Chapter 20

Measures and Integration

20.1 Extended real numbers

Measure Theory deals with measuring sets which is the act of assigning to a set a non negative number, the measure of the set, that somehow quantify how big this set is. This process of measuring satisfies countable additivity meaning that the measure of the countable unit of sets is the sum of the measures of each set in this union. Hence we have to deal with series of non negative numbers which are not always convergent in $\mathbb{R}$ meaning that the measure of a set would not be always defined as a real number. To solve this problem we extend the set of real numbers by adding two elements, ∞ which will be bigger then every real number and $-\infty$ which will be smaller then every real number.

Definition 20.1. *The set of extended real numbers $\overline{\mathbb{R}}$ is defined as*

$$\overline{\mathbb{R}} = \mathbb{R} \bigcup \{\infty, -\infty\}$$

where $\infty, -\infty \notin \mathbb{R}$ and $\infty \neq -\infty$. In other words we have that

$$\overline{\mathbb{R}} = \mathbb{R} \bigsqcup \{\infty\} \bigsqcup \{-\infty\}$$

***Note* 20.2.** *Using [lemma: 10.97] there exist two elements $(\varnothing, 0)$, $(\mathbb{Q}, 0)$ such that $(\varnothing, 0) \notin \mathbb{R}$, $(\mathbb{Q}, 0) \notin \mathbb{R}$ and $(\varnothing, 0) \neq (\mathbb{Q}, 0)$. So we can choose $\infty = (\varnothing, 0)$ and $-\infty = (\mathbb{Q}, 0)$. However we never make use of the exact nature of ∞, $-\infty$ the only thing that matters is that $\infty, -\infty \notin \mathbb{R}$ and that $\infty \neq -\infty$.*

20.1.1 Order relation in $\overline{\mathbb{R}}$

Definition 20.3. *A $x \in \overline{\mathbb{R}}$ is a called a finite real number if $x \in \mathbb{R}$ so $\mathbb{R}$ is the set of finite real numbers.*

We introduce now a fully order on $\overline{\mathbb{R}}$ based on the order defined on the set of real numbers. To avoid confusion with notation, $\leqslant$ will be the order relation on $\overline{\mathbb{R}}$ and $\leqslant_{\mathbb{R}}$ is the order relation on $\mathbb{R}$.

Definition 20.4. *The relation $\leqslant \subseteq \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ is defined by*

$$\leqslant = \leqslant_{\mathbb{R}} \bigcup (\bar{\mathbb{R}} \times \{\infty\}) \bigcup (\{-\infty\} \times \overline{\mathbb{R}})$$

where $\leqslant_{\mathbb{R}} \subseteq \mathbb{R} \times \mathbb{R}$ is the order relation on $\mathbb{R}$.

***Notation* 20.5.** *As usual $x \leqslant y$ is another notation for $(x, y) \in \leqslant$*

***Note* 20.6.** *As $\infty \notin \mathbb{R}$, $-\infty \notin \mathbb{R}$ and $-\infty \neq \infty$ we have that*

$$\leqslant_{\mathbb{R}} \bigcap (\bar{\mathbb{R}} \times \{\infty\}) = \varnothing$$
$$\leqslant_{\mathbb{R}} \bigcap (\{-\infty\} \times \overline{\mathbb{R}}) = \varnothing$$
$$(\bar{\mathbb{R}} \times \{\infty\}) \bigcap (\{-\infty\} \times \overline{\mathbb{R}}) = \{(-\infty, \infty)\}$$

Note 20.7. *Using the definition we have that* $\forall x\in\overline{\mathbb{R}}$ *we have that* $(x,\infty)\in\overline{\mathbb{R}}\times\{\infty\}$ *and* $(-\infty,x)\in\{-\infty\}\times\overline{\mathbb{R}}$ *so that*

$$\forall x\in\overline{\mathbb{R}}\,\text{we have}-\infty\leqslant x \text{ and } x\leqslant\infty$$

So in essence ∞ *is a number that is bigger then all numbers in* $\overline{\mathbb{R}}$ *and* $-\infty$ *is a number that is less than every number in* $\overline{\mathbb{R}}$.

Note 20.8. *If* $x,y\in\mathbb{R}$ *then if* $x\leqslant y$ *it follows that* $x\leqslant_{\mathbb{R}}y$

Proof. *As* $x,y\in\mathbb{R}$ $\infty,-\infty\notin\mathbb{R}$ $(x,y)\notin(\bar{\mathbb{R}}\times\{\infty\})\bigcup(\{-\infty\}\times\overline{\mathbb{R}})$ *so we must have that* $(x,y)\in\leqslant_{\mathbb{R}}$ *or* $x\leqslant_{\mathbb{R}}y$. □

We proof now that $\langle\overline{\mathbb{R}},\leqslant\rangle$ is fully ordered partial ordered set

Theorem 20.9. $\langle\overline{\mathbb{R}},\leqslant\rangle$ *is totally ordered.*

Proof. First we have to prove that $\leqslant$ is a partial ordered.

reflectivity. Let $x\in\overline{\mathbb{R}}$ then we have either:

- $\boldsymbol{x=\infty}$**.** Then $(x,x)=(\infty,\infty)\in\bar{\mathbb{R}}\times\{\infty\}\subseteq\leqslant$ proving that $x\leqslant x$.
- $\boldsymbol{x=-\infty}$**.** Then $(x,x)=(-\infty,-\infty)\in\{-\infty\}\times\overline{\mathbb{R}}\subseteq\leqslant$ proving that $x\leqslant x$.
- $\boldsymbol{x\in\mathbb{R}}$**.** Then as $\leqslant_{\mathbb{R}}$ is a partial order it follows that $(x,x)\in\leqslant_{\mathbb{R}}\subseteq\leqslant$ proving that $x\leqslant x$.

anti-symmetry. If $x\leqslant y\wedge y\leqslant x$ then $(x,y),(y,x)\in\leqslant_{\mathbb{R}}\bigsqcup(\bar{\mathbb{R}}\times\{\infty\})\bigsqcup(\{-\infty\}\times\overline{\mathbb{R}})$ and we have for (x,y) the following cases:

- $\boldsymbol{(x,y)\in\leqslant_{\mathbb{R}}}$**.** Then as $(y,x)\in\leqslant_{\mathbb{R}}\bigsqcup(\bar{\mathbb{R}}\times\{\infty\})\bigsqcup(\{-\infty\}\times\overline{\mathbb{R}})$ we have either:
  - $\boldsymbol{(y,x)\in\leqslant_{\mathbb{R}}}$**.** As $\leqslant_{\mathbb{R}}$ is a partial order we have that $x=y$.
  - $\boldsymbol{(y,x)\in\overline{\mathbb{R}}\times\{\infty\}}$**.** This can not occur because $(x,y)\in\leqslant_{\mathbb{R}}\subseteq\mathbb{R}\times\mathbb{R}$ so that $x\neq\infty\wedge x=\infty$.
  - $\boldsymbol{(y,x)\in\{-\infty\}\times\overline{\mathbb{R}}}$**.** This can not occur because $(x,y)\in\leqslant_{\mathbb{R}}\subseteq\mathbb{R}\times\mathbb{R}$ so that $y\neq-\infty\wedge y=-\infty$.
- $\boldsymbol{(x,y)\in\overline{\mathbb{R}}\times\{\infty\}}$**.** Then as $(y,x)\in\leqslant_{\mathbb{R}}\bigsqcup(\bar{\mathbb{R}}\times\{\infty\})\bigsqcup(\{-\infty\}\times\overline{\mathbb{R}})$ we have either:
  - $\boldsymbol{(y,x)\in\leqslant_{\mathbb{R}}}$**.** As $(y,x)\in\leqslant_{\mathbb{R}}\subseteq\mathbb{R}\times\mathbb{R}$ $y\neq\infty$ so this can not occur.
  - $\boldsymbol{(y,x)\in\overline{\mathbb{R}}\times\{\infty\}}$**.** Then $y=\infty\wedge x=\infty$ so that $x=y$.
  - $\boldsymbol{(y,x)\in\{-\infty\}\times\overline{\mathbb{R}}}$**.** Then $y=\infty\wedge y=-\infty$ which as $\infty\neq-\infty$ can not occur.
- $\boldsymbol{(x,y)\in\{-\infty\}\times\overline{\mathbb{R}}}$**.** Then as $(y,x)\in\leqslant_{\mathbb{R}}\bigsqcup(\bar{\mathbb{R}}\times\{\infty\})\bigsqcup(\{-\infty\}\times\overline{\mathbb{R}})$ we have either:
  - $\boldsymbol{(y,x)\in\leqslant_{\mathbb{R}}}$**.** As $(y,x)\in\leqslant_{\mathbb{R}}\subseteq\mathbb{R}\times\mathbb{R}$ $y\neq-\infty\wedge y=\infty$ so this can not occur.
  - $\boldsymbol{(y,x)\in\overline{\mathbb{R}}\times\{\infty\}}$**.** Then $x=-\infty\wedge x=\infty$ which can not occur as $\infty\neq-\infty$.
  - $\boldsymbol{(y,x)\in\{-\infty\}\times\overline{\mathbb{R}}}$**.** Then $x=-\infty\wedge y=-\infty$ so that $x=y$.

Hence in all valid cases we have that $x=y$.

transitivity. If $x\leqslant y\wedge y\leqslant z$ then $(x,y),(y,z)\in\leqslant_{\mathbb{R}}\bigsqcup(\bar{\mathbb{R}}\times\{\infty\})\bigsqcup(\{-\infty\}\times\overline{\mathbb{R}})$ and we have for (x,y) either:

- $\boldsymbol{(x,y)\in\leqslant_{\mathbb{R}}}$**.** Then for (y,z) we have either:
  - $\boldsymbol{(y,z)\in\leqslant_{\mathbb{R}}}$**.** Then $x\leqslant_{\mathbb{R}}y\wedge y\leqslant_{\mathbb{R}}z$ and as $\leqslant_{\mathbb{R}}$ is a partial order it follows that $x\leqslant_{\mathbb{R}}z$ hence $(x,z)\in\leqslant$ so that $x\leqslant z$.
  - $\boldsymbol{(y,z)\in\overline{\mathbb{R}}\times\{\infty\}}$**.** As $z=\infty$ we have that $(x,z)\in\overline{\mathbb{R}}\times\{\infty\}\subseteq\leqslant$ so that $x\leqslant z$.
  - $\boldsymbol{(y,z)\in\{-\infty\}\times\overline{\mathbb{R}}}$**.** As $(x,y)\in\leqslant_{\mathbb{R}}\subseteq\mathbb{R}\times\mathbb{R}$ we have that $y\neq-\infty\wedge y=-\infty$ so this case can never occur.

$(x,y)\in\overline{\mathbb{R}}\times\{\infty\}$. Then for (y,z) we have either:

$(y,z)\in\leqslant_{\mathbb{R}}$. Then as $(y,z)\in\leqslant_{\mathbb{R}}\subseteq\mathbb{R}\times\mathbb{R}$ we have that $y\neq\infty\wedge y=\infty$ so this case can never occur.

$(y,z)\in\overline{\mathbb{R}}\times\{\infty\}$. As $z=\infty$ we have $(x,z)\in\overline{\mathbb{R}}\times\{\infty\}\subseteq\leqslant$ so that $x\leqslant z$.

$(y,z)\in\{-\infty\}\times\overline{\mathbb{R}}$. As $y=\infty\wedge y=-\infty$ and $-\infty\neq\infty$ so that this case never occurs.

$(x,y)\in\{-\infty\}\times\overline{\mathbb{R}}$. Then for (y,z) we have either:

$(y,z)\in\leqslant_{\mathbb{R}}$. As $x=-\infty$ we have that $(x,z)\in\{-\infty\}\times\overline{\mathbb{R}}\subseteq\leqslant$ so that $x\leqslant z$.

$(y,z)\in\overline{\mathbb{R}}\times\{\infty\}$. As $z=\infty$ we have that $(x,z)\in\overline{\mathbb{R}}\times\{\infty\}\subseteq\leqslant$ so that $x\leqslant z$.

$(y,z)\in\{-\infty\}\times\overline{\mathbb{R}}$. As $x=-\infty$ we have that $(x,z)\in\{-\infty\}\times\overline{\mathbb{R}}\subseteq\leqslant$ so that $x\leqslant z$.

Hence in all valid cases we have that $x\leqslant z$.

Next we have to prove that $\leqslant$ is a total order. So let $x,y\in\overline{\mathbb{R}}$ then we have to consider the following cases:

$x\in\mathbb{R}\wedge y\in\mathbb{R}$. Then as $\leqslant_{\mathbb{R}}$ is a total order we have $(x,y)\in\leqslant_{\mathbb{R}}\subseteq\leqslant$ so that $x\leqslant y$.

$x=\infty\wedge y\in\mathbb{R}$. Then $(y,x)\in\overline{\mathbb{R}}\times\{\infty\}\subseteq\leqslant$ so that $y\leqslant x$.

$x=-\infty\wedge y\in\mathbb{R}$. Then $(x,y)\in\{-\infty\}\times\overline{\mathbb{R}}\subseteq\leqslant$ so that $x\leqslant y$.

$x\in\mathbb{R}\wedge y=\infty$. Then $(x,y)\in\overline{\mathbb{R}}\times\{\infty\}\subseteq\leqslant$ so that $x\leqslant y$.

$x=\infty\wedge y=\infty$. Then $(x,y)\in\overline{\mathbb{R}}\times\{\infty\}\subseteq\leqslant$ so that $x\leqslant y$.

$x=-\infty\wedge y=\infty$. Then $(x,y)\in\overline{\mathbb{R}}\times\{\infty\}\subseteq\leqslant$ so that $x\leqslant y$.

$x\in\mathbb{R}\wedge y=-\infty$. Then $(y,x)\in\{-\infty\}\times\overline{\mathbb{R}}\subseteq\leqslant$ so that $y\leqslant x$.

$x=\infty\wedge y=-\infty$. Then $(y,x)\in\{-\infty\}\times\overline{\mathbb{R}}\subseteq\leqslant$ so that $y\leqslant x$.

$x=-\infty\wedge y=-\infty$. Then $(y,x)\in\{-\infty\}\times\overline{\mathbb{R}}\subseteq\leqslant$ so that $y\leqslant x$. □

Theorem 20.10. *Let $x\in\mathbb{R}$ then we have $-\infty<x<\infty$*

Proof. As $(x,\infty)\in\mathbb{R}\times\{\infty\}\subseteq\leqslant$ we have $x\leqslant\infty$ and $x\neq\infty$ so that $x<\infty$. Further as $(-\infty,x)\in\{-\infty\}\times\mathbb{R}$ we have that $-\infty\leqslant x$ and $x\neq-\infty$ so that $-\infty<x$. □

Theorem 20.11. *Let $x,y\in\overline{\mathbb{R}}$ with $x<y$ then $x\neq\infty$ and $y\neq-\infty$.*

Proof. Assume that $x=\infty$ then by definition $y\leqslant x\underset{x<y}{\Rightarrow}y<y$ a contradiction so we must have that $x\neq\infty$. Assume that $y=-\infty$ then by definiton $y\leqslant x\underset{x<y}{\Rightarrow}y<y$ a contradiction so we must have that $y\neq-\infty$. □

The rational numbers are still dense in $\overline{\mathbb{R}}$ as the following theorem shows.

Theorem 20.12. *Let $x,y\in\overline{\mathbb{R}}$ with $x<y$ then $\exists q\in\mathbb{Q}$ such that $x<q<y$.*

Proof. As $x<y$ we have by the previous theorem [theorem: 20.11] $x\neq\infty\wedge y\neq-\infty$ so we must consider only the following cases for x,y:

$x\in\mathbb{R}\wedge y=\infty$. Then as $x,x+1\in\mathbb{R}$ and $x<x+1$ we can use [theorem: 10.32] to find a $q\in\mathbb{Q}$ such that $x<q<x+1<\infty=y$ hence $x<q<y$.

$x=-\infty\wedge y=\infty$. Then for $q=0\in\mathbb{Q}$ we have $x<q<y$.

$x\in\mathbb{R}\wedge y\in\mathbb{R}$. Then using [theorem: 10.32] there exist a $q\in\mathbb{Q}$ such that $x<q<y$.

$x=-\infty\wedge y\in\mathbb{R}$. Then $y,y-1\in\mathbb{R}$ and $y-1<y$ we can use [theorem: 10.32] to find a $q\in\mathbb{Q}$ such that $x=-\infty<y-1<q<y$ hence $x<q<y$. □

The primary reason for the extended real is that every non empty set has a supremum and infinum.

Theorem 20.13. *Let* $\varnothing \neq A \subseteq \overline{\mathbb{R}}$ *then* $\sup(A)$ *and* $\inf(A)$ *exist. Further we have:*

1. $\sup(A)$ *has the following values for the possible cases of* A*:*

   $\boldsymbol{A = \{-\infty\}}$**.** *Then* $\sup(A) = -\infty$

   $\boldsymbol{A \neq \{-\infty\}}$**.** *Then we have the following possible sub-cases:*

   $\boldsymbol{\infty \in A}$**.** *Then* $\sup(A) = \infty$

   $\boldsymbol{\infty \notin A}$**.** *Then we have the following sub-cases:*

   $\boldsymbol{\exists u \in \mathbb{R} \vdash \forall x \in A}$ ***we have*** $\boldsymbol{x \leqslant u}$**.** *Then* $\sup(A) = \sup_{\mathbb{R}}(A \bigcap \mathbb{R}) \in \mathbb{R}$

   $\boldsymbol{\forall u \in R \vdash \exists x \in A}$ ***we have*** $\boldsymbol{x \nleqslant u}$**.** *Then* $\sup(A) = \infty$

2. $\inf(A)$ *has the following values for the possible cases of* A*:*

   $\boldsymbol{A = \{\infty\}}$**.** *Then* $\inf(A) = \infty$

   $\boldsymbol{A \neq \{\infty\}}$**.** *Then we have the following possible sub-cases:*

   $\boldsymbol{-\infty \in A}$**.** *Then* $\inf(A) = -\infty$

   $\boldsymbol{-\infty \notin A}$**.** *Then we have the following sub-cases:*

   $\boldsymbol{\exists l \in \mathbb{R} \vdash \forall x \in A}$ ***we have*** $\boldsymbol{l \leqslant x}$**.** *Then* $\inf(A) = \inf_{\mathbb{R}}(A \bigcap \mathbb{R}) \in \mathbb{R}$

   $\boldsymbol{\forall l \in R \vdash \exists x \in A}$ ***we have*** $\boldsymbol{l \nleqslant x}$**.** *Then* $\inf(A) = -\infty$

Where $\sup(A)$ *is the supremum of* A *using* $\leqslant$, $\sup_{\mathbb{R}}(A \bigcap \mathbb{R})$ *is the supremum of* $A \bigcap \mathbb{R}$ *using* $\leqslant_{\mathbb{R}}$, $\inf(A)$ *is the infinum of* A *using* $\leqslant$ *and* $\inf_{\mathbb{R}}(A \bigcap \mathbb{R})$ *is the infinum of* $A \bigcap \mathbb{R}$ *using* $\leqslant_{\mathbb{R}0}$.

Proof. First we proof that $\sup(A)$ exist. For A we have either:

$\boldsymbol{A = \{-\infty\}}$**.** As $-\infty \leqslant -\infty$ $-\infty$ is a upper bound of A. If $u \in \overline{\mathbb{R}}$ is another upper bound then, as $-\infty \leqslant u$ it follows that $-\infty$ is the lowest upper bound of A hence $\sup(A) = -\infty$ and

$$\sup(\{-\infty\}) = -\infty$$

$\boldsymbol{A \neq \{-\infty\}}$**.** Then we have either:

$\boldsymbol{\infty \in A}$**.** As $\forall x \in A$ we have $x \leqslant \infty$ so ∞ is clearly a upper bound for A. If u is a another upper bound for A then, as $\infty \in A$, we must have that $\infty \leqslant u$ so ∞ is the lowest upper bound of A. Hence $\sup(A) = \infty$ or

$$\text{If } \infty \in A \text{ then } \sup(A) = \infty$$

$\boldsymbol{\infty \notin A}$**.** As $A \neq \varnothing$ and $A \neq \{-\infty\}$ there must be a $z \in A$ with $z \neq -\infty$, as $\infty \notin A$ it follows that $z \in \mathbb{R}$. Hence we conclude that

$$A \bigcap \mathbb{R} \neq \varnothing$$

Consider now for A the following cases:

$\boldsymbol{\exists u \in \mathbb{R} \vdash \forall x \in A}$ **we have** $\boldsymbol{x \leqslant u}$**.** Then $\forall x \in A \bigcap \mathbb{R}$ we have $x \leqslant u$ or, as $x, u \in \mathbb{R}$ that $x \leqslant_{\mathbb{R}} u$, so $\varnothing \neq A \bigcap \mathbb{R}$ is bounded above by u. As $\langle \mathbb{R}, \leqslant_{\mathbb{R}} \rangle$ is conditional complete [see theorem: 10.18] $A \bigcap \mathbb{R}$ has a supremum $S = \sup_{\mathbb{R}}(A \bigcap \mathbb{R}) \in \mathbb{R}$. Let $x \in A$, then, given that $\infty \notin A$, we have either:

$\boldsymbol{x \in A \bigcap \mathbb{R}}$**.** Then $x \leqslant_{\mathbb{R}} S$ so that $x \leqslant S$

$\boldsymbol{x = -\infty}$**.** Then $-\infty \leqslant S$

So in all cases we have $x \leqslant S$ which proves that

$$S \text{ is a upper bound of } A \, [\text{using} \leqslant]$$

Let U be another upper bound of A [using $\leqslant$] then we have either:

$U \in \mathbb{R}$. Then $\forall x \in A \bigcap \mathbb{R} \subseteq A$ we have $x \leqslant U$, hence, as $x, U \in \mathbb{R}$, $x \leqslant_{\mathbb{R}} U$ so that U is another upper bound of $A \bigcap \mathbb{R}$ using $\leqslant_{\mathbb{R}}$. So we must have that $S \leqslant U$.

$U = \infty$. Then trivially $S \leqslant U$.

$U = -\infty$. Then $\forall x \in A$ we have $-\infty \leqslant x \leqslant -\infty \Rightarrow x = -\infty$ so that $A \subseteq \{-\infty\}$ or $A \bigcap \mathbb{R} \subseteq \{-\infty\} \bigcap \mathbb{R} = \varnothing$ contradicting the fact that $A \bigcap \mathbb{R} = \varnothing$. So this case will never occur.

As in all valid cases $S \leqslant U$ it follows that S is the lowest upper bound of A or S is the supremum of A. So

$$\sup(A) = \sup_{\mathbb{R}}\left(A \bigcap \mathbb{R}\right)$$

$\forall u \in \mathbb{R} \vdash \exists x \in A$ with $x \nleqslant u$. As $\forall x \in \overline{\mathbb{R}}$ we have that $x \leqslant \infty$ it follows that ∞ is a upper bound of A [using $\leqslant$]. Let u be another upper bound of A. Assume that $\infty \nleqslant u$ then by [theorem: 3.39] $u < \infty$ or $u \neq \infty$ so that u is either:

$u = -\infty$. Then $\forall x \in A$ we have $-\infty \leqslant x \leqslant u = -\infty \Rightarrow x = -\infty$ so that $A \subseteq \{-\infty\}$ or $A \bigcap \mathbb{R} \subseteq \{-\infty\} \bigcap \mathbb{R} = \varnothing$ contradicting the fact that $A \bigcap \mathbb{R} = \varnothing$.

$u \in \mathbb{R}$. Then $\exists x \in A$ such that $x \nleqslant u$ contradicting the fact that u is a upper bound.

As in all cases we have a contradiction we must have that the assumption is wrong and that $\infty \leqslant u$. Hence ∞ is the lowest upper bound of A [using $\leqslant$] or ∞ is the supremum of A, so

$$\sup(A) = \infty$$

So in all possible cases of A we have found that $\sup(A)$ exists. Next we prove that $\inf(A)$ exist, for A we have either:

$A = \{\infty\}$. As $\infty \leqslant \infty$ ∞ is a lower bound of A. If $l \in \overline{\mathbb{R}}$ is another lower bound then, as $l \leqslant \infty$ it follows that ∞ is the greatest upper bound of A hence $\inf(A) = \infty$ and

$$\inf(\{\infty\}) = \infty$$

$A \neq \{\infty\}$. Then we have either:

$-\infty \in A$. As $\forall x \in A$ we have $-\infty \leqslant x$ so $-\infty$ is clearly a lower bound for A. If l is a another lower bound for A then, as $-\infty \in A$, we must have that $u \leqslant -\infty$ so $-\infty$ is the greatest lower bound of A. Hence $\inf(A) = -\infty$ or

$$\text{If } -\infty \in A \text{ then } \inf(A) = -\infty$$

$-\infty \notin A$. As $A \neq \varnothing$ and $A \neq \{\infty\}$ there must be a $z \in A$ with $z \neq \infty$, as $-\infty \notin A$ it follows that $z \in \mathbb{R}$. Hence we conclude that

$$A \bigcap \mathbb{R} \neq \varnothing$$

Consider now for A the following cases:

$\exists l \in \mathbb{R} \vdash \forall x \in A$ we have $l \leqslant x$. Then $\forall x \in A \bigcap \mathbb{R}$ we have $l \leqslant x$ or, as $x, l \in \mathbb{R}$ that $l \leqslant_{\mathbb{R}} x$, so $\varnothing \neq A \bigcap \mathbb{R}$ is bounded below by l. As $\langle \mathbb{R}, \leqslant_{\mathbb{R}} \rangle$ is conditional complete [see theorem: 10.18] $A \bigcap \mathbb{R}$ has a infinum $I = \inf_{\mathbb{R}}(A \bigcap \mathbb{R}) \in \mathbb{R}$. Let $x \in A$, then, given that $-\infty \notin A$, we have either:

$x \in A \bigcap \mathbb{R}$. Then $I \leqslant_{\mathbb{R}} x$ so that $I \leqslant x$

$x = \infty$. Then $I \leqslant x$

So in all cases we have $I \leqslant x$ which proves that

$$I \text{ is a lower bound of } A \, [\text{using} \leqslant]$$

Let L be another lower bound of A [using $\leqslant$] then we have either:

$L \in \mathbb{R}$. Then $\forall x \in A \bigcap \mathbb{R} \subseteq A$ we have $L \leqslant x$, hence, as $x, L \in \mathbb{R}$, $L \leqslant_{\mathbb{R}} x$ so that L is another lower bound of $A \bigcap \mathbb{R}$ using $\leqslant_{\mathbb{R}}$. So we must have that $L \leqslant I$.

$L = \infty$. Then $\forall x \in A$ we have $\infty \leqslant x \leqslant \infty \Rightarrow x = \infty$ so that $A \subseteq \{\infty\}$ or $A \bigcap \mathbb{R} \subseteq \{\infty\} \bigcap \mathbb{R} = \varnothing$ contradicting the fact that $A \bigcap \mathbb{R} = \varnothing$. So this case will never occur.

$L = -\infty$. Then $L \leqslant I$

As in all valid cases $L \leqslant I$ it follows that I is the greatest lower bound of A or I is the infinum of A. So

$$\inf(A) = \inf_{\mathbb{R}}\left(A \bigcap \mathbb{R}\right)$$

$\forall l \in \mathbb{R} \vdash \exists x \in A$ with $l \nleqslant x$. As $\forall x \in \overline{\mathbb{R}}$ we have that $-\infty \leqslant x$ it follows that $-\infty$ is a lower bound of A [using $\leqslant$]. Let l be another lower bound of A. Assume that $l \nleqslant -\infty$ then by [theorem: 3.39] $-\infty < l$ or $l \neq -\infty$ so that l is either:

$l = \infty$. Then $\forall x \in A$ we have $\infty = l \leqslant x \leqslant \infty \Rightarrow x = \infty$ so that $A \subseteq \{\infty\}$ or $A \bigcap \mathbb{R} \subseteq \{\infty\} \bigcap \mathbb{R} = \varnothing$ contradicting the fact that $A \bigcap \mathbb{R} = \varnothing$.

$l \in \mathbb{R}$. Then $\exists x \in A$ such that $l \nleqslant x$ contradicting the fact that l is a lower bound.

As in all cases we have a contradiction we must have that the assumption is wrong and that $l \leqslant -\infty$. Hence $-\infty$ is the greatest lower bound of A [using $\leqslant$] or $-\infty$ is the infinum of A. So

$$\inf(A) = -\infty$$

So in all possible cases of A we have proved that $\inf(A)$ exist. □

Corollary 20.14. *If* $\varnothing \neq A \subseteq \mathbb{R}$ *then* $\sup(A) \neq -\infty$ *and* $\inf(A) \neq \infty$

Proof. If $\sup(A) = -\infty$ then $\forall x \in A$ we have $-\infty \leqslant x \leqslant -\infty$ so that $x = -\infty$, hence $A \subseteq \{-\infty\}$ so that we get $\varnothing \neq A = A \bigcap \mathbb{R} \subseteq \{-\infty\} \bigcap \mathbb{R} = \varnothing$ a contradiction. Hence

$$\sup(A) \neq -\infty$$

Likewise if $\inf(A) = \infty$ then $\forall x \in A$ we have $\infty \leqslant x \leqslant \infty$ so that $x = \infty$, hence $A \subseteq \{\infty\}$ so that we get $\varnothing \neq A = A \bigcap \mathbb{R} \subseteq \{\infty\} \bigcap \mathbb{R} = \varnothing$ a contradiction. Hence

$$\inf(A) \neq \infty$$ □

The next theorem shows the necessary and sufficient constraint for the supremum and infinum to be not finite.

Corollary 20.15. *Let* $\varnothing \neq A \subseteq \overline{\mathbb{R}}$ *then*

1. $\sup(A) = \infty \Leftrightarrow \infty \in A$ *or* $\forall u \in \mathbb{R}$ *there exist a* $x \in A \bigcap \mathbb{R}$ *with* $x \nleqslant u$ *or* $u < x$.
2. $\inf(A) = -\infty \Leftrightarrow -\infty \in A$ *or* $\forall \in \mathbb{R}$ *there exist a* $x \in A \bigcap \mathbb{R}$ *with* $l \nleqslant x$ *or* $x < l$.
3. $\sup(A) = -\infty \Leftrightarrow A = \{-\infty\}$
4. $\inf(A) = \infty \Leftrightarrow A = \{\infty\}$

Proof.

1. From [theorem: 20.13 (1)] it follows that the only possible cases where $\sup(A) = \infty$ are that either $\infty \in A$ or $\forall u \in \mathbb{R}$ there exist a $x \in A \bigcap \mathbb{R}$ with $x \nleqslant u$.
2. From [theorem: 20.13 (2)] it follows that the only possible cases where $\inf(A) = -\infty$ are that either $-\infty \in A$ or $\forall l \in \mathbb{R}$ there exist a $x \in A \bigcap \mathbb{R}$ with $l \nleqslant x$.

3. From [theorem: 20.13 (1)] it follows that the only possible cases where $\sup(A) = -\infty$ is the case where $A = \{-\infty\}$.
4. From [theorem: 20.13 (2)] it follows that the only possible cases where $\inf(A) = \infty$ is the case where $A = \{\infty\}$. □

20.1.2 Arithmetic operations on $\overline{\mathbb{R}}$

The sum operator on $\overline{\mathbb{R}}$ is a extension of the sum on $\mathbb{R}$ such that most inequalities involving sums are still satisfied.

Definition 20.16. *The sum operator is defined by*

$$(+): (\overline{\mathbb{R}} \times \overline{\mathbb{R}}) \setminus \{(\infty, -\infty), (-\infty, \infty)\} \rightarrow \overline{\mathbb{R}} \text{ by } (+)(x, y) \underset{\text{notation}}{=} x + y$$

where

$$x + y = \begin{cases} \infty \text{ if } x = \infty \wedge y \in \overline{\mathbb{R}} \setminus \{-\infty\} \\ \infty \text{ if } y = \infty \wedge x \in \overline{\mathbb{R}} \setminus \{-\infty\} \\ -\infty \text{ if } x = -\infty \wedge y \in \overline{\mathbb{R}} \setminus \{\infty\} \\ -\infty \text{ if } y = -\infty \wedge x \in \overline{\mathbb{R}} \setminus \{\infty\} \\ x +_{\mathbb{R}} y \text{ if } x, y \in \mathbb{R} \text{ where } +_{\mathbb{R}} \text{ is the sum defined for the real numbers} \end{cases}$$

***Note* 20.17.** $\infty + (-\infty)$ *and* $-\infty + \infty$ *are not defined at all.*

Note 20.18. Let $x \in \overline{\mathbb{R}}$ then for $0 \in \mathbb{R} \subseteq \overline{\mathbb{R}}$ we have that $0 + x = x = x + 0$

Proof. For $x \in \overline{\mathbb{R}}$ we have either:

$\boldsymbol{x \in \mathbb{R}}$**.** Then $x + 0 = x +_{\mathbb{R}} 0 = x$ and $0 + x = 0 +_R x = x$.

$\boldsymbol{x = -\infty}$**.** Then $x + 0 = -\infty + 0 = -\infty = x$ and $0 + x = 0 + (-\infty) = -\infty = x$.

$\boldsymbol{x = \infty}$**.** Then $x + 0 = \infty + 0 = \infty = x$ and $0 + x = 0 + \infty = \infty = x$. □

Definition 20.19. *The multiplication operator* $\cdot$ *is defined by*

$$(\cdot): \overline{\mathbb{R}} \times \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}} \text{ by } (\cdot)(x, y) \underset{\text{notation}}{=} x \cdot y$$

where for $(x, y) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ *we define* $x \cdot y$ *as follows:*

$\boldsymbol{x = 0 \vee y = 0}$**.** *Then* $x \cdot y = 0$

$\boldsymbol{x < 0 \wedge y = \infty}$**.** *Then* $x \cdot y = -\infty$

$\boldsymbol{0 < x \wedge y = \infty}$**.** *Then* $x \cdot y = \infty$

$\boldsymbol{x < 0 \wedge y = -\infty}$**.** *Then* $x \cdot y = \infty$

$\boldsymbol{0 < x \wedge y = -\infty}$**.** *Then* $x \cdot y = -\infty$

$\boldsymbol{x = -\infty \wedge y \in]-\infty, 0[}$**.** *Then* $x \cdot y = \infty$

$\boldsymbol{x = \infty \wedge y \in]-\infty, 0[}$**.** *Then* $x \cdot y = -\infty$

$\boldsymbol{x \in \mathbb{R} \setminus \{0\} \wedge y \in]-\infty, 0[}$**.** *Then* $x \cdot y = x \cdot_{\mathbb{R}} y$

$\boldsymbol{x = -\infty \wedge y \in]0, \infty[}$**.** *Then* $x \cdot y = -\infty$

$\boldsymbol{x = \infty \wedge y \in]0, \infty[}$**.** *Then* $x \cdot y = \infty$

$\boldsymbol{x \in \mathbb{R} \setminus \{0\} \wedge y \in]0, \infty[}$**.** *Then* $x \cdot y = x \cdot_{\mathbb{R}} y$

where $x \cdot_{\mathbb{R}} y$ *is the product in* $\mathbb{R}$.

***Note* 20.20.** *In contrast with the sum the product in* $\overline{\mathbb{R}}$ *is defined* $\forall x, y \in \overline{\mathbb{R}}$.

Next we define the inverse operator.

Definition 20.21. *The inverse operator* $(\cdot)^{-1}$ *is defined by*

$$(\cdot)^{-1}: \overline{\mathbb{R}} \setminus \{0\} \rightarrow \overline{\mathbb{R}} \text{ by } (\cdot)^{-1} = \begin{cases} 0 \text{ if } x = -\infty \\ 0 \text{ if } x = \infty \\ x^{-1} \text{ if } x \in \mathbb{R} \text{ where } x^{-1} \text{ is the inverse defined in } \mathbb{R} \end{cases}$$

***Notation* 20.22.** *As usually we note* $(\cdot)^{-1}(x)$ *as* x^{-1}, $1/x$ *or* $\frac{1}{x}$

Note 20.23. In contrast with real numbers where for $x \in \mathbb{R} \setminus \{0\}$ we have $x \cdot \frac{1}{x} = 1 = \frac{1}{x} \cdot x$ this is not the case for $x = -\infty$ and $x = \infty$ because we have $\infty \cdot \frac{1}{\infty} = \infty \cdot 0 = 0$ and $(-\infty) \cdot \frac{1}{-\infty} = (-\infty) \cdot 0 = 0$.

Definition 20.24. *The negate operator* $-$ *is defined by*

$$(-): \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}} \text{ where } (-)(x) \underset{notation}{=} -x \underset{\text{defined}}{=} (-1) \cdot x$$

So as $(-1) \in]0, \infty[$ *we have that*

$$-x = \begin{cases} \infty \text{ if } x = -\infty \\ -\infty \text{ if } x = \infty \\ (-1) \cdot_{\mathbb{R}} x \text{ if } x \in \mathbb{R} \end{cases}$$

As the sum is not defined for every $x, y \in \overline{\mathbb{R}}$ $\langle \overline{\mathbb{R}}, + \rangle$ is not a group however if we restrict ourselves to non negative numbers then we have a Abelian semi group which allows us to work with generalized sums [see section: 11.1].

Definition 20.25. *The set of positive extended real numbers noted as* $\overline{\mathbb{R}}^+$ *and the set of non negative extended real numbers noted as* $\overline{\mathbb{R}}_0^+$ *are defined by*

$$\overline{\mathbb{R}}^+ =]0, \infty] = \{x \in \overline{\mathbb{R}} | 0 < x\}$$

and

$$\overline{\mathbb{R}}_0^+ = [0, \infty] = \{x \in \overline{\mathbb{R}} | 0 \leqslant x\}$$

Theorem 20.26. $\langle \overline{\mathbb{R}}_0^+, + \rangle$ *is a Abelian semi-group with neutral element* $0 \in \mathbb{R}$ *where* $+$ *is the restriction to* $\overline{\mathbb{R}}_0^+$ *of the sum defined in [definition: 20.16].*

Proof. First for $x, y \in \overline{\mathbb{R}}_0^+$ we have $0 \leqslant x \wedge 0 \leqslant y$ so that $0 \leqslant y \underset{\text{[note: 20.18]}}{=} 0 + y \underset{\text{[theorem: 20.29]}}{\leqslant} x + y$ proving that $x + y \in \overline{\mathbb{R}}_0^+$. Hence

$$+: \overline{\mathbb{R}}_0^+ \times \overline{\mathbb{R}}_0^+ \rightarrow \overline{\mathbb{R}}_0^+ \text{ defined by } +(x, y) \in x + y$$

is a well defined function. Next we need to prove the Abelian semi-group axioms:

neutral element. This was proved in [note: 20.18].

associativity. Let $x, y, z \in \overline{\mathbb{R}}_0^+$ then we have as $-\infty \notin \overline{\mathbb{R}}_0^+$ [because $-\infty < 0$] that

$\boldsymbol{x \in \mathbb{R} \wedge y \in \mathbb{R} \wedge z \in \mathbb{R}}$**.** Then $x + (y + z) = x +_{\mathbb{R}} (y +_{\mathbb{R}} z) = (x +_{\mathbb{R}} y) +_{\mathbb{R}} z = (x + y) + z$.

$\boldsymbol{x = \infty \wedge y \in \mathbb{R} \wedge z \in \mathbb{R}}$**.** Then

$$x + (y + z) = \infty + (y + z) = \infty = \infty + z = (x + \infty) + z = (x + y) + z$$

$\boldsymbol{x \in \mathbb{R} \wedge y = \infty \wedge z \in \mathbb{R}}$**.** Then

$$x + (y + y) = x + (\infty + z) = x + \infty = \infty + z = (x + \infty) + z = (x + y) + z$$

$\boldsymbol{x = \infty \wedge y = \infty \wedge z \in \mathbb{R}}$**.** Then

$$x + (y + z) = \infty + (y + z) = \infty = \infty + z = (x + y) + z$$

$\boldsymbol{x \in \mathbb{R} \wedge y \in \mathbb{R} \wedge z = \infty}$**.** Then

$$x + (y + z) = x + (y + \infty) = x + \infty = \infty = (x + y) + \infty = (x + y) + z$$

$x=\infty \wedge y\in\mathbb{R} \wedge z=\infty$. Then

$$x+(y+z)=\infty+(y+z)=\infty=(x+y)+\infty=(x+y)+z$$

$x\in\mathbb{R} \wedge y=\infty \wedge z=\infty$. Then

$$x+(y+z)=x+(\infty+\infty)=x+\infty=\infty=(x+y)+\infty=(x+y)+z$$

$x=\infty \wedge y=\infty \wedge z=\infty$. Then

$$x+(y+z)=x+(\infty+\infty)=x+\infty=\infty=(x+y)+\infty=(x+y)+z$$

So in all cases we have $x+(y+z)=(x+y)+z$.

commutativity. Let $x,y\in\overline{\mathbb{R}}_0^+$ then we have either:

$x\in\mathbb{R} \wedge y\in\mathbb{R}$. Then $x+y=x+_{\mathbb{R}}y=y+_{\mathbb{R}}x=y+x$.

$x=\infty \wedge y\in\mathbb{R}$. Then $x+y=\infty+y=\infty=y+\infty=y+x$.

$x\in\mathbb{R} \wedge y=\infty$. Then $x+y=x+\infty=\infty=\infty+x=y+x$.

$x=\infty \wedge y=\infty$. Then $x+y=\infty+\infty=y+x$.

So in all valid cases we have $x+y=y+x$. □

We have to be very careful if we combine sums and inequalities on $\overline{\mathbb{R}}$ the reasoning that works in $\mathbb{R}$ does not work always in $\overline{\mathbb{R}}$. For example if $x=10, y=4$ and $z=\infty$ then $x+z=y+z=\infty$ but $x\neq y$, also we have that $x+z\leqslant y+z$ but $x\nleqslant y$. The following theorems shows what is possible.

Theorem 20.27. *Let $x,y\in\overline{\mathbb{R}}$ be such that $0\leqslant x\wedge 0\leqslant y$ and one of x,y is finite [so that $x-y$ is well defined] then $x-y\leqslant x$*

Proof. As either x or y are finite we have to consider two cases:

$x\in\mathbb{R}$. then for y we have either:

$y\in\mathbb{R}$. then by the properties of the real numbers we have $x-y\leqslant x$

$y=\infty$. then $x-y=x+(-y)=x+(-\infty)=-\infty<x$ so that $x-y\leqslant y$

$y\in\mathbb{R}$. then for x we have either:

$x\in\mathbb{R}$. then by the properties of the real numbers we have $x-y\leqslant x$

$x=\infty$. then $x-y=x+(-y)=\infty\leqslant\infty=x$ giving $x-y\leqslant x$ □

Theorem 20.28. *If $x,y\in\overline{\mathbb{R}}$ and $z\in\mathbb{R}$ [so that $x+z$ and $y+z$ are well defined] then we have*

1. *if $x+z=y+z$ we have $x=y$*
2. *if $x+z\leqslant y+z$ we have $x\leqslant y$*
3. *if $x+z<y+z$ we have $x<y$*
4. *if $x+z=y$ we have $x=y-z\underset{\text{def}}{=}y+(-z)$ [where $y-z$ is well defined]*
5. *if $x+z\leqslant y$ we have $x\leqslant y-z\underset{\text{def}}{=}y+(-z)$ [where $y-z$ is well defined]*
6. *if $x+z<y$ we have $x<y-x\underset{\text{def}}{=}y+(-z)$ [where $y-z$ is well defined]*
7. *if $x\leqslant y+z$ we have $x+(-z)\leqslant y$*
8. *if $x<y+z$ we have $x+(-z)<y$*

Proof.

1. For x,y we have either:

   $x\in\mathbb{R} \wedge y\in\mathbb{R}$. Then using the properties of the real numbers we have $x=y$.

   $x=\infty \wedge y\in\mathbb{R}$. Then $\infty=x+z=y+z\in\mathbb{R}$ which as $\infty\notin\mathbb{R}$ is not a valid case.

$x = -\infty \wedge y \in \mathbb{R}$. Then $-\infty = x + z = y + z \in \mathbb{R}$ which as $-\infty \notin \mathbb{R}$ is not a valid case.

$x \in \mathbb{R} \wedge y = \infty$. Then $\infty = y + z = x + z \in \mathbb{R}$ which as $\infty \notin \mathbb{R}$ is not a valid case.

$x = \infty \wedge y = \infty$. Then $x = y$.

$x = -\infty \wedge y = \infty$. Then $-\infty = x + z = y + z = \infty$ which as $-\infty \neq \infty$ is not a valid case.

$x \in \mathbb{R} \wedge y = -\infty$. Then $-\infty = y + z = x + z \in \mathbb{R}$ which as $-\infty \notin \mathbb{R}$ is not a valid case.

$x = \infty \wedge y = -\infty$. Then $\infty = x + z = y + z = -\infty$ which as $-\infty \neq \infty$ is not a valid case.

$x = -\infty \wedge y = -\infty$. Then $x = y$

So in all valid cases we have $x = y$.

2. For x, y we have either:

$x \in \mathbb{R} \wedge y \in \mathbb{R}$. Then using the properties of the real numbers we have $x \leqslant y$.

$x = \infty \wedge y \in \mathbb{R}$. Then $\infty = x + z \leqslant y + z \in \mathbb{R}$ which as $y + z < \infty$ proves that $\infty = x + y \subset \mathbb{R}$ which as $\infty \notin \mathbb{R}$ is not a valid case.

$x = -\infty \wedge y \in \mathbb{R}$. Then $x = -\infty \leqslant y$.

$x \in \mathbb{R} \wedge y = \infty$. Then $x \leqslant \infty = y$,

$x = \infty \wedge y = \infty$. Then $x = \infty = y$ so that $x \leqslant y$.

$x = -\infty \wedge y = \infty$. Then $x = -\infty \leqslant \infty = y$.

$x \in \mathbb{R} \wedge y = -\infty$. As $x + z \in \mathbb{R}$ we have $-\infty \leqslant x + z \leqslant y + z \leqslant -\infty$ so that $-\infty = x + z \in \mathbb{R}$ which as $-\infty \notin \mathbb{R}$ is not a valid case.

$x = \infty \wedge y = -\infty$. As $\infty = x + z \leqslant y + z = -\infty$ and $-\infty \leqslant \infty$ we would have $-\infty = \infty$ so this is not a valid case.

$x = -\infty \wedge y = -\infty$. Then $x = -\infty = y$ so that $x \leqslant y$.

So in all valid cases we have $x \leqslant y$.

3. For x, y we have either:

$x \in \mathbb{R} \wedge y \in \mathbb{R}$. Then using the properties of the real numbers we have $x \leqslant y$.

$x = \infty \wedge y \in \mathbb{R}$. Then $\infty = x + z < y + z \in \mathbb{R}$ which as $y + z < \infty$ proves that $\infty = x + y \subset \mathbb{R}$ which as $\infty \notin \mathbb{R}$ is not a valid case.

$x = -\infty \wedge y \in \mathbb{R}$. Then $x = -\infty < y$.

$x \in \mathbb{R} \wedge y = \infty$. Then $x < \infty = y$.

$x = \infty \wedge y = \infty$. Then $\infty = x + z < y + z = \infty$ so this case does not apply.

$x = -\infty \wedge y = \infty$. Then $x = -\infty < \infty = y$.

$x \in \mathbb{R} \wedge y = -\infty$. Then $x + z \in \mathbb{R}$ so that $-\infty < x + z < y + z = -\infty$ so this case does not apply.

$x = \infty \wedge y = -\infty$. Then $\infty = x + z < y + z - \infty$ so this case does not apply.

$x = -\infty \wedge y = -\infty$. Then $-\infty = x + z < y + z = -\infty$ so this case does not apply.

So in all valid cases we have $x < y$.

4. For x, y we have either:

$x \in \mathbb{R} \wedge y \in \mathbb{R}$. Then using the properties of the real numbers we have $x = y - z$.

$x = \infty \wedge y \in \mathbb{R}$. Then $\infty = x + z = y \in \mathbb{R}$ which as $\infty \notin \mathbb{R}$ is not a valid case.

$x = -\infty \wedge y \in \mathbb{R}$. Then $-\infty = x + z = y \in \mathbb{R}$ which as $-\infty \notin \mathbb{R}$ is not a valid case.

$x \in \mathbb{R} \wedge y = \infty$. Then $\infty = y = x + z \in \mathbb{R}$ which as $\infty \notin \mathbb{R}$ is not a valid case.

$x = \infty \wedge y = \infty$. Then $y - z = \infty = x$.

$x = -\infty \wedge y = \infty$. Then $-\infty = x + z = y = \infty$ which as $-\infty \neq \infty$ is not a valid case.

$x \in \mathbb{R} \wedge y = -\infty$. Then $-\infty = y = x + z \in \mathbb{R}$ which as $-\infty \notin \mathbb{R}$ is not a valid case.

$x = \infty \wedge y = -\infty$. Then $\infty = x + z = y = -\infty$ which as $-\infty \neq \infty$ is not a valid case.

$x = -\infty \wedge y = -\infty$. Then $y - z = -\infty = x$ so that $x = y - z$.

So in all valid cases we have $x = y - z$.

5. For x, y we have either:

$x \in \mathbb{R} \wedge y \in \mathbb{R}$. Then using the properties of the real numbers we have $x \leqslant y - z$.

$x = \infty \wedge y \in \mathbb{R}$. Then $\infty = x + z \leqslant y \leqslant \infty$ [as $y \in \mathbb{R}$] so $\infty = y \in \mathbb{R}$ which as $\infty \notin \mathbb{R}$ is not a valid case.

$x = -\infty \wedge y \in \mathbb{R}$. Then $x = -\infty < y - z$ so that $x \leqslant y - z$.

$x \in \mathbb{R} \wedge y = \infty$. Then $x \leqslant \infty = y - z$.

$x = \infty \wedge y = \infty$. Then $y - z = \infty = x$ so that $x \leqslant y - z$.

$x = -\infty \wedge y = \infty$. Then $x = -\infty < \infty = y - z$ so that $x \leqslant y - z$.

$x \in \mathbb{R} \wedge y = -\infty$. As $x + z \in \mathbb{R}$ we have $-\infty < x + z \leqslant y = -\infty$ so this is not a valid case.

$x = \infty \wedge y = -\infty$. Then $\infty = x + z \leqslant y = -\infty < \infty$ so this is not a valid case.

$x = -\infty \wedge y = -\infty$. Then $x = -\infty = y - z$ so that $x \leqslant y - z$.

So in all valid cases we have $x \leqslant y - z$.

6. For x, y we have either:

$x \in \mathbb{R} \wedge y \in \mathbb{R}$. Then using the properties of the real numbers we have $x < y - z$.

$x = \infty \wedge y \in \mathbb{R}$. Then $\infty = x + z < y \leqslant \infty$ [as $y \in \mathbb{R}$] so this case is not valid.

$x = -\infty \wedge y \in \mathbb{R}$. Then $x = -\infty < y - z$.

$x \in \mathbb{R} \wedge y = \infty$. Then $x < \infty = y - z$ so that $x < y - z$.

$x = \infty \wedge y = \infty$. Then $\infty = x + z < y = \infty$ so this case is not valid.

$x = -\infty \wedge y = \infty$. Then $x = -\infty < \infty = y - z$ so that $x < y - z$.

$x \in \mathbb{R} \wedge y = -\infty$. As $x \in \mathbb{R}$ we have $-\infty < x + z < y = -\infty$ so this case is not valid.

$x = \infty \wedge y = -\infty$. Then $-\infty < \infty = x + z < y = -\infty$ so this case is not valid.

$x = -\infty \wedge y = -\infty$. Then $-\infty = x + z < y = -\infty$ so this case is not valid.

So in all valid cases we have$x < y - z$.

7. For x, y we have either:

$x \in \mathbb{R} \wedge y \in \mathbb{R}$. Then using the properties of the real numbers we have $x - z \leqslant y$.

$x = \infty \wedge y \in \mathbb{R}$. Then $\infty = x \leqslant y + z < \infty$ [as $x + z \in \mathbb{R}$] so this case is not valid.

$x = -\infty \wedge y \in \mathbb{R}$. Then $x - z = -\infty < y$ so that $x - z \leqslant y$.

$x \in \mathbb{R} \wedge y = \infty$. Then $x - z < \infty = y$.

$x = \infty \wedge y = \infty$. Then $x - z = \infty = y$ so that $x - z \leqslant y$.

$x = -\infty \wedge y = \infty$. Then $x - z = -\infty < \infty = y$ so that $x - z \leqslant y$.

$x \in \mathbb{R} \wedge y = -\infty$. Then $-\infty < x \leqslant y + z = -\infty$ so this case is not valid.

$x = \infty \wedge y = -\infty$. Then $-\infty < \infty = x \leqslant y + z = -\infty$ so this case is not valid.

$x = -\infty \wedge y = -\infty$. Then $-\infty = x \leqslant y + z = -\infty$ so this case is not valid.

So in all valid cases we have $x - z \leqslant y$.

8. For x, y we have either:

$x \in \mathbb{R} \wedge y \in \mathbb{R}$. Then using the properties of the real numbers we have $x - z < y$.

$x=\infty\wedge y\in\mathbb{R}$. Then $\infty=x<y+z<\infty$ [as $x+z\in\mathbb{R}$] so this case is not valid.

$x=-\infty\wedge y\in\mathbb{R}$. Then $x-z=-\infty<y$.

$x\in\mathbb{R}\wedge y=\infty$. Then $x-z<\infty$.

$x=\infty\wedge y=\infty$. Then $\infty=x<y+z=\infty$ so this case is not valid.

$x=-\infty\wedge y=\infty$. Then $x-z=-\infty<\infty=y$.

$x\in\mathbb{R}\wedge y=-\infty$. Then $x<\infty<y+z=-\infty<\infty$ so this case is not valid.

$x=\infty\wedge y=-\infty$. Then $-\infty<\infty=x<y+z=-\infty$ so this case is not valid.

$x=-\infty\wedge y=-\infty$. Then $-\infty=x<y+z=-\infty$ so this case is not valid.

So in all valid cases we have $x-z<y$. □

Theorem 20.29. *Let*

$$(x,z)\in\overline{\mathbb{R}}\times\overline{\mathbb{R}}\backslash\{(-\infty,\infty),(\infty,-\infty)\}$$

and

$$(y,z)\in\overline{\mathbb{R}}\times\overline{\mathbb{R}}\backslash\{(-\infty,\infty),(\infty,-\infty)\}$$

[so that $x+z$ and $y+z$ are well defined] then we have:

1. *If $x\leqslant y$ then $x+z\leqslant y+z$.*
2. *If $0\leqslant x$ then $z\leqslant z+x$*
3. *If $x<y$ and $z\in\mathbb{R}$ then $x+z<y+z$*

***Note* 20.30.** *The condition $z\in\mathbb{R}$ in (3) is necessary, for example if $x=2$, $y=3$ and $z=\infty$ then $x<y$ but $x+z=\infty=y+z$ so that $x+z\nless y+z$.*

Proof.

1. We have the following cases to check for x,y,z:

   $x\in\mathbb{R}\wedge y\in\mathbb{R}\wedge z\in\mathbb{R}$. Then using the properties of $\mathbb{R}$ we have $x+z\leqslant y+z$.

   $x=-\infty\wedge y\in\mathbb{R}\wedge z\in\mathbb{R}$. Then $x+y=-\infty<y+z$ [as $y+z\in\mathbb{R}$].

   $x=\infty\wedge y\in\mathbb{R}\wedge z\in\mathbb{R}$. Then $-\infty<\infty=x\leqslant y<\infty$ so this is not a valid case.

   $x\in\mathbb{R}\wedge y=-\infty\wedge z\in\mathbb{R}$. Then $-\infty<x\leqslant y=-\infty$ so this is not a valid case.

   $x\in-\infty\wedge y=-\infty\wedge z\in\mathbb{R}$. Then $x+z=-\infty=y+z$ so that $x+z\leqslant y+z$

   $x\in\infty\wedge y=-\infty\wedge z\in\mathbb{R}$. Then $\infty=x\leqslant y=-\infty<\infty$ so this is not a valid case.

   $x\in\mathbb{R}\wedge y=\infty\wedge z\in\mathbb{R}$. Then as $x+z\in\mathbb{R}$ we have $x+z<\infty=y+z$.

   $x=-\infty\wedge y=\infty\wedge z\in\mathbb{R}$. Then $x+z=-\infty<\infty=y+z$,

   $x=\infty\wedge y=\infty\wedge z\in\mathbb{R}$. Then $x+z=\infty=y+z$ so that $x+y\leqslant y+z$.

   $x\in\mathbb{R}\wedge y\in\mathbb{R}\wedge z=-\infty$. Then $x+z=-\infty=y+z$ so that

   $x=-\infty\wedge y\in\mathbb{R}\wedge z=-\infty$. Then $x+z=-\infty=y+z$ so that $x+y\leqslant y+z$.

   $x=\infty\wedge y\in\mathbb{R}\wedge z=-\infty$. This cases is excluded because

   $$(x,z)\in\overline{\mathbb{R}}\backslash\{(-\infty,\infty),(\infty,-\infty)\}.$$

   $x\in\mathbb{R}\wedge y=-\infty\wedge z=-\infty$. Then $-\infty<x\leqslant y=-\infty$ so this is not a valid case.

   $x=-\infty\wedge y=-\infty\wedge z=-\infty$. Then $x+z=-\infty=y+z$ so that $x+z\leqslant y+z$.

   $x=\infty\wedge y=-\infty\wedge z=-\infty$. This cases is excluded because

   $$(x,z)\in\overline{\mathbb{R}}\backslash\{(-\infty,\infty),(\infty,-\infty)\}.$$

$x \in \mathbb{R} \wedge y = \infty \wedge z = -\infty$. This cases is excluded because

$$(y, z) \in \overline{\mathbb{R}} \backslash \{(-\infty, \infty), (\infty, -\infty)\}.$$

$x = -\infty \wedge y = \infty \wedge z = -\infty$. This cases is excluded because

$$(y, z) \in \overline{\mathbb{R}} \backslash \{(-\infty, \infty), (\infty, -\infty)\}.$$

$x = \infty \wedge y = \infty \wedge z = -\infty$. This cases is excluded because

$$(x, z), (y, z) \in \overline{\mathbb{R}} \backslash \{(-\infty, \infty), (\infty, -\infty)\}.$$

$x \in \mathbb{R} \wedge y \in \mathbb{R} \wedge z = \infty$. Then $x + z = \infty = y + z$ so that $x + z \leqslant y + z$.

$x = -\infty \wedge y \in \mathbb{R} \wedge z = \infty$. This cases is excluded because

$$(x, z) \in \overline{\mathbb{R}} \backslash \{(-\infty, \infty), (\infty, -\infty)\}.$$

$x = \infty \wedge y \in \mathbb{R} \wedge z = \infty$. Then $\infty = x < y < \infty$ so this is not a valid case.

$x \in \mathbb{R} \wedge y = -\infty \wedge z = \infty$. This cases is excluded because

$$(y, z) \in \overline{\mathbb{R}} \backslash \{(-\infty, \infty), (\infty, -\infty)\}.$$

$x = -\infty \wedge y = -\infty \wedge z = \infty$. This cases is excluded because

$$(x, z)(y, z) \in \overline{\mathbb{R}} \backslash \{(-\infty, \infty), (\infty, -\infty)\}.$$

$x = \infty \wedge y = -\infty \wedge z = \infty$. This cases is excluded because

$$(y, z) \in \overline{\mathbb{R}} \backslash \{(-\infty, \infty), (\infty, -\infty)\}.$$

$x \in \mathbb{R} \wedge y = \infty \wedge z = \infty$. Then $x + z = \infty - y + z$ so that $x + z \leqslant y + z$.

$x = -\infty \wedge y = \infty \wedge z = \infty$. This cases is excluded because

$$(x, z) \in \overline{\mathbb{R}} \backslash \{(-\infty, \infty), (\infty, -\infty)\}.$$

$x = \infty \wedge y = \infty \wedge z = \infty$. Then $x + z = \infty = y + z$ so that $x + y \leqslant y + z$.

Hence in all valid cases we have $x + y \leqslant y + z$.

2. As $0 \leqslant x$ we have $z = 0 + z \underset{(1)}{\leqslant} x + z$

3. As $x < y$ implies $x \leqslant y$ it follows from (1) that

$$x + z \leqslant y + z$$

Assume that $x + z = y + z$ then, as $z \in \mathbb{R}$, it follows from [theorem: 20.28(1)] that $x = y$ contradicting $x < y$. Hence the assumption is wrong and $x + y \neq y + z$ so that

$$x + y < y + z$$

□

Theorem 20.31. *Let $x, y \in \overline{\mathbb{R}}$ with $0 < x \wedge 0 \leqslant y$ then $0 < x + y$*

Proof. As $0 < x$ and $0 \leqslant y$ then we have that $x \neq -\infty \neq y$ so that for x, y we have either:

$x \in \mathbb{R} \wedge y \in \mathbb{R}$. Then by the properties of the real numbers we have $0 < x + y$.

$x = \infty \wedge y \in \mathbb{R}$. Then $0 < \infty = x + y$.

$x \in \mathbb{R} \wedge y = \infty$. Then $0 < \infty = x + y$

$x = \infty \wedge y = \infty$. Then $0 < \infty = x + y$ □

Theorem 20.32. *Let $x, y \in \overline{\mathbb{R}}$ such that $0 \leqslant x \wedge 0 \leqslant y \wedge z = x + y$ then $x \leqslant z$.*

Proof. As $0 \leqslant x$ and $0 \leqslant y$ then we have that $x \neq -\infty \neq y$ so that for x, y we have either:

$x \in \mathbb{R} \wedge y \in \mathbb{R}$. Then by the properties of the real numbers we have $x \leqslant x + y = z$.

$x=\infty \wedge y \in \mathbb{R}$. Then $x=\infty=\infty+y=z$.

$x \in \mathbb{R} \wedge y=\infty$. Then $x<\infty=x+\infty=x+y=z$.

$x=\infty \wedge y=\infty$. Then $x=\infty=x+y=z$. □

Theorem 20.33. *Let $x, y \in \overline{\mathbb{R}}$ then we have:*

1. *If $0<x \wedge 0<y$ then $0<x \cdot y$.*
2. *If $0 \leqslant x \wedge 0 \leqslant y$ then $0 \leqslant x \cdot y$.*
3. *If $x<0 \wedge 0<y$ then $x \cdot y<0$.*
4. *If $0<x \wedge y<0$ then $x \cdot y<0$.*
5. *If $x<0 \wedge y<0$ then $0<x \cdot y$.*
6. *If $x, y \in \overline{\mathbb{R}}$ with $0<x \leqslant y$ then $\frac{1}{y} \leqslant \frac{1}{x}$*

Proof.

1. For $0<x \wedge 0<y$ we have either:

   $x \in \mathbb{R}^+ \wedge y \in \mathbb{R}^+$. Then by the properties of the real numbers we have $0<x \cdot y$.

   $x=\infty \wedge y \in \mathbb{R}^+$. Then $0<\infty=\infty \cdot y=x \cdot y$.

   $x=\mathbb{R}^+ \wedge y=\infty$. Then $0<\infty=x \cdot \infty=x \cdot y$.

   $x=\infty \wedge y=\infty$. Then $0<\infty=\infty \cdot \infty=x \cdot y$.

2. For $0 \leqslant x \wedge 0 \leqslant y$ we have either:

   $x=0 \vee y=0$. Then $x \cdot y=0$ so that $0 \leqslant x \cdot y$

   $x \neq 0 \wedge y \neq 0$. Then $0<x$ and $0<y$ so that by (1) $0<x \cdot y$ hence $0 \leqslant x \cdot y$.

3. For $x<0 \wedge 0<y$ we have either:

   $x \in \mathbb{R}^- \wedge y \in \mathbb{R}^+$. Then by the properties of the real numbers we have $x \cdot y<0$.

   $x=-\infty \wedge y \in \mathbb{R}^+$. Then $x \cdot y=-\infty<0$.

   $x \in \mathbb{R}^- \wedge y=\infty$. Then $x \cdot y=-\infty<0$

   $x=-\infty \wedge y=\infty$. Then $x \cdot y=-\infty<0$

4. For $0<x \wedge y<0$ we have either:

   $x \in \mathbb{R}^+ \wedge y \in \mathbb{R}^-$. Then by the properties of the real numbers $x \cdot y<0$.

   $x=\infty \wedge y \in \mathbb{R}^-$. Then $x \cdot y=-\infty<0$.

   $x \in \mathbb{R}^+ \wedge y=-\infty$. Then $x \cdot y=-\infty<0$.

   $x=\infty \wedge y=-\infty$. Then $x \cdot y=-\infty<0$.

5. For $x<0 \wedge y<0$ we have either:

   $x \in \mathbb{R}^- \wedge y \in \mathbb{R}^-$. Then by the properties of the real numbers $0<x \cdot y$.

   $x=-\infty \wedge y \in \mathbb{R}^-$. Then $0<\infty=x \cdot y$.

   $x \in \mathbb{R}^- \wedge y=-\infty$. Then $0<\infty=x \cdot y$.

   $x=-\infty \wedge y=-\infty$. Then $0<\infty=x \cdot y$.

6. If $x, y \in \overline{\mathbb{R}}$ with $0<x \leqslant y$ we have for x either:

   $x \in \mathbb{R}^+$. As $x \leqslant y$ we have for y either:

   $y \in \mathbb{R}^+$. Then by the properties of the real numbers we have $\frac{1}{y} \leqslant \frac{1}{x}$.

   $y=\infty$. Then $\frac{1}{y}=\frac{1}{\infty}=0<\frac{1}{x}$.

   $x=\infty$. Then $y=\infty$ so that $\frac{1}{y}=\frac{1}{\infty}=\frac{1}{x}$ so that $\frac{1}{y} \leqslant \frac{1}{x}$. □

Corollary 20.34. *If* $x \in \overline{\mathbb{R}}$ *then we have:*

1. *If* $0 < x$ *then* $-x < 0$
2. *If* $0 \leqslant x$ *then* $-x \leqslant 0$
3. *If* $x < 0$ *then* $0 < -x$
4. *If* $x \leqslant 0$ *then* $0 \leqslant -x$

Proof. As $-1 < 0$ we have:

1. $-x = (-1) \cdot x \underset{[\text{theorem: } 20.33]}{<} (-1) \cdot 0 = 0$
2. $-x = (-1) \cdot x \underset{[\text{theorem: } 20.33]}{\leqslant} (-1) \cdot 0 = 0$
3. $-x = (-1) \cdot x \underset{[\text{theorem: } 20.33]}{>} (-1) \cdot 0 = 0$
4. $-x = (-1) \cdot x \underset{[\text{theorem: } 20.33]}{\geqslant} (-1) \cdot 0 = 0$ □

Although the product $x \cdot y$ is defined for every $x, y \in \overline{\mathbb{R}}$ $\langle \overline{\mathbb{R}}, \cdot \rangle$ is not a group, for example 0 has still no inverse. However $\langle \overline{\mathbb{R}}, \cdot \rangle$ is a Abelian semi group.

Theorem 20.35. $\langle \overline{\mathbb{R}}, \cdot \rangle$ *is a Abelian semi-group with neutral element* 1

Proof. We have to prove the semi-group axioms for $(\cdot): \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

associativity. Let x, y, z then we have either:

$\boldsymbol{x = 0 \vee y = 0}$**.** Then $x \cdot (y \cdot z) = 0 \cdot (y \cdot z) = 0 = 0 \cdot z = (x \cdot 0) \cdot z = (x \cdot y) \cdot z$

$\boldsymbol{x < 0 \wedge y = \infty}$**.** Then for z we have either:

$\boldsymbol{z = 0}$**.** Then $x \cdot (y \cdot z) = x \cdot (y \cdot 0) = x \cdot 0 = 0 = (x \cdot y) \cdot 0 = (x \cdot y) \cdot z$

$\boldsymbol{z < 0}$**.** Then

$$x \cdot (y \cdot z) = x \cdot (-\infty) = (-\infty) \cdot (-\infty) = \infty = (-\infty) \cdot z = (x \cdot y) \cdot z$$

$\boldsymbol{0 < z}$**.** Then

$$x \cdot (y \cdot z) = x \cdot \infty = -\infty = \infty \cdot z = (x \cdot y) \cdot z$$

$\boldsymbol{0 < x \wedge y = \infty}$**.** Then for z we have either:

$\boldsymbol{z = 0}$**.** Then $x \cdot (y \cdot z) = x \cdot (y \cdot 0) = x \cdot 0 = 0 = (x \cdot y) \cdot 0 = (x \cdot y) \cdot z$

$\boldsymbol{z < 0}$**.** Then $x \cdot (y \cdot z) = x \cdot (-\infty) = -\infty = \infty \cdot z = (x \cdot y) \cdot z$

$\boldsymbol{0 < z}$**.** Then $x \cdot (y \cdot z) = x \cdot \infty = \infty = \infty \cdot z = (x \cdot y) \cdot z$

$\boldsymbol{x < 0 \wedge y = -\infty}$**.** Then for z we have either:

$\boldsymbol{z = 0}$**.** Then $x \cdot (y \cdot z) = x \cdot 0 = 0 = (x \cdot y) \cdot 0 = (x \cdot y) \cdot z$

$\boldsymbol{z < 0}$**.** Then $x \cdot (y \cdot z) = x \cdot \infty = -\infty = \infty \cdot z = (x \cdot y) \cdot z$

$\boldsymbol{0 < z}$**.** Then $x \cdot (y \cdot z) = x \cdot (-\infty) = \infty = \infty \cdot z = (x \cdot y) \cdot z$

$\boldsymbol{0 < x \wedge y = -\infty}$**.** Then for z we have either:

$\boldsymbol{z = 0}$**.** Then $x \cdot (y \cdot z) = x \cdot 0 = 0 = (x \cdot y) \cdot 0 = (x \cdot y) \cdot z$

$\boldsymbol{z < 0}$**.** Then $x \cdot (y \cdot z) = x \cdot \infty = \infty = (-\infty) \cdot z = (x \cdot y) \cdot z$

$\boldsymbol{0 < z}$**.** Then $x \cdot (y \cdot z) = x \cdot (-\infty) = -\infty = (-\infty) \cdot z = (x \cdot y) \cdot z$

$\boldsymbol{x = -\infty \wedge y \in]-\infty, 0[}$**.** for z we have either:

$\boldsymbol{z = 0}$**.** Then $x \cdot (y \cdot z) = x \cdot 0 = 0 = (x \cdot y) \cdot 0 = (x \cdot y) \cdot z$

$z<0$. Then by [theorem: 20.33] $0<y\cdot z$ so that

$$x\cdot(y\cdot z)=-\infty=\infty\cdot z=(x\cdot y)\cdot z$$

$0<z$. Then by [theorem: 20.33] $y\cdot z<0$ so that

$$x\cdot(y\cdot z)=\infty=\infty\cdot z=(x\cdot y)\cdot z$$

$x=\infty\wedge y\in]-\infty,0[$. Then for z we have either

$z=0$. Then $x\cdot(y\cdot z)=x\cdot 0=0=(x\cdot y)\cdot 0=(x\cdot y)\cdot z$

$z<0$. Then by [theorem: 20.33] $0<y\cdot z$ so that

$$x\cdot(y\cdot z)=\infty=(-\infty)\cdot z=(x\cdot y)\cdot z$$

$0<z$. Then by [theorem: 20.33] $y\cdot z<0$ so that

$$x\cdot(y\cdot z)=-\infty=(-\infty)\cdot z=(x\cdot y)\cdot z$$

$x\in\mathbb{R}\setminus\{0\}\wedge y\in]-\infty,0[$. Then for z we have either:

$z=\infty$. Then for x we have either:

$x<0$. Then by [theorem: 20.33] $0<x\cdot y$ so that

$$x\cdot(y\cdot z)=x\cdot(-\infty)=\infty=(x\cdot y)\cdot\infty=(x\cdot y)\cdot z$$

$0<x$. Then by [theorem: 20.33] $x\cdot y<0$ so that

$$x\cdot(y\cdot z)=x\cdot(-\infty)=-\infty=(x\cdot y)\cdot\infty=(x\cdot y)\cdot z$$

$z=-\infty$. Then for x we have either:

$x<0$. Then by [theorem: 20.33] $0<x\cdot y$ so that

$$x\cdot(y\cdot z)=x\cdot\infty=-\infty=(x\cdot y)\cdot(-\infty)=(x\cdot y)\cdot z$$

$0<x$. Then by [theorem: 20.33] $x\cdot y<0$ so that

$$x\cdot(y\cdot z)=x\cdot\infty=\infty=(x\cdot y)\cdot(-\infty)=(x\cdot y)\cdot z$$

$z\in\mathbb{R}$. Then

$$x\cdot(y\cdot z)=x\cdot_{|\mathbb{R}}(y\cdot_{\mathbb{R}}z)=(x\cdot_{\mathbb{R}}y)\cdot_{\mathbb{R}}z=(x\cdot y)\cdot z$$

$x=-\infty\wedge y\in]0,\infty[$. Then for z we have either:

$z=0$. Then $x\cdot(y\cdot z)=x\cdot 0=0=(x\cdot y)\cdot 0=(x\cdot y)\cdot z$

$z<0$. Then by [theorem: 20.33] $y\cdot z<0$ so that

$$x\cdot(y\cdot z)=\infty=(-\infty)\cdot z=(x\cdot y)\cdot z$$

$0<z$. Then by [theorem: 20.33] $0<y\cdot z$ so that

$$x\cdot(y\cdot z)=-\infty=-\infty\cdot z=(x\cdot y)\cdot z$$

$x=\infty\wedge y\in]0,\infty[$. Then for z we have either:

$z=0$. Then $x\cdot(y\cdot z)=x\cdot 0=0=(x\cdot y)\cdot 0=(x\cdot y)\cdot z$

$z<0$. Then by [theorem: 20.33] $y\cdot z<0$ so that

$$x\cdot(y\cdot z)=-\infty=\infty\cdot z=(x\cdot y)\cdot z$$

$0<z$. Then by [theorem: 20.33] $0<y\cdot z$ so that

$$x\cdot(y\cdot z)=\infty=\infty\cdot z=(x\cdot y)\cdot z$$

$\boldsymbol{x \in \mathbb{R} \setminus \{0\} \wedge y \in]0, \infty[}$. Then for z we have either:

$\boldsymbol{z = \infty}$. Then for x we have either:

$\boldsymbol{x < 0}$. Then by [theorem: 20.33] $x \cdot y < 0$ so that

$$x \cdot (y \cdot z) = x \cdot \infty = -\infty = (x \cdot y) \cdot \infty = (x \cdot y) \cdot z$$

$\boldsymbol{0 < x}$. Then by [theorem: 20.33] $0 < x \cdot y$ so that

$$x \cdot (y \cdot z) = x \cdot \infty = \infty = (x \cdot y) \cdot \infty = (x \cdot y) \cdot z$$

$\boldsymbol{z = -\infty}$. Then for x we have either:

$\boldsymbol{x < 0}$. Then by [theorem: 20.33] $x \cdot y < 0$ so that

$$x \cdot (y \cdot z) = x \cdot (-\infty) = \infty = (x \cdot y) \cdot (-\infty) = (x \cdot y) \cdot z$$

$\boldsymbol{0 < x}$. Then by [theorem: 20.33] $0 < x \cdot y$ so that

$$x \cdot (y \cdot z) = x \cdot (-\infty) = -\infty = (x \cdot y) \cdot (-\infty) = (x \cdot y) \cdot z$$

$\boldsymbol{z \in \mathbb{R}}$. Then

$$x \cdot (y \cdot z) = x \cdot_{|\mathbb{R}} (y \cdot_{\mathbb{R}} z) = (x \cdot_{\mathbb{R}} y) \cdot_{\mathbb{R}} z = (x \cdot y) \cdot z$$

So in all cases we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

neutral element. Let $x \in \overline{\mathbb{R}}$ then we have for x either:

$\boldsymbol{x = \infty}$. Then $1 \cdot \infty = \infty = \infty \cdot 1$

$\boldsymbol{x = -\infty}$. Then $1 \cdot (-\infty) = -\infty = (-\infty) \cdot 1$

$\boldsymbol{x \in \mathbb{R}}$. Then $1 \cdot x = 1 \cdot_{\mathbb{R}} x = x = x \cdot_{\mathbb{R}} 1 = x \cdot 1$

commutativity. Then for $x, y \in \overline{\mathbb{R}}$ we have either

$\boldsymbol{x = 0 \vee y = 0}$. Then $x \cdot y = 0 = y \cdot x$

$\boldsymbol{x < 0 \wedge y = \infty}$. Then $x \cdot y \underset{\text{def}}{=} -\infty$ and for x we have either:

$\boldsymbol{x = -\infty}$. Then $y \cdot x \underset{0 < \infty = y}{=} -\infty = x \cdot y$

$\boldsymbol{x \in]-\infty, 0[}$. Then $y \cdot x = -\infty = x \cdot y$.

