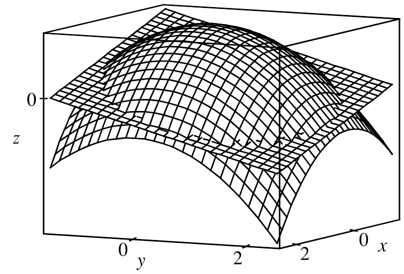
70. To find the equations of the boundary curves, we require that the z-values of the two surfaces be the same. In Maple, we use the command `solve(4-x^2-y^2=1-x-y,y);` and in Mathematica, we use `Solve[4-x^2-y^2==1-x-y,y]`. We find that the curves have equations $y = \dfrac{1 \pm \sqrt{13+4x-4x^2}}{2}$. To find the two points of intersection of these curves, we use the CAS to solve $13+4x-4x^2=0$, finding that $x = \frac{1 \pm \sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is



$$V = \int_{\left(1-\sqrt{14}\right)/2}^{\left(1+\sqrt{14}\right)/2} \int_{\left(1-\sqrt{13+4x-4x^2}\right)/2}^{\left(1+\sqrt{13+4x-4x^2}\right)/2} \left[(4-x^2-y^2)-(1-x-y)\right] dy\,dx = \frac{49\pi}{8}$$

15.3 Double Integrals in Polar Coordinates

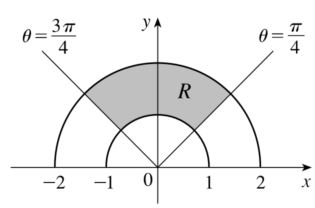
1. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 5, 0 \le \theta \le 2\pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^{2\pi} \int_2^5 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

2. The region R is more easily described by rectangular coordinates: $R = \{(x,y) \mid -1 \le x \le 1,\ -x \le y \le 1\}$. Thus $\iint_R f(x,y)\,dA = \int_{-1}^{1} \int_{-x}^{1} f(x,y)\,dy\,dx$.

3. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 0 \le r \le 1, \pi \le \theta \le 2\pi\}$. Thus $\iint_R f(x,y)\,dA = \int_\pi^{2\pi} \int_0^1 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

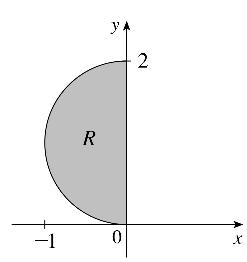
4. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 0 \le r \le 3,\ -\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\right\}$. Thus $\iint_R f(x,y)\,dA = \int_{-\pi/4}^{3\pi/4} \int_0^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

5. The integral $\int_{\pi/4}^{3\pi/4} \int_1^2 r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 1 \le r \le 2, \pi/4 \le \theta \le 3\pi/4\}$, the top quarter portion of a ring (annulus).



$$\int_{\pi/4}^{3\pi/4} \int_1^2 r\,dr\,d\theta = \left(\int_{\pi/4}^{3\pi/4} d\theta\right)\left(\int_1^2 r\,dr\right)$$
$$= \left[\theta\right]_{\pi/4}^{3\pi/4} \left[\tfrac{1}{2}r^2\right]_1^2 = \left(\tfrac{3\pi}{4} - \tfrac{\pi}{4}\right) \cdot \tfrac{1}{2}(4-1) = \tfrac{\pi}{2} \cdot \tfrac{3}{2} = \tfrac{3\pi}{4}$$

6. The integral $\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 0 \le r \le 2\sin\theta, \pi/2 \le \theta \le \pi\}$. Since $r = 2\sin\theta \quad \Rightarrow \quad r^2 = 2r\sin\theta \quad \Leftrightarrow \quad x^2+y^2 = 2y \quad \Leftrightarrow$ $x^2 + (y-1)^2 = 1$, R is the portion in the second quadrant of a disk of radius 1 with center $(0,1)$.



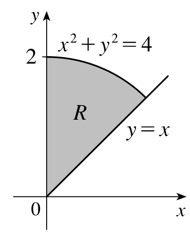
$$\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\,dr\,d\theta = \int_{\pi/2}^{\pi} \left[\tfrac{1}{2}r^2\right]_{r=0}^{r=2\sin\theta} d\theta = \int_{\pi/2}^{\pi} 2\sin^2\theta\,d\theta$$
$$= \int_{\pi/2}^{\pi} 2 \cdot \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/2}^{\pi}$$
$$= \pi - 0 - \tfrac{\pi}{2} + 0 = \tfrac{\pi}{2}$$

7. The half disk D can be described in polar coordinates as $D = \{(r,\theta) \mid 0 \le r \le 5, 0 \le \theta \le \pi\}$. Then

$$\iint_D x^2 y\, dA = \int_0^\pi \int_0^5 (r\cos\theta)^2 (r\sin\theta)\, r\, dr\, d\theta = \left(\int_0^\pi \cos^2\theta \sin\theta\, d\theta\right)\left(\int_0^5 r^4\, dr\right)$$
$$= \left[-\tfrac{1}{3}\cos^3\theta\right]_0^\pi \left[\tfrac{1}{5}r^5\right]_0^5 = -\tfrac{1}{3}(-1-1)\cdot 625 = \tfrac{1250}{3}$$

8. The region R is $\frac{1}{8}$ of a disk, as shown in the figure, and can be described by $R = \{(r,\theta) \mid 0 \le r \le 2, \pi/4 \le \theta \le \pi/2\}$. Thus

$$\iint_R (2x - y)\, dA = \int_{\pi/4}^{\pi/2} \int_0^2 (2r\cos\theta - r\sin\theta)\, r\, dr\, d\theta$$
$$= \int_{\pi/4}^{\pi/2} (2\cos\theta - \sin\theta)\, d\theta \int_0^2 r^2\, dr$$
$$= \left[2\sin\theta + \cos\theta\right]_{\pi/4}^{\pi/2} \left[\tfrac{1}{3}r^3\right]_0^2$$
$$= \left(2 + 0 - \sqrt{2} - \tfrac{\sqrt{2}}{2}\right)\left(\tfrac{8}{3}\right) = \tfrac{16}{3} - 4\sqrt{2}$$



9. $\iint_R \sin(x^2+y^2)\, dA = \int_0^{\pi/2} \int_1^3 \sin(r^2)\, r\, dr\, d\theta = \int_0^{\pi/2} d\theta \int_1^3 r\sin(r^2)\, dr = \left[\theta\right]_0^{\pi/2} \left[-\tfrac{1}{2}\cos(r^2)\right]_1^3$

$$= \left(\tfrac{\pi}{2}\right)\left[-\tfrac{1}{2}(\cos 9 - \cos 1)\right] = \tfrac{\pi}{4}(\cos 1 - \cos 9)$$

10. $\displaystyle\iint_R \frac{y^2}{x^2+y^2}\, dA = \int_0^{2\pi} \int_a^b \frac{(r\sin\theta)^2}{r^2}\, r\, dr\, d\theta = \int_0^{2\pi} \sin^2\theta\, d\theta \int_a^b r\, dr = \int_0^{2\pi} \frac{1}{2}(1-\cos 2\theta)\, d\theta \int_a^b r\, dr$

$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{1}{2}r^2\right]_a^b = \tfrac{1}{2}(2\pi - 0 - 0)\cdot\tfrac{1}{2}\left(b^2 - a^2\right) = \tfrac{\pi}{2}(b^2 - a^2)$$

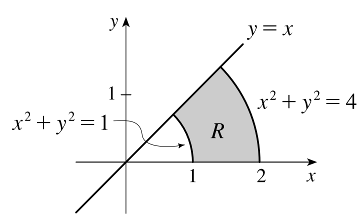
11. $\iint_D e^{-x^2-y^2}\, dA = \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2}\, r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^2 re^{-r^2}\, dr$

$$= \left[\theta\right]_{-\pi/2}^{\pi/2} \left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)(e^{-4} - e^0) = \tfrac{\pi}{2}(1 - e^{-4})$$

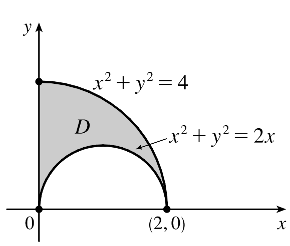
12. $\iint_D \cos\sqrt{x^2+y^2}\, dA = \int_0^{2\pi} \int_0^2 \cos\sqrt{r^2}\; r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 r\cos r\, dr$. For the second integral, integrate by parts with $u = r$, $dv = \cos r\, dr$. Then $\iint_D \cos\sqrt{x^2+y^2}\, dA = \left[\theta\right]_0^{2\pi} \left[r\sin r + \cos r\right]_0^2 = 2\pi(2\sin 2 + \cos 2 - 1)$.

13. R is the region shown in the figure, and can be described by $R = \{(r,\theta) \mid 0 \le \theta \le \pi/4, 1 \le r \le 2\}$. Thus $\iint_R \arctan(y/x)\, dA = \int_0^{\pi/4} \int_1^2 \arctan(\tan\theta)\, r\, dr\, d\theta$ since $y/x = \tan\theta$. Also, $\arctan(\tan\theta) = \theta$ for $0 \le \theta \le \pi/4$, so the integral becomes

$\int_0^{\pi/4} \int_1^2 \theta\, r\, dr\, d\theta = \int_0^{\pi/4} \theta\, d\theta \int_1^2 r\, dr = \left[\tfrac{1}{2}\theta^2\right]_0^{\pi/4} \left[\tfrac{1}{2}r^2\right]_1^2 = \tfrac{\pi^2}{32}\cdot\tfrac{3}{2} = \tfrac{3}{64}\pi^2$.



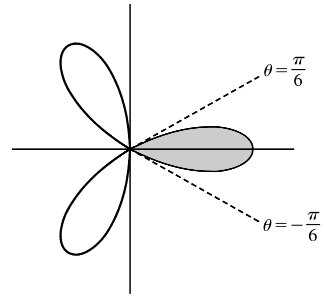
14.



$$\iint_D x\, dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge 0,\, y\ge 0}} x\, dA - \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\, dA$$
$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\, dr\, d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\, dr\, d\theta$$
$$= \int_0^{\pi/2} \tfrac{1}{3}(8\cos\theta)\, d\theta - \int_0^{\pi/2} \tfrac{1}{3}(8\cos^4\theta)\, d\theta$$
$$= \tfrac{8}{3} - \tfrac{8}{12}\left[\cos^3\theta\sin\theta + \tfrac{3}{2}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2}$$
$$= \tfrac{8}{3} - \tfrac{2}{3}\left[0 + \tfrac{3}{2}\left(\tfrac{\pi}{2}\right)\right] = \tfrac{16 - 3\pi}{6}$$

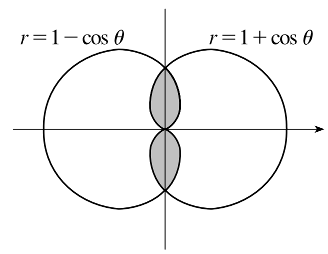
15. One loop is given by the region

$D = \{(r, \theta) \,|\, -\pi/6 \le \theta \le \pi/6, 0 \le r \le \cos 3\theta\}$, so the area is



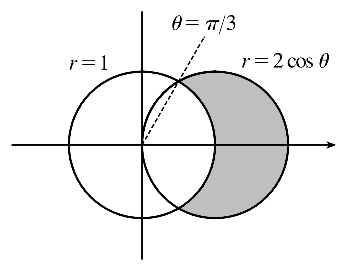
$$\iint_D dA = \int_{-\pi/6}^{\pi/6} \int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6} \left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$
$$= \int_{-\pi/6}^{\pi/6} \frac{1}{2}\cos^2 3\theta\,d\theta = 2\int_0^{\pi/6} \frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right) d\theta$$
$$= \frac{1}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{12}$$

16. By symmetry, the area of the region is 4 times the area of the region D in the first quadrant enclosed by the cardiod $r = 1 - \cos\theta$ (see the figure). Here $D = \{(r, \theta) \mid 0 \le r \le 1 - \cos\theta, 0 \le \theta \le \pi/2\}$, so the total area is



