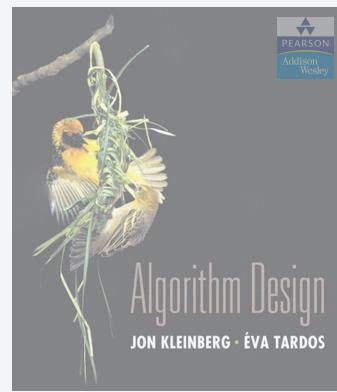


7. NETWORK FLOW I

- ▶ *max-flow and min-cut problems*
- ▶ *Ford–Fulkerson algorithm*
- ▶ *max-flow min-cut theorem*
- ▶ *capacity-scaling algorithm*
- ▶ *shortest augmenting paths*
- ▶ *Dinitz’ algorithm*
- ▶ *simple unit-capacity networks*

Lecture slides by Kevin Wayne
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<http://www.cs.princeton.edu/~wayne/kleinberg-tardos>

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7. NETWORK FLOW I

- ▶ *max-flow and min-cut problems*
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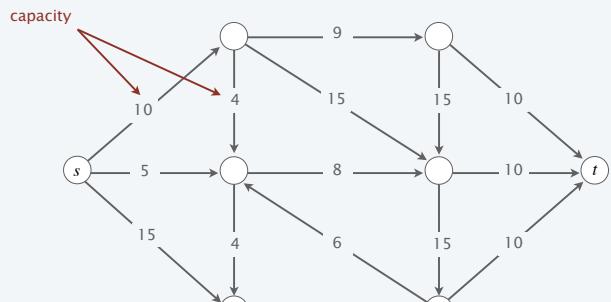
SECTION 7.1

Flow network

A **flow network** is a tuple $G = (V, E, s, t, c)$.

- Digraph (V, E) with source $s \in V$ and sink $t \in V$.
- Capacity $c(e) \geq 0$ for each $e \in E$. assume all nodes are reachable from s

Intuition. Material flowing through a transportation network;
material originates at source and is sent to sink.



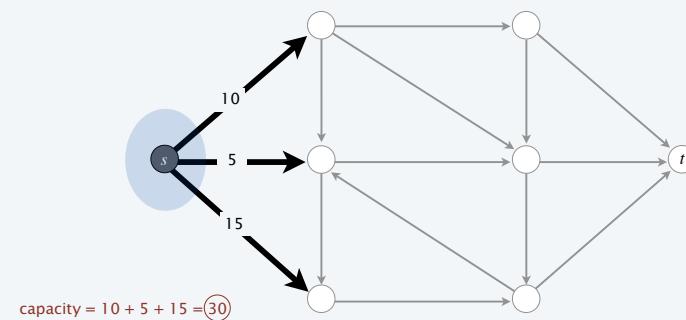
3

Minimum-cut problem

Def. An **st -cut (cut)** is a partition (A, B) of the nodes with $s \in A$ and $t \in B$.

Def. Its **capacity** is the sum of the capacities of the edges from A to B .

$$\text{cap}(A, B) = \sum_{e \text{ out of } A} c(e)$$



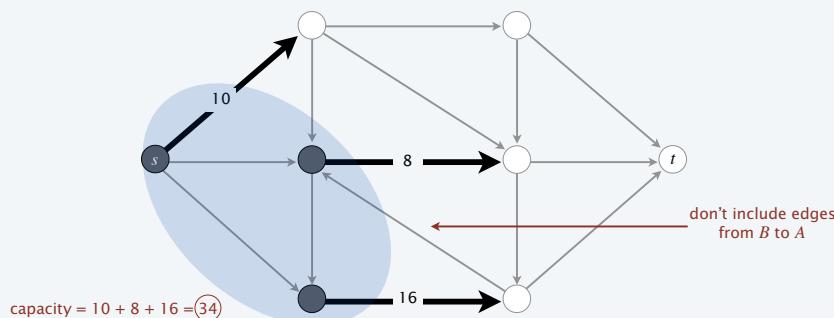
4

Minimum-cut problem

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5

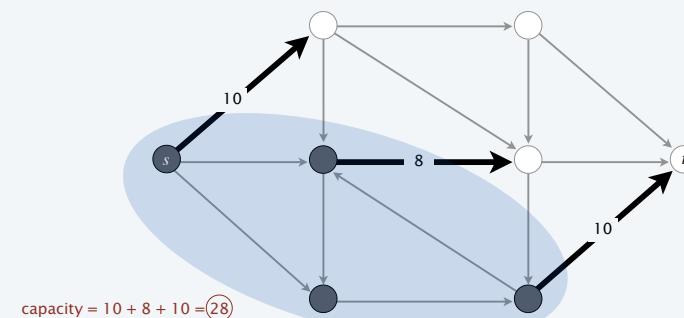
Minimum-cut problem

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Def. Its *capacity* is the sum of the capacities of the edges from A to B .

$$cap(A, B) = \sum_{e \text{ out of } A} c(e)$$

Min-cut problem. Find a cut of minimum capacity.



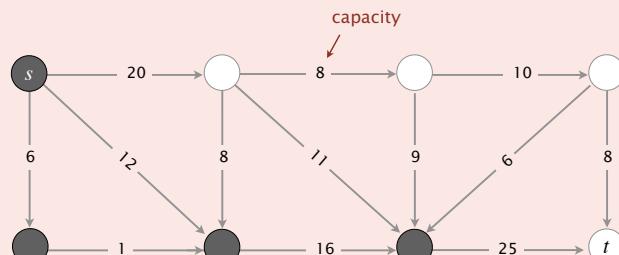
6

Network flow: quiz 1



Which is the capacity of the given *st-cut*?

- A. 11 ($20 + 25 - 8 - 11 - 9 - 6$)
- B. 34 ($8 + 11 + 9 + 6$)
- C. 45 ($20 + 25$)
- D. 79 ($20 + 25 + 8 + 11 + 9 + 6$)

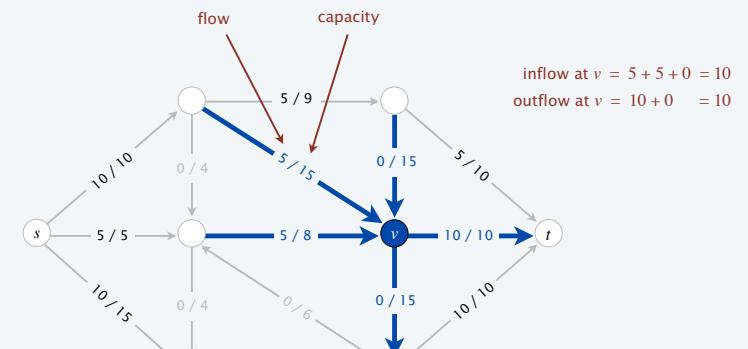


7

Maximum-flow problem

Def. An *st-flow (flow)* f is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ [capacity]
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [flow conservation]



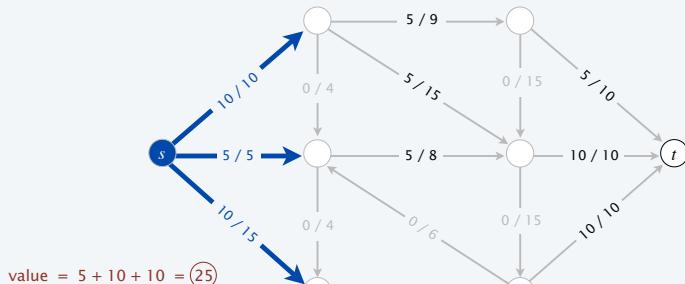
8

Maximum-flow problem

Def. An *st*-flow (flow) f is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ [capacity]
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [flow conservation]

Def. The value of a flow f is: $val(f) = \sum_{e \text{ out of } s} f(e) - \sum_{e \text{ in to } s} f(e)$



9

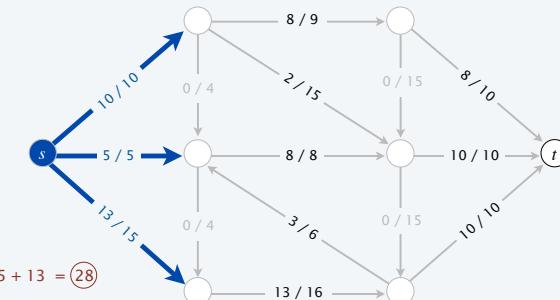
Maximum-flow problem

Def. An *st*-flow (flow) f is a function that satisfies:

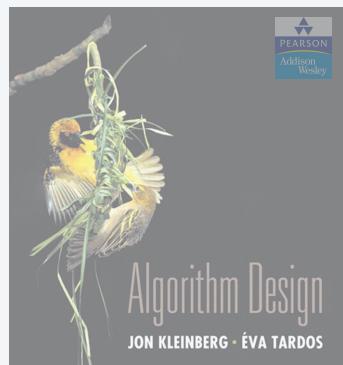
- For each $e \in E$: $0 \leq f(e) \leq c(e)$ [capacity]
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [flow conservation]

Def. The value of a flow f is: $val(f) = \sum_{e \text{ out of } s} f(e) - \sum_{e \text{ in to } s} f(e)$

Max-flow problem. Find a flow of maximum value.



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7. NETWORK FLOW I

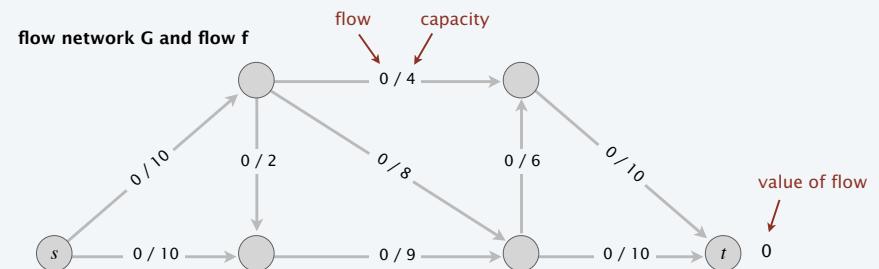
- ▶ max-flow and min-cut problems
- ▶ Ford–Fulkerson algorithm
- ▶ max-flow min-cut theorem
- ▶ capacity-scaling algorithm
- ▶ shortest augmenting paths
- ▶ Dinitz’ algorithm
- ▶ simple unit-capacity networks

SECTION 7.1

Toward a max-flow algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$.
- Augment flow along path P .
- Repeat until you get stuck.

