SPR 2003 **MATH 53 – FINAL EXAM** L. Evans

Each problem counts 14 points. Be sure to show all your work. Good luck.

Problem #1. Find the length of the curve

$$\mathbf{r}(t) =< \cos t + t\sin t, \sin t - t\cos t >$$

for $0 \le t \le 10$.

Problem #2. Compute $|\mathbf{r}_u \times \mathbf{r}_v|$ for

$$\mathbf{r}(u, v) =< v, uv, u + v > .$$

Problem #3. Let C denote the circle $(x-2)^2+(y-3)^2 = 4$ in the xy–plane, oriented counterclockwise. Compute

$$\int_C (2x + y^2)dx + (2xy + 3x)dy.$$

Problem #4. Find the critical points of

$$f(x, y) = x^4 - 2x^2 + y^2 - 2,$$

and classify each as a local maximum, a local minimum, or a saddle point.

Problem #5. Calculate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F} =< x, y + z^3, e^y >$ and S is the boundary of the solid region E determined by $0 \le x^2 + y^2 \le 1, 0 \le z \le 1$. Orient S by the outward pointing unit normal field.

Problem #6. Use Lagrange multipliers to find the point on the plane

$$x + 2y + 3z = 14$$

that is closest to the origin.

Problem #7. Let S denote that part of the surface $z = 4 - (x^2 + y^2)$ lying in the half-space $z \ge 0$. Orient S by the upward pointing unit normal field. Compute

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

for the vector field $\mathbf{F} =< x + z, x + y, x^3 >$.

(OVER)

Problem #8. Assume $b > a > 0$. Calculate the value of the integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx$$

in terms of a and b. (Hint: $\frac{e^{-ax}-e^{-bx}}{x} = \int_a^b e^{-xy}\, dy$.)

Problem #9. Let $\mathbf{u}$ and $\mathbf{v}$ be two parametric curves in three dimensions that satisfy

$$\begin{cases} \mathbf{u}' = \mathbf{v} - \mathbf{u} \\ \mathbf{v}' = \mathbf{v} + \mathbf{u} \end{cases}$$

for all times t, where $' = \frac{d}{dt}$. Show that $\mathbf{u} \times \mathbf{v}$ is constant in time.

Problem #10. A point moves along the curve of intersection of the paraboloid $z = x^2 + \frac{1}{4}y^2$ and circular cylinder $x^2 + y^2 = 25$. We are given that $x = 3, y = 4$ and $x' = 4$ at time $t = 0$, where $' = \frac{d}{dt}$. Calculate y' and z' at time $t = 0$.

Problem #11. Suppose that f is a function of a single variable and that the expression

$$u = f(x - ut)$$

implicitly defines u as a function of x and t. Show that

$$u_t + uu_x = 0.$$

Problem #12. Prove the Divergence Theorem in the special case that $\mathbf{F} =< 0, 0, R >$ and E is a solid region lying between two graphs in the z–direction:

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leq z \leq u_2(x, y)\}.$$

Draw a picture to illustrate the various terms in your calculation.

Problem #13. Let S be a surface whose boundary is the positively oriented curve C. Suppose also that f, g are real–valued functions. Show that

$$\iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S} = \int_C (f \nabla g) \cdot d\mathbf{r}.$$

Problem #14. Find the volume of the solid enclosed by the surface

$$(x^2 + y^2 + z^2)^2 = 2z(x^2 + y^2).$$

(Hints: Note that we must have $z \geq 0$ in this formula. You will need to use the identity $\cos^3 \phi = \cos \phi (1 - \sin^2 \phi)$.)