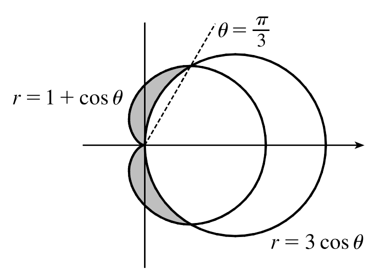
$$4A(D) = 4\iint_D dA = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta = 4\int_0^{\pi/2}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=1-\cos\theta} d\theta$$
$$= 2\int_0^{\pi/2}(1-\cos\theta)^2 d\theta = 2\int_0^{\pi/2}(1 - 2\cos\theta + \cos^2\theta)\,d\theta$$
$$= 2\int_0^{\pi/2}\left[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right] d\theta$$
$$= 2\left[\theta - 2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$
$$= 2\left(\tfrac{\pi}{2} - 2 + \tfrac{\pi}{4}\right) = \tfrac{3\pi}{2} - 4$$

17. In polar coordinates the circle $(x-1)^2 + y^2 = 1 \quad\Leftrightarrow\quad x^2 + y^2 = 2x$ is $r^2 = 2r\cos\theta \quad\Rightarrow\quad r = 2\cos\theta$, and the circle $x^2 + y^2 = 1$ is $r = 1$. The curves intersect in the first quadrant when $2\cos\theta = 1 \quad\Rightarrow\quad \cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \pi/3$, so the portion of the region in the first quadrant is given by $D = \{(r, \theta) \mid 1 \le r \le 2\cos\theta, 0 \le \theta \le \pi/3\}$. By symmetry, the total area is twice the area of D:



$$2A(D) = 2\iint_D dA = 2\int_0^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3}\left[\tfrac{1}{2}r^2\right]_{r=1}^{r=2\cos\theta} d\theta$$
$$= \int_0^{\pi/3}\left(4\cos^2\theta - 1\right) d\theta = \int_0^{\pi/3}\left[4\cdot\tfrac{1}{2}(1+\cos 2\theta) - 1\right] d\theta$$
$$= \int_0^{\pi/3}(1 + 2\cos 2\theta)\,d\theta = \left[\theta + \sin 2\theta\right]_0^{\pi/3} = \tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}$$

18. The region lies between the two polar curves in quadrants I and IV, but in quadrants II and III the region is enclosed by the cardioid. In the first quadrant, $1 + \cos\theta = 3\cos\theta$ when $\cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{3}$, so the area of the region inside the cardioid and outside the circle is



$$A_1 = \int_{\pi/3}^{\pi/2}\int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta = \int_{\pi/3}^{\pi/2}\left[\tfrac{1}{2}r^2\right]_{r=3\cos\theta}^{r=1+\cos\theta} d\theta$$
$$= \tfrac{1}{2}\int_{\pi/3}^{\pi/2}(1 + 2\cos\theta - 8\cos^2\theta)d\theta = \tfrac{1}{2}\left[\theta + 2\sin\theta - 8\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right)\right]_{\pi/3}^{\pi/2}$$
$$= \left[-\tfrac{3}{2}\theta + \sin\theta - \sin 2\theta\right]_{\pi/3}^{\pi/2} = \left(-\tfrac{3\pi}{4} + 1 - 0\right) - \left(-\tfrac{\pi}{2} + \tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{2}\right) = 1 - \tfrac{\pi}{4}.$$

[continued]

The area of the region in the second quadrant is

$$\begin{aligned} A_2 &= \int_{\pi/2}^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_{\pi/2}^{\pi}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=1+\cos\theta} d\theta = \tfrac{1}{2}\int_{\pi/2}^{\pi}(1+2\cos\theta+\cos^2\theta)d\theta \\ &= \tfrac{1}{2}\left[\theta+2\sin\theta+\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi} = \tfrac{1}{2}\left(\tfrac{3\pi}{4}-2\right) = \tfrac{3\pi}{8}-1. \end{aligned}$$

By symmetry, the total area is $A = 2(A_1+A_2) = 2\left(1-\frac{\pi}{4}+\frac{3\pi}{8}-1\right) = \frac{\pi}{4}$.

19. $V = \iint_{x^2+y^2\le 25}\left(x^2+y^2\right)dA = \int_0^{2\pi}\int_0^5 r^2\cdot r\,dr\,d\theta = \int_0^{2\pi}d\theta\,\int_0^5 r^3\,dr = \left[\,\theta\,\right]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^5 = 2\pi\left(\frac{625}{4}\right) = \frac{625}{2}\pi$

20. $V = \iint_{1\le x^2+y^2\le 4}\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_1^2\sqrt{r^2}\;r\,dr\,d\theta = \int_0^{2\pi}d\theta\,\int_1^2 r^2\,dr = \left[\,\theta\,\right]_0^{2\pi}\left[\frac{1}{3}r^3\right]_1^2 = 2\pi\left(\frac{8}{3}-\frac{1}{3}\right) = \frac{14}{3}\pi$

21. $2x+y+z=4 \quad\Leftrightarrow\quad z=4-2x-y$, so the volume of the solid is

$$\begin{aligned} V &= \iint_{x^2+y^2\le 1}(4-2x-y)\,dA = \int_0^{2\pi}\int_0^1(4-2r\cos\theta-r\sin\theta)\,r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1\left[4r-r^2\left(2\cos\theta+\sin\theta\right)\right]dr\,d\theta = \int_0^{2\pi}\left[2r^2-\tfrac{1}{3}r^3\left(2\cos\theta+\sin\theta\right)\right]_{r=0}^{r=1}d\theta \\ &= \int_0^{2\pi}\left[2-\tfrac{1}{3}\left(2\cos\theta+\sin\theta\right)\right]d\theta = \left[2\theta-\tfrac{1}{3}\left(2\sin\theta-\cos\theta\right)\right]_0^{2\pi} = 4\pi+\tfrac{1}{3}-0-\tfrac{1}{3} = 4\pi \end{aligned}$$

22. The sphere $x^2+y^2+z^2=16$ intersects the xy-plane in the circle $x^2+y^2=16$, so

$$\begin{aligned} V &= 2\iint\limits_{4\le x^2+y^2\le 16}\sqrt{16-x^2-y^2}\,dA \quad\text{[by symmetry]}\quad = 2\int_0^{2\pi}\int_2^4\sqrt{16-r^2}\;r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_2^4 r(16-r^2)^{1/2}dr \\ &= 2\left[\,\theta\,\right]_0^{2\pi}\left[-\tfrac{1}{3}(16-r^2)^{3/2}\right]_2^4 = -\tfrac{2}{3}(2\pi)(0-12^{3/2}) = \tfrac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi \end{aligned}$$

23. By symmetry,

$$\begin{aligned} V &= 2\iint\limits_{x^2+y^2\le a^2}\sqrt{a^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_0^a\sqrt{a^2-r^2}\;r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_0^a r\sqrt{a^2-r^2}\,dr \\ &= 2\left[\,\theta\,\right]_0^{2\pi}\left[-\tfrac{1}{3}(a^2-r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0+\tfrac{1}{3}a^3\right) = \tfrac{4}{3}\pi a^3 \end{aligned}$$

24. The paraboloid $z=1+2x^2+2y^2$ intersects the plane $z=7$ when $7=1+2x^2+2y^2$ or $x^2+y^2=3$ and we are restricted to the first octant, so

$$\begin{aligned} V &= \iint\limits_{\substack{x^2+y^2\le 3,\\ x\ge 0,y\ge 0}}\left[7-\left(1+2x^2+2y^2\right)\right]dA = \int_0^{\pi/2}\int_0^{\sqrt{3}}\left[7-\left(1+2r^2\right)\right]r\,dr\,d\theta \\ &= \int_0^{\pi/2}d\theta\,\int_0^{\sqrt{3}}\left(6r-2r^3\right)dr = \left[\,\theta\,\right]_0^{\pi/2}\left[3r^2-\tfrac{1}{2}r^4\right]_0^{\sqrt{3}} = \tfrac{\pi}{2}\cdot\tfrac{9}{2} = \tfrac{9}{4}\pi \end{aligned}$$

25. The cone $z=\sqrt{x^2+y^2}$ intersects the sphere $x^2+y^2+z^2=1$ when $x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=1$ or $x^2+y^2=\frac{1}{2}$. So

$$\begin{aligned} V &= \iint\limits_{x^2+y^2\le 1/2}\left(\sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}\right)dA = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\left(\sqrt{1-r^2}-r\right)r\,dr\,d\theta \\ &= \int_0^{2\pi}d\theta\,\int_0^{1/\sqrt{2}}\left(r\sqrt{1-r^2}-r^2\right)dr = \left[\,\theta\,\right]_0^{2\pi}\left[-\tfrac{1}{3}(1-r^2)^{3/2}-\tfrac{1}{3}r^3\right]_0^{1/\sqrt{2}} = 2\pi\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{\sqrt{2}}-1\right) = \tfrac{\pi}{3}\left(2-\sqrt{2}\right) \end{aligned}$$

26. The two paraboloids intersect when $6 - x^2 - y^2 = 2x^2 + 2y^2$ or $x^2 + y^2 = 2$. For $x^2 + y^2 \le 2$, the paraboloid $z = 6 - x^2 - y^2$ is above $z = 2x^2 + 2y^2$ so

$$V = \iint\limits_{x^2+y^2\le 2} [(6 - x^2 - y^2) - (2x^2 + 2y^2)]\, dA = \iint\limits_{x^2+y^2\le 2} [6 - 3(x^2 + y^2)]\, dA = \int_0^{2\pi}\int_0^{\sqrt{2}} (6 - 3r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} (6r - 3r^3)\, dr = \big[\,\theta\,\big]_0^{2\pi} \big[3r^2 - \tfrac{3}{4}r^4\big]_0^{\sqrt{2}} = 2\pi\,(6 - 3) = 6\pi$$

27. The given solid is the region inside the cylinder $x^2 + y^2 = 4$ between the surfaces $z = \sqrt{64 - 4x^2 - 4y^2}$ and $z = -\sqrt{64 - 4x^2 - 4y^2}$. So

$$V = \iint\limits_{x^2+y^2\le 4} \Big[\sqrt{64 - 4x^2 - 4y^2} - \Big(-\sqrt{64 - 4x^2 - 4y^2}\Big)\Big]\, dA = \iint\limits_{x^2+y^2\le 4} 2\cdot 2\,\sqrt{16 - x^2 - y^2}\, dA$$

$$= 4\int_0^{2\pi}\int_0^2 \sqrt{16 - r^2}\; r\, dr\, d\theta = 4\int_0^{2\pi} d\theta \int_0^2 r\,\sqrt{16 - r^2}\, dr = 4\,\big[\,\theta\,\big]_0^{2\pi} \Big[-\tfrac{1}{3}(16 - r^2)^{3/2}\Big]_0^2$$

$$= 8\pi\big(-\tfrac{1}{3}\big)(12^{3/2} - 16^{2/3}) = \tfrac{8\pi}{3}\big(64 - 24\sqrt{3}\,\big)$$

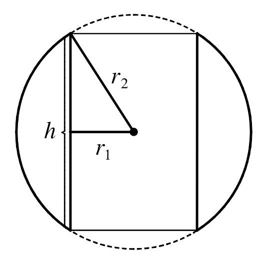
28. (a) Here the region in the xy-plane is the annular region $r_1^2 \le x^2 + y^2 \le r_2^2$ and the desired volume is twice that above the xy-plane. Hence

$$V = 2\iint\limits_{r_1^2\le x^2+y^2\le r_2^2} \sqrt{r_2^2 - x^2 - y^2}\, dA = 2\int_0^{2\pi}\int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\; r\, dr\, d\theta = 2\int_0^{2\pi} d\theta \int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\; r\, dr$$

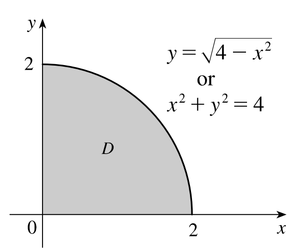
$$= 2\,(2\pi)\Big[-\tfrac{1}{3}(r_2^2 - r^2)^{3/2}\Big]_{r_1}^{r_2} = \tfrac{4\pi}{3}(r_2^2 - r_1^2)^{3/2}$$

(b) A cross-sectional cut is shown in the figure. So $r_2^2 = \big(\tfrac{1}{2}h\big)^2 + r_1^2$ or $\tfrac{1}{4}h^2 = r_2^2 - r_1^2$.