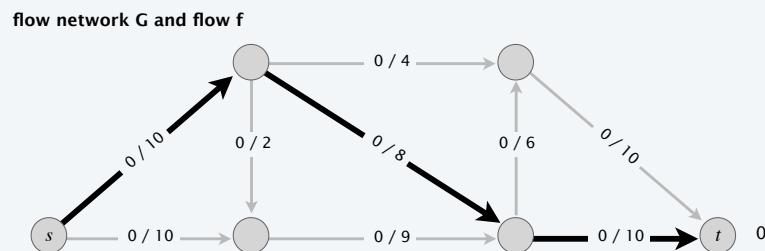


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Toward a max-flow algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for each edge $e \in E$.
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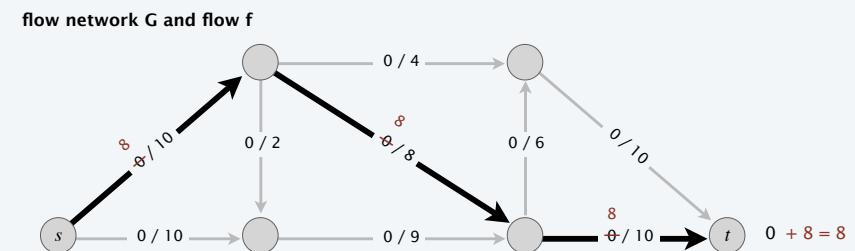


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Toward a max-flow algorithm

Greedy algorithm.

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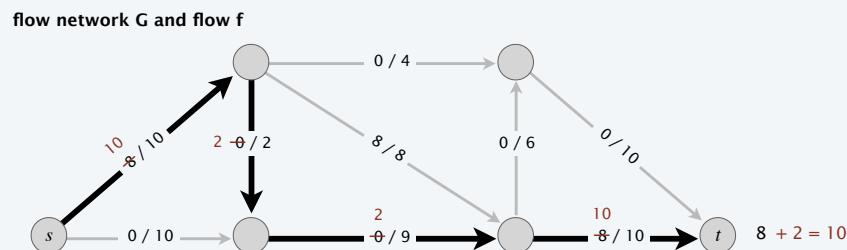


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Toward a max-flow algorithm

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- Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$.
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- Repeat until you get stuck.

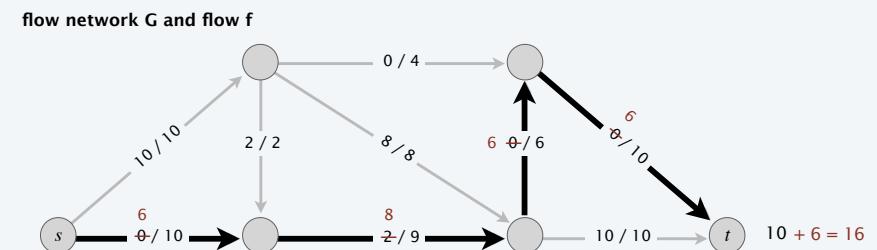


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Toward a max-flow algorithm

Greedy algorithm.

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- Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$.
- Augment flow along path P .
- Repeat until you get stuck.

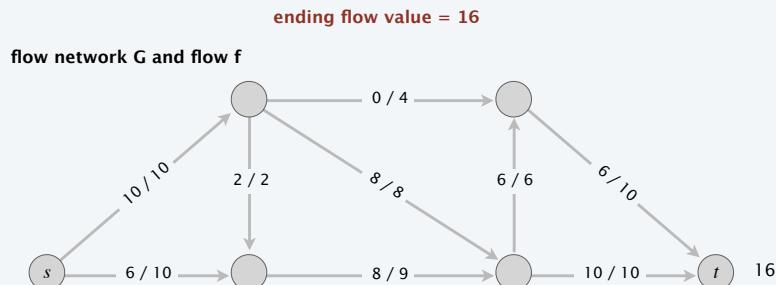


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Toward a max-flow algorithm

Greedy algorithm.

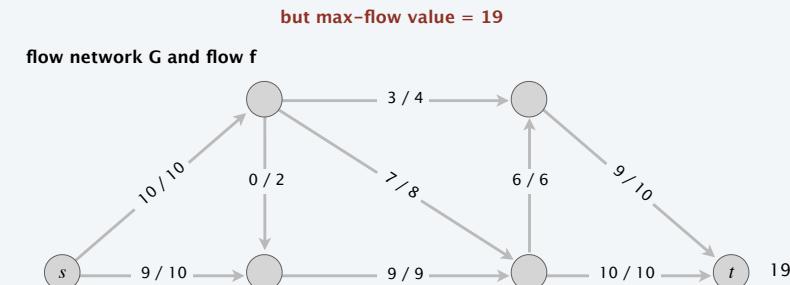
- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$.
- Augment flow along path P .
- Repeat until you get stuck.



Toward a max-flow algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$.
- Augment flow along path P .
- Repeat until you get stuck.



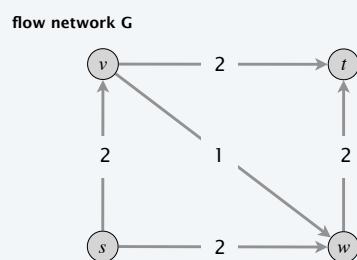
Why the greedy algorithm fails

Q. Why does the greedy algorithm fail?

A. Once greedy algorithm increases flow on an edge, it never decreases it.

Ex. Consider flow network G .

- The unique max flow f^* has $f^*(v, w) = 0$.
- Greedy algorithm could choose $s \rightarrow v \rightarrow w \rightarrow t$ as first path.



Bottom line. Need some mechanism to “undo” a bad decision.

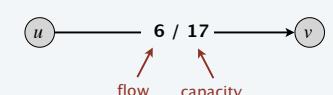
19

Residual network

Original edge. $e = (u, v) \in E$.

- Flow $f(e)$.
- Capacity $c(e)$.

original flow network G



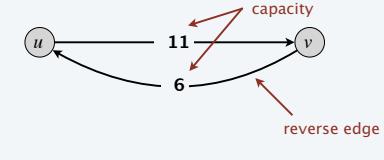
Reverse edge. $e^{\text{reverse}} = (v, u)$.

- “Undo” flow sent.

Residual capacity.

$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^{\text{reverse}} \in E \end{cases}$$

residual network G_f



Residual network. $G_f = (V, E_f, s, t, c_f)$.

- $E_f = \{e : f(e) < c(e)\} \cup \{e^{\text{reverse}} : f(e) > 0\}$.
- Key property: f' is a flow in G_f iff $f + f'$ is a flow in G .

where flow on a reverse edge negates flow on corresponding forward edge

20

Augmenting path

Def. An **augmenting path** is a simple $s \rightsquigarrow t$ path in the residual network G_f .

Def. The **bottleneck capacity** of an augmenting path P is the minimum residual capacity of any edge in P .

Key property. Let f be a flow and let P be an augmenting path in G_f . Then, after calling $f' \leftarrow \text{AUGMENT}(f, c, P)$, the resulting f' is a flow and $\text{val}(f') = \text{val}(f) + \text{bottleneck}(G_f, P)$.

```
AUGMENT( $f, c, P$ )  
     $\delta \leftarrow \text{bottleneck capacity of augmenting path } P$ .  
    FOREACH edge  $e \in P$  :  
        IF ( $e \in E$ )  $f(e) \leftarrow f(e) + \delta$ .  
        ELSE  $f(e^{\text{reverse}}) \leftarrow f(e^{\text{reverse}}) - \delta$ .  
    RETURN  $f$ .
```

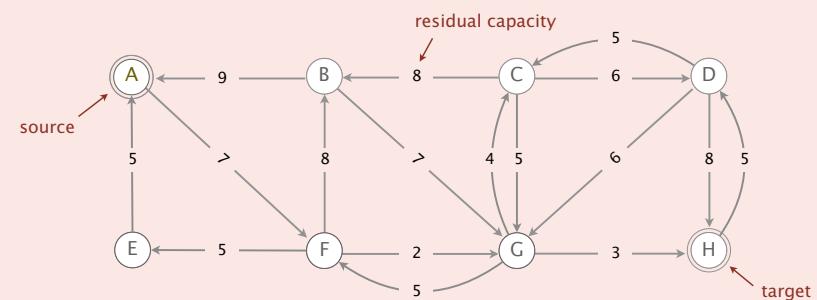
21

Network flow: quiz 2



Which is the augmenting path of highest bottleneck capacity?

- A. $A \rightarrow F \rightarrow G \rightarrow H$
- B. $A \rightarrow B \rightarrow C \rightarrow D \rightarrow H$
- C. $A \rightarrow F \rightarrow B \rightarrow G \rightarrow H$
- D. $A \rightarrow F \rightarrow B \rightarrow G \rightarrow C \rightarrow D \rightarrow H$



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Ford–Fulkerson algorithm

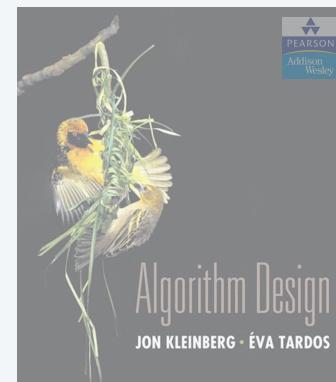
Ford–Fulkerson augmenting path algorithm.

- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \rightsquigarrow t$ path P in the residual network G_f .
- Augment flow along path P .
- Repeat until you get stuck.



```
FORD–FULKERSON( $G$ )  
    FOREACH edge  $e \in E$  :  $f(e) \leftarrow 0$ .  
     $G_f \leftarrow \text{residual network of } G \text{ with respect to flow } f$ .  
    WHILE (there exists an  $s \rightsquigarrow t$  path  $P$  in  $G_f$ )  
         $f \leftarrow \text{AUGMENT}(f, c, P)$ .  
        Update  $G_f$ .  
    RETURN  $f$ .
```

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SECTION 7.2

7. NETWORK FLOW I

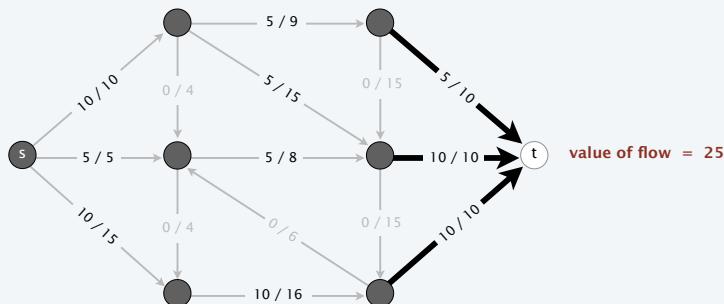
- ▶ max-flow and min-cut problems
- ▶ Ford–Fulkerson algorithm
- ▶ max-flow min-cut theorem
- ▶ capacity-scaling algorithm
- ▶ shortest augmenting paths
- ▶ Dinitz' algorithm
- ▶ simple unit-capacity networks

Relationship between flows and cuts

Flow value lemma. Let f be any flow and let (A, B) be any cut. Then, the value of the flow f equals the net flow across the cut (A, B) .

$$val(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

net flow across cut = $5 + 10 + 10 = 25$



25

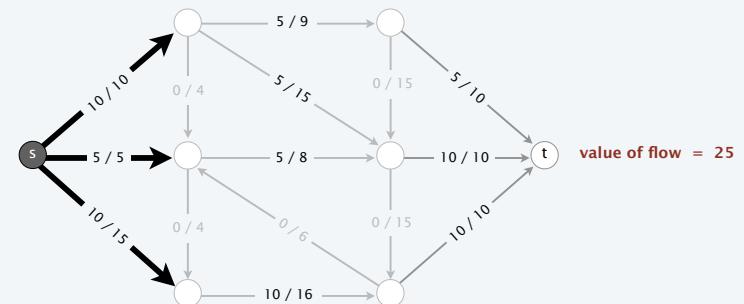
value of flow = 25

Relationship between flows and cuts

Flow value lemma. Let f be any flow and let (A, B) be any cut. Then, the value of the flow f equals the net flow across the cut (A, B) .

$$val(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

net flow across cut = $10 + 5 + 10 = 25$



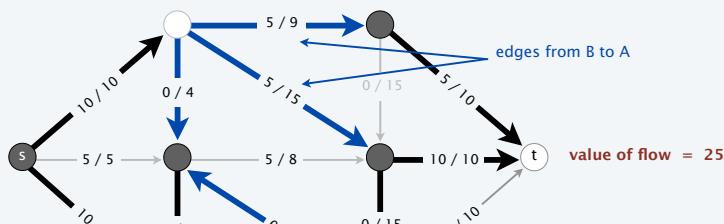
26

Relationship between flows and cuts

Flow value lemma. Let f be any flow and let (A, B) be any cut. Then, the value of the flow f equals the net flow across the cut (A, B) .

$$val(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

net flow across cut = $(10 + 10 + 5 + 10 + 0 + 0) - (5 + 5 + 0 + 0) = 25$



27

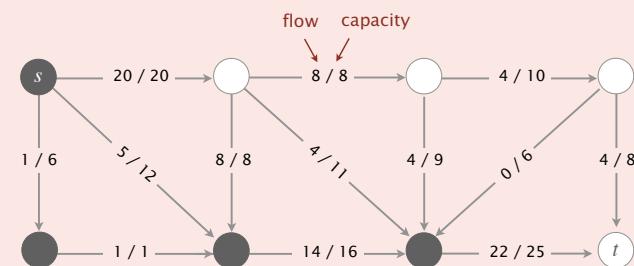
value of flow = 25

Network flow: quiz 3



Which is the net flow across the given cut?

- A. 11 ($20 + 25 - 8 - 11 - 9 - 6$)
- B. 26 ($20 + 22 - 8 - 4 - 4$)
- C. 42 ($20 + 22$)
- D. 45 ($20 + 25$)



28

Relationship between flows and cuts

Flow value lemma. Let f be any flow and let (A, B) be any cut. Then, the value of the flow f equals the net flow across the cut (A, B) .

$$val(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

Pf.

$$val(f) = \sum_{e \text{ out of } s} f(e) - \sum_{e \text{ in to } s} f(e)$$

by flow conservation, all terms except for $v = s$ are 0

$$\rightarrow = \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$$

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \blacksquare$$

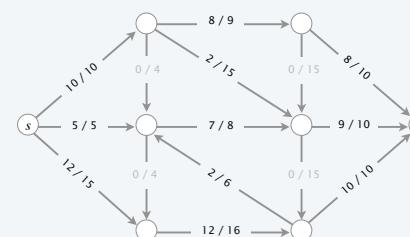
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Relationship between flows and cuts

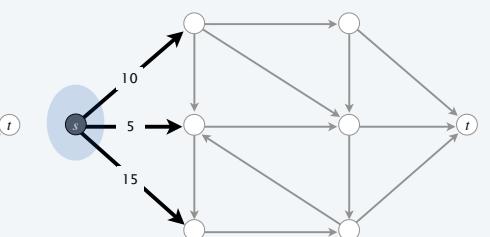
Weak duality. Let f be any flow and (A, B) be any cut. Then, $val(f) \leq cap(A, B)$.

Pf.

$$\begin{aligned} val(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &\stackrel{\text{flow value lemma}}{\leq} \sum_{e \text{ out of } A} f(e) \\ &\leq \sum_{e \text{ out of } A} c(e) \\ &= cap(A, B) \blacksquare \end{aligned}$$



value of flow = 27



capacity of cut = 30

30

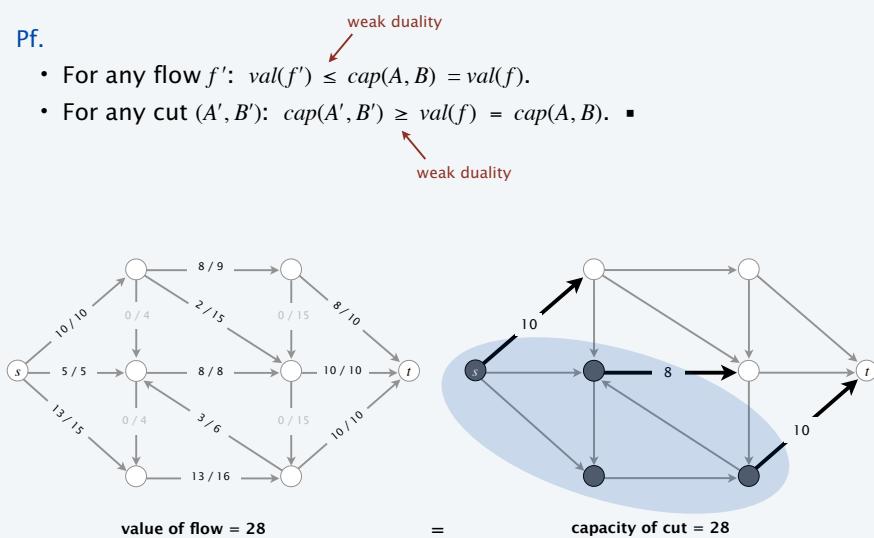
Certificate of optimality

Corollary. Let f be a flow and let (A, B) be any cut.

If $val(f) = cap(A, B)$, then f is a max flow and (A, B) is a min cut.

Pf.

- For any flow f' : $val(f') \leq cap(A, B) = val(f)$. weak duality
- For any cut (A', B') : $cap(A', B') \geq val(f) = cap(A, B)$. weak duality \blacksquare



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Max-flow min-cut theorem

Max-flow min-cut theorem. Value of a max flow = capacity of a min cut.

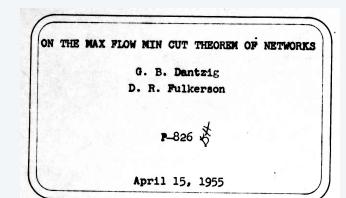
strong duality

MAXIMAL FLOW THROUGH A NETWORK

L. R. FORD, JR. AND D. R. FULKERSON

Introduction. The problem discussed in this paper was formulated by T. Harris as follows:

"Consider a rail network connecting two cities by way of a number of intermediate cities, where each link of the network has a number assigned to it representing its capacity. Assuming a steady state condition, find a maximal flow from one given city to the other."



A Note on the Maximum Flow Through a Network*

P. ELLIAS, A. FEINSTEIN†, AND C. E. SHANNON‡

Summary. This note discusses the problem of maximizing the rate of flow through a network, a problem which consists of a number of branches, each of which has a limited capacity. The main result is a theorem: The maximum possible flow from left to right through a network is equal to the minimum value among all its cut-sets. This theorem is applied to solve a more general problem, in which a number of input nodes and a number of output nodes are used.

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Max-flow min-cut theorem

Max-flow min-cut theorem. Value of a max flow = capacity of a min cut.

Augmenting path theorem. A flow f is a max flow iff no augmenting paths.

Pf. The following three conditions are equivalent for any flow f :

- There exists a cut (A, B) such that $\text{cap}(A, B) = \text{val}(f)$.
- f is a max flow.
- There is no augmenting path with respect to f . ← if Ford-Fulkerson terminates, then f is max flow

[i \Rightarrow ii]

- This is the weak duality corollary. ■

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Max-flow min-cut theorem

Max-flow min-cut theorem. Value of a max flow = capacity of a min cut.

Augmenting path theorem. A flow f is a max flow iff no augmenting paths.

Pf. The following three conditions are equivalent for any flow f :

- There exists a cut (A, B) such that $\text{cap}(A, B) = \text{val}(f)$.
- f is a max flow.
- There is no augmenting path with respect to f .

[ii \Rightarrow iii] We prove contrapositive: \neg iii \Rightarrow \neg ii.

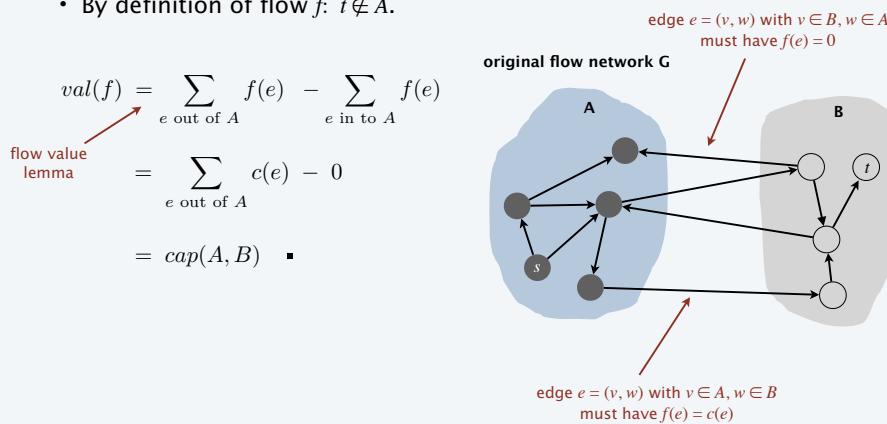
- Suppose that there is an augmenting path with respect to f .
- Can improve flow f by sending flow along this path.
- Thus, f is not a max flow. ■

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Max-flow min-cut theorem

[iii \Rightarrow i]

- Let f be a flow with no augmenting paths.
- Let A = set of nodes reachable from s in residual network G_f .
- By definition of A : $s \in A$.
- By definition of flow f : $t \notin A$.



