{81}\left[\tfrac{2}{5}u^{5/2} - \tfrac{8}{3}u^{3/2}\right]_4^{13} = \tfrac{\pi}{81}\cdot\tfrac{2}{15}\left[3u^{5/2} - 20u^{3/2}\right]_4^{13}$$

$$= \tfrac{2\pi}{1215}\left[\left(3\cdot 13^2\sqrt{13} - 20\cdot 13\sqrt{13}\right) - (3\cdot 32 - 20\cdot 8)\right] = \tfrac{2\pi}{1215}\left(247\sqrt{13} + 64\right)$$

62. $x = 2t^2 + 1/t,\ y = 8\sqrt{t},\ 1 \le t \le 3.$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(4t - \frac{1}{t^2}\right)^2 + \left(\frac{4}{\sqrt{t}}\right)^2 = 16t^2 - \frac{8}{t} + \frac{1}{t^4} + \frac{16}{t} = 16t^2 + \frac{8}{t} + \frac{1}{t^4} = \left(4t + \frac{1}{t^2}\right)^2.$$

$$S = \int_1^3 2\pi y\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt = \int_1^3 2\pi\left(8\sqrt{t}\right)\sqrt{\left(4t + \frac{1}{t^2}\right)^2}\,dt = 16\pi\int_1^3 t^{1/2}\left(4t + t^{-2}\right)dt$$

$$= 16\pi\int_1^3\left(4t^{3/2} + t^{-3/2}\right)dt = 16\pi\left[\tfrac{8}{5}t^{5/2} - 2t^{-1/2}\right]_1^3 = 16\pi\left[\left(\tfrac{72}{5}\sqrt{3} - \tfrac{2}{3}\sqrt{3}\right) - \left(\tfrac{8}{5} - 2\right)\right]$$

$$= 16\pi\left(\tfrac{206}{15}\sqrt{3} + \tfrac{6}{15}\right) = \tfrac{32\pi}{15}\left(103\sqrt{3} + 3\right)$$

63. $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \le \theta \le \frac{\pi}{2}.\quad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(-3a\cos^2\theta\,\sin\theta\right)^2 + \left(3a\sin^2\theta\,\cos\theta\right)^2 = 9a^2\sin^2\theta\,\cos^2\theta.$

$$S = \int_0^{\pi/2} 2\pi\cdot a\sin^3\theta\cdot 3a\sin\theta\,\cos\theta\,d\theta = 6\pi a^2\int_0^{\pi/2}\sin^4\theta\,\cos\theta\,d\theta = \tfrac{6}{5}\pi a^2\left[\sin^5\theta\right]_0^{\pi/2} = \tfrac{6}{5}\pi a^2$$

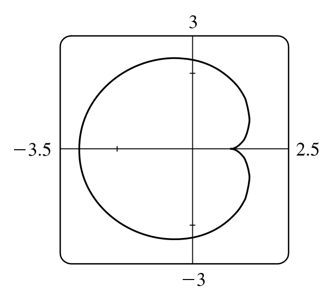
64. $x = 2\cos\theta - \cos 2\theta,\ y = 2\sin\theta - \sin 2\theta \quad\Rightarrow$

$$\left(\tfrac{dx}{d\theta}\right)^2 + \left(\tfrac{dy}{d\theta}\right)^2 = (-2\sin\theta + 2\sin 2\theta)^2 + (2\cos\theta - 2\cos 2\theta)^2$$

$$= 4[(\sin^2\theta - 2\sin\theta\,\sin 2\theta + \sin^2 2\theta) + (\cos^2\theta - 2\cos\theta\,\cos 2\theta + \cos^2 2\theta)]$$

$$= 4[1 + 1 - 2(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta)] = 8[1 - \cos(2\theta - \theta)] = 8(1 - \cos\theta)$$