Thus the volume in terms of h is $V = \tfrac{4\pi}{3}\big(\tfrac{1}{4}h^2\big)^{3/2} = \tfrac{\pi}{6}h^3$.



29.

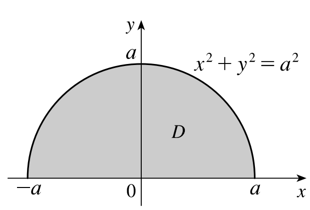


$$\int_0^2\int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx = \int_0^{\pi/2}\int_0^2 e^{-r^2} r\, dr\, d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^2 re^{-r^2} dr = \big[\,\theta\,\big]_0^{\pi/2} \Big[-\tfrac{1}{2}e^{-r^2}\Big]_0^2$$

$$= \tfrac{\pi}{2}\left[-\tfrac{1}{2}\left(e^{-4} - 1\right)\right] = \tfrac{\pi}{4}\left(1 - e^{-4}\right)$$

30.



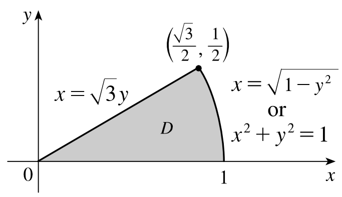
$$\int_0^a\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x + y)\, dx\, dy = \int_0^{\pi}\int_0^a (2r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi} (2\cos\theta + \sin\theta)\, d\theta \int_0^a r^2\, dr$$

$$= [2\sin\theta - \cos\theta]_0^{\pi}\,\big[\tfrac{1}{3}r^3\big]_0^a$$

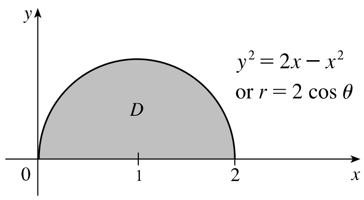
$$= [(0 + 1) - (0 - 1)]\cdot\tfrac{1}{3}(a^3 - 0) = \tfrac{2}{3}a^3$$

31. The region D of integration is shown in the figure. In polar coordinates the line $x = \sqrt{3}\,y$ is $\theta = \pi/6$, so



$$\begin{aligned}\int_0^{1/2}\int_{\sqrt{3}\,y}^{\sqrt{1-y^2}} xy^2\,dx\,dy &= \int_0^{\pi/6}\int_0^1 (r\cos\theta)(r\sin\theta)^2\,r\,dr\,d\theta \\ &= \int_0^{\pi/6}\sin^2\theta\cos\theta\,d\theta\,\int_0^1 r^4\,dr \\ &= \left[\tfrac{1}{3}\sin^3\theta\right]_0^{\pi/6}\left[\tfrac{1}{5}r^5\right]_0^1 \\ &= \left[\tfrac{1}{3}\left(\tfrac{1}{2}\right)^3 - 0\right]\left[\tfrac{1}{5} - 0\right] = \tfrac{1}{120}\end{aligned}$$

32.



$$\begin{aligned}\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta &= \int_0^{\pi/2}\left[\tfrac{1}{3}r^3\right]_{r=0}^{r=2\cos\theta}d\theta = \int_0^{\pi/2}\left(\tfrac{8}{3}\cos^3\theta\right)d\theta \\ &= \tfrac{8}{3}\int_0^{\pi/2}(1-\sin^2\theta)\cos\theta\,d\theta \\ &= \tfrac{8}{3}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right]_0^{\pi/2} = \tfrac{16}{9}\end{aligned}$$

33. $D = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$, so

$\iint_D e^{(x^2+y^2)^2}\,dA = \int_0^{2\pi}\int_0^1 e^{(r^2)^2}\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^1 re^{r^4}\,dr = 2\pi\int_0^1 re^{r^4}\,dr$. Using a calculator, we estimate $2\pi\int_0^1 re^{r^4}\,dr \approx 4.5951$.

34. $D = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le \pi/2\}$, so

$$\begin{aligned}\iint_D xy\sqrt{1+x^2+y^2}\,dA &= \int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)\sqrt{1+r^2}\;r\,dr\,d\theta \\ &= \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\,\int_0^1 r^3\sqrt{1+r^2}\,dr = \left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2}\int_0^1 r^3\sqrt{1+r^2}\,dr \\ &= \tfrac{1}{2}\int_0^1 r^3\sqrt{1+r^2}\,dr \approx 0.1609\end{aligned}$$

35. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x,y)$ to be the depth of the water at (x,y), then the volume of water in the pool is the volume of the solid that lies above $D = \{(x,y) \mid x^2+y^2 \le 400\}$ and below the graph of $f(x,y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0,-20) = 2$ to $f(0,20) = 7$. The trace in the yz-plane is a line segment from $(0,-20,2)$ to $(0,20,7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z - 7 = \frac{1}{8}(y-20) \;\Rightarrow\; z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x,y)$ is independent of x, $f(x,y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x,y)\,dA$, which is most conveniently evaluated using polar coordinates. Then $D = \{(r,\theta) \mid 0 \le r \le 20, 0 \le \theta \le 2\pi\}$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\begin{aligned}\int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta + \tfrac{9}{2}\right)r\,dr\,d\theta &= \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta + \tfrac{9}{4}r^2\right]_{r=0}^{r=20}d\theta = \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta + 900\right)d\theta \\ &= \left[-\tfrac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi} = 1800\pi\end{aligned}$$

Thus the pool contains $1800\pi \approx 5655$ ft^3 of water.

36. (a) If $R \le 100$, the total amount of water supplied each hour to the region within R feet of the sprinkler is

$$V = \int_0^{2\pi}\int_0^R e^{-r} r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^R re^{-r}\,dr = \big[\,\theta\,\big]_0^{2\pi}\big[-re^{-r} - e^{-r}\big]_0^R$$
$$= 2\pi[-Re^{-R} - e^{-R} + 0 + 1] = 2\pi(1 - Re^{-R} - e^{-R})\text{ ft}^3$$

(b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is

$\dfrac{V}{\text{area of region}} = \dfrac{V}{\pi R^2} = \dfrac{2\left(1 - Re^{-R} - e^{-R}\right)}{R^2}$ ft^3 (per hour per square foot). See the definition of the average value of a function on page 1037 [ET 997].

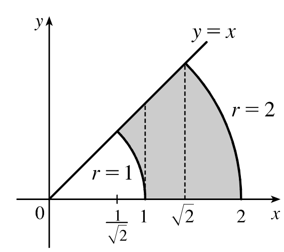
37. As in Exercise 15.2.61, $f_{\text{ave}} = \frac{1}{A(D)}\iint_D f(x,y)dA$. Here $D = \{(r,\theta) \mid a \le r \le b, 0 \le \theta \le 2\pi\}$, so $A(D) = \pi b^2 - \pi a^2 = \pi(b^2 - a^2)$ and

$$f_{\text{ave}} = \frac{1}{A(D)}\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dA = \frac{1}{\pi(b^2-a^2)}\int_0^{2\pi}\int_a^b \frac{1}{\sqrt{r^2}}\,r\,dr\,d\theta = \frac{1}{\pi(b^2-a^2)}\int_0^{2\pi} d\theta \int_a^b dr$$
$$= \frac{1}{\pi(b^2-a^2)}\,\big[\theta\big]_0^{2\pi}\,\big[r\big]_a^b = \frac{1}{\pi(b^2-a^2)}\,(2\pi)(b-a) = \frac{2(b-a)}{(b+a)(b-a)} = \frac{2}{a+b}$$

38. The distance from a point (x,y) to the origin is $f(x,y) = \sqrt{x^2+y^2}$, so the average distance from points in D to the origin is

$$f_{\text{ave}} = \tfrac{1}{A(D)}\iint_D \sqrt{x^2+y^2}\,dA = \tfrac{1}{\pi a^2}\int_0^{2\pi}\int_0^a \sqrt{r^2}\;r\,dr\,d\theta$$
$$= \tfrac{1}{\pi a^2}\int_0^{2\pi} d\theta \int_0^a r^2\,dr = \tfrac{1}{\pi a^2}\,[\theta]_0^{2\pi}\,\left[\tfrac13 r^3\right]_0^a = \tfrac{1}{\pi a^2}\cdot 2\pi\cdot\tfrac13 a^3 = \tfrac23 a$$

39. $$\int_{1/\sqrt2}^1\int_{\sqrt{1-x^2}}^x xy\,dy\,dx + \int_1^{\sqrt2}\int_0^x xy\,dy\,dx + \int_{\sqrt2}^2\int_0^{\sqrt{4-x^2}} xy\,dy\,dx$$
$$= \int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=1}^{r=2} d\theta$$
$$= \frac{15}{4}\int_0^{\pi/4}\sin\theta\cos\theta\,d\theta = \frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{15}{16}$$



40. (a) $\iint_{D_a} e^{-(x^2+y^2)}dA = \int_0^{2\pi}\int_0^a re^{-r^2}\,dr\,d\theta = 2\pi\left[-\frac12 e^{-r^2}\right]_0^a = \pi\left(1 - e^{-a^2}\right)$ for each a. Then $\lim\limits_{a\to\infty}\pi\left(1 - e^{-a^2}\right) = \pi$ since $e^{-a^2} \to 0$ as $a \to \infty$. Hence $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi$.

(b) $\iint_{S_a} e^{-(x^2+y^2)}\,dA = \int_{-a}^a\int_{-a}^a e^{-x^2}e^{-y^2}\,dx\,dy = \left(\int_{-a}^a e^{-x^2}\,dx\right)\left(\int_{-a}^a e^{-y^2}\,dy\right)$ for each a.

Then, from (a), $\pi = \iint_{\mathbb{R}^2} -(x^2+y^2)\,dA$, so

$\pi = \lim\limits_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA = \lim\limits_{a\to\infty}\left(\int_{-a}^a e^{-x^2}\,dx\right)\left(\int_{-a}^a e^{-y^2}\,dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right).$

To evaluate $\lim\limits_{a\to\infty}\left(\int_{-a}^a e^{-x^2}\,dx\right)\left(\int_{-a}^a e^{-y^2}\,dy\right)$, we are using the fact that these integrals are bounded. This is true since on $[-1,1]$, $0 < e^{-x^2} \le 1$ while on $(-\infty,-1)$, $0 < e^{-x^2} \le e^x$ and on $(1,\infty)$, $0 < e^{-x^2} < e^{-x}$. Hence $0 \le \int_{-\infty}^{\infty} e^{-x^2}\,dx \le \int_{-\infty}^{-1} e^x\,dx + \int_{-1}^1 dx + \int_1^{\infty} e^{-x}\,dx = 2(e^{-1}+1)$.

(c) Since $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \pi$ and y can be replaced by x, $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$ implies that

$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \pm\sqrt{\pi}$. But $e^{-x^2} \geq 0$ for all x, so $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

(d) Letting $t = \sqrt{2}\,x$, $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\left(e^{-t^2/2}\right) dt$, so that $\sqrt{\pi} = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt$ or $\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}$.