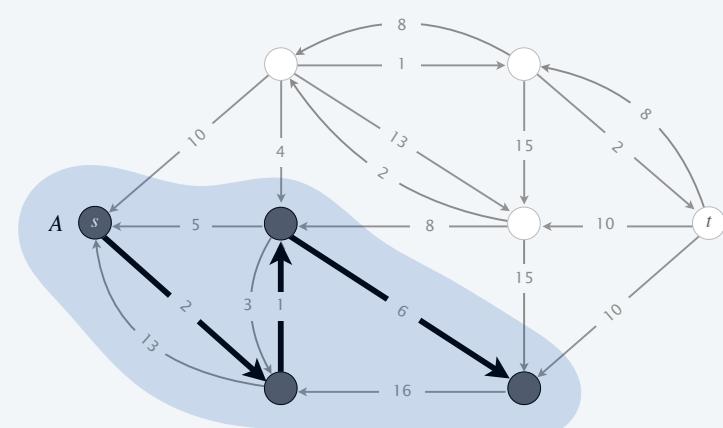
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Computing a minimum cut from a maximum flow

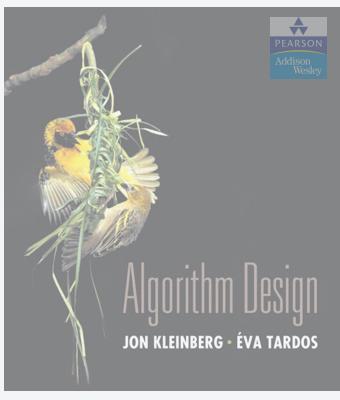
Theorem. Given any max flow f , can compute a min cut (A, B) in $O(m)$ time.

Pf. Let A = set of nodes reachable from s in residual network G_f . ■

argument from previous slide implies that
capacity of (A, B) = value of flow f



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SECTION 7.3

7. NETWORK FLOW I

- ▶ max-flow and min-cut problems
- ▶ Ford–Fulkerson algorithm
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- ▶ simple unit-capacity networks

Analysis of Ford–Fulkerson algorithm (when capacities are integral)

Assumption. Every edge capacity $c(e)$ is an integer between 1 and C .

Integrality invariant. Throughout Ford–Fulkerson, every edge flow $f(e)$ and residual capacity $c_f(e)$ is an integer.

Pf. By induction on the number of augmenting paths. ▀

consider cut $A = \{s\}$
(assumes no parallel edges)

Theorem. Ford–Fulkerson terminates after at most $\text{val}(f^*) \leq nC$ augmenting paths, where f^* is a max flow.

Pf. Each augmentation increases the value of the flow by at least 1. ▀

Corollary. The running time of Ford–Fulkerson is $O(mnC)$.

Pf. Can use either BFS or DFS to find an augmenting path in $O(m)$ time. ▀

$f(e)$ is an integer for every e

Integrality theorem. There exists an integral max flow f^* .

Pf. Since Ford–Fulkerson terminates, theorem follows from integrality invariant (and augmenting path theorem). ▀

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Ford–Fulkerson: exponential example

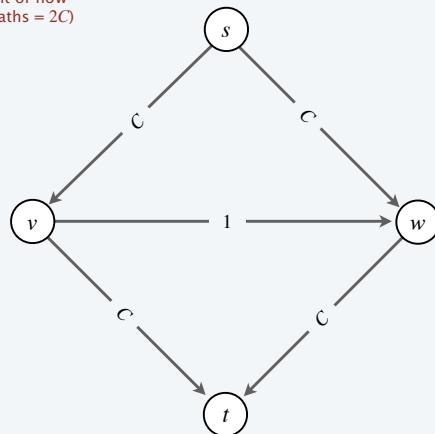
Q. Is generic Ford–Fulkerson algorithm poly-time in input size?

m, n , and $\log C$



A. No. If max capacity is C , then algorithm can take $\geq C$ iterations.

- $s \rightarrow v \rightarrow w \rightarrow t$
- $s \rightarrow w \rightarrow v \rightarrow t$
- $s \rightarrow v \rightarrow w \rightarrow t$
- $s \rightarrow w \rightarrow v \rightarrow t$
- ...
- $s \rightarrow v \rightarrow w \rightarrow t$
- $s \rightarrow w \rightarrow v \rightarrow t$



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Network flow: quiz 4



The Ford–Fulkerson algorithm is guaranteed to terminate if the edge capacities are ...

- A. Rational numbers.
- B. Real numbers.
- C. Both A and B.
- D. Neither A nor B.

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Choosing good augmenting paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.

Pathology. When edge capacities can be irrational, no guarantee that Ford–Fulkerson terminates (or converges to a maximum flow)!



Goal. Choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

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Choosing good augmenting paths

Choose augmenting paths with:

- Max bottleneck capacity (“fattest”). how to find?
- Sufficiently large bottleneck capacity. next
- Fewest edges. ahead

Theoretical Improvements in Algorithmic Efficiency
for Network Flow Problems

Dokl. Akad. Nauk SSSR
Ton 194 (1970), No. 4

Soviet Math. Dokl.
Vol. 11 (1970), No. 5

JACK EDMONDS
University of Waterloo, Waterloo, Ontario, Canada

AND

RICHARD M. KARP
University of California, Berkeley, California

Abstract. This paper presents new algorithms for the maximum flow problem, the Hitchcock transportation problem, and the general minimum-cost flow problem. Upper bounds on the numbers of steps in these algorithms are derived, and are shown to compare favorably with upper bounds on the numbers of steps required by earlier algorithms.

ALGORITHM FOR SOLUTION OF A PROBLEM OF MAXIMUM FLOW IN A NETWORK WITH
POWER ESTIMATION
UDC 518.5
E. A. DINITZ

Different variants of the formulation of the problem of maximal stationary flow in a network and its many applications are given in [1]. There also is given an algorithm solving the problem in the case where the initial data are integers (or, what is equivalent, commensurable). In the general case this algorithm requires preliminary rounding off of the initial data, i.e. only an approximate solution of the problem is possible. In this connection the rapidity of convergence of the algorithm is inversely proportional to the relative precision.

Edmonds-Karp 1972 (USA)

Dinitz 1970 (Soviet Union)



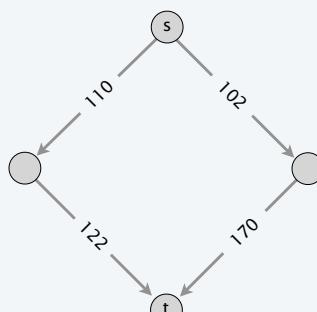
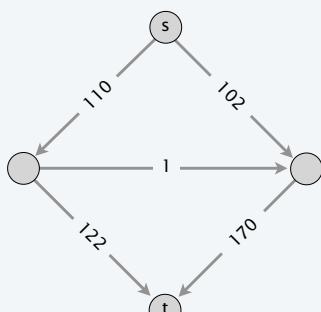
invented in response to a class
exercises by Adel'son-Vel'skiĭ

42

Capacity-scaling algorithm

Overview. Choosing augmenting paths with “large” bottleneck capacity.

- Maintain scaling parameter Δ . though not necessarily largest
- Let $G_f(\Delta)$ be the part of the residual network containing only those edges with capacity $\geq \Delta$.
- Any augmenting path in $G_f(\Delta)$ has bottleneck capacity $\geq \Delta$.



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Capacity-scaling algorithm

CAPACITY-SCALING(G)

FOREACH edge $e \in E$: $f(e) \leftarrow 0$.

$\Delta \leftarrow$ largest power of 2 $\leq C$.

WHILE ($\Delta \geq 1$)

$G_f(\Delta) \leftarrow \Delta$ -residual network of G with respect to flow f .
WHILE (there exists an $s \rightsquigarrow t$ path P in $G_f(\Delta)$)

$f \leftarrow$ AUGMENT(f, c, P).

Update $G_f(\Delta)$.

$\Delta \leftarrow \Delta / 2$.

Δ -scaling phase

RETURN f .

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Capacity-scaling algorithm: proof of correctness

Assumption. All edge capacities are integers between 1 and C .

Invariant. The scaling parameter Δ is a power of 2.

Pf. Initially a power of 2; each phase divides Δ by exactly 2. ▀

Integrality invariant. Throughout the algorithm, every edge flow $f(e)$ and residual capacity $c_f(e)$ is an integer.

Pf. Same as for generic Ford–Fulkerson. ▀

Theorem. If capacity-scaling algorithm terminates, then f is a max flow.

Pf.

- By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$.
- Upon termination of $\Delta=1$ phase, there are no augmenting paths.
- Result follows augmenting path theorem ▀

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Capacity-scaling algorithm: analysis of running time

Lemma 1. There are $1 + \lfloor \log_2 C \rfloor$ scaling phases.

Pf. Initially $C/2 < \Delta \leq C$; Δ decreases by a factor of 2 in each iteration. ▀

Lemma 2. Let f be the flow at the end of a Δ -scaling phase.

Then, the max-flow value $\leq \text{val}(f) + m\Delta$.

Pf. Next slide.

Lemma 3. There are $\leq 2m$ augmentations per scaling phase.

Pf.

- Let f be the flow at the beginning of a Δ -scaling phase.
- Lemma 2 \Rightarrow max-flow value $\leq \text{val}(f) + m(2\Delta)$.
- Each augmentation in a Δ -phase increases $\text{val}(f)$ by at least Δ . ▀

or equivalently,
at the end
of a 2Δ -scaling phase

Theorem. The capacity-scaling algorithm takes $O(m^2 \log C)$ time.

Pf.

- Lemma 1 + Lemma 3 $\Rightarrow O(m \log C)$ augmentations.
- Finding an augmenting path takes $O(m)$ time. ▀

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Capacity-scaling algorithm: analysis of running time

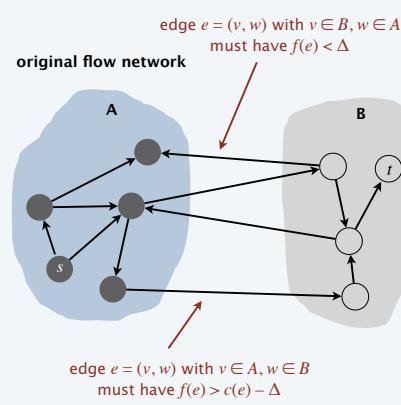
Lemma 2. Let f be the flow at the end of a Δ -scaling phase.

Then, the max-flow value $\leq \text{val}(f) + m\Delta$.