$\boldsymbol{0 < x \wedge y = \infty}$. Then $x \cdot y \underset{\text{def}}{=} \infty$ and for x we have either:

$\boldsymbol{x = \infty}$. Then $y \cdot x \underset{0 < \infty = y}{=} \infty = x \cdot y$.

$\boldsymbol{x \in]0, \infty[}$. Then $y \cdot x = \infty = x \cdot y$.

$\boldsymbol{x < 0 \wedge y = -\infty}$. Then $x \cdot y \underset{\text{def}}{=} \infty$ and for x we have either:

$\boldsymbol{x = -\infty}$. Then $y \cdot x \underset{y = -\infty < 0}{=} \infty = x \cdot y$.

$\boldsymbol{x \in]-\infty, 0[}$. Then $y \cdot x = \infty = x \cdot y$.

$\boldsymbol{0 < x \wedge y = -\infty}$. Then $x \cdot y = -\infty$ and for x we have either:

$\boldsymbol{x = \infty}$. Then $y \cdot x \underset{y = -\infty < 0}{=} -\infty = x \cdot y$

$\boldsymbol{x \in]0, \infty[}$. Then $y \cdot x = -\infty = x \cdot y$

$\boldsymbol{x = -\infty \wedge y \in]-\infty, 0[}$. Then $x \cdot y = \infty \underset{y < 0}{=} y \cdot x$

$\boldsymbol{x = \infty \wedge y \in]-\infty, 0[}$. Then $x \cdot y = -\infty \underset{y < 0}{=} y \cdot x$

$\boldsymbol{x \in \mathbb{R} \setminus \{0\} \wedge y \in]-\infty, 0[}$. Then $x \cdot y = x \cdot_{\mathbb{R}} y = y \cdot_{\mathbb{R}} x = y \cdot x$

$\boldsymbol{x = -\infty \wedge y \in]0, \infty[}$. Then $x \cdot y = -\infty \underset{0 < y}{=} y \cdot x$

$x=\infty \wedge y\in]0,\infty[$. Then $x\cdot y=\infty \underset{0<y}{=} y\cdot x$

$x\in\mathbb{R}\setminus\{0\}\wedge y\in]0,\infty[$. Then $x\cdot y=x\cdot_{\mathbb{R}} y=y\cdot_{\mathbb{R}} x=y\cdot x$ □

Corollary 20.36. *Let $x,y\in\overline{\mathbb{R}}$ then $-(x\cdot y)=(-x)\cdot y=x\cdot(-y)$*

Proof. We have

$$-(x\cdot y)=(-1)\cdot(x\cdot y)\underset{[\text{theorem: }20.35]}{=}((-1)\cdot x)\cdot y=(-x)\cdot y$$

and

$$-(x\cdot y)=(-1)\cdot(x\cdot y)\underset{[\text{theorem: }20.35]}{=}(-1)\cdot(y\cdot x)\underset{[\text{theorem: }20.35]}{=}((-1)\cdot y)\cdot x\underset{[\text{theorem: }20.35]}{=}x\cdot(-y)$$

□

Theorem 20.37. *If $x,y\in\overline{\mathbb{R}}$ with $x\leqslant y$ then for $\lambda\in\overline{\mathbb{R}}$ we have*

1. *If $0\leqslant\lambda$ then $\lambda\cdot x\leqslant\lambda\cdot y$*
2. *If $\lambda\leqslant 0$ then $\lambda\cdot y\leqslant\lambda\cdot x$*

Proof.

1. For $0\leqslant\lambda$ we have either:

   $\lambda=0$. Then $\lambda\cdot x=0\cdot x=0=0\cdot y=\lambda\cdot y$ so that $\lambda\cdot x\leqslant\lambda\cdot y$.

   $\lambda\in\mathbb{R}^+$. Then for x we have either:

   $x\in\mathbb{R}$. Then for y we have as $-\infty<x\leqslant y$ that

   $y\in\mathbb{R}$. Then using [theorem: 10.14] we have

   $$\lambda\cdot x=\lambda\cdot_{\mathbb{R}} x\underset{[\text{theorem: }10.14]}{\leqslant_{\mathbb{R}}}\lambda\cdot_{\mathbb{R}} y=\lambda\cdot y$$

   so that

   $$\lambda\cdot x\leqslant\lambda\cdot y$$

   $y=\infty$. Then $\lambda\cdot x\in\mathbb{R}$ so that $\lambda\cdot x\leqslant\infty=\lambda\cdot\infty=\lambda\cdot y$ so that $\lambda\cdot x\leqslant\lambda\cdot y$.

   $x=-\infty$. Then for y we have either:

   $y\in\mathbb{R}$. Then $\lambda\cdot y\in\mathbb{R}$ so that $\lambda\cdot x=\lambda\cdot(-\infty)=-\infty<\lambda\cdot y$ hence $\lambda\cdot x\leqslant\lambda\cdot y$.

   $y=-\infty$. Then $\lambda\cdot x=\lambda\cdot(-\infty)=-\infty=\lambda\cdot y$ so that $\lambda\cdot x\leqslant\lambda\cdot y$.

   $y=\infty$. Then $\lambda\cdot x=\lambda\cdot(-\infty)=-\infty<\infty=\lambda\cdot y$ so that $\lambda\cdot x\leqslant\lambda\cdot y$.

   $x=\infty$. Then as $x\leqslant y$ we have that $y=\infty$ hence $\lambda\cdot x=\infty=\lambda\cdot y$ so that $\lambda\cdot x\leqslant\lambda\cdot y$.

   $\lambda=\infty$. Then for x we have either:

   $x<0$. Then for y we have either:

   $y<0$. Then $\lambda\cdot x=-\infty=\lambda\cdot y$ so that $\lambda\cdot x\leqslant\lambda\cdot y$.

   $y=0$. Then $\lambda\cdot x=-\infty<0=\lambda\cdot\infty=\lambda\cdot y$ so that $\lambda\cdot x\leqslant\lambda\cdot y$.

   $0<y$. Then $\lambda\cdot x=-\infty<\infty=\lambda\cdot\infty=\lambda\cdot y$ so that $\lambda\cdot x\leqslant\lambda\cdot y$.

   $x=0$. Then as $x\leqslant y$ we have that y is either:

   $y=0$. Then $\lambda\cdot x=0=\lambda\cdot y$ so that $\lambda\cdot x\leqslant\lambda\cdot y$.

   $0<y$. Then $\lambda\cdot x=0<\infty=\lambda\cdot y$ so that $\lambda\cdot x\leqslant\lambda\cdot y$.

   $0<x$. Then as $x\leqslant y$ we have that $0<y$ so that $\lambda\cdot x=\infty=\lambda\cdot y$ so that $\lambda\cdot x\leqslant\lambda\cdot y$.

2. For $\lambda\leqslant 0$ we have either:

   $\lambda=0$. Then $\lambda\cdot x=0\cdot x=0=0\cdot y=\lambda\cdot y$ so that $\lambda\cdot x\leqslant\lambda\cdot y$.

$\lambda \in \mathbb{R}^-$. Then for x we have either:

$x \in \mathbb{R}$. Then for y we have as $-\infty < x \leqslant y$ that

$y \in \mathbb{R}$. Then using [theorem: 10.14] we have

$$\lambda \cdot y = \lambda \cdot_{\mathbb{R}} y \underset{\text{[theorem: 10.14]}}{\leqslant_{\mathbb{R}}} \lambda \cdot_{\mathbb{R}} x = \lambda \cdot x$$

so that

$$\lambda \cdot y \leqslant \lambda \cdot x$$

$y = \infty$. Then $\lambda \cdot x \in \mathbb{R}$ so that $\lambda \cdot y = -\infty < \lambda \cdot x$ so that $\lambda \cdot y \leqslant \lambda \cdot x$.

$x = -\infty$. Then for y we have either:

$y \in \mathbb{R}$. Then $\lambda \cdot y \in \mathbb{R}$ so that $\lambda \cdot y < \infty = \lambda \cdot x$ hence $\lambda \cdot y \leqslant \lambda \cdot x$.

$y = -\infty$. Then $\lambda \cdot y = \infty = \lambda \cdot x$ so that $\lambda \cdot y \leqslant \lambda \cdot x$.

$y = \infty$. Then $\lambda \cdot y = -\infty < \infty = \lambda \cdot x$ so that $\lambda \cdot y \leqslant \lambda \cdot x$.

$x = \infty$. Then as $x \leqslant y$ we have that $y = \infty$ hence $\lambda \cdot y = -\infty = \lambda \cdot x$ so that $\lambda \cdot y \leqslant \lambda \cdot x$.

$\lambda = -\infty$. Then for x we have either:

$x < 0$. Then for y we have either:

$y < 0$. Then $\lambda \cdot y = \infty = \lambda \cdot x$ so that $\lambda \cdot y \leqslant \lambda \cdot x$.

$y = 0$. Then $\lambda \cdot y = 0 < \infty = \lambda \cdot (-\infty) = \lambda \cdot x$ so that $\lambda \cdot y \leqslant \lambda \cdot x$.

$0 < y$. Then $\lambda \cdot y = -\infty < \infty = \lambda \cdot x$ so that $\lambda \cdot y \leqslant \lambda \cdot x$.

$x = 0$. Then as $x \leqslant y$ we have that y is either:

$y = 0$. Then $\lambda \cdot y = 0 = \lambda \cdot x$ so that $\lambda \cdot y \leqslant \lambda \cdot x$.

$0 < y$. Then $\lambda \cdot y = -\infty < 0 = \lambda \cdot x$ so that $\lambda \cdot y \leqslant \lambda \cdot x$.

$0 < x$. Then as $x \leqslant y$ we have that $0 < y$ so that $\lambda \cdot y = -\infty = \lambda \cdot x$ so that $\lambda \cdot y \leqslant \lambda \cdot x$. □

Corollary 20.38. *If $x, y \in \overline{\mathbb{R}}$ then we have*

1. *If $-x = -y$ then $x = y$*
2. *If $x \leqslant y$ then $-y \leqslant -x$*
3. *If $x < y$ then $-y < -x$*

Proof.

1. We have

$$\begin{aligned} x &\underset{\text{[theorem: 20.35]}}{=} 1 \cdot x \\ &= ((-1) \cdot (-1)) \cdot x \\ &\underset{\text{[theorem: 20.35]}}{=} (-1) \cdot ((-1) \cdot x) \\ &= (-1) \cdot (-x) \\ &= (-1) \cdot (-y) \\ &= (-1) \cdot ((-1) \cdot y) \\ &\underset{\text{[theorem: 20.35]}}{=} ((-1) \cdot (-1)) \cdot y \\ &= 1 \cdot y \\ &= y \end{aligned}$$

2. Take $\lambda = -1 \leqslant 0$ then $-y = \lambda \cdot y \underset{\text{[theorem: 20.37]}}{\leqslant} \lambda \cdot x = -x$

3. If $x<y$ then $x\neq y$ and $x<y$, using (2) we have then that $(-y)\leqslant -x$ and using (1) that $-x\neq -y$ proving that $-y<-x$ □

We have also a variant of the distributive law in $\overline{\mathbb{R}}$ but again we must be careful. For example $\infty\cdot(1-3)=\infty\cdot(-2)=-\infty$ but $\infty\cdot 1+\infty\cdot(-3)$ is not defined because $\infty\cdot 1=\infty$ and $\infty\cdot(-3)=-\infty$.

Theorem 20.39. *Let $\alpha\in\mathbb{R}$, $(x,y)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty)\}$ then $\alpha\cdot(x+y)=\alpha\cdot x+\alpha\cdot y$*

Proof. For (x,y) we have either:

$\boldsymbol{x\in\mathbb{R}\wedge y\in\mathbb{R}}$. then $\alpha\cdot(x+y)=\alpha\cdot_{\mathbb{R}}(x+_{\mathbb{R}}y)=\alpha\cdot_{\mathbb{R}}x+_{\mathbb{R}}\alpha\cdot_{\mathbb{R}}y=\alpha\cdot x+\alpha\cdot y$

$\boldsymbol{x\in\mathbb{R}\wedge y=\infty}$. then for α we have either

- $\boldsymbol{\alpha=0}$. then $\alpha\cdot(x+y)=0=0+0=\alpha\cdot x+\alpha\cdot y$.
- $\boldsymbol{0<\alpha}$. then $\alpha\cdot(x+y)=\alpha\cdot\infty=\infty=\alpha\cdot x+\infty=\alpha\cdot x+\alpha\cdot y$
- $\boldsymbol{\alpha<0}$. then $\alpha\cdot(x+y)=\alpha\cdot\infty=-\infty=\alpha\cdot x+(-\infty)=\alpha\cdot x+\alpha\cdot y$

$\boldsymbol{x\in\mathbb{R}\wedge y=-\infty}$. then for α we have either

- $\boldsymbol{\alpha=0}$. then $\alpha\cdot(x+y)=0=0+0=\alpha\cdot x+\alpha\cdot y$.
- $\boldsymbol{0<\alpha}$. then $\alpha\cdot(x+y)=\alpha\cdot(-\infty)=-\infty=\alpha\cdot x+(-\infty)=\alpha\cdot x+\alpha\cdot y$
- $\boldsymbol{\alpha<0}$. then $\alpha\cdot(x+y)=\alpha\cdot(-\infty)=\infty=\alpha\cdot x+\infty=\alpha\cdot x+\alpha\cdot y$

$\boldsymbol{x=\infty\wedge y\in\mathbb{R}}$. then for α we have either

- $\boldsymbol{\alpha=0}$. then $\alpha\cdot(x+y)=0=0+0=\alpha\cdot x+\alpha\cdot y$.
- $\boldsymbol{0<\alpha}$. then $\alpha\cdot(x+y)=\alpha\cdot\infty=\infty=\infty+\alpha\cdot y=\alpha\cdot x+\alpha\cdot y$
- $\boldsymbol{\alpha<0}$. then $\alpha\cdot(x+y)=\alpha\cdot\infty=-\infty=-\infty+\alpha\cdot y=\alpha\cdot x+\alpha\cdot y$

$\boldsymbol{x=\infty\wedge y=\infty}$. then for α we have either

- $\boldsymbol{\alpha=0}$. then $\alpha\cdot(x+y)=0=0+0=\alpha\cdot x+\alpha\cdot y$.
- $\boldsymbol{0<\alpha}$. then $\alpha\cdot(x+y)=\alpha\cdot\infty=\infty=\infty+\infty=\alpha\cdot x+\alpha\cdot y$
- $\boldsymbol{\alpha<0}$. then $\alpha\cdot(x+y)=\alpha\cdot\infty=-\infty=-\infty+(-\infty)=\alpha\cdot x+\alpha\cdot y$

$\boldsymbol{x=-\infty\wedge y\in\mathbb{R}}$. then for α we have either

- $\boldsymbol{\alpha=0}$. then $\alpha\cdot(x+y)=0=0+0=\alpha\cdot x+\alpha\cdot y$.
- $\boldsymbol{0<\alpha}$. then $\alpha\cdot(x+y)=\alpha\cdot(-\infty)=-\infty=-\infty+\alpha\cdot y=\alpha\cdot x+\alpha\cdot y$
- $\boldsymbol{\alpha<0}$. then $\alpha\cdot(x+y)=\alpha\cdot(-\infty)=\infty=\infty+\alpha\cdot y=\alpha\cdot x+\alpha\cdot y$

$\boldsymbol{x=-\infty\wedge y=-\infty}$. then for α we have either

- $\boldsymbol{\alpha=0}$. then $\alpha\cdot(x+y)=0=0+0=\alpha\cdot x+\alpha\cdot y$.
- $\boldsymbol{0<\alpha}$. then $\alpha\cdot(x+y)=\alpha\cdot(-\infty)=-\infty=-\infty+(-\infty)=\alpha\cdot x+\alpha\cdot y$
- $\boldsymbol{\alpha<0}$. then $\alpha\cdot(x+y)=\alpha\cdot(-\infty)=\infty=-\infty+\infty=\alpha\cdot x+\alpha\cdot y$

so in all cases we have $\alpha\cdot(x+y)=\alpha\cdot x+\alpha\cdot y$. □

Theorem 20.40. *Let $x,y\in\overline{\mathbb{R}}$ such that $\forall\varepsilon\in\mathbb{R}^+$ we have $x\leqslant y+\varepsilon$ then $x\leqslant y$.*

Proof. If $\forall\varepsilon\in\mathbb{R}^+$ we have $x\leqslant y+\varepsilon$ then we have for x,y to verify the following cases:

$\boldsymbol{x=-\infty\wedge y=-\infty}$. Then $x=y$ so that $x\leqslant y$

$\boldsymbol{x\in\mathbb{R}\wedge y=-\infty}$. Then $-\infty<x\leqslant y+\varepsilon=(-\infty)+\varepsilon=-\infty$ leading to the contradiction $-\infty<-\infty$, hence this cases never materilize.

$\boldsymbol{x=\infty\wedge y=-\infty}$. Then $\infty=x\leqslant y+\varepsilon=-\infty+\varepsilon=-\infty$ leading to the contradiction $\infty<-\infty$ so this case never occurs.

$\boldsymbol{x=-\infty\wedge y\in\mathbb{R}}$. Then $x=-\infty<y$ so that $x\leqslant y$.

$x\in\mathbb{R}\wedge y\in\mathbb{R}$. Then by [theorem: 10.31 (2)] it follows that $x\leqslant y$

$x=\infty\wedge y\in\mathbb{R}$. Then $\infty=x\leqslant y+\varepsilon<\infty$ leading to the contradiction $\infty<\infty$, hence this cases never occurs.

$x=-\infty\wedge y=\infty$. Then $x=-\infty<\infty=y$ so that $x\leqslant y$.

$x\in\mathbb{R}\wedge y=\infty$. Then $x<\infty=y$ so that $x\leqslant y$

$x=\infty\wedge y=\infty$. Then $x=y$ so that $x\leqslant y$

So in all valid cases we have that $x\leqslant y$ □

20.1.3 Topology on $\overline{\mathbb{R}}$

First we define the absolute value on $\overline{\mathbb{R}}$.

Definition 20.41. *Let $x\in\overline{\mathbb{R}}$ then the absolute value noted as $|x|$ is defined in analogy of the absolute value on $\mathbb{R}$ as follows*

$$|x|=\begin{cases} x \text{ if } 0\leqslant x\\ -x \text{ if } x<0\end{cases}$$

Note 20.42. $\forall x\in\overline{\mathbb{R}}$ we have that $x\leqslant|x|$

Note 20.43. If $x\in\mathbb{R}$ then $|x|=|x|_{\mathbb{R}}$ where $||_{\mathbb{R}}$ is the absolute value defined in $\mathbb{R}$ [see example: 14.97].

Proof. For $x\in\overline{\mathbb{R}}$ we have either:

$0\leqslant x$. Then $x=|x|$ so that $x\leqslant|x|$

$x<0$. Then by [theorem: 20.34] $0<-x=|x|$ so that $x<0<|x|$ or $x\leqslant|x|$ □

The absolute value of the extended real numbers has similar properties as the absolute value on the real numbers.

Theorem 20.44. *The absolute value has the following properties:*

1. $|\infty|=\infty$
2. $|-\infty|=\infty$
3. *For $x\in\overline{\mathbb{R}}$ we have $|x|=0\Leftrightarrow x=0$.*
4. *$\forall x,y\in\overline{\mathbb{R}}$ we have $|x\cdot y|=|x|\cdot|y|$.*
5. *$\forall(x,y)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$ we have $|x+y|\leqslant|x|+|y|$*

Proof.

1. As $0<\infty$ it follows that $|\infty|=\infty$.
2. As $-\infty<0$ it follows that $|-\infty|=-(-\infty)=(-1)\cdot(-\infty)=\infty$
3. We have

   $\Rightarrow$. Assume that $x\neq 0$ then we have either $0<x\Rightarrow 0\neq x=|x|\neq 0$ leading to the contradiction $0\neq 0$ or $x<0\underset{\text{[theorem: 20.34]}}{\Rightarrow}0<-x=|x|=0$ leading to the contradiction $0<0$. Hence the assumption is false and we must have that $x=0$.

   $\Leftarrow$. As $0\leqslant 0$ it follows that $|0|=0$.
4. For $x,y\in\overline{\mathbb{R}}$ we have the following cases to consider:

   $x=0\wedge y=0$. Then $|x\cdot y|=|0|\underset{(3)}{=}0$.

   $x<0\wedge y=0$. Then $|x\cdot y|=|0|\underset{(3)}{=}0$.

$0<x\wedge y=0$. Then $|x\cdot y|=|0|\underset{(3)}{=}0$.

$x=0\wedge y<0$. Then $|x\cdot y|=|0|\underset{(3)}{=}0$.

$x<0\wedge y<0$. Then by [theorem: 20.33] $0<x\cdot y$ so that $|x\cdot y|=x\cdot y=|x|\cdot|y|$.

$0<x\wedge y<0$. Then by [theorem: 20.33] $x\cdot y<0$ so that

$$|x\cdot y|=-(x\cdot y)\underset{\text{[theorem: 20.36]}}{=}x\cdot(-y)=|x|\cdot|y|$$

$x=0\wedge 0<y$. Then $|x\cdot y|=|0|\underset{(3)}{=}0$.

$x<0\wedge 0<y$. Then by [theorem: 20.33] $x\cdot y<0$ so that

$$|x\cdot y|=-(x\cdot y)=(-x)\cdot y=|x|\cdot|y|$$

$0<x\wedge 0<y$. Then by [theorem: 20.33] $0<x\cdot y$ so that

$$|x\cdot y|=x\cdot y=|x|\cdot|y|$$

5. For $(x,y)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$ we have either:

   $x=-\infty\wedge y=-\infty$. Then $|x+y|=|-\infty|\underset{(2)}{=}\infty=\infty+\infty=|-\infty|+|-\infty|=|x|+|y|$

   $x\in\mathbb{R}\wedge y=-\infty$. Then $|x+y|=|-\infty|\underset{(2)}{=}|x|+\infty\underset{(2)}{=}|x|+|-\infty|=|x|+|y|$

   $x=\infty\wedge y=\infty$. Then $|x+y|=|\infty|\underset{(1)}{=}\infty=\infty+\infty\underset{(1)}{=}|\infty|+|\infty|=|x|+|y|$

   $x\in\mathbb{R}\wedge y=\infty$. Then $|x+y|=|\infty|\underset{(1)}{=}\infty=|x|+\infty\underset{(1)}{=}|x|+|\infty|=|x|+|y|$

   $x=\infty\wedge y\in\mathbb{R}$. Then $|x+y|=|\infty|\underset{(1)}{=}\infty=\infty+|y|\underset{(1)}{=}|\infty|+|y|=|x|+|y|$

   $x=-\infty\wedge y\in\mathbb{R}$. Then $|x+y|=|-\infty|\underset{(2)}{=}\infty=\infty+|y|\underset{(2)}{=}|-\infty|+|y|=|x|+|y|$

   $x\in\mathbb{R}\wedge y\in\mathbb{R}$. Then $|x+y|=|x+y|_{\mathbb{R}}\leqslant|x|_{\mathbb{R}}+|y|_{\mathbb{R}}=|x|+|y|$ □

Unfortunately $\langle\overline{\mathbb{R}},\|\rangle$ is not a normed space because $\overline{\mathbb{R}}$ is not a vector space [remember $\infty+(-\infty)$ and $(-\infty)+\infty$ are not defined. So we can not base a topology on a norm instead we use a generating basis [see theorem: 14.39].

Definition 20.45. *The following sets are defined*

1. $\mathcal{B}_{||}=\{]a,b[|a,b\in\mathbb{R}\wedge a<b\}$ *which is the generating basis for the natural topology* $\mathcal{T}_{||}$ *on* $\mathbb{R}$ *[see theorem 14.99].*
2. $\mathcal{B}_{\infty}=\{]x,\infty]|x\in\mathbb{R}\}$
3. $\mathcal{B}_{-\infty}=\{[-\infty,x[|x\in\mathbb{R}\}$
4. $\mathcal{B}_{\overline{\mathbb{R}}}=\mathcal{B}_{||||}\bigcup\mathcal{B}_{\infty}\bigcup\mathcal{B}_{-\infty}$

We prove now that $\mathcal{B}_{\overline{\mathbb{R}}}$ satisfies the conditions to be a generating basis [see theorem: 14.39].

Lemma 20.46. *The set* $\mathcal{B}_{\overline{\mathbb{R}}}$ *satisfies*

1. $\forall x\in\overline{\mathbb{R}}$ *there exist a* $B\in\mathcal{B}_{\overline{\mathbb{R}}}$ *such that* $x\in B$.
2. $\forall B_1,B_2\in\mathcal{B}_{\overline{\mathbb{R}}}$ *we have that* $\forall x\in B_1\bigcap B_2$ *there exist a* $B\in\mathcal{B}_{\overline{\mathbb{R}}}$ *such that* $x\in B\subseteq B_1\bigcap B_2$

Proof.

1. If $x\in\overline{\mathbb{R}}$ then we have either

   $x\in\mathbb{R}$. Then $x\in]x-1,x+1[\in\mathcal{B}_{\overline{\mathbb{R}}}$

$\boldsymbol{x=-\infty}$**.** Then $x=-\infty\in[-\infty,0[\in\mathcal{B}_{\overline{\mathbb{R}}}$

$\boldsymbol{x=\infty}$**.** Then $x=\infty\in]0,\infty]\in\mathcal{B}_{\overline{\mathbb{R}}}$

2. If $B_1,B_2\in\mathcal{B}_{\overline{\mathbb{R}}}$ and $x\in B_1\bigcap B_2$ then we have for B_1,B_2 either:

   $\boldsymbol{B_1\in\mathcal{B}_{||}\wedge B_2\in\mathcal{B}_{||}}$**.** Then there exists a_1, b_1, a_2, b_2 with $B_1=]a_1,b_1[\wedge B_2=]a_2,b_2[$, as $x\in B_1\bigcap B_2$ $a_1<x<b_1$ and $a_2<x<b_2$. Define $B=]a,b[$ where $a=\max(a_1,a_2)$ and $b=\min(b_1,b_2)$ then $a<x<b$ or $x\in B\in\mathcal{B}_{||}$. Further if $y\in B$ then $a_1,a_2\leqslant a<y<b\leqslant b_1$, b_2 so that $y\in B_1\bigcap B_2$. Hence $x\in B\subseteq B_1\bigcap B_2$ where $B\in\mathcal{B}_{||}\subset-\mathcal{B}_{\overline{\mathbb{R}}}$.

   $\boldsymbol{B_1\in\mathcal{B}_{-\infty}\wedge B_2\in\mathcal{B}_{||}}$**.** Then there exists $a,b,c\in\mathbb{R}$ such that $B_2=]a,b[$ and $B_1=[-\infty,c[$, as $x\in B_1\bigcap B_2$ we have that $x<c$ and $a<x<b$. Define $B=]a,d[$ where $d=\min(c,b)$ then $a<x<d$ so that $x\in B\in\mathcal{B}_{||}$. Further if $y\in B$ then $a<y<d\leqslant c,b$ so that $y\in B_1\bigcap B_2$. Hence $x\in B\subseteq B_1\bigcap B_2$ where $B\in\mathcal{B}_{||}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$.

   $\boldsymbol{B_1\in\mathcal{B}_{\infty}\wedge B_2\in\mathcal{B}_{||}}$**.** Then there exists $a,b,c\in\mathbb{R}$ such that $B_2=]a,b[$ and $B_1=]c,\infty]$, as $x\in B_1\bigcap B_2$ we have that $c<x$ and $a<x<b$. Define $B=]d,b[$ where $d=\max(c,a)$ then $d<x<b$ so that $x\in B\in\mathcal{B}_{||}$. Further if $y\in B$ then $c,a\leqslant d<y<b$ so that $y\in B_1\bigcap B_2$. Hence $x\in B\subseteq B_1\bigcap B_2$ where $B\in\mathcal{B}_{||}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$.

   $\boldsymbol{B_1\in\mathcal{B}_{||}\wedge B_2\in\mathcal{B}_{-\infty}}$**.** Then there exists $a,b,c\in\mathbb{R}$ such that $B_1=]a,b[$ and $B_2=[-\infty,c[$, as $x\in B_1\bigcap B_2$ we have $a<x<b$ and $x<c$. Define $B=]a,d[$ where $d=\min(b,c)$ then $a<x<d$ so that $x\in B\in\mathcal{B}_{||}$. Further if $y\in B$ then $a<y<d\leqslant b,c$ so that $y\in B_1\bigcap B_2$. Hence $x\in B\subseteq B_1\bigcap B_2$ where $B\in\mathcal{B}_{||}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$.

   $\boldsymbol{B_1\in\mathcal{B}_{-\infty}\wedge B_2\in\mathcal{B}_{-\infty}}$**.** Then there exists $c_1,c_2\in\mathbb{R}$ such that $B_1=[-\infty,c_1[$ and $B_2=[-\infty,c_2[$, as $x\in B_1\bigcap B_2$ we have that $x<c_1,c_2$ hence if we define $B=[-\infty,c[$ where $c=\min(c_1,c_2)\in\mathbb{R}$ then $x<c$ so that $x\in B\in\mathcal{B}_{-\infty}$. Further if $y\in B$ then $y<c\leqslant c_1,c_2$ so that $y\in B_1\bigcap B_2$. Hence $x\in B\subseteq B_1\bigcap B_2$ where $B\in\mathcal{B}_{-\infty}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$.

   $\boldsymbol{B_1\in\mathcal{B}_{\infty}\wedge B_2\in\mathcal{B}_{-\infty}}$**.** Then there exists $c_1,c_2\in\mathbb{R}$ such that $B_1=]c_1,\infty]$ and $B_2=[-\infty,c_2[$, as $x\in B_1\bigcap B_2$ we have that $c_1<x<c_2$. Define $B=]c_1,c_2[$ then $x\in B\in\mathcal{B}_{||}$. Further if $y\in B$ then $c_1<y<c_2$ so that $y\in B_1\bigcap B_2$. Hence $x\in B\subseteq B_1\bigcap B_2$ where $B\in\mathcal{B}_{||}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$.

   $\boldsymbol{B_1\in\mathcal{B}_{||}\wedge B_2\in\mathcal{B}_{\infty}}$**.** Then there exists $a,b,c\in\mathbb{R}$ such that $B_1=]a,b[$ and $B_2=]c,\infty]$, as $x\in B_1\bigcap B_2$ we have that $a<x<b$ and $c<x$. Define $B=]d,b[$ where $d=\max(a,c)$ then we have $d<x<b$ so that $x\in B\in\mathcal{B}_{||}$. Further if $y\in B$ then $a,c\leqslant d<y<b$ so that $y\in B_1\bigcap B_2$. Hence $x\in B\subseteq B_1\bigcap B_2$ where $B\in\mathcal{B}_{||}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$.

   $\boldsymbol{B_1\in\mathcal{B}_{-\infty}\wedge B_2\in\mathcal{B}_{\infty}}$**.** Then there exists $c_1,c_2\in\mathbb{R}$ such that $B_1=[-\infty,c_1[$ and $B_2=]c_2,\infty]$, as $x\in B_1\bigcap B_2$ we have $c_2<x<c_1$. Define $B=]c_2,c_1[$ then $x\in B\in\mathcal{B}_{||}$. Further if $y\in B$ then $c_2<y<c_1$ so that $y\in B_1\bigcap B_2$. Hence $x\in B\subseteq B_1\bigcap B_2$ where $B\in\mathcal{B}_{||}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$.

   $\boldsymbol{B_1\in\mathcal{B}_{\infty}\wedge B_2\in\mathcal{B}_{\infty}}$**.** Then there exists $c_1,c_2\in\mathbb{R}$ such that $B_1=]c_1,\infty]\wedge B_2=]c_2,\infty]$, as $x\in B_1\bigcap B_2$ we have $c_1<x\wedge c_2<x$. Define $B=]c,\infty]$ where $c=\max(c_1,c_2)\in\mathbb{R}$ then $c<x$ so that $x\in B_{\infty}$. Further if $y\in B$ then $c_1,c_2\leqslant c<y$ so that $y\in B_1\bigcap B_2$. Hence $x\in B\subseteq B_1\bigcap B_2$ where $B\in\mathcal{B}_{||}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$. □

We use now $\mathcal{B}_{\overline{\mathbb{R}}}$ to generate the canonical topology on $\overline{\mathbb{R}}$.

Definition 20.47. *The canonical topology on $\overline{\mathbb{R}}$ noted as $\mathcal{T}_{\overline{\mathbb{R}}}$ is defined as*

$$\mathcal{T}_{\overline{\mathbb{R}}}=\{U\subseteq\overline{\mathbb{R}}|\forall x\in U \text{ we have that } \exists B\in\mathcal{B}_{\overline{\mathbb{R}}} \text{ such that } x\in B\subseteq U\}$$

which is indeed a topology by [theorem: 14.39] and [lemma: 20.46].

It turns out that the subspace topology of $\mathcal{T}_{\overline{\mathbb{R}}}$ [see theorem: 14.8] is equal to the canonical topology $\mathcal{T}_{||}$ on $\mathbb{R}$.

Theorem 20.48. *Let $\mathcal{T}_{\overline{\mathbb{R}}}$ be the canonical topology on $\overline{\mathbb{R}}$ and $\mathcal{T}_{||}$ the canonical topology on $\mathbb{R}$ [see theorem: 14.99] then*

$$\mathcal{T}_{||} \subseteq \mathcal{T}_{\overline{\mathbb{R}}}$$

and

$$\mathcal{T}_{||} = \left\{U \bigcap \mathbb{R} | U \in \mathcal{T}_{\overline{\mathbb{R}}}\right\} \underset{[theorem:\ 14.8]}{=} (\mathcal{T}_{\overline{\mathbb{R}}})_{|\mathbb{R}}$$

Proof. By definition

$$\mathcal{T}_{\overline{\mathbb{R}}} = \{U \subseteq \overline{\mathbb{R}} | \forall x \in U \text{ we have that } \exists B \in \mathcal{B}_{\overline{\mathbb{R}}} \text{ such that } x \in B \subseteq U\}$$

Further by [theorem: 14.99]

$$\mathcal{T}_{||} = \{U \subseteq \mathbb{R} | \forall x \in U \text{ we have that } \exists B \in \mathcal{B}_{||} \text{ such that } x \in B \subseteq U\}$$

where

$$\mathcal{B}_{||} = \{]a, b[|a, b \in \mathbb{R} \wedge a < b\} \subseteq \mathcal{B}_{\overline{\mathbb{R}}}$$

Let $U \in \mathcal{T}_{||}$ then $\forall x \in U$ $\exists B \in \mathcal{B}_{||} \subseteq \mathcal{B}_{\overline{\mathbb{R}}}$ such that $x \in B \subseteq U$ proving that $U \in \mathcal{T}_{\overline{\mathbb{R}}}$, hence we have

$$\mathcal{T}_{||} \subseteq \mathcal{T}_{\overline{\mathbb{R}}} \tag{20.1}$$

If $U \in \mathcal{T}_{||}$ then $U \subseteq \mathbb{R} \Rightarrow U = U \bigcap \mathbb{R}$ and by [eq: 20.1] $U \in \mathcal{T}_{\overline{\mathbb{R}}}$ so that $U \in \{U \bigcap \mathbb{R} | U \in \mathcal{T}_{\overline{\mathbb{R}}}\}$ hence

$$\mathcal{T}_{||} \subseteq \left\{U \bigcap \mathbb{R} | U \in \mathcal{T}_{\overline{\mathbb{R}}}\right\} \tag{20.2}$$

If $U \in \{U \bigcap \mathbb{R} | U \in \mathcal{T}_{\overline{\mathbb{R}}}\}$ then there exist a $V \in \mathcal{T}_{\overline{\mathbb{R}}}$ such that

$$U = V \bigcap \mathbb{R} \tag{20.3}$$

If $x \in U$ then $x \in \mathbb{R}$ and $x \in V \in \mathcal{T}_{\overline{\mathbb{R}}}$, so there exists a $B'_x \in \mathcal{B}_{\overline{\mathbb{R}}}$ such that $x \in B'_x \subseteq V$. For B'_x we have then either:

$B'_x \in \mathcal{B}_{||}$. Then there exist a $a, b \in \mathbb{R}$ such that $B'_x =]a, b[\subseteq \mathbb{R}$ hence if we take $B_x = B'_x$ then

$$x \in B_x = B_x \bigcap \mathbb{R} \subseteq V \bigcap \mathbb{R} = U$$

$B'_x \in \mathcal{B}_{-\infty}$. Then $\exists c \in \mathbb{R}$ such that $B'_x = [-\infty, c[$. Define $B_x =]x-1, c[\subseteq \mathbb{R}$ then $x-1 < x < c$ so that $x \in B_x \in \mathcal{B}_{||}$. Further for $y \in B_x$ we have $y < c$ so that $y \in B'_x$ proving that $x \in B_x \subseteq B'_x$. Hence

$$x \in B_x = B_x \bigcap \mathbb{R} \subseteq V \bigcap \mathbb{R} = U$$

$B'_x \in \mathcal{B}_{\infty}$. Then $\exists c \in \mathbb{R}$ such that $B'_\infty =]c, \infty]$. Define $B_x =]c, x+1[\subseteq \mathbb{R}$ then $c < x < x+1$ so that $x \in B_x \in \mathcal{B}_{||}$. Further for $y \in B_x$ we have $c < y$ so that $y \in B'_x$ proving that $x \in B_x \subseteq B'_x$. Hence

$$x \in B_x = B_x \bigcap \mathbb{R} \subseteq V \bigcap \mathbb{R} = U$$

so in all cases we found a $B_x \in \mathcal{B}_{||}$ such that $x \in B_x \subseteq U$. As x was choose arbitrary it follows that $U \in \mathcal{T}_{||}$. Hence $\{U \bigcap \mathbb{R} | U \in \mathcal{T}_{\overline{\mathbb{R}}}\} \subseteq \mathcal{T}_{||}$ which combined with [eq: 20.2] gives

$$\mathcal{T}_{||} = \left\{U \bigcap \mathbb{R} | U \in \mathcal{T}_{\overline{\mathbb{R}}}\right\}$$

□

Theorem 20.49. *We have the following closed sets in $\mathcal{T}_{\overline{\mathbb{R}}}$*

1. *$\forall x \in \mathbb{R}$ $[-\infty, x]$ is closed.*
2. *$\forall x \in \mathbb{R}$ $[x, \infty]$ is closed,*
3. *$\forall x \in \mathbb{R}$ $\{x\}$ is closed.*
4. *$[-\infty, \infty]$ is closed.*

5. *$\{-\infty,\infty\}$ is closed*
6. *$\{-\infty\}$ is closed.*
7. *$\{\infty\}$ is closed.*

Proof.

1. As

$$\begin{array}{rcl} y\in\overline{\mathbb{R}}\setminus[-\infty,x] & \Leftrightarrow & y\in\overline{\mathbb{R}}\wedge\neg(-\infty\leqslant y\leqslant x)\\ & \Leftrightarrow & y\in\overline{\mathbb{R}}\wedge(y<-\infty\vee x<y)\\ & \Leftrightarrow & y\in\overline{\mathbb{R}}\wedge x<y\\ & \Leftrightarrow & y\in]x,\infty] \end{array}$$

so that $\overline{\mathbb{R}}\setminus[-\infty,x]=]x,\infty]\in\mathcal{T}_{\overline{\mathbb{R}}}$, hence $[-\infty,x]$ is closed.

2. As

$$\begin{array}{rcl} y\in\overline{\mathbb{R}}\setminus[x,\infty] & \Leftrightarrow & y\in\overline{\mathbb{R}}\wedge\neg(x\leqslant y\geqslant\infty)\\ & \Leftrightarrow & y\in\overline{\mathbb{R}}\wedge(y<x\vee\infty<y)\\ & \Leftrightarrow & y\in\overline{\mathbb{R}}\wedge y<x\\ & \Leftrightarrow & y\in[-\infty,x[\end{array}$$

so that $\overline{\mathbb{R}}\setminus[x,\infty]=[-\infty,x[\in\mathcal{T}_{\overline{\mathbb{R}}}$ which proves that $[x,\infty]$ is closed.

3. As $\{x\}=[-\infty,x]\bigcap[x,\infty]$ it follows from (1) and (2) that $\{x\}$ is closed.
4. This is trivial as $\overline{\mathbb{R}}=[-\infty,\infty]$.
5. As $\overline{\mathbb{R}}\setminus\{-\infty,\infty\}=\mathbb{R}\in\mathcal{T}_{||}\subseteq\mathcal{T}_{\overline{\mathbb{R}}}$ it follows that $\{-\infty,\infty\}$ is closed.
6. As $\{-\infty\}=\{-\infty,\infty\}\bigcap[-\infty,0]$ it follows from (2) and (5) that $\{-\infty\}$ is closed.
7. As $\{\infty\}=\{-\infty,\infty\}\bigcap[0,\infty]$ it follows form (1) and (5) that $\{\infty\}$ is closed. □

Theorem 20.50.

1. *If $x\in\mathbb{R}$ then and $U\in\mathcal{T}_{\overline{\mathbb{R}}}$ with $x\in U$ then $\exists\varepsilon\in\mathbb{R}^+$ such that $x\in]x-\varepsilon,x+\varepsilon[\subseteq U$.*
2. *If $U\in\mathcal{T}_{\overline{\mathbb{R}}}$ with $-\infty\in U$ then $\exists\varepsilon\in\mathbb{R}^+$ such that $-\infty\in[-\infty,-\varepsilon[\subseteq U$.*
3. *f $U\in\mathcal{T}_{\overline{\mathbb{R}}}$ with $\infty\in U$ then $\exists\varepsilon\in\mathbb{R}^+$ such that $\infty\in]\varepsilon,\infty]\subseteq U$.*

Proof.

1. If $x\in U$ then as $\mathcal{B}_{\overline{\mathbb{R}}}$ is the generating basis for $\mathcal{T}_{\overline{\mathbb{R}}}$ there exist a $B\in\mathcal{B}_{\overline{\mathbb{R}}}$ such that $x\in B\subseteq U$. For B we have either:

   $B\in\mathcal{B}_{||}$. As $\mathcal{B}_{||}\underset{\text{[theorem: 14.99]}}{=}\{]y-\varepsilon,y+\varepsilon[|y\in\mathbb{R}\wedge\varepsilon\in\mathbb{R}^+\}$ there exists a $y\in\mathbb{R}$ and a $\delta\in\mathbb{R}^+$ so that $B=]y-\delta,y+\delta[$. As $x\in B$ we have $y-\delta<x<y+\delta$, define $\varepsilon=\min(x-(y-\delta),(y+\delta)-x)$ then if $z\in]x-\varepsilon,x+\varepsilon[$ we have

   $$y-\delta=x-(x-(y-\delta))\leqslant x-\varepsilon<z<x+\varepsilon\leqslant x+(y+\delta)-x=y+\delta$$

   so that $x\in B$. Hence $x\in]x-\varepsilon,x+\varepsilon[\subseteq B\subseteq U$.

   $B\in\mathcal{B}_{-\infty}$. Then there exist a $c\in\mathbb{R}$ such that $B=[-\infty,c[$, as $x\in\mathbb{R}$ and $x\in B$ we have $-\infty<x<c$. Take $\varepsilon=c-x\in\mathbb{R}^+$ then if $y\in]x-\varepsilon,x+\varepsilon[\subseteq\mathbb{R}$ we have $-\infty<y<x+\varepsilon=x+(c-x)=c$ so that $x\in]x-\varepsilon,x+\varepsilon[\subseteq]-\infty,c[=B\subseteq U$.

   $B\in\mathcal{B}_{||}$. Then there exist a $c\in\mathbb{R}$ such that $B=]c,\infty]$, as $x\in\mathbb{R}$ and $x\in B$ we have $c<x<\infty$. Take $\varepsilon=x-c\in\mathbb{R}$ then if $y\in]x-\varepsilon,x+\varepsilon[\subseteq\mathbb{R}$ we have $c=x-(x-c)=x-\varepsilon<y<\infty$ so that $x\in]x-\varepsilon,x+\varepsilon[\subseteq]c,\infty]=B\subseteq U$.

2. As $\mathcal{B}_{\overline{\mathbb{R}}}$ is the generating basis for $\mathcal{T}_{\overline{\mathbb{R}}}$ there exists a $B \in \mathcal{B}_{\overline{\mathbb{R}}}$ such that $-\infty \in B \subseteq U$. For B we have either:

   $B \in \mathcal{B}_{||}$. Then there exists $a, b \in \mathbb{R}$ such that $B =]a, b[$, as $-\infty \in B$ we have $-\infty < a < -\infty$ a contradiction. So this case does not apply.

   $B \in \mathcal{B}_{-\infty}$. Then there exists a $c \in \mathbb{R}$ such that $B = [-\infty, c[$, take $\varepsilon = -\min(c, -1) \in \mathbb{R}^+$. If $y \in [-\infty, -\varepsilon[$ we have that $-\infty \leqslant y < -\varepsilon = \min(c, -1) \leqslant c$ so that $-\infty \in [-\infty, -\varepsilon[\subseteq B \subseteq U$.

   $B \in \mathcal{B}_{\infty}$. Then there exists a $c \in \mathbb{R}$ such that $B =]c, \infty]$, as $-\infty \in B$ we have that $-\infty < c < -\infty$ a contradiction. So this case does not apply.

3. As $\mathcal{B}_{\overline{\mathbb{R}}}$ is the generating basis for $\mathcal{T}_{\overline{\mathbb{R}}}$ there exists a $B \in \mathcal{B}_{\overline{\mathbb{R}}}$ such that $\infty \in B \subseteq U$. For B we have either:

   $B \in \mathcal{B}_{||}$. Then there exists $a, b \in \mathbb{R}$ such that $B =]a, b[$, as $\infty \in B$ we have $\infty < b < \infty$ a contradiction. So this case does not apply.

   $B \in \mathcal{B}_{-\infty}$. Then there exists a $c \in \mathbb{R}$ such that $B = [-\infty, c[$, as $\infty \in B$ we have that $\infty < c < \infty$ a contradiction. So this case does not apply.

   $B \in \mathcal{B}_{\infty}$. Then there exists a $c \in \mathbb{R}$ such that $B =]c, \infty]$, take $\varepsilon = \max(1, c) \in \mathbb{R}^+$. If $y \in]\varepsilon, \infty]$ then $c \leqslant \varepsilon < y \leqslant \infty$ so that $\infty \in]\varepsilon, \infty] \subseteq B \subseteq U$. □

We show now that continuous functions with codomain $\mathbb{R}$ are also continuous if we use the codomain $\overline{\mathbb{R}}$.

Theorem 20.51. *Let $\langle X, \mathcal{T} \rangle$ be a topological space and let $f: X \rightarrow \mathbb{R}$ be a continuous function [using the topology $\mathcal{T}_{||}$] then $f: X \rightarrow \overline{\mathbb{R}}$ is also continuous using the topology $\mathcal{T}_{\overline{\mathbb{R}}}$.*

Proof. Let $x \in X$ and let $U \in \mathcal{T}_{\overline{\mathbb{R}}}$ be such that $f(x) \in U$. As $f(X) \subseteq \mathbb{R}$ we have that $f(x) \in \mathbb{R}$, then by [theorem: 20.50] there exist a $\varepsilon \in \mathbb{R}^+$ such that $f(x) \in]f(x) - \varepsilon, f(x) + \varepsilon[\subseteq U$. As $f: X \rightarrow \mathbb{R}$ is continuous and $]f(x) - \varepsilon, f(x) + \varepsilon[$ is open in $\mathbb{R}$ it follows that $V = f^{-1}(]f(x) - \varepsilon, f(x) + \varepsilon[) \in \mathcal{T}$. As $f(x) \in]f(x) - \varepsilon, f(x) + \varepsilon[$ we have that $x \in V$, further

$$f(V) = f(f^{-1}(]f(x) - \varepsilon, f(x) + \varepsilon[)) \subseteq]f(x) - \varepsilon, f(x) + \varepsilon[\subseteq U$$

So using [theorem: 14.124] f is continuous at x, as $x \in X$ was chosen arbitrary it follows that $f: X \rightarrow \overline{\mathbb{R}}$ is continuous. □

Theorem 20.52. *If $U \in \mathcal{T}_{\overline{\mathbb{R}}}$ then for $x \in \mathbb{R}$ we have that*

1. *$x + U = \{x + u | u \in U\} \in \mathcal{T}_{\overline{\mathbb{R}}}$ [in other words if U is open then $x + U$ is open]*
2. *If $x \neq 0$ then $x \cdot U = \{x \cdot u | u \in U\} \in \mathcal{T}_{\overline{\mathbb{R}}}$ [in other words is U is open then $x \cdot U$ is open]*

Proof.

1. Let $y \in x + U$ then there exist a $u \in U$ such that $y = x + u$. For $u \in U$ we have either:

   $u \in \mathbb{R}$. Using [theorem: 20.50] there exists a $\varepsilon \in \mathbb{R}^+$ such that

   $$u \in]u - \varepsilon, u + \varepsilon[\subseteq U$$

   Let $z \in]y - \varepsilon, y + \varepsilon[$ then $(x + u) - \varepsilon = y - \varepsilon < z < y + \varepsilon = (x + u) + \varepsilon$ hence $u - \varepsilon < z - x < u + \varepsilon$ or $z - x \in]u - \varepsilon, u + \varepsilon[\subseteq U$ so that $z = z + (z - x) \in x + U$, hence $y \in]y - \varepsilon, y + \varepsilon[\subseteq x + U$ As $]y - \varepsilon, y + \varepsilon[\in \mathcal{B}_{||} \subseteq B_{\overline{\mathbb{R}}}$ we have by taking $B =]y - \varepsilon, y + \varepsilon[$ we have that

   $$\exists B \in \mathcal{B}_{\overline{\mathbb{R}}} \text{ such that } y \in B \subseteq x + U$$

   $u = -\infty$. Using [theorem: 20.50] there exists a $\varepsilon \in \mathbb{R}^+$ such that

   $$u = -\infty \in [-\infty, -\varepsilon[\subseteq U$$

Let $z\in[-\infty,-\varepsilon+x[$ then $-\infty\leqslant z<-\varepsilon+x$ so that $-\infty\leqslant z-x<-\varepsilon$ proving that $z-x\in[-\infty,-\varepsilon[\subseteq U$, hence $z=x+(z-x)\in x+U$ or $[-\infty,-\varepsilon+x[\subseteq x+U$. As $y=x+u=x+(-\infty)=-\infty$ it follows that $y\in[-\infty,-\varepsilon+x[\subseteq x+U$. Hence if we take $B=[-\infty,-\varepsilon+x[\in\mathcal{B}_{-\infty}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$ we have that

$$\exists B\in\mathcal{B}_{\overline{\mathbb{R}}}\text{ such that }y\in B\subseteq x+U$$

$u\in\infty$. Using [theorem: 20.50] there exists a $\varepsilon\in\mathbb{R}^+$ such that

$$u=\infty\in]\varepsilon,\infty]\subseteq U$$

Let $z\in]\varepsilon+x,\infty]$ then $\varepsilon+x<z\leqslant\infty$ so that $\varepsilon<z-x\leqslant\infty$ proving that $z-x\in]\varepsilon,\infty]\subseteq U$, hence $z=x+(z-x)\in x+U$ or $]\varepsilon+x,\infty]\subseteq x+U$. As $y=x+u=x+\infty=\infty$ it follows that $y\in]\varepsilon+x,\infty]\subseteq x+U$. Hence if we take $B=]\varepsilon+x,\infty]\in\mathcal{B}_{-\infty}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$ we have that

$$\exists B\in\mathcal{B}_{\overline{\mathbb{R}}}\text{ such that }y\in B\subseteq x+U$$

As $y\in x+U$ was chosen arbitrary and in all cases we found a $B\in\mathcal{B}_{\overline{\mathbb{R}}}$ [the generating basis of $\mathcal{T}_{\overline{\mathbb{R}}}$] such that $y\in B\subseteq x+U$ it follows that $x+U\in\mathcal{T}_{\overline{\mathbb{R}}}$.

2. If $y\in x\cdot U$ then there exist a $u\in U$ such that $y=x\cdot y$. For $u\in U$ we have the following possibilities.

$u\in\mathbb{R}$. Using [theorem: 20.50] there exist a $\varepsilon\in\mathbb{R}^+$ such that

$$u\in]u-\varepsilon,u+\varepsilon[\subseteq U$$

For $x\neq 0$ we have now either:

$0<x$. Then $x\cdot\varepsilon\in\mathbb{R}^+$ so that $x\cdot u-x\cdot\varepsilon<x\cdot u<x\cdot u+x\cdot\varepsilon$. so that

$$y=x\cdot u\in]x\cdot u-x\cdot\varepsilon,x\cdot u+x\cdot\varepsilon[$$

Further if $z\in]x\cdot u-x\cdot\varepsilon,x\cdot u+x\cdot\varepsilon[$ then $x\cdot u-x\cdot\varepsilon<z<x\cdot u+x\cdot\varepsilon$ so, as $0<x$, $u-\varepsilon<\frac{z}{x}<x+\varepsilon$. Hence $\frac{z}{x}\in]u-\varepsilon,u+\varepsilon[\subseteq U$ so that $z=x\cdot\frac{z}{x}\in x\cdot U$ proving that

$$y=x\cdot u\in]x\cdot u-x\cdot\varepsilon,x\cdot u+x\cdot\varepsilon[\subseteq x\cdot U$$

Take $B=]x\cdot u-x\cdot\varepsilon,x\cdot u+x\cdot\varepsilon[\in\mathcal{B}_{||}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$ then we have that

$$\exists B\in\mathcal{B}_{\overline{\mathbb{R}}}\text{ such that }y\in B\subseteq x\cdot U$$

$x<0$. Then $-x\cdot\varepsilon\in\mathbb{R}^+$ so that $x\cdot u+x\cdot\varepsilon<x\cdot u<x\cdot u-x\cdot\varepsilon$, so that

$$y=x\cdot u\in]x\cdot u+x\cdot\varepsilon,x\cdot u-x\cdot\varepsilon[$$

Further if $z\in]x\cdot u+x\cdot\varepsilon,x\cdot u-x\cdot\varepsilon[$ then $x\cdot u+x\cdot\varepsilon<z<x\cdot u-x\cdot\varepsilon$ or, as $x<0\Rightarrow\frac{1}{x}<0$ that $u-\varepsilon<\frac{z}{x}<u+\varepsilon$. Hence $\frac{z}{x}\in]u-\varepsilon,u+\varepsilon[\subseteq U$ so that $z=x\cdot\frac{z}{x}\in x\cdot U$ proving that

$$y\in]x\cdot u+x\cdot\varepsilon,x\cdot u-x\cdot\varepsilon[\subseteq x\cdot U$$

Take $B=]x\cdot u+x\cdot\varepsilon,x\cdot u-x\cdot\varepsilon[\in\mathcal{B}_{||}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$ then we have that

$$\exists B\in\mathcal{B}_{\overline{\mathbb{R}}}\text{ such that }y\in B\subseteq x\cdot U$$

As $y\in x\cdot U$ was chosen arbitrary and in all cases we found a $B\in\mathcal{B}_{\overline{\mathbb{R}}}$ [the generating basis of $\mathcal{T}_{\overline{\mathbb{R}}}$] such that $y\in B\subseteq x\cdot U$ it follows that $x\cdot U\in\mathcal{T}_{\overline{\mathbb{R}}}$.

$u=-\infty$. Using [theorem: 20.50] there exist a $\varepsilon\in\mathbb{R}^+$ such that

$$-\infty\in[-\infty,-\varepsilon[\subseteq U$$

For $x\neq 0$ we have either:

$0<x$. then

$$y=x\cdot(-\infty)=-\infty\in[-\infty,-x\cdot\varepsilon[\text{ where }x\cdot\varepsilon\in\mathbb{R}^+$$

further if $z\in[-\infty,-x\cdot\varepsilon[$ then $-\infty\leqslant z<-x\cdot\varepsilon$ so that $-\infty\leqslant\frac{z}{x}<-\varepsilon$ proving that $\frac{z}{x}\in[-\infty,-\varepsilon[\subseteq U$ or that $z=x\cdot\frac{z}{x}\in x\cdot U$. Hence

$$y\in[-\infty,-x\cdot\varepsilon[\subset x\cdot U$$

Take $B=[-\infty,-x\cdot\varepsilon[\in\mathcal{B}_{-\infty}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$ then we have

$$\exists B\in\mathcal{B}_{\overline{\mathbb{R}}}\text{ such that }y\in B\subseteq x\cdot U$$

$x<0$. then

$$y=x\cdot(-\infty)=\infty\in]-x\cdot\varepsilon,\infty]\text{ where }-x\cdot\varepsilon\in\mathbb{R}^+$$

further if $z\in]-x\cdot\varepsilon,\infty]$ then $-x\cdot\varepsilon<z\leqslant\infty$ so that $-\infty=\frac{1}{x}\cdot\infty\leqslant\frac{z}{x}<-x\cdot\varepsilon$ proving that $\frac{z}{x}\in[-\infty,-x\cdot\varepsilon[\subseteq U$ so that $z=x\cdot\frac{z}{x}\in x\cdot U$. Hence

$$y\in]-x\cdot\varepsilon,\infty]\subset x\cdot U$$

Take $B=]-x\cdot\varepsilon,\infty]\in\mathcal{B}_{\infty}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$ then we have

$$\exists B\in\mathcal{B}_{\overline{\mathbb{R}}}\text{ such that }y\in B\subseteq x\cdot U$$

As $y\in x\cdot U$ was chosen arbitrary and in all cases we found a $B\in\mathcal{B}_{\overline{\mathbb{R}}}$ [the generating basis of $\mathcal{T}_{\overline{\mathbb{R}}}$] such that $y\in B\subseteq x\cdot U$ it follows that $x\cdot U\in\mathcal{T}_{\overline{\mathbb{R}}}$.

$u=\infty$. Using [theorem: 20.50] there exist a $\varepsilon\in\mathbb{R}^+$ such that

$$\infty\in]\varepsilon,\infty]\subseteq U$$

For $x\neq 0$ we have either:

$0<x$. then

$$y=x\cdot\infty=\infty\in]x\cdot\varepsilon,\infty]\text{ where }x\cdot\varepsilon\in\mathbb{R}^+$$

further if $z\in]x\cdot\varepsilon,\infty]$ then $x\cdot\varepsilon<z\leqslant\infty$ so that $\varepsilon<\frac{z}{x}\leqslant\infty$, proving that $\frac{z}{x}\in]\varepsilon,\infty]\subseteq U$ or that $z=x\cdot\frac{z}{x}\in x\cdot U$. Hence

$$y\in]x\cdot\varepsilon,\infty]\subset x\cdot U$$

Take $B=]x\cdot\varepsilon,\infty]\in\mathcal{B}_{\infty}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$ then we have

$$\exists B\in\mathcal{B}_{\overline{\mathbb{R}}}\text{ such that }y\in B\subseteq x\cdot U$$

$x<0$. then

$$y=x\cdot\infty=-\infty\in[-\infty,x\cdot\varepsilon[\text{ where }-x\cdot\varepsilon\in\mathbb{R}^+$$

further if $z\in[-\infty,x\cdot\varepsilon[$ then $-\infty\leqslant z<x\cdot\varepsilon$ so that $\varepsilon<\frac{z}{x}\leqslant\infty$, proving that $\frac{z}{x}\in]\varepsilon,\infty]\subseteq U$ so that $z=x\cdot\frac{z}{x}\in x\cdot U$. Hence

$$y\in[-\infty,x\cdot\varepsilon[\subset x\cdot U$$

Take $B=[-\infty,x\cdot\varepsilon[\in\mathcal{B}_{-\infty}\subseteq\mathcal{B}_{\overline{\mathbb{R}}}$ then we have that

$$\exists B\in\mathcal{B}_{\overline{\mathbb{R}}}\text{ such that }y\in B\subseteq x\cdot U$$

As $y\in x\cdot U$ was chosen arbitrary and in all cases we found a $B\in\mathcal{B}_{\overline{\mathbb{R}}}$ [the generating basis of $\mathcal{T}_{\overline{\mathbb{R}}}$] such that $y\in B\subseteq x\cdot U$ it follows that $x\cdot U\in\mathcal{T}_{\overline{\mathbb{R}}}$. □

Theorem 20.53. *$\mathfrak{T}_{\overline{\mathbb{R}}}$ is Hausdorff [see definition: 14.203].*

Proof. Let $x,y\in\overline{\mathbb{R}}$ with $x\neq y$ then we have to check the following cases for x,y:

$x\in\mathbb{R}\wedge y\in\mathbb{R}$. Then we have as $x\neq y$ we have either:

$x<y$. Take $\varepsilon=\frac{y-x}{2}\in\mathbb{R}^+$ then $x\in]x-\varepsilon,x+\varepsilon[\in\mathcal{T}_{\mathbb{R}}$ and $y\in]y-\varepsilon,y+\varepsilon[\in\mathcal{T}_{\mathbb{R}}$. Further

$$]x-\varepsilon,x+\varepsilon[=\left]x-\varepsilon,x+\frac{y-x}{2}\right[=\left]x-\varepsilon,\frac{x+y}{2}\right[$$

and

$$]y-\varepsilon, y+\varepsilon[=\left]y-\frac{y-x}{2}, y+\varepsilon\right[=\left]\frac{x+y}{2}, y+\varepsilon\right[$$

so that $]x-\varepsilon, x+\varepsilon[\bigcap]y-\varepsilon, y+\varepsilon[=\varnothing$.

$\boldsymbol{y<x}$**.** Take $\varepsilon=\frac{x-y}{2}\in\mathbb{R}^+$ then $x\in]x-\varepsilon, x+\varepsilon[\in\mathcal{T}_{\overline{\mathbb{R}}}$ and $y\in]y-\varepsilon, y+\varepsilon[\in\mathcal{T}_{\overline{\mathbb{R}}}$. Further

$$]x-\varepsilon, x+\varepsilon[=\left]x-\frac{x-y}{2}, x+y\right[=\left]\frac{x+y}{2}, x+y\right[$$

and

$$]y-\varepsilon, y+\varepsilon[=\left]y-\varepsilon, y+\frac{x-y}{2}\right[=\left]y-\varepsilon, \frac{x+y}{2}\right[$$

so that $]x-\varepsilon, x+\varepsilon[\bigcap]y-\varepsilon, y+\varepsilon[=\varnothing$.

$\boldsymbol{x=-\infty\wedge y\in\mathbb{R}}$**.** Then $x=-\infty\in[-\infty, y-1[\in\mathcal{T}_{\overline{\mathbb{R}}}$, $y\in]y-1, y+1[\in\mathcal{T}_{\overline{\mathbb{R}}}$ and $[-\infty, y-1[\bigcap]y-1, y+1[=\varnothing$.

$\boldsymbol{x=\infty\wedge y\in\mathbb{R}}$**.** Then $x=\infty\in]y+1, \infty]\in\mathcal{T}_{\overline{\mathbb{R}}}$, $y\in]y-1, y+1[\in\mathcal{T}_{\overline{\mathbb{R}}}$ and $]y+1, \infty[\bigcap]y-1, y+1[=\varnothing$.

$\boldsymbol{x\in\mathbb{R}\wedge y=-\infty}$**.** Then $x\in]x-1, x+1[\in\mathcal{T}_{\overline{\mathbb{R}}}$, $y=-\infty\in[-\infty, x-1[\in\mathcal{T}_{\overline{\mathbb{R}}}$ and $]x-1, x+1[\bigcap[-\infty, x-1[=\infty$.

$\boldsymbol{x=-\infty\wedge y=-\infty}$**.** As $x\neq y$ this is not a valid case.

$\boldsymbol{x=\infty\wedge y=-\infty}$**.** Then $x\in]1,\infty]\in\mathcal{T}_{\overline{\mathbb{R}}}$, $y\in[-\infty, 1[\in\mathcal{T}_{\overline{\mathbb{R}}}$ and $]1,\infty]\bigcap[-\infty, 1[=\infty$.

$\boldsymbol{x\in\mathbb{R}\wedge y=\infty}$**.** Then $x\in]x-1, x+1[\in\mathcal{T}_{\overline{\mathbb{R}}}$, $y\in]x+1, \infty]\in\mathcal{T}_{\overline{\mathbb{R}}}$ and $]x-1, x+1[\bigcap]x+1, \infty]=\varnothing$.

$\boldsymbol{x=-\infty\wedge y=\infty}$**.** Then $x\in[-\infty, 1[\in\mathcal{T}_{\overline{\mathbb{R}}}$, $\in]1,\infty]\in\mathcal{T}_{\overline{\mathbb{R}}}$ and $[-\infty, 1[\bigcap]1,\infty]=\varnothing$.

$\boldsymbol{x=\infty\wedge y=\infty}$**.** As $x\neq y$ this is not a valid case □

20.1.4 Sequences in $\overline{\mathbb{R}}$

We will now extend the concept of a limit of sequences in $\mathbb{R}$ to the limit of sequences in $\overline{\mathbb{R}}$ in such a way that the limit of increasing/decreasing sequences always exists. So if we have a series of non negative numbers then the limit of the partial sums will always have a limit, in other words series of non negative numbers always converges. To do this we make use of the fact that in $\overline{\mathbb{R}}$ every non empty subset has a supremum and infinum and define the limit of a sequence in terms of supremum and infinum.

Theorem 20.54. *Let $k\in\mathbb{N}_0$, $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ be a sequence of extended real numbers then we have:*

1. $\inf(\{x_i|i\in\{k,\ldots,\infty\}\})$ *and* $\sup(\{x_i|i\in\{k,\ldots,\infty\}\})$ *exists*
2. $\inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})$ *exists*
3. $\sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})$ *exists*
4. $\forall m\in\{k,\ldots,\infty\}$ *we have*

   $$\sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})=\sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{m,\ldots,\infty\}\}$$

5. $\forall m\in\{k,\ldots,\infty\}$ *we have*

   $$\inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})=\inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{m,\ldots,\infty\}\}$$

Proof.

1. As $\{x_i|i\in\{k,\ldots,\infty\}\}\neq\varnothing$ this follows from [theorem: 20.13].
2. As $\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}\neq\varnothing$ this this follows from [theorem: 20.13].