41. (a) We integrate by parts with $u = x$ and $dv = xe^{-x^2}\,dx$. Then $du = dx$ and $v = -\frac{1}{2}e^{-x^2}$, so

$$\begin{aligned}\int_0^{\infty} x^2e^{-x^2}\,dx &= \lim_{t\to\infty}\int_0^t x^2e^{-x^2}\,dx = \lim_{t\to\infty}\left(\left[-\tfrac{1}{2}xe^{-x^2}\right]_0^t + \int_0^t \tfrac{1}{2}e^{-x^2}\,dx\right)\\ &= \lim_{t\to\infty}\left(-\tfrac{1}{2}te^{-t^2}\right) + \tfrac{1}{2}\int_0^{\infty} e^{-x^2}\,dx = 0 + \tfrac{1}{2}\int_0^{\infty} e^{-x^2}\,dx \qquad \text{[by l'Hospital's Rule]}\\ &= \tfrac{1}{4}\int_{-\infty}^{\infty} e^{-x^2}\,dx \qquad \text{[since } e^{-x^2} \text{ is an even function]}\\ &= \tfrac{1}{4}\sqrt{\pi} \qquad \text{[by Exercise 40(c)]}\end{aligned}$$

(b) Let $u = \sqrt{x}$. Then $u^2 = x \quad\Rightarrow\quad dx = 2u\,du \quad\Rightarrow$

$\int_0^{\infty} \sqrt{x}\,e^{-x}\,dx = \lim_{t\to\infty}\int_0^t \sqrt{x}\,e^{-x}\,dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u^2}2u\,du = 2\int_0^{\infty} u^2e^{-u^2}\,du = 2\left(\frac{1}{4}\sqrt{\pi}\right)$ [by part(a)] $= \frac{1}{2}\sqrt{\pi}$.

15.4 Applications of Double Integrals

1. $Q = \iint_D \sigma(x,y)\,dA = \int_0^5\int_2^5 (2x+4y)\,dy\,dx = \int_0^5 \left[2xy + 2y^2\right]_{y=2}^{y=5} dx$

$= \int_0^5 (10x + 50 - 4x - 8)\,dx = \int_0^5 (6x+42)\,dx = \left[3x^2 + 42x\right]_0^5 = 75 + 210 = 285$ C

2. $Q = \iint_D \sigma(x,y)\,dA = \iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2}\;r\,dr\,d\theta$

$= \int_0^{2\pi} d\theta\;\int_0^1 r^2\,dr = [\theta]_0^{2\pi}\;\left[\frac{1}{3}r^3\right]_0^1 = 2\pi\cdot\frac{1}{3} = \frac{2\pi}{3}$ C

3. $m = \iint_D \rho(x,y)\,dA = \int_1^3\int_1^4 ky^2\,dy\,dx = k\int_1^3 dx\;\int_1^4 y^2\,dy = k\,[x]_1^3\;\left[\frac{1}{3}y^3\right]_1^4 = k(2)(21) = 42k$,

$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{1}{42k}\int_1^3\int_1^4 kxy^2\,dy\,dx = \frac{1}{42}\int_1^3 x\,dx\;\int_1^4 y^2\,dy = \frac{1}{42}\left[\frac{1}{2}x^2\right]_1^3\;\left[\frac{1}{3}y^3\right]_1^4 = \frac{1}{42}(4)(21) = 2$,

$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{1}{42k}\int_1^3\int_1^4 ky^3\,dy\,dx = \frac{1}{42}\int_1^3 dx\;\int_1^4 y^3\,dy = \frac{1}{42}\,[x]_1^3\;\left[\frac{1}{4}y^4\right]_1^4 = \frac{1}{42}(2)\left(\frac{255}{4}\right) = \frac{85}{28}$

Hence $m = 42k$, $(\overline{x},\overline{y}) = \left(2, \frac{85}{28}\right)$.

4. $m = \iint_D \rho(x,y)\,dA = \int_0^a\int_0^b (1 + x^2 + y^2)\,dy\,dx = \int_0^a \left[y + x^2y + \frac{1}{3}y^3\right]_{y=0}^{y=b} dx = \int_0^a \left(b + bx^2 + \frac{1}{3}b^3\right) dx$

$= \left[bx + \frac{1}{3}bx^3 + \frac{1}{3}b^3x\right]_0^a = ab + \frac{1}{3}a^3b + \frac{1}{3}ab^3 = \frac{1}{3}ab(3 + a^2 + b^2)$,

$M_y = \iint_D x\rho(x,y)\,dA = \int_0^a\int_0^b (x + x^3 + xy^2)\,dy\,dx = \int_0^a \left[xy + x^3y + \frac{1}{3}xy^3\right]_{y=0}^{y=b} dx = \int_0^a \left(bx + bx^3 + \frac{1}{3}b^3x\right) dx$

$= \left[\frac{1}{2}bx^2 + \frac{1}{4}bx^4 + \frac{1}{6}b^3x^2\right]_0^a = \frac{1}{2}a^2b + \frac{1}{4}a^4b + \frac{1}{6}a^2b^3 = \frac{1}{12}a^2b(6 + 3a^2 + 2b^2)$, and

$M_x = \iint_D y\rho(x,y)\,dA = \int_0^a\int_0^b (y + x^2y + y^3)\,dy\,dx = \int_0^a \left[\frac{1}{2}y^2 + \frac{1}{2}x^2y^2 + \frac{1}{4}y^4\right]_{y=0}^{y=b} dx = \int_0^a \left(\frac{1}{2}b^2 + \frac{1}{2}b^2x^2 + \frac{1}{4}b^4\right) dx$

$= \left[\frac{1}{2}b^2x + \frac{1}{6}b^2x^3 + \frac{1}{4}b^4x\right]_0^a = \frac{1}{2}ab^2 + \frac{1}{6}a^3b^2 + \frac{1}{4}ab^4 = \frac{1}{12}ab^2(6 + 2a^2 + 3b^2)$.

Hence, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m},\dfrac{M_x}{m}\right) = \left(\dfrac{\frac{1}{12}a^2b(6+3a^2+2b^2)}{\frac{1}{3}ab(3+a^2+b^2)},\dfrac{\frac{1}{12}ab^2(6+2a^2+3b^2)}{\frac{1}{3}ab(3+a^2+b^2)}\right)$

$= \left(\dfrac{a(6+3a^2+2b^2)}{4(3+a^2+b^2)},\dfrac{b(6+2a^2+3b^2)}{4(3+a^2+b^2)}\right).$

5. $m = \int_0^2\int_{x/2}^{3-x}(x+y)\,dy\,dx = \int_0^2\left[xy+\frac{1}{2}y^2\right]_{y=x/2}^{y=3-x}dx = \int_0^2\left[x(3-x)+\frac{1}{2}(3-x)^2-\frac{1}{2}x^2-\frac{1}{8}x^2\right]dx$

$= \int_0^2\left(-\frac{9}{8}x^2+\frac{9}{2}\right)dx = \left[-\frac{9}{8}\left(\frac{1}{3}x^3\right)+\frac{9}{2}x\right]_0^2 = 6,$

$M_y = \int_0^2\int_{x/2}^{3-x}(x^2+xy)\,dy\,dx = \int_0^2\left[x^2y+\frac{1}{2}xy^2\right]_{y=x/2}^{y=3-x}dx = \int_0^2\left(\frac{9}{2}x-\frac{9}{8}x^3\right)dx = \frac{9}{2},$

$M_x = \int_0^2\int_{x/2}^{3-y}(xy+y^2)\,dy\,dx = \int_0^2\left[\frac{1}{2}xy^2+\frac{1}{3}y^3\right]_{y=x/2}^{y=3-x}dx = \int_0^2\left(9-\frac{9}{2}x\right)dx = 9.$

Hence $m=6$, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m},\dfrac{M_x}{m}\right) = \left(\dfrac{3}{4},\dfrac{3}{2}\right).$

6. Here $D = \left\{(x,y)\mid 0\le y\le\frac{2}{5},\ \ y/2\le x\le 1-2y\right\}.$

$m = \int_0^{2/5}\int_{y/2}^{1-2y}x\,dx\,dy = \int_0^{2/5}\left[\frac{1}{2}x^2\right]_{x=y/2}^{x=1-2y}dy = \frac{1}{2}\int_0^{2/5}\left[(1-2y)^2-\left(\frac{1}{2}y\right)^2\right]dy$

$= \frac{1}{2}\int_0^{2/5}\left(\frac{15}{4}y^2-4y+1\right)dy = \frac{1}{2}\left[\frac{5}{4}y^3-2y^2+y\right]_0^{2/5} = \frac{1}{2}\left[\frac{2}{25}-\frac{8}{25}+\frac{2}{5}\right] = \frac{2}{25},$

$M_y = \int_0^{2/5}\int_{y/2}^{1-2y}x\cdot x\,dx\,dy = \int_0^{2/5}\left[\frac{1}{3}x^3\right]_{x=y/2}^{x=1-2y}dy = \frac{1}{3}\int_0^{2/5}\left[(1-2y)^3-\left(\frac{1}{2}y\right)^3\right]dy$

$= \frac{1}{3}\int_0^{2/5}\left(-\frac{65}{8}y^3+12y^2-6y+1\right)dy = \frac{1}{3}\left[-\frac{65}{32}y^4+4y^3-3y^2+y\right]_0^{2/5} = \frac{1}{3}\left[-\frac{13}{250}+\frac{32}{125}-\frac{12}{25}+\frac{2}{5}\right] = \frac{31}{750},$

$M_x = \int_0^{2/5}\int_{y/2}^{1-2y}y\cdot x\,dx\,dy = \int_0^{2/5}y\left[\frac{1}{2}x^2\right]_{x=y/2}^{x=1-2y}dy = \frac{1}{2}\int_0^{2/5}y\left(\frac{15}{4}y^2-4y+1\right)dy$

$= \frac{1}{2}\int_0^{2/5}\left(\frac{15}{4}y^3-4y^2+y\right)dy = \frac{1}{2}\left[\frac{15}{16}y^4-\frac{4}{3}y^3+\frac{1}{2}y^2\right]_0^{2/5} = \frac{1}{2}\left[\frac{3}{125}-\frac{32}{375}+\frac{2}{25}\right] = \frac{7}{750}.$

Hence $m=\frac{2}{25}$, $(\overline{x},\overline{y}) = \left(\frac{31/750}{2/25},\frac{7/750}{2/25}\right) = \left(\frac{31}{60},\frac{7}{60}\right).$

7. $m = \int_{-1}^1\int_0^{1-x^2}ky\,dy\,dx = k\int_{-1}^1\left[\frac{1}{2}y^2\right]_{y=0}^{y=1-x^2}dx = \frac{1}{2}k\int_{-1}^1(1-x^2)^2\,dx = \frac{1}{2}k\int_{-1}^1(1-2x^2+x^4)\,dx$

$= \frac{1}{2}k\left[x-\frac{2}{3}x^3+\frac{1}{5}x^5\right]_{-1}^1 = \frac{1}{2}k\left(1-\frac{2}{3}+\frac{1}{5}+1-\frac{2}{3}+\frac{1}{5}\right) = \frac{8}{15}k,$

$M_y = \int_{-1}^1\int_0^{1-x^2}kxy\,dy\,dx = k\int_{-1}^1\left[\frac{1}{2}xy^2\right]_{y=0}^{y=1-x^2}dx = \frac{1}{2}k\int_{-1}^1x\,(1-x^2)^2\,dx = \frac{1}{2}k\int_{-1}^1(x-2x^3+x^5)\,dx$

$= \frac{1}{2}k\left[\frac{1}{2}x^2-\frac{1}{2}x^4+\frac{1}{6}x^6\right]_{-1}^1 = \frac{1}{2}k\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}-\frac{1}{2}+\frac{1}{2}-\frac{1}{6}\right) = 0,$

$M_x = \int_{-1}^1\int_0^{1-x^2}ky^2\,dy\,dx = k\int_{-1}^1\left[\frac{1}{3}y^3\right]_{y=0}^{y=1-x^2}dx = \frac{1}{3}k\int_{-1}^1(1-x^2)^3\,dx = \frac{1}{3}k\int_{-1}^1(1-3x^2+3x^4-x^6)\,dx$

$= \frac{1}{3}k\left[x-x^3+\frac{3}{5}x^5-\frac{1}{7}x^7\right]_{-1}^1 = \frac{1}{3}k\left(1-1+\frac{3}{5}-\frac{1}{7}+1-1+\frac{3}{5}-\frac{1}{7}\right) = \frac{32}{105}k.$

Hence $m=\frac{8}{15}k$, $(\overline{x},\overline{y}) = \left(0,\frac{32k/105}{8k/15}\right) = \left(0,\frac{4}{7}\right).$

