2-BayesFilterDerivative

`navigation-algorithm`

---

1 Basic Knowledge

- Multivariate normal distributions are characterized by PDF of the following form
$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}$$
- If X and Y are independent, we have
$$p(x,y) = p(x)p(y)$$
- Conditional probability are described as
$$p(x|y) = \frac{p(x,y)}{p(y)}$$
- If X and Y are independent, we have
$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{p(x)p(y)}{p(y)} = p(x)$$
- Total probability
$$p(x) = \sum_y p(x|y)p(y)$$
- Conditional probability with total probability
$$p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\sum_{x'} p(y|x')p(x')}$$

An important observation is that the denominator of Bayes rule, $p(y)$, does not depend on x. **Thus, the factor $p(y)$ will be the same for any value x in the posterior $p(x|y)$.** For this reason, $p(y)$ is often written as a normalizer in Bayes rule variable, and generically denoted η

$$p(x|y) = \eta p(y|x)p(x)$$

- Conditional independence

$$p(x, y|z) = p(x|z)p(y|z)$$

is equivalent to

$$p(x|z) = p(x|z, y)$$
$$p(y|z) = p(y|z, x)$$

but

$$p(x, y|z) = p(x|z)p(y|z) \quad \neq \quad p(x, y) = p(x)p(y)$$

$$p(x, y) = p(x)p(y) \quad \neq \quad p(x, y|z) = p(x|z)p(y|z)$$

- The expectation of a random variable X is given by

$$E[X] = \sum_x xp(x) \quad (\text{ discrete })$$

$$E[aX + b] = aE[X] + b$$

- The covariance of X is obtained as follows

$$\mathrm{Cov}[X] = E[X - E[X]]^2 = E\left[X^2\right] - E[X]^2$$

- Entropy

$$H_p(x) = E[-\log_2 p(x)]$$

which resolves to

$$H_p(x) = -\sum_x p(x) \log_2 p(x) \quad (\text{ discrete })$$

2 Mathematical Derivation of the Bayes Filter

- Environment measurement data

The notation

$$z_{t_1:t_2} = z_{t_1}, z_{t_1+1}, z_{t_1+2}, \ldots, z_{t_2}$$

denotes the set of all measurements acquired from time t_1 to time t_2 , for $t_1 \leq t_2$

- Control data

  As before, we will denote sequences of control data by $u_{t_1:t_2}$, for $t_1 \leq t_2$

$$u_{t_1:t_2} = u_{t_1}, u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2}$$

- An important insight

$$p(x_t | x_{0:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t | x_{t-1}, u_t)$$

$$p(z_t | x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t | x_t)$$

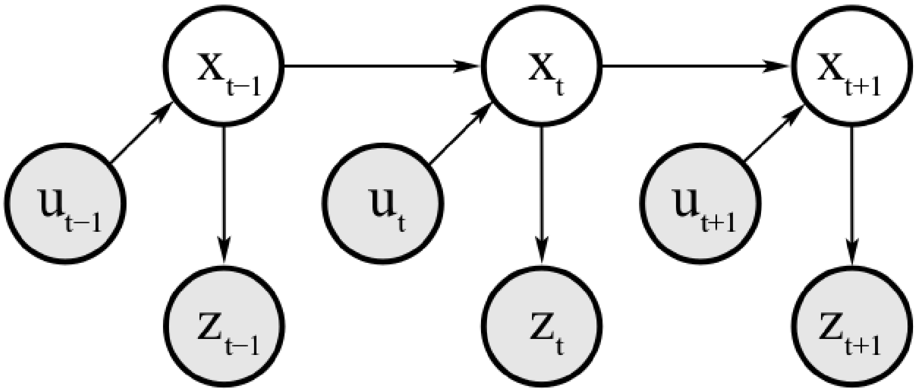

Figure 2.2 The dynamic Bayes network that characterizes the evolution of controls, states, and measurements.

- Belief Distributions

$$\text{bel}(x_t) = p(x_t | z_{1:t}, u_{1:t})$$

$$\overline{bel}(x_t) = p(x_t | z_{1:t-1}, u_{1:t})$$

- The Bayes Filter Algorithm

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$)**:**
for all x_t *do*
$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$
$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$
endfor
return $bel(x_t)$

target posterior

$$\begin{aligned} p(x_t|z_{1:t}, u_{1:t}) &= \frac{p(z_t|x_t, z_{1:t-1}, u_{1:t})p(x_t|z_{1:t-1}, u_{1:t})}{p(z_t|z_{1:t-1}, u_{1:t})} \\ &= \eta p(z_t|x_t, z_{1:t-1}, u_{1:t})p(x_t|z_{1:t-1}, u_{1:t}) \end{aligned}$$

conditional indepen-dence

$$p(z_t|x_t, z_{1:t-1}, u_{1:t}) = p(z_t|x_t)$$

simplify as follows

$$p(x_t|z_{1:t}, u_{1:t}) = \eta p(z_t|x_t)p(x_t|z_{1:t-1}, u_{1:t})$$

$$\begin{aligned} \overline{\text{bel}}(x_t) &= p(x_t|z_{1:t-1}, u_{1:t}) \\ &= \int p(x_t|x_{t-1}, z_{1:t-1}, u_{1:t})p(x_{t-1}|z_{1:t-1}, u_{1:t})dx_{t-1} \end{aligned}$$

$$p(x_t|x_{t-1}, z_{1:t-1}, u_{1:t}) = p(x_t|x_{t-1}, u_t)$$

$$\overline{bel}(x_t) = \int p(x_t|x_{t-1}, u_t)p(x_{t-1}|z_{1:t-1}, u_{1:t-1})dx_{t-1}$$