3. As $\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}\neq\varnothing$ this follows from [theorem: 20.13].
4. Given $l\in\{k,\ldots,\infty\}$ take $S_l=\inf(\{x_i|i\in\{l,\ldots,\infty\}\})$ then we have

$$\{S_l|l\in\{m,\ldots,\infty\}\}\subseteq\{S_l|l\in\{k,\ldots,\infty\}\}$$

so that by [theorem: 3.77]

$$\sup(\{S_l|l\in\{m,\ldots,\infty\}\})\leqslant\sup(\{S_l|l\in\{k,\ldots,\infty\}\}) \tag{20.4}$$

For the opposite inclusion, given $l\in\{k,\ldots,\infty\}$ take $n=\max(m,l)$ then $m,l\leqslant n$ so that

$$\{x_i|i\in\{l,\ldots,\infty\}\}\subseteq\{x_i|i\in\{n,\ldots,\infty\}\}$$

or using [theorem: 20.13]

$$S_n=\inf(\{x_i|i\in\{n,\ldots,\infty\}\})\leqslant\inf(\{x_i|i\in\{l,\ldots,\infty\}\})=S_l$$

Hence for every $S_l\in\{S_l|l\in\{k,\ldots,\infty\}\}$ we find a $S_n\in\{S_l|l\in\{m,\ldots,\infty\}\}$ such that $S_n\leqslant S_l$ which by [theorem: 3.78] proves that $\sup(\{S_l|l\in\{k,\ldots,\infty\}\})\leqslant\sup(\{S_l|l\in\{m,\ldots,\infty\}\})$. Combining this result with [eq: 20.4] gives

$$\sup(\{S_l|l\in\{m,\ldots,\infty\}\})=\sup(\{S_l|l\in\{k,\ldots,\infty\}\})$$

or

$$\sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})=\sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{m,\ldots,\infty\}\}$$

5. Given $l\in\{k,\ldots,\infty\}$ take $S_l=\sup(\{x_i|i\in\{l,\ldots,\infty\}\})$ then we have

$$\{S_l|l\in\{m,\ldots,\infty\}\}\subseteq\{S_l|l\in\{k,\ldots,\infty\}\}$$

so that by [theorem: 3.77]

$$\inf(\{S_l|l\in\{k,\ldots,\infty\}\})\leqslant\inf(\{S_l|l\in\{m,\ldots,\infty\}\}) \tag{20.5}$$

For the opposite inclusion, given $l\in\{k,\ldots,\infty\}$ take $n=\max(m,l)$ then $m,l\leqslant n$ so that

$$\{x_i|i\in\{l,\ldots,\infty\}\}\subseteq\{x_i|i\in\{n,\ldots,\infty\}\}$$

or using [theorem: 20.13]

$$S_n=\sup(\{x_i|i\in\{l,\ldots,\infty\}\})\leqslant\sup(\{x_i|i\in\{n,\ldots,\infty\}\})=S_l$$

Hence for every $S_l\in\{S_l|l\in\{k,\ldots,\infty\}\}$ we find a $S_n\in\{S_l|l\in\{m,\ldots,\infty\}\}$ such that $S_l\leqslant S_n$ which by [theorem: 3.78] proves that $\inf(\{S_l|l\in\{m,\ldots,\infty\}\})\leqslant\inf(\{S_l|l\in\{k,\ldots,\infty\}\})$. Combining this result with [eq: 20.5] gives

$$\inf(\{S_l|l\in\{m,\ldots,\infty\}\})=\inf(\{S_l|l\in\{k,\ldots,\infty\}\})$$

or

$$\inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})=\inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{m,\ldots,\infty\}\}\quad\square$$

Using the above theorem we can then define:

Definition 20.55. *Let $k\in\mathbb{N}_0$, $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ be a sequence of extended real numbers then we define:*

$$\liminf_{i\to\infty}x_i=\sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})$$

$$\limsup_{i\to\infty}x_i=\inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})l\in\{k,\ldots,\infty\}\})$$

***Note* 20.56.** () *The previous theorem [theorem: 20.54] ensures that the above definition is well defined for every sequence of extended real numbers. Note also that the notations $\liminf_{i\to\infty}x_i$, $\limsup_{i\to\infty}x_i$ do not mention the starting index k, which is indeed not needed because of [theorem: 20.54 (4),(5)].*

Example 20.57. Let $k\in\mathbb{N}_0$, $x\in\overline{\mathbb{R}}$ then for $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ defined by $x_i=x$ we have that

$$\liminf_{i\rightarrow\infty} x_i = x = \limsup_{i\rightarrow\infty} x_i$$

Proof.

$$\begin{aligned}
\liminf_{i\rightarrow\infty} x_i &= \sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\
&= \sup(\{\inf(\{x\})|l\in\{k,\ldots,\infty\}\}) \\
&= \sup(\{x|n\in\{k,\ldots,\infty\}\}) \\
&= \sup(\{x\}) \\
&= x \\
&= \inf(\{x\}) \\
&= \inf(\{x|n\in\{k,\ldots,\infty\}\}) \\
&= \inf(\{\sup(\{x\})|l\in\{k,\ldots,\infty\}\}) \\
&= \inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\
&= \limsup_{n\rightarrow\infty} x_n
\end{aligned}$$

□

Theorem 20.58. *Let $k\in\mathbb{N}_0$, $x\in\overline{\mathbb{R}}$ then for $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ we have $\forall n\in\{k,\ldots,\infty\}$ that*

$$\liminf_{i\rightarrow\infty} x_i = \liminf_{i\rightarrow\infty} x_{i+n} \text{ and } \limsup_{i\rightarrow\infty} x_i = \limsup_{i\rightarrow\infty} x_{i+n}$$

Proof. Let $l\in\{k,\ldots,\infty\}$ then we have

$$\begin{aligned}
x\in\{x_{i+n}|i\in\{l,\ldots,\infty\}\} &\Leftrightarrow x=x_{i+n} \text{ where } i\in\{l,\ldots,\infty\} \\
&\Leftrightarrow x=x_i \text{ where } i\in\{l+n,\ldots,\infty\} \\
&\Leftrightarrow x\in\{x_i|i\in\{l+n,\ldots,\infty\}\}
\end{aligned}$$

so that

$$\forall l\in\{k,\ldots,\infty\}\ \{x_{i+n}|i\in\{l,\ldots,\infty\}\}=\{x_i|i\in\{l+n,\ldots,\infty\}\} \tag{20.6}$$

Let $l\in\{k,\ldots,\infty\}$ and define $I_l=\inf(\{x_i|i\in\{l,\ldots,\infty\}\})$ and $S_l=\sup(\{x_i|i\in\{l,\ldots,\infty\}\})$ then we have

$$\begin{aligned}
x\in\{I_{l+n}|l\in\{k,\ldots,\infty\}\} &\Leftrightarrow x=I_{l+n} \text{ where } l\in\{k,\ldots,\infty\} \\
&\Leftrightarrow x=I_l \text{ where } l\in\{k+n,\ldots,\infty\} \\
&\Leftrightarrow x\in\{I_l|l\in\{k+n,\ldots,\infty\}\} \\
x\in\{S_{l+n}|l\in\{k,\ldots,\infty\}\} &\Leftrightarrow x=S_{l+n} \text{ where } l\in\{k,\ldots,\infty\} \\
&\Leftrightarrow x=S_l \text{ where } l\in\{k+n,\ldots,\infty\} \\
&\Leftrightarrow x\in\{S_l|l\in\{k+n,\ldots,\infty\}\}
\end{aligned}$$

so that

$$\{I_{l+n}|l\in\{k,\ldots,\infty\}\}=\{I_l|l\in\{k+n,\ldots,\infty\}\}\wedge\{S_{l+n}|l\in\{k,\ldots,n\}\}=\{S_l|l\in\{k+n,\ldots,\infty\}\} \tag{20.7}$$

So

$$\begin{aligned}
\liminf_{i\rightarrow\infty} x_{i+n} &= \sup(\{\inf(\{x_{i+n}|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\
&\underset{[\text{eq: } 20.6]}{=} \sup(\{\inf(\{x_i|i\in\{l+n,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\
&\underset{[\text{eq: } 20.7]}{=} \sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k+n,\ldots,\infty\}\}) \\
&\underset{[\text{theorem: } 20.54]}{=} \sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\
&= \liminf_{i\rightarrow\infty} x_i
\end{aligned}$$

and

$$\begin{aligned}\limsup_{i\rightarrow\infty} x_{i+n} &= \inf(\{\sup(\{x_{i+n}|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ &\underset{[\text{eq: }20.6]}{=} \inf(\{\sup(\{x_i|i\in\{l+n,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ &\underset{[\text{eq: }20.7]}{=} \inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k+n,\ldots,\infty\}\}) \\ &\underset{[\text{theorem: }20.54]}{=} \inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ &= \limsup_{i\rightarrow\infty} x_i\end{aligned}$$

□

Theorem 20.59. *Let $k\in\mathbb{N}_0$, $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ and $\{y_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ be such that $\forall i\in\{k,\ldots,\infty\}$ $x_i\leqslant y_i$ then*

1. $\liminf_{i\rightarrow\infty} x_i\leqslant\liminf_{i\rightarrow\infty} y_i$.
2. $\limsup_{i\rightarrow\infty} x_i\leqslant\limsup_{i\rightarrow\infty} y_i$.

Proof.

1. Let $l\in\{k,\ldots,\infty\}$ then using [theorem: 3.78] we have that

$$\inf(\{x_i|i\in\{l,\ldots,\infty\}\})\leqslant\inf(\{y_i|i\in\{l,\ldots,\infty\}\})$$

Applying [theorem: 3.78] again we have that

$$\begin{aligned}\liminf_{n\rightarrow\infty} x_n &\underset{\text{def}}{=} \sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ &\leqslant \sup(\{\inf(\{y_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}|\}) \\ &= \liminf_{n\rightarrow\infty} y_n\end{aligned}$$

2. Let $l\in\{k,\ldots,\infty\}$ then using [theorem: 3.78] we have that

$$\sup(\{x_i|i\in\{l,\ldots,\infty\}\})\leqslant\sup(\{y_i|i\in\{l,\ldots,\infty\}\})$$

Applying [theorem: 3.78] again we have that

$$\begin{aligned}\liminf_{n\rightarrow\infty} x_n &\underset{\text{def}}{=} \sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ &\leqslant \sup(\{\inf(\{y_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}|\}) \\ &= \liminf_{n\rightarrow\infty} y_n\end{aligned}$$

□

Theorem 20.60. *Let $k\in\mathbb{N}_0$, $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ and $x\in\overline{\mathbb{R}}$ then we have:*

1. *If* $\forall i\in\{k,\ldots,\infty\}$ *we have* $x\leqslant x_i$ *then* $x\leqslant\liminf_{i\rightarrow\infty} x_i$ *and* $x\leqslant\limsup_{i\rightarrow\infty} x_i$.
2. *If* $\forall i\in\{k,\ldots,\infty\}$ *we have* $x_i\leqslant x$ *then* $\liminf_{i\rightarrow\infty} x_i\leqslant x$ *and* $\limsup_{i\rightarrow\infty} x_i\leqslant x$.

Proof.

1. Let $l\in\{k,\ldots,\infty\}$ then $\forall i\in\{l,\ldots,\infty\}\subseteq\{k,\ldots,\infty\}$ we have $x\leqslant x_i$ so that

$$x\leqslant\inf(\{x_i|i\in\{l,\ldots,\infty\}\}) \text{ and } x\leqslant\sup(\{x_i|i\in\{l,\ldots,\infty\}\})$$

hence we have that

$$x\leqslant\sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})=\liminf_{i\rightarrow\infty} x_i$$

and

$$x\leqslant\inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})=\limsup_{i\rightarrow\infty} x_i$$

2. Let $l \in \{k, \ldots, \infty\}$ then $\forall i \in \{l, \ldots, \infty\} \subseteq \{k, \ldots, \infty\}$ we have $x_i \leqslant x$ so that

$$\inf(\{x_i | i \in \{l, \ldots, \infty\}\}) \leqslant x \text{ and } \sup(\{x_i | i \in \{l, \ldots, \infty\}\}) \leqslant x$$

hence we have that

$$\liminf_{i \to \infty} x_i = \sup(\{\inf(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\}) \leqslant x$$

and

$$\limsup_{i \to \infty} x_i = \inf(\{\sup(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\}) \leqslant x$$

□

Lemma 20.61. *Let $\varnothing \neq S \subseteq \overline{\mathbb{R}}$ then we have:*

1. *If $y \in \mathbb{R}$ then*
   a. $\sup(S) + y$ *is well defined then for $S + y = \{s + y = s \in S\}$ we have that*
   $$\sup(S + y) = \sup(S) + y$$
   b. $\inf(S) + y$ *is well defined then for $S + y = \{s + y = s \in S\}$ we have that*
   $$\inf(S + y) = \inf(S) + y$$
2. *If $y \in [0, \infty]$ and $S \subseteq [0, \infty]$ we have that $\sup(S) + y$ is well defined and for $S + y = \{s + y | s \in S\}$ we have that*
$$\sup(S + y) = \sup(S) + y$$
3. *If $y \in [0, \infty]$ and $S \subseteq [0, \infty]$ we have that $\inf(S) + y$ is well defined and for $S + y = \{s + y | s \in S\}$ we have that*
$$\inf(S + y) = \inf(S) + y$$

Proof.

1. As $y \in \mathbb{R}$ we have $\forall x \in \overline{\mathbb{R}}$ that $x + y$ is well defined, $\sup(S)$ exists and $\sup(S) \in \overline{\mathbb{R}}$ it follows that $\sup(S + y)$ is well defined and $S + y$ is well defined.
   a. $\forall s \in S$ we have that $s \leqslant \sup(S)$ so by [theorem: 20.29] $s + y \leqslant \sup(S)$ so that
   $$\sup(\{s + y | s \in S\}) \leqslant \sup(S) + y \tag{20.8}$$
   Assume that $\sup(\{s + y | s \in S\}) < \sup(S) + y$ then by [theorem: 20.28]
   $$\sup(\{s + y | s \in S\}) - y < \sup(S)$$
   so there exist a $s \in S$ such that $\sup(\{s + y | s \in S\}) - y < s$ so that by [theorem: 20.29] $\sup(\{s + y | s \in S\}) < s + y \leqslant \sup(\{s + y | s \in S\})$ leading to the contradiction $\sup(\{s + y | s \in S\}) < \sup(\{s + y | s \in S\})$. Hence the assumption is wrong and we must have $\sup(S) + y \leqslant \sup(\{s + y | s \in S\})$. Combining this with [eq: 20.8] gives
   $$\sup(S + y) = \sup(\{s + y | s \in S\}) = \sup(S) + y$$
   b. $\forall s \in S$ we have that $\inf(S) \leqslant s$ so by [theorem: 20.29] $\inf(S) + y \leqslant s + y$ so that
   $$\inf(S) + y \leqslant \inf(\{s + y | s \in S\}) \tag{20.9}$$
   Assume that $\inf(S) + y < \inf(\{s + y | s \in S\})$ then by [theorem: 20.28]
   $$\inf(S) < \inf(\{s + y | s \in S\}) - y$$
   so there exist a $s \in S$ such that $s < \inf(\{s + y | s \in S\}) - y$ so that by [theorem: 20.29] $\inf(\{s + y | s \in S\}) \leqslant s + y < \inf(\{s + y | s \in S\})$ leading to the contradiction $\inf(\{s + y | s \in S\}) < \inf(\{s + y | s \in S\})$. Hence the assumption is wrong and we must have $\inf(\{s + y | s \in S\}) \leqslant \inf(S) + y$. Combining this with [eq: 20.9] gives
   $$\inf(S + y) = \inf(\{s + y | s \in S\}) = \inf(S) + y$$

2. For $y \in [0, \infty]$ we have either:

   $y \in \mathbb{R}$. Then by (1.a) we have that $\sup(A) + y$ is well defined and

$$\sup(\{s + y | s \in S\}) = \sup(S) + y$$

   $y = \infty$. Then as $S \subseteq [0, \infty]$ we have $\forall s \in S$ that $s + y = \infty$ so that $\{s + y | s \in S\} = \{\infty\}$ so that $\sup(\{s + y | s \in S\}) = \infty$. Further as $\forall s \in S$ we have $0 \leqslant s$ it follows that $-\infty < 0 \leqslant \sup(S)$ so that $\infty = \sup(S) + \infty = \sup(S) + y$ is well defined. Hence

$$\sup(\{s + y | s \in S\}) = \infty = \sup(S) + \infty = \sup(S) + y$$

3. For $y \in [-\infty, 0]$ we have either:

   $y \in \mathbb{R}$. Then by (1.b) we have that $\inf(A) + y$ is well defined and

$$\inf(\{s + y | s \in S\}) = \inf(S) + y$$

   $y = -\infty$. Then as $S \subseteq [-\infty, 0]$ we have $\forall s \in S$ that $s + y = -\infty$ so that $\{s + y | s \in S\} = \{-\infty\}$ so that $\inf(\{s + y | s \in S\}) = -\infty$. Further as $\forall s \in S$ we have $s \leqslant 0$ it follows that $\inf(S) \leqslant 0 < \infty$ so that $-\infty = \inf(S) + (-\infty) = \inf(S) + y$ is well defined. Hence

$$\inf(\{s + y | s \in S\}) = -\infty = \inf(S) + (-\infty) = \sup(S) + y \qquad \square$$

Theorem 20.62. *Let $k \in \mathbb{N}_0$, $\{x_i\}_{i \in \{k, \ldots, \infty\}} \subseteq \overline{\mathbb{R}}$, $x \in \mathbb{R}$ [hence $\forall i \in \{k, \ldots, \infty\}$ we have that $x_i + x$ is well defined] then for $\{x_i + x\}_{i \in \{k, \ldots, \infty\}} \subseteq \overline{\mathbb{R}}$ we have*

1. $\liminf_{i \to \infty} (x_i + x) = \left(\liminf_{i \to \infty} x_i\right) + x$
2. $\limsup_{i \to \infty} (x_i + x) = \left(\limsup_{i \to \infty} x_i\right) + x$

Proof.

1. We have

$$\begin{aligned}
\liminf_{i \to \infty} (x_i + x) &= \sup(\{\inf(\{x_i + x | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\}) \\
&\underset{[\text{theorem: } 20.61]}{=} \sup(\{\inf(\{x_i | i \in \{l, \ldots, \infty\}\}) + x | l \in \{k, \ldots, \infty\}\}) \\
&\underset{[\text{theorem: } 20.61]}{=} \sup(\{\inf(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\}) + x \\
&= \left(\liminf_{i \to \infty} x_i\right) + x
\end{aligned}$$

2. We have

$$\begin{aligned}
\limsup_{i \to \infty} (x_i + x) &= \inf(\{\sup(\{x_i + x | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\}) \\
&\underset{[\text{theorem: } 20.61]}{=} \inf(\{\sup(\{x_i | i \in \{l, \ldots, \infty\}\}) + x | l \in \{k, \ldots, \infty\}\}) \\
&\underset{[\text{theorem: } 20.61]}{=} \inf(\{\sup(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\}) + x \\
&= \left(\limsup_{i \to \infty} x_i\right) + x
\end{aligned}$$

$\square$

Theorem 20.63. *Let $\varnothing \neq S \subseteq \overline{\mathbb{R}}$ then*

1. $-\sup(S) = \inf(\{-s | s \in S\})$
2. $-\inf(S) = \sup(\{-s | s \in S\})$
3. *If $\alpha \in [0, \infty[$ then* $\alpha \cdot \sup(S) = \sup(\{\alpha \cdot s | s \in S\})$
4. *If $\alpha \in [0, \infty[$ then* $\alpha \cdot \inf(S) = \inf(\{\alpha \cdot s | s \in S\})$

Proof.

1. Let $s \in S$ then by [theorem: 20.34 $s \leqslant \sup(S)$ so that $-\sup(S) \leqslant -s$ so that
$$-\sup(S) \leqslant \inf(\{-s|s \in S\} \tag{20.10}$$
Assume that $-\sup(S) < \inf(\{-s|s \in S\}$ then by [corollary: 20.38] $-\inf(\{-s|s \in S\}) < \sup(S)$ so that there exist a $s \in S$ such that $-\inf(\{-s|s \in S\}) < s$ so that by [corollary: 20.38] again we have $-s < \inf(\{-s|s \in S\})$, hence $\inf(\{-s|s \in S\}) < -s < \inf(\{-s|s \in S\})$ a contradiction. So the assumption must be false and we have $\inf(\{-s|s \in S\}) \leqslant -\sup(S)$, combining this with [eq: 20.10] proves that
$$-\sup(S) = \inf(\{-s|s \in S\})$$

2. Take $A = \{-s|s \in S\}$ then $S = \{-s|s \in A\}$ then
$$-\inf(S) = -\inf(\{-s|s \in A\}) \underset{(1)}{=} -(-\sup(A)) = \sup(\{-s|s \in S\})$$

3. For $\alpha \in [0, \infty[$ we have either:

   $\boldsymbol{\alpha = 0}$**.** Then $\{\alpha \cdot s|s \in S\} = \{0\}$ so that
$$\sup(\{\alpha \cdot s|s \in S\}) = \sup(\{0\}) = 0 = 0 \cdot \sup(S) = \alpha \cdot \sup(S)$$
   proving that $\alpha \cdot \sup(S) = \sup(\{\alpha \cdot s|s \in S\})$.

   $\boldsymbol{\alpha \neq 0}$**.** Let $s \in S$ then $s \leqslant \sup(s)$ then, as $0 < \alpha$, we have by [theorem: 20.37] that $\alpha \cdot s \leqslant \alpha \cdot \sup(S)$. Hence
$$\sup(\{\alpha \cdot s|s \in S\}) \leqslant \alpha \cdot \sup(S) \tag{20.11}$$
   Assume that $\sup(\{\alpha \cdot s|s \in S\}) < \alpha \cdot \sup(S)$ then as $0 < \alpha \in \mathbb{R}$ we have $0 < \frac{1}{\alpha}$ so that by [theorem: 20.37] $\frac{1}{\alpha} \cdot \sup(\{\alpha \cdot s|s \in S\}) < \sup(S)$. Hence there exist a $s \in S$ such that $\frac{1}{\alpha} \cdot \sup(\{\alpha \cdot s|s \in S\}) < s$ so that by [theorem: 20.37] again we have $\sup(\{\alpha \cdot s|s \in S\}) < \alpha \cdot s \leqslant \sup(\{\alpha \cdot s|s \in S\})$ a contradiction. Hence the assumption must be false and we have that $\alpha \cdot \sup(S) \leqslant \sup(\{\alpha \cdot s|s \in S\})$, combining this with [eq: 20.11] we have
$$\alpha \cdot \sup(S) = \sup(\{\alpha \cdot s|s \in S\})$$

4. For $\alpha \in [0, \infty[$ we have either:

   $\boldsymbol{\alpha = 0}$**.** Then $\{\alpha \cdot s|s \in S\} = \{0\}$ so that
$$\inf(\{\alpha \cdot s|s \in S\}) = \inf(\{0\}) = 0 = 0 \cdot \inf(S) = \alpha \cdot \inf(S)$$
   proving that $\alpha \cdot \inf(S) = \inf(\{\alpha \cdot s|s \in S\})$.

   $\boldsymbol{\alpha \neq 0}$**.** Let $s \in S$ then $\inf(s) \leqslant s$ then, as $0 < \alpha$, we have by [theorem: 20.37] that $\alpha \cdot \inf(S) \leqslant \alpha \cdot s$. Hence
$$\alpha \cdot \inf(S) \leqslant \inf(\{\alpha \cdot s|s \in S\}) \tag{20.12}$$
   Assume that $\alpha \cdot \inf(S) < \inf(\{\alpha \cdot s|s \in S\})$ then as $0 < \alpha \in \mathbb{R}$ we have $0 < \frac{1}{\alpha}$ so that by [theorem: 20.37] $\inf(S) < \frac{1}{\alpha} \cdot \inf(\{\alpha \cdot s|s \in S\})$. Hence there exist a $s \in S$ such that $s < \frac{1}{\alpha} \cdot \inf(\{\alpha \cdot s|s \in S\})$ so that by [theorem: 20.37] again we have $\inf(\{\alpha \cdot s|s \in S\}) \leqslant \alpha \cdot s < \inf(\{\alpha \cdot s|s \in S\})$ a contradiction. Hence the assumption must be false and we have that $\inf(\{\alpha \cdot s|s \in S\}) \leqslant \alpha \cdot \inf(S)$, combining this with [eq: 20.12] we have
$$\alpha \cdot \inf(S) = \inf(\{\alpha \cdot s|s \in S\})$$
□

Theorem 20.64. *Let $k \in \mathbb{N}_0$, $\{x_i\}_{i \in \{k, \ldots, \infty\}} \subseteq \overline{\mathbb{R}}$ then we have:*

1. $\liminf_{i \to \infty}(-x_i) = -\limsup_{i \to \infty} x_i$.

2. $\limsup_{i\rightarrow\infty}(-x_i) = -\liminf_{i\rightarrow\infty} x_i$.

Proof.

1.

$$\begin{aligned}\liminf_{i\rightarrow\infty}(-x_i) &= \sup(\{\inf(\{-x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ &\underset{[\text{theorem: } 20.63]}{=} \sup(\{-\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ &\underset{[\text{theorem: } 20.63]}{=} -\inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ &= -\limsup_{i\rightarrow\infty} x_i\end{aligned}$$

2.

$$\begin{aligned}\limsup_{i\rightarrow\infty}(-x_i) &= \inf(\{\sup(\{-x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ &\underset{[\text{theorem: } 20.63]}{=} \inf(\{-\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ &\underset{[\text{theorem: } 20.63]}{=} -\sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ &= -\liminf_{i\rightarrow\infty} x_i\end{aligned}$$

□

Lemma 20.65. *Let $k\in\mathbb{N}_0$, $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\mathbb{R}$, $\{y_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\mathbb{R}$ such that $\forall i\in\{k,\ldots,\infty\}$ we have [so that x_i+y_i is well defined] then we have:*

1. *If $\forall i\in\{k,\ldots,\infty\}$ $x_i\leqslant x_{i+1}\wedge y_i\leqslant y_{i+1}$ then*

$$\sup(\{x_i|i\in\{k,\ldots,\infty\}\})+\sup(\{y_i|i\in\{k,\ldots,\infty\}\})=\sup(\{x_i+y_i|i\in\{k,\ldots,\infty\}\})$$

2. *If $\forall i\in\{k,\ldots,\infty\}$ $x_{i+1}\leqslant x_i\wedge y_{i+1}\leqslant y_i$ then*

$$\inf(\{x_i|i\in\{k,\ldots,\infty\}\})+\inf(\{y_i|i\in\{k,\ldots,\infty\}\})=\inf(\{x_i+y_i|i\in\{k,\ldots,\infty\}\})$$

Proof.

1. As $\forall i\in\{k,\ldots,\infty\}$ $x_i\leqslant\sup(\{x_i|i\in\{k,\ldots,\infty\}\})$ and $y_i\leqslant\sup(\{y_i|i\in\{k,\ldots,\infty\}\})$ so that by [theorem: 20.29] that

$$x_i+y_i\leqslant\sup(\{x_i|i\in\{k,\ldots,\infty\}\})+y_i \tag{20.13}$$

By [theorem: 20.14] and the fact that $\{x_i|i\in\{k,\ldots,\infty\}\},\{y_i|i\in\{k,\ldots,\infty\}\}\neq-\infty$ so that

$$(y_i,\sup(\{x_i|i\in\{k,\ldots,\infty\}\}))\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$$

and

$$(\sup(\{y_i|i\in\{k,\ldots,\infty\}\}),\sup(\{x_i|i\in\{k,\ldots,\infty\}\}))\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$$

Hence by [theorem: 20.29] we have that

$$y_i+\sup(\{x_i|i\in\{k,\ldots,\infty\}\})\leqslant\sup(\{y_i|i\in\{k,\ldots,\infty\}\})+\sup(\{x_i|i\in\{k,\ldots,\infty\}\})$$

Using the above, [eq: 20.13] and commutativity we have that

$$x_i+y_i\leqslant\sup(\{y_i|i\in\{k,\ldots,\infty\}\})+\sup(\{x_i|i\in\{k,\ldots,\infty\}\})$$

$$\sup(\{x_i+y_i|i\in\{k,\ldots,\infty\}\})\leqslant\sup(\{x_i|i\in\{k,\ldots,\infty\}\})+\sup(\{y_i|i\in\{k,\ldots,\infty\}\}) \tag{20.14}$$

Let $m\in\{k,\ldots,\infty\}$ then for every $n\in\{k,\ldots,\infty\}$ we have either:

$n<m$. Then $x_m+y_n\leqslant x_m+y_m\leqslant\sup(\{x_i+y_i|i\in\{k,\ldots,n\}\})$ so that

$$y_n\leqslant\sup(\{x_i+y_i|i\in\{k,\ldots,\infty\}\})-x_m$$

$m \leqslant n$. Then $x_m + y_n \leqslant x_n + y_n \leqslant \sup(\{x_i + y_i | i \in \{k, \ldots, n\}\})$ so that

$$y_n \leqslant \sup(\{x_i + y_i | i \in \{k, \ldots, \infty\}\}) - x_m$$

As in all cases we have $y_n \leqslant \sup(\{x_i + y_i | i \in \{k, \ldots, \infty\}\}) - x_m$ it follows that

$$\forall m \in \{k, \ldots, \infty\} \ \sup(\{y_i | i \in \{k, \ldots, \infty\}\}) \leqslant \sup(\{x_i + y_i | i \in \{k, \ldots, \infty\}\}) - x_m \tag{20.15}$$

As $\forall i \in \{k, \ldots, \infty\}$ $x_i \in \mathbb{R}$ we have by [theorem: 20.14] only the following possibilities for $\sup(\{y_i | i \in \{k, \ldots, \infty\}\})$:

$\sup(\{y_i | i \in \{k, \ldots, \infty\}\}) = \infty$. Then by [eq: 20.15] $\sup(\{x_i + y_i | i \in \{k, \ldots, \infty\}\}) = \infty$ so that

$$\sup(\{x_i | i \in \{k, \ldots, \infty\}\}) + \sup(\{y_i | i \in \{k, \ldots, \infty\}\}) = \infty = \sup(\{x_i + y_i | i \in \{k, \ldots, \infty\}\})$$

hence

$$\sup(\{x_i | i \in \{k, \ldots, \infty\}\}) + \sup(\{y_i | i \in \{k, \ldots, \infty\}\}) \leqslant \sup(\{x_i + y_i | i \in \{k, \ldots, \infty\}\})$$

$\sup(\{y_i | i \in \{k, \ldots, \infty\}\}) \in \mathbb{R}$. Then by [theorems: 20.28, 20.29] and [eq: 20.15] we have

$$x_m \leqslant \sup(\{x_i + y_i | i \in \{k, \ldots, \infty\}\}) - \sup(\{y_i | i \in \{k, \ldots, \infty\}\})$$

so that

$$\sup(\{x_i | i \in \{k, \ldots, \infty\}\}) \leqslant \sup(\{x_i + y_i | i \in \{k, \ldots, \infty\}\}) - \sup(\{y_i | i \in \{k, \ldots, \infty\}\})$$

which by [theorems: 20.28] proves that

$$\sup(\{x_i | i \in \{k, \ldots, \infty\}\}) + \sup(\{y_i | i \in \{k, \ldots, \infty\}\}) \leqslant \sup(\{x_i + y_i | i \in \{k, \ldots, \infty\}\})$$

So in all cases we have

$$\sup(\{x_i | i \in \{k, \ldots, \infty\}\}) + \sup(\{y_i | i \in \{k, \ldots, \infty\}\}) \leqslant \sup(\{x_i + y_i | i \in \{k, \ldots, \infty\}\})$$

which combined with [eq: 20.14] proves that

$$\sup(\{x_i | i \in \{k, \ldots, \infty\}\}) + \sup(\{y_i | i \in \{k, \ldots, \infty\}\}) = \sup(\{x_i + y_i | i \in \{k, \ldots, \infty\}\})$$

2. Consider the sequences $\{-x_i\}_{i \in \{k, \ldots, \infty\}} \subseteq \mathbb{R}$, $\{-y_i\}_{i \in \{k, \ldots, \infty\}} \subseteq \mathbb{R}$ then as $\forall i \in \{k, \ldots, \infty\}$ $x_{i+1} \leqslant x_i \wedge y_{i+1} \leqslant y_i$ it follows from [theorem: 20.38] that $-x_i \leqslant -x_{i+1} \wedge -y_i \leqslant -y_{i+1}$. Hence we can use (1) to get

$$\sup(\{-x_i | i \in \{k, \ldots, \infty\}\}) + \sup(\{-y_i | i \in \{k, \ldots, \infty\}\}) = \sup(\{-(x_i + y_i) | i \in \{k, \ldots, \infty\}\})$$

Then using [theorem: 20.63] on the above we have

$$-\inf(\{x_i | i \in \{k, \ldots, \infty\}\}) - \inf(\{x_i | i \in \{k, \ldots, \infty\}\}) = -\inf(\{x_i + y_i | i \in \{k, \ldots, \infty\}\})$$

hence

$$\inf(\{x_i | i \in \{k, \ldots, \infty\}\}) + \inf(\{x_i | i \in \{k, \ldots, \infty\}\}) = \inf(\{x_i + y_i | i \in \{k, \ldots, \infty\}\})$$

□

Theorem 20.66. *Let $k \in \mathbb{N}_0$ then $\liminf_{i \to \infty}$ and $\limsup_{i \to \infty}$ have the following properties*

1. *If $\{x_i\}_{i \in \{k, \ldots, \infty\}} \subseteq \overline{\mathbb{R}}$ is a sequence of extended reals and $0 \leqslant \alpha \in \mathbb{R}$ then*

   a. $\liminf_{i \to \infty} (\alpha \cdot x_i) = \alpha \cdot \liminf_{i \to \infty} x_i$

   b. $\limsup_{i \to \infty} (\alpha \cdot x_i) = \alpha \cdot \limsup_{i \to \infty} x_i$

2. *If* $\{x_i\}_{i\in\{k,\ldots,n\}},\{y_i\}_{i\in\{k,\ldots,n\}}\subseteq\mathbb{R}$ *are sequences of* ***real numbers*** *such that* $\left(\liminf_{i\rightarrow\infty} x_i\right)+\left(\liminf_{i\rightarrow\infty} y_i\right)$ *are well defined then*

   a. $\left(\liminf_{i\rightarrow\infty} x_i\right)+\left(\liminf_{i\rightarrow\infty} y_i\right)\leqslant\liminf_{i\rightarrow\infty}(x_i+y_i)$

   b. $\limsup_{i\rightarrow\infty}(x_i+y_i)\leqslant\left(\liminf_{i\rightarrow\infty} x_i\right)+\left(\liminf_{i\rightarrow\infty} y_i\right)$

Proof.

1.

$$\begin{aligned}\liminf_{i\rightarrow\infty}(\alpha\cdot x_i) &= \sup(\{\inf(\{\alpha\cdot x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})\\ &\underset{[\text{theorem: }20.63]}{=} \sup(\{\alpha\cdot\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})\\ &\underset{[\text{theorem: }20.63]}{=} \alpha\cdot\sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})\\ &= \alpha\cdot\liminf_{i\rightarrow\infty}\quad x_i\end{aligned}$$

and

$$\begin{aligned}\limsup_{i\rightarrow\infty}(\alpha\cdot x_i) &= \inf(\{\sup(\{\alpha\cdot x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})\\ &\underset{[\text{theorem: }20.63]}{=} \inf(\{\alpha\cdot\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})\\ &\underset{[\text{theorem: }20.63]}{=} \alpha\cdot\inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\})\\ &= \alpha\cdot\liminf_{i\rightarrow\infty}\quad x_i\end{aligned}$$

2. Let $l\in\{k,\ldots,\infty\}$ then we have that $\forall i\in\{l,\ldots,\infty\}$ $\inf(\{x_i|i\in\{k,\ldots,\infty\}\})\leqslant x_i\in\mathbb{R}$ and $\inf(\{y_i|i\in\{k,\ldots,\infty\}\})\leqslant y_i\in\mathbb{R}$ so that by [theorem: 20.29]

$$\inf(\{x_i|i\in\{l,\ldots,\infty\}\})+y_i\leqslant x_i+y_i \tag{20.16}$$

Further by [theorem: 20.14] we have that $\inf(\{x_i|i\in\{l,\ldots,\infty\}\}),\inf(\{y_i|i\in\{l,\ldots,\infty\}\})\neq\infty$ so that

$$(y_i,\inf(\{x_i|i\in\{l,\ldots,\infty\}\}))\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$$

and

$$(\inf(\{y_i|i\in\{l,\ldots,\infty\}\}),\inf(\{x_i|i\in\{l,\ldots,\infty\}\}))\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$$

Hence using [theorem: 20.29] we have that

$$\inf(\{y_i|i\in\{l,\ldots,\infty\}\})+\inf(\{x_i|i\in\{l,\ldots,\infty\}\})\leqslant y_i+\inf(\{x_i|i\in\{l,\ldots,\infty\}\})$$

Using the above, [eq: 20.16] and commutativity we have that

$$\inf(\{x_i|i\in\{l,\ldots,\infty\}\})+\inf(\{y_i|i\in\{l,\ldots,\infty\}\})\leqslant x_i+y_i$$

So that

$$\inf(\{x_i|i\in\{l,\ldots,\infty\}\})+\inf(\{y_i|i\in\{l,\ldots,\infty\}\})\leqslant\inf(\{x_i+y_i|i\in\{l,\ldots,\infty\}\})$$

Hence

$$\begin{aligned}\sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})+\inf(\{y_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) &\leqslant\\ \sup(\{\inf(\{x_i+y_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) &=\\ \liminf_{i\rightarrow\infty}(x_i+y_i)\end{aligned} \tag{20.17}$$

Now $\forall l\in\{k,\ldots,\infty\}$ we have that

$$\inf(\{x_i|i\in\{l,\ldots,\infty\}\})\leqslant\inf(\{x_i|i\in\{l+1,\ldots,\infty\}\})$$

and

$$\inf(\{y_i|i\in\{l,\ldots,\infty\}\})\leqslant\inf(\{y_i|i\in\{l+1,\ldots,\infty\}\})$$

so that by [theorem: 20.65]

$$\sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})+\inf(\{y_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) = \sup(\{\{x_i|i\in\{l+1,\ldots,\infty\}\}|l\in\{k,\ldots,\infty\}\})+\sup(\{\{y_i|i\in\{l+1,\ldots,\infty\}\}|l\in\{k,\ldots,\infty\}\}) = \liminf_{i\rightarrow\infty} x_i+\liminf_{i\rightarrow\infty} y_i$$

Substituting the above in [eq: 20.17] gives finally

$$\forall\{x_i\}_{i\in\{k,\ldots,\infty\}},\{y_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\mathbb{R}\ \liminf_{i\rightarrow\infty} x_i+\liminf_{i\rightarrow\infty} y_i\leqslant\liminf_{i\rightarrow\infty}(x_i+y_i) \tag{20.18}$$

Consider the sequences $\{-x_i\}_{i\in\{k,\ldots,\infty\}},\{-y_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\mathbb{R}$ then by [eq: 20.18] we have that

$$\liminf_{i\rightarrow\infty}(-x_i)+\liminf_{i\rightarrow\infty}(-y_i)\leqslant\liminf_{i\rightarrow\infty}(-(x_i+y_i))$$

Then using [theorem: 20.63] on the above gives

$$-\limsup_{i\rightarrow\infty} x_i+\left(-\limsup_{i\rightarrow\infty} y_i\right)\leqslant-\limsup_{i\rightarrow\infty}(x_i+y_i)$$

hence

$$\limsup_{i\rightarrow\infty}(x_i+y_i)\leqslant\limsup_{i\rightarrow\infty} x_i+\limsup_{i\rightarrow\infty} y_i$$

□

Theorem 20.67. *Let $k\in\mathbb{N}_0$, $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ then $\liminf_{i\rightarrow\infty} x_i\leqslant\limsup_{i\rightarrow\infty} x_i$*

Proof. Fix $n\in\{k,\ldots,\infty\}$ then $\forall m\in\{k,\ldots,\infty\}$ we have either:

$\boldsymbol{m\in\{k,\ldots,n-1\}}$**.** then $m<n$ so that $\{x_i|i\in\{n,\ldots,\infty\}\}\subseteq\{x_i|i\in\{m,\ldots,\infty\}|\}$ hence

$$\inf(\{x_i|i\in\{m,\ldots,\infty\}\})\leqslant\inf(\{x_i|i\in\{n,\ldots,\infty\}\})\leqslant x_n\leqslant\sup(\{x_i|i\in\{n,\ldots,\infty\}\})$$

$\boldsymbol{m\in\{n,\ldots,\infty\}}$**.** then $\inf(\{x_i|i\in\{m,\ldots,\infty\}\})\underset{m\in\{n,\ldots,\infty\}}{\leqslant}x_n\leqslant\sup(\{x_i|i\in\{n,\ldots,\infty\}\})$

hence we have

$$\forall m\in\{k,\ldots,\infty\}\text{ we have }\inf(\{x_i|i\in\{m,\ldots,\infty\}\})\leqslant\sup(\{x_i|i\in\{n,\ldots,\infty\}\})$$

so we have

$$\sup(\{\inf(\{x_i|i\in\{m,\ldots,\infty\}\})|m\in\{k,\ldots,\infty\}\})\leqslant\sup(\{x_i|i\in\{n,\ldots,\infty\}\})$$

or

$$\forall n\in\{k,\ldots,\infty\}\ \liminf_{i\rightarrow\infty} x_i\leqslant\sup(\{x_i|i\in\{n,\ldots,\infty\}\})$$

so that

$$\liminf_{i\rightarrow\infty} x_i\leqslant\inf(\{\sup(\{x_i|i\in\{n,\ldots,\infty\}\})|n\in\{k,\ldots,\infty\}\})=\limsup_{i\rightarrow\infty} x_i$$

□

Remark 20.68. If we note $\leqslant_{\mathbb{R}}$ as the order relation on $\mathbb{R}$, $\sup_{\mathbb{R}}$ and $\inf_{\mathbb{R}}$ the supremum and infinum in $\mathbb{R}$ and $\lim_{\mathbb{R}}$ the limit in $\langle\mathbb{R},||_{\mathbb{R}}\rangle$ where $||_{\mathbb{R}}$ is the absolute value on $\mathbb{R}$ then we have in [definition: 14.318] defined $\underset{i\rightarrow\infty}{\lim_{\mathbb{R}}}\inf$ and $\underset{i\rightarrow\infty}{\lim_{\mathbb{R}}}\sup$ defined by

1. $\underset{n\rightarrow\infty}{\lim_{\mathbb{R}}}\sup x_n$ exist if
   a. $\forall n\in\{k,\ldots,\infty\}\ \sup_{\mathbb{R}}(\{x_i|i\in\{n,\ldots,\infty\}\})$ exist
   b. $\underset{n\rightarrow\infty}{\lim_{\mathbb{R}}}\sup_{\mathbb{R}}(\{x_i|i\in\{n,\ldots,\infty\}\})$ exist

   If $\underset{n\rightarrow\infty}{\lim_{\mathbb{R}}}\sup x_n$ exist then $\underset{n\rightarrow\infty}{\lim_{\mathbb{R}}}\sup x_n=\underset{m\rightarrow\infty}{\lim_{\mathbb{R}}}\sup_{\mathbb{R}}(\{x_i|i\in\{n,\ldots,\infty\}\})$
2. $\underset{n\rightarrow\infty}{\lim_{\mathbb{R}}}\inf x_n$ exist if
   a. $\forall n\in\{k,\ldots,\infty\}\ \inf_R(\{x_i|i\in\{n,\ldots,\infty\}\})$ exist
   b. $\underset{n\rightarrow\infty}{\lim_{\mathbb{R}}}\inf_{\mathbb{R}}(\{x_i|i\in\{n,\ldots,\infty\}\})$ exist

If $\lim_{\mathbb{R}} \inf_{n \rightarrow \infty} x_n$ exist then $\lim_{\mathbb{R}} \inf_{n \rightarrow \infty} x_n = \lim_{\mathbb{R}, m \rightarrow \infty} \inf_{\mathbb{R}}(\{x_i | i \in \{n, \ldots, \infty\}\})$

Using [theorem: 14.320] the above definition is equivalent with

1.
$$\lim_{\mathbb{R}} \sup_{n \rightarrow \infty} x_n \text{ exist}$$
$$\Updownarrow$$
$$\forall n \in \{k, \ldots, \infty\} \ \sup_{\mathbb{R}}(\{x_i | i \in \{n, \ldots, \infty\}\}) \text{ exist}$$
$$\text{and}$$
$$\inf_{\mathbb{R}} (\sup_{\mathbb{R}}(\{x_i | i \in \{n, \ldots., \infty\}\}) | n \in \{k, \ldots, \infty\}\}) \text{ exist}$$

   Further if $\lim_{\mathbb{R}} \sup_{n \rightarrow \infty} x_n$ exist then
$$\lim_{\mathbb{R}} \sup_{n \rightarrow \infty} x_n = \inf (\{\sup_{\mathbb{R}}(\{x_i | i \in \{n, \ldots, \infty\}\}) | n \in \{k, \ldots, \infty\}\})$$

2.
$$\lim_{\mathbb{R}} \inf_{n \rightarrow \infty} x_n \text{ exist}$$
$$\Updownarrow$$
$$\forall n \in \{k, \ldots, \infty\} \ \inf_{\mathbb{R}}(\{x_i | i \in \{n, \ldots, \infty\}\}) \text{ exist}$$
$$\text{and}$$
$$\sup_{\mathbb{R}}(\{\inf (\{x_i | i \in \{n, \ldots., \infty\}\}) | n \in \{k, \ldots, \infty\}\}) \text{ exist}$$

   Further if $\lim_{\mathbb{R}} \inf_{n \rightarrow \infty} x_n$ exist then
$$\lim_{\mathbb{R}} \inf_{n \rightarrow \infty} x_n = \sup_{\mathbb{R}}(\{\inf_{\mathbb{R}}(\{x_i | i \in \{n, \ldots, \infty\}\}) | n \in \{k, \ldots, \infty\}\})$$

Using [theorem: 14.321] we have that

$$\lim_{\mathbb{R}, n \rightarrow \infty} x_n \text{ exist} \Leftrightarrow \lim_{\mathbb{R}} \inf_{n \rightarrow \infty} x_n \text{ and } \lim_{\mathbb{R}} \sup_{n \rightarrow \infty} x_n \text{ exists and } \lim_{\mathbb{R}} \inf_{n \rightarrow \infty} x_n = \lim_{\mathbb{R}} \sup_{n \rightarrow \infty} x_n$$

Further if $\lim_{\mathbb{R}, n \rightarrow \infty} x_i$ exist then
$$\lim_{\mathbb{R}, n \rightarrow \infty} x_n = \lim_{\mathbb{R}} \inf_{n \rightarrow \infty} x_i = \lim_{\mathbb{R}} \sup_{n \rightarrow \infty} x_n$$

This suggest the following extension of a limit in $\mathbb{R}$ to the limit in $\overline{\mathbb{R}}$

Definition 20.69. *Let $k \in \mathbb{N}_0$ and $\{x_i\}_{i \in \{k, \ldots, \infty\}} \subseteq \overline{\mathbb{R}}$ such that $\liminf_{i \rightarrow \infty} x_i = \limsup_{i \rightarrow \infty} x_i$ then we say that $\lim_{i \rightarrow \infty} x_i$ exist and we have by definition that*

$$\lim_{i \rightarrow \infty} x_i = \liminf_{i \rightarrow \infty} x_i = \limsup_{i \rightarrow \infty} x_i$$

***Note* 20.70.** *In this chapter we use* $\lim$ *for the* $\lim$ *in* $\overline{\mathbb{R}}$ *which is defined using only the order relation* $\leqslant$*. As for* $\mathbb{R}$ $\lim_{\mathbb{R}}$ *represents the limit in* $\mathbb{R}$ *which is based on the absolute value* $||_{\mathbb{R}}$ *in* $\mathbb{R}$ *but can be defined by the order relation* $\leqslant_{\mathbb{R}}$ *in* $\mathbb{R}$ *where* $\leqslant_{\mathbb{R}} \subseteq \leqslant$*. Further* $\sup$, $\inf$, $\liminf$, $\limsup$ *are all defined in* $\overline{\mathbb{R}}$ *using the order relation* $\leqslant$ *on* $\overline{\mathbb{R}}$ *and* $\sup_{\mathbb{R}}$, $\inf_{\mathbb{R}}$ *are defined on* $\mathbb{R}$ *using the order relation* $\leqslant_{\mathbb{R}}$ *on* $\mathbb{R}$ *and* $\lim_{\mathbb{R}} \inf$, $\lim_{\mathbb{R}} \sup$ *are defined in [definition: 14.318] or [theorem: 14.320].*

Example 20.71. Let $k \in \mathbb{N}_0$ and $\{x_i\}_{i \in \{k, \ldots, \infty\}} \subseteq \overline{\mathbb{R}}$ is defined by $x_i = x$ where $x \in \overline{\mathbb{R}}$ then $\lim_{i \rightarrow \infty} x_i = x$.

Proof. This follows from the definition and [example: 20.57]. □

Example 20.72. Let $k \in \mathbb{N}_0$ then the sequence $\{(-\infty)^i\}_{i \in \{k, \ldots, \infty\}}$ has no limit

Proof. Let $l \in \{1, \ldots, \infty\}$ then we have $\{(-\infty)^i | i \in \{l, \ldots, \infty\}\} = \{-\infty, \infty\}$ so that

$$\{\sup(\{(-\infty)^i | i \in \{l, \ldots, \infty\}\})\} = \{\infty\} \text{ and } \{\sup(\{(-\infty)^i | i \in \{l, \ldots, \infty\}\})\} = \{-\infty\}$$

hence

$$\limsup_{i \to \infty} (-\infty)^i = \inf(\{\sup(\{(-\infty)^i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\} = \infty$$

and

$$\liminf_{i \to \infty} (-\infty)^i = \sup(\{\inf(\{(-\infty)^i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\} = -\infty$$

so that

$$\limsup_{i \to \infty} (-\infty)^i \neq \liminf_{i \to \infty} (-\infty)^i$$

□

We have the following relation between $\lim_{\mathbb{R}}$ the limit in $\mathbb{R}$ and $\lim$ the limit in $\overline{\mathbb{R}}$.

Theorem 20.73. *Let $k \in \mathbb{N}_0$, $\{x_i\}_{i \in \{k, \ldots, \infty\}} \subseteq \mathbb{R}$ a sequence of **real** numbers then we have:*

1. *If $\lim_{\mathbb{R}\, i \to \infty} x_i$ exists then $\liminf_{i \to \infty} x_i = \lim_{\mathbb{R}\, i \to \infty} x_i = \limsup_{i \to \infty} x_i$*
2. *If $\liminf_{i \to \infty} x_i = \limsup_{i \to \infty} x_i \in \mathbb{R}$ then $\lim_{\mathbb{R}\, i \to \infty} x_i$ exists and $\liminf_{i \to \infty} x_i = \lim_{\mathbb{R}\, i \to \infty} x_i = \limsup_{i \to \infty} x_i$*

Proof.

1. Let $x = \lim_{\mathbb{R}\, i \to \infty} x_i \in \mathbb{R}$. Assume that $\liminf_{i \to \infty} x_i < x$ then as $x \in \mathbb{R} \Rightarrow x < \infty$ we have either:

   $\liminf_{i \to \infty} x_i = -\infty$. Then $\sup(\{\inf(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\}) = -\infty$ so that by [theorem: 20.15] it follows that $\{\inf(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\} = \{-\infty\}$ hence

   $$\inf(\{x_i | i \in \{k, \ldots, \infty\}\}) = -\infty \tag{20.19}$$

   As $\lim_{\mathbb{R}\, i \to \infty} x_i$ exist it follows from [theorem: 14.306] that there exists a $L \in \mathbb{R}$ such that $\forall i \in \{k, \ldots, \infty\}$ $|x_i|_{\mathbb{R}} \leqslant L < L + 1$. So if we take $K = L + 1 \in \mathbb{R}$ then we have

   $$\forall i \in \{k, \ldots, \infty\} \text{ we have } -K < x_i < K$$

   So $-\infty < -K < x_i$ $\forall i \in \{k, \ldots, \infty\}$ hence $-\infty < -K \leqslant \inf(\{x_i | i \in \{k, \ldots, \infty\}\})$ contradicting [eq: 20.19].

   $\liminf_{i \to \infty} x_i \in \mathbb{R}$. As $\mathbb{R} \ni \liminf_{i \to \infty} x_i < x \in \mathbb{R}$ we have that $\varepsilon = x - \liminf_{i \to \infty} x_i \in \mathbb{R}^+$. As $\lim_{\mathbb{R}\, i \to \infty} = x$ there exist a $N \in \{k, \ldots, \infty\}$ such that

   $$\forall n \in \{N, \ldots, \infty\} \text{ we have } |x_n - x|_{\mathbb{R}} <_{\mathbb{R}} \varepsilon \tag{20.20}$$

   Now

   $$\begin{aligned} \inf(\{x_i | i \in \{N, \ldots, \infty\}\}) &\leqslant \sup(\{\inf(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\}) \\ &= \liminf_{i \to \infty} x_i \\ &= x - \varepsilon \\ &< x - \frac{\varepsilon}{2} \end{aligned}$$

   Hence by the definition of the infinum there exist a $M \in \{N, \ldots, \infty\}$ such that $x_M < x - \frac{\varepsilon}{2} \underset{x_M, x \in \mathbb{R}}{\Rightarrow} x_M <_{\mathbb{R}} x - \frac{\varepsilon}{2}$. Hence $|x_M - x|_{\mathbb{R}} \geqslant_{\mathbb{R}} x - x_M >_{\mathbb{R}} \frac{\varepsilon}{2}$ contradicting [20.20].

   So in all cases we reach a a contradiction proving that the assumption is wrong, hence we have that

   $$x \leqslant \liminf_{i \to \infty} x_i \underset{[\text{theorem: } 20.67]}{\leqslant} \limsup_{i \to \infty} x_i \tag{20.21}$$

Assume that $x < \limsup_{i \rightarrow \infty} x_i$. Then as $-\infty < x < \limsup_{i \rightarrow \infty} x_i$ we have for $\limsup_{i \rightarrow \infty} x_i$ either:

$\limsup_{i\rightarrow\infty} x_i = \infty$. Then $\inf(\{\sup(\{x_i|i \in \{l, \ldots, \infty\}\})|l \in \{k, \ldots, \infty\}\}) = \infty$ so that by [theorem: 20.15] it follows that $\{\sup(\{x_i|i \in \{l,\ldots,\infty\}\})|l \in \{k,\ldots,\infty\}\} = \{\infty\}$. Hence

$$\sup(\{x_i|i \in \{k, \ldots, \infty\}\}) = \infty \tag{20.22}$$

As $\lim_{i\rightarrow\infty}^{\mathbb{R}} x_i$ exist it follows from [theorem: 14.306] that there exists a $L \in \mathbb{R}$ such that $\forall i \in \{k, \ldots, \infty\}$ $|x_i|_{\mathbb{R}} \leqslant L < L + 1$. So if we take $K = L + 1 \in \mathbb{R}$ then we have

$$\forall i \in \{k, \ldots, \infty\} \text{ we have} - K < x_i < K$$

So $\sup(\{x_i|i \in \{k, \ldots, \infty\}\}) \leqslant K < \infty$ contradiction [eq: 20.22].

$\limsup_{i\rightarrow\infty} x_i \in \mathbb{R}$. As $\mathbb{R} \ni x < \limsup_{i\rightarrow\infty} x_i \in \mathbb{R}$ we have that $\varepsilon = \limsup_{i\rightarrow\infty} x_i - x \in \mathbb{R}^+$. As $\lim_{i\rightarrow\infty}^{\mathbb{R}} x_i = x$ there exist a $N \in \{k, \ldots, \infty\}$ such that

$$\forall n \in \{N, \ldots, \infty\} \text{ we have } |x_n - x|_{\mathbb{R}} < \varepsilon \tag{20.23}$$

Now

$$\begin{aligned}
\sup(\{x_i|i \in \{N, \ldots, \infty\}\}) &\geqslant \inf(\{\sup(\{x_i|i \in \{l, \ldots, \infty\}\})|l \in \{k, \ldots, \infty\}\}) \\
&= \limsup_{i\rightarrow\infty} x_i \\
&= x + \varepsilon \\
&> x + \frac{\varepsilon}{2}
\end{aligned}$$

So using the definition of the supremum there exist a $M \in \{N, \ldots, \infty\}$ such that $x_M > x + \frac{\varepsilon}{2} \underset{x_M, x \in \mathbb{R}}{\Rightarrow} x_M >_{\mathbb{R}} x + \frac{\varepsilon}{2}$. Hence $|x_M - x| \geqslant_{\mathbb{R}} x_M - x >_{\mathbb{R}} \frac{\varepsilon}{2}$ contradicting [eq: 20.23].

As in all cases we have a contradiction the assumption must be wrong so that

$$\limsup_{i\rightarrow\infty} x_i \leqslant x.$$

Combining this with [eq: 20.21] we have

$$\liminf_{i\rightarrow\infty} x_i \underset{\text{[theorem: 20.67]}}{\leqslant} \limsup_{i\rightarrow\infty} x_i \leqslant x \leqslant \liminf_{i\rightarrow\infty} x_i \underset{\text{[theorem: 20.67]}}{\leqslant} \limsup_{i\rightarrow\infty} x_i$$

so that

$$\liminf_{i\rightarrow\infty} x_i = \lim_{i\rightarrow\infty}^{\mathbb{R}} x_i = \limsup_{i\rightarrow\infty} x_i$$

2. Let $x = \liminf_{i\rightarrow\infty} x_i = \limsup_{i\rightarrow\infty} x_i \in \mathbb{R}$. Take $\varepsilon \in \mathbb{R}^+$ then we have

$$x - \varepsilon < x = \liminf_{i\rightarrow\infty} x_i = \sup(\{\inf(\{x_i|i \in \{l, \ldots, \infty\}\})|l \in \{k, \ldots, \infty\}\})$$

Hence by the definition of the supremum there exist a $M_1 \in \{k, \ldots, \infty\}$ such that

$$x - \varepsilon < \inf(\{x_i|i \in \{M_1, \ldots, \infty\}\})$$

Hence

$$\forall n \in \{M_1, \ldots, \infty\} \text{ we have } x - \varepsilon < x_n \tag{20.24}$$

Further as $x = \limsup_{i\rightarrow\infty} x_i$ we have that $\inf(\{\sup(\{x_i|i \in \{l,\ldots,\infty\}\})|l \in \{k,\ldots,\infty\}\}) = x < x + \varepsilon$ so that by the definition of the infinum there exist a $M_2 \in \{k, \ldots, \infty\}$ such that

$$\sup(\{x_i|i \in \{M_2, \ldots, \infty\}\}) < x + \varepsilon$$

Hence

$$\forall n \in \{M_2, \ldots, \infty\} \text{ we have } x_n < x + \varepsilon \tag{20.25}$$

Take $M = \max(M_1, M_2)$ then we have by [eqs: 20.24, 20.25] that $\forall n \in \{M, \ldots, \infty\}$ $x - \varepsilon < x_n < x + \varepsilon$ or as $x, \varepsilon \in \mathbb{R}$ that $x - \varepsilon <_{\mathbb{R}} x_n <_{\mathbb{R}} x + \varepsilon \Rightarrow |x_n - x|_R <_{\mathbb{R}} \varepsilon$ proving that

$$\lim_{i \rightarrow \infty \mathbb{R}} x_i \text{ exists and } \lim_{i \rightarrow \infty \mathbb{R}} x_i = x = \liminf_{i \rightarrow \infty} x_i = \limsup_{i \rightarrow \infty} x_i$$ □

A consequence of the above theorem is that $\lim_{i \rightarrow \infty}$ is a extension of $\lim_{i \rightarrow \infty \mathbb{R}}$ to $\overline{\mathbb{R}}$.

Corollary 20.74. *Let $k \in \mathbb{N}_0$, $\{x_i\}_{i \in \{k, \ldots, \infty\}} \subseteq \mathbb{R}$ a sequence of **real** numbers then we have*

$$\lim_{i \rightarrow \infty \mathbb{R}} x_i \textit{ exists} \Leftrightarrow \lim_{i \rightarrow \infty} x_i \textit{ exist and } \lim_{i \rightarrow \infty} x_i \in \mathbb{R}$$

Further if $\lim_{i \rightarrow \infty \mathbb{R}} x_i$ exists or $\lim_{i \rightarrow \infty} x_i$ exists then $\lim_{i \rightarrow \infty \mathbb{R}} x_i = \lim_{i \rightarrow \infty} x_i$

Proof.

⇒. Using [theorem: 20.73] it follows that $\liminf_{i \rightarrow \infty} x_i = \lim_{i \rightarrow \infty \mathbb{R}} x_i = \limsup_{i \rightarrow \infty} x_i$ so that by definition $\lim_{i \rightarrow \infty} x_i$ exists and $\mathbb{R} \ni \lim_{i \rightarrow \infty \mathbb{R}} x_i = \lim_{i \rightarrow \infty} x_i$.

⇐. If $\lim_{i \rightarrow \infty} x_i$ exists and $\lim_{i \rightarrow \infty} x_i \in \mathbb{R}$ we have that $\liminf_{i \rightarrow \infty} x_i = \limsup_{i \rightarrow \infty} x_i = \lim_{i \rightarrow \infty} x_i \in \mathbb{R}$. Hence by [theorem: 20.73] $\lim_{i \rightarrow \infty \mathbb{R}} x_i$ exists and $\lim_{i \rightarrow \infty} x_i = \liminf_{i \rightarrow \infty} x_i = \limsup_{i \rightarrow \infty} x_i$. □

Example 20.75. We have

1. For $\{\frac{1}{i}\}_{i \in \{1, \ldots, \infty\}}$ we have that $\lim_{i \rightarrow \infty} \frac{1}{i}$ exist and $\lim_{i \rightarrow \infty} \frac{1}{i} = 0$
2. For $\{\frac{1}{i}\}_{i \in \{1, \ldots, \infty\}}$ we have that $\lim_{i \rightarrow \infty} \frac{(-1)^i}{i}$ exist and $\lim_{i \rightarrow \infty} \frac{(-1)^i}{i} = 0$

Proof.

1. Note that $\{\frac{1}{i}\}_{i \in \{1, \ldots, \infty\}} \subseteq \mathbb{R}$ and by [example: 14.297] $\lim_{i \rightarrow \infty \mathbb{R}} \frac{1}{i} = 0$ exist. Hence by the previous corollary [corollary: 20.74] $\lim_{i \rightarrow \infty} \frac{1}{i} = 0$ exist.
2. Note that $\{\frac{1}{i}\}_{i \in \{1, \ldots, \infty\}} \subseteq \mathbb{R}$ and by [example: 14.308] $\lim_{i \rightarrow \infty \mathbb{R}} \frac{(-1)^i}{i} = 0$ exist. Hence by the previous corollary [corollary: 20.74] $\lim_{i \rightarrow \infty} \frac{(-1)^i}{i} = 0$ exist. □

The idea of a limit in $\mathbb{R}$ is that the sequence approaches its limit the higher the index is, we show now that the limit in $\overline{\mathbb{R}}$.

Theorem 20.76. *Let $k \in \mathbb{N}_0$, $\{x_i\}_{i \in \{k, \ldots, \infty\}} \subseteq \overline{\mathbb{R}}$ then we have the following equivalences.*

1. $$\lim_{i \rightarrow \infty} x_i = \infty \Leftrightarrow \forall C \in \mathbb{R}^+ \ \exists N \in \{k, \ldots, \infty\} \textit{ such that } \forall n \in \{N, \ldots, \infty\} \textit{ we have } C < x_n$$
2. $$\lim_{i \rightarrow \infty} x_i = -\infty \Leftrightarrow \forall C \in \mathbb{R}^+ \ \exists N \in \{k, \ldots, \infty\} \textit{ such that } \forall n \in \{N, \ldots, \infty\} \textit{ we have } x_n < -C$$
3. *If $x \in \mathbb{R}$ then*
   $$\lim_{i \rightarrow \infty} x_i = x \in \mathbb{R} \Leftrightarrow \forall \varepsilon \in \mathbb{R}^+ \ \exists N \in \{k, \ldots, \infty\} \textit{ such that } \forall n \in \{N, \ldots, \infty\} \ |x_n - x| < \varepsilon$$

Proof.

1.

   ⇒. If $\lim_{i \rightarrow \infty} x_i = \infty$ then $\liminf_{i \rightarrow \infty} x_i = \limsup_{i \rightarrow \infty} x_i = \infty$ so that

   $$\sup(\{\inf(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\}) = \infty$$

Hence using [theorem: 20.15] we have that either:

$\infty \in \{\inf(\{x_i|i \in \{l,\ldots,\infty\}\})|l \in \{k,\ldots,\infty\}\}$. Then there exist a $N \in \{k,\ldots,\infty\}$ such that $\inf(\{x_i|i \in \{N,\ldots,\infty\}\}) = \infty$. Using [theorem: 20.15] again it follows that $\{x_i|i \in \{N,\ldots,\infty\}\} = \{\infty\}$ so that $\forall n \in \{N,\ldots,\infty\}$ we have $x_n = \infty$. Hence if $C \in \mathbb{R}^+$ then $\forall n \in \{N,\ldots,\infty\}$ then $C < \infty = x_n$.

$\forall u \in \mathbb{R} \exists x \in \{\inf(\{x_i|i \in \{l,\ldots,\infty\}\})|l \in \{k,\ldots,\infty\}\} \bigcap \mathbb{R} \vdash u < x$. Then given $C \in \mathbb{R}^+$ there exist a $N \in \{k,\ldots,\infty\}$ such that $C < \inf(\{x_i|i \in \{N,\ldots,\infty\}\})$ hence $\forall n \in \{N,\ldots,\infty\}$ we have $C < \inf(\{x_i|i \in \{N,\ldots,\infty\}\}) \leqslant x_n$.

so in all cases we have that

$$\forall C \in \mathbb{R}_+ \text{ there } \exists N \in \{k,\ldots,\infty\} \text{ such that } \forall n \geqslant N \text{ we have } C < x_n$$

$\Leftarrow$. Assume that $\liminf_{i\rightarrow\infty} x_i < \infty$. Take $C = \max\left(\liminf_{i\rightarrow\infty} x_i, 1\right) \in \mathbb{R}^+$ then by the hypothesis there exists a $N \in \{k,\ldots,n\}$ such that $\forall n \in \{N,\ldots,\infty\}$ we have $C + 1 < x_n$. Hence $C + 1$ is a lower bound of $\{x_i|i \in \{N,\ldots,\infty\}\}$ so that

$$C + 1 \leqslant \inf(\{x_i|i \in \{N,\ldots,\infty\}\}) \tag{20.26}$$

So

$$\begin{aligned}
\liminf_{i\rightarrow\infty} x_i &\leqslant \max\left(\liminf_{i\rightarrow\infty} x_i, 1\right) \\
&= C \\
&< C + 1 \\
&\underset{\text{[eq: 20.26]}}{\leqslant} \inf(\{x_i|i \in \{N,\ldots,\infty\}\}) \\
&\leqslant \sup(\{\inf(\{x_i|i \in \{l,\ldots,\infty\}\})|l \in \{k,\ldots,\infty\}\}) \\
&= \liminf_{i\rightarrow\infty} x_i
\end{aligned}$$

leading to the contradiction $\liminf_{i\rightarrow\infty} x_i < \liminf_{i\rightarrow\infty} x_i$. Hence

$$\infty \leqslant \liminf_{i\rightarrow\infty} x_i \underset{\text{[theorem: 20.67]}}{\leqslant} \limsup_{i\rightarrow\infty} x_i \leqslant \infty$$

proving that $\liminf_{i\rightarrow\infty} x_i = \limsup_{i\rightarrow\infty} x_i = \infty$ so that

$$\lim_{i\rightarrow\infty} x_i \text{ exist and } \lim_{i\rightarrow\infty} x_i = \infty$$

2.

$\Rightarrow$. If $\lim_{i\rightarrow\infty} x_i = -\infty$ then $\liminf_{i\rightarrow\infty} x_i = \limsup_{i\rightarrow\infty} x_i = -\infty$ so that

$$\inf(\{\sup(\{x_i|i \in \{l,\ldots,\infty\}\})|l \in \{k,\ldots,\infty\}\}) = -\infty$$

Hence using [theorem: 20.15] we have that either:

$-\infty \in \{\sup(\{x_i|i \in \{l,\ldots,\infty\}\})|l \in \{k,\ldots,\infty\}\}$. Then there exist a $N \in \{k,\ldots,\infty\}$ such that $\sup(\{x_i|i \in \{N,\ldots,\infty\}\}) = -\infty$. Using [theorem: 20.15] again it follows that $\{x_i|i \in \{N,\ldots,\infty\}\} = \{-\infty\}$ so that $\forall n \in \{N,\ldots,\infty\}$ we have $x_n = -\infty$. Hence if $C \in \mathbb{R}^+$ then $\forall n \in \{N,\ldots,\infty\}$ then $x_n = -\infty < -C$.

$\forall u \in \mathbb{R} \exists x \in \{\sup(\{x_i|i \in \{l,\ldots,\infty\}\})|l \in \{k,\ldots,\infty\}\} \bigcap \mathbb{R} \vdash x < u$. Then given $C \in \mathbb{R}^+$ there exist a $N \in \{k,\ldots,\infty\}$ such that $\sup(\{x_i|i \in \{N,\ldots,\infty\}\}) < -C$ hence $\forall n \in \{N,\ldots,\infty\}$ we have $x_n < \sup(\{x_i|i \in \{N,\ldots,\infty\}\}) < -C$.

so in all cases we have that

$$\forall C \in \mathbb{R}_+ \text{ there } \exists N \in \{k,\ldots,\infty\} \text{ such that } \forall n \geqslant N \text{ we have } x_n < -C$$

$\Leftarrow$. Assume that $-\infty < \limsup_{i \rightarrow \infty} x_i$. Take $C = -\min\left(\limsup_{i \rightarrow \infty} x_i, -1\right) \in \mathbb{R}^+$ then by the hypothesis there exists a $N \in \{k, \ldots, n\}$ such that $\forall n \in \{N, \ldots, \infty\}$ we have $x_n < -(C+1)$. Hence $-(C+1)$ is a upper bound for $\{x_i | i \in \{N, \ldots, \infty\}\}$ so that

$$\sup(\{x_i | i \in \{N, \ldots, \infty\}\}) \leqslant -(C+1) \tag{20.27}$$

hence

$$\begin{array}{rcl} \limsup_{i \rightarrow \infty} x_i & \geqslant & \min\left(\limsup_{i \rightarrow \infty} x_i, -1\right) \\ & = & -C \\ & > & -(C+1) \\ & \underset{\text{[eq: 20.27]}}{\geqslant} & \sup(\{x_i | i \in \{N, \ldots, \infty\}\}) \\ & \geqslant & \inf(\{\sup(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\}) \\ & = & \limsup_{i \rightarrow \infty} x_i \end{array}$$

leading to the contradiction $\limsup_{i \rightarrow \infty} x_i < \limsup_{i \rightarrow \infty} x_i$. Hence

$$-\infty \leqslant \liminf_{i \rightarrow \infty} x_i \underset{\text{[theorem: 20.67]}}{\leqslant} \limsup_{i \rightarrow \infty} x_i \leqslant -\infty$$

proving that $\liminf_{i \rightarrow \infty} x_i = \limsup_{i \rightarrow \infty} x_i = -\infty$ so that

$$\lim_{i \rightarrow \infty} x_i \text{ exist and } \lim_{i \rightarrow \infty} x_i = -\infty$$

3.

$\Rightarrow$. If $\lim_{i \rightarrow \infty} x_i = x \in \mathbb{R}$ exist then $\liminf_{i \rightarrow \infty} x_i = \limsup_{i \rightarrow \infty} x_i$ so that

$$x = \inf(\{\sup(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\})$$

and

$$x = \sup(\{\inf(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\})$$

Let $\varepsilon \in \mathbb{R}^+$ then we have $x - \varepsilon < x = \sup(\{\inf(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\})$ so that there exist a $N_1 \in \{k, \ldots, \infty\}$ such that

$$x - \varepsilon < \inf(\{x_i | i \in \{N_1, \ldots, \infty\}\})$$

hence for $n \in \{N_1, \ldots, \infty\}$ we have that

$$x - \varepsilon < \inf(\{x_i | i \in \{N_1, \ldots, \infty\}\}) < x_n$$

Likewise, as $\inf(\{\sup(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\}) = x < x + \varepsilon$ there exist a $N_2 \in \{k, \ldots, \infty\}$ such that $\sup(\{x_i | i \in \{N, \ldots, \infty\}\}) < x + \varepsilon$ hence for $n \in \{N_2, \ldots, \infty\}$we have

$$x_n \leqslant \sup(\{x_i | i \in \{N, \ldots, \infty\}\}) < x + \varepsilon$$

Take $N = \max(N_1, N_2)$ then for $n \in \{N, \ldots, \infty\}$ we have

$$x - \varepsilon < x_n < x + \varepsilon \text{ or equivalently } |x_n - x| < \varepsilon$$

$\Leftarrow$. Let $\varepsilon \in \mathbb{R}^+$ then by the hypothesis there exist a N_ε such that $\forall n \in \{N_\varepsilon, \ldots, \infty\}$ we have $x - \varepsilon < x_n < x + \varepsilon$. Hence we have that $\{x_i | i \in \{N_\varepsilon, \ldots, \infty\}\}$ has a lower bound $x - \varepsilon$ and a upper bound $x + \varepsilon$. Hence

$$x - \varepsilon \leqslant \inf(\{x_i | i \in \{N_\varepsilon, \ldots, \infty\}\}) \text{ and } \sup(\{x_i | i \in \{N_\varepsilon, \ldots, \infty\}\}) < x + \varepsilon$$

As

$$\inf(\{x_i | i \in \{N_\varepsilon, \ldots, \infty\}\}) \leqslant \sup(\{\inf(\{x_i | i \in \{l, \ldots, \infty\}\}) | l \in \{k, \ldots, \infty\}\}) = \liminf_{i \rightarrow \infty} x_i$$

and

$$\limsup_{i\rightarrow\infty} x_i = \inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \leqslant \sup(\{x_i|i\in\{N_\varepsilon,\ldots,\infty\}\})$$

it follows that $x-\varepsilon < \liminf_{i\rightarrow\infty} x_i \underset{\text{[theorem: 20.67]}}{\leqslant} \limsup_{i\rightarrow\infty} x_i < x+\varepsilon$. Given that $x\in\mathbb{R}$ we have that

$$x-\varepsilon <_{\mathbb{R}} \liminf_{i\rightarrow\infty} x_i \underset{\text{[theorem: 20.67]}}{\leqslant_{\mathbb{R}}} \limsup_{i\rightarrow\infty} x_i <_{\mathbb{R}} x+\varepsilon$$

So as $\varepsilon\in\mathbb{R}^+$ was chosen arbitrary it follows from [theorem: 10.31] that

$$x \leqslant_{\mathbb{R}} \liminf_{i\rightarrow\infty} x_i \leqslant_{\mathbb{R}} \limsup_{i\rightarrow\infty} x_i \leqslant_{\mathbb{R}} x$$

so that $\liminf_{i\rightarrow\infty} x_i = \limsup_{i\rightarrow\infty} x_i = x$ proving that

$$\lim_{i\rightarrow\infty} x_i \text{ exists and that } \lim_{i\rightarrow\infty} x_i = x$$ □

We can now derive the different properties of the limit in $\overline{\mathbb{R}}$ based on the properties of $\limsup_{i\rightarrow\infty}$ and $\liminf_{i\rightarrow\infty}$.