8. The boundary curves intersect when $x+2=x^2 \quad\Leftrightarrow\quad x^2-x-2=0 \quad\Leftrightarrow\quad x=-1, x=2$. Thus here $D = \left\{(x,y)\mid -1\le x\le 2,\ \ x^2\le y\le x+2\right\}.$

$m = \int_{-1}^2\int_{x^2}^{x+2}kx^2\,dy\,dx = k\int_{-1}^2x^2\left[y\right]_{y=x^2}^{y=x+2}dx = k\int_{-1}^2(x^3+2x^2-x^4)\,dx$

$= k\left[\frac{1}{4}x^4+\frac{2}{3}x^3-\frac{1}{5}x^5\right]_{-1}^2 = k\left(\frac{44}{15}+\frac{13}{60}\right) = \frac{63}{20}k,$

$$M_y = \int_{-1}^{2}\int_{x^2}^{x+2} kx^3\,dy\,dx = k\int_{-1}^{2} x^3\,[y]_{y=x^2}^{y=x+2}\,dx = k\int_{-1}^{2}(x^4+2x^3-x^5)\,dx$$
$$= k\left[\tfrac15 x^5 + \tfrac12 x^4 - \tfrac16 x^6\right]_{-1}^{2} = k\left(\tfrac{56}{15} - \tfrac{2}{15}\right) = \tfrac{18}{5}k,$$
$$M_x = \int_{-1}^{2}\int_{x^2}^{x+2} kx^2 y\,dy\,dx = k\int_{-1}^{2} x^2\left[\tfrac12 y^2\right]_{y=x^2}^{y=x+2}dx = \tfrac12 k\int_{-1}^{2} x^2\left(x^2+4x+4-x^4\right)dx$$
$$= \tfrac12 k\int_{-1}^{2}(x^4+4x^3+4x^2-x^6)\,dx = \tfrac12 k\left[\tfrac15 x^5 + x^4 + \tfrac43 x^3 - \tfrac17 x^7\right]_{-1}^{2} = \tfrac12 k\left(\tfrac{1552}{105} + \tfrac{41}{105}\right) = \tfrac{531}{70}k.$$

Hence $m = \frac{63}{20}k$, $(\overline{x},\overline{y}) = \left(\frac{18k/5}{63k/20}, \frac{531k/70}{63k/20}\right) = \left(\frac87, \frac{118}{49}\right)$.

9. $m = \int_0^1\int_0^{e^{-x}} xy\,dy\,dx = \int_0^1 x\left[\tfrac12 y^2\right]_{y=0}^{y=e^{-x}}dx = \tfrac12\int_0^1 x\left(e^{-x}\right)^2 dx = \tfrac12\int_0^1 xe^{-2x}\,dx$ $\left[\begin{array}{c}\text{integrate by parts with}\\ u=x,\,dv=e^{-2x}\,dx\end{array}\right]$

$$= \tfrac12\left[-\tfrac14(2x+1)e^{-2x}\right]_0^1 = -\tfrac18\left(3e^{-2}-1\right) = \tfrac18 - \tfrac38 e^{-2},$$

$M_y = \int_0^1\int_0^{e^{-x}} x^2 y\,dy\,dx = \int_0^1 x^2\left[\tfrac12 y^2\right]_{y=0}^{y=e^{-x}}dx = \tfrac12\int_0^1 x^2 e^{-2x}\,dx$ [integrate by parts twice]

$$= \tfrac12\left[-\tfrac14\left(2x^2+2x+1\right)e^{-2x}\right]_0^1 = -\tfrac18\left(5e^{-2}-1\right) = \tfrac18 - \tfrac58 e^{-2},$$
$$M_x = \int_0^1\int_0^{e^{-x}} xy^2\,dy\,dx = \int_0^1 x\left[\tfrac13 y^3\right]_{y=0}^{y=e^{-x}}dx = \tfrac13\int_0^1 xe^{-3x}\,dx$$
$$= \tfrac13\left[-\tfrac19(3x+1)e^{-3x}\right]_0^1 = -\tfrac{1}{27}\left(4e^{-3}-1\right) = \tfrac{1}{27} - \tfrac{4}{27}e^{-3}.$$

Hence $m = \frac18\left(1-3e^{-2}\right)$, $(\overline{x},\overline{y}) = \left(\dfrac{\frac18\left(1-5e^{-2}\right)}{\frac18\left(1-3e^{-2}\right)}, \dfrac{\frac{1}{27}\left(1-4e^{-3}\right)}{\frac18\left(1-3e^{-2}\right)}\right) = \left(\dfrac{e^2-5}{e^2-3}, \dfrac{8\left(e^3-4\right)}{27\left(e^3-3e\right)}\right)$.

10. Note that $\cos x \geq 0$ for $-\pi/2 \leq x \leq \pi/2$.

$$m = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} y\,dy\,dx = \int_{-\pi/2}^{\pi/2}\left[\tfrac12 y^2\right]_{y=0}^{y=\cos x}dx = \tfrac12\int_{-\pi/2}^{\pi/2}\cos^2 x\,dx = \tfrac12\left[\tfrac12 x + \tfrac14\sin 2x\right]_{-\pi/2}^{\pi/2} = \tfrac{\pi}{4},$$

$M_y = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} xy\,dy\,dx = \int_{-\pi/2}^{\pi/2} x\left[\tfrac12 y^2\right]_{y=0}^{y=\cos x}dx = \tfrac12\int_{-\pi/2}^{\pi/2} x\cos^2 x\,dx$ $\left[\begin{array}{c}\text{integrate by parts with}\\ u=x,\,dv=\cos^2 x\,dx\end{array}\right]$

$$= \tfrac12\left[x\left(\tfrac12 x + \tfrac14\sin 2x\right)\Big|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2}\left(\tfrac12 x + \tfrac14\sin 2x\right)dx\right]$$
$$= \tfrac12\left(\tfrac18\pi^2 - \tfrac18\pi^2 - \left[\tfrac14 x^2 - \tfrac18\cos 2x\right]_{-\pi/2}^{\pi/2}\right) = \tfrac12\left(0 - \left[\tfrac{1}{16}\pi^2 + \tfrac18 - \tfrac{1}{16}\pi^2 - \tfrac18\right]\right) = 0,$$

$$M_x = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} y^2\,dy\,dx = \int_{-\pi/2}^{\pi/2}\left[\tfrac13 y^3\right]_{y=0}^{y=\cos x}dx = \tfrac13\int_{-\pi/2}^{\pi/2}\cos^3 x\,dx = \tfrac13\int_{-\pi/2}^{\pi/2}(1-\sin^2 x)\cos x\,dx$$

[substitute $u = \sin x \quad\Rightarrow\quad du = \cos x\,dx$]

$$= \tfrac13\left[\sin x - \tfrac13\sin^3 x\right]_{-\pi/2}^{\pi/2} = \tfrac13\left(1 - \tfrac13 + 1 - \tfrac13\right) = \tfrac49.$$

Hence $m = \frac{\pi}{4}$, $(\overline{x},\overline{y}) = \left(0, \frac{4/9}{\pi/4}\right) = \left(0, \frac{16}{9\pi}\right)$.

11. $\rho(x,y) = ky$, $\quad m = \iint_D ky\,dA = \int_0^{\pi/2}\int_0^1 k(r\sin\theta)\,r\,dr\,d\theta = k\int_0^{\pi/2}\sin\theta\,d\theta\,\int_0^1 r^2\,dr$

$$= k\left[-\cos\theta\right]_0^{\pi/2}\,\left[\tfrac13 r^3\right]_0^1 = k(1)\left(\tfrac13\right) = \tfrac13 k,$$
$$M_y = \iint_D x\cdot ky\,dA = \int_0^{\pi/2}\int_0^1 k(r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta = k\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\,\int_0^1 r^3\,dr$$
$$= k\left[\tfrac12\sin^2\theta\right]_0^{\pi/2}\,\left[\tfrac14 r^4\right]_0^1 = k\left(\tfrac12\right)\left(\tfrac14\right) = \tfrac18 k,$$

$$M_x = \iint_D y \cdot ky\, dA = \int_0^{\pi/2}\int_0^1 k(r\sin\theta)^2\, r\, dr\, d\theta = k\int_0^{\pi/2}\sin^2\theta\, d\theta\, \int_0^1 r^3\, dr$$
$$= k\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}\,\left[\tfrac{1}{4}r^4\right]_0^1 = k\left(\tfrac{\pi}{4}\right)\left(\tfrac{1}{4}\right) = \tfrac{\pi}{16}k.$$

Hence $(\overline{x}, \overline{y}) = \left(\frac{k/8}{k/3}, \frac{k\pi/16}{k/3}\right) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$.

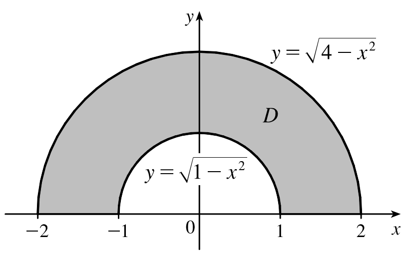
12. $\rho(x,y) = k(x^2+y^2) = kr^2$, $m = \int_0^{\pi/2}\int_0^1 kr^3\, dr\, d\theta = \frac{\pi}{8}k$,

$M_y = \int_0^{\pi/2}\int_0^1 kr^4\cos\theta\, dr\, d\theta = \frac{1}{5}k\int_0^{\pi/2}\cos\theta\, d\theta = \frac{1}{5}k\big[\sin\theta\big]_0^{\pi/2} = \frac{1}{5}k$,

$M_x = \int_0^{\pi/2}\int_0^1 kr^4\sin\theta\, dr\, d\theta = \frac{1}{5}k\int_0^{\pi/2}\sin\theta\, d\theta = \frac{1}{5}k\big[-\cos\theta\big]_0^{\pi/2} = \frac{1}{5}k$.

Hence $(\overline{x}, \overline{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi}\right)$.

13.



$\rho(x,y) = k\sqrt{x^2+y^2} = kr$,

$$m = \iint_D \rho(x,y)dA = \int_0^{\pi}\int_1^2 kr\cdot r\, dr\, d\theta$$
$$= k\int_0^{\pi} d\theta\, \int_1^2 r^2\, dr = k(\pi)\left[\tfrac{1}{3}r^3\right]_1^2 = \tfrac{7}{3}\pi k,$$

$$M_y = \iint_D x\rho(x,y)dA = \int_0^{\pi}\int_1^2 (r\cos\theta)(kr)\, r\, dr\, d\theta = k\int_0^{\pi}\cos\theta\, d\theta\, \int_1^2 r^3\, dr$$
$$= k\left[\sin\theta\right]_0^{\pi}\,\left[\tfrac{1}{4}r^4\right]_1^2 = k(0)\left(\tfrac{15}{4}\right) = 0$$

[this is to be expected as the region and density function are symmetric about the y-axis]

$$M_x = \iint_D y\rho(x,y)dA = \int_0^{\pi}\int_1^2 (r\sin\theta)(kr)\, r\, dr\, d\theta = k\int_0^{\pi}\sin\theta\, d\theta\, \int_1^2 r^3\, dr$$
$$= k\left[-\cos\theta\right]_0^{\pi}\,\left[\tfrac{1}{4}r^4\right]_1^2 = k(1+1)\left(\tfrac{15}{4}\right) = \tfrac{15}{2}k.$$

Hence $(\overline{x}, \overline{y}) = \left(0, \frac{15k/2}{7\pi k/3}\right) = \left(0, \frac{45}{14\pi}\right)$.