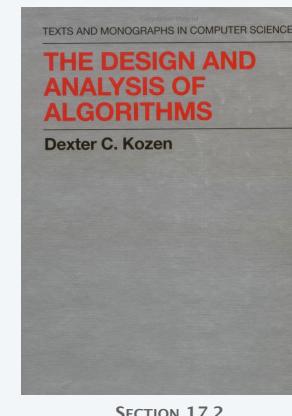
Pf.

- We show there exists a cut (A, B) such that $\text{cap}(A, B) \leq \text{val}(f) + m\Delta$.
- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
- By definition of A : $s \in A$.
- By definition of flow f : $t \notin A$.

$$\begin{aligned} \text{val}(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &\stackrel{\text{flow value lemma}}{\geq} \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\ &\geq \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta \\ &\geq \text{cap}(A, B) - m\Delta \quad \blacksquare \end{aligned}$$



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7. NETWORK FLOW I

- ▶ max-flow and min-cut problems
- ▶ Ford–Fulkerson algorithm
- ▶ max-flow min-cut theorem
- ▶ capacity-scaling algorithm
- ▶ shortest augmenting paths
- ▶ Dinitz' algorithm
- ▶ simple unit-capacity networks

Shortest augmenting path

Q. How to choose next augmenting path in Ford–Fulkerson?

- A. Pick one that uses the fewest edges.

can find via BFS

SHORTEST-AUGMENTING-PATH(G)

FOREACH $e \in E : f(e) \leftarrow 0$.

$G_f \leftarrow$ residual network of G with respect to flow f .

WHILE (there exists an $s \rightsquigarrow t$ path in G_f)

$P \leftarrow$ BREADTH-FIRST-SEARCH(G_f).

$f \leftarrow$ AUGMENT(f, c, P).

Update G_f .

RETURN f .

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Shortest augmenting path: overview of analysis

Lemma 1. The length of a shortest augmenting path never decreases.

Pf. Ahead.

number of edges

Lemma 2. After at most m shortest-path augmentations, the length of a shortest augmenting path strictly increases.

Pf. Ahead.

Theorem. The shortest-augmenting-path algorithm takes $O(m^2 n)$ time.

Pf.

- $O(m)$ time to find a shortest augmenting path via BFS.

- There are $\leq m n$ augmentations.

- at most m augmenting paths of length k

- at most $n-1$ different lengths ■

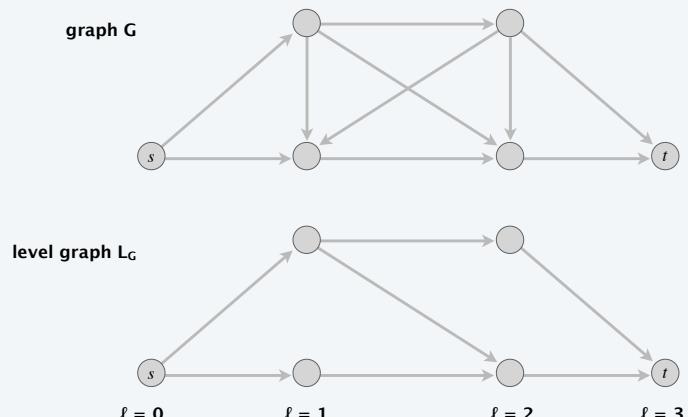
augmenting paths are simple paths

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Shortest augmenting path: analysis

Def. Given a digraph $G = (V, E)$ with source s , its **level graph** is defined by:

- $\ell(v) =$ number of edges in shortest $s \rightsquigarrow v$ path.
- $L_G = (V, E_G)$ is the subgraph of G that contains only those edges $(v, w) \in E$ with $\ell(w) = \ell(v) + 1$.



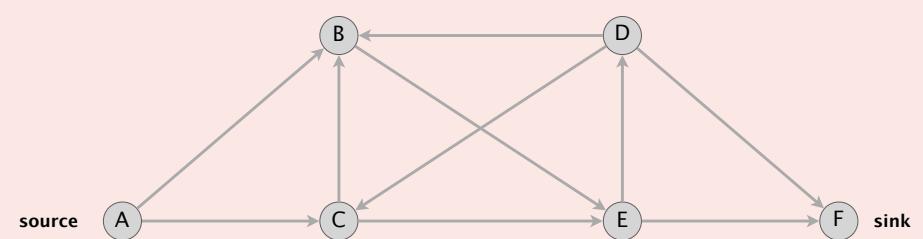
51

Network flow: quiz 5



Which edges are in the level graph of the following digraph?

- A. $D \rightarrow F$.
 B. $E \rightarrow F$.
 C. Both A and B.
 D. Neither A nor B.



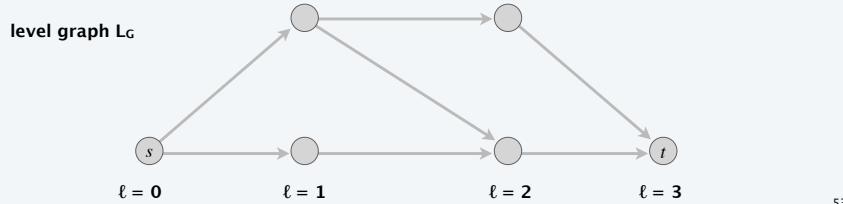
52

Shortest augmenting path: analysis

Def. Given a digraph $G = (V, E)$ with source s , its **level graph** is defined by:

- $\ell(v) = \text{number of edges in shortest } s \rightsquigarrow v \text{ path}$.
- $L_G = (V, E_G)$ is the subgraph of G that contains only those edges $(v, w) \in E$ with $\ell(w) = \ell(v) + 1$.

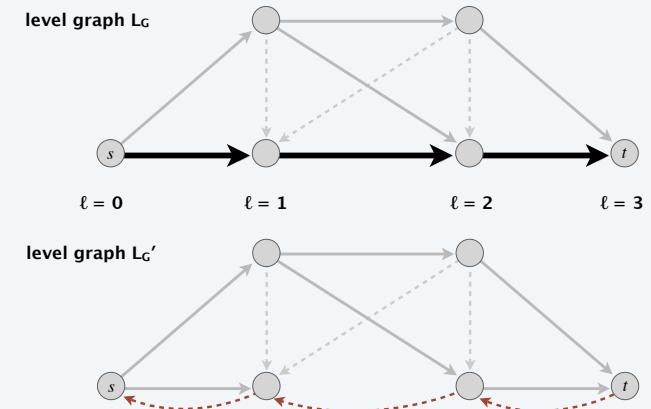
Key property. P is a shortest $s \rightsquigarrow v$ path in G iff P is an $s \rightsquigarrow v$ path in L_G .



Shortest augmenting path: analysis

Lemma 1. The length of a shortest augmenting path never decreases.

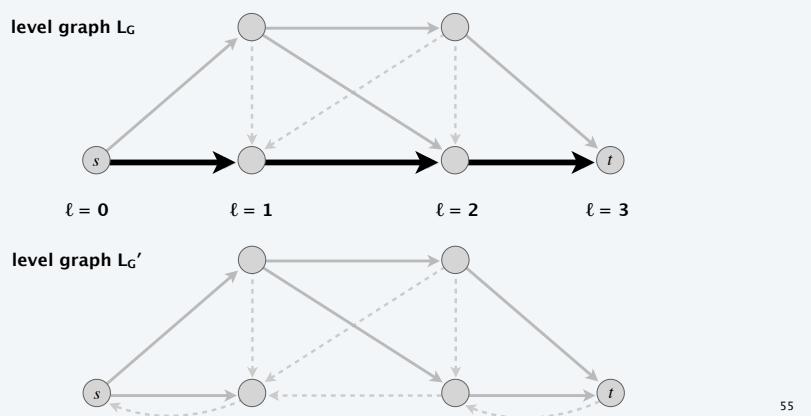
- Let f and f' be flow before and after a shortest-path augmentation.
- Let L_G and $L_{G'}$ be level graphs of G_f and $G_{f'}$.
- Only back edges added to $G_{f'}$
- (any $s \rightsquigarrow t$ path that uses a back edge is longer than previous length) ▀



Shortest augmenting path: analysis

Lemma 2. After at most m shortest-path augmentations, the length of a shortest augmenting path strictly increases.

- At least one (bottleneck) edge is deleted from L_G per augmentation.
- No new edge added to L_G until shortest path length strictly increases. ▀



Shortest augmenting path: review of analysis

Lemma 1. Throughout the algorithm, the length of a shortest augmenting path never decreases.

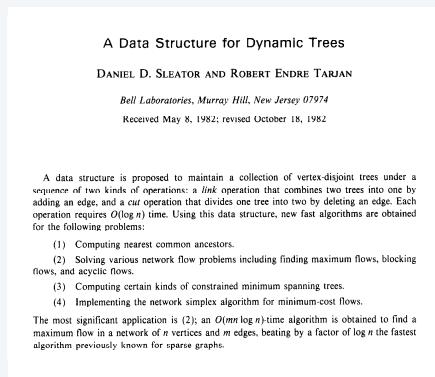
Lemma 2. After at most m shortest-path augmentations, the length of a shortest augmenting path strictly increases.

Theorem. The shortest-augmenting-path algorithm takes $O(m^2 n)$ time.

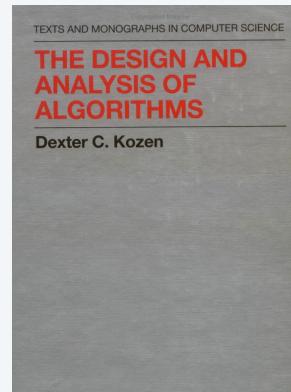
Shortest augmenting path: improving the running time

Note. $\Theta(mn)$ augmentations necessary for some flow networks.

- Try to decrease time per augmentation instead.
- Simple idea $\Rightarrow O(mn^2)$ [Dinitz 1970] ← ahead
- Dynamic trees $\Rightarrow O(mn \log n)$ [Sleator–Tarjan 1983]



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SECTION 18.1

7. NETWORK FLOW I

- ▶ max-flow and min-cut problems
- ▶ Ford–Fulkerson algorithm
- ▶ max-flow min-cut theorem
- ▶ capacity-scaling algorithm
- ▶ shortest augmenting paths
- ▶ Dinitz' algorithm
- ▶ simple unit-capacity networks

Dinitz' algorithm

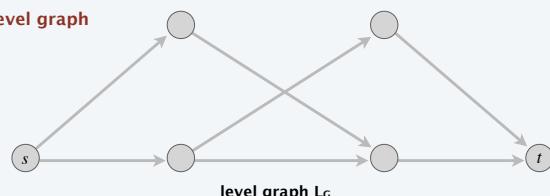
Two types of augmentations.