Theorem 20.77. *Let $k\in\mathbb{N}_0$, $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ and $n\in\mathbb{N}_0$ then if $\lim_{i\rightarrow\infty} x_i$ exists we have that $\lim_{i\rightarrow\infty} x_{i+n}$ exists and $\lim_{i\rightarrow\infty} x_{i+n} = \lim_{i\rightarrow\infty} x_i$.*

Proof. As $\lim_{i\rightarrow\infty} x_i$ we have that $\liminf_{i\rightarrow\infty} x_i = \limsup_{i\rightarrow\infty} x_i = \lim_{i\rightarrow\infty} x_i$. Using [theorem: 20.58] it follows that $\liminf_{i\rightarrow\infty} x_{i+n} = \limsup_{i\rightarrow\infty} x_{i+n}$ so that $\lim_{i\rightarrow\infty} x_{i+n}$ exists and $\lim_{i\rightarrow\infty} x_{i+n} = \lim_{i\rightarrow\infty} x_{i+n}$. □

Theorem 20.78. *The limit in $\overline{\mathbb{R}}$ has the following properties where $k\in\mathbb{N}_0$*

1. *Let $x\in\overline{\mathbb{R}}$ and $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ then we have*

   a. *If $\forall i\in\{k,\ldots,\infty\}$ $x\leqslant x_i$ then $x\leqslant\lim_{i\rightarrow\infty} x_i$*

   b. *If $\forall i\in\{k,\ldots,\infty\}$ $x_i\leqslant x$ then $\lim_{i\rightarrow\infty} x_i\leqslant x$*

2. *Let $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ and $\{y_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ be such that $\lim_{i\rightarrow\infty} x_i$, $\lim_{i\rightarrow\infty} y_i$ exist and $\forall i\in\{k,\ldots,\infty\}$ $x_i\leqslant y_i$ then $\lim_{i\rightarrow\infty} x_i\leqslant\lim_{i\rightarrow\infty} y_i$.*

3. *If $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ has a limit and $\alpha\in\mathbb{R}$ then $\{\alpha\cdot x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ has a limit and $\lim_{i\rightarrow\infty}(\alpha\cdot x_i) = \alpha\cdot\lim_{i\rightarrow\infty} x_i$.*

4. *If $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ has a limit, $x\in\overline{\mathbb{R}}$ and we have that $\forall i\in\{k,\ldots,\infty\}$ $(x_i,x)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$ then $\lim_{i\rightarrow\infty}(x_i+x)$ exists and $\lim_{i\rightarrow\infty}(x_i+x) = \lim_{i\rightarrow\infty} x_i + x$.*

5. *Let $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ and $\{y_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ be sequences with limits, $\left(\lim_{i\rightarrow\infty} x_i, \lim_{i\rightarrow\infty} y_i\right)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$ [so that $\lim_{i\rightarrow\infty} x_i + \lim_{i\rightarrow\infty} y_i$ is well defined] then there exist a $n\in\{k,\ldots,\infty\}$ such that $\forall i\in\{n,\ldots,\infty\}$ $(x_i,y_i)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$ [hence x_i+y_i is defined] and $\{x_i+y_i\}_{i\in\{n,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ has a limit with*

$$\lim_{i\rightarrow\infty}(x_i+y_i) = \lim_{i\rightarrow\infty} x_i + \lim_{i\rightarrow\infty} y_i$$

6. *Let $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ and $\{y_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ be sequences of extended reals with limits such that for $x=\lim_{i\rightarrow\infty} x_i$ and $y=\lim_{i\rightarrow\infty}$ we have*

$$(x,y)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(0,-\infty),(0,\infty),(-\infty,0),(\infty,0)\}$$

then $\lim_{i\rightarrow\infty}(x_i\cdot y_i)=\left(\lim_{i\rightarrow\infty}x_i\right)\cdot\left(\lim_{i\rightarrow\infty}y_i\right)$

7. *Let* $\{x_i\}_{i\in\{k,\ldots,\infty\}},\{y_i\}_{i\in\{k,\ldots,\infty\}}$ *be sequences of extended reals with limits then we have*

   a. $\lim_{i\rightarrow\infty}(\min(x_i,y_i))=\min\left(\lim_{i\rightarrow\infty}x_i,\lim_{i\rightarrow\infty}y_i\right)$

   b. $\lim_{i\rightarrow\infty}(\max(x_i,y_i))=\max\left(\lim_{i\rightarrow\infty}x_i,\lim_{i\rightarrow\infty}y_i\right)$

***Note* 20.79.** *The condition in (3) that* $\alpha\in\mathbb{R}$ *is required. For example the sequence* $\left\{\frac{(-1)^i}{i}\right\}_{i\in\{1,\ldots,\infty\}}$ *converges to* 0 *but* $\left\{\infty\cdot\frac{(-1)^i}{i}\right\}_i=\{(-\infty)^i\}_{i\in\{1,\ldots,\infty\}}$ *does not converges [see examples: 20.72, 20.75].*

***Note* 20.80.** *The extra conditions in (5) are needed, for example take* $\{x_i\}_{i\in\mathbb{N}_0}$ *defined by* $x_i=\frac{1}{i}$ *and* $\{y_i\}_{i\in\mathbb{N}_0}$ *defined by* $y_i=i^2$ *then* $\lim_{i\rightarrow\infty}x_i=0$ *and* $\lim_{i\rightarrow\infty}y_i=\infty$ *but* $\lim_{i\rightarrow\infty}(x_i\cdot y_i)=\lim_{i\rightarrow\infty}i=\infty\neq\infty\cdot 0=0$

Proof.

1.
   a. As $\forall i\in\{k,\ldots,\infty\}$ $x\leqslant x_i$ it follows from [theorem: 20.60] that $x\leqslant\liminf_{i\rightarrow\infty}x_i$ proving, as $\liminf_{i\rightarrow\infty}x_i=\lim_{i\rightarrow\infty}x_i$, that $x\leqslant\lim_{i\rightarrow\infty}x_i$.

   b. As $\forall i\in\{k,\ldots,\infty\}$ $x_i\leqslant x$ it follows from [theorem: 20.60] that $\liminf_{i\rightarrow\infty}x_i\leqslant x$ proving, as $\liminf_{i\rightarrow\infty}x_i=\lim_{i\rightarrow\infty}x_i$, that $x\leqslant\lim_{i\rightarrow\infty}x_i$.

2. As $\forall i\in\{k,\ldots,\infty\}$ $x_i\leqslant y_i$ we have by [theorem: 20.59] that $\liminf_{i\rightarrow\infty}x_i\leqslant\liminf_{i\rightarrow\infty}y_i$. Hence

$$\lim_{i\rightarrow\infty}x_i\underset{\text{def}}{=}\liminf_{i\rightarrow\infty}x_i\leqslant\liminf_{i\rightarrow\infty}y_i\underset{\text{def}}{=}\lim_{i\rightarrow\infty}y_i$$

3. For $\alpha\in\mathbb{R}$ we have the following cases to consider:

   $\boldsymbol{0\leqslant\alpha}$**.** Then we have

$$\begin{aligned}\liminf_{i\rightarrow\infty}(\alpha\cdot x_i) &\underset{\text{[theorem: 20.66]}}{=} \alpha\cdot\liminf_{i\rightarrow\infty}x_i\\ &\underset{\text{def}}{=} \alpha\cdot\lim_{i\rightarrow\infty}x_i\\ &\underset{\text{def}}{=} \alpha\cdot\limsup_{i\rightarrow\infty}x_i\\ &\underset{\text{[theorem: 20.66]}}{=} \limsup_{i\rightarrow\infty}(\alpha\cdot x_i)\end{aligned}$$

   so that $\lim_{i\rightarrow\infty}(\alpha\cdot x_i)$ exists and $\lim_{i\rightarrow\infty}(\alpha\cdot x_i)=\alpha\cdot\lim_{i\rightarrow\infty}x_i$

   $\boldsymbol{\alpha<0}$**.** Then $0<-\alpha$ so that

$$\begin{aligned}\liminf_{i\rightarrow\infty}(\alpha\cdot x_i) &\underset{\text{[theorem: 20.64]}}{=} -\limsup_{i\rightarrow\infty}((-\alpha)\cdot x_i)\\ &\underset{\text{[theorem: 20.66]}}{=} -\left((-\alpha)\cdot\limsup_{i\rightarrow\infty}x_i\right)\\ &= \alpha\cdot\limsup_{i\rightarrow\infty}x_i\\ &\underset{\text{def}}{=} \alpha\cdot\lim_{i\rightarrow\infty}x_i\\ &\underset{\text{def}}{=} \alpha\cdot\liminf_{i\rightarrow\infty}x_i\\ &= -\left((-\alpha)\cdot\liminf_{i\rightarrow\infty}x_i\right)\\ &\underset{\text{[theorem: 20.66]}}{=} -\liminf_{i\rightarrow\infty}((-\alpha)\cdot x_i)\\ &\underset{\text{[theorem: 20.64]}}{=} \limsup_{i\rightarrow\infty}(\alpha\cdot x_i)\end{aligned}$$

so that $\liminf_{i\rightarrow\infty}(\alpha\cdot x_i)=\limsup_{i\rightarrow\infty}(\alpha\cdot x_i)=\alpha\cdot\lim_{i\rightarrow\infty}x_i$. Hence

$$\lim_{i\rightarrow\infty}(\alpha\cdot x_i)\text{ exist and }\lim_{i\rightarrow\infty}(\alpha\cdot x_i)=\alpha\cdot\lim_{i\rightarrow\infty}x_i$$

4. Using [theorem: 20.62] we have that

$$\begin{aligned}\liminf_{i\rightarrow\infty}(x_i+x) &\underset{[\text{theorem: }20.62]}{=} \left(\liminf_{i\rightarrow\infty}x_i\right)+x\\ &\underset{\text{def}}{=} \left(\lim_{i\rightarrow\infty}x_i\right)+x\\ &\underset{\text{def}}{=} \left(\limsup_{i\rightarrow\infty}x_i\right)+x\\ &\underset{[\text{theorem: }20.62]}{=} \limsup_{i\rightarrow\infty}(x_i+x)\end{aligned}$$

Hence

$$\lim_{i\rightarrow\infty}(x_i+x)\text{ exist and }\lim_{i\rightarrow\infty}(x_i+x)=\left(\lim_{i\rightarrow\infty}x_i\right)+x$$

5. As $\left(\lim_{i\rightarrow\infty}x_i,\lim_{i\rightarrow\infty}y_i\right)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$ we have the following cases to consider:

$\lim_{i\rightarrow\infty}x_i\in\mathbb{R}\wedge\lim_{i\rightarrow\infty}y_i\in\mathbb{R}$. Let $x=\lim_{i\rightarrow\infty}x_i$ and $y=\lim_{i\rightarrow\infty}y_i$ then by [theorem: 20.76] there exists $N_1,N_2\in\{k,\ldots,\infty\}$ such that

$$\forall i\in\{N_1,\ldots,\infty\}\ |x_i-x|<1\Rightarrow-\infty<x-1<x_i<x+1<\infty$$

and

$$\forall i\in\{N_2,\ldots,\infty\}\ |y_i-y|<1\Rightarrow-\infty<y-1<x=y_i<y+1<\infty$$

Hence if take $n=\max(N_1,N_2)$ then we have

$$\forall i\in\{n,\ldots,\infty\}\text{ that }(x_i,y_i)\in\mathbb{R}\times\mathbb{R}\subseteq(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$$

Let $\varepsilon\in\mathbb{R}^+$ then [theorem: 20.76] there exists $M_1,M_2\in\{k,\ldots,\infty\}$ such that

$$\forall i\in\{M_1,\ldots,\infty\}\ |y_i-y|<\frac{\varepsilon}{2}$$

and

$$\forall i\in\{M_1,\ldots,\infty\}\ |y_i-y|<\frac{\varepsilon}{2}$$

Take $N=\max(M_1,M_2,n)$ then $\forall i\in\{N,...,\infty\}$ we have that x_i+y_i is well defined and

$$|x_i+y_i-(x+y)|\leqslant|x_i-x|+|y_i-y|=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

Hence using [theorem: 20.76] it follows that $\{x_i+y_i\}_{i\in\{n,\ldots,\infty\}}$ has a limit

$$\lim_{i\rightarrow\infty}(x_i+y_i)=x+y=\lim_{i\rightarrow\infty}x_i+\lim_{i\rightarrow\infty}y_i$$

$\lim_{i\rightarrow\infty}x_i=-\infty\wedge\lim_{i\rightarrow\infty}y_i\in\mathbb{R}$. Let $y=\lim_{i\rightarrow\infty}y_i$ then by [theorem: 20.76] there exists $N_1,N_2\in\{k,\ldots,\infty\}$ such that

$$\forall i\in\{N_1,\ldots,\infty\}\text{ we have }x_i<-1<\infty$$

$$\forall i\in\{N_2,\ldots,\infty\}\text{ we have }|y-1|<1\Rightarrow-\infty<y-1<y_i<y+1<\infty$$

Hence if we take $n=\max(N_1,N_2)$ we have that

$$\forall i\in\{n,\ldots,\infty\}\text{ that }(x_i,y_i)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$$

Further take $C\in\mathbb{R}^+$ and take $C'=\max(C+(y+1),1)\in\mathbb{R}^+$ then $C+(y+1)\leqslant C'$ or $-C'\leqslant -C-(y+1)$. By [theorem: 20.76] there exist a $N_3\in\{k,\ldots,\infty\}$ such that $\forall i\in\{N_3,\ldots,\infty\}$ we have that $x_i<-C'$. Hence if $N=\max(n,N_3)\in\{n,\ldots,\infty\}$ then we have for $i\in\{N,\ldots,\infty\}$ that $x_i<-C'$ and as $y_i\in\mathbb{R}$ it follows from [theorem: 20.29] that $x_i+y_i<-C'+y_i$. Hence

$$\begin{array}{rcl} x_i+y_i & < & -C'+y_i \\ & \leqslant & -C-(y+1)+y_i \\ & < & -C-(y+1)+y+1 \\ & = & -C \end{array}$$

Using [theorem: 20.76] it follows then that $\{x_i+y_i\}_{i\in\{n,\ldots,\infty\}}$ has a limit and $\lim_{i\rightarrow\infty}(x_i+y_i)=-\infty$. As $\lim_{i\rightarrow\infty}x_i=-\infty$ and $\lim_{i\rightarrow\infty}y_i\in\mathbb{R}$ it follows that $\lim_{i\rightarrow\infty}x_i+\lim_{i\rightarrow\infty}y_i=-\infty$. So

$$\lim_{i\rightarrow\infty}(x_i+y_i) \text{ exist and } \lim_{i\rightarrow\infty}(x_i+y_i)=\lim_{i\rightarrow\infty}x_i+\lim_{i\rightarrow\infty}y_i$$

$\boldsymbol{\lim_{i\rightarrow\infty}x_i=\infty\wedge\lim_{i\rightarrow\infty}y_i\in\mathbb{R}}$**.** Let $y=\lim_{i\rightarrow\infty}y_i$ then by [theorem: 20.76] there exists $N_1,N_2\in\{k,\ldots,\infty\}$ such that

$$\forall i\in\{N_1,\ldots,\infty\} \text{ we have } -\infty<1<x_i$$

$$\forall i\in\{N_2,\ldots,\infty\} \text{ we have } |y-1|<1\Rightarrow-\infty<y-1<y_i<y+1<\infty$$

Hence if we take $n=\max(N_1,N_2)$ we have that

$$\forall i\in\{n,\ldots,\infty\} \text{ that } (x_i,y_i)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$$

Further take $C\in\mathbb{R}^+$ and take $C'=\max(C-(y-1),1)\in\mathbb{R}^+$ then $C-(y-1)\leqslant C'$. By [theorem: 20.76] there exist a $N_3\in\{k,\ldots,\infty\}$ such that $\forall i\in\{N_3,\ldots,\infty\}$ we have that $C'<x$. Hence if $N=\max(n,N_3)\in\{n,\ldots,\infty\}$ then we have for $i\in\{N,\ldots,\infty\}$ that $C'<x_i$ and as $y_i\in\mathbb{R}$ it follows from [theorem: 20.29] that $C'+y_i<x_i+y_i$. Hence

$$\begin{array}{rcl} x_i+y_i & > & C'+y_i \\ & \geqslant & C-(y-1)+y_i \\ & > & C-(y-1)+(y-1) \\ & = & C \end{array}$$

Using [theorem: 20.76] it follows then that $\{x_i+y_i\}_{i\in\{n,\ldots,\infty\}}$ has a limit and $\lim_{i\rightarrow\infty}(x_i+y_i)=\infty$. As $\lim_{i\rightarrow\infty}x_i=\infty$ and $\lim_{i\rightarrow\infty}y_i\in\mathbb{R}$ it follows that $\lim_{i\rightarrow\infty}x_i+\lim_{i\rightarrow\infty}y_i=\infty$. So

$$\lim_{i\rightarrow\infty}(x_i+y_i) \text{ exist and } \lim_{i\rightarrow\infty}(x_i+y_i)=\lim_{i\rightarrow\infty}x_i+\lim_{i\rightarrow\infty}y_i$$

$\boldsymbol{\lim_{i\rightarrow\infty}x_i\in\mathbb{R}\wedge\lim_{i\rightarrow\infty}y_i=-\infty}$**.** Let $x=\lim_{i\rightarrow\infty}x_i$ then by [theorem: 20.76] there exists $N_1,N_2\in\{k,\ldots,\infty\}$ such that

$$\forall i\in\{N_1,\ldots,\infty\} \text{ we have } y_i<-1<\infty$$

$$\forall i\in\{N_2,\ldots,\infty\} \text{ we have } |x-1|<1\Rightarrow-\infty<x-1<x_i<x+1<\infty$$

Hence if we take $n=\max(N_1,N_2)$ we have that

$$\forall i\in\{n,\ldots,\infty\} \text{ that } (x_i,y_i)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$$

Further take $C\in\mathbb{R}^+$ and take $C'=\max(C+(x+1),1)\in\mathbb{R}^+$ then $C+(x+1)\leqslant C'$ or $-C'\leqslant -C-(x+1)$. By [theorem: 20.76] there exist a $N_3\in\{k,\ldots,\infty\}$ such that $\forall i\in\{N_3,\ldots,\infty\}$ we have that $y_i<-C'$. Hence if $N=\max(n,N_3)\in\{n,\ldots,\infty\}$ then we have for $i\in\{N,\ldots,\infty\}$ that $y_i<-C'$ and as $y_i\in\mathbb{R}$ it follows from [theorem: 20.29] that $x_i+y_i<-C'+x_i$. Hence

$$\begin{array}{rcl} x_i+y_i & < & -C'+x_i\\ & \leqslant & -C-(x+1)+x_i\\ & < & -C-(x+1)+x+1\\ & = & -C \end{array}$$

Using [theorem: 20.76] it follows then that $\{x_i+y_i\}_{i\in\{n,\ldots,\infty\}}$ has a limit and $\lim_{i\rightarrow\infty}(x_i+y_i)=-\infty$. As $\lim_{i\rightarrow\infty} y_i=-\infty$ and $\lim_{i\rightarrow\infty} x_i\in\mathbb{R}$ it follows that $\lim_{i\rightarrow\infty} x_i+\lim_{i\rightarrow\infty} y_i=-\infty$. So

$$\lim_{i\rightarrow\infty}(x_i+y_i) \text{ exist and } \lim_{i\rightarrow\infty}(x_i+y_i)=\lim_{i\rightarrow\infty} x_i+\lim_{i\rightarrow\infty} y_i$$

$\boldsymbol{\lim_{i\rightarrow\infty} x_i=-\infty\wedge\lim_{i\rightarrow\infty} y_i=-\infty.}$ Using [theorem: 20.76] there exists $N_1,N_2\in\{k,\ldots,\infty\}$ such that

$$\forall i\in\{N_1,\ldots,\infty\}\ x_i<-1<\infty \text{ and } \forall i\in\{N_2,\ldots,\infty\}\ y_i<-1<\infty$$

So if we take $n=\max(N_1,N_2)$ then for $i\in\{n,\ldots,\infty\}$ we have

$$(x_i,y_i)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$$

Further if $C\in\mathbb{R}^+$ then there exists $N_3,N_4\in\{n,\ldots,\infty\}$ such that

$$\forall i\in\{N_3,\ldots,\infty\}\ x_i<-\frac{C}{2} \text{ and } \forall i\in\{N_4,\ldots,\infty\}\ y_i<-\frac{C}{2}$$

Take $N=\max(N_3,N_4)\in\{n,\ldots,\infty\}$ then $\forall i\in\{N,\ldots,\infty\}$ we have by [theorem: 20.29] that

$$x_i+y_i<-\frac{C}{2}+\left(-\frac{C}{2}\right)=-C$$

So that by [theorem: 20.76] $\{x_i+y_i\}_{i\in\{n,\ldots,\infty\}}$ has a limit and $\lim_{i\rightarrow\infty}(x_i+y_i)=-\infty$, hence, as $\lim_{i\rightarrow\infty} x_i+\lim_{i\rightarrow\infty} y_i=-\infty+(-\infty)=-\infty$, it follows that

$$\lim_{i\rightarrow\infty}(x_i+y_i) \text{ exist and } \lim_{i\rightarrow\infty}(x_i+y_i)=\lim_{i\rightarrow\infty} x_i+\lim_{i\rightarrow\infty} y_i$$

$\boldsymbol{\lim_{i\rightarrow\infty} x_i\in\mathbb{R}\wedge\lim_{i\rightarrow\infty} y_i=\infty.}$ Let $x=\lim_{i\rightarrow\infty} x_i$ then by [theorem: 20.76] there exists $N_1,N_2\in\{k,\ldots,\infty\}$ such that

$$\forall i\in\{N_1,\ldots,\infty\} \text{ we have } -\infty<1<y_i$$

$$\forall i\in\{N_2,\ldots,\infty\} \text{ we have } |x-1|<1\Rightarrow -\infty<x-1<x_i<x+1<\infty$$

Hence if we take $n=\max(N_1,N_2)$ we have that

$$\forall i\in\{n,\ldots,\infty\} \text{ that } (x_i,y_i)\in(\overline{\mathbb{R}}\times\overline{\mathbb{R}})\setminus\{(-\infty,\infty),(\infty,-\infty)\}$$

Further take $C\in\mathbb{R}^+$ and take $C'=\max(C-(x-1),1)\in\mathbb{R}^+$ then $C-(x-1)\leqslant C'$. By [theorem: 20.76] there exist a $N_3\in\{k,\ldots,\infty\}$ such that $\forall i\in\{N_3,\ldots,\infty\}$ we have that $C'<y$. Hence if $N=\max(n,N_3)\in\{n,\ldots,\infty\}$ then we have for $i\in\{N,\ldots,\infty\}$ that $C'<y_i$ and as $x_i\in\mathbb{R}$ it follows from [theorem: 20.29] that $C'+x_i<x_i+y_i$. Hence

$$\begin{array}{rcl} x_i+y_i & > & C'+x_i\\ & \geqslant & C-(x-1)+x_i\\ & > & C-(x-1)+(x-1)\\ & = & C \end{array}$$

Using [theorem: 20.76] it follows then that $\{x_i + y_i\}_{i \in \{n, \ldots, \infty\}}$ has a limit and $\lim_{i \rightarrow \infty} (x_i + y_i) = \infty$. As $\lim_{i \rightarrow \infty} y_i = \infty$ and $\lim_{i \rightarrow \infty} x_i \in \mathbb{R}$ it follows that $\lim_{i \rightarrow \infty} x_i + \lim_{i \rightarrow \infty} y_i = \infty$. So

$$\lim_{i \rightarrow \infty} (x_i + y_i) \text{ exist and } \lim_{i \rightarrow \infty} (x_i + y_i) = \lim_{i \rightarrow \infty} x_i + \lim_{i \rightarrow \infty} y_i$$

$\boldsymbol{\lim_{i \rightarrow \infty} x_i = \infty \wedge \lim_{i \rightarrow \infty} y_i = \infty}$**.** Using [theorem: 20.76] there exists $N_1, N_2 \in \{k, \ldots, \infty\}$ such that

$$\forall i \in \{N_1, \ldots, \infty\} \ -\infty < 1 < x_i \text{ and } \forall i \in \{N_2, \ldots, \infty\} \ -\infty < 1 < y_i$$

So if we take $n = \max(N_1, N_2)$ then for $i \in \{n, \ldots, \infty\}$ we have

$$(x_i, y_i) \in (\overline{\mathbb{R}} \times \overline{\mathbb{R}}) \setminus \{(-\infty, \infty), (\infty, -\infty)\}$$

Further if $C \in \mathbb{R}^+$ then there exists $N_3, N_4 \in \{n, \ldots, \infty\}$ such that

$$\forall i \in \{N_3, \ldots, \infty\} \ \frac{C}{2} < x_i \text{ and } \forall i \in \{N_4, \ldots, \infty\} \ \frac{C}{2} < y_i$$

Take $N = \max(N_3, N_4) \in \{n, \ldots, \infty\}$ then $\forall i \in \{N, \ldots, \infty\}$ we have by [theorem: 20.29] that

$$C = \frac{C}{2} + \frac{C}{2} < x_i + y_i$$

So that by [theorem: 20.76] $\{x_i + y_i\}_{i \in \{n, \ldots, \infty\}}$ has a limit and $\lim_{i \rightarrow \infty} (x_i + y_i) = \infty$, hence, as $\lim_{i \rightarrow \infty} x_i + \lim_{i \rightarrow \infty} y_i = \infty + \infty = \infty$, it follows that

$$\lim_{i \rightarrow \infty} (x_i + y_i) \text{ exist and } \lim_{i \rightarrow \infty} (x_i + y_i) = \lim_{i \rightarrow \infty} x_i + \lim_{i \rightarrow \infty} y_i$$

6. As $\left(\lim_{i \rightarrow \infty} x_i, \lim_{i \rightarrow \infty} y_i\right) \in (\overline{\mathbb{R}} \times \overline{\mathbb{R}}) \setminus \{(0, -\infty), (0, \infty), (-\infty, 0), (\infty, 0)\}$ we have either:

$\boldsymbol{\lim_{i \rightarrow \infty} x_i \in \mathbb{R} \wedge \lim_{i \rightarrow \infty} y_i \in \mathbb{R}}$**.** Let $x = \lim_{i \rightarrow \infty} x_i$ and $y = \lim_{i \rightarrow \infty} y_i$. Take $\varepsilon \in \mathbb{R}^+$. Using [theorem: 20.76] there exists a $N_1 \in \{k, \ldots, \infty\}$ such that

$$\forall i \in \{N_1, \ldots, \infty\} \text{ we have } |x - x_i| < 1 \Rightarrow |x_i| = |x - x_i| + |x| < 1 + |x|$$

Further as $1 + |x| \in \mathbb{R}^+$ there exists a $N_2 \in \{k, \ldots, \infty\}$ such that

$$\forall i \in \{N_2, \ldots, \infty\} \ |x_i - x| < \frac{\varepsilon}{2 \cdot (1 + |y|)} < \infty$$

and there exist a $N_3 \in \{k, \ldots, \infty\}$ such that

$$\forall i \in \{N_3, \ldots, \infty\} \ |y_i - y| < \frac{\varepsilon}{2 \cdot (1 + |x|)} < \infty$$

Let $N = \max(N_1, N_2, N_3)$ then we have for $i \in \{N, \ldots, \infty\}$ that $-\infty < x - \varepsilon < x_i < x + \varepsilon < \infty$ and $-\infty < y - \varepsilon < y_i < y + \varepsilon < \infty$ so that $x_i, y_i \in \mathbb{R}$ hence

$$0 \leqslant |x_i|, |y_i|, |x|, |y| \in \mathbb{R}$$

$$\begin{aligned}
|x_i \cdot y_i - x \cdot y| &= |x_i \cdot y_i - x_i \cdot y + x_i \cdot y - x \cdot y| \\
&= |x_i \cdot (y_i - y) + (x_i - x) \cdot y| \\
&\leqslant |x_i \cdot (y_i - y)| + |(x_i - x) \cdot y| \\
&= |x_i| \cdot |y_i - y| + |x_i - x| \cdot |y| \\
&\leqslant |x_i| \cdot \frac{\varepsilon}{2 \cdot (1 + |x|)} + \frac{\varepsilon}{2 \cdot (1 + |y|)} \cdot |y| \\
&< (1 + |x_i|) \cdot \frac{\varepsilon}{2 \cdot (1 + |x|)} + \frac{\varepsilon}{2 \cdot (1 + |y|)} \cdot |y| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Using [theorem: 20.76] again it follows that

$$\lim_{i \to \infty} (x_i \cdot y_i) \text{ exists and } \lim_{i \to \infty} (x_i \cdot y_i) = \left(\lim_{i \to \infty} x_i\right) \cdot \left(\lim_{i \to \infty} y_i\right)$$

$\boldsymbol{\lim_{i \to \infty} x_i = \infty \wedge \lim_{i \to \infty} y_i \in \mathbb{R} \setminus \{0\}}$**.** Let $C \in \mathbb{R}^+$. For $y = \lim_{i \to \infty} y_i$ we have either:

$\boldsymbol{0 < y}$**.** For $\frac{y}{2} \in \mathbb{R}^+$ we have by [theorem: 20.76] that there exist a N_1 such that

$$\forall i \in \{N_1, \ldots, \infty\} \text{ we have } -\infty < \frac{y}{2} = y - \frac{y}{2} < y_i < y + \frac{y}{2} < \infty \tag{20.28}$$

Let $C \in \mathbb{R}^+$ then $\frac{2 \cdot (C+1)}{y} \in \mathbb{R}^+$ so by [theorem: 20.76] there exist a $N_2 \in \{k, \ldots, \infty\}$ such that

$$\forall i \in \{N_2, \ldots, \infty\} \text{ we have } \frac{2 \cdot (C+1)}{y} < x_i \tag{20.29}$$

Let $N = \max(N_1, N_2)$ then $\forall i \in \{N, \ldots, \infty\}$ we have as $\frac{2 \cdot (C+1)}{y} \in \mathbb{R}^+$ by [theorem: 20.37] that

$$C + 1 = \frac{y}{2} \cdot \frac{2 \cdot (C+1)}{y} \underset{[\text{eq: } 20.28]}{\leqslant} \frac{2 \cdot (C+1)}{y} \cdot y_i$$

Further as by [eq: 20.28] $0 < y_i$ we have by [theorem: 20.37] and [eq: 20.29] that

$$\frac{2 \cdot (C+1)}{y} \cdot y_i < x_i \cdot y_i = \left(\lim_{i \to \infty} x_i\right) \cdot \left(\lim_{i \to \infty} y_i\right)$$

So that $C < C + 1 < x_i \cdot y_i$ $\forall i \in \{N, \ldots, \infty\}$, hence using [theorem: 20.76] we have that

$$\lim_{i \to \infty} (x_i \cdot y_i) \text{ exists and } \lim_{i \to \infty} (x_i \cdot y_i) = \infty = \infty \cdot y = x \cdot y$$

$\boldsymbol{y < 0}$**.** By (3) we have that $-y \in \lim_{i \to \infty} (-y_1)$. For $-\frac{y}{2} \in \mathbb{R}^+$ we have by [theorem: 20.76] that there exist a N_1 such that

$$\forall i \in \{N_1, \ldots, \infty\} \text{ we have } -\infty < -\frac{y}{2} = -y - \left(-\frac{y}{2}\right) < -y_i < -y + \left(-\frac{y}{2}\right) \tag{20.30}$$

Let $C \in \mathbb{R}^+$ then $-\frac{2 \cdot (C+1)}{y} \in \mathbb{R}^+$ so by [theorem: 20.76] there exist a $N_2 \in \{k, \ldots, \infty\}$ such that

$$\forall i \in \{N_2, \ldots, \infty\} \text{ we have } -\frac{2 \cdot (C+1)}{y} < x_i \tag{20.31}$$

Let $N = \max(N_1, N_2)$ then $\forall i \in \{N, \ldots, \infty\}$ we have as $-\frac{2 \cdot (C+1)}{y} \in \mathbb{R}^+$ by [theorem: 20.37] that

$$C + 1 = \left(-\frac{y}{2}\right) \cdot \left(-\frac{2 \cdot (C+1)}{y}\right) \underset{[\text{eq: } 20.30]}{<} \left(-\frac{2 \cdot (C+1)}{y}\right) \cdot (-y_i)$$

Further as by [eq: 20.30] $0 < -y_i$ so that by [theorem: 20.37] and [eq: 20.31]

$$-\frac{2 \cdot (C+1)}{y} \cdot (-y_i) < x_i \cdot (-y_i) = -(x_i \cdot y_i)$$

hence

$$C < C + 1 < -(x_i \cdot y_i)$$

Hence by [theorem: 20.76] $\lim_{i \to \infty} (-(x_i \cdot y_i))$ exists and $\lim_{i \to \infty} (-(x_i \cdot y_i)) = \infty$, using (3) it follows that $\lim_{i \to \infty} (x_i \cdot y_i)$ exist and

$$\lim_{i \to \infty} (x_i \cdot y_i) = -\lim_{i \to \infty} (-(x_i \cdot y_i)) = -\infty = \infty \cdot y = x \cdot y$$

or

$$\lim_{i\rightarrow\infty}(x_i\cdot y_i) \text{ exists and } \lim_{i\rightarrow\infty}(x_i\cdot y_i)=x\cdot y=\left(\lim_{i\rightarrow\infty}x_i\right)\cdot\left(\lim_{i\rightarrow\infty}y_i\right)$$

$\boldsymbol{\lim_{i\rightarrow\infty}x_i=-\infty\wedge\lim_{i\rightarrow\infty}y_i\in\mathbb{R}\setminus\{0\}}$**.** Define $\{x_i'\}_{i\in\{k,\ldots,\infty\}}$ by $x_i'=-x_i$ then $\lim_{i\rightarrow\infty}x_i' \underset{(2)}{=} -\lim_{i\rightarrow\infty}x_i=\infty$. Hence this reduces to the case $\lim_{i\rightarrow\infty}x_i=-\infty\wedge\lim_{i\rightarrow\infty}y_i\in\mathbb{R}\setminus\{0\}$ by substituting x_i by x_i'. So we have that $\lim_{i\rightarrow\infty}(x_i'\cdot y_i)=\left(\lim_{i\rightarrow\infty}x_i'\right)\cdot\left(\lim_{i\rightarrow\infty}y_i\right)$ so

$$\begin{aligned}
\lim_{i\rightarrow\infty}(x_i\cdot y_i) &\underset{(3)}{=} -\lim_{i\rightarrow\infty}(-(x_i\cdot y_i))\\
&= \lim_{i\rightarrow\infty}(x_i'\cdot y_i)\\
&= \left(\lim_{i\rightarrow\infty}x_i'\right)\cdot\left(\lim_{i\rightarrow\infty}y_i\right)\\
&= \left(\lim_{i\rightarrow\infty}(-x_i)\right)\cdot\left(\lim_{i\rightarrow\infty}y_i\right)\\
&= -\left(\lim_{i\rightarrow\infty}x_i\right)\cdot\left(\lim_{i\rightarrow\infty}y_i\right)
\end{aligned}$$

Hence

$$\lim_{i\rightarrow\infty}(x_i\cdot y_i) \text{ exists and } \lim_{i\rightarrow\infty}(x_i\cdot y_i)=\left(\lim_{i\rightarrow\infty}x_i\right)\cdot\left(\lim_{i\rightarrow\infty}y_i\right)$$

$\boldsymbol{\lim_{i\rightarrow\infty}x_i\in\mathbb{R}\setminus\{0\}\wedge\lim_{i\rightarrow\infty}y_i=\infty}$**.** Define $\{x_i'\}_{i\in\{k,\ldots,\infty\}}$, $\{y_i'\}_{i\in\{k,\ldots,\infty\}}$ by $x_i'=y_i\wedge y_i'=x_i$. Then this cases reduces to the case $\lim_{i\rightarrow\infty}x_i=\infty\wedge\lim_{i\rightarrow\infty}y_i\in\mathbb{R}\setminus\{0\}$ hence

$$\lim_{i\rightarrow\infty}(x_i'\cdot y_i') \text{ exists and } \lim_{i\rightarrow\infty}(x_i'\cdot y_i')=\left(\lim_{i\rightarrow\infty}x_i'\right)\cdot\left(\lim_{i\rightarrow\infty}y_i'\right)=\left(\lim_{i\rightarrow\infty}y_i\right)\cdot\left(\lim_{i\rightarrow\infty}x_i\right)$$

so that

$$\lim_{i\rightarrow\infty}(x_i\cdot y_i)=\lim_{i\rightarrow\infty}(y_i\cdot x_i)=\lim_{i\rightarrow\infty}(x_i'\cdot y_i') \text{ exist and } \lim_{i\rightarrow\infty}(x_i\cdot y_i)=\left(\lim_{i\rightarrow\infty}y_i\right)\cdot\left(\lim_{i\rightarrow\infty}x_i\right)$$

$\boldsymbol{\lim_{i\rightarrow\infty}x_i=\infty\wedge\lim_{i\rightarrow\infty}y_i=\infty}$**.** Let $C\in\mathbb{R}^+$ then by [theorem: 20.76] there exists $N_1,N_2\in\{k,\ldots,\infty\}$ such that

$$\forall i\in\{N_1,\ldots,\infty\}\ 1<x_i \text{ and } \forall i\in\{N_2,\ldots,\infty\}\ (C+1)<y_i$$

Take $N=\max(N_1,N_2)$ then we have by [theorem: 20.37] that $C+1\leqslant(C+1)\cdot x_i$ and $(C+1)\cdot x_i\leqslant x_i\cdot y_i$ so that

$$C<C+1\leqslant x_i\cdot y_i$$

Hence using [theorem: 20.76] it follows that $\lim_{i\rightarrow\infty}(x_i\cdot y_i)$ exists and

$$\lim_{i\rightarrow\infty}(x_i\cdot y_i)=\infty=\infty\cdot\infty=\left(\lim_{i\rightarrow\infty}x_i\right)\cdot\left(\lim_{i\rightarrow\infty}y_i\right)$$

$\boldsymbol{\lim_{i\rightarrow\infty}x_i-\infty\wedge\lim_{i\rightarrow\infty}y_i=\infty}$**.** Let $C\in\mathbb{R}^+$ then by [theorem: 20.76] there exists $N_1,N_2\in\{k,\ldots,\infty\}$ such that

$$\forall i\in\{N_1,\ldots,\infty\}\ x_i<-1\Rightarrow 1<-x_i \text{ and } \forall i\in\{N_2,\ldots,\infty\}\ (C+1)<y_i$$

Let $N=\max(N_1,N_2)$ then by [theorem: 20.37] $(C+1)\leqslant(-x_i)\cdot(C+1)$ and $(-x_i)\cdot(C+1)\leqslant(-x_i)\cdot y_i=-(x_i\cdot y_i)$, hence

$$C<C+1\leqslant-(x_i\cdot y_i)$$

By [theorem: 20.76] it follows that $\lim_{i\rightarrow\infty}(-(x_i\cdot y_i))=\infty$, hence by (3) we have that

$$\lim_{i\rightarrow\infty}(x_i\cdot y_i)=-\infty=-\infty\cdot\infty=\left(\lim_{i\rightarrow\infty}x_i\right)\cdot\left(\lim_{i\rightarrow\infty}y_i\right)$$

$\lim_{i\to\infty} x_i \in \mathbb{R}\setminus\{0\} \wedge \lim_{i\to\infty} y_i = -\infty$. Define $\{x'_i\}_{i\in\{k,\ldots,\infty\}}$, $\{y'_i\}_{i\in\{k,\ldots,\infty\}}$ by $x'_i = y_i \wedge y'_i = x_i$. Then this cases reduces to the case $\lim_{i\to\infty} x_i = -\infty \wedge \lim_{i\to\infty} y_i \in \mathbb{R}\setminus\{0\}$ hence

$$\lim_{i\to\infty}(x'_i\cdot y'_i) \text{ exists and } \lim_{i\to\infty}(x'_i\cdot y'_i) = \left(\lim_{i\to\infty} x'_i\right)\cdot\left(\lim_{i\to\infty} y'_i\right) = \left(\lim_{i\to\infty} y_i\right)\cdot\left(\lim_{i\to\infty} x_i\right)$$

so that

$$\lim_{i\to\infty}(x_i\cdot y_i) = \lim_{i\to\infty}(y_i\cdot x_i) = \lim_{i\to\infty}(x'_i\cdot y'_i) \text{ exist and } \lim_{i\to\infty}(x_i\cdot y_i) = \left(\lim_{i\to\infty} y_i\right)\cdot\left(\lim_{i\to\infty} x_i\right)$$

$\lim_{i\to\infty} x_i = -\infty \wedge \lim_{i\to\infty} y_i = -\infty$. Let $C\in\mathbb{R}^+$ then by [theorem: 20.76] there exists N_1, $N_2\in\{k,\ldots,\infty\}$ such that

$$\forall i\in\{N_1,\ldots,\infty\}\ x_i < -1 \Rightarrow 1 < -x_i \text{ and } \forall i\in\{N_2,\ldots,\infty\}\ y_i < -(C+1) \Rightarrow C+1 < -y_i$$

Take $N = \max(N_1, N_2)$ then we have by [theorem: 20.37] that $C+1 \leqslant (C+1)\cdot(-x_i)$ and $(C+1)\cdot(-x_i) \leqslant (-x_i)\cdot(-y_i) = x_i\cdot y_i$ so that

$$C < C+1 \leqslant x_i\cdot y_i$$

Hence using [theorem: 20.76] it follows that $\lim_{i\to\infty}(x_i\cdot y_i)$ exists and

$$\lim_{i\to\infty}(x_i\cdot y_i) = \infty = (-\infty)\cdot(-\infty) = \left(\lim_{i\to\infty} x_i\right)\cdot\left(\lim_{i\to\infty} y_i\right)$$

$\lim_{i\to\infty} x_i = \infty \wedge \lim_{i\to\infty} y_i = -\infty$. Define $\{x'_i\}_{i\in\{k,\ldots,\infty\}}$, $\{y'_i\}_{i\in\{k,\ldots,\infty\}}$ by $x'_i = y_i \wedge y'_i = x_i$. Then this cases reduces to the case $\lim_{i\to\infty} x_i = -\infty \wedge \lim_{i\to\infty} y_i = \infty$ hence

$$\lim_{i\to\infty}(x'_i\cdot y'_i) \text{ exists and } \lim_{i\to\infty}(x'_i\cdot y'_i) = \left(\lim_{i\to\infty} x'_i\right)\cdot\left(\lim_{i\to\infty} y'_i\right) = \left(\lim_{i\to\infty} y_i\right)\cdot\left(\lim_{i\to\infty} x_i\right)$$

so that

$$\lim_{i\to\infty}(x_i\cdot y_i) = \lim_{i\to\infty}(y_i\cdot x_i) = \lim_{i\to\infty}(x'_i\cdot y'_i) \text{ exist and } \lim_{i\to\infty}(x_i\cdot y_i) = \left(\lim_{i\to\infty} y_i\right)\cdot\left(\lim_{i\to\infty} x_i\right) \quad \square$$

Next we prove that if $\lim_{i\to\infty} x_i$ exists then $\lim_{i\to\infty}\frac{1}{x^i}$ exist and $\lim_{i\to\infty}\frac{1}{x^i} = \frac{1}{\lim_{i\to\infty} x_i}$. To do this we must first prove a little lemma.

Lemma 20.81. *Let $k\in\mathbb{N}_0$ and $\{x_i\}_{i\in\{k,\ldots,\infty\}} \subseteq \overline{\mathbb{R}}$ be sequence of extended real numbers such that $\forall i\in\{k,\ldots,\infty\}\ x_i \neq 0$ and $\lim_{i\to\infty} x_i$ exists and $0 \neq \lim_{i\to\infty} x_i \in \mathbb{R}$ then $0 < \inf(\{|x_i| | i\in\{k,\ldots,\infty\}\})$*

Proof. Take $x = \lim_{i\to\infty} x_i$ and define $\varepsilon = \frac{|x|}{2} \in \mathbb{R}^+$. As $\lim_{i\to\infty} x_i \in \mathbb{R}$ if follows from [theorem: 20.76] that there exist a $N\in\{k,\ldots,\infty\}$ such that

$$\forall i\in\{N,\ldots,\infty\} \text{ we have } |x_i - x| < \varepsilon$$

Assume that $\exists i\in\{N,\ldots,\infty\}$ such that $|x_i| < \varepsilon$ then we have

$$|x| = |x - x_i + x_i| \leqslant |x - x_i| + |x_i| < \varepsilon + \varepsilon = 2\cdot\varepsilon = 2\cdot\frac{|x|}{2} = |x|$$

giving the contradiction $|x| <|x$. Hence the assumption is wrong and we must have that

$$\forall i\in\{N,\ldots,\infty\}\ \varepsilon \leqslant |x_i|$$

Take $M' = \min(\{|x_i| | i\in\{k,\ldots,N\}\})$ then as $\forall i\in\{k,\ldots,\infty\}\ x_i \neq 0$ it follows that $M'\in\mathbb{R}^+$. Take $M = \min(M', \varepsilon)\in\mathbb{R}^+$ then we have for $i\in\{k,\ldots,\infty\}$ either:

$i\in\{k,\ldots,N\}$. Then $|x_i| \geqslant M' \geqslant M$

$i\in\{N+1,\ldots,\infty\}$. Then $i\in\{N,\ldots,\infty\}$ so that $|x_i| \geqslant \varepsilon \geqslant M$

hence $\forall i \in \{k, \ldots, \infty\}$ $|x_i| \geqslant M$ or M is a lower bound of $\{|x_i|| i \in \{k, \ldots, \infty\}\}$ proving that

$$\inf(\{|x_i|| i \in \{k, \ldots, \infty\}\}) \geqslant M > 0$$ □

Theorem 20.82. *Let $k \in \mathbb{N}_0$ and $\{x_i\}_{i \in \{k, \ldots, \infty\}} \subseteq \overline{\mathbb{R}}$ be sequence of extended real numbers such that $\lim_{i \to \infty} x_i$ exists and $\lim_{i \to \infty} x_i \neq 0$ then there exist a $n \in \{k, \ldots, \infty\}$ such that $\forall i \in \{n, \ldots, \infty\}$ $x_i \neq 0$ and for $\left\{\frac{1}{x_i}\right\}_{i \in \{n, \ldots, \infty\}} \subseteq \overline{\mathbb{R}}$ we have that*

$$\lim_{i \to \infty} \frac{1}{x_i} \textit{ exists and } \lim_{i \to \infty} \frac{1}{x_i} = \frac{1}{\lim_{i \to \infty} x_i}$$

Proof. Let $x = \lim_{i \to \infty} x_i$ then we have either:

$x = \infty$. Then by [theorem: 20.76] there exists a $n \in \{k, \ldots, \infty\}$ such that $\forall i \in \{n, \ldots, \infty\}$ we have $0 < 1 < x_i$ hence $\left\{\frac{1}{x_i}\right\}_{i \in \{n, \ldots, \infty\}}$ is well defined. Take $\varepsilon \in \mathbb{R}^+$ then by [theorem: 10.30] there exist a $C \in \mathbb{N}$ such that $0 < \frac{1}{C} < \varepsilon$. As $\lim_{i \to \infty} x_i = \infty$ there exist a $m \in \{k, \ldots, \infty\}$ such that $\forall i \in \{m, \ldots, \infty\}$ $0 < C < x_i$. Take $N = \max(m, n)$ then for $i \in \{N, \ldots, \infty\}$ we have by [theorem: 20.33] that $0 < \frac{1}{x_i} \leqslant \frac{1}{C} < \varepsilon$, as $\frac{1}{x_i} = \left|\frac{1}{x_i} - 0\right|$, that $\left|\frac{1}{x_i} - 0\right| < \varepsilon$. Hence by [theorem: 20.76] it follows that

$$\lim_{i \to \infty} \frac{1}{x_i} \text{ exist and } \lim_{i \to \infty} \frac{1}{x_i} = 0 = \frac{1}{\infty} = \frac{1}{\lim_{i \to \infty} x_i}$$

$x = -\infty$. Let $\varepsilon \in \mathbb{R}^+$. Then by [theorem: 20.76] there exists a $n \in \{k, \ldots, \infty\}$ such that $\forall i \in \{n, \ldots, \infty\}$ we have $x_i < -1 < 0$ hence $\left\{\frac{1}{x_i}\right\}_{i \in \{n, \ldots, \infty\}}$ is well defined. Take $\varepsilon \in \mathbb{R}^+$ then by [theorem: 10.30] there exist a $C \in \mathbb{N}$ such that $0 < \frac{1}{C} < \varepsilon$. As $\lim_{i \to \infty} x_i = -\infty$ there exist a $m \in \{k, \ldots, \infty\}$ such that $x_i < -C$. Take $N = \max(m, n)$ then for $i \in \{N, \ldots, \infty\}$ we have that $x_i < 0 \Rightarrow 0 < -x_i$ and $x_i < -C \Rightarrow 0 < C < -x_i$. Applying [theorem: 20.33] it follows that $0 < -\frac{1}{x_i} < \frac{1}{C}$ so that $\left|\frac{1}{x_i} - 0\right| = -\frac{1}{x_i} < \frac{1}{C} = \varepsilon$. Hence by [theorem: 20.76] it follows that

$$\lim_{i \to \infty} \frac{1}{x_i} \text{ exist and } \lim_{i \to \infty} \frac{1}{x_i} = 0 = \frac{1}{-\infty} = \frac{1}{\lim_{i \to \infty} x_i}$$

$x \in \mathbb{R} \setminus \{0\}$. Let $\delta = \frac{|x|}{2}$ then as $x \neq 0$ we have that $\delta \in \mathbb{R}^+$so that by [theorem: 20.76] there exist a $n \in \{k, \ldots, \infty\}$ such that $\forall i \in \{n, \ldots, \infty\}$ we have $|x - x_i| < \delta = \frac{|x|}{2}$. Assume that there exist a $i \in \{n, \ldots, \infty\}$ such that $x_i = 0$ then $|x| = |x - 0| = |x - x_i| < \delta = \frac{|x|}{2}$ a contradiction. So the assumption is false and we must have that $\forall i \in \{n, \ldots, \infty\}$ $|x_i| \neq 0$. By [lemma: 20.81] we have that $I = \inf(\{|x_i|| i \in \{n, \ldots, \infty\}\}) \in \mathbb{R}^+$. Hence

$$\forall i \in \{n, \ldots, \infty\}\, \text{0<I<=|x\_i}$$

Take now $\varepsilon \in \mathbb{R}^+$ then by [theorem: 20.76] there exist a $M \in \{n, \ldots, \infty\}$ such that

$$\forall i \in \{M, \ldots, \infty\}\, |x_i - x| < \varepsilon \cdot |x| \cdot I$$

Take $M = \max(n, M)$ then $\forall i \in \{N, \ldots, \infty\}$ we have $|x_i - x| < \varepsilon \cdot |x| \cdot I < \varepsilon \cdot |x| \cdot |x_i|$ so that

$$\left|\frac{1}{x_i} - \frac{1}{x}\right| = \left|\frac{x - x_i}{x_i \cdot x}\right| = \frac{|x - x_i|}{|x| \cdot |x_i|} \leqslant \frac{|x - x_i|}{|x| \cdot I} < \frac{\varepsilon \cdot |x| \cdot |x_i|}{|x| \cdot |x_i|} = \varepsilon$$

which by [theorem: 20.76] proves that

$$\lim_{i \to \infty} \frac{1}{x_i} \text{ exist and } \lim_{i \to \infty} \frac{1}{x_i} = \frac{1}{x} = \frac{1}{\lim_{i \to \infty} x_o}$$ □

20.1.5 Series of non negative sequences

We want to deal with series in $\overline{\mathbb{R}}$ however there is a complication $\langle\overline{\mathbb{R}},+\rangle$ is not a semi-group so that a partial sum $\sum_{i=0}^{n} x_i$ is not always defined. However by [theorem: 20.26] $\langle\overline{\mathbb{R}}_0^+,+\rangle$ is a commutative semi-group so if we limit ourselves to non negative extended real numbers than we can use everything defined in [Section: 11.1]. If we limit ourselves to non negative extended real numbers then if $\{x_i\}_{i\in\mathbb{N}_0}\subseteq\overline{\mathbb{R}}_0^+$ $\{\sum_{i=0}^{n} x_i\}_{n\in\mathbb{N}_0}$ is a increasing sequence, as series are defined as $\lim_{n\rightarrow\infty}\sum_{i=0}^{\infty} x_i$ it makes sense to study first limits of increasing/decreasing sequences.

Lemma 20.83. *Let $k\in\mathbb{N}_0$, $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ then we have:*

1. *If $\{x_i\}_{i\in\{k,\ldots,\infty\}}$ is increasing [in other words $\forall i\in\{k,\ldots,\infty\}$ we have $x_i\leqslant x_{i+1}$] then*
$$\forall n\in\{k,\ldots,\infty\} \text{ we have } \sup(\{x_i|i\in\{n,\ldots,\infty\}\})=\sup(\{x_i|i\in\{k,\ldots,\infty\}\})$$
2. *If $\{x_i\}_{i\in\{k,\ldots,\infty\}}$ is decreasing [in other words $\forall i\in\{k,\ldots,\infty\}$ we have $x_{i+1}\leqslant x_i$] then*
$$\forall n\in\{k,\ldots,\infty\} \text{ we have } \inf(\{x_i|i\in\{n,\ldots,\infty\}\})=\inf(\{x_i|i\in\{k,\ldots,\infty\}\})$$

Proof. Let $n\in\{k,\ldots,\infty\}$ then we have:

1. As $\{x_i|i\in\{n,\ldots,\infty\}\}\subseteq\{x_i|i\in\{k,\ldots,\infty\}\}$we have by [theorem: 3.77] that
$$\sup(\{x_i|i\in\{n,\ldots,\infty\}\})\leqslant\sup(\{x_i|i\in\{k,\ldots,\infty\}\}) \tag{20.32}$$
Next if $x\in\{x_i|i\in\{k,\ldots,\infty\}\}$ then there exist a $i\in\{k,\ldots,\infty\}$ such that $x=x_i$, for i we have either:

   $\boldsymbol{i\in\{k,\ldots,n-1\}}$**.** Then $x=x_i\underset{i\leqslant n-1<n}{\leqslant}x_n\in\{x_i|i\in\{n,\ldots,\infty\}\}$

   $\boldsymbol{i\in\{n,\ldots,\infty\}}$**.** Then $x=x_i\in\{x_i|i\in\{n,\ldots,\infty\}\}$

   so in all cases there exist a $y\in\{x_i|i\in\{n,\ldots,\infty\}\}$ such that $x\leqslant y$. Hence by [theorem: 3.78] it follows that $\sup(\{x_i|i\in\{k,\ldots,\infty\}\})\leqslant\sup(\{x_i|i\in\{n,\ldots,\infty\}\})$. Combining this with [eq: 20.32] proves that
$$\sup(\{x_i|i\in\{n,\ldots,\infty\}\})=\sup(\{x_i|i\in\{k,\ldots,\infty\}\})$$
2. As $\{x_i|i\in\{n,\ldots,\infty\}\}\subseteq\{x_i|i\in\{k,\ldots,\infty\}\}$we have by [theorem: 3.77] that
$$\inf(\{x_i|i\in\{k,\ldots,\infty\}\})\leqslant\inf(\{x_i|i\in\{n,\ldots,\infty\}\}) \tag{20.33}$$
Next if $x\in\{x_i|i\in\{k,\ldots,\infty\}\}$ then there exist a $i\in\{k,\ldots,\infty\}$ such that $x=x_i$, for i we have either:

   $\boldsymbol{i\in\{k,\ldots,n-1\}}$**.** Then $x=x_i\underset{i\leqslant n-1<n}{\geqslant}x_n\in\{x_i|i\in\{n,\ldots,\infty\}\}$

   $\boldsymbol{i\in\{n,\ldots,\infty\}}$**.** Then $x=x_i\in\{x_i|i\in\{n,\ldots,\infty\}\}$

   so in all cases there exist a $y\in\{x_i|i\in\{n,\ldots,\infty\}\}$ such that $y\leqslant x$. Hence by [theorem: 3.78] it follows that $\inf(\{x_i|i\in\{k,\ldots,\infty\}\})\geqslant\inf(\{x_i|i\in\{n,\ldots,\infty\}\})$. Combining this with [eq: 20.33] proves that
$$\inf(\{x_i|i\in\{n,\ldots,\infty\}\})=\inf(\{x_i|i\in\{k,\ldots,\infty\}\})$$ □

Theorem 20.84. *Let $k\in\mathbb{N}_0$, $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ then we have:*

1. *If $\{x_i\}_{i\in\{k,\ldots,\infty\}}$ is increasing [in other words $\forall i\in\{k,\ldots,\infty\}$ we have $x_i\leqslant x_{i+1}$] then*
$$\lim_{i\rightarrow\infty} x_i \text{ exist and } \lim_{i\rightarrow\infty} x_i=\sup(\{x_i|i\in\{k,\ldots,\infty\}\})$$
2. *If $\{x_i\}_{i\in\{k,\ldots,\infty\}}$ is decreasing [in other words $\forall i\in\{k,\ldots,\infty\}$ we have $x_{i+1}\leqslant x_i$] then*
$$\lim_{i\rightarrow\infty} x_i \text{ exist and } \lim_{i\rightarrow\infty} x_i=\inf(\{x_i|i\in\{k,\ldots,\infty\}\})$$

Proof.

1. First

$$\begin{array}{rcl}\limsup_{i\rightarrow\infty} x_i & = & \inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ & \underset{[\text{lemma: }20.83]}{=} & \inf(\{\sup(\{x_i|i\in\{k,\ldots,\infty\}\})\}) \\ & = & \sup(\{x_i|i\in\{k,\ldots,\infty\}\})\end{array} \tag{20.34}$$

Next as $\{x_i\}_{i\in\{k,\ldots,\infty\}}$ is increasing we have for $l\in\{k,\ldots,\infty\}$ that $\forall i\in\{l,\ldots,\infty\}$ that $x_l\leqslant x_i$ so that $\inf(\{x_i|i\in\{l,\ldots,\infty\}\})\leqslant x_l\leqslant\inf(\{x_i|i\in\{l,\ldots,\infty\}\})$ proving that

$$x_l=\inf(\{x_i|i\in\{l,\ldots,\infty\}\}) \tag{20.35}$$

So

$$\begin{array}{rcl}\liminf_{i\rightarrow\infty} x_i & = & \sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ & \underset{[\text{eq: }20.35]}{=} & \sup(\{x_l|l\in\{k,\ldots,\infty\}\}) \\ & \underset{[\text{eq: }20.34]}{=} & \limsup_{i\rightarrow\infty} x_i\end{array}$$

so that

$$\lim_{i\rightarrow\infty} x_i \text{ exist and } \lim_{i\rightarrow\infty} x_i=\sup(\{x_l|l\in\{k,\ldots,\infty\}\})$$

2. First

$$\begin{array}{rcl}\liminf_{i\rightarrow\infty} x_i & = & \sup(\{\inf(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ & \underset{[\text{lemma: }20.83]}{=} & \sup(\{\inf(\{x_i|i\in\{k,\ldots,\infty\}\})\}) \\ & = & \inf(\{x_i|i\in\{k,\ldots,\infty\}\})\end{array} \tag{20.36}$$

Next as $\{x_i\}_{i\in\{k,\ldots,\infty\}}$ is decreasing we have for $l\in\{k,\ldots,\infty\}$ that $\forall i\in\{l,\ldots,\infty\}$ that $x_i\leqslant x_l$ so that $\sup(\{x_i|i\in\{l,\ldots,\infty\}\})\leqslant x_l\leqslant\sup(\{x_i|i\in\{l,\ldots,\infty\}\})$ proving that

$$x_l=\sup(\{x_i|i\in\{l,\ldots,\infty\}\}) \tag{20.37}$$

So

$$\begin{array}{rcl}\limsup_{i\rightarrow\infty} x_i & = & \inf(\{\sup(\{x_i|i\in\{l,\ldots,\infty\}\})|l\in\{k,\ldots,\infty\}\}) \\ & \underset{[\text{eq: }20.37]}{=} & \inf(\{x_l|l\in\{k,\ldots,\infty\}\}) \\ & \underset{[\text{eq: }20.36]}{=} & \liminf_{i\rightarrow\infty} x_i\end{array}$$

so that

$$\lim_{i\rightarrow\infty} x_i \text{ exist and } \lim_{i\rightarrow\infty} x_i=\inf(\{x_l|l\in\{k,\ldots,\infty\}\})$$ □

To be able to define the infinite sum of non negative extended we must prove that the partial sums area increasing. For this we extend [theorem: 11.55] to be valid for the extended real numbers.

Theorem 20.85. *We have for families of non negative numbers [members of $[0,\infty]=\mathbb{R}^+\bigcup\{0\}$ that:*

1. *If $n\in\mathbb{N}_0$ and $\{x_i\}_{i\in\{0,\ldots,n\}}\subseteq[0,\infty]$ a finite family of non negative extended real numbers then*

$$0\leqslant\sum_{i=0}^{n} x_i$$

2. *If I is a finite set and $\{x_i\}_{i\in I}\subseteq[0,\infty]$ a finite family of non negative extended real numbers then*

$$0\leqslant\sum_{i\in I} x_i$$

3. *If* $n \in \mathbb{N}_0$ *and* $\{x_i\}_{i \in \{0,\ldots,n\}} \subseteq [0,\infty]$, $\{y_i\}_{i \in \{0,\ldots,n\}} \subseteq [0,\infty]$ *be finite families of non negative extended real numbers such that* $\forall i \in \{0,\ldots,n\}$ $x_i \leqslant y_i$ *then*

$$\sum_{i=0}^{n} x_i \leqslant \sum_{i=0}^{n} y_i$$

4. *If* I *is a finite set and* $\{x_i\}_{i \in I} \subseteq [0,\infty]$, $\{y_i\}_{i \in I} \subseteq [0,\infty]$ *be finite families of non negative extended real numbers such that* $\forall i \in I$ $x_i \leqslant y_i$ *then*

$$\sum_{i=0}^{n} x_i \leqslant \sum_{i=0}^{n} y_i$$

5. *If* $n, m \in \mathbb{N}_0$ *and* $\{x_i\}_{i \in n+m} \subseteq [0,\ldots,\infty]$ *then*

$$\sum_{i=0}^{n} x_i \leqslant \sum_{i=0}^{n+m} x_i$$

6. *If* I *is a finite set,* $J \subseteq I$ *and* $\{x_i\}_{i \in I} \subseteq [0,\infty]$ *a finite family of non negative extended real numbers then*

$$\sum_{i \in J} x_i \leqslant \sum_{i \in I} x_i$$

Proof.

1. We use induction to prove this, so define

$$S = \left\{ n \in \mathbb{N}_0 | \text{For every } \{x_i\}_{i \in \{0,\ldots,\infty\}} \subseteq [0,\infty] \text{ we have } 0 \leqslant \sum_{i=0}^{n} x_i \right\}$$

then we have:

$0 \in S$. For $\{x_i\}_{i \in \{0\}} \subseteq [0,\infty]$ we have $0 \leqslant x_0$ so that $\sum_{i=0}^{0} x_i = x_0 \geqslant 0$ hence $0 \in S$.

$n \in S \Rightarrow n+1 \in S$. Let $\{x_i\}_{i \in \{0,\ldots,n+1\}} \subseteq [0,\infty]$ be a family of non negative numbers then we have $0 \leqslant x_{n+1}$ and, as $n \in S$, that $0 \leqslant \sum_{i=0}^{n} x_i$. Hence we have

$$0 \underset{[\text{theorem: } 20.29]}{\leqslant} x_{n+1} + \sum_{i=0}^{n} x_i = \sum_{i=0}^{n+1} x_i$$

proving that $n+1 \in S$.

2. For I we have either

$I = \varnothing$. Then by [definition: 11.32] $0 = \sum_{i \in I} x_i$.

$I \neq \varnothing$. Then by [definition: 11.32] there exist a $n \in \mathbb{N}$ and a bijection $\beta: \{0,\ldots,n-1\} \rightarrow I$ such that $\sum_{i \in I} x_i = \sum_{i=0}^{n-1} x_{\beta(i)}$. As by (1) we have that $0 \leqslant \sum_{i=0}^{n-1} x_{\beta(i)}$ so that $0 \leqslant \sum_{i \in I} x_i$.

Hence in all cases we have $0 \leqslant \sum_{i \in I} x_i$.

3. We use induction to prove this, so let

$$S = \left\{ n \in \mathbb{N}_0 | \text{For every } \{x_i\}_{i \in \{0,\ldots,\infty\}} \subseteq [0,\infty], \{y_i\}_{i \in \{0,\ldots,\infty\}} \subseteq [0,\infty] \text{ with } \forall i \in \{0,\ldots,n\} \; x_i \leqslant y_i \text{ we have } \sum_{i=0}^{n} x_i \leqslant \sum_{i=0}^{n} y_i \right\}$$

then we have:

$0 \in S$. For $\{x_i\}_{i \in \{0\}} \subseteq [0,\infty]$, $\{y_i\}_{i \in \{0\}} \subseteq [0,\infty]$ we have that $\sum_{i=0}^{0} x_i = x_0 \leqslant y_0 = \sum_{i=0}^{0} y_i$ proving that $0 \in S$.

$n\in S\Rightarrow n+1\in S$. As $x_{n+1}\leqslant y_{n+1}$ and $\sum_{i=0}^n x_i \underset{n\in S}{\leqslant} \sum_{i=0}^n y_i$ we have

$$\sum_{i=0}^{n+1} x_i = x_{n+1} + \sum_{i=0}^n x_i \underset{[\text{theorem: } 20.29]}{\leqslant} y_{n+1} + \sum_{i=0}^n y_i = \sum_{i=0}^{n+1} y_i$$

proving that $n+1\in S$.

4. For I we have either

$I=\varnothing$. Then by [definition: 11.32] $\sum_{i\in I} x_i = \sum_{i\in I} y_i$.

$I\neq\varnothing$. Then by [definition: 11.32] there exist a $n\in\mathbb{N}$ and a bijection $\beta:\{0,\ldots,n-1\}\to I$ such that $\sum_{i\in I} x_i = \sum_{i=0}^{n-1} x_{\beta(i)}$ and $\sum_{i\in I} y_i = \sum_{i=0}^{n-1} y_{\beta(i)}$ As by (3) we have that $\sum_{i=0}^{n-1} x_{\beta(i)} \leqslant \sum_{i=0}^{n-1} y_{\beta(i)}$ so that $\sum_{i\in I} x_i \leqslant \sum_{i\in I} y_i$.

5.

$$\begin{aligned}\sum_{i=0}^n x_i &\underset{[\text{theorem: } 11.35]}{=} \sum_{i\in\{0,\ldots,n\}} x_i \\ &\underset{(2)\wedge[\text{theorem: } 20.29]}{\leqslant} \sum_{i\in\{0,\ldots,n\}} x_i + \sum_{i\in\{n+1,\ldots,n+m\}} x_i \\ &\underset{[\text{theorem: } 11.43]}{=} \sum_{i\in\{0,\ldots,n+m\}} x_i \\ &\underset{[\text{theorem: } 11.35]}{=} \sum_{i=0}^{n+m} x_i\end{aligned}$$

6.

$$\begin{aligned}\sum_{i\in J} x_i &\underset{(2)\ [\text{theorem: } 20.29]}{\leqslant} \sum_{i\in J} x_i + \sum_{i\in I\setminus J} x_i \\ &\underset{[\text{theorem: } 11.43]}{=} \sum_{i\in I} x_i\end{aligned}$$

□

Theorem 20.86. *Let $k\in\mathbb{N}_0$ and $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq[0,\infty]$ then for $\{\sum_{i=k}^n x_k\}_{n\in\{k,\ldots,\infty\}}\subseteq\overline{\mathbb{R}}$ we have*

$$\left\{\sum_{i=k}^n x_k\right\}_{n\in\{k,\ldots,\infty\}} \subseteq [0,\infty]$$

and

$$\lim_{n\to\infty}\sum_{i=k}^n x_i \text{ exist and } \lim_{n\to\infty}\sum_{i=k}^n x_i = \sup\left(\left\{\sum_{i=k}^n x_i | n\in\{k,\ldots,\infty\}\right\}\right)$$

Proof. Let $n\in\{k,\ldots,\infty\}$ then we have by [theorem: 20.85] that

$$0\leqslant\sum_{i=0}^{n-k} x_{i+k} \underset{[\text{definition: } 11.12]}{=} \sum_{i=k}^n x_i$$

and

$$\begin{aligned}\sum_{i=k}^n x_i &\underset{[\text{definition: } 11.12]}{=} \sum_{i=0}^{n-k} x_{i+k} \\ &\underset{[\text{theorem: } 20.85]}{\leqslant} \sum_{i=0}^{(n-k)+1} x_{i+k} \\ &= \sum_{i=0}^{(n+1)-k} x_{i+k} \\ &= \sum_{i=k}^{n+1} x_i\end{aligned}$$

proving that $\{\sum_{i=k}^{n} x_i\}_{n\in\{k,\ldots,\infty\}} \subseteq [0,\infty]$ is a increasing sequence. By [theorem: 20.84] it follows then that

$$\lim_{n\rightarrow\infty}\sum_{i=k}^{n} x_i \text{ exist and } \lim_{n\rightarrow\infty}\sum_{i=k}^{n} x_n = \sup\left(\left\{\sum_{i=k}^{n} x_i | n\in\{k,\ldots,\infty\}\right\}\right)$$ □

The above theorem motivates the following definition for a denumerable sums.

Definition 20.87. *Let $k\in\mathbb{N}_0$ and let $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq[0,\infty]$ be a family of non negative extended real numbers then $\sum_{i=k}^{\infty} x_i$ is defined by*

$$\sum_{i=k}^{\infty} x_i = \lim_{n\rightarrow\infty}\sum_{i=k}^{n} x_i$$

***Note* 20.88.** *By [theorem: 20.86] $\lim\limits_{n\rightarrow\infty}\sum_{i=k}^{n} x_i$ always exist so that $\sum_{i=k}^{\infty} x_i$ is always defined and that*

$$\sum_{i=k}^{\infty} x_i = \sup\left(\left\{\sum_{i=k}^{n} x_i | n\in\{k,\ldots,\infty\}\right\}\right)\in[0,\infty]$$

Example 20.89. Let $\{x_i\}_{i\in\{1,\ldots,\infty\}}\subseteq[0,\infty]$ defined by $x_i=0$ then $\sum_{i=1}^{\infty} x_i = 0$

Proof. We have

$$\sum_{i=1}^{\infty} x_i = \sup\left(\left\{\sum_{i=k}^{n} x_i | n\in\{k,\ldots,\infty\}\right\}\right) = \sup(\{0\}) = 0$$

□

We prove now the usual properties for $\sum_{i=k}^{\infty} x_i$, first we extend [theorem: 11.8] to the set of extended real numbers:

Theorem 20.90. *Let $\alpha\in\mathbb{R}$ then we have:*

1. *If $k\in\mathbb{N}_0$, $n\in\{k,\ldots,\infty\}$ and $\{x_i\}_{i\in\{k,\ldots,n\}}\subseteq[0,\infty]$ then $\sum_{i=k}^{n}(\alpha\cdot x_i) = \alpha\cdot\sum_{i=k}^{\infty} x_i$*
2. *If I is a finite set, $\{x_i\}_{i\in I}\subseteq[0,\infty]$ then $\sum_{i\in I}(\alpha\cdot x_i) = \alpha\cdot\sum_{i\in I} x_i$*

Proof.

1. We use induction in the proof, so let

$$S=\left\{n\in\mathbb{N}_0 | \text{For every } \{x_i\}_{i\in\{0,\ldots,n\}}\subseteq[0,\infty] \text{ we have } \sum_{i=0}^{n}(\alpha\cdot x_i) = \alpha\cdot\sum_{i=0}^{n} x_i\right\}$$

then we have:

$0\in S$. If $\{x_i\}_{i\in\{0\}}\subseteq[0,\infty]$ then $\sum_{i=0}^{0}(\alpha\cdot x_i) = \alpha\cdot x_0 = \alpha\cdot\sum_{i=0}^{0} x_i$ proving that $0\in S$.

$n\in S\Rightarrow n+1\in S$. If $\{x_i\}_{i\in\{0,\ldots,n+1\}}\subseteq[0,\infty]$ then we have

$$\begin{aligned}
\sum_{i=0}^{n+1}(\alpha\cdot x_i) &= \alpha\cdot x_{n+1} + \sum_{i=0}^{n}(\alpha\cdot x_i)\\
&\underset{n\in S}{=} \alpha\cdot x_{n+1} + \alpha\cdot\sum_{i=0}^{n} x_i\\
&\underset{\text{[theorem: 20.39]}}{=} \alpha\cdot\left(x_{n+1} + \sum_{i=0}^{n} x_i\right)\\
&= \alpha\cdot\sum_{i=0}^{n+1} x_i
\end{aligned}$$

Hence using mathematical induction we have that

$$\forall\{x_i\}_{i\in\{0,\ldots,n\}}\subseteq[0,\infty],\ n\in\mathbb{N}_0 \text{ we have } \sum_{i=0}^{n}(\alpha\cdot x_i) = \alpha\cdot\sum_{i=0}^{n} x_i \tag{20.38}$$

Let $k\in\mathbb{N}_0, n\in\{k,\ldots,\infty\}$ then we have

$$\sum_{i=k}^{n}(\alpha\cdot x_i)\underset{[\text{definition: }11.12]}{=}\sum_{i=0}^{n-k}(\alpha\cdot x_{i+k})\underset{[\text{eq: }20.38]}{=}\alpha\cdot\sum_{i=0}^{n-k}x_{i+k}\underset{[\text{definition: }11.12]}{=}\alpha\cdot\sum_{i=k}^{n}x_i$$

2. For I we have either

   $\boldsymbol{I=\varnothing}$**.** Then by [definition: 11.32]

$$\alpha\cdot\sum_{i\in I}x_i=\alpha\cdot\sum_{i\in\varnothing}x_i=0=\sum_{i\in\varnothing}(\alpha\cdot x_i)=\sum_{i\in I}(\alpha\cdot x_i)$$

   $\boldsymbol{I\neq\varnothing}$**.** Then by [definition: 11.32] there exist a $n\in\mathbb{N}$ and a bijection $\beta:\{0,\ldots,n-1\}\to I$ such that $\sum_{i\in I}(\alpha\cdot x_i)=\sum_{i=0}^{n-1}(\alpha\cdot x_{\beta(i)})$ and $\sum_{i\in I}x_i=\sum_{i=0}^{n-1}x_{\beta(i)}$. As by (1) we have $\sum_{i=0}^{n-1}(\alpha\cdot x_{\beta(i)})=\alpha\cdot\sum_{i=0}^{n-1}x_{\beta(i)}$ it follows that $\sum_{i\in I}(\alpha\cdot x_i)=\alpha\cdot\sum_{i\in I}x_i$. □

Theorem 20.91. *Let $k\in\mathbb{N}_0$ then we have*

1. *If $\alpha\in\mathbb{R}$ and $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq[0,\infty]$ then $\sum_{i=k}^{\infty}(\alpha\cdot x_i)=\alpha\cdot\sum_{i=k}^{\infty}x_i$*
2. *If $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq[0,\infty]$, $\{y_i\}_{i\in\{k,\ldots,\infty\}}\subseteq[0,\infty]$ then $\sum_{i=k}^{\infty}(x_i+y_i)=\sum_{i=k}^{\infty}x_i+\sum_{i=k}^{\infty}y_i$.*
3. *If $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq[0,\infty]$, $\{y_i\}_{i\in\{k,\ldots,\infty\}}\subseteq[0,\infty]$ are such that $\forall i\in\{k,\ldots,\infty\}$ we have $x_i\leqslant y_i$ then $\sum_{i=k}^{\infty}x_i\leqslant\sum_{i=k}^{\infty}y_i$.*

Proof.