14. Now $\rho(x,y) = k/\sqrt{x^2+y^2} = k/r$, so

$$m = \iint_D \rho(x,y)dA = \int_0^{\pi}\int_1^2 (k/r)\, r\, dr\, d\theta = k\int_0^{\pi} d\theta\, \int_1^2 dr = k(\pi)(1) = \pi k,$$

$$M_y = \iint_D x\rho(x,y)dA = \int_0^{\pi}\int_1^2 (r\cos\theta)(k/r)\, r\, dr\, d\theta = k\int_0^{\pi}\cos\theta\, d\theta\, \int_1^2 r\, dr$$
$$= k\left[\sin\theta\right]_0^{\pi}\,\left[\tfrac{1}{2}r^2\right]_1^2 = k(0)\left(\tfrac{3}{2}\right) = 0,$$

$$M_x = \iint_D y\rho(x,y)dA = \int_0^{\pi}\int_1^2 (r\sin\theta)(k/r)\, r\, dr\, d\theta = k\int_0^{\pi}\sin\theta\, d\theta\, \int_1^2 r\, dr$$
$$= k\left[-\cos\theta\right]_0^{\pi}\,\left[\tfrac{1}{2}r^2\right]_1^2 = k(1+1)\left(\tfrac{3}{2}\right) = 3k.$$

Hence $(\overline{x}, \overline{y}) = \left(0, \frac{3k}{\pi k}\right) = \left(0, \frac{3}{\pi}\right)$.

15. Placing the vertex opposite the hypotenuse at $(0,0)$, $\rho(x,y) = k(x^2+y^2)$. Then

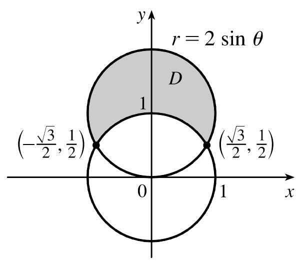
$$m = \int_0^a\int_0^{a-x} k\big(x^2+y^2\big)\, dy\, dx = k\int_0^a \left[ax^2 - x^3 + \tfrac{1}{3}(a-x)^3\right] dx = k\big[\tfrac{1}{3}ax^3 - \tfrac{1}{4}x^4 - \tfrac{1}{12}(a-x)^4\big]_0^a = \tfrac{1}{6}ka^4.$$

[continued]

By symmetry, $$M_y = M_x = \int_0^a \int_0^{a-x} ky(x^2+y^2)\,dy\,dx = k\int_0^a \left[\tfrac{1}{2}(a-x)^2x^2 + \tfrac{1}{4}(a-x)^4\right] dx$$
$$= k\left[\tfrac{1}{6}a^2x^3 - \tfrac{1}{4}ax^4 + \tfrac{1}{10}x^5 - \tfrac{1}{20}(a-x)^5\right]_0^a = \tfrac{1}{15}ka^5$$

Hence $(\overline{x},\overline{y}) = \left(\frac{2}{5}a, \frac{2}{5}a\right)$.

16. $\rho(x,y) = k/\sqrt{x^2+y^2} = k/r$.



$$m = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} \frac{k}{r}\,r\,dr\,d\theta = k\int_{\pi/6}^{5\pi/6}[(2\sin\theta)-1]\,d\theta$$
$$= k\left[-2\cos\theta - \theta\right]_{\pi/6}^{5\pi/6} = 2k\left(\sqrt{3} - \tfrac{\pi}{3}\right)$$

By symmetry of D and $f(x) = x$, $M_y = 0$, and

$$M_x = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} kr\sin\theta\,dr\,d\theta = \tfrac{1}{2}k\int_{\pi/6}^{5\pi/6}(4\sin^3\theta - \sin\theta)\,d\theta$$
$$= \tfrac{1}{2}k\left[-3\cos\theta + \tfrac{4}{3}\cos^3\theta\right]_{\pi/6}^{5\pi/6} = \sqrt{3}\,k$$

Hence $(\overline{x},\overline{y}) = \left(0, \frac{3\sqrt{3}}{2\left(3\sqrt{3}-\pi\right)}\right)$.

17. $I_x = \iint_D y^2\rho(x,y)dA = \int_1^3\int_1^4 y^2\cdot ky^2\,dy\,dx = k\int_1^3 dx\int_1^4 y^4\,dy = k\,[x]_1^3\,\left[\frac{1}{5}y^5\right]_1^4 = k(2)\left(\frac{1023}{5}\right) = 409.2k$,

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_1^3\int_1^4 x^2\cdot ky^2\,dy\,dx = k\int_1^3 x^2\,dx\int_1^4 y^2\,dy = k\left[\frac{1}{3}x^3\right]_1^3\left[\frac{1}{3}y^3\right]_1^4 = k\left(\frac{26}{3}\right)(21) = 182k$,

and $I_0 = I_x + I_y = 409.2k + 182k = 591.2k$.

18. $I_x = \iint_D y^2\rho(x,y)dA = \int_0^{2/5}\int_{y/2}^{1-2y} y^2\cdot x\,dx\,dy = \int_0^{2/5} y^2\left[\frac{1}{2}x^2\right]_{x=y/2}^{x=1-2y}dy = \frac{1}{2}\int_0^{2/5} y^2\left(\frac{15}{4}y^2 - 4y + 1\right)dy$

$= \frac{1}{2}\int_0^{2/5}\left(\frac{15}{4}y^4 - 4y^3 + y^2\right)dy = \frac{1}{2}\left[\frac{3}{4}y^5 - y^4 + \frac{1}{3}y^3\right]_0^{2/5} = \frac{16}{9375}$,

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^{2/5}\int_{y/2}^{1-2y} x^2\cdot x\,dx\,dy = \int_0^{2/5}\left[\frac{1}{4}x^4\right]_{x=y/2}^{x=1-2y}dy = \frac{1}{4}\int_0^{2/5}\left[(1-2y)^4 - \frac{1}{16}y^4\right]dy$

$= \frac{1}{4}\int_0^{2/5}\left(\frac{255}{16}y^4 - 32y^3 + 24y^2 - 8y + 1\right)dy = \frac{1}{4}\left[\frac{51}{16}y^5 - 8y^4 + 8y^3 - 4y^2 + y\right]_0^{2/5} = \frac{78}{3125}$,

and $I_0 = I_x + I_y = \frac{16}{9375} + \frac{78}{3125} = \frac{2}{75}$.

19. As in Exercise 15, we place the vertex opposite the hypotenuse at $(0,0)$ and the equal sides along the positive axes.

$I_x = \int_0^a\int_0^{a-x} y^2k(x^2+y^2)\,dy\,dx = k\int_0^a\int_0^{a-x}(x^2y^2+y^4)\,dy\,dx = k\int_0^a\left[\frac{1}{3}x^2y^3 + \frac{1}{5}y^5\right]_{y=0}^{y=a-x}\,dx$

$= k\int_0^a\left[\frac{1}{3}x^2(a-x)^3 + \frac{1}{5}(a-x)^5\right]dx = k\left[\frac{1}{3}\left(\frac{1}{3}a^3x^3 - \frac{3}{4}a^2x^4 + \frac{3}{5}ax^5 - \frac{1}{6}x^6\right) - \frac{1}{30}(a-x)^6\right]_0^a = \frac{7}{180}ka^6$,

$I_y = \int_0^a\int_0^{a-x} x^2k(x^2+y^2)\,dy\,dx = k\int_0^a\int_0^{a-x}(x^4+x^2y^2)\,dy\,dx = k\int_0^a\left[x^4y + \frac{1}{3}x^2y^3\right]_{y=0}^{y=a-x}\,dx$

$= k\int_0^a\left[x^4(a-x) + \frac{1}{3}x^2(a-x)^3\right]dx = k\left[\frac{1}{5}ax^5 - \frac{1}{6}x^6 + \frac{1}{3}\left(\frac{1}{3}a^3x^3 - \frac{3}{4}a^2x^4 + \frac{3}{5}ax^5 - \frac{1}{6}x^6\right)\right]_0^a = \frac{7}{180}ka^6$,

and $I_0 = I_x + I_y = \frac{7}{90}ka^6$.

20. If we find the moments of inertia about the x- and y-axes, we can determine in which direction rotation will be more difficult. (See the explanation following Example 4.) The moment of inertia about the x-axis is given by

$$I_x = \iint_D y^2 \rho(x,y)\, dA = \int_0^2 \int_0^2 y^2(1+0.1x)\, dy\, dx = \int_0^2 (1+0.1x)\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=2} dx$$
$$= \tfrac{8}{3}\int_0^2 (1+0.1x)\, dx = \tfrac{8}{3}\left[x + 0.1\cdot\tfrac{1}{2}x^2\right]_0^2 = \tfrac{8}{3}(2.2) \approx 5.87$$

Similarly, the moment of inertia about the y-axis is given by

$$I_y = \iint_D x^2 \rho(x,y)\, dA = \int_0^2 \int_0^2 x^2(1+0.1x)\, dy\, dx = \int_0^2 x^2(1+0.1x)\left[\, y\,\right]_{y=0}^{y=2} dx$$
$$= 2\int_0^2 (x^2+0.1x^3)\, dx = 2\left[\tfrac{1}{3}x^3 + 0.1\cdot\tfrac{1}{4}x^4\right]_0^2 = 2\left(\tfrac{8}{3}+0.4\right) \approx 6.13$$

Since $I_y > I_x$, more force is required to rotate the fan blade about the y-axis.

21. $I_x = \iint_D y^2\rho(x,y)dA = \int_0^h \int_0^b \rho y^2\, dx\, dy = \rho\int_0^b dx\, \int_0^h y^2\, dy = \rho\left[\, x\,\right]_0^b \left[\tfrac{1}{3}y^3\right]_0^h = \rho b\left(\tfrac{1}{3}h^3\right) = \tfrac{1}{3}\rho b h^3$,

$I_y = \iint_D x^2\rho(x,y)dA = \int_0^h \int_0^b \rho x^2\, dx\, dy = \rho\int_0^b x^2\, dx\, \int_0^h dy = \rho\left[\tfrac{1}{3}x^3\right]_0^b [y]_0^h = \tfrac{1}{3}\rho b^3 h$,

and $m = \rho\,(\text{area of rectangle}) = \rho bh$ since the lamina is homogeneous. Hence $\overline{\overline{x}}^2 = \dfrac{I_y}{m} = \dfrac{\frac{1}{3}\rho b^3 h}{\rho bh} = \dfrac{b^2}{3} \quad\Rightarrow\quad \overline{\overline{x}} = \dfrac{b}{\sqrt{3}}$

and $\overline{\overline{y}}^2 = \dfrac{I_x}{m} = \dfrac{\frac{1}{3}\rho b h^3}{\rho bh} = \dfrac{h^2}{3} \quad\Rightarrow\quad \overline{\overline{y}} = \dfrac{h}{\sqrt{3}}$.

22. Here we assume $b > 0$, $h > 0$ but note that we arrive at the same results if $b < 0$ or $h < 0$. We have $D = \left\{(x,y) \mid 0 \le x \le b, 0 \le y \le h - \frac{h}{b}x\right\}$, so

$I_x = \int_0^b \int_0^{h-hx/b} y^2\rho\, dy\, dx = \rho\int_0^b \left[\tfrac{1}{3}y^3\right]_{y=0}^{y=h-hx/b} dx = \tfrac{1}{3}\rho\int_0^b \left(h - \tfrac{h}{b}x\right)^3 dx$

$\quad = \tfrac{1}{3}\rho\left[-\tfrac{b}{h}\left(\tfrac{1}{4}\right)\left(h-\tfrac{h}{b}x\right)^4\right]_0^b = -\tfrac{b}{12h}\rho(0-h^4) = \tfrac{1}{12}\rho b h^3$,

$I_y = \int_0^b \int_0^{h-hx/b} x^2\rho\, dy\, dx = \rho\int_0^b x^2\left(h - \tfrac{h}{b}x\right) dx = \rho\int_0^b \left(hx^2 - \tfrac{h}{b}x^3\right) dx$

$\quad = \rho\left[\tfrac{h}{3}x^3 - \tfrac{h}{4b}x^4\right]_0^b = \rho\left(\tfrac{hb^3}{3} - \tfrac{hb^3}{4}\right) = \tfrac{1}{12}\rho b^3 h$,

and $m = \int_0^b \int_0^{h-hx/b} \rho\, dy\, dx = \rho\int_0^b \left(h - \tfrac{h}{b}x\right) dx = \rho\left[hx - \tfrac{h}{2b}x^2\right]_0^b = \tfrac{1}{2}\rho bh$. Hence $\overline{\overline{x}}^2 = \dfrac{I_y}{m} = \dfrac{\frac{1}{12}\rho b^3 h}{\frac{1}{2}\rho bh} = \dfrac{b^2}{6} \quad\Rightarrow$

$\overline{\overline{x}} = \dfrac{b}{\sqrt{6}}$ and $\overline{\overline{y}}^2 = \dfrac{I_x}{m} = \dfrac{\frac{1}{12}\rho b h^3}{\frac{1}{2}\rho bh} = \dfrac{h^2}{6} \quad\Rightarrow\quad \overline{\overline{y}} = \dfrac{h}{\sqrt{6}}$.

