Approximate Laplace-Beltrami

1. Form the matrix $\mathbf{K}_1$ with entries $\exp(-\frac{\|x_i - x_j\|^2}{\varepsilon})$
2. Set $\mathbf{p} = \mathbf{K}_1 * \mathbf{1}$ where $\mathbf{1} = (1\ 1\ ...\ 1)'$
3. Define $\mathbf{K}_2 = \mathbf{K}_1./(\mathbf{p} * \mathbf{p}\prime)$
4. Set $\mathbf{v} = \mathrm{sqrt}(\mathbf{K}_2 * \mathbf{1})$
5. Define $\mathbf{K} = \mathbf{K}_2./(\mathbf{v} * \mathbf{v}')$
6. Diagonalize $\mathbf{K}$ by $[\mathbf{U}, \mathbf{S}, \mathbf{V}] = \mathrm{svd}(\mathbf{K})$
7. The eigenvalues of Δ are approximated by those of $\mathbf{K}$, and its eigenfunctions ϕ_i are approximated by $\mathbf{U}(:, i)./\mathbf{U}(:, 1)$

2.4.1 Curves

Closed curves

We first discuss the case of closed curves in $\mathbb{R}^n$. We assume that Γ is a C^∞ simple curve (it has no double points) of length 1. Since Γ has no boundary, the Neumann heat kernel is merely the heat kernel.

This case is degenerate as from the heat diffusion point of view, all such curves are the same: the amount of heat that has propagated from x to y at a given time t only depends on the initial distribution of temperature and the length of the curve between x and y. Equivalently, every curve is isometric to a circle and the heat kernel is a function of the geodesic distance. As a consequence, all closed simple curves can be identified to a circle of the same length, and for the circle, the eigenfunctions of the Laplace-Beltrami operator are known to be the Fourier basis. For these curves, the heat kernel is

$$p_t(\alpha, \beta) = \frac{1}{\sqrt{4\pi t}} \sum_{j \geq 0} e^{-\frac{(\alpha - \beta + 2\pi j)^2}{4t}} = \sum_{j \in \mathbb{Z}} e^{-j^2 t} e^{2i\pi j(\alpha - \beta)}$$

where α and β are the curvilinear abscissas of two points on Γ. Thus

$$p_t(\alpha, \beta) = 1 + 2 \sum_{j \geq 1} e^{-j^2 t} (\cos(2\pi j \alpha) \cos(2\pi j \beta) + \sin(2\pi j \alpha) \sin(2\pi j \beta))$$

which constitutes the spectral decomposition of this kernel.

This identity shows that for very moderate values of t, only the first terms contribute to this sum, and the heat flow can be accurately computed using the embedding $\alpha \mapsto (\cos(2\pi\alpha), \sin(2\pi\alpha))$. In other words, the curve Γ is mapped onto a circle in the plane. We therefore have shown that the heat metric can be computed on a closed simple curve as the cord length of a circle to any accuracy:

$$e^t D_t^2(x, y) = \sum_{j \geq 1} e^{-(j^2 - 1)t} \left| e^{2i\pi j \alpha} - e^{2i\pi j \beta} \right|^2 = \left| e^{2i\pi \alpha} - e^{2i\pi \beta} \right|^2 \left(1 + e^{-3t} r_t(x, y)\right)$$

where $r_t(x, y)$ is a bounded function.