We plot the graph with parameter interval $[0, 2\pi]$, and see that we should only integrate between 0 and π. (If the interval $[0, 2\pi]$ were taken, the surface of revolution would be generated twice.) Also note that $y = 2\sin\theta - \sin 2\theta = 2\sin\theta(1 - \cos\theta)$. So



$$S = \int_0^\pi 2\pi\cdot 2\sin\theta(1 - \cos\theta)\,2\sqrt{2}\sqrt{1 - \cos\theta}\,d\theta$$

$$= 8\sqrt{2}\pi\int_0^\pi(1 - \cos\theta)^{3/2}\sin\theta\,d\theta = 8\sqrt{2}\pi\int_0^2\sqrt{u^3}\,du \quad \begin{bmatrix} u = 1 - \cos\theta, \\ du = \sin\theta\,d\theta \end{bmatrix}$$

$$= 8\sqrt{2}\pi\left[\left(\tfrac{2}{5}\right)u^{5/2}\right]_0^2 = \tfrac{16}{5}\sqrt{2}\pi\left(2^{5/2}\right) = \tfrac{128}{5}\pi$$

65. $x = 3t^2,\ y = 2t^3,\ 0 \le t \le 5 \quad\Rightarrow\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (6t)^2 + (6t^2)^2 = 36t^2(1 + t^2) \quad\Rightarrow$

$$S = \int_0^5 2\pi x\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^5 2\pi(3t^2)6t\sqrt{1 + t^2}\,dt = 18\pi\int_0^5 t^2\sqrt{1 + t^2}\,2t\,dt$$

$$= 18\pi\int_1^{26}(u - 1)\sqrt{u}\,du \quad \begin{bmatrix} u = 1 + t^2, \\ du = 2t\,dt \end{bmatrix} \quad = 18\pi\int_1^{26}\left(u^{3/2} - u^{1/2}\right)du = 18\pi\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^{26}$$

$$= 18\pi\left[\left(\tfrac{2}{5}\cdot 676\sqrt{26} - \tfrac{2}{3}\cdot 26\sqrt{26}\right) - \left(\tfrac{2}{5} - \tfrac{2}{3}\right)\right] = \tfrac{24}{5}\pi\left(949\sqrt{26} + 1\right)$$

66. $x = e^t - t$, $y = 4e^{t/2}$, $0 \le t \le 1$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} + 2e^t + 1 = (e^t + 1)^2$.

$$\begin{aligned} S &= \int_0^1 2\pi(e^t - t)\sqrt{(e^t-1)^2 + (2e^{t/2})^2}\,dt = \int_0^1 2\pi(e^t - t)(e^t + 1)d \\ &= 2\pi\left[\tfrac{1}{2}e^{2t} + e^t - (t-1)e^t - \tfrac{1}{2}t^2\right]_0^1 = \pi(e^2 + 2e - 6) \end{aligned}$$

67. If f' is continuous and $f'(t) \neq 0$ for $a \le t \le b$, then either $f'(t) > 0$ for all t in $[a, b]$ or $f'(t) < 0$ for all t in $[a, b]$. Thus, f is monotonic (in fact, strictly increasing or strictly decreasing) on $[a, b]$. It follows that f has an inverse. Set $F = g \circ f^{-1}$, that is, define F by $F(x) = g(f^{-1}(x))$. Then $x = f(t) \quad \Rightarrow \quad f^{-1}(x) = t$, so $y = g(t) = g(f^{-1}(x)) = F(x)$.

68. By Formula 8.2.5 with $y = F(x)$, $S = \int_a^b 2\pi F(x)\sqrt{1 + [F'(x)]^2}\,dx$. But by Formula 10.2.1,

$1 + [F'(x)]^2 = 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \left(\dfrac{dy/dt}{dx/dt}\right)^2 = \dfrac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}$. Using the Substitution Rule with $x = x(t)$, where $a = x(\alpha)$ and $b = x(\beta)$, we have $\left[\text{since } dx = \dfrac{dx}{dt}\,dt\right]$

$S = \displaystyle\int_\alpha^\beta 2\pi\, F(x(t))\sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}}\,\frac{dx}{dt}\,dt = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$, which is Formula 10.2.6.