23. In polar coordinates, the region is $D = \left\{(r,\theta) \mid 0 \le r \le a, 0 \le \theta \le \frac{\pi}{2}\right\}$, so

$I_x = \iint_D y^2\rho\, dA = \int_0^{\pi/2}\int_0^a \rho(r\sin\theta)^2\, r\, dr\, d\theta = \rho\int_0^{\pi/2} \sin^2 d\theta\, \int_0^a r^3\, dr$

$\quad = \rho\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\tfrac{1}{4}r^4\right]_0^a = \rho\left(\tfrac{\pi}{4}\right)\left(\tfrac{1}{4}a^4\right) = \tfrac{1}{16}\rho a^4\pi$,

$I_y = \iint_D x^2\rho\, dA = \int_0^{\pi/2}\int_0^a \rho(r\cos\theta)^2\, r\, dr\, d\theta = \rho\int_0^{\pi/2} \cos^2 d\theta\, \int_0^a r^3\, dr$

$\quad = \rho\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\tfrac{1}{4}r^4\right]_0^a = \rho\left(\tfrac{\pi}{4}\right)\left(\tfrac{1}{4}a^4\right) = \tfrac{1}{16}\rho a^4\pi$,

and $m = \rho\cdot A(D) = \rho\cdot\frac{1}{4}\pi a^2$ since the lamina is homogeneous. Hence $\overline{\overline{x}}^2 = \overline{\overline{y}}^2 = \dfrac{\frac{1}{16}\rho a^4\pi}{\frac{1}{4}\rho a^2\pi} = \dfrac{a^2}{4} \quad\Rightarrow\quad \overline{\overline{x}} = \overline{\overline{y}} = \dfrac{a}{2}$.

24. $m = \int_0^\pi \int_0^{\sin x} \rho\, dy\, dx = \rho \int_0^\pi \sin x\, dx = \rho\big[-\cos x\big]_0^\pi = 2\rho$,

$I_x = \int_0^\pi \int_0^{\sin x} \rho y^2\, dy\, dx = \frac{1}{3}\rho \int_0^\pi \sin^3 x\, dx = \frac{1}{3}\rho \int_0^\pi (1-\cos^2 x)\sin x\, dx = \frac{1}{3}\rho\big[-\cos x + \frac{1}{3}\cos^3 x\big]_0^\pi = \frac{4}{9}\rho$,

$I_y = \int_0^\pi \int_0^{\sin x} \rho x^2\, dy\, dx = \rho \int_0^\pi x^2 \sin x\, dx = \rho\big[-x^2\cos x + 2x\sin x + 2\cos x\big]_0^\pi$ [by integrating by parts twice]
$= \rho(\pi^2 - 4)$.

Then $\overline{\overline{y}}^{\,2} = \dfrac{I_x}{m} = \dfrac{2}{9}$, so $\overline{\overline{y}} = \dfrac{\sqrt{2}}{3}$ and $\overline{\overline{x}}^{\,2} = \dfrac{I_y}{m} = \dfrac{\pi^2 - 4}{2}$, so $\overline{\overline{x}} = \sqrt{\dfrac{\pi^2-4}{2}}$.

25. The right loop of the curve is given by $D = \{(r,\theta) \mid 0 \le r \le \cos 2\theta,\ -\pi/4 \le \theta \le \pi/4\}$. Using a CAS, we find $m = \iint_D \rho(x,y)\, dA = \iint_D (x^2+y^2)\, dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^2\, r\, dr\, d\theta = \dfrac{3\pi}{64}$. Then

$\overline{x} = \dfrac{1}{m} \displaystyle\iint_D x\rho(x,y)\, dA = \dfrac{64}{3\pi} \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} (r\cos\theta)\, r^2\, r\, dr\, d\theta = \dfrac{64}{3\pi} \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^4 \cos\theta\, dr\, d\theta = \dfrac{16384\sqrt{2}}{10395\pi}$ and

$\overline{y} = \dfrac{1}{m} \displaystyle\iint_D y\rho(x,y)\, dA = \dfrac{64}{3\pi} \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} (r\sin\theta)\, r^2\, r\, dr\, d\theta = \dfrac{64}{3\pi} \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^4 \sin\theta\, dr\, d\theta = 0$, so

$(\overline{x}, \overline{y}) = \left(\dfrac{16384\sqrt{2}}{10395\pi}, 0\right)$.

The moments of inertia are

$I_x = \iint_D y^2 \rho(x,y)\, dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} (r\sin\theta)^2\, r^2\, r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^5 \sin^2\theta\, dr\, d\theta = \dfrac{5\pi}{384} - \dfrac{4}{105}$,

$I_y = \iint_D x^2 \rho(x,y)\, dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} (r\cos\theta)^2\, r^2\, r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r^5 \cos^2\theta\, dr\, d\theta = \dfrac{5\pi}{384} + \dfrac{4}{105}$, and

$I_0 = I_x + I_y = \dfrac{5\pi}{192}$.

26. Using a CAS, we find $m = \iint_D \rho(x,y)\, dA = \int_0^2 \int_0^{xe^{-x}} x^2 y^2\, dy\, dx = \frac{8}{729}(5 - 899e^{-6})$. Then

$\overline{x} = \dfrac{1}{m} \displaystyle\iint_D x\rho(x,y)\, dA = \dfrac{729}{8(5-899e^{-6})} \int_0^2 \int_0^{xe^{-x}} x^3 y^2\, dy\, dx = \dfrac{2(5e^6 - 1223)}{5e^6 - 899}$ and

$\overline{y} = \dfrac{1}{m} \displaystyle\iint_D y\rho(x,y)\, dA = \dfrac{729}{8(5-899e^{-6})} \int_0^2 \int_0^{xe^{-x}} x^2 y^3\, dy\, dx = \dfrac{729(45e^6 - 42037e^{-2})}{32768(5e^6 - 899)}$, so

$(\overline{x}, \overline{y}) = \left(\dfrac{2(5e^6 - 1223)}{5e^6 - 899}, \dfrac{729(45e^6 - 42037e^{-2})}{32768(5e^6 - 899)}\right)$.

The moments of inertia are $I_x = \iint_D y^2 \rho(x,y)\, dA = \int_0^2 \int_0^{xe^{-x}} x^2 y^4\, dy\, dx = \frac{16}{390625}(63 - 305593e^{-10})$,

$I_y = \iint_D x^2 \rho(x,y)\, dA = \int_0^2 \int_0^{xe^{-x}} x^4 y^2\, dy\, dx = \frac{80}{2187}(7 - 2101e^{-6})$, and

$I_0 = I_x + I_y = \frac{16}{854296875}(13809656 - 4103515625e^{-6} - 668331891e^{-10})$.

27. (a) $f(x,y)$ is a joint density function, so we know $\iint_{\mathbb{R}^2} f(x,y)\, dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,1] \times [0,2]$, we can say

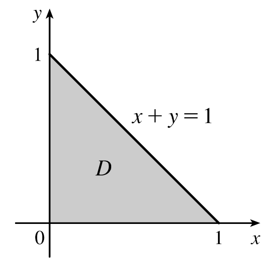
$$\iint_{\mathbb{R}^2} f(x,y)\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dy\, dx = \int_0^1 \int_0^2 Cx(1+y)\, dy\, dx$$
$$= C \int_0^1 x\big[y + \tfrac{1}{2}y^2\big]_{y=0}^{y=2}\, dx = C \int_0^1 4x\, dx = C\big[2x^2\big]_0^1 = 2C$$

Then $2C = 1 \quad \Rightarrow \quad C = \frac{1}{2}$.

(b) $P(X \le 1, Y \le 1) = \int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y)\,dy\,dx = \int_0^1\int_0^1 \frac{1}{2}x(1+y)\,dy\,dx$

$= \int_0^1 \frac{1}{2}x\left[y + \frac{1}{2}y^2\right]_{y=0}^{y=1} dx = \int_0^1 \frac{1}{2}x\left(\frac{3}{2}\right) dx = \frac{3}{4}\left[\frac{1}{2}x^2\right]_0^1 = \frac{3}{8}$ or 0.375

(c) $P(X+Y \le 1) = P((X,Y) \in D)$ where D is the triangular region shown in the figure. Thus



$$P(X+Y \le 1) = \iint_D f(x,y)\,dA = \int_0^1\int_0^{1-x} \tfrac{1}{2}x(1+y)\,dy\,dx$$

$$= \int_0^1 \tfrac{1}{2}x\left[y + \tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \int_0^1 \tfrac{1}{2}x\left(\tfrac{1}{2}x^2 - 2x + \tfrac{3}{2}\right) dx$$

$$= \tfrac{1}{4}\int_0^1 \left(x^3 - 4x^2 + 3x\right) dx = \tfrac{1}{4}\left[\tfrac{x^4}{4} - 4\tfrac{x^3}{3} + 3\tfrac{x^2}{2}\right]_0^1$$

$$= \tfrac{5}{48} \approx 0.1042$$

28. (a) $f(x,y) \ge 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Here, $f(x,y) = 0$ outside the square $[0,1] \times [0,1]$, so $\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_0^1\int_0^1 4xy\,dy\,dx = \int_0^1 \left[2xy^2\right]_{y=0}^{y=1} dx = \int_0^1 2x\,dx = \left.x^2\right]_0^1 = 1$.

Thus, $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on Y, so

$$P\left(X \ge \tfrac{1}{2}\right) = \int_{1/2}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_{1/2}^{1}\int_0^1 4xy\,dy\,dx = \int_{1/2}^{1}\left[2xy^2\right]_{y=0}^{y=1} dx = \int_{1/2}^{1} 2x\,dx = \left.x^2\right]_{1/2}^{1} = \tfrac{3}{4}.$$

(ii) $P\left(X \ge \frac{1}{2}, Y \le \frac{1}{2}\right) = \int_{1/2}^{\infty}\int_{-\infty}^{1/2} f(x,y)\,dy\,dx = \int_{1/2}^{1}\int_0^{1/2} 4xy\,dy\,dx$

$$= \int_{1/2}^{1}\left[2xy^2\right]_{y=0}^{y=1/2} dx = \int_{1/2}^{1} \tfrac{1}{2}x\,dx = \left.\tfrac{1}{2}\cdot\tfrac{1}{2}x^2\right]_{1/2}^{1} = \tfrac{3}{16}$$

(c) The expected value of X is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x\,f(x,y)\,dA = \int_0^1\int_0^1 x(4xy)\,dy\,dx = \int_0^1 2x^2\left[y^2\right]_{y=0}^{y=1} dx = 2\int_0^1 x^2\,dx = 2\left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{2}{3}$$

The expected value of Y is

$$\mu_2 = \iint_{\mathbb{R}^2} y\,f(x,y)\,dA = \int_0^1\int_0^1 y(4xy)\,dy\,dx = \int_0^1 4x\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=1} dx = \tfrac{4}{3}\int_0^1 x\,dx = \tfrac{4}{3}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{2}{3}$$

29. (a) $f(x,y) \ge 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Here, $f(x,y) = 0$ outside the first quadrant, so

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_0^\infty\int_0^\infty 0.1e^{-(0.5x+0.2y)}\,dy\,dx = 0.1\int_0^\infty\int_0^\infty e^{-0.5x}e^{-0.2y}\,dy\,dx = 0.1\int_0^\infty e^{-0.5x}\,dx\int_0^\infty e^{-0.2y}\,dy$$

$$= 0.1\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx\lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy = 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t$$

$$= 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right] = (0.1)\cdot(-2)(0-1)\cdot(-5)(0-1) = 1$$