1. We have

$$\begin{aligned}\sum_{i=k}^{\infty}(\alpha\cdot x_i) &= \lim_{n\to\infty}\sum_{i=k}^{n}(\alpha\cdot x_i)\\ &\underset{[\text{theorem: }20.90]}{=} \lim_{n\to\infty}\left(\alpha\cdot\sum_{i=k}^{n}x_i\right)\\ &\underset{[\text{theorem: }20.78]}{=} \alpha\cdot\lim_{n\to\infty}\sum_{i=k}^{n}x_i\\ &= \alpha\cdot\sum_{i=k}^{\infty}x_i\end{aligned}$$

2. We have

$$\begin{aligned}\sum_{i=k}^{\infty}(x_i+y_i) &= \lim_{n\to\infty}\sum_{i=k}^{n}(x_i+y_i)\\ &\underset{[\text{theorem: }11.17]}{=} \lim_{n\to\infty}\left(\sum_{i=k}^{n}x_i+\sum_{i=k}^{n}y_i\right)\\ &\underset{[\text{theorem: }20.78]}{=} \lim_{n\to\infty}\sum_{i=k}^{\infty}x_i+\lim_{n\to\infty}\sum_{i=k}^{n}y_i\\ &= \sum_{i=k}^{\infty}x_i+\sum_{i=k}^{\infty}y_i\end{aligned}$$

3. Let $n\in\{k,\ldots,\infty\}$ then we have by [theorem: 20.85] that $\sum_{i=k}^{n}x_i\leqslant\sum_{i=k}^{n}y_k$ so that

$$\sum_{i=k}^{\infty}x_i=\lim_{n\to\infty}\sum_{i=k}^{n}x_i\underset{[\text{theorem: }20.78]}{\leqslant}\lim_{n\to\infty}\sum_{i=k}^{n}y_i=\sum_{i=k}^{\infty}y_i$$ □

Theorem 20.92. *Let $k\in\mathbb{N}_0$, $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq[0,\infty]$ then for $l\in\{k,\ldots,\infty\}$ we have*

$$\sum_{i=k}^{\infty}x_i=\sum_{i=k}^{l}x_i+\sum_{i=l+1}^{\infty}x_i$$

Proof. Let $n \in \{l+1, \ldots, \infty\}$ then by [theorem: 11.23]

$$\sum_{i=k}^{n} x_i = \sum_{i=k}^{l} x_i + \sum_{i=l+1}^{n} x_i \tag{20.39}$$

Next we have

$$\begin{aligned}
\sum_{i=k}^{\infty} x_i &= \lim_{n\to\infty} \sum_{i=k}^{n} x_i \\
&\underset{[\text{theorem: } 20.77]}{=} \lim_{n\to\infty} \sum_{i=k}^{n+(l+1)} x_i \\
&\underset{[\text{eq: } 20.39]}{=} \lim_{n\to\infty} \left(\sum_{i=k}^{l} x_i + \sum_{i=l+1}^{n+(l+1)} x_i \right) \\
&\underset{[\text{theorem: } 20.78\ (4)]}{=} \sum_{i=k}^{l} x_i + \lim_{n\to\infty} \sum_{i=l+1}^{n+(l+1)} x_i \\
&\underset{[\text{theorem: } 20.77]}{=} \sum_{i=k}^{l} x_i + \lim_{n\to\infty} \sum_{i=l+1}^{n} x_i \\
&= \sum_{i=k}^{l} x_i + \sum_{i=l+1}^{\infty} x_i
\end{aligned}$$

□

Theorem 20.93. *Let $k \in \mathbb{N}_0$, $\{x_i\}_{i\in\{k,\ldots,\infty\}} \subseteq [0, \infty]$ then if $\sigma: \{k, \ldots, \infty\} \rightarrow \{k, \ldots, \infty\}$ is a permutation then*

$$\sum_{i=k}^{\infty} x_i = \sum_{i=k}^{\infty} x_{\sigma(i)}$$

Proof. Let $\sigma: \{k, \ldots, \infty\} \rightarrow \{k, \ldots, \infty\}$ be a bijection. Let $s \in \{\sum_{i=k}^{n} x_{\sigma(i)} | n \in \{k, \ldots, \infty\}\}$ then there exist a $n \in \{k, \ldots, \infty\}$ such that

$$s = \sum_{i=k}^{n} x_{\sigma(i)}$$

Let $m_n = \max(\{\sigma(i) | i \in \{k, \ldots, n\}\})$ then $\forall i \in \{k, \ldots, n\}$ we have $\sigma(i) \leqslant m_n \Rightarrow \sigma(i) \in \{k, \ldots, m_n\}$. Hence we have that

$$\sigma(\{k, \ldots, n\}) = \{\sigma(i) | i \in \{k, \ldots, n\}\} \subseteq \{k, \ldots, m_n\} \tag{20.40}$$

As σ is injective it follows that

$$\sigma_{|\{k,\ldots,n\}}: \{k, \ldots, n\} \rightarrow \sigma(\{k, \ldots, n\}) \text{ is a bijection}$$

$$\begin{aligned}
s &= \sum_{i=k}^{n} x_{\sigma(i)} \\
&\underset{[\text{theorem: } 11.35]}{=} \sum_{i\in\{k,\ldots,n\}} x_{\sigma(i)} \\
&= \sum_{i\in\{k,\ldots,m\}} x_{\sigma_{|\{k,\ldots,n\}}(i)} \\
&\underset{[\text{theorem: } 11.36]}{=} \sum_{i\in\sigma(\{x,\ldots,n\})} x_i \\
&\underset{[\text{eq: } 20.40]\&[\text{theorem: } 20.85]}{\leqslant} \sum_{i\in\{k,\ldots,m_n\}} x_i \in \left\{ \sum_{i=k}^{n} x_i | n \in \{k, \ldots, \infty\} \right\}
\end{aligned}$$

So $\forall s \in \{\sum_{i=k}^{n} x_{\sigma(i)} | n \in \{k,\ldots,\infty\}\}$ we found a $\sum_{i\in\{k,\ldots,m_n\}} x_i \in \{\sum_{i=k}^{n} x_i | n \in \{k,\ldots,\infty\}\}$ such that

$$s \leqslant \sum_{i\in\{k,\ldots,m_n\}} x_i$$

Hence using [theorem: 3.78] it follows that

$$\sum_{i=k}^{\infty} x_{\sigma(i)} = \sup\left(\left\{\sum_{i=k}^{n} x_{\sigma(i)} | n \in \{k, \ldots, \infty\}\right\}\right) \leqslant \sup\left(\left\{\sum_{i=k}^{n} x_i | n \in \{k, \ldots, \infty\}\right\}\right) = \sum_{i=k}^{\infty} x_i$$

So we have proved that

$$\text{If } \sigma: \{k,\ldots,\infty\} \to \{k,\ldots,\infty\} \text{ is a bijection, } \{x_i\}_{i\in\{k,\ldots,\infty\}} \subseteq [0,\infty] \text{ we have } \sum_{i=k}^{\infty} x_{\sigma(i)} \leqslant \sum_{i=k}^{\infty} x_i \tag{20.41}$$

For the opposite inequality if $\sigma: \{k, \ldots, \infty\} \to \{k, \ldots, \infty\}$ is a bijection then

$$\sum_{i=k}^{\infty} x_{\sigma(i)} \leqslant \sum_{i=k}^{\infty} x_i \tag{20.42}$$

and as $\sigma^{-1}: \{k, \ldots, \infty\} \to \{k, \ldots, \infty\}$ is a bijection we have also

$$\sum_{i=k}^{\infty} x_i = \sum_{i=k}^{\infty} x_{\sigma^{-1}(\sigma(i))} \leqslant \sum_{i=k}^{\infty} x_{\sigma(i)}$$

which combined with [eq: 20.42] gives

$$\sum_{i=k}^{\infty} x_i = \sum_{i=k}^{\infty} x_{\sigma(i)}$$

□

In analogy with [definition: 11.32] the above theorem allows us to define the sum of a denumerable set of extended non negative numbers.

Definition 20.94. *Let I be a denumerable set and $\{x_n\}_{n\in I}$ then $\sum_{i\in I} x_i$ is defined by*

$$\sum_{i\in I} x_i = \sum_{i=0}^{\infty} x_{\beta(i)}$$

where

$$\beta: \mathbb{N}_0 \to I \text{ is a bijection}$$

***Note* 20.95.** *As I is denumerable there exist a bijection from $\mathbb{N}_0$ to I. Further if $\gamma: \mathbb{N}_0 \to I$ is another bijection then we have that $\beta^{-1} \circ \gamma: \mathbb{N}_0 \to \mathbb{N}_0$ is a bijection so that*

$$\sum_{i=0}^{\infty} x_{\beta(i)} \underset{\text{[theorem: 20.93]}}{=} \sum_{i=0}^{\infty} x_{\beta((\beta^{-1}\circ\gamma)(i))} = \sum_{i=0}^{\infty} x_{\gamma(i)}$$

so that this definition is independent of the chosen bijection.

The next theorem shows how we can interchange sums.

Theorem 20.96. *Let $k, l \in \mathbb{N}$ then we have:*

1. *If $n \in \{k, \ldots, \infty\}$, $m \in \{l, \ldots, \infty\}$ and $\{x_{i,j}\}_{(i,j)\in\{k,\ldots,n\}\times\{l,\ldots,m\}} \subseteq [0, \infty]$ then*

$$\sum_{i=k}^{n}\left(\sum_{j=l}^{m} x_{i,j}\right) = \sum_{j=k}^{m}\left(\sum_{i=l}^{n} x_{i,j}\right)$$

2. *Let $n \in \{l, \ldots, \infty\}$ and $\{x_{i,j}\}_{(i,j)\in\{k,\ldots,\infty\}\times\{l,\ldots,n\}} \subseteq [0, \infty]$ then*

$$\sum_{i=k}^{\infty}\left(\sum_{j=l}^{n} x_{i,j}\right) = \sum_{j=k}^{n}\left(\sum_{i=l}^{\infty} x_{i,j}\right)$$

3. *Let* $n\in\{l,\ldots,\infty\}$ *and* $\{x_{i,j}\}_{(i,j)\in\{l,\ldots,n\}\times\{k,\ldots,\infty\}}\subseteq[0,\infty]$ *then*

$$\sum_{i=l}^{n}\left(\sum_{j=k}^{\infty}x_{i,j}\right)=\sum_{j=k}^{\infty}\left(\sum_{i=l}^{n}x_{i,j}\right)$$

4. *Let* $\{x_{i,j}\}_{(i,j)\in\{k,\ldots,\infty\}\times\{l,\ldots,\infty\}}\subseteq[0,\infty]$ *then*

$$\sum_{i=k}^{\infty}\left(\sum_{j=l}^{\infty}x_{i,j}\right)=\sum_{j=k}^{\infty}\left(\sum_{i=l}^{\infty}x_{i,j}\right)$$

Proof.

1. This was already proved in [theorem: 11.19].

2. We prove this by induction, so let

$$S=\left\{n\in\{l,\ldots,\infty\}\text{If }\{x_{i,j}\}_{(i,j)\in\{k,\ldots,\infty\}\times\{l,\ldots,n\}}\subseteq[0,\infty]\text{ then }\sum_{i=k}^{\infty}\left(\sum_{j=l}^{n}x_{i,j}\right)=\sum_{j=k}^{n}\left(\sum_{i=l}^{\infty}x_{i,j}\right)\right\}$$

then we have:

***l* ∈ *S*.** Let $\{x_{i,j}\}_{(i,j)\in\{k,\ldots,\infty\}\times\{l\}}\subseteq[0,\infty]$ then we have

$$\begin{aligned}\sum_{i=k}^{\infty}\left(\sum_{j=l}^{l}x_{i,j}\right)&=\sum_{i=k}^{\infty}x_{i,l}\\&=\sum_{j=l}^{l}\left(\sum_{i=k}^{\infty}x_{i,j}\right)\end{aligned}$$

so that $l\in S$.

***n* ∈ *S* → *n* + 1 ∈ *S*.** Let $\{x_{i,j}\}_{(i,j)\in\{k,\ldots,\infty\}\times\{l,\ldots,n+1\}}\subseteq[0,\infty]$ then we have

$$\begin{aligned}\sum_{i=k}^{\infty}\left(\sum_{j=l}^{n+1}x_{i,j}\right)&=\sum_{i=k}^{\infty}\left(\sum_{j=l}^{n}x_{i,j}+x_{i,n+1}\right)\\&\underset{[\text{theorem: }20.91]}{=}\sum_{i=k}^{\infty}\left(\sum_{j=l}^{n}x_{i,j}\right)+\sum_{i=k}^{\infty}x_{i,n+1}\\&\underset{n\in S}{=}\sum_{j=l}^{n}\left(\sum_{i=k}^{\infty}x_{i,j}\right)+\sum_{i=k}^{\infty}x_{i,n+1}\\&=\sum_{j-l}^{n+1}\left(\sum_{i=k}^{\infty}x_{i,j}\right)\end{aligned}$$

proving that $n+1\in S$

3. Define $\{y_{(i,j)}\}_{(i,j)\in\{k,\ldots,n\}\times\{l,\ldots,\infty\}}$ by $y_{(i,j)}=x_{(j,i)}$ then we have

$$\begin{aligned}\sum_{i=k}^{n}\left(\sum_{j=l}^{\infty}x_{i,j}\right) &= \sum_{i=k}^{n}\left(\sum_{j=l}^{\infty}y_{j,i}\right)\\ &\underset{(2)}{=} \sum_{j=l}^{\infty}\left(\sum_{i=k}^{n}y_{j,i}\right)\\ &= \sum_{j=l}^{\infty}\left(\sum_{i=k}^{n}x_{i,j}\right)\end{aligned}$$

4. Let $n\in\{k,\ldots,\infty\}$ then

$$\begin{aligned}\sum_{i=k}^{n}\left(\sum_{j=l}^{\infty}x_{i,j}\right) &\underset{(3)}{=} \sum_{j=l}^{\infty}\left(\sum_{i=k}^{n}x_{i,j}\right)\\ &\underset{[\text{theorem: } 20.91]}{\leqslant} \sum_{j=l}^{\infty}\sup\left(\left\{\sum_{i=k}^{n}x_{i,j}|n\in\{k,\ldots,\infty\}\right\}\right)\\ &\underset{[\text{definition: } 20.87]}{=} \sum_{j=l}^{\infty}\left(\sum_{i=k}^{\infty}x_{i,j}\right)\end{aligned}$$

So $\sum_{j=l}^{\infty}(\sum_{i=k}^{\infty}x_{i,j})$ is a upper bound of $\{\sum_{i=k}^{n}(\sum_{j=l}^{\infty}x_{i,j})|n\in\{k,\ldots,\infty\}\}$ from which it follows that

$$\sum_{i=k}^{\infty}\left(\sum_{j=l}^{\infty}x_{i,j}\right)=\sup\left(\sum_{i=k}^{n}\left(\sum_{j=l}^{\infty}x_{i,j}\right)|n\in\{k,\ldots,\infty\}\right)\leqslant\sum_{j=l}^{\infty}\left(\sum_{i=k}^{\infty}x_{i,j}\right) \tag{20.43}$$

For the opposite inequality take $m\in\{l,\ldots,\infty\}$ then we have

$$\begin{aligned}\sum_{j=l}^{m}\left(\sum_{i=k}^{\infty}x_{i,j}\right) &\underset{(2)}{=} \sum_{i=k}^{\infty}\left(\sum_{j=l}^{m}x_{i,j}\right)\\ &\underset{[\text{theorem: } 20.91]}{\leqslant} \sum_{i=k}^{\infty}\sup\left(\left\{\sum_{j=l}^{m}x_{i,j}|m\in\{l,\ldots,\infty\}\right\}\right)\\ &\underset{[\text{definition: } 20.87]}{=} \sum_{i=k}^{\infty}\left(\sum_{j=l}^{\infty}x_{i,j}\right)\end{aligned}$$

So $\sum_{i=k}^{\infty}(\sum_{j=l}^{\infty}x_{i,j})$ is a upper bound of $\{\sum_{j=l}^{m}(\sum_{i=k}^{\infty}x_{i,j})|m\in\{l,\ldots,\infty\}\}$ proving that

$$\sum_{j=l}^{\infty}\left(\sum_{i=k}^{\infty}x_{i,j}\right)=\sup\left(\left\{\sum_{j=l}^{m}\left(\sum_{i=k}^{\infty}x_{i,j}\right)|m\in\{l,\ldots,\infty\}\right\}\right)\leqslant\sum_{i=k}^{\infty}\left(\sum_{j=l}^{\infty}x_{i,j}\right)$$

Combining this with [eq: 20.43] proves

$$\sum_{i=k}^{\infty}\left(\sum_{j=l}^{\infty}x_{i,j}\right)=\sum_{j=l}^{\infty}\left(\sum_{i=k}^{\infty}x_{i,j}\right)$$ □

Every finite sum of denumerable sums can be written as a denumerable sum.

Lemma 20.97. *Let* $k,l,m\in\mathbb{N}_0$ *and* $n\in\{k,\ldots,\infty\}$, $\{x_{i,j}\}_{(i,j)\in\{k,\ldots,n\}\times\{l,\ldots,\infty\}}\subseteq[0,\infty]$ *and*

$$\beta:\{m,\ldots,\infty\}\to\{k,\ldots,n\}\times\{l,\ldots,\infty\}$$

a bijection then

$$\sum_{i=m}^{\infty} x_{\sigma(i)_1,\sigma(i)_2} = \sum_{j=l}^{\infty} \left(\sum_{i=k}^{n} x_{i,j} \right) \underset{\text{[theorem: 20.96]}}{=} \sum_{i=k}^{n} \left(\sum_{j=l}^{\infty} x_{i,j} \right)$$

Note 20.98. *By [theorem: 6.65] there exist always a bijection between* $\{m,\ldots,\infty\}$ *and* $\{k,\ldots,n\} \times \{l,\ldots,\infty\}$ *for a given* $m \in \mathbb{N}_0$.

Proof. Let $r \in \{m,\ldots,\infty\}$ and define $N_r = \max(\{\beta(i)_2 | i \in \{m,\ldots,r\}\}) \in \{l,\ldots,\infty\}$ then if $i \in \{m,\ldots,r\}$ we have that $\beta(i)_2 \in \{k,\ldots,N_r\}$ so that $\beta(i) = (\beta(i)_1, \beta(i)_2) \in \{k,\ldots,n\} \times \{l,\ldots,N_r\}$. Hence

$$\beta(\{m,\ldots,r\}) \subseteq \{k,\ldots,n\} \times \{l,\ldots,N_r\}$$

Further as β is a bijection

$$\beta_{|\{m,\ldots,r\}} \colon \{m,\ldots,r\} \to \beta(\{m,\ldots,r\}) \text{ is a bijection}$$

Next

$$\begin{aligned}
\sum_{i=m}^{r} x_{\beta(i)_1,\beta(i)_2} &\underset{\text{[theorem: 11.35]}}{=} \sum_{i \in \{m,\ldots,r\}} x_{\beta(i)_1,\beta(i)_2} \\
&= \sum_{i \in \{m,\ldots,r\}} x_{\sigma_{|\{m,\ldots,r\}}(i)_1,\sigma_{|\{m,\ldots,r\}}(i)_2} \\
&\underset{\text{[theorem: 11.36]}}{=} \sum_{(i,j) \in \beta(\{m,\ldots,r\})} x_{(i,j)_1,(i,j)_2} \\
&= \sum_{(i,j) \in \beta(\{m,\ldots,r\})} x_{i,j} \\
&\underset{\text{[theorem: 20.85]}}{\leqslant} \sum_{(i,j) \in \{k,\ldots,n\} \times \{l,\ldots,N_r\}} x_{i,j} \\
&\underset{\text{[theorem: 11.45]}}{=} \sum_{i \in \{k,\ldots,n\}} \left(\sum_{j \in \{l,\ldots,N_r\}} x_{i,j} \right) \\
&\underset{\text{[theorem: 11.35]}}{=} \sum_{i=k}^{n} \left(\sum_{j=l}^{N_r} x_{i,j} \right) \\
&\underset{\text{[theorem: 20.91]}}{\leqslant} \sum_{i=k}^{n} \sup\left(\left\{ \sum_{j=l}^{s} x_{i,j} | s \in \{l,\ldots,\infty\} \right\} \right) \\
&= \sum_{i=k}^{n} \left(\sum_{j=l}^{\infty} x_{i,j} \right)
\end{aligned}$$

Hence as $r \in \{m,\ldots,\infty\}$ was chosen arbitrary it follows that

$$\sum_{i=m}^{\infty} x_{\beta(i)_1,\beta(i)_2} = \sup\left(\left\{ \sum_{i=m}^{r} x_{\beta(i)_1,\beta(i)_2} | r \in \{m,\ldots,\infty\} \right\} \right) \leqslant \sum_{i=k}^{n} \left(\sum_{j=l}^{\infty} x_{i,j} \right) \tag{20.44}$$

For the opposite inequality. Let $r \in \{l,\ldots,\infty\}$ and take

$$M_r = \max((\beta^{-1})(\{k,\ldots,n\} \times \{l,\ldots,r\})) \in \{m,\ldots,\infty\}$$

So $(\beta^{-1})(\{k,\ldots,n\} \times \{l,\ldots,r\}) \subseteq \{m,\ldots,M_r\}$ hence

$$\{k,\ldots,n\} \times \{l,\ldots,r\} = \beta((\beta^{-1})(\{k,\ldots,n\} \times \{l,\ldots,r\})) \subseteq \beta(\{m,\ldots,M_r\}) \tag{20.45}$$

As β is a bijection it follows that

$$\beta_{|\{m,\ldots,M_r\}}\colon \{m,\ldots,M_r\} \rightarrow \beta(\{m,\ldots,M_r\})$$

$$\begin{array}{rcl}
\sum_{j=l}^{r} \left(\sum_{i=k}^{n} x_{i,j}\right) & \underset{\text{[theorem: 20.96]}}{=} & \sum_{i=k}^{n} \left(\sum_{j=l}^{r} x_{i,j}\right)\\
& \underset{\text{[theorem: 11.35]}}{=} & \sum_{i\in\{k,\ldots,n\}} \left(\sum_{j\in\{l,\ldots,r\}} x_{i,j}\right)\\
& \underset{\text{[theorem: 11.45]}}{=} & \sum_{(i,j)\in\{k,\ldots,n\}\times\{l,\ldots,r\}} x_{i,j}\\
& \underset{\text{[theorem: 20.91]}\wedge\text{[eq: 20.45]}}{\leqslant} & \sum_{(i,j)\in\beta(m,\ldots,M_r)} x_{i,j}\\
& = & \sum_{(i,j)\in\beta(m,\ldots,M_r)} x_{(i,j)_1,(i,j)_2}\\
& \underset{\text{[theorem: 11.36]}}{=} & \sum_{i\in\{m,\ldots,M_r\}} x_{\beta_{|\{m,\ldots,M_r\}}(i)_1,\beta_{|\{m,\ldots,M_r\}}(i)_2}\\
& = & \sum_{i\in\{m,\ldots,M_r\}} x_{\beta(i)_1,\beta(i)_2}\\
& \underset{\text{[theorem: 11.35]}}{=} & \sum_{i=m}^{M_r} x_{\beta(i)_1,\beta(i)_2}\\
& \leqslant & \sup\left(\left\{\sum_{i=m}^{s} x_{\beta(i)_1,\beta(i)_2} | s\in\{m,\ldots,\infty\}\right\}\right)\\
& = & \sum_{i=m}^{\infty} x_{\beta(i)_1,\beta(i)_2}
\end{array}$$

Hence as $r \in \{l, \ldots, \infty\}$ was chosen arbitrary we have that

$$\sum_{j=l}^{\infty} \left(\sum_{i=k}^{n} x_{i,j}\right) = \sup\left(\left\{\sum_{j=l}^{s} \left(\sum_{i=k}^{n} x_{i,j}\right) | s\in\{l,\ldots,\infty\}\right\}\right) \leqslant \sum_{i=m}^{\infty} x_{\beta(i)_1,\beta(i)_2}$$

which combined with [eq: 20.44] proves that

$$\sum_{i=m}^{\infty} x_{\beta(i)_1,\beta(i)_2} = \sum_{j=l}^{\infty} \left(\sum_{i=k}^{n} x_{i,j}\right)$$

□

Next we extend the above theorem for arbitrary finite sums of different size.

Theorem 20.99. *Let $k, l \in \mathbb{N}_0$ and $\{x_{(i,j)}\}_{(i,j)\in\bigcup_{i\in\{k,\ldots,\infty\}}\{i\}\times\{n_i,\ldots,m_i\}} \subseteq [0,\infty]$ [where $\forall i \in \{k,\ldots,\infty\}$ we have $n_i \leqslant m_i$] then if*

$$\beta\colon \{l,\ldots,\infty\} \rightarrow \bigcup_{i\in\{k,\ldots,\infty\}} \{i\}\times\{n_i,\ldots,m_i\}$$

is a bijection then we have

$$\sum_{i=l}^{\infty} x_{\beta(i)_1,\beta(i)_2} = \sum_{i=k}^{\infty} \left(\sum_{j=n_i}^{m_i} x_{(i,j)}\right)$$

Proof. Let $i \in \{k, \ldots, \infty\}$ and define

$$\tau_i\colon \{n_i,\ldots,m_i\} \rightarrow \{i\}\times\{n_i,\ldots,m_i\} \text{ where } \tau_i(j) = (i,j) \text{ which is a bijection}$$

Proof.

injectivity. If $\tau_i(r)=\tau_i(s)$ then $(i,r)=(i,s)$ so that $r=s$.

surjectivity. If $(r,s)\in\{i\}\times\{n_i,...,m_i\}$ then $s\in\{n_i,...,m_i\}$ and $r=i$ so that $\tau_i(s)=(i,s)=(r,s)$ proving subjectivity. □

Further

$$\begin{aligned}\sum_{(k,l)\in\tau_i(\{n_i,\ldots,m_i\})} x_{k,l} &\underset{[\text{theorem: } 11.36]}{=} \sum_{j\in\{n_i,\ldots,m_i\}} x_{\tau_i(j)_1,\tau_i(j)_2}\\ &= \sum_{j\in\{n_i,\ldots,m_i\}} x_{i,j}\\ &= \sum_{j=n_i}^{m_i} x_{i,j} \end{aligned}\tag{20.46}$$

Let $m\in\{l,\ldots,\infty\}$ and define $N_m=\max(\beta(i)_1|i\in\{l,\ldots,m\})$. If $i\in\{l,\ldots,m\}$ then, as $\beta(i)\in\bigcup_{j\in\{k,\ldots,\infty\}}\{j\}\times\{n_i,\ldots,m_i\}$ there exist a $j\in\{k,\ldots,\infty\}$ such that $j=\beta(i)_1\leqslant N_m$ and $\beta(i)_2\in\{n_j,\ldots,m_j\}$ proving that

$$\beta(\{l,\ldots,m\})\subseteq\bigcup_{i\in\{k,\ldots,N_m\}}\{i\}\times\{n_i,\ldots,m_i\}$$

Further if $i,j\in\{k,...,N_m\}$ with $i\neq j$ then $\{i\}\times\{n_i,...,m_i\}\bigcap\{j\}\times\{n_j,...,m_j\}=\varnothing$ hence we have

$$\beta(\{l,\ldots,m\})\subseteq\bigsqcup_{i\in\{k,\ldots,N_m\}}\{i\}\times\{n_i,\ldots,m_i\} \text{ a disjoint union}$$

and as β is a bijection

$$\beta_{|\{l,\ldots,m\}}:\{l,\ldots,m\}\to\beta(\{l,\ldots,m\}) \text{ is a bijection}$$

$$\begin{aligned}\sum_{i=l}^{m} x_{\beta(i)_1,\beta(i)_2} &\underset{[\text{theorem: } 11.35]}{=} \sum_{i\in\{l,\ldots,m\}} x_{\beta(i)_1,\beta(i)_2}\\ &= \sum_{i\in\{l,\ldots,m\}} x_{\beta_{|\{l,\ldots,m\}}(i)_1,\beta_{|\{l,\ldots,m\}}(i)_2}\\ &\underset{[\text{theorem: } 11.36]}{=} \sum_{(i,j)\in\beta(\{l,\ldots,m\})} x_{(i,j)_1,(i,j)_2}\\ &\underset{[\text{theorem: } 20.85]}{\leqslant} \sum_{(i,j)\in\bigsqcup_{i\in\{k,\ldots,N_m\}}\{i\}\times\{n_i,\ldots,m_i\}} x_{(i,j)_1,(i,j)_2}\\ &= \sum_{(i,j)\in\bigsqcup_{i\in\{k,\ldots,N_m\}}\{i\}\times\{n_i,\ldots,m_i\}} x_{i,j}\\ &\underset{[\text{theorem: } 11.44]}{=} \sum_{r\in\{k,\ldots,N_m\}}\left(\sum_{(i,j)\in\{r\}\times\{n_r,\ldots,m_r\}} x_{i,j}\right)\\ &= \sum_{r\in\{k,\ldots,N_m\}}\left(\sum_{j\in\{n_r,\ldots,m_r\}} x_{r,j}\right)\\ &\underset{[\text{theorem: } 11.35]}{=} \sum_{r=k}^{N_m}\left(\sum_{j=n_r}^{m_r} x_{r,j}\right)\\ &\underset{[\text{theorem: } 20.85]}{\leqslant} \sup\left(\left\{\sum_{r=k}^{n}\left(\sum_{j=n_r}^{m_r} x_{r,j}\right)|n\in\{k,\ldots,\infty\}\right\}\right)\\ &= \sum_{r=k}^{\infty}\left(\sum_{j=n_r}^{m_r} x_{r,j}\right)\end{aligned}$$

So $m \in \{l, \ldots, \infty\}$ was chosen arbitrary we have that

$$\sum_{i=l}^{\infty} x_{\beta(i)_1,\beta(i)_2} = \sup\left(\left\{\sum_{i=l}^{m} x_{\beta(i)_1,\beta(i)_2} | m \in \{l, \ldots, \infty\}\right\}\right) \leqslant \sum_{r=k}^{\infty}\left(\sum_{j=n_r}^{m_r} x_{r,j}\right) \tag{20.47}$$

For the opposite inequality take $m \in \{k, \ldots, \infty\}$ and take

$$M_m = \max\left(\left\{(\beta^{-1})\left(\bigcup_{i \in \{k, \ldots, m\}} \{i\} \times \{n_i, \ldots, m_i\}\right)\right\}\right)$$

then $(\beta^{-1})\left(\bigcup_{i \in \{k, \ldots, m\}} \{i\} \times \{n_i, \ldots, m_i\}\right) \subseteq \{l, \ldots, M_m\}$ then

$$\bigcup_{i \in \{k, \ldots, m\}} \{i\} \times \{n_i,\ldots,m_i\} = \beta\left((\beta^{-1})\left(\bigcup_{i \in \{k, \ldots, m\}} \{i\} \times \{n_i,\ldots,m_i\}\right)\right) \subseteq \beta(\{l,\ldots,M_m\}) \tag{20.48}$$

Next

$$\begin{aligned}
\sum_{r=k}^{m}\left(\sum_{j=n_r}^{m_r} x_{r,j}\right) &\underset{[\text{eq: } 20.46]}{=} \sum_{r=k}^{m}\left(\sum_{(i,j) \in \{r\} \times \{n_r, \ldots, m_r\}} x_{i,j}\right) \\
&= \sum_{r \in \{k, \ldots, m\}}\left(\sum_{(i,j) \in \{r\} \times \{n_r, \ldots, m_r\}} x_{i,j}\right) \\
&\underset{[\text{theorem: } 11.44]}{=} \sum_{(i,j) \in \bigcup_{r \in \{k, \ldots, m\}} (r) \times \{n_r, \ldots, m_r\}} x_{i,j} \\
&\underset{[\text{theorem: } 20.85] \wedge [\text{eq: } 20.48]}{\leqslant} \sum_{(i,j) \in \beta(\{l, \ldots, M_m\})} x_{i.j} \\
&= \sum_{(i,j) \in \beta(\{l, \ldots, M_m\})} x_{(i,j)_1.(i,j)_2} \\
&\underset{[\text{theorem: } 11.36]}{=} \sum_{i \in \{l, \ldots, M_m\}} x_{\beta(i)_1,\beta(i)_2} \\
&= \sum_{i=l}^{M_m} x_{\beta(i)_1,\beta(i)_2} \\
&\leqslant \sup\left(\left\{\sum_{i=l}^{n} x_{\beta(i)_1,\beta(i)_2} | n \in \{l, \ldots, \infty\}\right\}\right) \\
&= \sum_{i=l}^{\infty} x_{\beta(i)_1,\beta(i)_2}
\end{aligned}$$

As $m \in \{k, \ldots, \infty\}$ was chosen arbitrary it follows that

$$\sum_{r=k}^{\infty}\left(\sum_{j=n_r}^{m_r} x_{r,j}\right) = \sup\left(\left\{\sum_{r=k}^{n}\left(\sum_{j=n_r}^{m_r} x_{r,j}\right) | n \in \{r, \ldots, \infty\}\right\}\right) \leqslant \sum_{i=l}^{\infty} x_{\beta(i)_1,\beta(i)_2}$$

which combined with [eq: 20.47] proves that

$$\sum_{i=l}^{\infty} x_{\beta(i)_1,\beta(i)_2} = \sum_{r=k}^{\infty}\left(\sum_{j=n_r}^{m_r} x_{(r,j)}\right)$$

□

Theorem 20.100. *Let* $k, l, n \in \mathbb{N}_0$ *and* $\{x_{i,j}\}_{(i,j) \in \{k, \ldots, \infty\} \times \{l, \ldots, \infty\}} \subseteq [0, \infty]$ *and*

$$\beta: \{n, \ldots, \infty\} \to \{k, \ldots, \infty\} \times \{l, \ldots, \infty\} \text{ a bijection}$$

then

$$\sum_{i=0}^{\infty} x_{\beta(i)_1,\beta(i)} = \sum_{i=k}^{\infty}\left(\sum_{j=l}^{\infty} x_{i,j}\right)$$

Note 20.101. *By [theorem: 6.65] there exist always a bijection between $\{n,\ldots,\infty\}$ and $\{k,\ldots,\infty\}\times\{l,\ldots,\infty\}$ for a given $m\in\mathbb{N}_0$.*

Proof. Let $m\in\{n,\ldots,\infty\}$ and define $N_m=\max(\{\beta(i)_1|i\in\{n,\ldots,m\}\})$ and $M_m=\max(\{\beta(i)_2|i\in\{n,\ldots,m\}\})$ then we have that

$$\beta(\{n,\ldots,m\})\subseteq\{k,\ldots,N_m\}\times\{l,\ldots,M_m\} \tag{20.49}$$

Then we have

$$\begin{aligned}
\sum_{i=n}^{m} x_{\beta(i)_1,\beta(i)_2} &\underset{[\text{theorem: } 11.35]}{=} \sum_{i\in\{n,\ldots,m\}} x_{\beta(i)_1,\beta(i)_2}\\
&\underset{[\text{theorem: } 11.36]}{=} \sum_{(i,j)\in\beta(\{n,\ldots,m\})} x_{(i,j)_1,(i,j)_2}\\
&= \sum_{(i,j)\in\beta(\{n,\ldots,m\})} x_{i,j}\\
&\underset{[\text{theorem: } 20.85]\wedge[\text{eq: } 20.49]}{\leqslant} \sum_{(i,j)\in\{k,\ldots,N_m\}\times\{l,\ldots,M_m\}} x_{i,j}\\
&\underset{[\text{theorem: } 11.45]}{=} \sum_{i\in\{k,\ldots,N_m\}}\left(\sum_{j\in\{l,\ldots,M_m\}} x_{i,j}\right)\\
&\underset{[\text{theorem: } 11.35]}{=} \sum_{i=k}^{N_m}\left(\sum_{j=l}^{M_m} x_{i,j}\right)\\
&\underset{[\text{theorem: } 20.85]}{\leqslant} \sum_{i=k}^{N_m}\sup\left(\left\{\sum_{j=l}^{r} x_{i,j}|r\in\{l,\ldots,\infty\}\right\}\right)\\
&= \sum_{i=k}^{N_m}\left(\sum_{j=l}^{M_m} x_{i,j}\right)\\
&\leqslant \sup\left(\left\{\sum_{i=k}^{r}\left(\sum_{j=l}^{M_m} x_{i,j}\right)|r\in\{k,\ldots,\infty\}\right\}\right)\\
&= \sum_{i=k}^{\infty}\left(\sum_{j=l}^{\infty} x_{i,j}\right)
\end{aligned}$$

As $m\in\{n,\ldots,\infty\}$ was chosen arbitrary it follows that

$$\sum_{i=n}^{\infty} x_{\beta(i)_1,\beta(i)_2} = \sup\left(\left\{\sum_{i=n}^{m} x_{\beta(i)_1,\beta(i)_2}|m\in\{n,\ldots,\infty\}\right\}\right)\leqslant\sum_{i=k}^{\infty}\left(\sum_{j=l}^{\infty} x_{i,j}\right) \tag{20.50}$$

For the opposite inequality let $r\in\{k,\ldots,\infty\}$, $s\in\{l,\ldots,\infty\}$ and take

$$N=\max((\beta^{-1})(\{k,\ldots,r\}\times\{l,\ldots,s\}))$$

then $(\beta^{-1})(\{k,\ldots,r\}\times\{l,\ldots,s\})\subseteq\{n,\ldots,N\}$ so that

$$\{k,\ldots,r\}\times\{l,\ldots,s\}=\beta((\beta^{-1})(\{k,\ldots,r\}\times\{l,\ldots,s\}))\subseteq\beta(n,\ldots,N) \tag{20.51}$$

Next

$$
\begin{aligned}
\sum_{i=k}^{r}\left(\sum_{j=l}^{s} x_{i,j}\right) &\underset{[\text{theorem: } 11.35]}{=} \sum_{i\in\{k,\ldots,r\}}\left(\sum_{j\in\{l,\ldots,s\}} x_{i,j}\right)\\
&\underset{[\text{theorem: } 11.45]}{=} \sum_{(i,j)\in\{k,\ldots,r\}\times\{l,\ldots,s\}} x_{i,j}\\
&\underset{[\text{theorem: } 20.85]\wedge[\text{eq: } 20.51]}{\leqslant} \sum_{(i,j)\in\beta(n,\ldots,N)} x_{i,j}\\
&= \sum_{(i,j)\in\beta(n,\ldots,N)} x_{(i,j)_1,(i,j)_2}\\
&\underset{[\text{theorem: } 11.36]}{=} \sum_{i\in\{n,\ldots,N\}} x_{\beta(i)_1,\beta(i)_2}\\
&= \sum_{i=n}^{N} x_{\beta(i)_1,\beta(i)_2}\\
&\leqslant \sup\left(\left\{\sum_{i=n}^{m} x_{\beta(i)_1,\beta(i)_2} | m\in\{n,\ldots,\infty\}\right\}\right)\\
&= \sum_{i=n}^{\infty} x_{\beta(i)_1,\beta(i)_2}
\end{aligned}
$$

Hence as $r\in\{k,\ldots,\infty\}$ was chosen arbitrary we have that

$$
\sum_{i=k}^{\infty}\left(\sum_{j=l}^{s} x_{i,j}\right)=\sup\left(\left\{\sum_{i=k}^{m}\left(\sum_{j=l}^{s} x_{i,j}\right) | m\in\{k,\ldots,\infty\}\right\}\right)\leqslant\sum_{i=n}^{\infty} x_{\beta(i)_1,\beta(i)_2} \tag{20.52}
$$

Further

$$
\begin{aligned}
\sum_{j=l}^{s}\left(\sum_{i=k}^{\infty} x_{i,j}\right) &\underset{[\text{theorem: } 20.96]}{=} \sum_{i=k}^{\infty}\left(\sum_{j=l}^{s} x_{i,j}\right)\\
&\underset{[\text{eq: } 20.52]}{\leqslant} \sum_{i=n}^{\infty} x_{\beta(i)_1,\beta(i)_2}
\end{aligned}
$$

So as $s\in\{l,\ldots,\infty\}$ is chosen arbitrary we have that

$$
\sum_{j=l}^{\infty}\left(\sum_{i=k}^{\infty} x_{i,j}\right)=\sup\left(\left\{\sum_{j=l}^{m}\left(\sum_{i=k}^{\infty} x_{i,j}\right) | m\in\{l,\ldots,\infty\}\right\}\right)\leqslant\sum_{i=n}^{\infty} x_{\beta(i)_1,\beta(i)_2}
$$

hence

$$
\sum_{i=k}^{\infty}\left(\sum_{j=l}^{\infty}\right)\underset{[\text{theorem: } 20.96]}{=}\sum_{j=l}^{\infty}\left(\sum_{i=k}^{\infty} x_{i,j}\right)\leqslant\sum_{i=n}^{\infty} x_{\beta(i)_1,\beta(i)_2}
$$

which combined with [eq: 20.50] proves that

$$
\sum_{i=0}^{\infty} x_{\beta(i)_1,\beta(i)}=\sum_{i=k}^{\infty}\left(\sum_{j=l}^{\infty} x_{i,j}\right)
$$

□

If a series if finite then all the terms in the series are real numbers.

Lemma 20.102. *Let $k\in\mathbb{N}_0$, $n\in\{k,\ldots,\infty\}$ and $\{x_i\}_{i\in\{k,\ldots,n\}}\subseteq[0,\infty]$ then we have*

$$
\sum_{i=k}^{n} x_i=\infty \Leftrightarrow \infty\in\{x_i | i\in\{k,\ldots,n\}\}
$$

Proof.

⇒. We use induction to prove this, so let

$$S=\left\{n\in\{k,\ldots,\infty\}|\text{If }\{x_i\}_{i\in\{k,\ldots,n\}}\subseteq[0,\infty]\text{ satisfies }\sum_{i=k}^n x_i\text{ then }\infty\in\{x_i|i\in\{k,\ldots,\infty\}\}\right\}$$

then we have:

$k\in S$. For $\{x_i\}_{i\in\{k\}}\subseteq[0,\infty]$ we have $x_k=\sum_{i=k}^k x_i=\infty$ hence $\infty\in\{x_i|i\in\{k\}\}$ proving that $k\in S$.

$n\in S\Rightarrow n+1\in S$. Let $\{x_i\}_{i\in\{k,\ldots,n+1\}}\subseteq[0,\infty]$ be such that $\sum_{i=k}^{n+1}x_i=\infty$ then, as $\sum_{i=k}^{n+1}x_i=x_{n+1}+\sum_{i=k}^n x_i$, we have either:

$x_{n+1=\infty}$. Then $\infty\in\{x_i|i\in\{k,\ldots,n+1\}\}$

$\sum_{i=k}^n x_i$. Then as $n\in S$ $\infty\in\{x_i|i\in\{k,\ldots,n\}\}\subseteq\infty\in\{x_i|i\in\{k,\ldots,n+1\}\}$

So in all possible cases we have $\infty\in\{x_i|i\in\{k,\ldots,n+1\}\}$ which proves that $n+1\in S$

⇐. We use also induction for this, so let

$$S=\left\{n\in\{k,...,\infty\}|\text{If }\{x_i\}_{i\in\{k,\ldots,n\}}\subseteq[0,\infty]\text{ satisfies }\infty\in\left\{x_i|i\in\{k,...,n\}\text{ then }\sum_{i=k}^n x_i=\infty\right\}\right\}$$

the we have:

$k\in S$. If $\{x_i\}_{i\in\{k\}}\subseteq[0,\infty]$ satisfies $\infty\in\{x_i|i\in\{k,\ldots,k\}\}=\{x_i\}$ then $x_k=\infty$ so that $\sum_{i=k}^k x_i=x_k=\infty$. Hence $k\in S$.

$n\in S\Rightarrow n+1\in S$. If $\{x_i\}_{i\in\{k,\ldots,n+1\}}\subseteq[0,\infty]$ satisfies $\infty\in\{x_i|i\in\{k,\ldots,n+1\}$ then there exist a $i\in\{k,\ldots,n+1\}$ such that $x_i=\infty$. For i we have either:

$i=n+1$. Then $\sum_{i=k}^{n+1}x_i=x_{n+1}+\sum_{i=k}^n x_i=\infty+\sum_{i=k}^n x_i=\infty$

$i\in\{k,\ldots,n\}$. Then $\sum_{i=k}^{n+1}x_i=x_{n+1}+\sum_{i=k}^n x_i\underset{n\in S}{=}x_{n+1}+\infty=\infty$

So in all cases we have $\sum_{i=k}^{n+1}x_i=\infty$ proving that $n+1\in S$. ☐

Theorem 20.103. *Let $k\in\mathbb{N}_0$ and for $\{x_i\}_{i\in\{k,\ldots,\infty\}}\subseteq[0,\infty]$ we have $\sum_{i=k}^\infty x_i<\infty$ then $\forall i\in\{k,\ldots,\infty\}$ we have $x_i<\infty$*

Proof. As $\sup\left(\{\sum_{i=k}^n x_i|n\in\{k,\ldots,\infty\}\}\right)=\sum_{i=k}^\infty x_i<\infty$ it follows from [theorem: 20.15] that $\forall n\in\{k,\ldots,\infty\}$ we have that $\sum_{i=k}^n x_i<\infty\underset{[\text{lemma: }20.102]}{=}x_n=\infty$. ☐

20.2 Preliminaries

Theorem 20.104. *Let A,B,C,D be sets then*

1. $(A\cdot B)\setminus(C\cdot D)=((A\setminus C)\cdot B)\bigcup(A\cdot(B\setminus D))$
2. $(A\cdot B)\setminus(C\cdot D)=(A\setminus C)\cdot(B\setminus D)\bigcup(A\setminus C)\cdot(B\bigcap D)\bigcup(A\bigcap C)\cdot(B\setminus D)$ *and*
   a. $((A\setminus C)\cdot(B\setminus D))\bigcap((A\setminus C)\cdot(B\bigcap D))=\varnothing$
   b. $((A\setminus C)\cdot(B\setminus D))\bigcap((A\bigcap C)\cdot(B\setminus D))=\varnothing$
   c. $((A\setminus C)\cdot(B\bigcap D))\bigcap((A\bigcap C)\cdot(B\setminus D))=\varnothing$

Proof.

1. We have the following equivalences

$$\begin{aligned}(x,y)\in(A\cdot B)\backslash(C\cdot D) &\Leftrightarrow (x,y)\in A\cdot B\wedge(x,y)\notin C*D\\ &\Leftrightarrow x\in A\wedge y\in B\wedge\neg(x\in C\wedge y\in D)\\ &\Leftrightarrow x\in A\wedge y\in B\wedge(x\notin C\vee y\notin D)\\ &\Leftrightarrow (x\in A\wedge y\in B\wedge x\notin C)\vee(x\in A\wedge y\in B\wedge y\notin D)\\ &\Leftrightarrow (x\in A\backslash C\wedge y\in B)\vee(x\in A\wedge y\in B\backslash D)\\ &\Leftrightarrow (x,y)\in(A\backslash C)\cdot B\vee(x,y)\in A\cdot(B\backslash D)\\ &\Leftrightarrow (x,y)\in(A\backslash C)\cdot B\bigcup(A\cdot(B\backslash D))\end{aligned}$$

which proves that

$$(A\cdot B)\backslash(C\cdot D)=((A\backslash C)\cdot B)\bigcup(A\cdot(B\backslash D))$$

2. Next we have

$$\begin{aligned}(A\cdot B)\backslash(C\cdot D) &=\\ ((A\backslash C)\cdot B)\bigcup(A\cdot(B\backslash D)) &\underset{[\text{theorem: }1.35]}{=}\\ \left((A\backslash C)\cdot\left((B\backslash D)\bigcup\left(B\bigcap D\right)\right)\right)\bigcup\left(\left((A\backslash C)\bigcup\left(A\bigcap C\right)\right)\cdot(B\backslash D)\right) &\underset{[\text{theorem: }6.84]}{=}\\ \underbrace{(A\backslash C)\cdot(B\backslash D)}_{1}\bigcup(A\backslash C)\cdot\left(B\bigcap D\right)\bigcup\underbrace{(A\backslash C)\cdot(B\backslash D)}_{1}\bigcup\left(A\bigcap C\right)\cdot(B\backslash D) &=\\ (A\backslash C)\cdot(B\backslash D)\bigcup(A\backslash C)\cdot\left(B\bigcap D\right)\bigcup\left(A\bigcap C\right)\cdot(B\backslash D)\end{aligned}$$

proving

$$(A\cdot B)\backslash(C\cdot D)=(A\backslash C)\cdot(B\backslash D)\bigcup(A\backslash C)\cdot\left(B\bigcap D\right)\bigcup\left(A\bigcap C\right)\cdot(B\backslash D)$$

a.

$$\begin{aligned}((A\backslash C)\cdot(B\backslash D))\bigcap\left((A\backslash C)\cdot\left(B\bigcap D\right)\right) &\underset{[\text{theorem: }6.84]}{=}\\ (A\backslash C)\cdot\left((B\backslash D)\bigcap\left(B\bigcap D\right)\right) &=\\ (A\backslash C)\cdot\varnothing &\underset{[\text{theorem: }6.84]}{=}\\ \varnothing\end{aligned}$$

b.

$$\begin{aligned}((A\backslash C)\cdot(B\backslash D))\bigcap\left(\left(A\bigcap C\right)\cdot(B\backslash D)\right) &\underset{[\text{theorem: }6.84]}{=}\\ \left((A\backslash C)\bigcap\left(A\bigcap C\right)\right)\cdot(B\backslash D) &=\\ \varnothing\bigcap(B\backslash D) &\underset{[\text{theorem: }6.84]}{=}\\ \varnothing\end{aligned}$$

c.

$$\begin{aligned}\left((A\backslash C)\cdot\left(B\bigcap D\right)\right)\bigcap\left(\left(A\bigcap C\right)\cdot(B\backslash D)\right) &\underset{[\text{theorem: }6.84]}{\subseteq}\\ \left(A\cdot\left(B\bigcap D\right)\right)\bigcap(A\cdot(B\backslash D)) &\underset{[\text{theorem: }6.84]}{=}\\ A\cdot\left(\left(B\bigcap D\right)\bigcap(B\backslash D)\right) &=\\ A\cdot\varnothing &\underset{[\text{theorem: }6.84]}{=}\\ \varnothing\end{aligned}$$

□

Definition 20.105. *Let X be a set, $\mathcal{A}\subseteq\mathcal{P}(X)$, I a set then a family of sets $\{A_i\}_{i\in I}\subseteq\mathcal{A}$ is* ***pairwise disjoint*** *if*

$$\forall i,j\in I \text{ with } i\neq j \text{ we have } A_i\bigcap A_j=\varnothing$$

Notation 20.106. *Let X be a set, $\mathcal{A}\subseteq\mathcal{P}(X)$, I a set and $\{A_i\}_{i\in I}\subseteq\mathcal{A}$ then if we write*

$$\bigsqcup_{i\in I} A_i$$

we mean

$$\bigsqcup_{i\in I} A_i$$

and additional that $\{A_i\}_{i\in I}\subseteq\mathcal{A}$ is pairwise disjoint.

Theorem 20.107. *Let X be a set, $\mathcal{A}\subseteq\mathcal{P}(X)$, I,J sets and $\{A_i\}_{i\in J}\subseteq\mathcal{A}$ be a pairwise disjoint family and $\sigma: I\rightarrow J$ a bijection then $\{A_{\sigma(i)}\}_{i\in I}\subseteq\mathcal{A}$ is pairwise disjoint and $\bigcup_{i\in I}A_{\sigma(i)}=\bigcup_{i\in J}A_i$. In other words using the above theorem we have*

$$\bigsqcup_{i\in J} A_i=\bigsqcup_{i\in I} A_{\sigma(i)}$$

Proof. Using [theorem: 2.119] and the fact that a bijection is a surjection we have

$$\bigcup_{i\in I} A_{\sigma(i)}=\bigcup_{i\in J} A_i$$

Let $i,j\in I$ be such that $i\neq j$ then as σ is injective we have $\sigma(i)\neq\sigma(j)$ [if $\sigma(i)=\sigma(j)$ then $i=j$] so that $A_{\sigma(i)}\bigcap A_{\sigma(j)}=\varnothing$. Hence $\{A_{\sigma(i)}\}_{i\in\{1,\ldots,n\}}$ is pairwise disjoint. □

The following set based theorems and lemmas will be usefull later

Lemma 20.108. *Let be N be either ∞ or $N\in\mathbb{N}$ $\{A_i\}_{i\in\{1,\ldots,N\}}\subseteq\mathcal{A}$ be a sequence or finite family of sets then for*

$$\{B_i\}_{i\in\{1,\ldots,N\}} \text{ defined by } B_i=\begin{cases} A_i \text{ if } i=1\\ A_i\setminus\bigcup_{j\in\{1,\ldots,j-1\}}A_j \text{ if } i\in\{2,\ldots,N\}\end{cases}$$

we have

$$\forall i\in\{1,\ldots,N\}\quad B_i\subseteq A_i$$

$$\bigsqcup_{i\in\{1,\ldots,N\}} B_i=\bigcup_{i\in\{1,\ldots,N\}} A_i$$

and

$\forall i,j\in\{1,\ldots,N\}$ with $i\neq j$ then $B_i\bigcap B_j\neq\varnothing$ [in other words $\{B_i\}_{i\in\{1,\ldots,N\}}$ is pairwise disjoint

Proof. We have

$$\forall i\in\{1,\ldots,N\} \text{ that } B_i=\begin{cases} A_1\subseteq A_i \text{ if } i=1\\ A_i\setminus\bigcup_{j\in\{1,\ldots,i-1\}}A_j\subseteq A_i \text{ if } i\in\{2,\ldots,N\}\end{cases}\subseteq A_i$$

From the above it follows that

$$\bigcup_{i\in\{1,\ldots,N\}} B_i\subseteq\bigcup_{i\in\{1,\ldots,N\}} A_i \tag{20.53}$$

If $x\in\bigcup_{i\in\{1,\ldots,N\}}A_i$ then there exist a $k\in\{1,\ldots,N\}$ such that $x\in A_k$. Define $M=\{i\in\{1,\ldots,k\}|x\in A_i\}\subseteq\{1,\ldots,k\}$ then $k\in M$ and M is finite so that

$$m=\min(M) \text{ exist}$$

As $m \in M \subseteq \{1, \ldots, k\}$ we have $x \in A_m$ and for m either:

$m = 1$. Then $x \in A_1 = B_1$ so that $x \in \bigcup_{i \in \{1, \ldots, N\}} B_i$.

$1 < m$. Then for $\forall j \in \{1, \ldots, m-1\}$ we have $x \notin A_j$ so that $x \notin \bigcup_{j \in \{1, \ldots, m-1\}} A_j$ hence $x \in A_m \setminus \bigcup_{j \in \{1, \ldots, m-1\}} A_j = B_m$. So $x \in \bigcup_{i \in \{1, \ldots, N\}} B_i$.

As $x \in \bigcup_{i \in \{1, \ldots, N\}} A_i$ was chosen arbitrary it follows that $\bigcup_{i \in \{1, \ldots, N\}} A_i \subseteq \bigcup_{i \in \{1, \ldots, N\}} B_i$ which combined with [eq: 20.53] proves that

$$\bigcup_{i \in \{1, \ldots, N\}} B_i = \bigcup_{i \in \{1, \ldots, N\}} A_i$$

Let $i, j \in \{1, \ldots, N\}$ with $i \neq j$ then we may assume that $i < j$ [otherwise exchange i and j]. Assume that $x \in B_i \bigcap B_j$ then for i we have either:

$i = 1$. Then $x \in B_1 = A_1$ and $x \in B_j = A_j \setminus \bigcup_{k \in \{1, \ldots, j-1\}} A_k$. So $\forall k \in \{1, \ldots, j-1\}$ we have that $x \notin A_k$, hence, as $1 = i < j \Rightarrow 1 \in \{1, \ldots, j-1\}$, it follows that $x \notin A_1$ contradicting $x \in B_1 = A_1$.

$1 < j$. Then $x \in B_j = A_j \setminus \bigcup_{k \in \{1, \ldots, j-1\}} A_k \Rightarrow \forall k \in \{1, \ldots, j-1\}$ we have $x \notin A_k$. As $i < j \Rightarrow i \in \{1, \ldots, j-1\}$ it follows that $x \notin A_i$ contradicting $x \in A_i \setminus \bigcup_{k \in \{1, \ldots, i-1\}} A_k \subseteq A_i$.

As in all cases we have a contradiction the assumption is wrong and we must have that

$$B_i \bigcap B_j = \varnothing$$

□

Lemma 20.109. *Let $\{A_i\}_{i \in \mathbb{N}}$ be a sequence of sets then if we define $\{B_i\}_{i \in \mathbb{N}}$ by $B_i = \bigcup_{j \in \{1, \ldots, i\}} A_j$ we have*

$$\bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i \in \mathbb{N}} B_i \text{ and } \forall i \in \mathbb{N} \text{ we have } B_i \subseteq B_{i+1}$$

Proof. First if $i \in \mathbb{N}$ and $x \in B_i$ then there exists a $j \in \{1, \ldots, i\}$ such that

$$x \in A_j \subseteq \bigcup_{j \in \{1, \ldots, i+1\}} A_j = B_{i+1}$$

proving that

$$\forall i \in \mathbb{N}_0 \text{ we have } B_i \subseteq B_{i+1} \tag{20.54}$$

Second if $x \in \bigcup_{i \in \mathbb{N}_0} A_i$ then $\exists i \in \mathbb{N}$ such that $x \in A_i$. So as $i \in \{1, \ldots, i\}$ we have that $x \in \bigcup_{j \in \{1, \ldots, i\}} A_i = B_i$ which proves that $x \in \bigcup_{i \in \mathbb{N}} B_i$, or

$$\bigcup_{i \in \mathbb{N}_0} A_i \subseteq \bigcup_{i \in \mathbb{N}_0} B_i \tag{20.55}$$

Third if $x \in \bigcup_{i \in \mathbb{N}_0} B_i$ then $\exists i \in \mathbb{N}$ such that $x \in B_i = \bigcup_{j \in \{1, \ldots, i\}} A_j$, so there exists a $j \in \{1, \ldots, i\} \subseteq \mathbb{N}$ with $x \in A_j$, proving that $x \in \bigcup_{i \in \mathbb{N}} A_i$. Hence $\bigcup_{i \in \mathbb{N}_0} B_i \subseteq \bigcup_{i \in \mathbb{N}_0} A_i$ which combined with [eq: 20.55] gives

$$\bigcup_{i \in \mathbb{N}_0} A_i = \bigcup_{i \in \mathbb{N}_0} B_i \tag{20.56}$$

The theorem is then proved by [eqs: 20.54, 20.55 and 20.56]. □

Definition 20.110. *Let X be a set and $\mathcal{A} \subseteq \mathcal{P}(X)$ then*

1. *$\mathcal{A}$ is $\cup$-stable if $\forall A, B \in \mathcal{A}$ we have $A \bigcup B \in \mathcal{A}$*
2. *$\mathcal{A}$ is $\cap$-stable if $\forall A, B \in \mathcal{A}$ we have $A \bigcap B \in \mathcal{A}$*

Theorem 20.111. *Let X be a set and $\mathcal{A} \subseteq \mathcal{P}(X)$ then we have:*

1. *$\mathcal{A}$ is $\cup$-stable $\Leftrightarrow \forall \{A_i\}_{\in \{1, \ldots, n\}} \subseteq \mathcal{A}$, $n \in \mathbb{N}$ we have $\bigcup_{i \in \{1, \ldots, n\}} A_i \in \mathcal{A}$*

2. *$\mathcal{A}$ is $\cap$-stable $\Leftrightarrow$ $\forall\{A_i\}_{\in\{1,\ldots,n\}}\subseteq\mathcal{A}$, $n\in\mathbb{N}$ we have $\bigcap_{i\in\{1,\ldots,n\}}A_i\in\mathcal{A}$*

Proof. This is easely proved by induction

1.

$\Rightarrow$. We prove this by induction so let

$$S=\left\{n\in\mathbb{N}|\text{If } \{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{A} \text{ then } \bigcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{A}\right\}$$

then we have:

$1\in S$. If $\{A_i\}_{i\in\{1\}}\subseteq\mathcal{A}$ then $\bigcup_{i\in\{1\}}A_i=A_1\in\mathcal{A}$ so that $1\in S$.

$n\in S\Rightarrow n+1\in S$. Let $\{A_i\}_{i\in\{1,\ldots,n+1\}}\subseteq\mathcal{A}$ then $A_{n+1}\in\mathcal{A}$ and, as $n\in S$, $\bigcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{A}$. Hence $\bigcup_{i\in\{1,\ldots,n+1\}}A_i=A_{n+1}\bigcup\left(\bigcup_{i\in\{1,\ldots,n\}}A_i\right)\in\mathcal{A}$ which proves that $n+1\in S$

$\Leftarrow$. Let $A,B\in\mathcal{A}$ and define $\{A_i\}_{i\in\{1,2\}}\subseteq\mathcal{A}$ by $A_1=A$ and $A_2=B$ then $A\bigcup B=A_1\bigcup A_2=\bigcup_{i\in\{1,2\}}A_i\in\mathcal{A}$.

2.

$\Rightarrow$. $\Rightarrow$. We prove this by induction so let

$$S=\left\{n\in\mathbb{N}|\text{If } \{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{A} \text{ then } \bigcap_{i\in\{1,\ldots,n\}}A_i\in\mathcal{A}\right\}$$

then we have:

$1\in S$. If $\{A_i\}_{i\in\{1\}}\subseteq\mathcal{A}$ then $\bigcap_{i\in\{1\}}A_i=A_1\in\mathcal{A}$ so that $1\in S$.

$n\in S\Rightarrow n+1\in S$. Let $\{A_i\}_{i\in\{1,\ldots,n+1\}}\subseteq\mathcal{A}$ then $A_{n+1}\in\mathcal{A}$ and, as $n\in S$, $\bigcap_{i\in\{1,\ldots,n\}}A_i\in\mathcal{A}$. Hence $\bigcap_{i\in\{1,\ldots,n+1\}}A_i=A_{n+1}\bigcap\left(\bigcap_{i\in\{1,\ldots,n\}}A_i\right)\in\mathcal{A}$ which proves that $n+1\in S$

$\Leftarrow$. Let $A,B\in\mathcal{A}$ and define $\{A_i\}_{i\in\{1,2\}}\subseteq\mathcal{A}$ by $A_1=A$ and $A_2=B$ then $A\bigcap B=A_1\bigcap A_2=\bigcap_{i\in\{1,2\}}A_i\in\mathcal{A}$. □

Lemma 20.112. *Let $n\in\mathbb{N}$, $\{A_i\}_{i\in\{1,\ldots,n\}}$ a finite family of sets such that $\forall i\in\{1,\ldots,n\}$ there exist a family $\{B_{i,j}\}_{j\in I_i}$ such that $A_i=\bigcup_{j\in I_i}B_{i,j}$ then we have:*

1. *$\bigcap_{i\in\{1,\ldots,n\}}A_i=\bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}}I_i}\left(\bigcap_{k\in\{1,\ldots,n\}}B_{k,\sigma_k}\right)$*
2. *If $\forall i\in\{1,\ldots,n\}$ $\{B_{i,j}\}_{j\in I_i}$ is pairwise disjoint then we have that $\left\{\bigcap_{k\in\{1,\ldots,n\}}B_{k,\sigma_k}\right\}_{\sigma\in\prod_{i\in\{1,\ldots,n\}}I_i}$ is pairwise disjoint. In other words*

$$\bigcap_{i\in\{1,\ldots,n\}}A_i=\bigsqcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}}I_i}\left(\bigcap_{k\in\{1,\ldots,n\}}B_{k,\sigma_k}\right)$$

Proof.

1. We use induction to prove this, so define

$$S=\left\{n\in\mathbb{N}|\text{If } \{A_i\}_{i\in\{1,\ldots,n\}} \text{ is such that } \forall i\in\{1,\ldots,n\}\ \exists\{B_{i,j}\}_{j\in I_i} \text{ with } A_i=\bigcup_{j\in I_i}B_{i,j}\right.$$
$$\left.\text{then } \bigcap_{i\in\{1,\ldots,n\}}A_i=\bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}}I_i}\left(\bigcap_{k\in\{1,\ldots,n\}}B_{k,\sigma_k}\right)\right\}$$

then we have:

$1\in S$. If $\{A_i\}_{i\in\{1,1\}}$ satisfies that there exist a $\{B_{1,j}\}_{j\in I_1}$ such that $A_1=\bigcup_{j\in I_1} B_{1,j}$ then

$$\begin{aligned}\bigcap_{i\in\{1\}} A_i &= A_1\\ &= \bigcup_{j\in I_1} B_{1,j}\\ &= \bigcup_{j\in I_1}\left(\bigcap_{k\in\{1\}} B_{k,j}\right)\\ &= \bigcup_{(j)\in\prod_{i\in\{1\}} I_i}\left(\bigcap_{k\in\{1\}} B_{k,j}\right)\\ &= \bigcup_{\sigma\in\prod_{i\in\{1\}} I_i}\left(\bigcap_{k\in\{1\}} B_{k,\sigma_k}\right)\end{aligned}$$

proving that $1\in S$.

$n\in S\Rightarrow n+1\in S$. Let $\{A_i\}_{i\in\{1,\ldots,n+1\}}$ is such that $\forall i\in\{1,\ldots,n+1\}$ $\exists\{B_{i,j}\}_{j\in I_i}$ with $A_i=\bigcup_{j\in I_i} B_{i,j}$. Then as $n\in\mathcal{S}$ we have that

$$\bigcap_{i\in\{1,\ldots,n\}} A_i=\bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}} I_i}\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right) \tag{20.57}$$

Hence

$$\begin{gathered}\bigcap_{i\in\{1,\ldots,n+1\}} A_i \quad =\\ \left(\bigcap_{i\in\{1,\ldots,n\}} A_i\right)\bigcap A_{n+1} \quad \underset{n\in S}{=}\\ \left(\bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}} I_i}\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right)\right)\bigcap A_{n+1} \quad \underset{[\text{theorem: } 2.130]}{=}\\ \bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}} I_i}\left(\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right)\bigcap A_{n+1}\right) \quad =\\ \bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}} I_i}\left(\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right)\bigcap\left(\bigcup_{j\in I_{n+1}} B_{n+1,j}\right)\right) \quad \underset{[\text{theorem: } 2.130]}{=}\\ \bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}} I_i}\left(\left(\bigcup_{j\in I_{n+1}}\left(B_{n+1,j}\bigcap\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right)\right)\right)\right) \quad =\end{gathered}$$

So that

$$\begin{gathered}\bigcap_{i\in\{1,\ldots,n+1\}} A_i \;=\\ \bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}} I_i}\left(\left(\bigcup_{j\in I_{n+1}}\left(B_{n+1,j}\bigcap\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right)\right)\right)\right)\end{gathered} \tag{20.58}$$

If $x\in\bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}} I_i}\left(\left(\bigcup_{j\in I_{n+1}}\left(B_{n+1,j}\bigcap\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right)\right)\right)\right)$ then there exist a $\sigma=(\sigma_1,\ldots,\sigma_n)$ such that $x\in\left(\bigcup_{j\in I_{n+1}}\left(B_{n+1,j}\bigcap\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right)\right)\right)$, hence there exist a $j\in I_{n+1}$ such that

$$x\in B_{n+1,j}\bigcap\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right)$$

Define

$$\tau \in \bigcup_{i\in\{1,\ldots,n+1\}} I_i \text{ by } \tau_i = \begin{cases} \sigma_i \text{ if } i\in\{1,\ldots,n\} \\ j \text{ if } i=n+1 \end{cases}$$

then $x\in B_{n+1,\tau_{n+1}}\bigcap\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,t_k}\right)=\bigcap_{k\in\{1,\ldots,n+1\}} B_{k,\tau_k}$. Hence

$$x\in \bigcup_{\tau\in\prod_{i\in\{1,\ldots,n+1\}}}\left(\bigcap_{k\in\{1,\ldots,n+1\}} B_{k,\tau_k}\right)$$

whic proves that

$$\bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}} I_i}\left(\left(\bigcup_{j\in I_{n+1}}\left(B_{n+1,j}\bigcap\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right)\right)\right)\right) \subseteq \bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n+1\}}}\left(\bigcap_{k\in\{1,\ldots,n+1\}} B_{k,\sigma_k}\right) \tag{20.59}$$

If $x\in\bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n+1\}}}\left(\bigcap_{k\in\{1,\ldots,n+1\}} B_{k,\sigma_k}\right)$ then there exist a $\sigma\in\prod_{i\in\{1,\ldots,n+1\}} I_i$ such that $x\in\bigcap_{k\in\{1,\ldots,n+1\}} B_{k,\sigma_k}$. Define $\eta\in\prod_{i\in\{1,\ldots,n\}} I_i$ by $\eta_l=\sigma_l$ and $j=\sigma_{n+1}\in I_{n+1}$ then

$$x\in B_{n+1,j}\bigcap\left(\bigcap_{k\in\{1,\ldots,n+1\}} B_{k,\eta_k}\right)\subseteq\bigcup_{j\in I_{n+1}}\left(B_{n+1,j}\bigcap\left(\bigcap_{k\in\{1,\ldots,n+1\}} B_{k,\eta_k}\right)\right)$$

so that $x\in\bigcup_{j\in I_{n+1}}\left(B_{n+1,j}\bigcap\left(\bigcap_{k\in\{1,\ldots,n+1\}} B_{k,\eta_k}\right)\right)$ or as $\eta\in\prod_{i\in\{1,\ldots,n\}} I_i$ we have that $x\in\bigcup_{\eta\in\prod_{i\in\{1,\ldots,n\}} I_i}\left(\bigcup_{j\in I_{n+1}}\left(B_{n+1,j}\bigcap\left(\bigcap_{k\in\{1,\ldots,n+1\}} B_{k,\eta_k}\right)\right)\right)$. Hence

$$\bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n+1\}}}\left(\bigcap_{k\in\{1,\ldots,n+1\}} B_{k,\sigma_k}\right) \subseteq \bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}} I_i}\left(\left(\bigcup_{j\in I_{n+1}}\left(B_{n+1,j}\bigcap\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right)\right)\right)\right)$$

which combined with [eq: 20.59] proves that

$$\bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n\}} I_i}\left(\left(\bigcup_{j\in I_{n+1}}\left(B_{n+1,j}\bigcap\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right)\right)\right)\right) = \bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n+1\}}}\left(\bigcap_{k\in\{1,\ldots,n+1\}} B_{k,\sigma_k}\right)$$

Combining the above with [eq: 20.58] proves that

$$\bigcap_{i\in\{1,\ldots,n+1\}} A_i = \bigcup_{\sigma\in\prod_{i\in\{1,\ldots,n+1\}}}\left(\bigcap_{k\in\{1,\ldots,n+1\}} B_{k,\sigma_k}\right)$$

Hence we have that $n+1\in\mathcal{S}$

2. Assume that $\sigma,\tau\in\prod_{i\in\{1,\ldots,n\}} I_i$ with $\sigma\neq\tau$ then there exist $i\in\{1,\ldots,n\}$ such that $\sigma_i\neq\tau_i$. As $\{B_{i,j}\}_{j\in\{1,\ldots,I_i\}}$ is pairwise disjoint we have that $B_{i,\sigma_i}\bigcap B_{i,\tau_i}=\varnothing$. Hence

$$\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right)\bigcap\left(\bigcap_{k\in\{1,\ldots,n\}} B_{k,\tau_k}\right)\subseteq B_{i,\sigma_i}\bigcap B_{i,\tau_i}=\varnothing$$

proving that $\left\{\bigcap_{k\in\{1,\ldots,n\}} B_{k,\sigma_k}\right\}_{\sigma\in\prod_{i\in\{1,\ldots,n\}} I_i}$ is pairwise disjoint.

□

Theorem 20.113. *Let $\{A_i\}_{i\in I}$ be a family of sets such that $\forall i\in I$ there exist a $\{B_{i,j}\}_{j\in I_i}$ such that $A_i=\bigcup_{j\in I_i} B_{i,j}$ then*

1. $\bigcup_{i\in I} A_i=\bigcup_{i\in I}(\bigcup_{j\in I_i} B_{i,j})=\bigcup_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i} B_{\sigma_1,\sigma_2}$
2. *If additional $\{A_i\}_{i\in I}$ is pairwise disjoint and $\forall i\in I$ $\{B_{i,j}\}_{j\in I_i}$ is pairwise disjoint then*

$$\{B_{\sigma_1,\sigma_2}\}_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i} \text{ is pairwise disjoint}$$

so that we have

$$\bigsqcup_{i\in I} A_i=\bigsqcup_{i\in I}\left(\bigsqcup_{j\in I_i} B_{i,j}\right)=\bigsqcup_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i} B_{\sigma_1,\sigma_2}$$

Proof.