- Normal: length of shortest path does not change.
- Special: length of shortest path strictly increases.

Phase of normal augmentations. ← within a phase, length of shortest augmenting path does not change

- Construct level graph L_G .
- Start at s , advance along an edge in L_G until reach t or get stuck.
- If reach t , augment flow; update L_G ; and restart from s .
- If get stuck, delete node from L_G and retreat to previous node.

construct level graph



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Dinitz' algorithm

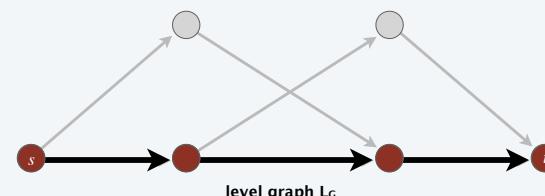
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- If reach t , augment flow; update L_G ; and restart from s .
- If get stuck, delete node from L_G and retreat to previous node.

advance



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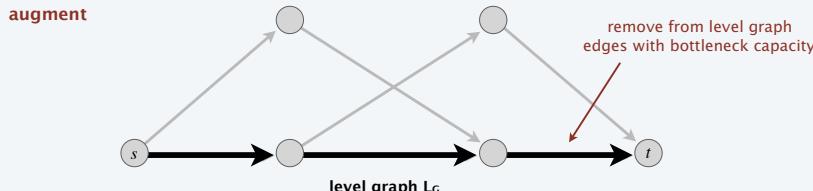
Dinitz' algorithm

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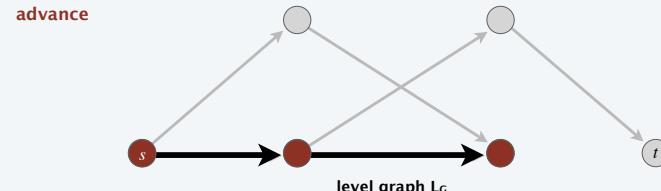
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62

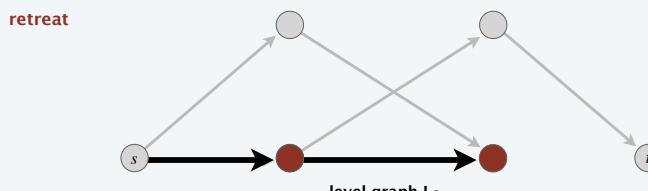
Dinitz' algorithm

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- If get stuck, delete node from L_G and retreat to previous node.



63

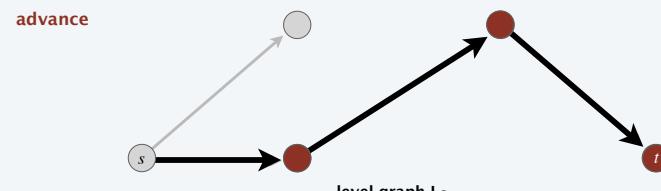
Dinitz' algorithm

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64

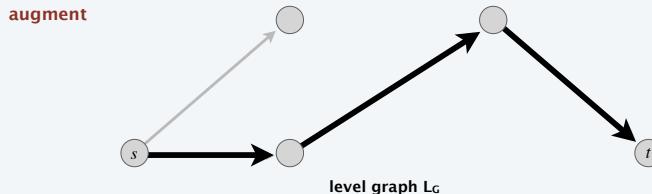
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65

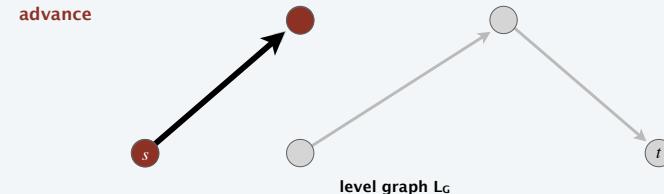
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66

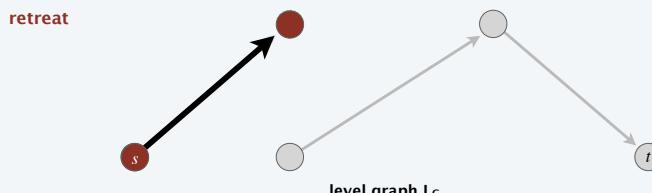
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67

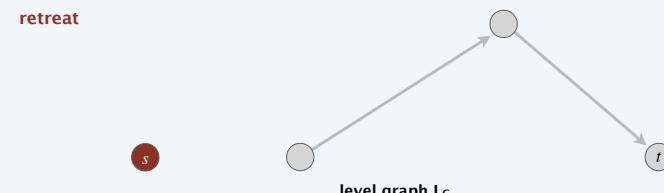
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Dinitz' algorithm

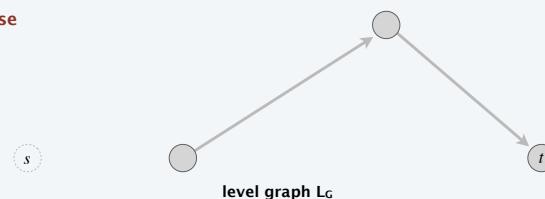
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Phase of normal augmentations.

- Construct level graph L_G .
- Start at s , advance along an edge in L_G until reach t or get stuck.
- If reach t , augment flow; update L_G ; and restart from s .
- If get stuck, delete node from L_G and retreat to previous node.

end of phase



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Dinitz' algorithm (as refined by Even and Itai)

INITIALIZE(G, f)

$L_G \leftarrow$ level-graph of G_f .

$P \leftarrow \emptyset$.

GOTO **ADVANCE**(s).

ADVANCE(v)

IF ($v = t$)

AUGMENT(P).

Remove saturated edges from L_G .

$P \leftarrow \emptyset$.

GOTO **ADVANCE**(s).

RETREAT(v)

IF ($v = s$)

STOP.

ELSE

Delete v (and all incident edges) from L_G .

Remove last edge (u, v) from P .

GOTO **ADVANCE**(u).

IF (there exists edge $(v, w) \in L_G$)

Add edge (v, w) to P .

GOTO **ADVANCE**(w).

ELSE

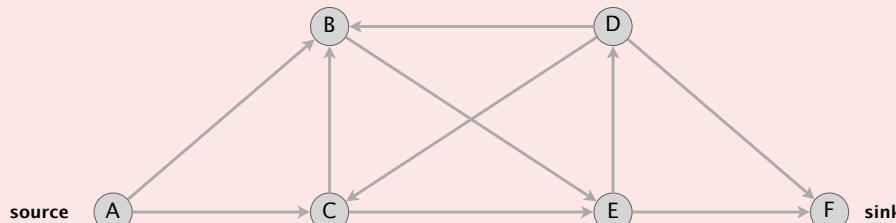
GOTO **RETREAT**(v).

Network flow: quiz 6



How to compute the level graph L_G efficiently?

- A. Depth-first search.
- B. Breadth-first search.
- C. Both A and B.
- D. Neither A nor B.



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Dinitz' algorithm: analysis

Lemma. A phase can be implemented to run in $O(mn)$ time.

Pf.

- Initialization happens once per phase. $\xleftarrow{\text{O}(m)}$ using BFS
- At most m augmentations per phase. $\xleftarrow{\text{O}(mn)}$ per phase
(because an augmentation deletes at least one edge from L_G)
- At most n retreats per phase. $\xleftarrow{\text{O}(m+n)}$ per phase
(because a retreat deletes one node from L_G)
- At most mn advances per phase. $\xleftarrow{\text{O}(mn)}$ per phase
(because at most n advances before retreat or augmentation) ▀

Theorem. [Dinitz 1970] Dinitz' algorithm runs in $O(mn^2)$ time.

Pf.

- By Lemma, $O(mn)$ time per phase.
- At most $n-1$ phases (as in shortest-augmenting-path analysis). ▀

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Augmenting-path algorithms: summary

year	method	# augmentations	running time
1955	augmenting path	$n C$	$O(mnC)$
1972	fattest path	$m \log(mC)$	$O(m^2 \log n \log(mC))$
1972	capacity scaling	$m \log C$	$O(m^2 \log C)$
1985	improved capacity scaling	$m \log C$	$O(mn \log C)$
1970	shortest augmenting path	$m n$	$O(m^2 n)$
1970	level graph	$m n$	$O(mn^2)$
1983	dynamic trees	$m n$	$O(mn \log n)$

augmenting-path algorithms with m edges, n nodes, and integer capacities between 1 and C



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Maximum-flow algorithms: theory highlights

year	method	worst case	discovered by
1951	simplex	$O(mn^2C)$	Dantzig
1955	augmenting paths	$O(mnC)$	Ford–Fulkerson
1970	shortest augmenting paths	$O(mn^2)$	Edmonds–Karp, Dinitz
1974	blocking flows	$O(n^3)$	Karzanov
1983	dynamic trees	$O(mn \log n)$	Sleator–Tarjan
1985	improved capacity scaling	$O(mn \log C)$	Gabow
1988	push-relabel	$O(mn \log(n^2/m))$	Goldberg–Tarjan
1998	binary blocking flows	$O(m^{3/2} \log(n^2/m) \log C)$	Goldberg–Rao
2013	compact networks	$O(mn)$	Orlin
2014	interior-point methods	$\tilde{O}(m^{1/2} \log C)$	Lee–Sidford
2016	electrical flows	$\tilde{O}(m^{10/7} C^{1/7})$	Mądry
20xx		???	

max-flow algorithms with m edges, n nodes, and integer capacities between 1 and C

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Maximum-flow algorithms: practice

Push–relabel algorithm (SECTION 7.4). [Goldberg–Tarjan 1988]

Increases flow one edge at a time instead of one augmenting path at a time.

A New Approach to the Maximum-Flow Problem

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AND

ROBERT E. TARJAN

Princeton University, Princeton, New Jersey, and AT&T Bell Laboratories, Murray Hill, New Jersey

Abstract. All previously known efficient maximum-flow algorithms work by finding augmenting paths, either one path at a time (as in the original Ford and Fulkerson algorithm) or all shortest-length augmenting paths at once (using the layered network approach of Dinic). An alternative method based on the *preflow* concept of Karzanov is introduced. A preflow is like a flow, except that the total amount flowing into a vertex is allowed to exceed the total amount flowing out. The method maintains a preflow in the original network and pushes local flow excess toward the sink along what are estimated to be shortest paths. The algorithm and its analysis are simple and intuitive, yet the algorithm runs as fast as any other known method on dense graphs, achieving an $O(n^2)$ time bound on an n -vertex graph. By incorporating the dynamic tree data structure of Sleator and Tarjan, we obtain a version of the algorithm running in $O(\alpha m \log(n^2/m))$ time on an n -vertex, m -edge graph. This is as fast as any known method for any graph density and faster on graphs of moderate density. The algorithm also admits efficient distributed and parallel implementations. A parallel implementation running in $O(n^2 \log n)$ time using n processors and $O(m)$ space is obtained. This time bound matches that of the Shiloach–Vishkin algorithm, which also uses n processors but requires $O(n^2)$ space.