Thus $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on X, so

$$P(Y \ge 1) = \int_{-\infty}^{\infty}\int_1^\infty f(x,y)\,dy\,dx = \int_0^\infty\int_1^\infty 0.1e^{-(0.5x+0.2y)}\,dy\,dx$$

$$= 0.1\int_0^\infty e^{-0.5x}\,dx\int_1^\infty e^{-0.2y}\,dy = 0.1\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx\lim_{t\to\infty}\int_1^t e^{-0.2y}\,dy$$

$$= 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_1^t = 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-e^{-0.2})\right]$$

$$(0.1)\cdot(-2)(0-1)\cdot(-5)(0-e^{-0.2}) = e^{-0.2} \approx 0.8187$$

(ii) $P(X \le 2, Y \le 4) = \int_{-\infty}^{2}\int_{-\infty}^{4} f(x,y)\,dy\,dx = \int_0^2\int_0^4 0.1e^{-(0.5x+0.2y)}\,dy\,dx$

$$= 0.1\int_0^2 e^{-0.5x}\,dx \int_0^4 e^{-0.2y}\,dy = 0.1\big[-2e^{-0.5x}\big]_0^2\big[-5e^{-0.2y}\big]_0^4$$
$$= (0.1)\cdot(-2)(e^{-1}-1)\cdot(-5)(e^{-0.8}-1)$$
$$= (e^{-1}-1)(e^{-0.8}-1) = 1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481$$

(c) The expected value of X is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x\,f(x,y)\,dA = \int_0^\infty\int_0^\infty x\Big[0.1e^{-(0.5x+0.2y)}\Big]\,dy\,dx$$
$$= 0.1\int_0^\infty xe^{-0.5x}\,dx \int_0^\infty e^{-0.2y}\,dy = 0.1\lim_{t\to\infty}\int_0^t xe^{-0.5x}\,dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy$$

To evaluate the first integral, we integrate by parts with $u = x$ and $dv = e^{-0.5x}\,dx$ (or we can use Formula 96 in the Table of Integrals): $\int xe^{-0.5x}\,dx = -2xe^{-0.5x} - \int -2e^{-0.5x}\,dx = -2xe^{-0.5x} - 4e^{-0.5x} = -2(x+2)e^{-0.5x}$. Thus

$$\mu_1 = 0.1\lim_{t\to\infty}\big[-2(x+2)e^{-0.5x}\big]_0^t \lim_{t\to\infty}\big[-5e^{-0.2y}\big]_0^t$$
$$= 0.1\lim_{t\to\infty}(-2)\big[(t+2)e^{-0.5t}-2\big]\lim_{t\to\infty}(-5)\big[e^{-0.2t}-1\big]$$
$$= 0.1(-2)\left(\lim_{t\to\infty}\frac{t+2}{e^{0.5t}}-2\right)(-5)(-1) = 2 \qquad \text{[by l'Hospital's Rule]}$$

The expected value of Y is given by

$$\mu_2 = \iint_{\mathbb{R}^2} y\,f(x,y)\,dA = \int_0^\infty\int_0^\infty y\Big[0.1e^{-(0.5\,+0.2y)}\Big]\,dy\,dx$$
$$= 0.1\int_0^\infty e^{-0.5x}\,dx \int_0^\infty ye^{-0.2y}\,dy = 0.1\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx \lim_{t\to\infty}\int_0^t ye^{-0.2y}\,dy$$

To evaluate the second integral, we integrate by parts with $u = y$ and $dv = e^{-0.2y}\,dy$ (or again we can use Formula 96 in the Table of Integrals) which gives $\int ye^{-0.2y}\,dy = -5ye^{-0.2y} + \int 5e^{-0.2y}\,dy = -5(y+5)e^{-0.2y}$. Then

$$\mu_2 = 0.1\lim_{t\to\infty}\big[-2e^{-0.5x}\big]_0^t \lim_{t\to\infty}\big[-5(y+5)e^{-0.2y}\big]_0^t$$
$$= 0.1\lim_{t\to\infty}\big[-2(e^{-0.5t}-1)\big]\lim_{t\to\infty}\big(-5\big[(t+5)e^{-0.2t}-5\big]\big)$$
$$= 0.1(-2)(-1)\cdot(-5)\left(\lim_{t\to\infty}\frac{t+5}{e^{0.2t}}-5\right) = 5 \qquad \text{[by l'Hospital's Rule]}$$

30. (a) The lifetime of each bulb has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \ge 0 \end{cases}$$

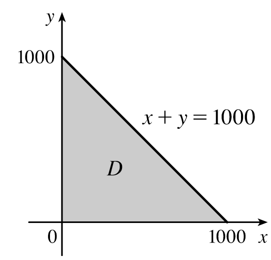
If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x,y) = \begin{cases} 10^{-6}e^{-(x+y)/1000} & \text{if } x \ge 0, y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

$$\begin{aligned}P(X \leq 1000, Y \leq 1000) &= \int_{-\infty}^{1000}\int_{-\infty}^{1000} f(x,y)\,dy\,dx = \int_0^{1000}\int_0^{1000} 10^{-6}e^{-(x+y)/1000}\,dy\,dx \\ &= 10^{-6}\int_0^{1000} e^{-x/1000}\,dx \int_0^{1000} e^{-y/1000}\,dy \\ &= 10^{-6}\left[-1000e^{-x/1000}\right]_0^{1000}\left[-1000e^{-y/1000}\right]_0^{1000} \\ &= \left(e^{-1}-1\right)^2 \approx 0.3996\end{aligned}$$

(b) Now we are asked for the probability that the combined lifetimes of both bulbs is 1000 hours or less. Thus we want to find $P(X+Y \leq 1000)$, or equivalently $P((X,Y) \in D)$ where D is the triangular region shown in the figure. Then



$$\begin{aligned}P(X+Y \leq 1000) &= \iint_D f(x,y)\ dA \\ &= \int_0^{1000}\int_0^{1000-x} 10^{-6}e^{-(x+y)/1000}\,dy\,dx \\ &= 10^{-6}\int_0^{1000}\left[-1000e^{-(x+y)/1000}\right]_{y=0}^{y=1000-x} dx = -10^{-3}\int_0^{1000}\left(e^{-1}-e^{-x/1000}\right)dx \\ &= -10^{-3}\left[e^{-1}x + 1000e^{-x/1000}\right]_0^{1000} = 1-2e^{-1} \approx 0.2642\end{aligned}$$

31. (a) The random variables X and Y are normally distributed with $\mu_1 = 45$, $\mu_2 = 20$, $\sigma_1 = 0.5$, and $\sigma_2 = 0.1$.

The individual density functions for X and Y, then, are $f_1(x) = \dfrac{1}{0.5\sqrt{2\pi}}\,e^{-(x-45)^2/0.5}$ and

$f_2(y) = \dfrac{1}{0.1\sqrt{2\pi}}\,e^{-(y-20)^2/0.02}$. Since X and Y are independent, the joint density function is the product

$f(x,y) = f_1(x)f_2(y) = \dfrac{1}{0.5\sqrt{2\pi}}\,e^{-(x-45)^2/0.5}\,\dfrac{1}{0.1\sqrt{2\pi}}\,e^{-(y-20)^2/0.02} = \frac{10}{\pi}e^{-2(x-45)^2-50(y-20)^2}$.

Then $P(40 \leq X \leq 50, 20 \leq Y \leq 25) = \int_{40}^{50}\int_{20}^{25} f(x,y)\,dy\,dx = \frac{10}{\pi}\int_{40}^{50}\int_{20}^{25} e^{-2(x-45)^2-50(y-20)^2}\,dy\,dx$.

Using a CAS or calculator to evaluate the integral, we get $P(40 \leq X \leq 50, 20 \leq Y \leq 25) \approx 0.500$.

(b) $P(4(X-45)^2 + 100(Y-20)^2 \leq 2) = \iint_D \frac{10}{\pi}e^{-2(x-45)^2-50(y-20)^2}\,dA$, where D is the region enclosed by the ellipse $4(x-45)^2 + 100(y-20)^2 = 2$. Solving for y gives $y = 20 \pm \frac{1}{10}\sqrt{2-4(x-45)^2}$, the upper and lower halves of the ellipse, and these two halves meet where $y = 20$ [since the ellipse is centered at $(45,20)$] $\Rightarrow$ $4(x-45)^2 = 2$ $\Rightarrow$ $x = 45 \pm \frac{1}{\sqrt{2}}$. Thus

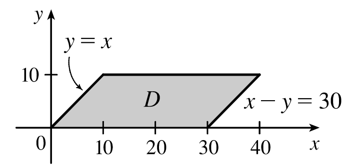
$$\iint_D \tfrac{10}{\pi}e^{-2(x-45)^2-50(y-20)^2}\,dA = \frac{10}{\pi}\int_{45-1/\sqrt{2}}^{45+1/\sqrt{2}}\int_{20-\frac{1}{10}\sqrt{2-4(x-45)^2}}^{20+\frac{1}{10}\sqrt{2-4(x-45)^2}} e^{-2(x-45)^2-50(y-20)^2}\,dy\,dx.$$

Using a CAS or calculator to evaluate the integral, we get $P(4(X-45)^2 + 100(Y-20)^2 \leq 2) \approx 0.632$.

32. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x,y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{50}e^{-x}y & \text{if } x \geq 0, 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless $X \geq Y$. Additionally, Yolanda will wait up to half an hour but no longer, so they won't meet unless $X - Y \leq 30$. Thus the probability that they meet is $P((X,Y) \in D)$ where D is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider D as a type II region, so



$$\begin{aligned} P((X,Y) \in D) &= \textstyle\iint_D f(x,y)\,dx\,dy = \int_0^{10}\int_y^{y+30} \frac{1}{50}e^{-x}y\,dx\,dy \\ &= \tfrac{1}{50}\textstyle\int_0^{10} y\big[-e^{-x}\big]_{x=y}^{x=y+30}\,dy = \frac{1}{50}\int_0^{10} y(-e^{-(y+30)} + e^{-y})\,dy \\ &= \tfrac{1}{50}(1 - e^{-30})\textstyle\int_0^{10} ye^{-y}\,dy \end{aligned}$$

By integration by parts (or Formula 96 in the Table of Integrals), this is

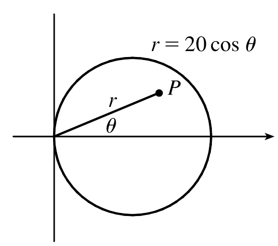
$\frac{1}{50}(1 - e^{-30})\big[-(y+1)e^{-y}\big]_0^{10} = \frac{1}{50}(1 - e^{-30})(1 - 11e^{-10}) \approx 0.020$. Thus there is only about a 2% chance they will meet. Such is student life!

33. (a) If $f(P,A)$ is the probability that an individual at A will be infected by an individual at P, and $k\,dA$ is the number of infected individuals in an element of area dA, then $f(P,A)k\,dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA. Integration over D gives the number of infections of the person at A due to all the infected people in D. In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

$$E = \iint_D kf(P,A)\,dA = k\iint_D \tfrac{1}{20}\,[20 - d(P,A)]\,\,dA = k\iint_D \left[1 - \tfrac{1}{20}\sqrt{(x-x_0)^2 + (y-y_0)^2}\right] dA$$

(b) If $A = (0,0)$, then

$$\begin{aligned} E &= k\iint_D \left[1 - \tfrac{1}{20}\sqrt{x^2+y^2}\right] dA \\ &= k\int_0^{2\pi}\int_0^{10} \left(1 - \tfrac{1}{20}r\right) r\,dr\,d\theta = 2\pi k\left[\tfrac{1}{2}r^2 - \tfrac{1}{60}r^3\right]_0^{10} \\ &= 2\pi k\left(50 - \tfrac{50}{3}\right) = \tfrac{200}{3}\pi k \approx 209k \end{aligned}$$



For A at the edge of the city, it is convenient to use a polar coordinate system centered at A. Then the polar equation for the circular boundary of the city becomes $r = 20\cos\theta$ instead of $r = 10$, and the distance from A to a point P in the city