1. Let $x\in\bigcup_{i\in I} A_i$ then there exist a $i\in I$ such that $x\in A_i$, hence, as $A_i=\bigcup_{j\in I_i} B_{i,j}$ there exist a $j\in I_i$ such that $x\in B_{i,j}$. As $\sigma=(i,j)\in\{i\}\times I_i\subseteq\bigcup_{i\in I}\{i\}\times I_i$ it follows $x\in B_{\sigma_1,\sigma_2}$, hence we have that $\bigcup_{i\in I} A_i\subseteq\bigcup_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i} B_{\sigma_1,\sigma_2}$. Further as for $\sigma\in\bigcup_{i\in I}\{i\}\times I$ we have that $B_{\sigma_1,\sigma_2}\subseteq\bigcup_{j\in I_{\sigma_1}} B_{\sigma_1,j}=A_{\sigma_1}\subseteq\bigcup_{i\in I} A_i$ it follows that $\bigcup_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i} B_{\sigma_1,\sigma_2}\subseteq\bigcup_{i\in I} A_i$. So it follows that

$$\bigcup_{i\in I} A_i=\bigcup_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i} B_{\sigma_1,\sigma_2}$$

2. Let $\sigma,\tau\in\bigcup_{i\in I}\{i\}\times I_i$ with $\sigma\neq\tau$ then we have either:

   $\sigma_1=\tau_1$. Then we must have that $\sigma_2\neq\tau_2$ so that $B_{\sigma_1,\sigma_2}\bigcap B_{\tau_1,\tau_2}=B_{\sigma_1,\sigma_2}\bigcap B_{\sigma_1,\tau_2}=\varnothing$ [because $\{B_{\sigma_1,j}\}_{j\in I_{\sigma_1}}$ is pairwise disjoint].

   $\sigma_1\neq\tau_1$. Then $B_{\sigma_1,\sigma_2}\bigcap B_{\tau_1,\tau_2}\subseteq(\bigcup_{j\in I_{\sigma_1}} B_{\sigma_1,j})\bigcap(\bigcup_{j\in I_{\tau_1}} B_{\tau_1,j})=A_{\sigma_1}\bigcap A_{\tau_1}=\varnothing$ [because $\{A_i\}_{i\in I}$ is pairwise disjoint.

   Hence $\{B_{\sigma_1,\sigma_2}\}_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i}$ is pairwise disjoint. □

Corollary 20.114. *Let $\{A_i\}_{i\in I}$ be a family of sets such that $\forall i\in I$ there exist a $\{B_{i,j}\}_{j\in I_i}$ such that $A_i=\bigcup_{j\in I_i} B_{i,j}$ and $\beta: J\rightarrow\bigcup_{i\in I}\{i\}\times I_i$ a bijection then*

1. $\bigcup_{i\in I} A_i=\bigcup_{i\in I}(\bigcup_{j\in I_i} B_{i,j})=\bigcup_{j\in J} B_{\beta(j)_1,\beta(j)_2}$
2. *If additional $\{A_i\}_{i\in I}$ is pairwise disjoint and $\forall i\in I$ $\{B_{i,j}\}_{j\in I_i}$ is pairwise disjoint then*

$$\{B_{\beta(j)_1,\beta(j)_2}\}_{j\in J} \text{ is pairwise disjoint}$$

so that

$$\bigsqcup_{i\in I} A_i=\bigsqcup_{i\in I}\left(\bigsqcup_{j\in I_i} B_{i,j}\right)=\bigsqcup_{j\in J} B_{\beta(j)_1,\beta(j)_2}$$

Proof.

1. Using [theorem: 20.113] we have that

$$\bigcup_{i\in I} A_i=\bigcup_{i\in I}\left(\bigcup_{j\in I_i} B_{i,j}\right)=\bigcup_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i} B_{\sigma_1,\sigma_2} \tag{20.60}$$

Let $x\in\bigcup_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i} B_{\sigma_1,\sigma_2}$ then there exist a $\sigma\in\bigcup_{i\in I}\{i\}\times I_i$ such that $x\in B_{\sigma_1,\sigma_2}$. As β is a bijection hence surjective there exist a $j\in J$ such that $\beta(j)=\sigma$ so that $x\in B_{\beta(j)_1,\beta(j)_2}$, hence $x\in\bigcup_{j\in J} B_{\beta(j)_1,\beta(j)_2}$. So

$$\bigcup_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i} B_{\sigma_1,\sigma_2}\subseteq\bigcup_{j\in J} B_{\beta(j)_1,\beta(j)_2} \tag{20.61}$$

If $x\in\bigcup_{j\in J}B_{\beta(j)_1,\beta(j)_2}$ then $\exists j\in J$ such that $x\in B_{\beta(j)_1,\beta(j)_2}$ which as $\beta(j)\in\bigcup_{i\in I}\{i\}\times I_i$ proves that $x\in\bigcup_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i}B_{\sigma_1,\sigma_2}$. Hence $\bigcup_{j\in J}B_{\beta(j)_1,\beta(j)_2}\subseteq\bigcup_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i}B_{\sigma_1,\sigma_2}$ which combined with [eq: 20.61] results in

$$\bigcup_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i}B_{\sigma_1,\sigma_2}=\bigcup_{j\in J}B_{\beta(j)_1,\beta(j)_2} \tag{20.62}$$

Substituting the above in [eq: 20.60] proves finally that

$$\bigcup_{i\in I}A_i=\bigcup_{i\in I}\left(\bigcup_{j\in I_i}B_{i,j}\right)=\bigcup_{j\in J}B_{\beta(j)_1,\beta(j)_2}$$

2. Assume that additional $\{A_i\}_{i\in I}$ is pairwise disjoint and $\forall i\in I$ $\{B_{i,j}\}_{j\in I_i}$ is pairwise disjoint then by [theorem: 20.113]

$$\{B_{\sigma_1,\sigma_2}\}_{\sigma\in\bigcup_{i\in I}\{i\}\times I_i}\text{ pairwise disjoint} \tag{20.63}$$

Let $i,j\in J$ with $i\neq j$ then as β is injective we have $\beta(i)\neq\beta(j)$ hence we have by the above $B_{\beta(i)_1,\beta(j)_2}=\varnothing$ proving that

$$\{B_{\beta(i)_1,\beta(i)_2}\}_{i\in J}\text{ is pairwise disjoint}$$ □

Definition 20.115. (Monotone set function) *Let X be a set, $\mathcal{A}\subseteq\mathcal{P}(X)$ then a set function $\mu:\mathcal{A}\rightarrow[0,\infty]$ is* ***monotone*** *if $\forall A,B\in\mathcal{A}$ then $\mu(A)\leqslant\mu(B)$.*

Definition 20.116. (Additive set function) *Let X be a set, $\mathcal{A}\subseteq\mathcal{P}(X)$ then a set function $\mu:\mathcal{A}\rightarrow[0,\infty]$ is* ***finite additive on $\mathcal{A}$*** *if $\forall\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{A}$, $n\in\mathbb{N}$ pairwise disjoint with $\bigcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{A}$ we have*

$$\mu\left(\bigcup_{i\in\{1,\ldots,n\}}A_i\right)=\sum_{i=1}^{n}\mu(A_i)$$

Definition 20.117. (Sub-additive set function) *Let X be a set, $\mathcal{A}\subseteq\mathcal{P}(X)$ then a set function $\mu:\mathcal{A}\rightarrow[0,\infty]$ is sub-additive if $\forall\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{A}$ with $\bigcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{A}$ we have*

$$\mu\left(\bigcup_{i\in\{1,\ldots,n\}}A\right)\leqslant\sum_{i=1}^{n}\mu(A_i)$$

Definition 20.118. (Countable additive set function) *Let X be a set, $\mathcal{A}\subseteq\mathcal{P}(X)$ then a set function $\mu:\mathcal{A}\rightarrow[0,\infty]$ is* ***countable additve on $\mathcal{A}$*** *if $\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$, $n\in\mathbb{N}$ pairwise disjoint with $\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$ we have*

$$\mu\left(\bigcup_{i\in\mathbb{N}}A_i\right)=\sum_{i=1}^{\infty}\mu(A_i)$$

Definition 20.119. (Countable sub-additive set function) *Let X be a set, $\mathcal{A}\subseteq\mathcal{P}(X)$ then a set function $\mu:\mathcal{A}\rightarrow[0,\infty]$ is countable sub-additive if $\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ with $\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$ we have*

$$\mu\left(\bigcup_{i\in\mathbb{N}}A\right)\leqslant\sum_{i=1}^{\infty}\mu(A_i)$$

Countable sets are either finite or denumerable,, $n\in\mathbb{N}$ so you can ask yourselve why we talk about countable additivity instead of denumerable additivity [because $\mathbb{N}$ is denumerable]. However the next theorem shows that countable additivity implies finite additivity justifying the use of the term countable additivity.

Theorem 20.120. *Let X be a set, $\mathcal{A}\subseteq\mathcal{P}(X)$ such that $\varnothing\in\mathcal{A}$ and $\mu:\mathcal{A}\rightarrow[0,\infty]$ a countable additive set function on $\mathcal{A}$ satisfying $\mu(\varnothing)=0$ is finite additive on $\mathcal{A}$.*

Note 20.121. The requirement that $\varnothing\in\mathcal{A}$ and $\mu(\varnothing)=0$ is essential for the proof to work.

Proof. Let $n\in\mathbb{N}$, $\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{A}$ be a pairwise disjoint finite family such that $\bigcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{A}$. Define

$$\{B_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}\text{ by }B_i=\begin{cases}A_i\text{ if }i\in\{1,\ldots,n\}\\ \varnothing\text{ if }i\in\{n+1,\ldots,\infty\}\end{cases}\underset{\varnothing,A_i\in\mathcal{A}}{\in}\mathcal{A}$$

Let $x\in\bigcup_{i\in\mathbb{N}}B_i$ then there exist a $i\in\mathbb{N}$ such that $x\in B_i$. Assume that $i\in\{n+1,\ldots,\infty\}$ then $x\in B_i=\varnothing$ a contradiction so we must have that $i\in\{1,\ldots,n\}$. Hence $x\in B_i=A_i$ proving that $x\in\bigcup_{i\in\{1,\ldots,n\}}A_i$, so

$$\bigcup_{i\in\mathbb{N}}B_i\subseteq\bigcup_{i\in\{1,\ldots,n\}}A_i \tag{20.64}$$

Further if $x\in\bigcup_{i\in\{1,\ldots,n\}}A_i$ then there exist a $i\in\{1,\ldots,n\}$ such that $x\in A_i=B_i$ so that $x\in\bigcup_{i\in\mathbb{N}}B_i$ proving that $\bigcup_{i\in\{1,\ldots,n\}}A_i\subseteq\bigcup_{i\in\{1,\ldots,n\}}B_i$. Combining this with [eq: 20.64] proves that

$$\bigcup_{i\in\mathbb{N}}B_i=\bigcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{A} \tag{20.65}$$

Let $i,j\in\mathbb{N}$ with $i\neq j$ then we have the following possible cases:

$\boldsymbol{i\in\{1,\ldots,n\}\wedge j\in\{1,\ldots,n\}}$**.** Then $B_i\bigcap B_j=A_i\bigcap A_j\underset{i\neq j}{=}\varnothing$

$\boldsymbol{i\in\{n+1,\ldots,\infty\}\wedge j\in\{1,\ldots,n\}}$**.** Then $B_i\bigcap B_j=\varnothing\bigcap A_j=\varnothing$

$\boldsymbol{i\in\{1,\ldots,n\}\wedge j\in\{n+1,\ldots,\infty\}}$**.** Then $B_i\bigcap B_j=A_i\bigcap\varnothing=\varnothing$

$\boldsymbol{i\in\{n+1,\ldots,\infty\}\wedge j\in\{n+1,\ldots,\infty\}}$**.** Then $B_i\bigcap B_j=\varnothing\bigcap\varnothing=\varnothing$

Hence we conclude that

$$\{B_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}\text{ is pairwise disjoint} \tag{20.66}$$

As by [eq: 20.65] $\bigcup_{i\in\mathbb{N}}B_i\in\mathcal{A}$ we have by the above and the countable additivity of μ that

$$\mu\left(\bigcup_{i\in\mathbb{N}}B_i\right)=\sum_{i=1}^{\infty}\mu(B_i) \tag{20.67}$$

Next we have

$$\begin{aligned}\mu\left(\bigcup_{i\in\{1,\ldots,n\}}A_i\right) &\underset{[\text{eq: }20.65]}{=} \mu\left(\bigcup_{i\in\mathbb{N}}B_i\right)\\ &\underset{[\text{eq: }20.67]}{=} \sum_{i=1}^{\infty}\mu(B_i)\\ &\underset{[\text{theorem: }20.92]}{=} \sum_{i=1}^{n}\mu(B_i)+\sum_{i=n+1}^{\infty}\mu(B_i)\\ &= \sum_{i=1}^{n}\mu(A_i)+\sum_{i=n+1}^{\infty}\mu(\varnothing)\\ &\underset{\mu(\varnothing)=0\wedge[\text{theorem: }20.89]}{=} \sum_{i=1}^{n}\mu(A_i)\end{aligned}$$

Hence $\forall\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{A}$, $n\in\mathbb{N}$ pairwise disjoint we have $\mu\left(\bigcup_{i\in\{1,\ldots,n\}}A_i\right)=\sum_{i=1}^{n}\mu(A_i)$ proving that μ is finite additive on $\mathcal{A}$. □

Theorem 20.122. *Let X be a set, $\mathcal{A}\subseteq\mathcal{P}(X)$ such that $\varnothing\in\mathcal{A}$ and $\mu:\mathcal{A}\rightarrow[0,\infty]$ a countable sub-additive function then μ is sub-additivity.*

Proof. Let $n\in\mathbb{N}$, $\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{A}$ be a finite family such that $\bigcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{A}$. Define

$$\{B_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A} \text{ by } B_i=\left\{\begin{array}{l}A_i \text{ if } i\in\{1,\ldots,n\}\\ \varnothing \text{ if } i\in\{n+1,\ldots,\infty\}\end{array}\right. \underset{\varnothing,A_i\in\mathcal{A}}{\in}\mathcal{A}$$

Let $x\in\bigcup_{i\in\mathbb{N}}B_i$ then there exist a $i\in\mathbb{N}$ such that $x\in B_i$. Assume that $i\in\{n+1,\ldots,\infty\}$ then $x\in B_i=\varnothing$ a contradiction so we must have that $i\in\{1,\ldots,n\}$. Hence $x\in B_i=A_i$ proving that $x\in\bigcup_{i\in\{1,\ldots,n\}}A_i$, so

$$\bigcup_{i\in\mathbb{N}}B_i\subseteq\bigcup_{i\in\{1,\ldots,n\}}A_i \tag{20.68}$$

Further if $x\in\bigcup_{i\in\{1,\ldots,n\}}A_i$ then there exist a $i\in\{1,\ldots,n\}$ such that $x\in A_i=B_i$ so that $x\in\bigcup_{i\in\mathbb{N}}B_i$ proving that $\bigcup_{i\in\{1,\ldots,n\}}A_i\subseteq\bigcup_{i\in\{1,\ldots,n\}}B_i$. Combining this with [eq: 20.68] proves that

$$\bigcup_{i\in\mathbb{N}}B_i=\bigcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{A} \tag{20.69}$$

As $\bigcup_{i\in\mathbb{N}}B_i\in\mathcal{A}$ and μ is countable additive it follows that

$$\mu\left(\bigcup_{i\in\mathbb{N}}B_i\right)\leqslant\sum_{i=1}^{\infty}\mu(B_i) \tag{20.70}$$

$$\begin{array}{rcl}\mu\left(\bigcup_{i\in\{1,\ldots,n\}}A_i\right) & \underset{[\text{eq: } 20.69]}{=} & \mu\left(\bigcup_{i\in\mathbb{N}}B_i\right)\\ & \underset{[\text{eq: } 20.70]}{\leqslant} & \sum_{i=1}^{\infty}\mu(B_i)\\ & \underset{[\text{theorem: } 20.92]}{=} & \sum_{i=1}^{n}\mu(B_i)+\sum_{i=n+1}^{\infty}\mu(B_i)\\ & = & \sum_{i=1}^{n}\mu(A_i)+\sum_{i=n+1}^{\infty}\mu(\varnothing)\\ & \underset{\mu(\varnothing)=0\wedge[\text{theorem: } 20.89]}{=} & \sum_{i=1}^{n}\mu(A_i)\end{array}$$

□

20.3 Rings, algebras and σ-algebras

Definition 20.123. (Ring) *Let X be a set then $\mathcal{R}\subseteq\mathcal{P}(X)$ [the set of subsets of X] is a **ring** on X if $\mathcal{R}$ satisfies:*

1. *$\mathcal{R}\neq\varnothing$*
2. *If $A,B\in\mathcal{R}$ then $A\bigcup B\in\mathcal{R}$ [in other words $\mathcal{R}$ is $\cup$-stable]*
3. *If A, $B\in\mathcal{R}$ then $A\setminus B\in\mathcal{R}$*

Rather trivial examples of rings are the following:

Example 20.124. Let X be a set then $\{\varnothing\}$ is a ring on X

Proof.

1. As $\varnothing\in\{\varnothing\}$ we have that $\{\varnothing\}\neq\varnothing$
2. If $A,B\in\{\varnothing\}$ then $A=B=\varnothing$ so that $A\bigcup B=\varnothing\in\{\varnothing\}$
3. If If $A,B\in\{\varnothing\}$ then $A=B=\varnothing$ so that $A\setminus B=\varnothing\setminus\varnothing=\varnothing\in\{\varnothing\}$ □

Example 20.125. Let X be a set then $\mathcal{P}(X)$ is a ring on X

Proof. As $X \subseteq X$ $X \in \mathcal{P}(X)$ so that

$$\mathcal{P}(X) \neq \varnothing$$

If $A, B \in \mathcal{P}(X)$ then $A, B \subseteq X$ so that $A \bigcup B \subseteq X$ and $A \setminus B \subseteq X$ so that

$$A \bigcup B \in \mathcal{P}(X) \text{ and } A \setminus B \in \mathcal{P}(X)$$

□

Theorem 20.126. *Let X be a set and $\mathcal{R}$ a ring on X then we have*

1. $\varnothing \in \mathcal{R}$
2. *If $n \in \mathbb{N}$ and $\{A_i\}_{i \in \{1,\ldots,n\}} \subseteq \mathcal{R}$ a finite family of sets in $\mathcal{R}$ then $\bigcup_{i \in \{1,\ldots,n\}} A_i \in \mathcal{R}$*
3. *If $A, B \in \mathcal{R}$ then $A \bigcap B \in \mathcal{R}$ [in other words $\mathcal{R}$ is $\cap$-stable]*
4. *If $n \in \mathbb{N}$ and $\{A_i\}_{i \in \{1,\ldots,n\}} \subseteq \mathcal{R}$ a finite family of sets in $\mathcal{R}$ then $\bigcap_{i \in \{1,\ldots,n\}} A_i \in \mathcal{R}$*

Proof.

1. As $\mathcal{R} \neq \varnothing$ there exist a $A \in \mathcal{R}$ so that $\varnothing = A \setminus A \in \mathcal{R}$
2. This follows from [theorem: 20.111].
3. If $A, B \in \mathcal{R}$ then $A \setminus B \in \mathcal{R}$ so that $A \setminus (A \setminus B) \in \mathcal{R}$ which as $A \setminus (A \setminus B) \underset{\text{[theorem: 1.33]}}{=} A \bigcap B$ proves that $A \bigcap B \in \mathcal{R}$.
4. This follows from [theorem: 20.111]. □

A algebra on X is a ring such that X is a element of the ring.

Definition 20.127. (Algebra) *Let X be a set then $\mathcal{A} \subseteq \mathcal{P}(X)$ is a **algebra** on X if $\mathcal{A}$ satisfies*

1. *$\mathcal{A}$ is a ring on X*
2. $X \in \mathcal{R}$

Example 20.128. Let $X \neq \varnothing$ be a non empty set then $\{\varnothing\}$ is a ring [see example: 20.124] but not a algebra because $X \notin \{\varnothing\}$, however $\{X, \varnothing\}$ is a algebra on X.

Proof.

1. As $X, \varnothing \in \{X, \varnothing\}$ $\{X, \varnothing\} \neq \varnothing$
2. If $A, B \in \{X, \varnothing\}$ then we have either:

   $\boldsymbol{A = X \wedge B = X}$**.** Then $A \bigcup B = X \in \{X, \varnothing\}$ and $A \setminus B = X \setminus X = \varnothing \in \{X, \varnothing\}$

   $\boldsymbol{A = \varnothing \wedge B = X}$**.** Then $A \bigcup B = X \in \{X, \varnothing\}$ and $A \setminus B = \varnothing \setminus X = \varnothing \in \{X, \varnothing\}$

   $\boldsymbol{A = X \wedge B = \varnothing}$**.** Then $A \bigcup B = X \in \{X, \varnothing\}$ and $A \setminus B = X \setminus \varnothing = X \in \{X, \varnothing\}$

   $\boldsymbol{A = \varnothing \wedge B = \varnothing}$**.** Then $A \bigcup B = \varnothing \in \{X, \varnothing\}$ and $A \setminus B = \varnothing \setminus \varnothing = \varnothing \in \{X, \varnothing\}$ □

Example 20.129. Let X be a set then $\mathcal{P}(X)$ is a ring on X

Proof. Using [example: 20.125] it follows that $\mathcal{A}$ is a ring. Further as $X \subset X$ it follows that $X \in \mathcal{P}(X)$. □

The following summarizes some trivial properties of a algebra.

Theorem 20.130. *Let X be a set and $\mathcal{A}$ a algebra on X then we have:*

1. $\varnothing \in \mathcal{A}$
2. $X \in \mathcal{A}$
3. *$\forall A, B \in \mathcal{A}$ we have that $A \bigcup B \in \mathcal{A}$, $A \bigcap B \in \mathcal{A}$ and $A \setminus B \in \mathcal{A}$*

4. *If* $n \in \mathbb{N}$ *and* $\{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{A}$ *a finite family of sets in* $\mathcal{A}$ *then* $\bigcup_{i \in \{1, \ldots, n\}} A_i \in \mathcal{A}$
5. *If* $n \in \mathbb{N}$ *and* $\{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{A}$ *a finite family of sets in* $\mathcal{A}$ *then* $\bigcap_{i \in \{1, \ldots, n\}} A_i \in \mathcal{A}$
6. $\forall A \in \mathcal{A}$ *we have that* $X \setminus A \in \mathcal{A}$

Proof.

1. As $\mathcal{A}$ is a ring this follows from [theorem: 20.126].
2. This is stated in the definition of a algebra.
3. As $\mathcal{A}$ is a ring this follows from [theorem: 20.126] and the definition of a ring.
4. As $\mathcal{A}$ is a ring this follows from [theorem: 20.126].
5. As $\mathcal{A}$ is a ring this follows from [theorem: 20.126].
6. As $X \in \mathcal{A}$ and $A \in \mathcal{A}$ we have by (3) that $X \setminus A \in \mathcal{A}$. □

In most books we have the following equivalent definitons of a algebra

Theorem 20.131. *Let* X *be a set and* $\mathcal{A} \subseteq \mathcal{P}(X)$ *then the following are equivalent:*

1. $\mathcal{A}$ *is a algebra on* X
2. $\mathcal{A}$ *satisifes:*
   a. $X \in \mathcal{A}$
   b. $\forall A \in \mathcal{A}$ *we have* $X \setminus A \in \mathcal{A}$
   c. $\forall A, B \in \mathcal{A}$ *we have* $A \bigcap B \in \mathcal{A}$ *[in other words* $\mathcal{A}$ *is* $\cap$*-stable]*
3. $\mathcal{A}$ *satisfies:*
   a. $X \in \mathcal{A}$
   b. $\forall A \in \mathcal{A}$ *we have* $X \setminus A \in \mathcal{A}$
   c. $\forall A, B \in \mathcal{A}$ *we have* $A \bigcup B \in \mathcal{A}$ *[in other words* $\mathcal{A}$ *is* $\cup$*-stable]*
4. $\mathcal{A}$ *satisfies:*
   a. $X \in \mathcal{A}$
   b. $\forall A, B \in \mathcal{A}$ *we have* $A \setminus B \in \mathcal{A}$

Proof.

1 ⇒ 2. If $\mathcal{A}$ is a algebra then by definition $\mathcal{A}$ is a ring and $X \in \mathcal{A}$ so that

1. $X \in \mathcal{A}$
2. $\forall A \in \mathcal{A}$ we have, as $X \in \mathcal{A}$, that $X \setminus A \in \mathcal{A}$ [by the definition of a ring].
3. $\forall A, B \in \mathcal{A}$ we have by the definition of a ring that $A \setminus B \in \mathcal{A}$.

2 ⇒ 3. We have:

1. $X \in \mathcal{A}$ by the hypothesis.
2. $X \setminus A$ by the hypothesis.
3. Let $A, B \in \mathcal{A}$ then we have that $X \setminus A \in \mathcal{A}$ and $X \setminus B \in \mathcal{A}$ so that $(X \setminus A) \bigcap (X \setminus B) \in \mathcal{A}$, hence

$$X \setminus \big((X \setminus A) \bigcap (X \setminus B)\big) \in \mathcal{A}$$

Further we have

$$\begin{aligned} X \setminus \big((X \setminus A) \bigcap (X \setminus B)\big) &\underset{[\text{theorem: } 1.31]}{=} (X \setminus (X \setminus A)) \bigcup (X \setminus (X \setminus B)) \\ &\underset{A, B \subseteq X \wedge [\text{theorem: } 1.33]}{=} A \bigcup B \end{aligned}$$

so that $A \bigcup B \in \mathcal{A}$.

3 ⇒ 4. We have

1. $X \in \mathcal{A}$ by the hypothesis
2. Let $A, B \in \mathcal{A}$ then $X \setminus A \in \mathcal{A}$ so that $(X \setminus A) \bigcup B \in \mathcal{A}$ hence

$$X \setminus \left((X \setminus A) \bigcup B\right) \in \mathcal{A}$$

Further we have

$$\begin{aligned} X \setminus \left((X \setminus A) \bigcup B\right) &\underset{[\text{theorem: } 1.31]}{=} (X \setminus (X \setminus A)) \bigcap (X \setminus B) \\ &\underset{A \subseteq X \wedge [\text{theorem: } 1.33]}{=} A \bigcap (X \setminus B) \\ &\underset{A,B \subseteq X \wedge [\text{theorem: } 1.33]}{=} A \setminus B \end{aligned}$$

so that $A \setminus B \in \mathcal{A}$

4 ⇒ 1. Then we have

1. $X \in \mathcal{A}$ by the hypothesis.
2. If $A, B \in \mathcal{A}$ then by the hypothesis $A \setminus B \in \mathcal{A}$ so that $A \setminus (A \setminus B) \in \mathcal{A}$, as by [theorem: 1.33] $A \setminus (A \setminus B) = A \bigcap B$ it follows that $A \bigcap B \in \mathcal{A}$ hence we have proved that

$$\forall A, B \in \mathcal{A} \text{ we have } A \bigcap B \in \mathcal{A}$$

Further if $A, B \in \mathcal{A}$ then as $X \in \mathcal{A}$ we have that $(X \setminus A), (X \setminus B) \in \mathcal{A}$ so that $(X \setminus A) \bigcap (X \setminus B) \in \mathcal{A}$, hence we have that

$$X \setminus \left((X \setminus A) \bigcap (X \setminus B)\right) \in \mathcal{A}$$

Now

$$\begin{aligned} X \setminus \left((X \setminus A) \bigcap (X \setminus B)\right) &\underset{[\text{theorem: } 1.31]}{=} (X \setminus (X \setminus A)) \bigcup (X \setminus (X \setminus B)) \\ &\underset{A,B \subseteq X \wedge [\text{theorem: } 1.33]}{=} A \bigcup B \end{aligned}$$

proving that

$$\forall A, B \in \mathcal{A} \text{ we have } A \bigcup B \in \mathcal{A}$$

So as $X \in \mathcal{A} \Rightarrow \mathcal{A} \neq \varnothing$ and $\forall A, B \in \mathcal{A}$ we have $A \bigcup B \in \mathcal{A}$ and $A \setminus B \in \mathcal{A}$ it follows that $\mathcal{A}$ is a algebra. □

We also want to deal with countable unions of sets and for this we define a sigma algebra.

Definition 20.132. *Let X be a set, $\mathcal{A} \subseteq \mathcal{P}(X)$ then $\mathcal{A}$ is a sigma algebra [noted as σ-algebra] on X if*

1. $X \in \mathcal{A}$
2. $\forall A \in \mathcal{A}$ *we have* $X \setminus A \in \mathcal{A}$
3. *If* $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ *is a countable family of sets in* $\mathcal{A}$ *then* $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$

***Note* 20.133.** *If $A, B \in \mathcal{A}$ then if we define $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ by $A_i = \left\{ \begin{smallmatrix} A \text{ if } i = 1 \\ B \text{ if } i \in \{, \ldots, \infty\} \end{smallmatrix} \right.$ then*

$$A \bigcup B = A_1 \bigcup \left(\bigcup_{i \in \{2, \ldots, \infty\}} A_i \right) = \bigcup_{i \in \mathbb{N}} A_i$$

so that every σ-algebra on X is a algebra. So we have

$$\sigma\text{-}algebra \underset{\text{is}\, a}{\Rightarrow} algebra \underset{\text{is}\, a}{\Rightarrow} ring$$

Definition 20.134. (Measurable Space) *Let X be a set and $\mathcal{A}$ a σ-algebra on X then the couple $\langle X, \mathcal{A} \rangle$ is called a **measurable space**.*

We have the following equivalent definitions of a σ-algebra

Theorem 20.135. *Let X be a set, $\mathcal{A}\subseteq\mathcal{P}(X)$ then we have the following equivalences:*

1. *$\mathcal{A}$ is a σ-algebra on X*
2. *$\mathcal{A}$ is a algebra and $\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ **pairwise disjoint** we have $\bigsqcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$*
3. *$\mathcal{A}$ is a algebra and $\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ we have $\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$*
4. *$\mathcal{A}$ satisfies:*
   a. *$\varnothing\in\mathcal{A}$*
   b. *$\forall A\in\mathcal{A}$ we have $X\setminus A\in\mathcal{A}$*
   c. *$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ we have $\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$*
5. *$\mathcal{A}$ satisfies:*
   a. *$\varnothing\in\mathcal{A}$*
   b. *$\forall A\in\mathcal{A}$ we have $X\setminus A\in\mathcal{A}$*
   c. *$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ we have $\bigcap_{i\in\mathbb{N}}A_i\in\mathcal{A}$*
6. *$\mathcal{A}$ satisfies:*
   a. *$X\in\mathcal{A}$*
   b. *$\forall A\in\mathcal{A}$ we have $X\setminus A\in\mathcal{A}$*
   c. *$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ we have $\bigcap_{i\in\mathbb{N}}A_i\in\mathcal{A}$*

Proof.

$1\Rightarrow 2$. As $\mathcal{A}$ is a sigma-algebra it follows that

$$\mathcal{A}\text{ is a algebra}$$

Further if $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ is pairwise disjoint then by the definition of a σ-algebra it follows that

$$\bigsqcup_{i\in\mathbb{N}}A_i\underset{\text{def}}{=}\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$$

$2\Rightarrow 3$. Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ and define

$$\{B_i\}_{i\in\mathbb{N}}\text{ by }B_i=\begin{cases}A_1\text{ if }i=1\\ A_i\setminus\bigcup_{j\in\{1,\ldots,i-1\}}A_j\text{ if }i\in\{2,\ldots,\infty\}\end{cases}$$

As $\mathcal{A}$ is a algbra we have by [theorem: 20.130] it follows that $\forall i\in\mathbb{N}$ we have $B_i\in\mathcal{A}$ so that

$$\{B_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$$

Further by [lemma: 20.108] $\{B_i\}_{i\in\mathbb{N}}$ is pairwise disjoint so that by (2)

$$\bigsqcup_{i\in\mathbb{N}}B_i\in\mathcal{A}$$

As further $\bigcup_{i\in\mathbb{N}}A_i\underset{\text{[lemma: 20.108]}}{=}\bigsqcup_{i\in\mathbb{N}}B_i$ it follows that

$$\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$$

$3\Rightarrow 4$. As $\mathcal{A}$ is a algebra we have by [theorem: 20.130] that

$$\varnothing\in\mathcal{A}$$

$$\forall A\in\mathcal{A}\text{ we have }X\setminus A\in\mathcal{A}$$

Finally by the definition of a σ-algebra we have

$$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}\text{ we have }\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$$

4 ⇒ 5. We have by (3) that

$$\varnothing\in\mathcal{A} \text{ and } \forall A\in\mathcal{A} \text{ we have } X\setminus A\in\mathcal{A}$$

Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$. If $i\in\mathbb{N}$ then as $A_i\in\mathcal{A}$ it follows that $X\setminus A_i\in\mathcal{A}$ we have by (3) that $\bigcup_{i\in\mathbb{N}}(X\setminus A_i)\in\mathcal{A}$ so that

$$X\setminus\left(\bigcup_{i\in\mathbb{N}}(X\setminus A_i)\right)\in\mathcal{A}$$

As

$$\begin{aligned}X\setminus\left(\bigcup_{i\in\mathbb{N}}(X\setminus A_i)\right) &\underset{[\text{theorem: } 2.132]}{=} \bigcap_{i\in\mathbb{N}}(X\setminus(X\setminus A_i))\\ &\underset{A_i\subseteq X\wedge[\text{theorem: } 1.33]}{=} \bigcap_{i\in\mathbb{N}} A_i\end{aligned}$$

it follows that

$$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A} \text{ we have } \bigcap_{i\in\mathbb{N}} A_i\in\mathcal{A}$$

5 ⇒ 6. By (4) we have that $\varnothing\in\mathcal{A}$ so that by (4) again we have

$$X=X\setminus\varnothing\in\mathcal{A}$$

further by (4) we have

$$\forall A\in\mathcal{A} \text{ we have } X\setminus A\in\mathcal{A} \text{ and } \forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A} \text{ we have } \bigcap_{i\in\mathbb{N}} A_i\in\mathcal{A}$$

6 ⇒ 1. By (1) we have

$$X\in\mathcal{A} \text{ and } \forall A\in\mathcal{A} \text{ we have } X\setminus A\in\mathcal{A}$$

Further if $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ then $\forall i\in\mathbb{N}$ we have $A_i\in\mathcal{A}$ so that $X\setminus A_i\in\mathcal{A}$. Hence we have

$$X\setminus\left(\bigcap_{i\in\mathbb{N}}(X\setminus A_i)\right)\in\mathcal{A}$$

Now

$$\begin{aligned}X\setminus\left(\bigcap_{i\in\mathbb{N}}(X\setminus A_i)\right) &\underset{[\text{theorem: } 2.132]}{=} \bigcup_{i\in\mathbb{N}}(X\setminus(X\setminus A_i))\\ &\underset{A_i\subseteq X\wedge[\text{theorem: } 1.33]}{=} \bigcup_{i\in\mathbb{N}} A_i\end{aligned}$$

so that

$$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A} \text{ we have } \bigcup_{i\in\mathbb{N}} A_i\in\mathcal{A}$$

This proves that $\mathcal{A}$ is a σ-algebra.

□

Example 20.136. Let $X\neq\varnothing$ be a set then $\{\varnothing\}$ is not a σ-algebra on X [see example: 20.124] but $\{\varnothing,X\}$ is a σ-algebra on X.

Proof. Using [example: 20.124] it follows that $\{\varnothing,X\}$ is a algebra on X so that

$$X\in\mathcal{A} \text{ and } \forall A\in\mathcal{A} \text{ we have } X\setminus A\in\mathcal{A}$$

Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\{\varnothing,X\}$ then we have two cases to consider:

$\forall i\in\mathbb{N}$ we have $A_i\neq\varnothing$. Then $\forall i\in\mathbb{N}$ $A_i=X$ so that $\bigcap_{i\in\mathbb{N}}A_i=X\in\mathcal{A}$

$\exists i\in\mathbb{N}$ with $A_i=\varnothing$. Then $\bigcap_{i\in\mathbb{N}}A_i=\varnothing\in\mathcal{A}$

so that $\bigcap_{i\in\mathbb{N}}A_i\in\mathcal{A}$. Applying then [theorem: 20.135] proves that $\{\varnothing,X\}$ is a σ-algebra on X. □

Example 20.137. Let X be a set then $\mathcal{P}(X)$ is a σ-algebra on X

Proof. Using [example: 20.129] $\mathcal{P}(X)$ is a algebra on X. Further for $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{P}(X)$ we have $\forall i\in\mathbb{N}$ that $A_i\subseteq X$ so that $\bigcup_{i\in\mathbb{N}}A_i\subseteq X$ proving that $\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{P}(X)$. Applying then [theorem: 20.135] proves that $\mathcal{P}(X)$ is a σ-algebra on X. □

Theorem 20.138. *Let X be a set, $\mathcal{A}$ a σ-algebra on X, I a countable set, $\{A\}_{i\in I}\subseteq\mathcal{A}$ a countable family of element of $\mathcal{A}$ then we have*

1. $\bigcup_{i\in I}A_i\in\mathcal{A}$
2. $\bigcap_{i\in I}A_i\in\mathcal{A}$

Proof.

1. For I we have either

   ***I* is finite.** Then there exist a bijection $\beta:\{1,\ldots,n\}\rightarrow I$ where $n\in\mathbb{N}_0$. Let $x\in\bigcup_{i\in I}A_i$ then there exist a $i\in I$ such that $x\in A_i$. As β is a bijection hence surjective there exist a $j\in\{1,\ldots,n\}$ such that $i=\beta(j)$ so that $x\in A_{\beta(j)}\subseteq\bigcup_{j\in\{1,\ldots,n\}}A_{\beta(j)}$. Hence we have that

   $$\bigcup_{i\in I}A_i\subseteq\bigcup_{j\in\{1,\ldots,n\}}A_{\beta(i)} \tag{20.71}$$

   Likewise if $x\in\bigcup_{j\in\{1,\ldots,n\}}A_{\beta(i)}$ then there exist a $j\in\{1,\ldots,n\}$ such that $x\in A_{\beta(j)}$ which as $\beta(j)\in I$ proves that $x\in\bigcup_{i\in I}A_i$, hence $\bigcup_{j\in\{1,\ldots,n\}}A_{\beta(i)}\subseteq\bigcup_{i\in I}A_i$. Combining this with [eq: 20.71] proves that

   $$\bigcup_{i\in I}A_i=\bigcup_{j\in\{1,\ldots,n\}}A_{\beta(i)}$$

   As $\forall j\in\{1,\ldots,n\}$ $A_{\beta(i)}\in\mathcal{A}$ [because $\beta(i)\in I$] it follows, as a σ-algebra is a ring, from [theorem: 20.126] that $\bigcup_{j\in\{1,\ldots,n\}}A_{\beta(i)}\in\mathcal{A}$. Hence

   $$\bigcup_{i\in I}A_i\in\mathcal{A}$$

   ***I* is denumerable.** Then there exist a bijection $\beta:\mathbb{N}\rightarrow I$. Let $x\in\bigcup_{i\in I}A_i$ then there exist a $i\in I$ such that $x\in A_i$. As β is a bijection hence surjective there exist a $j\in\mathbb{N}$ such that $i=\beta(j)$ so that $x\in A_{\beta(j)}\subseteq\bigcup_{j\in\mathbb{N}}A_{\beta(j)}$. Hence we have that

   $$\bigcup_{i\in I}A_i\subseteq\bigcup_{j\in\mathbb{N}}A_{\beta(i)} \tag{20.72}$$

   Likewise if $x\in\bigcup_{j\in\mathbb{N}}A_{\beta(i)}$ then there exist a $j\in\mathbb{N}$ such that $x\in A_{\beta(j)}$ which as $\beta(j)\in I$ proves that $x\in\bigcup_{i\in I}A_i$, hence $\bigcup_{j\in\mathbb{N}}A_{\beta(i)}\subseteq\bigcup_{i\in I}A_i$. Combining this with [eq: 20.72] proves that

   $$\bigcup_{i\in I}A_i=\bigcup_{j\in\mathbb{N}}A_{\beta(i)}$$

   As $\forall j\in\mathbb{N}$ $A_{\beta(i)}\in\mathcal{A}$ [because $\beta(i)\in I$] it follows, as a σ-algebra is a ring, from [theorem: 20.126] that $\bigcup_{j\in\mathbb{N}}A_{\beta(i)}\in\mathcal{A}$. Hence

   $$\bigcup_{i\in I}A_i\in\mathcal{A}$$

2. Let $i\in I$ then $A_i\in\mathcal{A}$ hence $X\setminus A_i\in\mathcal{A}$ so that by (1) $\bigcup_{i\in I}(X\setminus A_i)\in\mathcal{A}$ from which it follows that

   $$X\setminus\left(\bigcup_{i\in I}(X\setminus A_i)\right)\in\mathcal{A}$$

As we have

$$X\setminus\left(\bigcup_{i\in I}(X\setminus A_i)\right)\underset{[\text{theorem: } 2.132]}{=}\bigcap_{i\in I}(X\setminus(X\setminus A_i))$$
$$\underset{A_i\subseteq X\wedge[\text{theorem: } 1.33]}{=}\bigcap_{i\in I}A_i$$

it follows that

$$\bigcap_{i\in I}A_n\in\mathcal{A}$$ □

If we have already proved that a subset of $\mathcal{P}(X)$ is a algebra on X then the following theorem will be usefull to prove that this subset is actually a σ-algebra.

Theorem 20.139. *Let X be a set and let $\mathcal{A}$ be a algebra on X then we have:*

1. *If $\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ with $\forall i\in\mathbb{N}$ $A_i\subseteq A_{i+1}$ we have $\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$ then $\mathcal{A}$ is a σ-algebra on X.*
2. *If $\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ with $\forall i\in\mathbb{N}$ $A_{i+1}\subseteq A_i$ we have $\bigcap_{i\in\mathbb{N}}A_i\in\mathcal{A}$ then $\mathcal{A}$ is a σ-algebra on X.*

Proof.

1. Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$. Define $\{B_i\}_{i\in\mathbb{N}}$ by $B_i=\bigcup_{j\in\{1,\ldots,i\}}A_i$ then as $\mathcal{A}$ is a algebra we have by [theorem: 20.130] that $\{B_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$. Further for $i\in\mathbb{N}$ we have

$$B_i=\bigcup_{j\in\{1,\ldots,i\}}B_j\subseteq\bigcup_{j\in\{1,\ldots,i+1\}}B_j=B_{i+1}$$

hence by the hypothese we have

$$\bigcup_{i\in\mathbb{N}}B_i\in\mathcal{A}\tag{20.73}$$

As $\forall i\in\mathbb{N}$ we have $A_i\subseteq\bigcup_{j\in\{1,\ldots,i\}}A_j=B_i$ it follows that

$$\bigcup_{i\in\mathbb{N}}A_i\subseteq\bigcup_{i\in\mathbb{N}}B_i\tag{20.74}$$

Further if $x\in\bigcup_{i\in\mathbb{N}}B_i$ then there exist a $i\in\mathbb{N}$ such that $x\in B_i=\bigcup_{j\in\{1,\ldots,i\}}A_j$, hence there exist a $j\in\{1,\ldots,i\}$ such that $x\in A_j\subseteq\bigcup_{i\in\mathbb{N}}A_i$. This proves that $\bigcup_{i\in\mathbb{N}}B_i\subseteq\bigcup_{i\in\mathbb{N}}A_i$ which combined with [eq: 20.74] gives us $\bigcup_{i\in\mathbb{N}}A_i=\bigcup_{i\in\mathbb{N}}B_i$, hence using [eq: 20.73] we have that

$$\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$$

Applying then [theorem: 20.135] it follows that $\mathcal{A}$ is a σ-algebra.

2. Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ be such that $\forall i\in\mathbb{N}$ we have $A_i\subseteq A_{i+1}$. Let $i\in\mathbb{N}$ then as $\mathcal{A}$ is a algebra $X\setminus A_i\in\mathcal{A}$, further if $x\in X\setminus A_{i+1}$ then $x\in X$ and $x\notin A_i$ [as $A_i\subseteq A_{i+1}$] so that $x\in X\setminus A_i$. Hence $\{X\setminus A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ and $\forall i\in\mathbb{N}$ we have $X\setminus A_{i+1}\subseteq X\setminus A_i$ so by the hypothesis we have that $\bigcap_{i\in\mathbb{N}}(X\setminus A_i)\in\mathcal{A}$. As $\mathcal{A}$ is a algebra we have then that

$$X\setminus\left(\bigcap_{i\in\mathbb{N}}(X\setminus A_i)\right)\in\mathcal{A}$$

Further $X\setminus(\bigcap_{i\in\mathbb{N}}(X\setminus A_i))\underset{[\text{theorem: } 2.132]}{=}\bigcup_{i\in\mathbb{N}}(X\setminus(X\setminus A_i))\underset{A_i\subseteq X\wedge[\text{theorem: } 1.33]}{=}\bigcup_{i\in\mathbb{N}}A_i$ so that $\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$. Hence we have proved that $\forall\{A_i\}_{i\in\mathcal{A}}\subseteq\mathcal{A}$ with $\forall i\in\mathbb{N}$ $A_i\subseteq A_{i+1}$ $\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$ ao that by (1) $\mathcal{A}$ is a σ-algebra. □

We show now how we can generate a σ algebra based on a arbitrary subset of $\mathcal{P}(X)$. First we prove that for a non empty collection of σ-algebra's the intersection of this collection is again a σ-algebra.

Lemma 20.140. *Let X be a set and $\mathfrak{A}$ a non empty set of σ-algebra's on X then $\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}$ is a σ-algebra on X.*

Proof. As $\mathfrak{A}\neq\varnothing$ there exist a $\mathcal{S}\in\mathfrak{A}$ with $\mathcal{S}\subseteq\mathcal{P}(X)$ so that we have by [theorem: 1.63] that $\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}\subseteq\mathcal{S}\subseteq\mathcal{P}(X)$ proving that

$$\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}\subseteq\mathcal{P}(X)$$

As $\forall\mathcal{A}\in\mathfrak{A}$ $\mathcal{A}$ is a σ-algebra on X we have by [theorem: 20.135] that $X\in\mathcal{A}$ hence

$$X\in\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}$$

Let $A\in\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}$ then for $\mathcal{A}\in\mathfrak{A}$ we have $A\in\mathcal{A}$ so that $X\setminus A\in\mathcal{A}$ which as $\mathcal{A}\in\mathfrak{A}$ was chosen arbitrary proves that $X\setminus A\in\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}$. Hence as $A\in\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}$ was chosen arbitrary we have that

$$\forall A\in\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}\text{ we have }X\setminus A\in\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}$$

If $\{A_i\}_{i\in\mathbb{N}}\subseteq\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}$ then for $\mathcal{A}\in\mathfrak{A}$ we have $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ so that $\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{A}$ which as $\mathcal{A}\in\mathfrak{A}$ was chosen arbitrary proves that $\bigcup_{i\in\mathbb{N}}A\in\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}$. Hence as $\{A_i\}_{i\in\mathbb{N}}\subseteq\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}$ was chosen arbitrary we have that

$$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}\text{ we have }\bigcup_{i\in\mathbb{N}}A\in\bigcap_{\mathcal{A}\in\mathfrak{A}}\mathcal{A}$$

□

We use now the above theorem to show how we can generate a σ-algebra out of a non empty subset of $\mathcal{P}(X)$.

Theorem 20.141. *Let X be a set $\mathcal{A}\subseteq\mathcal{P}(X)$ then there exist a **unique** algebra $\sigma[\mathcal{A}]$ on X such that:*

1. *$\mathcal{A}\subseteq\sigma[\mathcal{A}]$*
2. *If $\mathcal{B}$ is another σ-algebra with $\mathcal{A}\subseteq\mathcal{B}$ then $\sigma[\mathcal{A}]\subseteq\mathcal{B}$*

This unique smallest σ-algebra that contains $\mathcal{A}$ is called the σ-algebra generated by $\mathcal{A}$

Proof. Define $\mathfrak{A}=\{\mathcal{T}|\mathcal{T}\text{ is a }\sigma\text{-algebra on }X\text{ with }\mathcal{A}\subseteq\mathcal{T}\}$. As $\mathcal{A}\subseteq\mathcal{P}(X)$ and $\mathcal{P}(X)$ and $\mathcal{P}(X)$ is a σ-algebra on X [see example: 20.137] it follows that $\mathcal{P}(X)\in\mathfrak{A}$ hence $\mathfrak{A}$ is not empty. Using then previous lemma [lemma: 20.140] it follows that

$$\bigcap_{\mathcal{T}\in\mathfrak{A}}\mathcal{T}\text{ is a }\sigma\text{-algebra on }X$$

As $\forall\mathcal{T}\in\mathfrak{A}$ $\mathcal{A}\subseteq\mathcal{T}$ it follows from [theorem: 1.63] that

$$\mathcal{A}\subseteq\bigcap_{\mathcal{T}\in\mathfrak{A}}\mathcal{T}$$

Finally assume that $\mathcal{B}$ is another σ-algebra on X such that $\mathcal{A}\subseteq\mathcal{B}$ then $\mathcal{B}\in\mathfrak{A}$ so that by [theorem: 1.63]

$$\bigcap_{\mathcal{T}\in\mathfrak{A}}\mathcal{T}\subseteq\mathcal{B}$$

Taking $\sigma[\mathcal{A}]=\bigcap_{\mathcal{T}\in\mathfrak{A}}\mathcal{T}$ then proves the theorem. □

A special case of a σ-algebra in the context of topological spaces is the Borel algebra.

Definition 20.142. *Let $\langle X,\mathcal{T}\rangle$ be a topological space then the Borel algebra on X noted as $\mathcal{B}[X,\mathcal{T}]$ is defined to be $\mathcal{B}[X,\mathcal{T}]=\sigma[\mathcal{T}]$. In other words the Borel algebra on a topological space is the σalgebra generated by $\mathcal{T}$.*

Theorem 20.143. *Let $\langle X,\mathcal{T}\rangle$ be a topological space then $\mathcal{B}[X,\mathcal{T}]=\sigma[\{C\subseteq X|C\text{ is closed}\}]$.*

Proof. If $C \in \{C \subseteq X | C \text{ is closed}\}$ then C is closed, hence there exist a open $U \in \mathcal{T}$ such that $C = X \setminus U$. As $U \in \mathcal{T} \in \sigma[\mathcal{T}]$ it follows that $C = X \setminus U \in \sigma[\mathcal{T}]$. So $\{C \subseteq X | C \text{ is closed}\} \subseteq \sigma[\mathcal{T}]$ a σ-algebra, hence using [theorem: 20.141[we have

$$\sigma[\{C \subseteq X | C \text{ is closed}\}] \subseteq \sigma[\mathcal{T}]$$

If $U \in \mathcal{T}$ then $X \setminus U$ is closed hence $X \setminus U \in \{C \subseteq X | C \text{ is closed}\} \in \sigma[\{C \subseteq X | C \text{ is closed}\}]$, hence $U = X \setminus (X \setminus U) \in \sigma[\{C \subseteq X | C \text{ is closed}\}]$ proving that $\mathcal{T} \subseteq \sigma[\{C \subseteq X | C \text{ is closed}\}]$. So using [theorem: 20.141[we have

$$\sigma[\mathcal{T}] \subseteq \sigma[\{C \subseteq X | C \text{ is closed}\}]$$

This proves that

$$\mathcal{B}[X, \mathcal{T}] = \sigma[\mathcal{T}] = \sigma[\{C \subseteq X | C \text{ is closed}\}]$$ □

Theorem 20.144. *Let $\langle X, \mathcal{A}\rangle$ be a measurable space and $B \in \mathcal{A}$ then $\langle B, \mathcal{A}_{|B}\rangle$ is a measurable space where $\mathcal{A}_{|B} = \{C \in \mathcal{A} | C \subseteq B\}$ is a σ-algebra on B.*

Proof.

1. As $B \in \mathcal{A}$ and $B \subseteq B$ it follows that $B \in \mathcal{A}_{|B}$.
2. Let $A \in \mathcal{A}_{|B}$ then $A \in \mathcal{A}$ and $A \subseteq B$. As also $B \in \mathcal{A}$ it follows from [theorem: 20.135] that $B \bigcap (X \setminus A) \in \mathcal{A}$. Further $B \bigcap (X \setminus A) \underset{[\text{theorem: } 1.25]}{=} (B \bigcap X) \setminus A \underset{B \subseteq X}{=} B \setminus A$ so that $B \setminus A \in \mathrm{A}$ which as $B \setminus A \subseteq B$ proves that
$$B \setminus A \in \mathcal{A}_{|B}$$
3. Let $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}_{|B}$ then $\forall i \in \mathbb{N}$ we have $A_i \in \mathcal{A}$ and $A_i \subseteq B$ so that $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$ and $\bigcup_{i \in \mathbb{N}} A_i \subseteq B$. Hence
$$\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}_{|B}$$ □

20.3.1 Borel algebra on $\mathbb{R}$

Definition 20.145. *Let $\langle \mathbb{R}, \mathcal{T}_{||}\rangle$ be the set of real numbers with the canonical topology $\mathcal{T}_{||}$ based on the absolute value norm [see theorem: 14.99] then then $\mathcal{B}[\mathbb{R}]$ is the Borel algebra based on $\mathcal{T}_{||}$. In other words*

$$\mathcal{B}[\mathbb{R}] = \mathcal{B}[\mathbb{R}, \mathcal{T}_{||}] = \sigma[\mathcal{T}_{||}]$$

We will now find equivalent definitions for $\mathcal{B}[\mathbb{R}]$ that will be usefull if we define a measure. First we introduce the concept of bounded intervals on $\mathbb{R}$.

Definition 20.146. *We define the following set of bounded intervals:*

1. $\mathcal{I}_{[]} = \{[a, b] | a, b \in \mathbb{R} \wedge [a, b] \neq \varnothing\}$ *[the set of non empty closed bounded intervals]*
2. $\mathcal{I}_{[[} = \{[a, b[| a, b \in \mathbb{R} \wedge [a, b[\neq \varnothing\}$ *[the set of non empty right open bounded intervals]*
3. $\mathcal{I}_{]]} = \{]a, b] | a, b \in \mathbb{R} \wedge]a, b] \neq \varnothing\}$ *[the set of non empty left open bounded intervals]*
4. $\mathcal{I}_{][} = \{]a, b[| a, b \in \mathbb{R} \wedge]a, b[\neq \varnothing\}$ *[the set of non empty open bounded intervals]*
5. $\mathcal{I} = \mathcal{I}_{[]} \bigcup \mathcal{I}_{[[} \bigcup \mathcal{I}_{]]} \bigcup \mathcal{I}_{][}$

Note 20.147. *If $b = a$ then $]a, b[= [a, b[=]a, b] = \varnothing$ and if $b < a$ then $[a, b] = [a, b[=]a, b] =]a, b[= \varnothing$ then we have valid bounded intervals but they are not elements of $\mathcal{I}_{[]}$, $\mathcal{I}_{[[}$, $\mathcal{I}_{]]}$ and $\mathcal{I}_{][}$. One reason to exclude these intervals is that the boundary points of these intevals are not well defined [for eample $[1, 0[= [2, 0[= [0, 0[$ which is needed for defining the length of the interval based on the boundary points.*

For case to case analysis the following lemma will be usefull.

Lemma 20.148. *We have that $\mathcal{I}_{[]}$, $\mathcal{I}_{[[}$, $\mathcal{I}_{]]}$ and $\mathcal{I}_{][}$ are pairwise disjoint. In other words we have*

$$\mathcal{I}_{[]}\bigcap \mathcal{I}_{[[} = \varnothing$$
$$\mathcal{I}_{[]}\bigcap \mathcal{I}_{]]} = \varnothing$$
$$\mathcal{I}_{[]}\bigcap \mathcal{I}_{][} = \varnothing$$
$$\mathcal{I}_{[[}\bigcap \mathcal{I}_{[]} = \varnothing$$
$$\mathcal{I}_{[[}\bigcap \mathcal{I}_{]]} = \varnothing$$
$$\mathcal{I}_{[[}\bigcap \mathcal{I}_{][} = \varnothing$$
$$\mathcal{I}_{]]}\bigcap \mathcal{I}_{[]} = \varnothing$$
$$\mathcal{I}_{]]}\bigcap \mathcal{I}_{[[} = \varnothing$$
$$\mathcal{I}_{]]}\bigcap \mathcal{I}_{][} = \varnothing$$
$$\mathcal{I}_{][}\bigcap \mathcal{I}_{[]} = \varnothing$$
$$\mathcal{I}_{][}\bigcap \mathcal{I}_{[[} = \varnothing$$
$$\mathcal{I}_{][}\bigcap \mathcal{I}_{]]} = \varnothing$$

so that

$$\mathcal{I} = \mathcal{I}_{[]}\bigsqcup \mathcal{I}_{[[}\bigsqcup \mathcal{I}_{]]}\bigsqcup \mathcal{I}_{][}$$

Proof. We have to check the following cases:

$\mathcal{I}_{[]}\bigcap \mathcal{I}_{[[}$. If $I\in\mathcal{I}_{[]}\bigcap \mathcal{I}_{[[}$ then $\varnothing\neq I=[a,b]=[c,d[$. As $[a,b]\neq\varnothing$ it follows from [theorem: 3.139] that $a\leqslant b$ so that $b\in[a,b]=[c,d[$ hence $c\leqslant b<d$. Using [theorem: 10.32] there exist a $x\in\mathbb{R}$ such that $c\leqslant b<x<d$ proving that $x\in[c,d[=[a,b]$ so that $b<x\leqslant b$ leading to the contradiction $b<b$. Hence we have

$$\mathcal{I}_{[]}\bigcap \mathcal{I}_{[[} = \varnothing \tag{20.75}$$

$\mathcal{I}_{[]}\bigcap \mathcal{I}_{]]}$. If $I\in\mathcal{I}_{[]}\bigcap \mathcal{I}_{]]}$ then $\varnothing=[a,b]=]c,d]$. As $[a,b]\neq\varnothing$ it follows from [theorem: 3.139] that $a\leqslant b$ so that $a\in[a,b]=]c,d]$ hence $c<a\leqslant d$. Using [theorem: 10.32] there exist a $x\in\mathbb{R}$ such that $c<x<a\leqslant d$ proving that $x\in]c,d]=[a,b]$ so that $a\leqslant x<a$ leading to the contradiction $a<a$. Hence we have

$$\mathcal{I}_{[]}\bigcap \mathcal{I}_{]]} = \varnothing \tag{20.76}$$

$\mathcal{I}_{[]}\bigcap \mathcal{I}_{][}$. If $I\in\mathcal{I}_{[]}\bigcap \mathcal{I}_{][}$ then $\varnothing\neq I=[a,b]=]c,d[$. As As $[a,b]\neq\varnothing$ it follows from [theorem: 3.139] that $a\leqslant b$ so that $b\in[a,b]=]c,d[$ hence $c<b<d$. Using [theorem: 10.32] there exist a $x\in\mathbb{R}$ such that $c<b<x<d$ proving that $x\in]c,d[=[a,b]$ so that $b<x\leqslant b$ leading to the contradiction $b<b$. Hence we have

$$\mathcal{I}_{[]}\bigcap \mathcal{I}_{][} = \varnothing \tag{20.77}$$

$\mathcal{I}_{[[}\bigcap \mathcal{I}_{[]}$. Then we have

$$\mathcal{I}_{[[}\bigcap \mathcal{I}_{[]} = \mathcal{I}_{[]}\bigcap \mathcal{I}_{[[} \underset{[\text{eq; } 20.75]}{=} \varnothing$$

$\mathcal{I}_{[[}\bigcap \mathcal{I}_{]]}$. If $I\in\mathcal{I}_{[[}\bigcap \mathcal{I}_{]]}$ then $\varnothing\neq I=[a,b[=]c,d]$. As $[a,b]\neq\varnothing$ it follows from [theorem: 3.139] that $a<b$ so that $a\in[a,b[=]c,d]$ hence $c<a\leqslant d$. Using [theorem: 10.32] there exist a $x\in\mathbb{R}$ such that $c<x<a\leqslant d$ proving that $x\in]c,d]=[a,b[$ so that $a\leqslant x<a$ leading to the contradiction $a<a$. Hence we have

$$\mathcal{I}_{[[}\bigcap \mathcal{I}_{]]} = \varnothing \tag{20.78}$$

$\mathcal{I}_{[[}\bigcap \mathcal{I}_{][}$. If $I\in\mathcal{I}_{[[}\bigcap \mathcal{I}_{][}$ then $\varnothing\neq I=[a,b[=]c,d[$. As $[a,b]\neq\varnothing$ it follows from [theorem: 3.139] that $a<b$ so that $a\in[a,b[=]c,d]$ hence $c<a<d$. Using [theorem: 10.32] there exist a $x\in\mathbb{R}$ such that $c<x<a<d$ proving that $x\in]c,d[=[a,b[$ so that $a\leqslant x<a$ leading to the contradiction $a<a$. Hence we have

$$\mathcal{I}_{[[}\bigcap \mathcal{I}_{][} = \varnothing \tag{20.79}$$

$\mathcal{I}_{]]}\bigcap\mathcal{I}_{[]}$. Then we have

$$\mathcal{I}_{]]}\bigcap\mathcal{I}_{[]}=\mathcal{I}_{[]}\bigcap\mathcal{I}_{]]}\underset{[\text{eq: } 20.76]}{=}\varnothing$$

$\mathcal{I}_{]]}\bigcap\mathcal{I}_{[[}$. If $I\in\mathcal{I}_{]]}\bigcap\mathcal{I}_{[[}$ then $I=]a,b]=[c,d[$. As $]a,b]\neq\varnothing$ we have by [theorem: 3.139] that $a<b$ so that $b\in]a,b]=[c,d[$ hence $c\leqslant b<d$. Using [theorem: 10.32] there exist a $x\in\mathbb{R}$ such that $c\leqslant b<x<d$ proving that $x\in[c,d[=]a,b]$ so that $b<x\leqslant b$ leading to the contradiction $b<b$. Hence we have

$$\mathcal{I}_{]]}\bigcap\mathcal{I}_{[[}=\varnothing \tag{20.80}$$

$\mathcal{I}_{]]}\bigcap\mathcal{I}_{][}$. If $I\in\mathcal{I}_{]]}\bigcap\mathcal{I}_{][}$ then $I=]a,b]=]c,d[$. As $]a,b]\neq\varnothing$ we have by [theorem: 3.139] that $a<b$ so that $b\in]a,b]=]c,d[$ hence $c<b<d$. Using [theorem: 10.32] there exist a $x\in\mathbb{R}$ such that $c<b<x<d$ proving that $x\in]c,d[=]a,b]$ so that $b<x\leqslant b$ leading to the contradiction $b<b$. Hence we have

$$\mathcal{I}_{]]}\bigcap\mathcal{I}_{][}=\varnothing \tag{20.81}$$

$\mathcal{I}_{][}\bigcap\mathcal{I}_{[]}$. Then we have

$$\mathcal{I}_{][}\bigcap\mathcal{I}_{[]}=\mathcal{I}_{[]}\bigcap\mathcal{I}_{][}\underset{[\text{eq: } 20.77]}{=}\varnothing$$

$\mathcal{I}_{][}\bigcap\mathcal{I}_{[[}$. Then we have

$$\mathcal{I}_{][}\bigcap\mathcal{I}_{[[}=\mathcal{I}_{[[}\bigcap\mathcal{I}_{][}\underset{[\text{eq: } 20.79]}{=}\varnothing$$

$\mathcal{I}_{][}\bigcup\mathcal{I}_{]]}$. Then we have

$$\mathcal{I}_{][}\bigcap\mathcal{I}_{]]}=\mathcal{I}_{]]}\bigcap\mathcal{I}_{][}\underset{[\text{eq: } 20.81]}{=}\varnothing$$

□

The following allows us to switch between the different types of non empty bounded intervals.

Lemma 20.149. *Let $a,b\in\mathbb{R}$ the we have*

1. $]a,b]=\bigcap_{n\in\mathbb{N}}]a,b+\frac{1}{n}[$
2. $[a,b[=\bigcap_{n\in\mathbb{N}}]a-\frac{1}{n},b[$
3. $[a,b]=\bigcap_{n\in\mathbb{N}}]a-\frac{1}{n},b]$
4. $[a,b]=\bigcap_{n\in\mathbb{N}}[a,b+\frac{1}{n}[$
5. $[a,b]=\bigcap_{n\in\mathbb{N}}]a-\frac{1}{n},b+\frac{1}{n}[$
6. $]-\infty,a]=\bigcap_{n\in\mathbb{N}}]-\infty,a+\frac{1}{n}[$
7. $[a,\infty[=\bigcap_{n\in\mathbb{N}}]a-\frac{1}{n},\infty[$
8. $]a,b]=\bigcup_{n\in\mathbb{N}}[a+\frac{1}{n},b]$
9. $[a,b[=\bigcup_{n\in\mathbb{N}}[a,b-\frac{1}{n}]$
10. $]a,b[=\bigcup_{n\in\mathbb{N}}]a,b-\frac{1}{n}]$
11. $]a,b[=\bigcup_{n\in\mathbb{N}}[a+\frac{1}{n},b[$
12. $]a,b[=\bigcup_{n\in\mathbb{N}}[a+\frac{1}{n},b-\frac{1}{n}]$
13. $]-\infty,a[=\bigcup_{n\in\mathbb{N}}]-\infty,a-\frac{1}{n}]$
14. $]a,\infty[=\bigcup_{n\in\mathbb{N}}[a+\frac{1}{n},\infty[$
15. $[a,b[=[a,\infty[\setminus[b,\infty[$
16. $]a,b]=]-\infty,b]\setminus]-\infty,a]$
17. $]a,\infty[=\mathbb{R}\setminus]-\infty,a]$

Proof.

1. Let $x \in]a, b]$ then $a < x \leqslant b$ and as $\forall n \in \mathbb{N}$ $b < b + \frac{1}{n}$ we have $a < x < b + \frac{1}{n}$ so it follows that $x \in \left]a, b + \frac{1}{n}\right[$. Hence
$$]a, b] \subseteq \bigcap_{n \in \mathbb{N}} \left]a, b + \frac{1}{n}\right[\tag{20.82}$$
If $x \in \bigcap_{n \in \mathbb{N}} \left]a, b + \frac{1}{n}\right[$ then $\forall n \in \mathbb{N}$ $x \in \left]a, b + \frac{1}{n}\right[$ or $a < x < b + \frac{1}{n}$ which by [corollary: 10.31] proves that $a < x \leqslant b$ hence $x \in]a, b]$. So it follows that $\bigcap_{n \in \mathbb{N}} \left]a, b + \frac{1}{n}\right[\subseteq]a, b]$ which using [eq: 20.82] proves that
$$]a, b] = \bigcap_{n \in \mathbb{N}} \left]a, b + \frac{1}{n}\right[$$

2. Let $x \in [a, b[$ then $a \leqslant x < b$ and as $\forall n \in \mathbb{N}$ we have $a - \frac{1}{n} < a$ it follows that $x \in \left]a - \frac{1}{n}, b\right[$. Hence
$$[a, b[\subseteq \bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, b\right[\tag{20.83}$$
If $x \in \bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, b\right[$ then $\forall n \in \mathbb{N}$ we have $a - \frac{1}{n} < x < b \Rightarrow a < x + \frac{1}{n} \wedge x < b$ which by [corollary: 10.31] proves that $a \leqslant x < b$ hence $x \in [a, b[$. So $\bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, b\right[\subseteq [a, b[$ which using [eq: 20.83] prove that
$$[a, b[= \bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, b\right[$$

3. Let $x \in [a, b]$ then $a \leqslant x < b$ and as $\forall n \in \mathbb{N}$ we have $a - \frac{1}{n} < a \Rightarrow x \in \left]a - \frac{1}{n}, b\right]$. Hence
$$[a, b] \subseteq \bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, b\right] \tag{20.84}$$
If $x \in \bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, b\right]$ then $\forall n \in \mathbb{N}$ then $a - \frac{1}{n} < x \leqslant b$ or $a < x + \frac{1}{n}$ which by [corollary: 10.31] proves that $a \leqslant x$ giving $x \in [a, b]$. So we have $\bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, b\right] \subseteq [a, b]$ which combined with [eq: 20.84] gives
$$[a, b] = \bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, b\right]$$

4. Let $x \in [a, b]$ then $a \leqslant x \leqslant b$ then $\forall n \in \mathbb{N}$ we have $a \leqslant x \leqslant b < b + \frac{1}{n}$. Which proves that
$$[a, b] \subseteq \bigcap_{n \in \mathbb{N}} \left[a, b + \frac{1}{n}\right[\tag{20.85}$$
If $x \in \bigcap_{n \in \mathbb{N}} \left[a, b + \frac{1}{n}\right[$ then $\forall n \in \mathbb{N}$ we have $a \leqslant x < b + \frac{1}{n}$. Using [corollary: 10.31] we have $x \leqslant b$ proving that $x \in [a, b]$. Hence $\bigcap_{n \in \mathbb{N}} \left[a, b + \frac{1}{n}\right[\subseteq [a, b]$ which combined with [eq: 20.85] gives
$$[a, b] = \bigcap_{n \in \mathbb{N}} \left[a, b + \frac{1}{n}\right[$$

5. Let $x \in [a, b]$ then $a \leqslant x \leqslant b$ and as $\forall n \in \mathbb{N}$ we have $a - \frac{1}{n} < x < b + \frac{1}{n}$ it follows that $x \in \left]a - \frac{1}{n}, b + \frac{1}{n}\right[$. Hence
$$[a, b] \subseteq \bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, b + \frac{1}{n}\right[\tag{20.86}$$
Let $x \in \bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, b + \frac{1}{n}\right[$ then $\forall n \in \mathbb{N}$ we have $a - \frac{1}{n} < x \wedge x < b + \frac{1}{n} \Rightarrow a < x + \frac{1}{n} \wedge x < b + \frac{1}{n}$. Using [corollary: 10.31] we have $a \leqslant x \wedge x \leqslant b$ or $x \in [a, b]$. Hence $\bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, b + \frac{1}{n}\right[\subseteq [a, b]$ which together with [eq: 20.86] proves
$$[a, b] = \bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, b + \frac{1}{n}\right[$$

6. Let $x \in]-\infty, a]$ then $x \leqslant a$ and as $\forall n \in \mathbb{N}$ we have $x < a + \frac{1}{n}$ it follows that $x \in \left]-\infty, a + \frac{1}{n}\right[$. Hence

$$]-\infty, a] \subseteq \bigcap_{n \in \mathbb{N}} \left]-\infty, a + \frac{1}{n}\right[\tag{20.87}$$

Let $x \in \bigcap_{n \in \mathbb{N}} \left]-\infty, a + \frac{1}{n}\right[$ then $\forall n \in \mathbb{N}$ we have $x < a + \frac{1}{n}$ which by [corollary: 10.31] gives $x \leqslant a$ or $x \in]-\infty, a]$. Hence $\bigcap_{n \in \mathbb{N}} \left]-\infty, a + \frac{1}{n}\right[\subseteq]-\infty, a]$ which together with [eq: 20.87] proves

$$\bigcap_{n \in \mathbb{N}} \left]-\infty, a + \frac{1}{n}\right[=]-\infty, a]$$

7. Let $x \in [a, \infty[$ then $a \leqslant x$ so that $\forall n \in \mathbb{N}_9$ we have $a - \frac{1}{n} < x$ giving $x \in \left]a - \frac{1}{n}, \infty\right[$ proving that

$$[a, \infty[\subseteq \bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, \infty\right[\tag{20.88}$$

Let $x \in \bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, \infty\right[$ then $\forall n \in \mathbb{N}$ we have $a - \frac{1}{n} < x \Rightarrow a < x + \frac{1}{n}$ which by [corollary: 10.31] proves that $x \in [a, \infty[$. Hence $\bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, \infty\right[\subseteq [a, \infty]$ which together with [eq: 20.88] proves that

$$[a, \infty[= \bigcap_{n \in \mathbb{N}} \left]a - \frac{1}{n}, \infty\right[$$

8. Let $x \in]a, b]$ then $a < x \leqslant b$ and using [corollary: 10.30] there exists a $n \in \mathbb{N}$ such that $a < a + \frac{1}{n} < x \leqslant b$ proving that $x \in \left[a + \frac{1}{n}, b\right]$. Hence

$$]a, b] \subseteq \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right] \tag{20.89}$$

Let $x \in \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right]$ then there exists a $n \in \mathbb{N}$ such that $x \in \left[a + \frac{1}{n}, b\right] \Rightarrow a < a + \frac{1}{n} \leqslant x \leqslant b$. Hence $x \in]a, b]$ proving that $\bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right] \subseteq]a, b]$ which by [eq: 20.89] gives

$$]a, b] = \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right]$$

9. Let $x \in [a, b[$ then $a \leqslant x < b$ and using [corollary: 10.30] there exists a $n \in \mathbb{N}$ such that $a \leqslant x < x + \frac{1}{n} < b \Rightarrow a \leqslant x \wedge x < b - \frac{1}{n}$ proving that $x \in \left[a, b - \frac{1}{n}\right]$. So

$$a, b[\subseteq \bigcup_{n \in \mathbb{N}} \left[a, b - \frac{1}{n}\right] \tag{20.90}$$

If $x \in \bigcup_{n \in \mathbb{N}} \left[a, b - \frac{1}{n}\right]$ then $\exists n \in \mathbb{N}$ such that $x \in \left[a, b - \frac{1}{n}\right] \Rightarrow a \leqslant x \leqslant b - \frac{1}{n} < b$ proving that $x \in [a, b[$. Hence $\bigcup_{n \in \mathbb{N}} \left[a, b - \frac{1}{n}\right] \subseteq [a, b[$ which combined with 20.90 gives

$$[a, b[= \bigcup_{n \in \mathbb{N}} \left[a, b - \frac{1}{n}\right]$$

10. Let $x \in]a, b[$ then $a < x < b$ and using [corollary: 10.30] there exists a $n \in \mathbb{N}$ such that $a < x < x + \frac{1}{n} < b$ hence $a < x < b - \frac{1}{n}$ proving that $x \in \left]a, b - \frac{1}{n}\right]$. So

$$]a, b[\subseteq \bigcup_{n \in \mathbb{N}} \left]a, b - \frac{1}{n}\right] \tag{20.91}$$

If $x \in \bigcup_{n \in \mathbb{N}} \left]a, b - \frac{1}{n}\right]$ then $\exists n \in \mathbb{N}$ such that $a < x \leqslant b - \frac{1}{n} < b \Rightarrow x \in]a, b[$ proving that $\bigcup_{n \in \mathbb{N}} \left]a, b - \frac{1}{n}\right] \subseteq]a, b[$. Combining this with [eq: 20.91] gives

$$]a, b[= \bigcup_{n \in \mathbb{N}} \left]a, b - \frac{1}{n}\right]$$

11. Let $x \in]a, b[$ then $a < x < b$ and using [corollary: 10.30] there exists a $n \in \mathbb{N}$ such that $a < a + \frac{1}{n} < x < b$ proving that $x \in \left[a + \frac{1}{n}, b\right[$. Hence

$$]a, b[\subseteq \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right[\tag{20.92}$$

If $x \in \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right[$ then $\exists n \in \mathbb{N}$ such that $a < a + \frac{1}{n} < x < b \Rightarrow x \in]a, b[$ proving that $\bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right[\subseteq]a, b[$. Combining this with [eq: 20.92] gives

$$]a, b[= \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}\right[$$

12. Let $x \in]a, b[$ then $a < x < b$ and using [corollary: 10.30] there exists $n_1, n_2 \in \mathbb{N}$ such that $a < a + \frac{1}{n_1} < x < x + \frac{1}{n_2} < b \Rightarrow a + \frac{1}{n_1} < x < b - \frac{1}{n_2}$. Take $m = \max(n_1, n_2)$ then $\frac{1}{m} \leqslant \frac{1}{n_1}, \frac{1}{n_2}$ so that $a + \frac{1}{m} \leqslant a + \frac{1}{n_1} < x < b - \frac{1}{n_2} \leqslant b - \frac{1}{m}$ proving that $x \in \left[a + \frac{1}{m}, b - \frac{1}{m}\right]$. Hence

$$]a, b[\subseteq \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{m}, b - \frac{1}{m}\right] \tag{20.93}$$

If $x \in \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$ then $\exists n \in \mathbb{N}$ such that $x \in \left[a + \frac{1}{n}, b - \frac{1}{n}\right] \Rightarrow a < a + \frac{1}{n} \leqslant x < b - \frac{1}{n} < b$ proving that $a < x < b$ or $x \in]a, b[$. So $\bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b - \frac{1}{n}\right] \subseteq]a, b[$ which together with [eq: 20.93] gives

$$]a, b[= \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{m}, b - \frac{1}{m}\right]$$

13. Let $x \in]-\infty, a[$ then $x < a$ and using [corollary: 10.30] there exists a $n \in \mathbb{N}$ such that $x + \frac{1}{n} < a \Rightarrow x < a - \frac{1}{n} \Rightarrow x \in \left]-\infty, a - \frac{1}{n}\right]$ proving that

$$]-\infty, a[\subseteq \bigcup_{n \in \mathbb{N}} \left]-\infty, a - \frac{1}{n}\right] \tag{20.94}$$

If $x \in \bigcup_{n \in \mathbb{N}} \left]-\infty, a - \frac{1}{n}\right]$ then $\exists n \in \mathbb{N}$ such that $x \in \left]-\infty, a - \frac{1}{n}\right] \Rightarrow x \leqslant a - \frac{1}{n} < a$ giving $x \in]-\infty, a[$ and proving that $\bigcup_{n \in \mathbb{N}}]-\infty, a] \subseteq]-\infty, a[$. Using this together with [eq: 20.94] proves

$$]-\infty, a[= \bigcup_{n \in \mathbb{N}} \left]-\infty, a - \frac{1}{n}\right]$$

14. Let $x \in]a, \infty[$ then $a < x$ and using [corollary: 10.30] there exists a $n \in \mathbb{N}$ such that $a + \frac{1}{n} < x \Rightarrow x \in \left[a + \frac{1}{n}, \infty\right[$. Hence

$$]a, \infty[\subseteq \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, \infty\right[\tag{20.95}$$

If $x \in \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, \infty\right[$ then $\exists n \in \mathbb{N}$ such that $x \in \left[a + \frac{1}{n}, \infty\right[\Rightarrow a < a + \frac{1}{n} \leqslant x$ hence $x \in]a, \infty[$. This proves that $\bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, \infty\right[\subseteq]a, \infty[$, combining this with [eq: 20.95] gives

$$]a, \infty[= \bigcup_{n \in \mathbb{N}+0} \left[a + \frac{1}{n}, \infty\right[$$

15. Let $x \in [a, b[$ then $a \leqslant x \wedge x < b \Rightarrow a \leqslant x \wedge \neg(b \leqslant x)$ so that $x \in [a, \infty[$ and $x \notin [b, \infty[$ proving that

$$[a, b[\subseteq [a, \infty[\setminus [b, \infty[\tag{20.96}$$

If $x \in [a, \infty[\setminus [b, \infty[$ then $a \leqslant x \wedge \neg(b \leqslant x)$ or $a \leqslant x$ and $x < b$ proving that $[a, \infty[\setminus [b, \infty[\subseteq [a, b[$. Combining this with [eq: 20.96] proves that

$$[a, b[= [a, \infty[\setminus [b, \infty[$$

16. Let $x \in]a, b]$ then $a < x \wedge x \leqslant b$ then $\neg(x \leqslant a) \wedge x \leqslant b$ so that $x \notin]-\infty, a] \wedge x \in]-\infty, b]$ proving that $x \in]-\infty, b[\backslash]-\infty, a]$. For the opposite inclusion let $x \in]-\infty, b]\backslash]-\infty, a]$ then $x \leqslant b$ and $\neg(x \leqslant a)$ proving that $x \leqslant b$ and $a < x$ or $x \in]a, b]$. Hence

$$]a, b] =]-\infty, b] \backslash]-\infty, a]$$

17. If $x \in]a, \infty[$ then $x \in \mathbb{R} \wedge a < x$ so that $x \nleqslant a \Rightarrow x \notin]-\infty, a]$ hence $x \in \mathbb{R}\backslash]-\infty, a]$, further if $x \in \mathbb{R}\backslash]-\infty, a]$ then $x \in \mathbb{R} \wedge x \notin]-\infty, a] \Rightarrow x \nleqslant a \Rightarrow a < x$ or $x \in]a, \infty[$ hence

$$]a, \infty[= \mathbb{R} \backslash]-\infty, a]$$

□

We are now ready to prove alternative forms for $\mathcal{B}[\mathbb{R}]$.