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Maximum-flow algorithms: practice

Caveat. Worst-case running time is generally not useful for predicting or comparing max-flow algorithm performance in practice.

Best in practice. Push–relabel method with gap relabeling: $O(m^{3/2})$ in practice.

On Implementing Push–Relabel Method for the Maximum Flow Problem

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European Journal of Operational Research 97 (1997) 509–542

Theory and Methodology

Computational investigations of maximum flow algorithms

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Received 30 August 1995; accepted 27 June 1996

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Maximum-flow algorithms: practice

Computer vision. Different algorithms work better for some dense problems that arise in applications to computer vision.

An Experimental Comparison of Min-Cut/Max-Flow Algorithms for Energy Minimization in Vision
Yuri Boykov and Vladimir Kolmogorov*

Abstract

After [15, 31, 19, 8, 25, 5] minimum cut/max-flow algorithms on graphs emerged as an increasingly useful tool for exact or approximate energy minimization in low-level vision. The combinatorial optimization literature provides many min-cut/max-flow algorithms with different polynomial time complexity. Their practical efficiency, however, has to date been studied mainly outside the scope of computer vision. The goal of this paper is to provide an experimental comparison of the efficiency of min-cut/max-flow algorithms for applications in vision. We compare the running times of several standard algorithms, as well as a new algorithm that we have recently developed. The algorithms we study include both Goldberg-Tarjan style “push-relabel” methods and algorithms based on Ford-Fulkerson style “augmenting paths”. We benchmark these algorithms on a number of typical graphs in the contexts of image restoration, stereo, and segmentation. In many cases our new algorithm works several times faster than any of the other methods making near real-time performance possible. An implementation of our max-flow/min-cut algorithm is available upon request for research purposes.

VERMA, BATRA: MAXFLOW REVISITED¹

MaxFlow Revisited: An Empirical Comparison of Maxflow Algorithms for Dense Vision Problems

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Abstract

Algorithms for finding the maximum amount of flow possible in a network (or max-flow) play a central role in computer vision problems. We present an empirical comparison of different max-flow algorithms on modern problems. Our problem instances arise from energy minimization problems in Object Category Segmentation, Image Deconvolution, Super Resolution, Texture Restoration, Character Completion and 3D Segmentation. We compare 14 different implementations and find that the most popularly used implementation of Kolmogorov [5] is no longer the fastest algorithm available, especially for dense graphs.

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Maximum-flow algorithms: Matlab



Documentation

CONTENTS

maxflow

Maximum flow in graph



R2018a

collapse all in page

Syntax

```
mf = maxflow(G,s,t)
mf = maxflow(G,s,t,algorithm)
[mf,GF] = maxflow(__)
[mf,GF,cs,ct] = maxflow(__)
```

Description

`mf = maxflow(G,s,t)` returns the **maximum flow** between nodes `s` and `t`. If graph `G` is unweighted (that is, `G.Edges` does not contain the variable `Weight`), then `maxflow` treats all graph edges as having a weight equal to 1.

example

`mf = maxflow(G,s,t,algorithm)` specifies the maximum flow algorithm to use. This syntax is only available if `G` is a directed graph.

example

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Maximum-flow algorithms: Google

Google Optimization Tools

Products > Optimization > Reference

Contents Classes

C++ Reference: max_flow

This documentation is automatically generated.

An implementation of a push-relabel algorithm for the max flow problem.

In the following, we consider a graph $G = (V, E, s, t)$ where V denotes the set of nodes (vertices) in the graph, E denotes the set of arcs (edges). s and t denote distinguished nodes in G called source and target. $n = |V|$ denotes the number of nodes in the graph, and $m = |E|$ denotes the number of arcs in the graph.

Each arc (v, w) is associated a capacity $c(v, w)$.

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7. NETWORK FLOW I

- ▶ *max-flow and min-cut problems*
- ▶ *Ford–Fulkerson algorithm*
- ▶ *max-flow min-cut theorem*
- ▶ *capacity-scaling algorithm*
- ▶ *shortest augmenting paths*
- ▶ *Dinitz' algorithm*
- ▶ *simple unit-capacity networks*



Which max-flow algorithm to use for bipartite matching?

- A. Ford-Fulkerson: $O(m n C)$.
- B. Capacity scaling: $O(m^2 \log C)$.
- C. Shortest augmenting path: $O(m^2 n)$.
- D. Dinitz' algorithm: $O(m n^2)$.

Simple unit-capacity networks

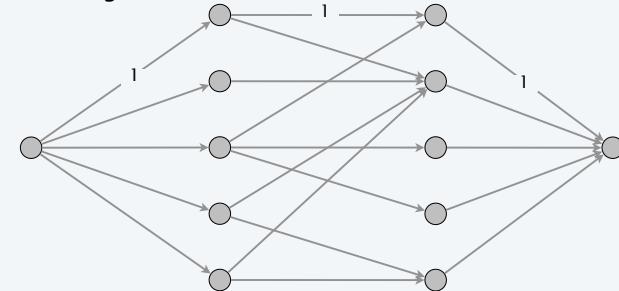
Def. A flow network is a **simple unit-capacity network** if:

- Every edge has capacity 1.
- Every node (other than s or t) has exactly one entering edge, or exactly one leaving edge, or both.

node capacity = 1

Property. Let G be a simple unit-capacity network and let f be a 0–1 flow. Then, residual network G_f is also a simple unit-capacity network.

Ex. Bipartite matching.



Simple unit-capacity networks

Shortest-augmenting-path algorithm.

- Normal augmentation: length of shortest path does not change.
- Special augmentation: length of shortest path strictly increases.

Theorem. [Even–Tarjan 1975] In simple unit-capacity networks, Dinitz' algorithm computes a maximum flow in $O(m n^{1/2})$ time.

Pf.

- Lemma 1. Each phase of normal augmentations takes $O(m)$ time.
- Lemma 2. After $n^{1/2}$ phases, $\text{val}(f) \geq \text{val}(f^*) - n^{1/2}$.
- Lemma 3. After $\leq n^{1/2}$ additional augmentations, flow is optimal. ▀

Lemma 3. After $\leq n^{1/2}$ additional augmentations, flow is optimal.

Pf. Each augmentation increases flow value by at least 1. ▀

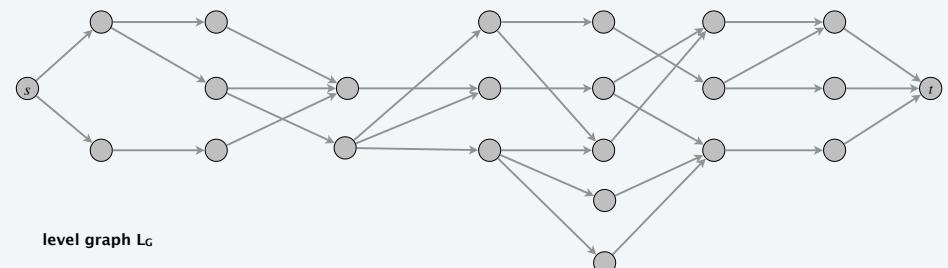
Lemma 1 and Lemma 2. Ahead.

Simple unit-capacity networks

Phase of normal augmentations. ← within a phase, length of shortest augmenting path does not change

- Construct level graph L_G .
- Start at s , advance along an edge in L_G until reach t or get stuck.
- If reach t , augment flow; update L_G ; and restart from s .
- If get stuck, delete node from L_G and go to previous node.

construct level graph

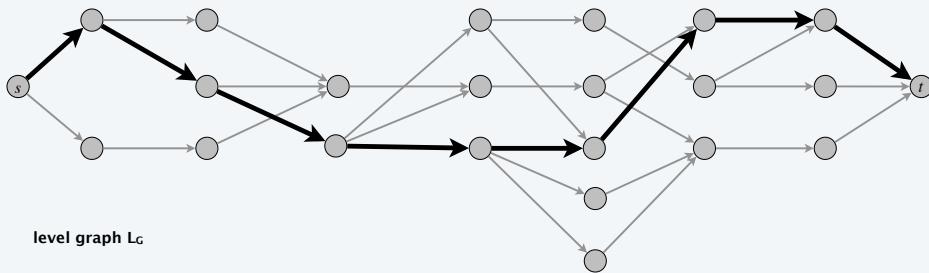
level graph L_G

Simple unit-capacity networks

Phase of normal augmentations.

- Construct level graph L_G .
- Start at s , advance along an edge in L_G until reach t or get stuck.
- If reach t , augment flow; update L_G ; and restart from s .
- If get stuck, delete node from L_G and go to previous node.

advance



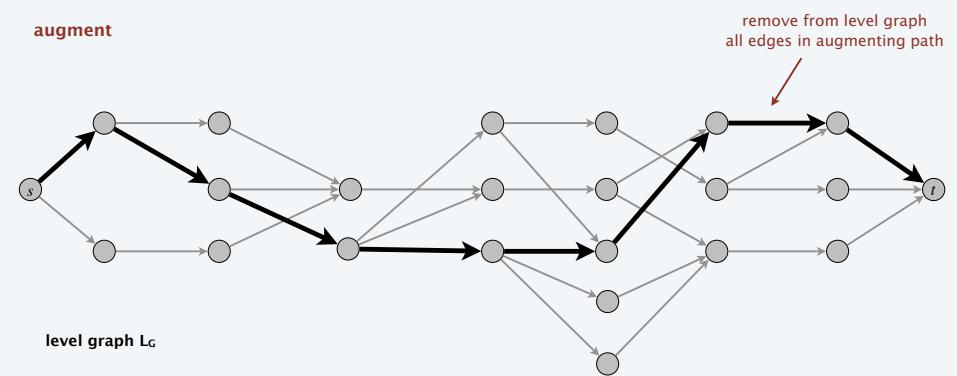
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Simple unit-capacity networks

Phase of normal augmentations.