69. (a) $\phi = \tan^{-1}\left(\dfrac{dy}{dx}\right) \quad \Rightarrow \quad \dfrac{d\phi}{dt} = \dfrac{d}{dt}\tan^{-1}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{1 + (dy/dx)^2}\left[\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\right]$. But $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\dot{y}}{\dot{x}} \quad \Rightarrow$

$\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{\dot{y}}{\dot{x}}\right) = \dfrac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2} \quad \Rightarrow \quad \dfrac{d\phi}{dt} = \dfrac{1}{1 + (\dot{y}/\dot{x})^2}\left(\dfrac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2}\right) = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}$. Using the Chain Rule, and the fact that $s = \displaystyle\int_0^t \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt \quad \Rightarrow \quad \tfrac{ds}{dt} = \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2} = \left(\dot{x}^2 + \dot{y}^2\right)^{1/2}$, we have that

$\dfrac{d\phi}{ds} = \dfrac{d\phi/dt}{ds/dt} = \left(\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}\right)\dfrac{1}{(\dot{x}^2 + \dot{y}^2)^{1/2}} = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$. So $\kappa = \left|\dfrac{d\phi}{ds}\right| = \left|\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\right| = \dfrac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$.

(b) $x = x$ and $y = f(x) \quad \Rightarrow \quad \dot{x} = 1$, $\ddot{x} = 0$ and $\dot{y} = \dfrac{dy}{dx}$, $\ddot{y} = \dfrac{d^2y}{dx^2}$.

So $\kappa = \dfrac{\left|1 \cdot (d^2y/dx^2) - 0 \cdot (dy/dx)\right|}{[1 + (dy/dx)^2]^{3/2}} = \dfrac{\left|d^2y/dx^2\right|}{[1 + (dy/dx)^2]^{3/2}}$.

70. (a) $y = x^2 \quad \Rightarrow \quad \dfrac{dy}{dx} = 2x \quad \Rightarrow \quad \dfrac{d^2y}{dx^2} = 2$. So $\kappa = \dfrac{\left|d^2y/dx^2\right|}{[1 + (dy/dx)^2]^{3/2}} = \dfrac{2}{(1 + 4x^2)^{3/2}}$, and at $(1, 1)$,

$\kappa = \dfrac{2}{5^{3/2}} = \dfrac{2}{5\sqrt{5}}$.

(b) $\kappa' = \dfrac{d\kappa}{dx} = -3(1 + 4x^2)^{-5/2}(8x) = 0 \quad \Leftrightarrow \quad x = 0 \quad \Rightarrow \quad y = 0$. This is a maximum since $\kappa' > 0$ for $x < 0$ and $\kappa' < 0$ for $x > 0$. So the parabola $y = x^2$ has maximum curvature at the origin.

71. $x = \theta - \sin\theta \quad \Rightarrow \quad \dot{x} = 1 - \cos\theta \quad \Rightarrow \quad \ddot{x} = \sin\theta$, and $y = 1 - \cos\theta \quad \Rightarrow \quad \dot{y} = \sin\theta \quad \Rightarrow \quad \ddot{y} = \cos\theta$. Therefore,

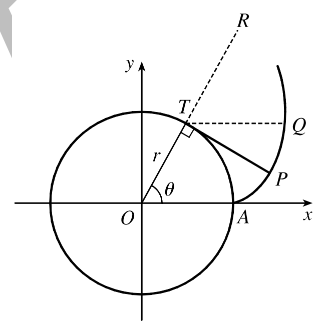
$\kappa = \dfrac{\left|\cos\theta - \cos^2\theta - \sin^2\theta\right|}{[(1 - \cos\theta)^2 + \sin^2\theta]^{3/2}} = \dfrac{\left|\cos\theta - (\cos^2\theta + \sin^2\theta)\right|}{(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)^{3/2}} = \dfrac{|\cos\theta - 1|}{(2 - 2\cos\theta)^{3/2}}$. The top of the arch is characterized by a horizontal tangent, and from Example 2(b) in Section 10.2, the tangent is horizontal when $\theta = (2n-1)\pi$, so take $n = 1$ and substitute $\theta = \pi$ into the expression for κ: $\kappa = \dfrac{|\cos\pi - 1|}{(2 - 2\cos\pi)^{3/2}} = \dfrac{|-1-1|}{[2-2(-1)]^{3/2}} = \dfrac{1}{4}$.