Theorem 20.150. *We have the following alternative forms for* $\mathcal{B}[\mathbb{R}] \underset{\text{definition}}{=} \sigma[\mathcal{T}_{||}]$

$$\begin{array}{rcl}
\sigma[\{C \subseteq \mathbb{R} | C \text{ is closed in } \mathcal{T}_{||}\}] & \underset{[\text{theorem: } 20.143]}{=} & \mathcal{B}[\mathbb{R}] \\
& = & \sigma[\mathcal{T}_{||}] \\
& = & \sigma[\mathcal{I}_{][}] \\
& = & \sigma[\mathcal{I}_{]]}] \\
& = & \sigma[\mathcal{I}_{[]}] \\
& = & \sigma[\mathcal{I}_{[[}] \\
& = & \sigma[\{[a, \infty[| a \in \mathbb{R}\}] \\
& = & \sigma[\{]a, \infty[| a \in \mathbb{R}\}] \\
& = & \sigma[\{]-\infty, a] | a \in \mathbb{R}\}] \\
& = & \sigma[\{]-\infty, a[| a \in \mathbb{R}\}]
\end{array}$$

Proof. Let $U \in \mathcal{T}_{||}$ then by [corollary: 14.105] there exist a countable family $\{]a_i, b_i[\}_{i \in I} \subseteq \mathcal{I}_{][}$ such that

$$\forall i \in I \ ]a_i, \beta_i[\neq \varnothing \text{ and } U = \bigcup_{i \in I}]a_i, \beta_i[\tag{20.97}$$

As $\mathcal{I}_{][} \subseteq \sigma[\mathcal{I}_{][}]$ which is a σ-algebra it follows from [theorem: 20.138] that $U = \bigcup_{i \in I}]a_i, \beta_i[\in \sigma(\mathcal{I}_{][})$. As $U \in \mathcal{T}_{||}$ was chosen arbitrary it follows that $\mathcal{T}_{||} \subseteq \sigma[\mathcal{I}_{][}]$. Using then [theorem: 20.141] it follows that

$$\mathcal{B}[\mathbb{R}] \underset{\text{def}}{=} \sigma[\mathcal{T}_{||}] \subseteq \sigma[\mathcal{I}_{][}] \tag{20.98}$$

If $]a, b[\in \mathcal{I}_{][}$ then by [lemma: 20.149] $]a, b[= \bigcup_{n \in \mathbb{N}} \left]a, b - \frac{1}{n}\right]$. Let $n \in \mathbb{N}$ then we have either:

$\left]a, b - \frac{1}{n}\right] = \varnothing$. Then $\left]a, b - \frac{1}{n}\right] = \varnothing \in \sigma[\mathcal{I}_{]]}]$

$\left]a, b - \frac{1}{n}\right] \neq \varnothing$. Then $\left]a, b - \frac{1}{n}\right] \in \sigma[\mathcal{I}_{]]}]$

so that $\left\{\left]a, b - \frac{1}{n}\right]\right\}_{n \in \mathbb{N}} \subseteq \sigma[\mathcal{I}_{]]}]$ from which it follows by [lemma: 20.138] that

$$]a, b[= \bigcup_{n \in \mathbb{N}} \left]a, b - \frac{1}{n}\right] \in \sigma[\mathcal{I}_{]]}]$$

. Hence we have $\mathcal{I}_{][} \subseteq \sigma[\mathcal{I}_{]]}]$ a σ-algebra on $\mathbb{R}$. Using [theorem: 20.141] it follows then that

$$\sigma[\mathcal{I}_{][}] \subseteq \sigma[\mathcal{I}_{]]}] \tag{20.99}$$

If $]a, b] \in \mathcal{I}_{]]}$ then by [lemma: 20.149] $]a, b] = \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right]$. For $n \in \mathbb{N}$ we have either:

$\left[a + \frac{1}{n}, b\right] = \varnothing$. Then $\left[a + \frac{1}{n}, b\right] = \varnothing \in \sigma[\mathcal{I}_{[]}]$

$\left[a + \frac{1}{n}, b\right] \neq \varnothing$. Then $\left[a + \frac{1}{n}, b\right] \in \mathcal{I}_{[]} \subseteq \sigma[\mathcal{I}_{[]})[]$

so that $\left\{\left[a+\frac{1}{n},b\right]\right\}_{n\in\mathbb{N}}\subseteq\sigma[\mathcal{I}_{[]}]$ from which it follows by [theorem: 20.138] that

$$]a,b]=\bigcup_{n\in\mathbb{N}}\left[a+\frac{1}{n},b\right]\in\sigma[\mathcal{I}_{[]}]$$

Hence we have $\mathcal{I}_{]]}\subseteq\sigma[\mathcal{I}_{[]}]$ a σ-algebra on $\mathbb{R}$ so that by [theorem: 20.141]

$$\sigma[\mathcal{I}_{]]}]\subseteq\sigma[\mathcal{I}_{[]}] \tag{20.100}$$

If $[a,b]\in\mathcal{I}_{[]}$ then by [lemma: 20.149] $[a,b]=\bigcap_{n\in\mathbb{N}}\left[a,b+\frac{1}{n}\right[$. For $n\in\mathbb{N}$ we have either:

$\left[a,b+\frac{1}{n}\right[=\varnothing$. Then $\left[a,b+\frac{1}{n}\right[=\varnothing\in\sigma[\mathcal{I}_{[[}]$

$\left[a,b+\frac{1}{n}\right[\neq\varnothing$. Then $\left[a,b+\frac{1}{n}\right[\in\sigma[\mathcal{I}_{[[}]$

so that $\left\{\left[a,b+\frac{1}{n}\right[\right\}_{n\in\mathbb{N}}\subseteq\sigma[\mathcal{I}_{[[}]$ form which it it follows by [theorem: 20.138] that

$$[a,b]=\bigcap_{n\in\mathbb{N}}\left[a,b+\frac{1}{n}\right[\in\sigma[\mathcal{I}_{[[}]$$

Hence we have $\mathcal{I}_{[]}\subseteq\sigma[\mathcal{I}_{[[}]$ a σ-algebra on $\mathbb{R}$ so that by [theorem: 20.141]

$$\sigma[\mathcal{I}_{[]}]\subseteq\sigma[\mathcal{I}_{[[}] \tag{20.101}$$

If $[a,b[\in\mathcal{I}_{[[}$ then by [lemma: 20.149] $[a,b[=\bigcap_{n\in\mathbb{N}}\left]a-\frac{1}{n},b\right[$. As $\forall n\in\mathbb{N}$ $\left]a-\frac{1}{n},b\right[$ is open it follows that $\left\{\left]a-\frac{1}{n},b\right[\right\}_{n\in\mathbb{N}}\subseteq\mathcal{T}_{||}$ so that by [theorem 20.138] $[a,b[=\bigcap_{n\in\mathbb{N}}\left]a-\frac{1}{n},b\right[\in\sigma[\mathcal{T}_{||}]$. Hence we have that $\mathcal{I}_{[[}\subseteq\sigma[\mathcal{T}_{||}]$ so that

$$\sigma[\mathcal{I}_{[[}]\subseteq\sigma[\mathcal{T}_{||}]\underset{\text{def}}{=}\mathcal{B}[\mathbb{R}] \tag{20.102}$$

Combining [eqs: 20.98, 20.99, 20.100, 20.101 and 20.102] it follows that

$$\mathcal{B}[\mathbb{R}]=\sigma[\mathcal{T}_{||}]=\sigma[\mathcal{I}_{][}]=\sigma[\mathcal{I}_{]]}]=\sigma[\mathcal{I}_{[]}]=\sigma[\mathcal{I}_{[[}] \tag{20.103}$$

If $[a,b[\in\mathcal{I}_{[[}$ then using [lemma: 20.149] we have that

$$[a,b]=[a,\infty]\setminus[b,\infty[\underset{\text{[theorem: 20.130]}}{\in}\sigma[\{[a,\infty[|a\in\mathbb{R}\}]$$

proving $\mathcal{I}_{[[}\subseteq\sigma[\{[a,\infty[|a\in\mathbb{R}\}]$ a σ-algebra. Hence using [theorem: 20.141] we have

$$\sigma[\mathcal{I}_{[[}]\subseteq\sigma[\{[a,\infty[|a\in\mathbb{R}\}] \tag{20.104}$$

If $[a,\infty[\in\{[a,\infty[|a\in\mathbb{R}\}$ then using [lemma: 20.149] we have that $[a,\infty[=\bigcap_{n\in\mathbb{N}}\left]a-\frac{1}{n},\infty\right[$. As $\left\{\left]a-\frac{1}{n},\infty\right[\right\}_{n\in\mathbb{N}}\subseteq\left\{\left]a-\frac{1}{n},\infty\right[|n\in\mathbb{N}\right\}\subseteq\sigma[\{]a,\infty[|a\in\mathbb{R}\}]$ it follows from [theorem: 20.138] that

$$[a,\infty[=\bigcap_{n\in\mathbb{N}}\left]a-\frac{1}{n},\infty\right[\in\sigma[\{]a,\infty[|a\in\mathbb{R}\}]$$

Hence $\{[a,\infty[|a\in\mathbb{R}\}\subseteq\sigma[\{]a,\infty[|a\in\mathbb{R}\}]$ so that by [theorem: 20.141]

$$\sigma[\{[a,\infty[|a\in\mathbb{R}\}]\subseteq\sigma[\{]a,\infty[|a\in\mathbb{R}\}] \tag{20.105}$$

Let $]a,\infty[\in\{]a,\infty[|a\in\mathbb{R}\}$ then by [lemma: 20.149] $]a,\infty[=\mathbb{R}\setminus]-\infty,a]\in\sigma[\{]-\infty,a]|a\in\mathbb{R}\}]$ so that $\{]a,\infty[|a\in\mathbb{R}\}\subseteq\sigma[\{]-\infty,a]|a\in\mathbb{R}\}]$. Using [theorem: 20.141] it follows that

$$\sigma[\{]a,\infty[|a\in\mathbb{R}\}]\subseteq\sigma[\{]-\infty,a]|a\in\mathbb{R}\}] \tag{20.106}$$

Let $]-\infty,a]\in\{]-\infty,a]|a\in\mathbb{R}\}$ then by [lemma: 20.149] $]-\infty,a]=\bigcap_{n\in\mathbb{N}}\left]-\infty,a+\frac{1}{n}\right[$. As $\left\{\left]-\infty,a+\frac{1}{n}\right[\right\}_{n\in\mathbb{N}}\subseteq\{]-\infty,a[|a\in\mathbb{R}\}\subseteq\sigma[\{]-\infty,a[|a\in\mathbb{R}\}]$ it follows from [theorem: 20.138] that $]-\infty,a]=\bigcap_{n\in\mathbb{N}}\left]-\infty,a+\frac{1}{n}\right[\in\sigma[\{]-\infty,a[|a\in\mathbb{R}\}]$. Hence

$$\{]-\infty,a]|a\in\mathbb{R}\}\subseteq\sigma[\{]-\infty,a[|a\in\mathbb{R}\}]$$

which by [theorem: 20.141] proves that

$$\sigma[\{]-\infty,a]|a\in\mathbb{R}\}]\subseteq\sigma[\{]-\infty,a[|a\in\mathbb{R}\}] \tag{20.107}$$

Let $]-\infty, a[\in\{]-\infty, a[|a\in\mathbb{R}\}$ then as $]-\infty, a[$ is open it follows that $\{]-\infty, a[|a\in\mathbb{R}\}\subseteq\mathcal{T}_{||}\subseteq\sigma[\mathcal{T}_{||}]$ and using [theorem: 20.141] it follows that

$$\sigma[\{]-\infty, a[|a\in\mathbb{R}\}]\subseteq\sigma[\mathcal{T}_{||}]\underset{\text{[eq: 20.103]}}{=}\sigma[\mathcal{I}_{[[}] \tag{20.108}$$

Combining [eqs: 20.104, 20.105, 20.106, 20.107 and 20.108] gives

$$\sigma[\mathcal{I}_{[[}]=\sigma[\{[a,\infty[|a\in\mathbb{R}\}]=\sigma[\{]a,\infty[|a\in\mathbb{R}\}]=\sigma[\{]-\infty,a]|a\in\mathbb{R}\}]=\sigma[\{]-\infty,a[|a\in\mathbb{R}\}]$$

Finally combining the above with [eq: 20.103] proves that

$$\begin{aligned}
\mathcal{B}[\mathbb{R}] &= \sigma[\mathcal{T}_{||}]\\
&= \sigma[\mathcal{I}_{][}]\\
&= \sigma[\mathcal{I}_{]]}]\\
&= \sigma[\mathcal{I}_{[]}]\\
&= \sigma[\mathcal{I}_{[[}]\\
&= \sigma[\{[a,\infty[|a\in\mathbb{R}\}]\\
&= \sigma[\{]a,\infty[|a\in\mathbb{R}\}]\\
&= \sigma[\{]-\infty,a]|a\in\mathbb{R}\}]\\
&= \sigma[\{]-\infty,a[|a\in\mathbb{R}\}]
\end{aligned}$$

□

20.3.2 Borel algebra on $\overline{\mathbb{R}}$

Definition 20.151. *Let $\langle\overline{\mathbb{R}},\mathcal{T}_{\overline{\mathbb{R}}}\rangle$ be the topological space consisting of the extended real numbers with the canonical topology $\mathcal{T}_{\overline{\mathbb{R}}}$ [see definition: 20.47] then the Borel algebra on $\overline{\mathbb{R}}$ is noted as $\mathcal{B}[\overline{\mathbb{R}}]$ hence*

$$\mathcal{B}[\overline{\mathbb{R}}]=\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$$

Just as in the real case we formulate some alternative definitions for the Borel algebra on the extended real numbers.

Lemma 20.152. *We have*

1. $\forall a\in\mathbb{R}$ *we have* $[a,\infty]=\bigcap_{n\in\mathbb{N}_0}\left]a-\frac{1}{n},\infty\right]$
2. $\{\infty\}=\bigcap_{n\in\mathbb{N}_0}]n,\infty]$
3. $\{-\infty\}=\bigcap_{n\in\mathbb{N}_0}[-\infty,-n[$

Proof.

1. Let $n\in\mathbb{N}$ then if $x\in[a,\infty]$ $a\leqslant x\leqslant\infty\Rightarrow a-\frac{1}{n}<a\leqslant x\leqslant\infty$ so that $[a,\infty]\subseteq\left]a-\frac{1}{n},\infty\right]$, hence

$$[a,\infty]\subseteq\bigcap_{n\in\mathbb{N}}\left]a-\frac{1}{n},\infty\right] \tag{20.109}$$

If $x\in\bigcap_{n\in\mathbb{N}}\left]a-\frac{1}{n},\infty\right]$. Assume that $x<a$ then using [theorem: 10.30] there exist a $n\in\mathbb{N}$ such that $x+\frac{1}{n}<a$ or $x<a-\frac{1}{n}$ so that $x\notin\left]a-\frac{1}{n},\infty\right]$ contradicting $x\in\bigcap_{n\in\mathbb{N}}\left]a-\frac{1}{n},\infty\right]$. Hence the assumption is wrong and we have $a\leqslant x\leqslant\infty$ so that $x\in[a,\infty]$ proving that $\bigcap_{n\in\mathbb{N}}\left]a-\frac{1}{n},\infty\right]\subseteq[a,\infty[$, combining this with [eq: 20.109] results in

$$[a,\infty]=\bigcap_{n\in\mathbb{N}}\left]a-\frac{1}{n},\infty\right]$$

2. As $\forall n\in\mathbb{N}$ we have $n<\infty\leqslant\infty$ it follows that $\infty\in\bigcap_{n\in\mathbb{N}}]n,\infty]$ so that

$$\{\infty\}\subseteq\bigcap_{n\in\mathbb{N}}]n,\infty] \tag{20.110}$$

Let $x \in \bigcap_{n \in \mathbb{N}}]n, \infty]$. Then $x \in [1, \infty] \Rightarrow -\infty < 1 < x \leqslant \infty$ so that $x \notin -\infty$. Assume that $x \neq \infty$ then $x \in \mathbb{R}$ and by [theorem: 10.30] there exist a $n \in \mathbb{N}$ such that $x < n \Rightarrow x \in \notin]n, \infty]$ contradicting $x \in \bigcap_{n \in \mathbb{N}}]n, \infty]$, hence the assumption is wrong and we must have that $x = \infty \in \{\infty\}$. So $\bigcap_{n \in \mathbb{N}}]n, \infty] \subseteq \{\infty\}$ which combined with [eq: 20.110] proves that

$$\{\infty\} = \bigcap_{n \in \mathbb{N}}]n, \infty]$$

3. As $\forall n \in \mathbb{N}$ we have $-\infty \leqslant -\infty < -n$ it follows that $-\infty \in \bigcap_{n \in \mathbb{N}} [-\infty, -n[$ so that

$$\{-\infty\} \subseteq \bigcap_{n \in \mathbb{N}} [-\infty, -n[\tag{20.111}$$

Let $x \in \bigcap_{n \in \mathbb{N}} [-\infty, -n[$. Then $x \in [-\infty, -1[\Rightarrow x < -1 < \infty$ so that $x \notin \infty$. Assume that $x \neq -\infty$ then $x \in \mathbb{R}$ and by [theorem: 10.30] there exist a $n \in \mathbb{N}$ such that $-x < n \Rightarrow -n < x$ so that $x \notin [-\infty, -n[$ contradicting $x \in \bigcap_{n \in \mathbb{N}} [-\infty, -n[$, hence the assumption is wrong and we must have that $x = -\infty \in \{-\infty\}$. So $\bigcap_{n \in \mathbb{N}} [-\infty, -n[\subseteq \{-\infty\}$ which combined with [eq: 20.111] proves that

$$\{\infty\} = \bigcap_{n \in \mathbb{N}} [-\infty, -n[$$

□

Lemma 20.153. *For $\sigma[\{]a, \infty] | a \in \mathbb{R}\}]$ we have the following*

1. $\forall a \in \mathbb{R}$ *we have that* $[a, \infty] \in \sigma[\{]a, \infty] | a \in \mathbb{R}\}]$
2. $\forall a \in \mathbb{R}$ *we have that* $[-\infty, a] \in \sigma[\{]a, \infty] | a \in \mathbb{R}\}]$
3. $\forall a, b \in \mathbb{R}$ *we have that* $]a, b[\in \sigma[\{]a, \infty] | a \in \mathbb{R}\}]$
4. $\forall U \in \mathcal{T}_{||}$ *we have that* $U \in \sigma[\{]a, \infty] | a \in \mathbb{R}\}]$
5. $\{\infty\} \in \sigma[\{]a, \infty] | a \in \mathbb{R}\}]$
6. $\{-\infty\} \in \sigma[\{]a, \infty] | a \in \mathbb{R}\}]$

Proof.

1. Let $a \in \mathbb{R}$ then $\left\{ \left]a - \frac{1}{n}, \infty\right] \right\}_{n \in \mathbb{N}_0} \subseteq \{]a, \infty] | a \in \mathbb{R}\} \subseteq \sigma(\{]a, \infty] | a \in \mathbb{R}\})$, hence $\bigcup_{n \in \mathbb{N}_0} \left]a - \frac{1}{n}, \infty\right] \in \sigma(\{]a, \infty] | a \in \mathbb{R}\})$. As $[a, \infty] \underset{[\text{lemma: } 20.152]}{=} \bigcup_{n \in \mathbb{N}_0} \left]a - \frac{1}{n}, \infty\right]$ we have that

$$[a, \infty] \in \sigma(\{]a, \infty] | a \in \mathbb{R}\})$$

2. Let $a \in \mathbb{R}$ then

$$\begin{array}{rcl} x \in \overline{\mathbb{R}} \setminus [a, \infty] & \Leftrightarrow & x \in [-\infty, \infty] \wedge x \notin [a, \infty] \\ & \Leftrightarrow & (-\infty \leqslant x \wedge x \leqslant \infty) \wedge \neg(a \leqslant x \leqslant \infty) \\ & \Leftrightarrow & (-\infty \leqslant x \wedge x \leqslant \infty) \wedge (x < a \vee \infty < x) \\ & \underset{\infty < x \text{ is false}}{\Leftrightarrow} & -\infty \leqslant x \wedge x \leqslant \infty \wedge x < a \\ & \Leftrightarrow & -\infty \leqslant x \wedge x < a \\ & \Leftrightarrow & x \in [-\infty, a[\end{array}$$

proving that $[-\infty, a[= \overline{\mathbb{R}} \setminus [a, \infty]$ which, as by (1) $[a, \infty] \in \sigma(\{]a, \infty] | a \in \mathbb{R}\})$ and $\sigma(\{]a, \infty] | a \in \mathbb{R}\})$ is a σ-algebra, proves

$$[-\infty, a[\in \sigma(\{]a, \infty] | a \in \mathbb{R}\})$$

3. Let $a, b \in \mathbb{R}$ with $a \leqslant b$ then

$$\begin{array}{rcl} x \in]a, \infty] \bigcap [-\infty, b[& \Leftrightarrow & x \in]a, \infty] \wedge x \in [-\infty, b[\\ & \Leftrightarrow & a < x \wedge x \leqslant \infty \wedge -\infty \leqslant x \wedge x < b \\ & \Leftrightarrow & a < x \wedge x < b \\ & \Leftrightarrow & x \in]a, b[\end{array}$$

proving that $]a,b[=]a,\infty]\bigcap[-\infty,b[$. As $]a,\infty]\in\sigma(\{]a,\infty]|a\in\mathbb{R}\})$ and by (2) $[-\infty,b[\in\sigma(\{]a,\infty]|a\in\mathbb{R}\})$ we have that $]a,\infty]\bigcap[-\infty,b[\in\sigma\{]a,\infty]|a\in\mathbb{R}\}$. So

$$]a,b[\in\sigma(\{]a,\infty]|a\in\mathbb{R}\})$$

4. Let U be a open set in $\mathbb{R}$ then by [corollary: 14.105] there exist a countable set I and family $\{]a_i,b_i[\}_{i\in I}$ such that $U=\bigcup_{i\in I}]a_i,b_i[$. By (3) it follows that $\{]a_i,b_i[\}_{i\in I}\subseteq\sigma(\{]a,\infty]|a\in\mathbb{R}\})$ so that using [theorem: 20.138] $U=\bigcup_{i\in I}]a_i,b_i[\in\sigma(\{]a,\infty]|a\in\mathbb{R}\})$.
5. Using [lemma: 20.152] we have that $\{\infty\}=\bigcap_{n\in\mathbb{N}_0}]n,\infty]$ which, as $\{]n,\infty]\}_{n\in\mathbb{N}_0}\subseteq(\{]a,\infty]|a\in\mathbb{R}\})$, proves that

$$\{\infty\}\in\sigma(\{]a,\infty]|a\in\mathbb{R}\})$$

6. Using [lemma: 20.152] we have that $\{-\infty\}=\bigcap_{n\in\mathbb{N}_0}[-\infty,-n[$ which, as by (2) $\{[-\infty,-n]\}_{n\in\mathbb{N}_0}\in\sigma(\{]a,\infty]|a\in\mathbb{R}\})$, proves by [theorem: 20.135] that

$$\{-\infty\}\in\sigma(\{]a,\infty]|a\in\mathbb{R}\})$$ □

Lemma 20.154. $\mathcal{B}[\overline{\mathbb{R}}]\underset{definition}{=}\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]=\sigma[\{]a,\infty]|a\in\mathbb{R}\}]$

Proof. Let $I\in\{]a,\infty]|a\in\mathbb{R}\}$ then there exist a $a\in\mathbb{R}$ such that $I=]a,\infty]$. By [definition: 20.45] it follows that $I\in\mathcal{B}_\infty\subseteq\mathcal{T}_{\overline{\mathbb{R}}}$, so that $\{]a,\infty]|a\in\mathbb{R}\}\subseteq\mathcal{T}_{\overline{\mathbb{R}}}\subseteq\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$. Hence using [theorem: 20.141]

$$\sigma[\{]a,\infty]|a\in\mathbb{R}\}]\subseteq\sigma[\mathcal{T}_{\overline{\mathbb{R}}}] \tag{20.112}$$

For the opposite inclusion, let $U\in\mathcal{T}_{\overline{\mathbb{R}}}$ then by [theorem: 20.48] we have that $U\bigcap\mathbb{R}\in\mathcal{T}_{||}$. From $\overline{\mathbb{R}}=\mathbb{R}\bigcup\{-\infty\}\bigcup\{\infty\}$ it follows that

$$U=U\bigcap\overline{\mathbb{R}}=(U\bigcap\mathbb{R})\bigcup(U\bigcap\{-\infty\})\bigcup(U\bigcap\{\infty\}) \tag{20.113}$$

Now as $U\bigcap\mathbb{R}\in\mathcal{T}_{||}$ we have by [lemma: 20.153] that

$$U\bigcap\mathbb{R}\in\sigma[\{]a,\infty]|a\in\mathbb{R}\}] \tag{20.114}$$

Also $U\bigcap\{-\infty\}$ is either $\emptyset\in\sigma[\{]a,\infty]|a\in\mathbb{R}\}]$ or $\{-\infty\}$ and as by [lemma: 20.153] $\{-\infty\}\in\sigma[\{]a,\infty]|a\in\mathbb{R}\}]$ we have that

$$U\bigcap\{-\infty\}\in\sigma[\{]a,\infty]|a\in\mathbb{R}\}] \tag{20.115}$$

Finally $U\bigcap\{\infty\}$ is either $\emptyset\in\sigma[\{]a,\infty]|a\in\mathbb{R}\}]$ or $\{\infty\}$ and as by [lemma: 20.153] $\{\infty\}\in\sigma[\{]a,\infty]|a\in\mathbb{R}\}]$ it follows that

$$U\bigcap\{\infty\}\in\sigma[\{]a,\infty]|a\in\mathbb{R}\}] \tag{20.116}$$

From [eqs: 20.113, 20.114, 20.115 and 20.116] it follows that $U\in\sigma[\{]a,\infty]|a\in\mathbb{R}\}]$. Hence $\mathcal{T}_{\overline{\mathbb{R}}}\subseteq\sigma[\{]a,\infty]|a\in\mathbb{R}\}]$ [a sigma algebra on $\overline{\mathbb{R}}$] so that $\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]\subseteq\sigma[\{]a,\infty]|a\in\mathbb{R}\}]$, combining this with [eq: 20.112] gives

$$\sigma[\{]a,\infty]|a\in\mathbb{R}\}]=\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$$ □

The relation between $\mathcal{B}[\mathbb{R}]$ and $\mathcal{B}[\overline{\mathbb{R}}]$ is expressed in the following theorem.

Lemma 20.155. $\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]=\{U\in\mathcal{P}[\overline{\mathbb{R}}]|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\}\underset{\mathcal{B}[\mathbb{R}]=\sigma[\mathcal{T}_{||}]}{=}\{U\in\mathcal{P}(\overline{\mathbb{R}})|U\bigcap\mathbb{R}\in\sigma[\mathcal{T}_{||}]\}$

Proof. First we prove that

$$\{U\in\mathcal{P}[\overline{\mathbb{R}}]|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\}\text{ is a }\sigma\text{-algebra on }\overline{\mathbb{R}} \tag{20.117}$$

Proof. As $\overline{\mathbb{R}}\bigcap\mathbb{R}=\mathbb{R}\in\mathcal{B}[\mathbb{R}]$ it follows that

$$\overline{\mathbb{R}}\in\{U\in\mathcal{P}[\overline{\mathbb{R}}]|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\} \tag{20.118}$$

Let $U\in\{U\in\mathcal{P}[\overline{\mathbb{R}}]|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\}$ then $U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]$ hence $\mathbb{R}\backslash(U\bigcap\mathbb{R})\in\mathcal{B}[\mathbb{R}]$, further

$$(\overline{\mathbb{R}}\backslash U)\bigcap\mathbb{R}\underset{[\text{theorem: }1.33]}{=}(\overline{\mathbb{R}}\bigcap\mathbb{R})\backslash U=\mathbb{R}\backslash U \text{ and } \mathbb{R}\backslash\left(U\bigcap\mathbb{R}\right)\underset{[\text{eq: }1.31]}{=}(\mathbb{R}\backslash U)\bigcup(\mathbb{R}\backslash\mathbb{R})=\mathbb{R}\backslash U$$

so that $(\overline{\mathbb{R}}\backslash U)\bigcap\mathbb{R}=\mathbb{R}\backslash(U\bigcap\mathbb{R})\in\mathcal{B}[\mathbb{R}]$, proving that $(\overline{\mathbb{R}}\backslash U)\in\{U\in\mathcal{P}[\overline{\mathbb{R}}]|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\}$. In other words

$$\forall U\in\left\{U\in\mathcal{P}[\overline{\mathbb{R}}]|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\right\} \text{ we have } \overline{\mathbb{R}}\backslash U\in\left\{U\in\mathcal{P}[\overline{\mathbb{R}}]|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\right\} \tag{20.119}$$

Let $\{U_i\}_{i\in\mathbb{N}}\subseteq\{U\in\mathcal{P}(\overline{\mathbb{R}})|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\}$ then $\forall i\in\mathbb{N}$ $U_i\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]$ so that $(\bigcup_{i\in\mathbb{N}}U_i)\bigcap\mathbb{R}=\bigcup_{i\in\mathbb{N}}(U_i\bigcap\mathbb{R})\in\mathcal{B}[\mathbb{R}]$. Hence

$$\forall\{U_i\}_{i\in\mathbb{N}}\subseteq\left\{U\in\mathcal{P}(\overline{\mathbb{R}})|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\right\}\bigcup_{i\in\mathbb{N}}\left(U_i\bigcap\mathbb{R}\right)\in\mathcal{B}[\mathbb{R}] \tag{20.120}$$

Finally [eqs: 20.118, 20.119 and 20.120] proves [eq: 20.117]. □

Take $U\in\mathcal{T}_{\overline{\mathbb{R}}}$ then by [theorem: 20.48] we have that $U\bigcap\mathbb{R}\in\mathcal{T}_{||}\subseteq\sigma[\mathcal{T}_{||}]\underset{\text{definition}}{=}\mathcal{B}[\mathbb{R}]$ proving that $U\in\{U\in\mathcal{P}(\overline{\mathbb{R}})|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\}$ so that

$$\mathcal{T}_{\overline{\mathbb{R}}}\subseteq\left\{U\in\mathcal{P}(\overline{\mathbb{R}})|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\right\}$$

and as $\{U\in\mathcal{P}(\overline{\mathbb{R}})|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\}$ is a σ-algebra on $\overline{\mathbb{R}}$ [see eq: 20.117] it follows from [theorem: 20.141] that

$$\mathcal{B}[\overline{\mathbb{R}}]\underset{\text{definition}}{=}\sigma(\mathcal{T}_{\overline{\mathbb{R}}})\subseteq\left\{U\in\mathcal{P}(\overline{\mathbb{R}})|U\bigcap\mathbb{R}\in\mathcal{B}[\mathbb{R}]\right\} \tag{20.121}$$

For the opposite inclusion define

$$S=\left\{U\bigcap\mathbb{R}|U\in\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]\right\}\subseteq\mathcal{P}(\mathbb{R}) \tag{20.122}$$

then as $\mathcal{T}_{\overline{\mathbb{R}}}\subseteq\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ we have

$$\mathcal{T}_{||}\underset{]\text{theorem: }20.48}{=}\left\{U\bigcap\mathbb{R}|U\in\mathcal{T}_{\overline{\mathbb{R}}}\right\}\subseteq\left\{U\bigcap\mathbb{R}|U\in\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]\right\}=S \tag{20.123}$$

Further as $\mathbb{R}\in\mathcal{T}_{||}\underset{[\text{theorem: }20.48]}{\subseteq}\mathcal{T}_{\overline{\mathbb{R}}}\subseteq\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ we have for $U\in\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ that $U\bigcap\mathbb{R}\in\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ proving

$$S\subseteq\sigma[\mathcal{T}_{\overline{\mathbb{R}}}] \tag{20.124}$$

Next we prove that

$$S \text{ is a } \sigma\text{-algebra on } \mathbb{R} \tag{20.125}$$

Proof. As $\mathbb{R}=\mathbb{R}\bigcap\mathbb{R}$ and $\mathbb{R}\in\mathcal{T}_{\mathbb{R}}\underset{[\text{theorem: }20.48]}{\subseteq}\mathcal{T}_{\overline{\mathbb{R}}}\subseteq\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ it follows that

$$\mathbb{R}\in\mathcal{S} \tag{20.126}$$

If $A\in\mathcal{S}$ then there exist a $U\in\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ such that $A=U\bigcap\mathbb{R}$ so that

$$\mathbb{R}\backslash A=\mathbb{R}\backslash\left(U\bigcap\mathbb{R}\right)\underset{[\text{eq: }1.31]}{=}(\mathbb{R}\backslash U)\bigcup(\mathbb{R}\backslash\mathbb{R})=\mathbb{R}\backslash U$$

which, as $\mathbb{R},U\in\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]\underset{[\text{eq: }20.131]}{\Rightarrow}\mathbb{R}\backslash U\in\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$, proves that

$$\mathbb{R}\backslash A\in\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$$

Hence $(\mathbb{R}\backslash A)\bigcap\mathbb{R}=\mathbb{R}\backslash A\in\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ so that $\mathbb{R}\backslash A\in\mathcal{S}$. Summarized

$$\forall A\in\mathcal{S} \text{ we have } \mathbb{R}\backslash A\in S \tag{20.127}$$

If $\{A_i\}_{i\in\mathbb{N}}\subseteq S$ then there exists a $\{U_i\}_{i\in\mathbb{N}}\subseteq\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ such that $\forall i\in\mathbb{N}$ we have $A_i=U_i\bigcap\mathbb{R}$, then $\bigcup_{i\in\mathbb{N}}A_i=(\bigcup_{i\in\mathbb{N}}(U_i\bigcap\mathbb{R}))=(\bigcup_{i\in\mathbb{N}}U_i)\bigcap\mathbb{R}$ which, as $\bigcup_{i\in\mathbb{N}}U_i\in\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$. proves that $\bigcup_{i\in\mathbb{N}}A_i\in S$. In other words

$$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq S \text{ we have } \bigcup_{i\in\mathbb{N}}A_i\in\mathcal{S} \tag{20.128}$$

Finally [eqs: 20.126, 20.127 and 20.128] proves that S is a σ-algebra on $\mathbb{R}$ thus proving [eq: 20.125]. $\square$

So we have proved that $\mathcal{T}_{||} \subseteq S$ [see eq: 20.123] and S is a σ-algebra on $\mathbb{R}$ from which it follows, using [theorem: 20.141], that

$$\mathcal{B}[\mathbb{R}] \underset{\text{definition}}{=} \sigma[\mathcal{T}_{\mathbb{R}}] \subseteq S \underset{\text{[eq: 20.124]}}{\subseteq} \sigma[\mathcal{T}_{\overline{\mathbb{R}}}] \tag{20.129}$$

Let $A \in \{U \in \mathcal{P}[\overline{\mathbb{R}}] | U \bigcap \mathbb{R} \in \mathcal{B}[\mathbb{R}]\}$ then $A \subseteq \overline{\mathbb{R}}$ so that $A = A \bigcap \bar{\mathbb{R}} = A \bigcap (\mathbb{R} \bigcup \{-\infty\}, \{\infty\})$ giving

$$A = (A \bigcap \mathbb{R}) \bigcap (A \bigcap \{-\infty\}) \bigcap (A \bigcap \{\infty\}) \tag{20.130}$$

where

1. $A \bigcap \mathbb{R} \in \mathcal{B}[\mathbb{R}] \underset{\text{[eq: 20.129]}}{\Rightarrow} A \bigcap \mathbb{R} \in \sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$
2. $A \bigcap \{-\infty\}$ is either $\emptyset \in \sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ or $\{-\infty\} \in \sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ [as $-\infty$ is closed [theorem: 20.49] and closed sets are part of $\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ by [theorem: 20.143]].
3. $A \bigcap \{-\infty\}$ is either $\emptyset \in \sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ or $\{\infty\} \in \sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ [as ∞ is closed by [theorem: 20.49]and closed sets are part of $\sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ by [theorem: 20.143]].

From the above we conclude that $\{U \in \mathcal{P}[\overline{\mathbb{R}}] | U \bigcap \mathbb{R} \in \mathcal{B}[\mathbb{R}]\} \subseteq \sigma[\mathcal{T}_{\overline{\mathbb{R}}}]$ which combined with [eq: 20.121] proves that

$$\sigma[\mathcal{T}_{\overline{\mathbb{R}}}] = \{U \in \mathcal{P}[\overline{\mathbb{R}}] | U \bigcap \mathbb{R} \in \mathcal{B}[\mathbb{R}]\}$$ $\square$

Summarizing the above two lemmas in a theorem we get:

Theorem 20.156. *The Borel algebra on $\overline{\mathbb{R}}$ satisfies*

$$\begin{array}{rcl} \mathcal{B}[\overline{\mathbb{R}}] & \underset{\text{defintion}}{=} & \sigma[\mathcal{T}_{\overline{\mathbb{R}}}] \\ & \underset{\text{[theorem: 20.143]}}{=} & \sigma[\{C | C \text{ is closed in } \mathcal{T}_{\overline{\mathbb{R}}}\}] \\ & = & \sigma[\{]a, \infty] | a \in \mathbb{R}\}] \\ & = & \{U \subseteq \mathcal{P}[\overline{\mathbb{R}}] | U \bigcap \mathbb{R} \in \mathcal{B}[\mathbb{R}]\} \end{array}$$

Proof. This follows from [lemma: 20.155] and [lemma: 20.155]. $\square$

20.3.3 Borel algebra on $\mathbb{R}^n$

Definition 20.157. *Let $n \in \mathbb{N}$ then the Borel algebra on $\mathbb{R}^n$ is noted as $\mathcal{B}[\mathbb{R}^n]$ so*

$$\mathcal{B}[\mathbb{R}^n] = \sigma[\mathbb{R}^n, \mathcal{T}_{\|\|n}]$$

where $\mathcal{T}_{\|\|_n}$ is the canonical topology on $\mathbb{R}^n$ [see example: 14.95]

First we extend the concept of generalized intervals to intervals on $\mathbb{R}^n$

Definition 20.158. *Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$ then $x \leqslant_n y$ if $\forall i \in \{1, \ldots, n\}$ $x_i < y_i$*

This relation makes $\langle \mathbb{R}^n, \leqslant_n \rangle$ a partial ordered set.

Theorem 20.159. *Let $n \in \mathbb{N}$ then $\langle \mathbb{R}^n, \leqslant_n \rangle$ is a partial ordered set.*

***Note* 20.160.** *As for $(1,2)$, $(2,1)$ we that $(1,2) \nleqslant_2 (2,1)$ and $(2,1) \nleqslant_2 (1,2)$ $\leqslant_n$ is not a total or full order.*

Proof. We have

reflexivity. If $x \in \mathbb{R}^n$ then $\forall i \in \{1, \ldots, n\}$ we have $x_i = x_i \Rightarrow x_i \leqslant x_i$ proving that $x \leqslant_n x$.

anti-symmetry. If $x, y \in \mathbb{R}^n$ satisfies $x \leqslant y \wedge y \leqslant x$ then we have $\forall i \in \mathbb{N}$ we have $x_i \leqslant y_i$ and $y_i \leqslant x_i$ so that $x_i = y_i$ from which it follows that $x = y$.

transitivity. If $x \leqslant_n y$ and $y \leqslant_n z$ then we have $\forall i \in \{1, \ldots, n\}$ that $x_i \leqslant y_i \wedge y_i \leqslant z_i \Rightarrow x_i \leqslant z_i$ from which it follows that $x \leqslant_n z$. □

Using this order relation we can extend the concept of intervals to intervals in $\mathbb{R}^n$.

Definition 20.161. *Let $n \in \mathbb{N}_0$, $a, b \in \mathbb{R}^n$ such that $a \leqslant_n b$ then we have*

$$\begin{aligned}
[a, b] &= \{x \in \mathbb{R}^n | a \leqslant_n x \wedge x \leqslant_n b\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } a_i \leqslant x_i \wedge x_i \leqslant b_i\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } x_i \in [a_i, b_i]\} \\
&= \prod_{i \in \{1, \ldots, n\}} [a_i, b_i] \\
[a, b[&= \{x \in \mathbb{R}^n | a \leqslant_n x \wedge x <_n b\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } a_i \leqslant x_i \wedge x_i < b_i\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } x_i \in [a_i, b_i[\} \\
&= \prod_{i < \{1, \ldots, n\}} [a_i, b_i[\\
]a, b] &= \{x \in \mathbb{R}^n | a <_n x \wedge x \leqslant_n b\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } a_i < x_i \wedge x_i \leqslant b_i\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } x_i \in]a_i, b_i]\} \\
&= \prod_{i \in \{1, \ldots, n\}}]a_i, b_i] \\
]a, b[&= \{x \in \mathbb{R}^n | a <_n x \wedge x <_n b\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } a_i < x_i \wedge x_i < b_i\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } x_i \in]a_i, b_i[\} \\
&= \prod_{i < \{1, \ldots, n\}}]a_i, b_i[\\
]-\infty, a] &= \{x \in \mathbb{R}^n | x \leqslant_n a\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } x_i \leqslant a_i\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } x_i \in]-\infty, a_i]\} \\
&= \prod_{i < \{1, \ldots, n\}}]-\infty, a_i] \\
]-\infty, a[&= \{x \in \mathbb{R}^n | x < a\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } x_i < a_i\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } x_i \in]-\infty, a_i[\} \\
&= \prod_{i < \{1, \ldots, n\}}]-\infty, a_i[\\
[a, \infty[&= \{x \in \mathbb{R}^n | a \leqslant_n x\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } a_i \leqslant x_i\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } x_i \in [a_i, \infty[\} \\
&= \prod_{i < \{1, \ldots, n\}} [a_i, \infty[\\
]a, \infty[&= \{x \in \mathbb{R}^n | a <_n x\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } a_i < x_i\} \\
&= \{x \in \mathbb{R}^n | \forall i \in \{1, \ldots, n\} \text{ we have } x_i \in]a_i, \infty[\} \\
&= \prod_{i < \{1, \ldots, n\}}]a_i, \infty[
\end{aligned}$$

To simply notation we have also the following definitions:

Definition 20.162. *Let $n \in \mathbb{N}_0$ and $a, b \in \mathbb{R}$ then we define the following sets*

1. $[a, b]^n = \{x \in \mathbb{R}^n | a \leqslant x_i \leqslant b\}$
2. $[a, b[^n = \{x \in \mathbb{R}^n | a \leqslant x_i < b\}$
3. $]a, b]^n = \{x \in \mathbb{R}^n | a < x_i \leqslant b\}$
4. $]a, b[^n = \{x \in \mathbb{R}^n | a < x_i < b\}$
5. $]-\infty, a]^n = \{x \in \mathbb{R}^n | x_i \leqslant a\}$
6. $]-\infty, a[^n = \{x \in \mathbb{R}^n | x_i < a\}$
7. $[a, \infty[^n = \{x \in \mathbb{R}^n | a \leqslant x_i\}$
8. $]a, \infty[^n = \{x \in \mathbb{R}^n | a < x_i\}$

We specify now the sufficient and necessary condition for a interval in $\mathbb{R}^n$ to be empty.

Lemma 20.163.
Let $n \in \mathbb{N}$ and $a, b \in \mathbb{R}^n$ then we have

1. $[a, b[= \emptyset \Leftrightarrow \exists i \in \{1, \ldots, n\}$ *such that* $a_i \geqslant b_i$
2. $]a, b] = \emptyset \Leftrightarrow \exists i \in \{1, \ldots, n\}$ *such that* $a_i \geqslant b_i$
3. $]a, b[= \emptyset \Leftrightarrow \exists i \in \{1, \ldots, n\}$ *such that* $a_i \geqslant b_i$

Proof.

1. As $[a, b[= \prod_{i \in \{1, \ldots, n\}} [a_i, b_i[$ we have

   $\Rightarrow$. If $[a, b[= \emptyset$ then by [theorem 3.110] $\exists i \in \{1, \ldots, n\}$ such that $[a_i, b_i[= \emptyset$. If $a_i < b_i$ then by [theorem: 10.32] there exists a $x \in \mathbb{R}$ such that $a_i < x < b_i \Rightarrow x \in [a_i, b_i[$ contradicting $[a_i, b_i[= \emptyset$, so we must have $a_i \geqslant b_i$.

   $\Leftarrow$. If there exists a $i \in \{1, \ldots, n\}$ with $a_i \geqslant b_i$ then if $x \in [a_i, b_i[$ we have $a_i \leqslant x \wedge x < b_i \leqslant a_i$ leading to the contradiction $a_i < a_i$ so $[a_i, b_i[= \emptyset$. Hence using [theorem 3.110] $[a, b[= \varnothing$.

2.

   $\Rightarrow$. If $]a, b] = \emptyset$ then by [theorem 3.110] $\exists i \in \{1, \ldots, n\}$ such that $]a_i, b_i] = \emptyset$. If $a_i < b_i$ then by [theorem: 10.32] there exists a $x \in \mathbb{R}$ such that $a_i < x < b_i \Rightarrow x \in]a_i, b_i]$ contradicting $]a_i, b_i] = \emptyset$, so we must have $a_i \geqslant b_i$.

   $\Leftarrow$. If there exists a $i \in \{1, \ldots, n\}$ with $a_i \geqslant b_i$ then if $x \in]a_i, b_i]$ we have $a_i < x \wedge x \leqslant b_i \leqslant a_i$ leading to the contradiction $a_i < a_i$ so $]a_i, b_i] = \emptyset$. Hence using [theorem 3.110] $]a, b] = \varnothing$.

3.

   $\Rightarrow$. If $]a, b[= \emptyset$ then by [theorem 3.110] $\exists i \in \{1, \ldots, n\}$ such that $]a_i, b_i[= \emptyset$. If $a_i < b_i$ then by [theorem: 10.32] there exists a $x \in \mathbb{R}$ such that $a_i < x < b_i \Rightarrow x \in]a_i, b_i[$ contradicting $]a_i, b_i[= \emptyset$, so we must have $a_i \geqslant b_i$.

   $\Leftarrow$. If there exists a $i \in \{1, \ldots, n\}$ with $a_i \geqslant b_i$ then if $x \in]a_i, b_i[$ we have $a_i < x \wedge x < b_i \leqslant a_i$ leading to the contradiction $a_i < a_i$ so $]a_i, b_i[= \emptyset$. Hence using [theorem 3.110] $]a, b] = \varnothing$. □

To be able to use half open intervals in $\mathbb{R}^n$ of the form $[a, b]$ to generate the Borel algebra on $\mathbb{R}^n$ we introduce Dyadic intervals. We prove then that any open sets in $\langle \mathbb{R}^n, \mathcal{T} \rangle$ is a countable union of Diadic intervals. First we define the set of half open intervals on $\mathbb{R}^n$

Definition 20.164. *Let $n \in \mathbb{N}$ then*

$$\mathcal{R}^n = \{[a, b[| a, b \in \mathbb{R}^n \wedge a < b\}$$

***Note* 20.165.** *Note as $a < b$ we have by [lemma: 20.163] that $\forall i \in \{1, \ldots, n\}$ $a_i < b_i$ so that $[a_i, b_i[\neq \varnothing$*

The following lemma allows us to define a function that maps a element of $\mathcal{R}^n$ to its endpoints. Using this we can define then the volume of elements of $\mathcal{R}^n$.

Lemma 20.166. *Let $n \in \mathbb{N}$ and $\{x_i\}_{i \in \{1,\ldots,n\}} \subseteq \mathbb{R}^+$ then $\prod_{i \in \{1,\ldots,n\}} x_i \in \mathbb{R}^+$*

Proof. We prove this by induction, so let

$$S = \left\{ n \in \mathbb{N} | \text{If } \{x_i\}_{i \in \{1,\ldots,n\}} \subseteq \mathbb{R}^+ \text{ then } \prod_{i \in \{1,\ldots,n\}} x_o \right\}$$

then we have:

$1 \in \boldsymbol{S}$. If $\{x_i\}_{i \in \{1\}} \subseteq \mathbb{R}^+$ then $\prod_{i \in \{1\}} x_i = x_1 \in \mathbb{R}^+$ which proves that $1 \in S$.

$\boldsymbol{n \in S \Rightarrow n+1 \in S}$. Let $\{x_i\}_{i \in \{1,\ldots,n+1\}} \subseteq [0, \infty]$ then as $n \in S$ we have $\prod_{i=1}^n x_i \in \mathbb{R}^+$, as also $x_i \in \mathbb{R}^+$ it follows from [theorem: 10.14] that

$$0 \leqslant \left(\prod_{i \in \{1,\ldots,n\}} x_i \right) \cdot x_{n+1} = \prod_{i \in \{1,\ldots,n+1\}} x_i$$

so that $n + 1 \in \mathbb{N}$ □

Lemma 20.167. *Let $n \in \mathbb{N}$ and $I \in \mathcal{R}^n$ then there exist a **unique** a $a, b \in \mathbb{R}^n$ with $a < b$ such that $I = [a, b[$*

Proof. If $I \in \mathcal{R}^n$ then by definition of $\mathcal{R}^n$ there exist a $a, b \in \mathbb{R}^n$ with $a < b$ such that $I = [a, b[$. Assume that there exists also a $a', b' \in \mathbb{R}^n$ such that $I = [a', b'[$. Then $\prod_{i \in \{1,\ldots,n\}} [a_i, b_i[= \prod_{i \in \{1,\ldots,n\}} [a'_i, b'_i[$ and $\forall i \in \{1, \ldots, n\}$ $[a_i, b_i[\neq \varnothing$, hence using [theorem: 3.113] we have that

$$\forall i \in \{1, \ldots, n\} \text{ that } \varnothing \neq [a_i, b_i[= [a'_i, b'_i[$$

Let $i \in \{1, \ldots, n\}$ then as $\varnothing \neq [a_i, b_i[= [a'_i, b'_i[$ it follows from [theorem: 3.144] that $a_i = a'_i$ and $b_i = b'_i$ proving uniqueness.. □

Thanks to the above lemmas the following is well defined.

Definition 20.168. *Let $n \in \mathbb{N}$ then we define:*

1. $\text{begin}: \mathcal{R}^n \rightarrow \mathbb{R}$ *by* $\text{begin}(I) = a$ *where* $I = [a, b[$ *and* $a < b$
2. $\text{end}: \mathcal{R}^n \rightarrow \mathbb{R}$ *by* $\text{end}(I) = b$ *where* $I = [a, b[$ *and* $a < b$
3. $v^n: \mathcal{R}^n \rightarrow]0, \infty[= \mathbb{R}^n$ *where* $v^n(I) = \prod_{i \in \{1,\ldots,n\}} (\text{begin}(I)_i - \text{end}(I)_i)$

***Note* 20.169.** *As $\text{begin}(I) < \text{end}(I)$ we have $\forall i \in \{1, \ldots, n\}$*

$$\text{begin}(I)_i < \text{end}(I)_i \Rightarrow \text{end}(I)_i - \text{begin}(I)_i \in [0, \infty] \text{ so that by [lemma: 20.166]}$$

$v^n(R) \in \mathbb{R}^+$ as is needed in (3).

We introduce now Dyadic cubes that forms a countable subset of $\mathcal{R}^n$.

Definition 20.170. (Dyadic Cubes) *Let $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ then we define $\mathcal{D}^n_m$ by*

$$\mathcal{D}^n_m = \left\{ \left[\frac{z}{2^m}, \frac{z+1}{2^m} \right[| z \in \mathbb{Z}^n \right\} \subseteq \mathcal{R}^n$$

Elements of $\mathcal{D}^n_m$ are called Dyadic cubes of order m. Further $\mathcal{D}^n$ defined by

$$\mathcal{D}^n = \bigcup_{m \in \mathbb{N}_0} \mathcal{D}^n_m \subseteq \mathcal{R}^n$$

is the set of all Dyadic cubes.

***Note* 20.171.** *As $\frac{z}{2^m} < \frac{z+1}{2^m}$ we have that $\frac{z}{2^m} \in \left[\frac{z}{2^m}, \frac{z+1}{2^m} \right[$ so that Dyadic cubes are not empty.*

Lemma 20.172. *Let $k,l\in\mathbb{N}_0$ with $k\leqslant l$, $r,q\in\mathbb{Z}$ then for $\left[\frac{r}{2^l},\frac{r+1}{2^l}\right]$, $\left[\frac{q}{2^k},\frac{q+1}{2^k}\right[$ we have*

$$\left[\frac{r}{2^l},\frac{r+1}{2^l}\right[\bigcap\left[\frac{q}{2^k},\frac{q+1}{2^k}\right[\neq\emptyset\;\Rightarrow\;\left[\frac{r}{2^l},\frac{r+1}{2^l}\right[\subseteq\left[\frac{q}{2^k},\frac{q+1}{2^k}\right[$$
$$\left[\frac{r}{2^l},\frac{r+1}{2^l}\right[\bigcap\left[\frac{q}{2^k},\frac{q+1}{2^k}\right[=\emptyset\;\Rightarrow\;\left[\frac{r}{2^l},\frac{r+1}{2^l}\right[\nsubseteq\left[\frac{q}{2^k},\frac{q+1}{2^k}\right[$$

in other words. We have either $\left[\frac{r}{2^l},\frac{r+1}{2^l}\right[\bigcap\left[\frac{q}{2^k},\frac{q+1}{2^k}\right[=\emptyset$ or $\left[\frac{r}{2^l},\frac{r+1}{2^l}\right[\subseteq\left[\frac{q}{2^k},\frac{q+1}{2^k}\right[$ but not both.

Proof. Take $I=\left[\frac{q}{2^l},\frac{q+1}{2^l}\right[$ and $J=\left[\frac{r}{2^k},\frac{r+1}{2^k}\right[$. As $k\leqslant l$ we have $0\leqslant l-k=n$. Next

$$\begin{aligned} J &= \left[\frac{r}{2^k},\frac{r+1}{2^k}\right[\\ &= \left[\frac{r}{2^k}\cdot\frac{2^{l-k}}{2^{l-k}},\frac{r+1}{2^k}\cdot\frac{2^{l-k}}{2^{l-k}}\right[\\ &= \left[\frac{r\cdot 2^{l-k}}{2^l},\frac{r\cdot 2^{l-k}+2^{l-k}}{2^l}\right[\\ &= \left[\frac{r\cdot 2^n}{2^l},\frac{r\cdot 2^n+2^n}{2^l}\right[\end{aligned}\tag{20.131}$$

Assume that $I\bigcap J\neq\emptyset$ so that $\exists x\in I\bigcap J$ giving

$$\frac{q}{2^l}\leqslant x<\frac{q+1}{2^l}\wedge\frac{r\cdot 2^n}{2^l}\leqslant x<\frac{r\cdot 2^n+2^n}{2^l}\tag{20.132}$$

If now $\frac{q+1}{2^l}\leqslant\frac{r\cdot 2^n}{2^l}$ then $x<\frac{q+1}{2^l}\leqslant\frac{r\cdot 2^n}{2^l}\leqslant x$ giving the contradiction $x<x$ so we must have

$$\frac{r\cdot 2^n}{2^l}<\frac{q+1}{2^l}\tag{20.133}$$

Further if $\frac{r\cdot 2^n+2^n}{2^l}\leqslant\frac{q}{2^l}$ then $x<\frac{r\cdot 2^n+2^n}{2^l}\leqslant\frac{q}{2^l}\leqslant x$ giving the contradiction $x<x$ again so we must have that

$$\frac{q}{2^l}<\frac{r\cdot 2^n+2^n}{2^l}\tag{20.134}$$

Multiplying 20.133 and 20.134 by 2^l gives $r\cdot 2^n<q+1\wedge q<r\cdot 2^n+2^n$. As $r\cdot 2^n, q+1, r\cdot 2^n+2^n\in\mathbb{Z}$ we have by [theorem: 10.28] that $r\cdot 2^n+1\leqslant q+1\wedge q+1\leqslant r\cdot 2^n+2^n\Rightarrow r\cdot 2^n\leqslant q\wedge q+1\leqslant r\cdot 2^n+2^n$. Dividing by 2^l gives:

$$\frac{r\cdot 2^n}{2^l}\leqslant\frac{q}{2^l}\wedge\frac{q+1}{2^l}\leqslant\frac{r\cdot 2^n+2^n}{2^l}\tag{20.135}$$

Next if $x\in I\bigcap J$ then $\frac{q}{2^l}\leqslant x<\frac{q+1}{2^l}\underset{20.135}{\Rightarrow}\frac{r\cdot 2^n}{2^l}\leqslant x<\frac{r\cdot 2^n+2^n}{2^l}$ proving using 20.131 that $x\in J$. Hence we have

$$\text{If } I\bigcap J\neq\emptyset \text{ then } I\subseteq J$$

Finally if $I\bigcap J=\emptyset$ then if $I\subseteq J$ we have $\emptyset=I\bigcap J=I$ a contradiction as $\frac{q}{2^l}\in I$. Hence we have either $I\bigcap J=\emptyset$ or $I\subseteq J$ but not both. □

Theorem 20.173. *Let $n\in\mathbb{N}$ then we have*

1. *$\forall m\in\mathbb{N}_0$ and $\forall R,Q\in\mathcal{D}_m^n$ with $R\neq Q$ we have $R\bigcap Q=\varnothing$*
2. *Let $k,l\in\mathbb{N}_0$ with $k\leqslant l$ then if $R\in\mathcal{D}_l^n$ and $R\in\mathcal{D}_k^n$ we have that*

$$R\bigcap Q\neq\varnothing\;\Rightarrow\;R\subseteq Q$$
$$R\bigcap Q=\varnothing\;\Rightarrow\;R\nsubseteq Q$$

3. *$\forall m\in\mathbb{N}_0$ we have that $\mathbb{R}^n=\bigsqcup_{R\in\mathcal{D}_m^n}R$ [here $\bigsqcup_{R\in\mathcal{D}_m^n}R$ is a disjoint union because of (1)].*

4. *$\forall m \in \mathbb{N}_0$ and $\forall R \in \mathcal{D}_m^n$ we have that $v^n(R) = \frac{1}{2^{n \cdot m}} < 1$*

Proof.

1. Let $m \in \mathbb{N}_0$ and $R, Q \in \mathcal{D}_m^n$ with $R \neq Q$. Then there exist $r, z \in \mathbb{Z}^n$ such that

$$R = \left[\frac{r}{2^m}, \frac{r+1}{2^m}\right[= \prod_{i \in \{1, \ldots, n\}} \left[\frac{r_i}{2^m}, \frac{r_i}{2^m}\right[\text{ and } Q = \left[\frac{z}{2^m}, \frac{z+1}{2^m}\right[= \prod_{i \in \{1, \ldots, n\}} \left[\frac{z_i}{2^m}, \frac{z_i+1}{2^m}\right[$$

As $R \neq Q$ there exist a $i \in \{1, \ldots, n\}$ such that $\left[\frac{r_i}{2^m}, \frac{r_i+1}{2^m}\right[\neq \left[\frac{z_i}{2^m}, \frac{z_i+1}{2^m}\right[$, hence we must have that $r_i \neq z_i$. We can always assume that $r_i < z_i$ [if not exchange R and Q] so that $r_i + 1 \leqslant z_i$ [as $r_i, z_i \in \mathbb{Z}$ we can use [theorem: 10.28]]. Assume that $x \in \left[\frac{r_i}{2^m}, \frac{r_i+1}{2^m}\right[\bigcap \left[\frac{z_i}{2^m}, \frac{z_i+1}{2^m}\right[$ then $\frac{r_i}{2^m} \leqslant x < \frac{r_i+1}{2^m} \leqslant \frac{z_i}{2^m} \leqslant x < \frac{z_i+1}{2^m}$ leading to the contradiction $x < x$ hence we must have that $\left[\frac{r_i}{2^m}, \frac{r_i+1}{2^m}\right[\bigcap \left[\frac{z_i}{2^m}, \frac{z_i+1}{2^m}\right[= \varnothing$. Using [theorem: 3.110] it follows that $\prod_{i \in \{1, \ldots, n\}} \left[\frac{r_i}{2^m}, \frac{r_i}{2^m}\right[\bigcap \prod_{i \in \{1, \ldots, n\}} \left[\frac{z_i}{2^m}, \frac{z_i+1}{2^m}\right[$ or

$$R \bigcap Q = \varnothing$$

2. Let $k, l \in \mathbb{N}_0$ with $k \leqslant l$ and $R \in \mathcal{D}_l^n$, $Q \in \mathcal{D}_k^n$ then there exists $r, q \in \mathbb{Z}^n$ with $R = \left[\frac{r}{2^m}, \frac{r+1}{2^m}\right[$, $Q = \left[\frac{z}{2^m}, \frac{z+1}{2^m}\right[$. Assume that $R \nsubseteq Q$ then we have either

   $\exists i \in \{1, \ldots, n\} \vDash \left[\frac{r_i}{2^m}, \frac{r_i+1}{2^m}\right[\bigcap \left[\frac{z_i}{2^m}, \frac{z_i+1}{2^m}\right[= \varnothing$. Then using [theorem: 3.110] we have $R \bigcap Q = \varnothing$

   $\exists i \in \{1, \ldots, n\} \vDash \left[\frac{r_i}{2^m}, \frac{r_i+1}{2^m}\right[\bigcap \left[\frac{z_i}{2^m}, \frac{z_i+1}{2^m}\right[\neq \varnothing$. Then using [lemma: 20.172] we have $\forall i \in \{1, \ldots, n\}$ that $\left[\frac{r_i}{2^m}, \frac{r_i+1}{2^m}\right[\subseteq \left[\frac{z_i}{2^m}, \frac{z_i+1}{2^m}\right[$. Applying [theorem: 3.111] proves that $R \subseteq Q$ contradicting $R \nsubseteq Q$. So this case never applies.

   So we conclude that if $R \nsubseteq \mathbb{Q}$ then $R \bigcap Q = \varnothing$ or using contraposition we have

$$\text{If } R \bigcap Q \neq \varnothing \text{ then } R \subseteq Q$$

   On the other hand if $R \bigcap Q = \varnothing$ and $R \subseteq Q$ then $\varnothing \neq R = R \bigcap Q = \varnothing$ s controdiction so we have

$$\text{If } R \bigcap Q = \varnothing \text{ then } R \nsubseteq Q$$

3. Let $m \in \mathbb{N}$ and $x \in \mathbb{R}^n$ then $\forall i \in \{1, \ldots, n\}$ we have $x_i \in \mathbb{R}$. Using [theorem: 10.30] there exist a $z_i \in \mathbb{Z}$ such that $z_i \leqslant x_i \cdot 2^m < z_i + 1$, hence $\frac{z_i}{2^m} \leqslant x_i < \frac{z_i+1}{2^m}$. So if we take $z = (z_1, \ldots, z_m)$ then $x \in \prod_{i \in \{1, \ldots, n\}} \left[\frac{z_i}{2^m}, \frac{z_i+1}{2^m}\right[= \left[\frac{z}{2^m}, \frac{z+1}{2^m}\right[\in \mathcal{D}_m^n$ proving that $\mathbb{R}^n \subseteq \bigcup_{R \in \mathcal{D}_m^n} R$. As $\forall R \in \mathcal{D}_m^n$ $R \subseteq R^n$ we have proved that

$$R^n = \bigcup_{R \in \mathcal{D}_m^n} R$$

   Finally by (1) we have that $\forall R, R' \in \mathcal{D}_m^n$ with $R \neq R'$ we have $R \bigcap R' = \varnothing$ so that

$$R^n = \bigsqcup_{R \in \mathcal{D}_m^n} R$$

4. If $R \in \mathcal{R}_m^n$ then $\exists z \in \mathbb{Z}^n$ such that $R = \prod_{i \in \{1, \ldots, n\}} \left[\frac{z_i}{2^n}, \frac{z_i+1}{2^n}\right[$ so that

$$v^n(R) = \prod_{i=1}^{n} \left(\frac{z_i + 1 - z_i}{2^m}\right) = \prod_{i=1}^{n} \frac{1}{2^m} = \frac{1}{2^{n \cdot m}}$$ □

One of the reasons that we introduce the Dyadic cubes as a subset of $\mathcal{R}^n$ is that $\mathcal{D}_m^n$ is a denumerable set in contrast with $\mathcal{R}^n$ which we need giving the definition of a σ-algebra. This is proved in the next lemma.

Lemma 20.174. *Let $n\in\mathbb{N}$ then*

1. *$\forall m\in\mathbb{N}_0$ $\mathcal{D}_m^n$ is denumerable.*
2. *$\mathcal{D}^n$ is denumerable*

Proof.

1. Let $m\in\mathbb{N}_0$ then as $\mathbb{Z}$ is denumerable [see theorem: 10.5] we have by [theorem: 6.67] that $\mathbb{Z}^n$ is also denumerable. Define now $\beta:\mathbb{Z}^n\rightarrow\mathcal{D}_m^n$ by $\beta(z)=\left[\frac{z}{2^m},\frac{z+1}{2^m}\right[$ then β is a bijection.

   injectivity. If $\beta(z)-\beta(w)$ then $\left[\frac{z}{2^m},\frac{z+1}{2^m}\right[=\left[\frac{w}{2^m},\frac{w+1}{2^m}\right[$. Using the fact that a Dyadic cube is not empty together with [lemma: 20.167] gives $\frac{z}{2^m}=\frac{w}{2^m}$ proving that $z=w$.

   surjectivity. This follows from the definition of $\mathcal{D}_m^n$

   So $\mathcal{D}_m^n$ is denumerable.

2. Using [theorem: 6.69] it follows that $\mathcal{D}^n=\bigcup_{m\in\mathbb{N}}\mathcal{D}_m^n$ is denumerable □

We are now finally ready to prove that every non empty open set on $\mathbb{R}^n$ can be written as a denumerable union of pairwise disjoint Dyadic cubes.