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- Start at s , advance along an edge in L_G until reach t or get stuck.
- If reach t , augment flow; update L_G ; and restart from s .
- If get stuck, delete node from L_G and go to previous node.

augment



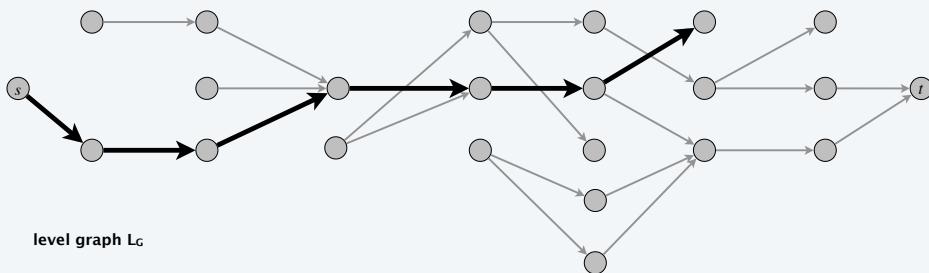
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Simple unit-capacity networks

Phase of normal augmentations.

- Construct level graph L_G .
- Start at s , advance along an edge in L_G until reach t or get stuck.
- If reach t , augment flow; update L_G ; and restart from s .
- If get stuck, delete node from L_G and go to previous node.

advance



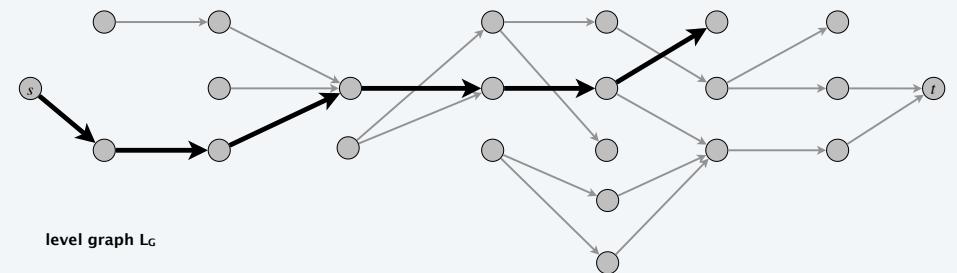
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Simple unit-capacity networks

Phase of normal augmentations.

- Construct level graph L_G .
- Start at s , advance along an edge in L_G until reach t or get stuck.
- If reach t , augment flow; update L_G ; and restart from s .
- If get stuck, delete node from L_G and go to previous node.

retreat



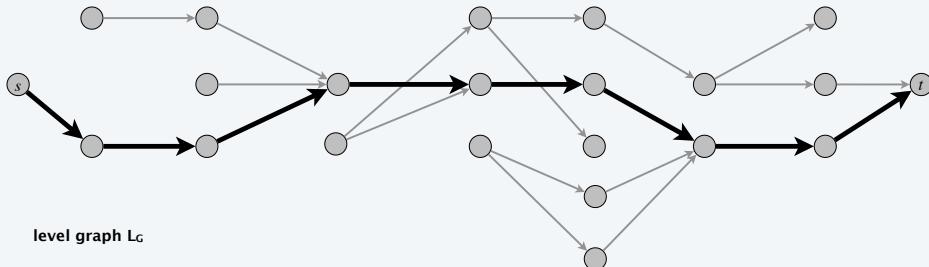
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Simple unit-capacity networks

Phase of normal augmentations.

- Construct level graph L_G .
- Start at s , advance along an edge in L_G until reach t or get stuck.
- If reach t , augment flow; update L_G ; and restart from s .
- If get stuck, delete node from L_G and go to previous node.

advance



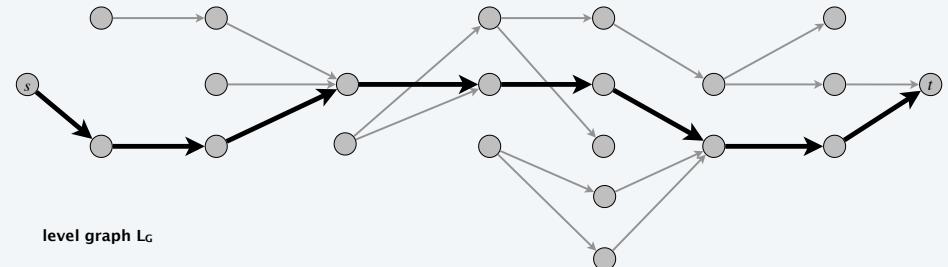
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Simple unit-capacity networks

Phase of normal augmentations.

- Construct level graph L_G .
- Start at s , advance along an edge in L_G until reach t or get stuck.
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augment



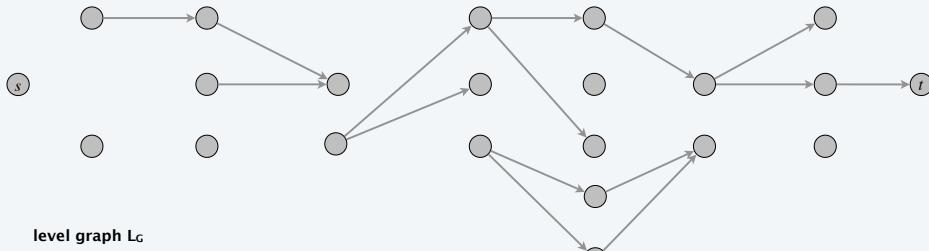
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Simple unit-capacity networks

Phase of normal augmentations.

- Construct level graph L_G .
- Start at s , advance along an edge in L_G until reach t or get stuck.
- If reach t , augment flow; update L_G ; and restart from s .
- If get stuck, delete node from L_G and go to previous node.

end of phase (length of shortest augmenting path has increased)



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Simple unit-capacity networks: analysis

Phase of normal augmentations.

- Construct level graph L_G .
- Start at s , advance along an edge in L_G until reach t or get stuck.
- If reach t , augment flow; update L_G ; and restart from s .
- If get stuck, delete node from L_G and go to previous node.

Lemma 1. A phase of normal augmentations takes $O(m)$ time.

Pf.

- $O(m)$ to create level graph L_G .
- $O(1)$ per edge (each edge involved in at most one advance, retreat, and augmentation).
- $O(1)$ per node (each node deleted at most once). ▀

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Network flow: quiz 8



Consider running advance-retreat algorithm in a unit-capacity network (but not necessarily a simple one). What is running time?

- A. $O(m)$.
- B. $O(m^{3/2})$.
- C. $O(m n)$.
- D. May not terminate.

both indegree and outdegree of a node can be larger than 1

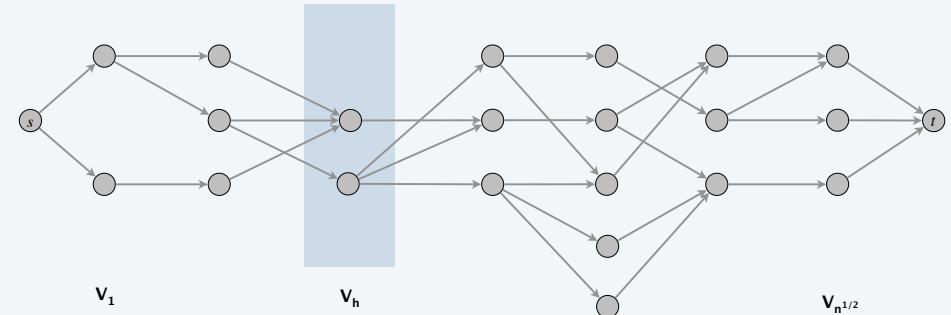
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Simple unit-capacity networks: analysis

Lemma 2. After $n^{1/2}$ phases, $\text{val}(f) \geq \text{val}(f^*) - n^{1/2}$.

- After $n^{1/2}$ phases, length of shortest augmenting path is $> n^{1/2}$.
- Thus, level graph has $\geq n^{1/2}$ levels (not including levels for s or t).
- Let $1 \leq h \leq n^{1/2}$ be a level with min number of nodes $\Rightarrow |V_h| \leq n^{1/2}$.

level graph L_G for flow f

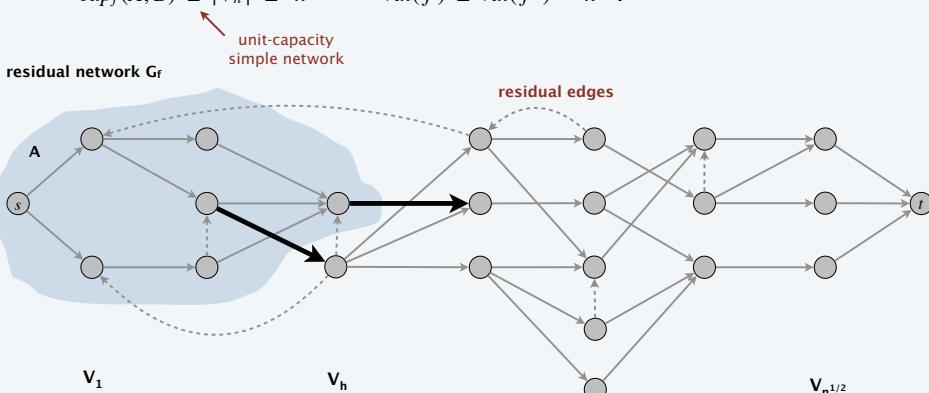


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Simple unit-capacity networks: analysis

Lemma 2. After $n^{1/2}$ phases, $\text{val}(f) \geq \text{val}(f^*) - n^{1/2}$.

- After $n^{1/2}$ phases, length of shortest augmenting path is $> n^{1/2}$.
- Thus, level graph has $\geq n^{1/2}$ levels (not including levels for s or t).
- Let $1 \leq h \leq n^{1/2}$ be a level with min number of nodes $\Rightarrow |V_h| \leq n^{1/2}$.
- Let $A = \{v : \ell(v) < h\} \cup \{v : \ell(v) = h \text{ and } v \text{ has } \leq 1 \text{ outgoing residual edge}\}$.
- $\text{cap}_f(A, B) \leq |V_h| \leq n^{1/2} \Rightarrow \text{val}(f) \geq \text{val}(f^*) - n^{1/2}$. ■



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Simple unit-capacity networks: review

Theorem. [Even–Tarjan 1975] In simple unit-capacity networks, Dinitz' algorithm computes a maximum flow in $O(m n^{1/2})$ time.

Pf.

- Lemma 1. Each phase takes $O(m)$ time.
- Lemma 2. After $n^{1/2}$ phases, $\text{val}(f) \geq \text{val}(f^*) - n^{1/2}$.
- Lemma 3. After $\leq n^{1/2}$ additional augmentations, flow is optimal. ■

Corollary. Dinitz' algorithm computes max-cardinality bipartite matching in $O(m n^{1/2})$ time.

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