72. (a) Every straight line has parametrizations of the form $x = a + vt$, $y = b + wt$, where a, b are arbitrary and $v, w \neq 0$. For example, a straight line passing through distinct points (a, b) and (c, d) can be described as the parametrized curve $x = a + (c-a)t$, $y = b + (d-b)t$. Starting with $x = a + vt$, $y = b + wt$, we compute $\dot{x} = v$, $\dot{y} = w$, $\ddot{x} = \ddot{y} = 0$, and $\kappa = \dfrac{|v \cdot 0 - w \cdot 0|}{(v^2 + w^2)^{3/2}} = 0$.

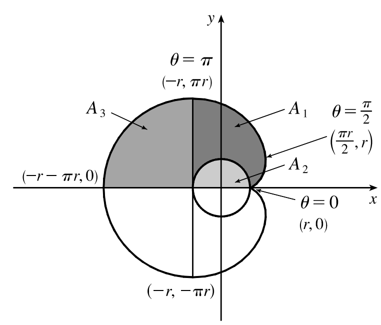
(b) Parametric equations for a circle of radius r are $x = r\cos\theta$ and $y = r\sin\theta$. We can take the center to be the origin. So $\dot{x} = -r\sin\theta \;\Rightarrow\; \ddot{x} = -r\cos\theta$ and $\dot{y} = r\cos\theta \;\Rightarrow\; \ddot{y} = -r\sin\theta$. Therefore,

$\kappa = \dfrac{\left|r^2\sin^2\theta + r^2\cos^2\theta\right|}{(r^2\sin^2\theta + r^2\cos^2\theta)^{3/2}} = \dfrac{r^2}{r^3} = \dfrac{1}{r}$. And so for any θ (and thus any point), $\kappa = \dfrac{1}{r}$.

73. The coordinates of T are $(r\cos\theta, r\sin\theta)$. Since TP was unwound from arc TA, TP has length $r\theta$. Also $\angle PTQ = \angle PTR - \angle QTR = \frac{1}{2}\pi - \theta$, so P has coordinates $x = r\cos\theta + r\theta\cos\left(\frac{1}{2}\pi - \theta\right) = r(\cos\theta + \theta\sin\theta)$, $y = r\sin\theta - r\theta\sin\left(\frac{1}{2}\pi - \theta\right) = r(\sin\theta - \theta\cos\theta)$.



74. If the cow walks with the rope taut, it traces out the portion of the involute in Exercise 73 corresponding to the range $0 \le \theta \le \pi$, arriving at the point $(-r, \pi r)$ when $\theta = \pi$. With the rope now fully extended, the cow walks in a semicircle of radius πr, arriving at $(-r, -\pi r)$. Finally, the cow traces out another portion of the involute, namely the reflection about the x-axis of the initial involute path. (This corresponds to the range $-\pi \le \theta \le 0$.) Referring to the figure, we see that the total grazing area is $2(A_1 + A_3)$. A_3 is one-quarter of the area of a circle of radius πr, so $A_3 = \frac{1}{4}\pi(\pi r)^2 = \frac{1}{4}\pi^3 r^2$. We will compute $A_1 + A_2$ and then subtract $A_2 = \frac{1}{2}\pi r^2$ to obtain A_1.



To find $A_1 + A_2$, first note that the rightmost point of the involute is $\left(\frac{\pi}{2}r, r\right)$. [To see this, note that $dx/d\theta = 0$ when $\theta = 0$ or $\frac{\pi}{2}$. $\theta = 0$ corresponds to the cusp at $(r, 0)$ and $\theta = \frac{\pi}{2}$ corresponds to $\left(\frac{\pi}{2}r, r\right)$.] The leftmost point of the involute is $(-r, \pi r)$. Thus, $A_1 + A_2 = \int_{\theta=\pi}^{\pi/2} y\,dx - \int_{\theta=0}^{\pi/2} y\,dx = \int_{\theta=\pi}^{0} y\,dx$.

Now $y\,dx = r(\sin\theta - \theta\cos\theta)\,r\theta\cos\theta\,d\theta = r^2(\theta\sin\theta\,\cos\theta - \theta^2\cos^2\theta)d\theta$. Integrate:

$(1/r^2)\int y\,dx = -\theta\cos^2\theta - \frac{1}{2}\left(\theta^2 - 1\right)\sin\theta\,\cos\theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta + C$. This enables us to compute