Lemma 20.175. *Let $n\in\mathbb{N}$, $\mathcal{T}_{\|\|_n}$ the canonical topology on $\mathbb{R}$ [see example: 14.95] then if $\varnothing\neq U\in\mathcal{T}_{\|\|_n}$ there exist a* ***pairwise disjoint family*** *$\{D_i\}_{i\in\mathbb{N}}\subseteq\mathcal{D}^n$ such that $U=\bigsqcup_{i\in\mathbb{N}}D_i$.*

Proof. Let $\varnothing\neq U\in\mathcal{T}_{\|\|_n}$ be a non empty open set in $\mathbb{R}^n$. Given $m\in\mathbb{N}_0$ define

$$\mathcal{S}_m=\{D\in\mathcal{D}_m^n|D\subseteq U\}\subseteq\mathcal{D}_m^n \tag{20.136}$$

and recursively

$$\mathcal{T}_m=\begin{cases} S_0 \text{ if } m=0\\ \{D\in\mathcal{S}_m|\forall i\in\{0,\ldots,m-1\} \text{ we have } \forall R\in\mathcal{T}_i \text{ that } R\bigcap D=\varnothing\} \text{ if } m\in\mathbb{N}\end{cases} \tag{20.137}$$

Using the above definitions we have the following inclusions

$$\forall m\in\mathbb{N}_0 \text{ we have } \mathcal{T}_m\subseteq\mathcal{S}_m\subseteq\mathcal{D}_m^n\subseteq\mathcal{D}^n \tag{20.138}$$

Further we define

$$\mathcal{T}=\bigcup_{m\in\mathbb{N}_0}\mathcal{T}_m \tag{20.139}$$

If $D\in\mathcal{D}$ then there exist a $i\in\mathbb{N}_0$ such that $D\in\mathcal{T}_i\subseteq\mathcal{S}_i$ proving by [eq: 20.136] that $D\subseteq U$, hence we have

$$\bigcup_{D\in\mathcal{T}}D\subseteq U \tag{20.140}$$

Proving the opposite inclusion is a little more difficult. Let $x\in U$ then there exist a $\varepsilon\in\mathbb{R}^+$ such that $x\in B_{\|\|_n}(x,\varepsilon)\subseteq U$. Using the Archimedean property of the real numbers [see theorem: 10.30] there exist $n_\varepsilon\in\mathbb{N}$ such that

$$0<\frac{1}{n_x}<\varepsilon \tag{20.141}$$

Using [theorem: 20.173 (3)] there exist a $D_x\in\mathcal{D}_{n_x}^n$ such that

$$D_x=\left[\frac{z}{2^{n_x}},\frac{z+1}{2^{n_x}}\right[=\prod_{i\in\{1,\ldots,n\}}\left[\frac{z_i}{2^{n_x}},\frac{z_i+1}{2^{n_x}}\right[\in\mathcal{D}_{n_x}^n \text{ such that } x\in D_x$$

from which it follows that

$$\forall i\in\{1,\ldots,n\} \text{ we have } \frac{z_i}{2^{n_x}}\leqslant x_i<\frac{z_i+1}{2^{n_x}} \tag{20.142}$$

Let $y\in D_x$ then $y\in\prod_{i\in\{1,\ldots,n\}}\left[\frac{z_i}{2^{n_x}},\frac{z_i+1}{2^{n_x}}\right[$ so that

$$\forall i\in\{1,\ldots,n\} \text{ we have } \frac{z_i}{2^{n_x}}\leqslant y_i<\frac{z_i+1}{2^{n_x}} \tag{20.143}$$

Hence we have

$$\begin{aligned} y_i - x_i &\underset{[\text{eq: } 20.143}{<} \frac{z_i+1}{2^{n_x}} - x_i \\ &\underset{[\text{eq: } 20.142]}{\leqslant} \frac{z_i+1}{2^{n_x}} - \frac{z_i}{2^{n_x}} \\ &= \frac{1}{2^{n_x}} \\ &\underset{[\text{eq: } 20.141}{<} \varepsilon \\ x_i - y_i &\underset{[\text{eq: } 20.142]}{<} \frac{z_i+1}{2^{n_x}} - y_i \\ &\underset{[\text{eq: } 20.143}{\leqslant} \frac{z_i+1}{2^{n_x}} - \frac{z_i}{2^{n_x}} \\ &\underset{[\text{eq: } 20.141}{<} \varepsilon \end{aligned}$$

which proves that $\forall i \in \{1, \ldots, n\}$ $|y_i - x_i| < \varepsilon$. Hence $\|y - x\|_n = \max(\{|y_i - x_i| | i \in \{1, \ldots, n\}\}) < \varepsilon$ so that $y \in B_{\|\|_n}(x, \varepsilon)$. In conclusion

$$x \in D_x \subseteq B_{\|\|_n}(x, \varepsilon) \subseteq U$$

As $D_x \in \mathcal{D}^n_{n_x}$ it follows from the above and [eq: 20.136] that

$$x \in D_x \in S_{n_x} \tag{20.144}$$

Define now

$$\mathcal{K}_x = \{i \in \{0, \ldots, n_x\} | \exists D \in \mathcal{S}_i | x \in D\} \subseteq \{0, \ldots, n_x\} \tag{20.145}$$

Then by [eq: 20.144] $n_x \in \mathcal{K}_x$ so that $\mathcal{K}_x \neq \varnothing$, as further $\mathcal{K}_x$ is finite it follows that

$$m_x = \min(\mathcal{K}_x) \text{ exist} \tag{20.146}$$

For $m_x \in \mathcal{K}_x \subseteq \{0, \ldots, n_x\}$ we have now two possible cases:

$\boldsymbol{m_x = 0}$**.** then $\exists D \in S_0$ such that $x \in D$. As by [eqs: 20.137, 20.139] $S_0 = \mathcal{T}_0$ and $\mathcal{T}_0 \in \mathcal{T}$ we that $x \in D \in \mathcal{T}$. In other words $x \in \bigcup_{D \in \mathcal{T}} D$.

$\boldsymbol{m_x \in \{1, \ldots, n_x\}}$**.** Then $\exists D \in \mathcal{S}_{m_x} \underset{[\text{eq: } 20.136]}{\subseteq} \mathcal{D}^n_{m_x}$ such that $x \in D$ and $\forall i \in \{0, \ldots, m_x - 1\}$ we have that $\forall R \in \mathcal{S}_i$ that $x \notin R$. Let $i \in \{0, \ldots, m_x - 1\}$ and assume that $\exists R \in \mathcal{S}_i \subseteq \mathcal{D}^n_i$ such that $R \bigcap D \neq \varnothing$ then by [theorem: 20.173 (2)] and the fact that $i \leqslant m_x$ it follows that $D \subseteq R$, as $x \in D$ it follows that $x \in R$ contradicing $x \notin R$ hence the assumption is wrong and we must have that $\forall R \in \mathcal{S}_i$ we have that $R \bigcap D = \varnothing$. As this is true $\forall i \in \{0, \ldots, m_x - 1\}$ it follows from [definition: 20.137] that $D \in \mathcal{T}_{m_x} \underset{[\text{eq: } 20.139]}{\subseteq} \mathcal{T}$. In other words we have $x \in D \in \mathcal{T}$ which proves that $x \in \bigcup_{D \in \mathcal{T}} D$.

So in all cases we have that $x \in \bigcup_{D \in \mathcal{T}} D$ which as $x \in U$ was chosen arbitrary proves that $U \subseteq \bigcup_{D \in \mathcal{T}} D$. Combining this with [eq: 20.139] proves that

$$U = \bigcup_{D \in \mathcal{T}} D \tag{20.147}$$

Next we must prove that U is a pairwise disjoint union. Let $D, E \in \mathcal{T} = \bigcup_{m \in \mathbb{N}_0} \mathcal{T}_m$ with $D \neq E$ then there exist $k, l \in \mathbb{N}_0$ such that $D \in \mathcal{T}_k \subseteq \mathcal{D}^n_k$ and $E \in \mathcal{T}_l \subseteq \mathcal{D}^n_l$. For k, l we have the following possible cases:

$\boldsymbol{k = l}$**.** Then using [theorem: 20.173 (1)] it follows that $D \bigcap E = \varnothing$.

$\boldsymbol{k < l}$**.** Then using [eq: 20.137] that $D \bigcap E = \varnothing$.

$\boldsymbol{l < k}$**.** Then using [eq: 20.137] that $D \bigcap E = \varnothing$.

combining the above with [eq: 20.147] gives then

$$\forall D, E \in \mathcal{T} \text{ with } D \neq E \text{ we have } D \bigcap E = \varnothing \tag{20.148}$$

Next we prove that $\mathcal{T}$ is denumerable, first we prove that $\mathcal{T}$ is not finite (hence by definition infinite). Assume that $\mathcal{T}$ is finite. As $U\neq\varnothing$ we must have that $\mathcal{T}\neq\varnothing$ [because $\bigcup_{D\in\varnothing} D=\varnothing$] so that $m=\min(\{(\text{begin}(D))_1|D\in\mathcal{T}\})$ exist. Hence there exist a $R=[a,b[\in\mathcal{T}$ such that $a_1=m$. As $a\in[a,b[=R$ and by [eq: 20.148] $R\subseteq U$ it follows that $a\in U$. Hence there exist a $\varepsilon\in\mathbb{R}^+$ such that $a\in B_{\|\|_n}(a,\varepsilon)\subseteq U$. Define e by $(e)_i=\left\{\begin{smallmatrix} 1 & \text{i=1} \\ 0 & \text{if } i\in\{2,\ldots,n\}\end{smallmatrix}\right.$ then for $i\in\{1,\ldots,n\}$ we have

$$\left|\left(a-\left(a-\frac{\varepsilon}{2}\cdot e\right)\right)_i\right|=\left|\frac{\varepsilon}{2}\cdot e_i\right|=\left|\frac{\varepsilon}{2}\right|\cdot\left|\begin{cases} 1 \text{ if } i=1 \\ 0 \text{ if } i\in\{2,\ldots,n\}\end{cases}\right|\leqslant\frac{\varepsilon}{2}<\varepsilon$$

so that $\left\|a-\left(a-\frac{\varepsilon}{2}\cdot e\right)\right\|_n<\varepsilon$. Hence $a-\frac{\varepsilon}{2}\cdot e\in B_{\|\|_n}(a,\varepsilon)\subseteq U$ then by [eq: 20.148] there exist a $D'\in\mathcal{T}$ such that $a-\frac{\varepsilon}{2}\cdot e\in D'$. As $D'=[\text{begin}(D'),\text{end}(D')[$ it follows that $\text{begin}(D')\leqslant a-\frac{\varepsilon}{2}\cdot e$ so that $(\text{begin}(D'))_1\leqslant\left(a-\frac{\varepsilon}{2}\cdot e\right)_1=a_1-\frac{\varepsilon}{2}\cdot 1=m-\frac{\varepsilon}{2}$. As $m=\min(\{(\text{begin}(D))_1|D\in\mathcal{T}\})$ $m\leqslant(\text{begin}(D'))_1$ which leads to the contradiction $m\leqslant m-\frac{\varepsilon}{2}<m$, so the assumption is wrong and we must have that

$$\mathcal{T} \text{ is not finite or } \mathcal{T} \text{ is infinite} \tag{20.149}$$

As $\mathcal{D}^n$ is denumerable [see theorem: 20.174] and $\mathcal{T}\underset{\text{[eq: 20.138]}}{\subseteq}\mathcal{D}^n$ it follows from the above and [theorem: 6.60] that

$$\mathcal{T} \text{ is denumerable}$$

So there exist a bijection $\beta:\mathbb{N}\rightarrow\mathcal{T}$. define then $\{D_i\}_{i\in\mathbb{N}}\subseteq\mathcal{T}\subseteq\mathcal{D}^n$ by $D_i=\beta(i)$ then we have

$$\bigcup_{i\in\mathbb{N}} D_i\underset{\text{[theorem: 2.119]}}{=}\bigcup_{D\in\mathcal{T}} D=U$$

Further if $i,j\in\mathbb{N}$ with $i\neq j$ then as β is a bijection, hence injective, it follows that $D_i\neq D_j$, so that using [eq: 20.148] we have $D_i\bigcap D_j=\varnothing$. Hence

$$U=\bigsqcup_{i\in\mathbb{N}} D_i$$

□

Lemma 20.176. *Let $n\in\mathbb{N}$ and $R\in\mathcal{R}^n$ with $R\neq\varnothing$ then there exist a sequence $\{U_i\}_{i\in\mathbb{N}}\subseteq\mathcal{T}_{\|\|_n}$ of open sets such that $R=\bigcap_{i\in\mathbb{N}} U_i$*

Proof. As $R\in\mathcal{R}^n$ there exists a $a,b\in\mathbb{R}^n$ with $a\leqslant b$ such that $R=[a,b[=\prod_{i\in\{1,\ldots,n\}}[a_i,b_i[$. Define then $\{U_i\}_{i\in\mathbb{N}}$ by $U_i=\prod_{j\in\{1,\ldots,n\}}\left]a_j-\frac{1}{j},b_j\right[$. As $\left]a_j-\frac{1}{i},b_j\right[\in\mathcal{T}_{||}$ [see theorem: 14.99] and $\mathcal{T}_{\|\|_n}$ is the product topology based on $\mathcal{T}_{||}$ [see 14.95] it follows that $\{U_i\}_{i\in\mathbb{N}}\subseteq\mathcal{T}_{\|\|_n}$. Let $i\in\mathbb{N}$ and $x\in R$ then we have $\forall j\in\{1,\ldots,n\}$ that $x_j\in[a_j,b_j[\Rightarrow a_j\leqslant x_j<b_j\Rightarrow a_j-\frac{1}{i}<a_j\leqslant x_j<b_j\Rightarrow x_j\in\left]a_j-\frac{1}{i},b_j\right[$, proving using [theorem: 3.111] that $R\subseteq U_i$. Hence we have

$$R\subseteq\bigcap_{i\in\mathbb{N}} U_i \tag{20.150}$$

For the opposite inequality let $x\in\bigcap_{i\in\mathbb{N}} U_i$ then $\forall i\in\mathbb{N}$ we have $x\in U_i$ so that $\forall j\in\{1,\ldots,n\}$ we have $x_j\in\left]a_j-\frac{1}{i},b_j\right[\Rightarrow a_j-\frac{1}{i}<x_j<b_j$. Assume that $\exists j\in\{1,\ldots,n\}$ such that $x_j<a_j$ then using [theorem: 10.30] there exists a $i\in\mathbb{N}$ such that $x_j<x_j+\frac{1}{i}<a_j\Rightarrow x_j<a_j-\frac{1}{i}$ contradicting the fact that $a_j-\frac{1}{i}<x_j$. Hence $\forall j\in\{1,\ldots,n\}$ we have $a_j\leqslant x_j<b_j$ proving that $x\in R$ so that $\bigcap_{i\in\mathbb{N}} U_i\subseteq R$. This together with [eq: 20.150] proves

$$R=\bigcap_{i\in\mathbb{N}} U_i$$

□

We are ready now to prove a alternative definition for the Borel algebra on $\mathbb{R}^n$.

Theorem 20.177. *Let $n\in\mathbb{N}$ then we have*

$$\sigma[\{C\subseteq\mathbb{R}^n|C \textit{ is close in } \mathcal{T}_{\|\|_n}\}]=\mathcal{B}[\mathbb{R}^n]=\sigma[\mathcal{R}^n]$$

Proof. Using [theorem: 20.143] we have

$$\sigma[\{C\subseteq\mathbb{R}^n|C \text{ is close in } \mathcal{T}_{\|\|_n}\}]\underset{\text{[theorem: 20.143]}}{=}\mathcal{B}[\mathbb{R}^n,\mathcal{T}_{\|\|_n}]\underset{\text{def}}{=}\mathcal{B}[\mathbb{R}^n]$$

so we only have to prove that $\mathcal{B}[\mathbb{R}^n, \mathcal{T}_{\|\|_n}] = \sigma[\mathcal{R}^n]$. Let $U \in \mathcal{T}_{\|\|_n}$ then we have either:

$U = \varnothing$. Then $U = \varnothing \in \sigma[\mathcal{R}^n]$.

$U \neq \varnothing$. Then using [lemma: 20.175] there exist a sequence $\{D_i\}_{i\in\mathbb{N}} \subseteq \mathcal{D}^n \subseteq \mathcal{R}^n$ such that $U = \bigcup_{i\in\mathbb{N}} D_i$ which as $\sigma[\mathcal{R}^n]$ is a σ-algebra proves that $U \in \sigma[\mathcal{R}^n]$

Hence we have $\mathcal{T}_{\|\|_n} \subseteq \sigma[\mathcal{R}^n]$ which by [theorem: 20.141] gives

$$\sigma[\mathcal{T}_{\|\|_n}] \subseteq \sigma[\mathcal{R}^n] \tag{20.151}$$

For the opposite inclusion let $R \in \mathcal{R}^n$ then by [lemma: 20.176] there exist a $\{U_i\}_{i\in\mathbb{N}} \subseteq \mathcal{T}_{\|\|_n}$ such that $R = \bigcap_{i\in\mathbb{N}} U_i$. As $\sigma[\mathcal{T}_{\|\|_n}]$ is a σ-alebra we have by [theorem: 20.138] that $R \in \sigma[\mathcal{T}_{\|\|_n}]$. Hence $\mathcal{R}^n \subseteq \sigma[\mathcal{T}_{\|\|_n}]$ which by [theorem: 20.141] gives $\sigma[\mathcal{R}^n] \subseteq \sigma[\mathcal{T}_{\|\|_n}]$. Combining this with [eq: 20.151] gives

$$\sigma[\mathcal{R}^n] = \sigma[\mathcal{T}_{\|\|_n}] \underset{\text{def}}{=} \mathcal{B}[\mathbb{R}^n, \mathcal{T}_{\|\|_n}] = \mathcal{B}[\mathbb{R}^n]$$ □

20.4 Measures

20.4.1 Measure definition and properties

For a measurable space we have some simplication in the definitions of additivity, sub-additivity, countable additivity and countable sub-additivity bexause of the extra properties of a σ-algebra.

Theorem 20.178. *Let $\langle X, \mathcal{A}\rangle$ be a measurable space [see definition: 20.134] and $\mu: \mathcal{A} \to [0, \infty[$ a function then we say that:*

1. *μ is **finite additive** $\Leftrightarrow$ for every **pairwise disjoint** families $\{A_i\}_{i\in\{1,\ldots,n\}} \subseteq \mathcal{A}$ where $n \in \mathbb{N}$ we have*

$$\mu\left(\bigsqcup_{i\in\{1,\ldots,n\}} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

2. *μ is **finite sub-additive** $\Leftrightarrow$ for every $\{A_i\}_{i\in\{1,\ldots,n\}} \subseteq \mathcal{A}$, $n \in \mathbb{N}$ we have that*

$$\mu\left(\bigcup_{i\in\{1,\ldots,n\}} A_i\right) \leqslant \sum_{i=1}^{n} \mu(A_i)$$

3. *μ is **countable additive** $\Leftrightarrow$ for all **pairwise disjoint** $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$ we have that*

$$\mu\left(\bigsqcup_{i\in\mathbb{N}} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

4. *μ is **countable sub-additive** if $\Leftrightarrow$ for all $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$ we have*

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leqslant \sum_{i=1}^{\infty} \mu(A_i)$$

Proof.

1. This follows from the fact that $\forall \{A_i\}_{i\in\{1,\ldots,n\}} \subseteq \mathcal{A}$ we have as $\mathcal{A}$ is a σ-algebra that $\bigcup_{i\in\{1,\ldots,n\}} A_i \in \mathcal{A}$ and the definition of additivity [see definition: 20.116].
2. This follows from the fact that $\forall \{A_i\}_{i\in\{1,\ldots,n\}} \subseteq \mathcal{A}$ we have as $\mathcal{A}$ is a σ-algebra that $\bigcup_{i\in\{1,\ldots,n\}} A_i \in \mathcal{A}$ and the definition of sub-additivity [see definition: 20.117].
3. This follows from the fact that $\forall \{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$ we have as $\mathcal{A}$ is a σ-algebra that $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{A}$ and the definition of countable additivity [see definition: 20.118].
4. This follows from the fact that $\forall \{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$ we have as $\mathcal{A}$ is a σ-algebra that $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{A}$ and the definition of countable sub-additivity [see definition: 20.119]. □

Using mathematical induction it is easy to specify a neccesary and sufficient condition for finite additivity and finite sub additivity.

Lemma 20.179. *Let $\langle X, \mathcal{A}\rangle$ be a measurable space and $\mu: \mathcal{A} \rightarrow [0, \infty]$ a function then we have*

1. *μ is finate additive $\Leftrightarrow$ $\forall A, B \in \mathcal{A}$ with $A \bigcap B = \emptyset$ we have $\mu(A \bigcup B) = \mu(A) + \mu(B)$*
2. *μ is finite sub additive $\Leftrightarrow$ $\forall A, B \in \mathcal{A}$ we have $\mu(A \bigcup B) \leqslant \mu(A) + \mu(B)$*

Proof.

1.

$\Rightarrow$. Let $A, B \in \mathcal{A}$ with $A \bigcap B = \emptyset$ then $\{A_i\}_{i \in \{1, \ldots, 2\}}$ defined by $A_i = \left\{ \begin{smallmatrix} A \text{ if } i = 1 \\ B \text{ if } i = 2 \end{smallmatrix} \right.$ is pairwise disjoint and $\{A_i\}_{i \in \{1, \ldots, 2\}} \subseteq \mathcal{A}$. So by assumption we have that

$$\mu\left(A \bigcup B\right) = \mu\left(\bigcup_{i \in \{1, \ldots, 2\}} A_i\right) = \sum_{i=1}^{2} \mu(A_i) = \mu(A_1) + \mu(A_2) = \mu(A) + \mu(B)$$

$\Leftarrow$. Let

$$\mathcal{S} = \left\{ n \in \mathbb{N}_0 | \text{If } \{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{A} \text{ is a pairwise disjoint family then we have that } \mu\left(\bigcup_{i \in \{1, \ldots, n\}} A_i\right) = \sum \mu(A_i) \right\}$$

then we have:

$1 \in \mathcal{S}$. Let $\{A_i\}_{i \in \{1, \ldots, 1\}} \subseteq \mathcal{A}$ then $\mu\left(\bigcup_{i \in \{1, \ldots, 1\}} A_i\right) = \mu(A_1) = \sum_{i=1}^{1} \mu(A_i)$ proving that $1 \in \mathcal{S}$

$n \in \mathcal{S} \Rightarrow n + 1 \in \mathcal{S}$. Let $\{A_i\}_{i \in \{1, \ldots, n+1\}} \subseteq \mathcal{A}$ be pairwise disjoint. Take $A = \bigcup_{i \in \{1, \ldots, n\}} A_i$ and $B = A_{n+1}$, then $A \bigcup B = \bigcup_{i \in \{1, \ldots, n+1\}} A_i$ and $A \bigcap B = \left(\bigcup_{i \in \{1, \ldots, n\}} A_i\right) \bigcap A_{n+1} = \bigcup_{i \in \{1, \ldots, n\}} (A_i \bigcap A_{i+1}) = \emptyset$. Hence we have

$$\begin{aligned}
\mu\left(\bigcup_{i \in \{1, \ldots, n\}} A_i\right) &= \mu\left(A \bigcup B\right) \\
&= \mu(A) + \mu(B) \\
&= \mu\left(\bigcup_{i \in \{1, \ldots, n\}} A_i\right) + \mu(A_{n+1}) \\
&\underset{n \in \mathcal{S}}{=} \left(\sum_{i=1}^{n} \mu(A_i)\right) + \mu(A_{n+1}) \\
&= \sum_{i=1}^{n+1} \mu(A_i)
\end{aligned}$$

proving that $n + 1 \in \mathcal{S}$.

Using mathematical induction we have that $\mathbb{N} = \mathcal{S}$. So $\forall \{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{A}$ pairwise disjoint we have $\mu\left(\bigcup_{i \in \{1, \ldots, n+1\}} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$

2.

$\Rightarrow$. Let $A, B \in \mathcal{A}$ then $\{A_i\}_{i \in \{1, \ldots, 2\}}$ defined by $A_i = \left\{ \begin{smallmatrix} A \text{ if } i = 1 \\ B \text{ if } i = 2 \end{smallmatrix} \right.$ satisfies $\{A_i\}_{i \in \{1, \ldots, 2\}} \subseteq \mathcal{A}$. So by assumption we have that

$$\mu\left(A \bigcup B\right) = \mu\left(\bigcup_{i \in \{1, \ldots, 2\}} A_i\right) \leqslant \sum_{i=1}^{2} \mu(A_i) = \mu(A_1) + \mu(A_2) = \mu(A) + \mu(B)$$

⇐. Let

$$\mathcal{S}=\left\{n\in\mathbb{N}_0|\forall\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{A}\text{ we have }\mu\left(\bigcup_{i\in\{1,\ldots,n\}}A_i\right)\leqslant\sum_{i=1}^{n}\mu(A_i)\right\}$$

then:

$1\in\mathcal{S}$. Let $\{A_i\}_{i\in\{1,\ldots,1\}}\subseteq\mathcal{A}$ then $\mu\left(\bigcup_{i\in\{1,\ldots,1\}}A_i\right)=\mu(A_1)=\sum_{i=1}^{1}\mu(A_i)$ proving that $1\in\mathcal{S}$

$n\in\mathcal{S}\Rightarrow n+1\in\mathcal{S}$. Let $\{A_i\}_{i\in\{1,\ldots,n+1\}}\subseteq\mathcal{A}$ pairwise disjoint. Take $A=\bigcup_{i\in\{1,\ldots,n\}}A_i$ and $B=A_{n+1}$, then $A\bigcup B=\bigcup_{i\in\{1,\ldots,n+1\}}A_i$. Hence we have

$$\begin{aligned}\mu\left(\bigcup_{i\in\{1,\ldots,n\}}A_i\right) &= \mu\left(A\bigcup B\right)\\ &\leqslant \mu(A)+\mu(B)\\ &= \mu\left(\bigcup_{i\in\{1,\ldots,n\}}A_i\right)+\mu(A_{n+1})\\ &\leqslant_{n\in\mathcal{S}} \left(\sum_{i=1}^{n}\mu(A_i)\right)+\mu(A_{n+1})\\ &= \sum_{i=1}^{n+1}\mu(A_i)\end{aligned}$$

proving that $n+1\in\mathcal{S}$.

Using mathematical induction we have that $\mathbb{N}=\mathcal{S}$. So $\forall\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{A}$ we have $\mu\left(\bigcup_{i\in\{1,\ldots,n+1\}}A_i\right)\leqslant\sum_{i=1}^{n}\mu(A_i)$ □

Theorem 20.180. *Let $\langle X,\mathcal{A}\rangle$ be a measurable space then if $\mu:\mathcal{A}\to[0,\infty]$ satisfies:*

1. $\mu(\varnothing)=0$
2. *μ is countable additive*

then μ is additive.

Proof. This was actually proved in [theorem: 20.180]. □

Definition 20.181. *Let $\langle X,\mathcal{A}\rangle$ be a measurable space then $\mu:\mathcal{A}\to[0,\infty]$ is a measure on $\mathcal{A}$ if*

1. $\mu(\varnothing)=0$
2. *For every* ***pairwise disjoint*** $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ *we have*

$$\mu\left(\bigcup_{i\in\mathbb{N}}A_i\right)=\sum_{i=1}^{\infty}\mu(A_i)$$

[in other word μ is countable additve].

A trivial example of a measure is the following.

Example 20.182. Let $\langle X,\mathcal{A}\rangle$ be a measurable space then $\mu:\mathcal{A}\to[0,\infty]$ defined by $\mu(A)=0$ is a measure on $\mathcal{A}$.

Proof. We have

1. $\mu(\varnothing)=0$
2. If $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ then $\sum_{i=1}^{\infty}\mu(A_i)=\sum_{i=1}^{\infty}0\underset{\text{[example: 20.89]}}{=}0=\mu\left(\bigcup_{i\in\mathbb{N}}A_i\right)$ □

Definition 20.183. (Measure Space) *A measure space is a triple $\langle X, \mathcal{A}, \mu\rangle$ where $\mathcal{A}$ is a σ-algebra on X and $\mu: \mathcal{A} \to [0, \infty]$ is a measure on $\mathcal{A}$.*

Theorem 20.184. *Let $\langle X, \mathcal{A}\rangle$ be a measureable space then we have:*

1. *If $\mu_1: \mathcal{A} \to [0, \infty]$ and $\mu_2: \mathcal{A} \to [0, \infty]$ are measures on $\mathcal{A}$ then*

$$\mu_1 + \mu_2: \mathcal{A} \to [0, \infty] \text{ defined by } (\mu_1 + \mu_2)(A) = \mu_1(A) + \mu_2(A)$$

   is a measure on $\mathcal{A}$.

2. *If $\mu: \mathcal{A} \to [0, \infty]$ is a measure on $\mathcal{A}$ and $\alpha \in [0, \infty]$ then*

$$\alpha \cdot \mu: \mathcal{A} \to [0, \infty] \text{ defined by } (\alpha \cdot \mu)(A) = \alpha \cdot \mu(A)$$

Proof.

1. First we have

$$(\mu_1 + \mu_2)(\emptyset) = \mu_1(\emptyset) + \mu_2(\emptyset) = 0 + 0 = 0$$

   Secondly if $\{A_i\}_{i \in \mathbb{N}_0} \subseteq \mathcal{A}$ is pairwise disjoint then

$$\begin{aligned} (\mu_1 + \mu_2)\left(\bigsqcup_{i \in \mathbb{N}_0} A_i\right) &= \mu_1\left(\bigsqcup_{i \in \mathbb{N}_0} A_i\right) + \mu_2\left(\bigsqcup_{i \in \mathbb{N}_0} A_i\right) \\ &= \sum_{i=1}^{\infty} \mu_1(A_i) + \sum_{i=1}^{\infty} \mu_2(A_i) \\ &\underset{[\text{theorem: } 20.91]}{=} \sum_{i=1}^{\infty} (\mu_1(A_i) + \mu_2(A_i)) \\ &= \sum_{i=1}^{\infty} (\mu_1 + \mu_2)(A_i) \end{aligned}$$

   proving that $\mu_1 + \mu_2$ is a measure.

2. First we have

$$(\alpha \cdot \mu)(\emptyset) = \alpha \cdot \mu(\emptyset) = \alpha \cdot 0 = 0$$

   Secondly if $\{A_i\}_{i \in \mathbb{N}_0} \subseteq \mathcal{A}$ is pairwise disjoint then

$$\begin{aligned} (\alpha \cdot \mu)\left(\bigsqcup_{i \in \mathbb{N}_0} A_i\right) &= \alpha \cdot \mu\left(\bigsqcup_{i \in \mathbb{N}_0} A_i\right) \\ &= \alpha \cdot \sum_{i=1}^{\infty} \mu(A_i) \\ &\underset{[\text{theorem: } 20.91]}{=} \sum_{i=1}^{\infty} (\alpha \cdot \mu(A_i)) \\ &= \sum_{i=1}^{\infty} (\alpha \cdot \mu)(A_i) \end{aligned}$$

   proving that $\alpha \cdot \mu$ is a measure. □

Theorem 20.185. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure and let $\mathcal{B}$ a σ-algebra on X such that $\mathcal{B} \subseteq \mathcal{A}$ then $\langle X, \mathcal{B}, \mu_{|\mathcal{B}}\rangle$ is a measure space.*

Proof. As $\mathcal{B}$ is already a σ-algebra on X we only have to proof that $\mu_{|\mathcal{B}}$ is a measure on $\mathcal{B}$.

1. As $\emptyset \in \mathcal{B} \subseteq \mathcal{A}$ we have $\mu_{|\mathcal{B}}(\emptyset) = \mu(\emptyset) = 0$
2. If $\{A_i\}_{i \in \mathbb{N}_0} \subseteq \mathcal{B} \subseteq \mathcal{A}$ is a pairwise disjoint family then

$$\mu_{|\mathcal{B}}\left(\bigsqcup_{i \in \mathbb{N}_0} A_i\right) = \mu\left(\bigsqcup_{i \in \mathbb{N}_0} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \mu_{|\mathcal{B}}(A_i)$$

□

Theorem 20.186. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space and $B \in \mathcal{A}$ then $\langle B, \mathcal{A}_{|B}, \mu_{|\mathcal{A}_{|B}}\rangle$ is a measure space. Here $\mathcal{A}_{|B} = \{C \in \mathcal{A} | C \subseteq B\}$ and $\mu_{|\mathcal{A}_{|B}}$ is μ restricted to $\mathcal{A}_{|B}$.*

Proof. Using [theorem: 20.144] $\mathcal{A}_{|B}$ is a σ-algebra on X. Further $\mathcal{A}_{|B} = \{C \in \mathcal{A} | C \subseteq B\} \subseteq \mathcal{A}$ so that by the previous theorem [theorem: 20.185] $\mu_{|\mathcal{A}_{|B}}$ is a measure on $\mathcal{A}_{|B}$. □

Theorem 20.187. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space then*

1. *$\forall A, B \in \mathcal{A}$ with $A \subseteq B$ we have $\mu(A) \leqslant \mu(B)$*
2. *$\forall A, B \in \mathcal{A}$ with $A \subseteq B$ and $\mu(A) < \infty$ we have $\mu(B \setminus A) = \mu(B) - \mu(A)$*

Proof.

1. If $A \subseteq B$ then $A \bigcap (B \setminus A) = \varnothing$ and $B = A \bigcup (B \setminus A)$ so that by [theorem: 20.180] we have

$$\mu(B) = \mu(A) + \mu(B \setminus A)$$

   Further as $0 \leqslant \mu(B \setminus A)$ we have by [theorem: 20.32] that $\mu(A) \leqslant \mu(B)$.

2. If $A \subseteq B$ and $\mu(A) < \infty$ then as $0 \leqslant \mu(A)$ we have that $\mu(A) \in \mathbb{R}$, further by (1)

$$\mu(B) = \mu(A) + \mu(B \setminus A)$$

   so that by [theorem: 20.28] $\mu(B \setminus A) = \mu(B) - \mu(A)$. □

Definition 20.188. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space then μ is a **finite measure** if $\mu(X) < \infty$*

We have the following equivalent definition of a finite measure.

Theorem 20.189. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space*

$$\mu \text{ is a finite measure} \Leftrightarrow \forall A \in \mathcal{A} \text{ we have } \mu(A) < \infty$$

Proof.

⇒. If $A \in \mathcal{A}$ then $A \subseteq X$ so that by [theorem: 20.187] $\mu(A) \leqslant \mu(X) < \infty$.

⇐. If $\forall A \in \mathcal{A}$ $\mu(A) < \infty$ we have as $X \in \mathcal{A}$ that $\mu(X) < \infty$. □

Definition 20.190. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space then μ is a **σ-finite measure** if there exist a $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ such that $\forall i \in \mathbb{N}$ $\mu(A_i) < \infty$ and $X = \bigcup_{i \in \mathbb{N}} A_i$.*

Next we prove that countable additivity implies sub additivity.

Theorem 20.191. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space and $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ then there exist a **pairwise disjoint family** $\{B_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ such that*

$$\forall i \in \mathbb{N} \text{ we have } B_i \subseteq A_i$$

and

$$\bigcup_{i \in \mathbb{N}} A_i = \bigsqcup_{i \in \mathbb{N}} B_i$$

Proof. Using [lemma: 20.108] we have for $\{B_i\}_{i \in \mathbb{N}}$ defined by

$$B_i = \begin{cases} A_1 \text{ if } i = 1 \\ A_i \setminus \bigcup_{j \in \{1, \ldots, i-1\}} A_j \text{ if } i \in \{2, \ldots, \infty\} \end{cases}$$

that

$$\forall i \in \mathbb{N} \text{ we have } B_i \subseteq A_i$$

$$\bigcup_{i \in \mathbb{N}} B_i = \bigcup_{i \in \mathbb{N}} A_i$$

and

$$\forall i, j \in \mathbb{N} \text{ with } i \neq j \text{ then } B_i \bigcap B_j \neq \varnothing$$

Let $i \in \mathbb{N}$ then we have either:

$i = 1$. Then $B_i = B_1 = A_1 \in \mathcal{A}$

$1 < i$. Then by [theorems: 20.135, 20.130] $B_i = A_i \setminus \bigcup_{j \in \{1,\ldots,i-1\}} A_j \in \mathcal{A}$

so that

$$\{B_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$$ □

Theorem 20.192. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space then for every $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ we have*

$$\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) \leqslant \sum_{i=1}^{\infty} \mu(A_i)$$

Proof. Using [theorem: 20.191] there exists a **pairwise disjoint** family $\{B_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ such that

$$\forall i \in \mathbb{N}\ B_i \subseteq A_i \text{ and } \bigsqcup_{i \in \mathbb{N}} B_i = \bigcup_{i \in \mathbb{N}} A_i \tag{20.152}$$

So

$$\begin{aligned} \mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) &\underset{\text{[eq: 20.152]}}{=} \mu\left(\bigsqcup_{i \in \mathbb{N}} B_i\right) \\ &= \sum_{i=1}^{\infty} \mu(B_i) \\ &\underset{\text{[eq: 20.152]}\wedge\text{[theorem: 20.91]}}{\leqslant} \sum_{i=1}^{\infty} \mu(A_i) \end{aligned}$$

□

Theorem 20.193. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space then*

1. *If $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ satisfies $\forall i \in \mathbb{N}\ A_i \leqslant A_{i+1}$ then*

$$\mu\left(\bigcup_{i \in \mathbb{N}_0} A_i\right) = \lim_{i \to \infty} \mu(A_i) = \sup(\{\mu(A_i) | i \in \mathbb{N}\})$$

2. *$\forall \{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ satisfies $\forall i \in \mathbb{N}\ A_{i+1} \leqslant A_i$ and $\exists k \in \mathbb{N}$ such that $\mu(A_k) < \infty$ then*

$$\mu\left(\bigcap_{i \in \mathbb{N}} A_i\right) = \lim_{i \to \infty} \mu(A_i) = \inf(\{\mu(A_i) | i \in \mathbb{N}\})$$

Proof.

1. Define

$$\{B_i\}_{i \in \mathbb{N}} \text{ by } B_i = \begin{cases} A_1 \text{ if } i = 1 \\ A_i \setminus A_{i-1} \text{ if } i \in \{2, \ldots, \infty\} \end{cases} \in \mathcal{A}$$

Let $i, j \in \mathbb{N}$ with $i \neq j$ then we may always assume that $i < j$ [otherwise exchange i and j]. Assume that $x \in B_i \bigcap B_j$ then we have for i either:

$i = 1$. Then $1 = i < j \Rightarrow 1 \leqslant j - 1$ so that $x \in B_i = B_1 = A_1 \subseteq A_{j-1}$. Further as $x \in B_j = A_j \setminus A_{j-1}$ we have $x \notin A_{j-1}$ contradicting that we have also $x \in A_{j-1}$.

$1 < i$. Then $i < j \Rightarrow i \leqslant j - 1$ so that $A_i \subseteq A_{j-1}$. As $x \in B_i = A_i \setminus A_{i-1}$ it follows that $x \in A_i \Rightarrow x \in A_{j-1}$ but as $x \in B_j = A_j \setminus A_{j-1}$ we have that $x \notin A_{j-1}$ hence we have a contradiction.

so the assumption must be wrong proving that $B_i \bigcap B_j = \varnothing$. Hence

$$\forall i, j \in \mathbb{N} \text{ we have } B_i \bigcap B_j = \varnothing \tag{20.153}$$

Next if $n \in \mathbb{N}$ then $\forall i \in \{1, \ldots, n\}$ we have that $B_i = \left\{ \begin{array}{l} A_1 \subseteq A_i \text{ if } i = 1 \\ A_i \setminus A_{i-1} \subseteq A_i \text{ if } i \in \{2, \ldots, \infty\} \end{array} \right. \subseteq A_i \subseteq A_n$ so that

$$\bigcup_{i \in \{1, \ldots, n\}} B_i \subseteq A_n \tag{20.154}$$

Further if $x \in A_n$ then $n \in \{i \in \{1, \ldots, n\} | x \in A_i\} \subseteq \{1, \ldots, n\}$ so that $m = \min(\{i \in \{1, \ldots, n\} | x \in A_i\})$ exists. We have now two cases to consider for m:

$\boldsymbol{m = 1.}$ then $x \in A_1 = B_1 \subseteq \bigcup_{i \in \{1, \ldots, n\}} B_i$

$\boldsymbol{1 < m.}$ then $x \notin A_{m-1}$ and $x \in A_m$ so that $x \in A_m \setminus A_{m-1} = B_m \subseteq \bigcup_{i \in \{1, \ldots, n\}} B_i$

proving that $A_n \subseteq \bigcup_{i \in \{1, \ldots, n\}} B_i$ which by [eq: 20.154] gives $A_n = \bigcup_{i \in \{1, \ldots, n\}} B_i$. Hence

$$\forall n \in \mathbb{N} \text{ we have } A_n = \bigsqcup_{i \in \{1, \ldots, n\}} B_i \tag{20.155}$$

From the above it follows that $\forall n \in \mathbb{N}$ $A_n \subseteq \bigcup_{i \in \{1, \ldots, n\}} B_i \subseteq \bigcup_{i \in \mathbb{N}} B_i$ proving that $\bigcup_{n \in \mathbb{N}} A_n \subseteq \bigcup_{n \in \mathbb{N}} B_n$, futher $\forall i \in \mathbb{N}$ we have $B_i = \left\{ \begin{array}{l} A_1 \subseteq A_i \text{ if } i = 1 \\ A_i \setminus A_{i-1} \subseteq A_i \text{ if } i \in \{2, \ldots, \infty\} \end{array} \right. \subseteq A_i$ so that $\bigcup_{i \in \mathbb{N}} B_i \subseteq \bigcup_{i \in \mathbb{N}} A_i$. Hence we have taking in account [eq: 20.153]

$$\bigcup_{i \in \mathbb{N}} A_i = \bigsqcup_{i \in \mathbb{N}} B_i \tag{20.156}$$

Next we have

$$\begin{aligned} \mu\left(\bigcup_{i \in \mathbb{N}_0} A_i\right) &= \mu\left(\bigsqcup_{i \in \mathbb{N}_0} B_i\right) \\ &= \sum_{i=1}^{\infty} \mu(B_i) \\ &\underset{\text{def}}{=} \lim_{n \to \infty} \left(\sum_{i=1}^{n} \mu(B_i)\right) \\ &\underset{[\text{theorem: } 20.180]}{=} \lim_{n \to \infty} \left(\mu\left(\bigsqcup_{i \in \{1, \ldots, n\}} B_i\right)\right) \\ &\underset{20.155}{=} \lim_{n \to \infty} \mu(A_n) \end{aligned}$$

So that

$$\mu\left(\bigcup_{i \in \mathbb{N}_0} A_i\right) = \lim_{n \to \infty} \mu(A_n)$$

As $\forall i \in \mathbb{N}$ we have $A_i \subseteq A_{i+1} \underset{[\text{theorem: } 20.187]}{\Rightarrow} \mu(A_i) \leqslant \mu(A_{i+1})$ so that by [theorem: 20.84]

$$\lim_{n \to \infty} \mu(A_n) = \sup(\{\mu(A_n) | n \in \mathbb{N}\})$$

2. First we consider the cases where $k = 1$. Then $\forall i \in \mathbb{N}$ we have $A_i \subseteq A_1$ so that by [theorem: 20.187] $\mu(A_i) \leqslant \mu(A_1) < \infty$ proving that

$$\forall i \in \mathbb{N} \text{ we have } \mu(A_i) < \infty \tag{20.157}$$

Define $\{B_i\}_{i \in \mathbb{N}}$ by $B_i = A \setminus A_i$. Let $i \in \mathbb{N}$ then for $x \in B_i$ we have $x \in A_1 \wedge x \notin A_i \supseteq A_{i+1}$ so that $x \in A_1 \setminus A_{i+1} = B_{i+1}$. Hence we have

$$\forall i \in \mathbb{N} \; B_i \subseteq B_{i+1} \tag{20.158}$$

So using (1) it follows that

$$\mu\left(\bigcup_{i \in \mathbb{N}} B_i\right) = \lim_{i \to \infty} \mu(B_i) \tag{20.159}$$

Now

$$A_1 \setminus \left(\bigcup_{i \in \mathbb{N}} B_i \right) \underset{[\text{theorem: } 2.132]}{=} \bigcap_{i \in \mathbb{N}} (A_1 \setminus B_i) = \bigcap_{i \in \mathbb{N}} (A_1 \setminus (A_1 \setminus A_i)) \underset{A_i \subseteq A_1 \wedge [\text{theorem: } 2.132]}{=} \bigcap_{i \in \mathbb{N}} A_i \tag{20.160}$$

As $\forall i \in \mathbb{N}$ $B_i = A_1 \setminus A_i \subseteq A_1$ it follows that

$$\mu(B_i) \underset{[\text{theorem: } 20.187]}{\leqslant} \mu(A_1) < \infty \tag{20.161}$$

and $\bigcup_{i \in \mathbb{N}} B_i \subseteq A_1$ so that

$$\mu\left(\bigcup_{i \in \mathbb{N}} B_i \right) \underset{[\text{theorem: } 20.187]}{\leqslant} \mu(A_1) < \infty \tag{20.162}$$

Next

$$\begin{aligned}
\mu\left(\bigcap_{i \in \mathbb{N}} A_i \right) & \underset{[\text{eq: } 20.160]}{=} \mu\left(A_1 \setminus \left(\bigcup_{i \in \mathbb{N}} B_i \right) \right) \\
& \underset{[\text{eq: } 20.162] \wedge [\text{theorem: } 20.187}{=} \mu(A_1) - \mu\left(\bigcup_{i \in \mathbb{N}} B_i \right) \\
& \underset{[\text{eq: } 20.159]}{=} \mu(A_1) - \lim_{i \rightarrow \infty} \mu(B_i) \\
& = \mu(A_1) - \lim_{i \rightarrow \infty} \mu(A_1 \setminus A_i) \\
& \underset{[\text{eq: } 20.161] \wedge [\text{theorem: } 20.187}{=} \mu(A_1) - \lim_{i \rightarrow \infty} (\mu(A_1) - \mu(A_i)) \\
& \underset{[\text{theorems: } 20.78,\ 20.71]]}{=} \mu(A_1) - \left(\mu(A_1) - \lim_{i \rightarrow \infty} \mu(A_i) \right) \\
& = \lim_{i \rightarrow \infty} \mu(A_i)
\end{aligned}$$

hence we have that

$$\forall \{A_i\}_{i \in \mathbb{N}} \text{ with } \mu(A_1) < \infty \text{ and } \forall i \in \mathbb{N}\ A_{i+1} \subseteq A \text{ we have } \mu\left(\bigcap_{i \in \mathbb{N}} A_i \right) = \lim_{i \rightarrow \infty} \mu(A_i) \tag{20.163}$$

Next we proof the general case where $1 < k$. Define $\{C_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ by $C_i = A_{i+(k-1)}$ then we have as $\mu(A_k) < \infty$ that $\mu(C_1) = A_{1+(k-1)} = A_1 < \infty$. Further $\forall i \in \mathbb{N}$ we have that $C_{i+1} = A_{(i+(k-1))+1} \subseteq A_{i+(k-1)} = C_i$. So all conditions of [eq: 20.163] are satisfied and we have that

$$\mu\left(\bigcap_{i \in \mathbb{N}} C_i \right) = \lim_{i \rightarrow \infty} \mu(C_i)$$

As $\lim_{i \rightarrow \infty} \mu(C_i) = \lim_{i \rightarrow \infty} A_{i+(k-1)} \underset{[\text{theorem: } 20.77]}{=} \lim_{i \rightarrow \infty} A_i$ so that we have

$$\mu\left(\bigcap_{i \in \mathbb{N}} C_i \right) = \lim_{i \rightarrow \infty} \mu(A_i) \tag{20.164}$$

Let $x \in \bigcap_{i \in \mathbb{N}} A_i$. Then $\forall i \in \mathbb{N}$ we have, as $i + (k-1) \in \mathbb{N}$ [because $1 < k$], that $x \in A_{i+(k-1)} = C_i$ hence

$$\bigcap_{i \in \mathbb{N}} A_i \subseteq \bigcap_{i \in \mathbb{N}} C_i \tag{20.165}$$

Let $x \in \bigcap_{i \in \mathbb{N}} C_i$. Let $i \in \mathbb{N}$ then we have for i either:

$i \in \{1, \ldots, k-1\}$. Then $x \in C_1 = A_{1+(k-1)} = A_k \underset{i \leqslant k-1 < k}{\subseteq} A_i$ so that $x \in A_i$.

$i\in\{k,\ldots,\infty\}$. Then $i-(k-1)\in\mathbb{N}$ so that $x\in C_{i-(k-1)}=A_{(i-(k-1))+(k-1)}=A_i$ hence $x\in A_i$

as $i\in\mathbb{N}$ was choosen arbitrary it follows that $x\in\bigcap_{i\in\mathbb{N}}A_i$, Hence $\bigcap_{i\in\mathbb{N}}C_i\subseteq\bigcap_{i\in\mathbb{N}}A_i$ which combined with [eq: 20.165] proves that

$$\bigcap_{i\in\mathbb{N}}A_i=\bigcap_{i\in\mathbb{N}}C_i \tag{20.166}$$

So combining [eq: 20.164] and [eq: 20.166] it follows that

$$\mu\left(\bigcap_{i\in\mathbb{N}}A_i\right)=\lim_{i\rightarrow\infty}\mu(A_i)$$

which proves the case $1<k$. □

The above theorem has the following inverse which provides ways to prove if a set function is actually a mesure.

Theorem 20.194. *Let $\langle X,\mathcal{A}\rangle$ be a measurable space and $\mu:\mathcal{A}\rightarrow[0,\infty]$ a funtion that satisfies:*

1. *$\mu(\varnothing)=0$*
2. *If $n\in\mathbb{N},\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{A}$ is a **pairwise disjoint family** then*

$$\mu\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)=\sum_{i=1}^n\mu(A_i)$$

then we have:

1. *If $\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ that satisfies $\forall i\in\mathbb{N}\ A_i\subseteq A_{i+1}$ we have that $\lim_{i\rightarrow\infty}\mu(A_i)$ exists and $\mu(\bigcup_{i\in\mathbb{N}}A_i)=\lim_{i\rightarrow\infty}\mu(A_i)$ then μ is a measure on $\mathcal{A}$.*
2. *If $\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{A}$ that satisfies $\forall i\in\mathbb{N}\ A_{i+1}\subseteq A_i$ and $\bigcap_{i\in\mathbb{N}}A_i=\varnothing$ we have that $\lim_{i\rightarrow\infty}\mu(A_i)$ exists and $\mu(\bigcap_{i\in\mathbb{N}}A_i)=0$ then μ is a measure on $\mathcal{A}$.*

Proof.

1. Let $\{A_i\}_{i\in\mathbb{N}_o}\subseteq\mathcal{A}$ be a pairwise disjoint family. Define

$$\{B_i\}_{i\in\mathbb{N}}\text{ by }B_i=\bigsqcup_{j\in\{1,\ldots,i\}}A_j\in\mathcal{A}$$

then by [lemma: 20.109] we have that

$$\bigsqcup_{i\in\mathbb{N}}A_i=\bigcup_{i\in\mathbb{N}}B_i\text{ and }\forall i\in\mathbb{N}\text{ we have }B_i\subseteq B_{i+1} \tag{20.167}$$

Then by the hypothesis

$$\lim_{i\rightarrow\infty}B_i\text{ exist and }\mu\left(\bigcup_{i\in\mathbb{N}}B_i\right)=\lim_{i\rightarrow\infty}\mu(B_i) \tag{20.168}$$

Next

$$\begin{aligned}\mu\left(\bigsqcup_{i\in\mathbb{N}}A_i\right) &\underset{\text{[eq: 20.167]}}{=} \mu\left(\bigcup_{i\in\mathbb{N}}B_i\right)\\ &\underset{\text{[eq: 20.168]}}{=} \lim_{i\rightarrow\infty}\mu(B_i)\\ &= \lim_{i\rightarrow\infty}\mu\left(\bigsqcup_{j\in\{1,\ldots,i\}}A_j\right)\\ &\underset{\text{hypothesis}}{=} \lim_{i\rightarrow\infty}\left(\sum_{j=1}^i\mu(A_j)\right)\\ &\underset{\text{def}}{=} \sum_{j=1}^\infty\mu(A_j)\end{aligned}$$

proving countable additivity. Hence μ is a measure on $\mathcal{A}$.

2. Let $\{A_i\}_{i\in\mathbb{N}_o}\subseteq\mathcal{A}$ be a pairwise disjoint family. Define

$$\{B_i\}_{i\in\mathbb{N}} \text{ by } B_i = \bigsqcup_{j\in\{i+1,\ldots,\infty\}} A_j \underset{[\text{theorem: } 20.180]}{\in} \mathcal{A}$$

Let $i\in\mathbb{N}$ then $B_{i+1}=\bigcup_{j\in\{i+2,\ldots,\infty\}} A_j \subseteq \bigcup_{j\in\{i+1,\ldots,n\}} A_j = B_i$ so that

$$\forall i\in\mathbb{N} \text{ we have } B_{i+1}\subseteq B_i$$

Let $x\in\bigcap_{i\in\mathbb{N}} B_i$ then $x\in B_1=\bigcup_{j\in\{2,\ldots,\infty\}} A_j$ so that $\exists k\in\{2,\ldots,\infty\}$ such that $x\in A_k$, as $x\in\bigcap_{i\in\mathbb{N}} B_i$ we must have that $x\in B_k=\bigcup_{j\in\{k+1,\ldots,\infty\}} A_j$ so that $\exists l\in\{k+1,\ldots,\infty\}$ such that $x\in A_l$. Hence it follows that $x\in A_k\bigcap A_l \underset{k\neq l}{=} \varnothing$ leading to the contradiction $x\in\varnothing$. Hence we have that

$$\bigcap_{i\in\mathbb{N}} B_i=\varnothing$$

Let $i\in\mathbb{N}$ then $\bigsqcup_{j\in\mathbb{N}} A_j = (\bigsqcup_{j\in\{1,\ldots,i\}} A_j)\bigcup(\bigsqcup_{j\in\{i+1,\ldots,\infty\}} A_j)$. Assume $x\in(\bigsqcup_{j\in\{1,\ldots,i\}} A_j)\bigcap(\bigsqcup_{j\in\{i+1,\ldots,\infty\}} A_j)$ then there exist a $k\in\{1,\ldots,i\}$ and a $l\in\{i+1,\ldots,\infty\}$ such that $x\in A_k\wedge x\in A_l\Rightarrow x\in A_k\bigcap A_l \underset{k\neq l}{=} \varnothing$ leading to the contradiction $x\in\varnothing$. Hence we must have that $(\bigsqcup_{j\in\{1,\ldots,i\}} A_j)\bigcap(\bigsqcup_{j\in\{i+1,\ldots,\infty\}} A_j)=\varnothing$. By the hypothesis we have then

$$\begin{aligned}
\mu\left(\bigcup_{j\in\mathbb{N}} A_j\right) &= \mu\left(\left(\bigsqcup_{j\in\{1,\ldots,i\}} A_j\right)\bigsqcup\left(\bigsqcup_{j\in\{i+1,\ldots,\infty\}} A_j\right)\right)\\
&\underset{\mu\text{ is finite additive}}{=} \mu\left(\bigsqcup_{j\in\{1,\ldots,i\}} A_j\right)+\mu\left(\bigsqcup_{j\in\{i+1,\ldots,\infty\}} A_j\right)\\
&\underset{\mu\text{ is finite additive}}{=} \sum_{j=1}^{i}\mu(A_j)+\mu\left(\bigsqcup_{j\in\{i+1,\ldots,\infty\}} A_j\right)\\
&= \sum_{j=1}^{i}\mu(A_j)+\mu(B_i) \qquad (20.169)
\end{aligned}$$

By [definition: 20.87]

$$\lim_{i\rightarrow\infty}\sum_{j=1}^{i}\mu(A_j) \text{ exist and } \sum_{i=1}^{\infty}\mu(A_i)=\lim_{i\rightarrow\infty}\sum_{j=1}^{i}\mu(A_j) \qquad (20.170)$$

and by the hypothesis

$$\lim_{i\rightarrow\infty}\mu(B_i) \text{ exist and } \lim_{i\rightarrow\infty}\mu(B_i)=0 \qquad (20.171)$$

Next we have

$$\begin{aligned}
\mu\left(\bigcup_{j\in\mathbb{N}} A_j\right) &\underset{[\text{theorem: } 20.71]}{=} \lim_{i\rightarrow\infty}\left(\mu\left(\bigcup_{j\in\mathbb{N}} A_j\right)\right)\\
&\underset{[\text{eq: } 20.169}{=} \lim_{i\rightarrow\infty}\left(\sum_{j=1}^{i}\mu(A_j)+\mu(B_i)\right)\\
&\underset{[\text{theorem: } 20.78]\wedge[\text{eqs: } 20.170,\ 20.171]}{=} \lim_{i\rightarrow\infty}\sum_{j=1}^{i}\mu(A_j)+\lim_{i\rightarrow\infty}\mu(B_i)\\
&\underset{[\text{eqs: } 20.170,\ 20.171]}{=} \sum_{i=1}^{\infty}\mu(A_i)+0\\
&= \sum_{i=1}^{\infty}\mu(A_i)
\end{aligned}$$

So μ is countable additive proving that μ is a measure.

$\square$

20.4.2 Carathéodory Theorem

The Carathéodory theorem provides a general method for constructing a measure. Given a set X and a non negative set function on $\mathcal{P}(X)$ the Carathéodory theorem allows us to reduce $\mathcal{P}(X)$ to a σ-algebra on X such that the restriction of the set function to this σ-algebra is a measure on this σ-algebra. The idea is that $\mathcal{P}(X)$ is in general to big to be used as a σ-algebra on which we can define a measure. As the restriction of the set function is a measure it must satisfies some extra conditions. For example every σ-algebra contains $\varnothing$ and a measure must map $\varnothing$ to 0 so the set function on $\mathcal{P}(X)$ must also map $\varnothing$ to 0. This is the idea of a outer-measure.

Definition 20.195. (Outer Measure) *Let X be a set then $\mu^*:\mathcal{P}(X)\rightarrow[0,\infty]$ is a* ***outer measure*** *on X if*

1. $\mu^*(\varnothing)=0$
2. *For $A,B\in\mathcal{P}(X)$ with $A\subseteq B$ we have $\mu^*(A)\leqslant\mu^*(B)$ [in other words μ^* is monoton]*
3. *$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{P}(X)$ we have $\mu^*(\bigcup_{i\in\mathbb{N}}A_i)\leqslant\sum_{i=1}^{\infty}\mu^*(A_i)$ [in otherwords $\mu^\star$ is countable sub-additive]*

It is easy to prove that a outer measure is finitely sub-additive.

Theorem 20.196. *Let X be a set and $\mu^*:\mathcal{P}(X)\rightarrow[0,\infty]$ is a outer measure on X, $n\in\mathbb{N}$ and $\{A_i\}_{i\in\{1,\ldots,n\}}\in\mathcal{P}(X)$ then $\mu^*\big(\bigcup_{i\in\{1,\ldots,n\}}A_i\big)\leqslant\sum_{i=1}^{n}\mu^\star(A_i)$*

Proof. Define $\{C_i\}_{i\in\mathbb{N}}\subseteq\mathcal{P}(X)$ by $C_i=\left\{\begin{smallmatrix}A_i \text{ if } i\in\{1,\ldots,n\}\\ \varnothing \text{ if } i\in\{n+1,\ldots,\infty\}\end{smallmatrix}\right.$ then we have

$$\begin{aligned}
\mu^*\left(\bigcup_{i\in\{1,\ldots,n\}}A_i\right) &= \mu^*\left(\bigcup_{i\in\{1,\ldots,n\}}C_i\right)\\
&\underset{\mu^*\text{ is a outer measure}}{\leqslant} \sum_{i=1}^{\infty}\mu^*(C_i)\\
&\underset{[\text{theorem: }20.92]}{=} \sum_{i=1}^{n}\mu^*(C_i)+\sum_{i=n+1}^{\infty}\mu^*(C_i)\\
&= \sum_{i=1}^{n}\mu^*(A_i)+\sum_{i=n+1}^{\infty}0\\
&\underset{[\text{theorem: }20.89]}{=} \sum_{i=1}^{n}\mu^*(A_i)
\end{aligned}$$

$\square$

We need now to reduce $\mathcal{P}(X)$ to a σ-algebra in such a way that the restriction of the outer-measure to this σ-algebra becomes a measure. This is the idea of μ^*-measurable sets.

Definition 20.197. (μ^*-measurable sets) *Let X be a set and $\mu^*:\mathcal{P}(X)\rightarrow[0,\infty]$ a outer measure on X then $A\in\mathcal{P}(X)$ is μ^*-measurable if*

$$\forall B\in\mathcal{P}(X) \text{ we have } \mu^*(B)=\mu^*\big(B\bigcap A\big)+\mu^*(B\setminus A)$$

The set of all μ^*-measurable sets in $\mathcal{P}(X)$ is noted as $\mathcal{M}[\mu^*]$, hence

$$\mathcal{M}[\mu^*]=\big\{A\in\mathcal{P}(X)|\forall B\in\mathcal{P}(X) \text{ we have } \mu^*(B)=\mu^*\big(B\bigcap A\big)+\mu^*(B\setminus A)\big\}$$

Using the properties of a outer-measure we can simplify the above definition.

Theorem 20.198. *Let X be a set and $\mu^{\ast}: \mathcal{P}(X) \rightarrow [0, \infty]$ and $A \in \mathcal{P}(X)$ then we have*

$$A \text{ is } \mu^{\ast}\text{-measurable} \Leftrightarrow \forall B \in \mathcal{P}(X) \text{ we have } \mu^{\ast}(B) \geqslant \mu^{\ast}\left(B \bigcap A\right) + \mu^{\ast}(B \setminus A)$$

Hence we have the following alternative definition for $\mathcal{M}[\mu^{\ast}]$

$$\mathcal{M}[\mu^{\ast}] = \left\{A \in \mathcal{P}(X) | \forall B \in \mathcal{P}(X) \text{ we have } \mu^{\ast}(B) \geqslant \mu^{\ast}\left(B \bigcap A\right) + \mu^{\ast}(B \setminus A)\right\}$$

Proof.

⇒. This is trivial as $\mu^{\ast}(B) = \mu^{\ast}(B \bigcap A) + \mu^{\ast}(B \setminus A)$ implies $\mu^{\ast}(B) \geqslant \mu^{\ast}(B \bigcap A) + \mu^{\ast}(B \setminus A)$.

⇐. Let $B \in \mathcal{P}(X)$ then as $B = (B \bigcap A) \bigcup (B \setminus A)$ we have by [theorem: 20.196] that $\mu^{\ast}(B) \leqslant \mu^{\ast}(B \bigcap A) + \mu^{\ast}(B \setminus A)$, combining this with the hypothesis it follows that

$$\mu^{\ast}(B) = \mu^{\ast}\left(B \bigcap A\right) + \mu^{\ast}(B \setminus A)$$ □

Theorem 20.199. *Let X be a set, $\mu^{\ast}: \mathcal{P}(X) \rightarrow [0, \infty]$ a outer measure and $A \in \mathcal{P}(X)$ then*

1. *If $\mu^{\ast}(A) = 0$ then $A \in \mathcal{M}[\mu^{\ast}]$.*
2. *If $\mu^{\ast}(X \setminus A) = 0$ then $A \in \mathcal{M}[\mu^{\ast}]$.*

Proof.

1. Let $B \in \mathcal{P}(X)$ then as $B \bigcap A \subseteq A$ we have $\mu^{\ast}(B \bigcap A) \leqslant \mu^{\ast}(A) = 0$ so that $\mu^{\ast}(B \bigcap A)$, further as $B \setminus A \subseteq B$ we have $\mu^{\ast}(B \setminus A) \leqslant \mu^{\ast}(B)$. Hence it follows that $\forall B \in \mathcal{P}(X)$ we have

$$\mu^{\ast}\left(B \bigcap A\right) + \mu^{\ast}(B \setminus A) \leqslant \mu^{\ast}(B)$$

proving by the previous theorem [theorem: 20.198] that $A \in \mathcal{M}[\mu^{\ast}]$.

2. Let $B \in \mathcal{P}(X)$ then as $B \setminus A \subseteq X \setminus A$ we have $\mu^{\ast}(B \setminus A) \subseteq \mu^{\ast}(X \setminus A) = 0$ so that $\mu^{\ast}(B \setminus A) = 0$, further as $B \bigcap A \subseteq B$ we have $\mu^{\ast}(B \bigcap A) \leqslant \mu^{\ast}(B)$. Hence it follows that

$$\mu^{\ast}\left(B \bigcap A\right) + \mu^{\ast}(B \setminus A) \leqslant \mu^{\ast}(B)$$

proving by the previous theorem [theorem: 20.198] that $A \in \mathcal{M}[\mu^{\ast}]$. □

We prove now that $\mathcal{M}[\mu^{\ast}]$ is a σ-algebra and the restriction of $\mu^{\ast}$ on $\mathcal{M}[\mu^{\ast}]$ is a measure on $\mathcal{M}[\mu^{\ast}]$.

Theorem 20.200. (Carathéodory Theorem) *Let X be a set and $\mu^{\ast}: \mathcal{P}(X) \rightarrow [0, \infty]$ a outer measure on X then we have:*

1. *$\mathcal{M}[\mu^{\ast}]$ is a σ-algebra on X.*
2. *$(\mu^{\ast})_{|\mathcal{M}[\mu^{\ast}]}: \mathcal{M}[\mu^{\ast}] \rightarrow [0, \infty]$ is a measure on $\mathcal{M}[\mu^{\ast}]$.*

Proof.

1. Let $B \in \mathcal{P}(X)$ then

$$\mu^{\ast}(B) = \mu^{\ast}\left(B \bigcap X\right) + 0 = \mu^{\ast}\left(B \bigcap X\right) + \mu^{\star}(\varnothing) = \mu^{\ast}\left(B \bigcap X\right) + \mu^{\star}(B \setminus X)$$

proving that

$$X \in \mathcal{M}[\mu^{\ast}] \tag{20.172}$$

If $A \in \mathcal{M}[\mu^{\ast}]$ then for $B \in \mathcal{P}(X)$ we have

$$\begin{aligned} \mu^{\ast}\left(B \bigcap (X \setminus A)\right) + \mu^{\ast}(B \setminus (X \setminus A)) & \underset{B \subseteq X \wedge [\text{theorem: } 1.33\ (6)]}{=} \mu^{\ast}\left(B \bigcap (X \setminus A)\right) + \mu^{\ast}\left(B \bigcap A\right) \\ & \underset{B \subseteq X \wedge [\text{theorem: } 1.33\ (8)]}{=} \mu^{\ast}(B \setminus A) + \mu^{\ast}\left(B \bigcap A\right) \\ & \underset{A \in \mathcal{M}[\mu^{\ast}]}{=} \mu^{\ast}(B) \end{aligned}$$

so that $X \setminus A \in \mathcal{M}[\mu^{\ast}]$. Hence we have

$$\forall A \in \mathcal{M}[\mu^{\ast}] \text{ we have } X \setminus A \in \mathcal{M}[\mu^{\ast}] \tag{20.173}$$

Let $A, B \in \mathcal{M}[\mu^*]$ then for $C \in \mathcal{P}(X)$ we have

$$\begin{array}{rcl}\mu^*(C\bigcap (A\bigcup B)) & \underset{A\in\mathcal{M}[\mu^*]}{=} & \mu^*((C\bigcap (A\bigcup B))\bigcap A)+\mu^*((C\bigcap (A\bigcup B))\setminus A)\\ & = & \mu^*(C\bigcap ((A\bigcup B)\bigcap A))+\mu^*((C\bigcap (A\bigcup B))\setminus A)\\ & = & \mu^*(C\bigcap A)+\mu^*((C\bigcap (A\bigcup B))\setminus A)\\ & \underset{[\text{theorem: } 1.25\ (7)]}{=} & \mu^*(C\bigcap A)+\mu^*(C\bigcap ((A\bigcup B)\setminus A))\\ & \underset{[\text{theorem: } 1.25\ (6)]}{=} & \mu^*(C\bigcap A)+\mu^*(C\bigcap ((A\setminus A)\bigcup (B\setminus A)))\\ & = & \mu^*(C\bigcap A)+\mu^*(C\bigcap (B\setminus A))\end{array}$$

proving that

$$\mu^*(C\bigcap (A\bigcup B))=\mu^*(C\bigcap A)+\mu^*(C\bigcap (B\setminus A)) \tag{20.174}$$

So we have

$$\begin{array}{rc}\mu^*(C\bigcap (A\bigcup B))+\mu^*(C\setminus (A\bigcup B)) & \underset{[\text{eq: } 20.174]}{=}\\ \mu^*(C\bigcap A)+\mu^*(C\bigcap (B\setminus A))+\mu^*(C\setminus (A\bigcup B)) & \underset{[\text{theorem: } 1.31(1)]}{=}\\ \mu^*(C\bigcap A)+\mu^*(C\bigcap (B\setminus A))+\mu^*((C\setminus A)\setminus B) & \underset{[\text{theorem: } 1.25(7)]}{=}\\ \mu^*(C\bigcap A)+\mu^*((C\setminus A)\bigcap B)+\mu^*((C\setminus A)\setminus B) & \underset{B\in\mathcal{M}[\mu^*]}{=}\\ \mu^*(C\bigcap A)+\mu^\star(C\setminus A) & \underset{A\in\mathcal{M}[\mu^*]}{=}\\ \mu^*(C) & \end{array}$$

proving that $A\bigcup B$ is μ^*-measurable. Hence we have

$$\forall A, B\in\mathcal{M}[\mu^*] \text{ we have that } A\bigcup B\in\mathcal{M}[\mu^*] \text{ [in other words } \mathcal{M}[\mu^*] \text{ is } \cup\text{-stable]} \tag{20.175}$$

From [eqs: 20.172, 20.173 and 20.175] and [theorem: 20.131] it follows that

$$\mathcal{M}[\mu^*] \text{ is a algebra} \tag{20.176}$$

Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{M}[\mu^*]$ **pairwise disjoint** and $B\in\mathcal{P}(X)$. We show then by mathematical induction that $\forall n\in\mathbb{N}$

$$\mu^*(B)=\sum_{i=1}^n \mu^*(B\bigcap A_i)+\mu^*\left(B\setminus \bigsqcup_{i\in\{1,\ldots,n\}} A_i\right) \tag{20.177}$$

Proof. Define

$$S=\left\{n\in\mathbb{N}|\mu^*(B)=\sum_{i=1}^n \mu^*(B\bigcap A_i)+\mu^*\left(B\setminus \bigsqcup_{i\in\{1,\ldots,n\}} A_i\right)\right\}$$

then we have:

$\boldsymbol{1\in S}$. As $A_1\in\mathcal{M}[\mu^*]$ we have

$$\begin{array}{rcl}\mu^*(B) & = & \mu^*(B\bigcap A_1)+\mu^*(B\setminus A_1)\\ & = & \sum_{i=1}^1 \mu^*[B\bigcap A_i]+\mu^*\left(B\setminus \bigsqcup_{i\in\{1\}} A_i\right)\end{array}$$

which proves that $1\in\mathcal{S}$

$\boldsymbol{n\in S\Rightarrow n+1\in S}$. If $x\in B\bigcap A_{n+1}$ then $x\in B$ and $x\in A_{n+1}$ then $\forall i\in\{1,\ldots,n\}$ we have by pairwise disjointedness $x\notin A_i$ so that $x\notin\bigsqcup_{i\in\{1,\ldots,n\}} A_i$ or $x\in(B\setminus\bigsqcup_{i\in\{1,\ldots,n\}} A_i)\bigcap A_{n+1}$. Hence $B\bigcap A_{n+1}\subseteq(B\setminus\bigsqcup_{i\in\{1,\ldots,n\}} A_i)\bigcap A_{n+1}\subseteq B\bigcap A_{n+1}$. Summarized

$$\left(B\setminus \bigsqcup_{i\in\{1,\ldots,n\}} A_i\right)\bigcap A_{n+1}=B\bigcap A_{n+1} \tag{20.178}$$

As $A_{n+1} \in \mathcal{M}[\mu^*]$ we have

$$\mu^*\left(B \setminus \bigsqcup_{i \in \{1,\ldots,n\}} A_i\right) =$$

$$\mu^*\left(\left(B \setminus \bigsqcup_{i \in \{1,\ldots,n\}} A_i\right) \bigcap A_{n+1}\right) + \mu^*\left(\left(B \setminus \bigsqcup_{i \in \{1,\ldots,n\}} A_i\right) \setminus A_{n+1}\right) \underset{\text{[eq: 20.178]}}{=}$$

$$\mu^*\left(B \bigcap A_{n+1}\right) + \mu^*\left(\left(B \setminus \bigsqcup_{i \in \{1,\ldots,n\}} A_i\right) \setminus A_{n+1}\right) \underset{\text{[theorem: 1.31]}}{=}$$

$$\mu^*\left(B \bigcap A_{n+1}\right) + \mu^*\left(B \setminus \left(\left(\bigsqcup_{i \in \{1,\ldots,n\}} A_i\right) \bigcup A_{n+1}\right)\right) =$$

$$\mu^*\left(B \bigcap A_{n+1}\right) + \mu^*\left(B \setminus \left(\bigsqcup_{i \in \{1,\ldots,n+1\}} A_i\right)\right)$$

proving that

$$\mu^*\left(B \setminus \bigsqcup_{i \in \{1,\ldots,n\}} A_i\right) = \mu^*\left(B \bigcap A_{n+1}\right) + \mu^*\left(B \setminus \left(\bigsqcup_{i \in \{1,\ldots,n+1\}} A_i\right)\right) \tag{20.179}$$

Finally

$$\begin{aligned}
\mu^*(B) \quad & \underset{n \in S}{=} \quad \sum_{i=1}^{n} \mu^*\left(B \bigcap A_i\right) + \mu^*\left(B \setminus \bigsqcup_{i \in \{1,\ldots,n\}} A_i\right) \\
& \underset{\text{[eq: 20.179]}}{=} \sum_{i=1}^{n} \mu^*\left(B \bigcap A_i\right) + \mu^*\left(B \bigcap A_{n+1}\right) + \mu^*\left(B \setminus \left(\bigsqcup_{i \in \{1,\ldots,n+1\}} A_i\right)\right) \\
& = \quad \sum_{i \asymp 1}^{n+1} \mu^*\left(B \bigcap A_i\right) + \mu^*\left(B \setminus \left(\bigsqcup_{i \in \{1,\ldots,n+1\}} A_i\right)\right)
\end{aligned}$$

which proves that $n+1 \in \mathcal{S}$.

Mathematical induction proves then [eq: 20.177]. □

Let $n \in \mathbb{N}$ then we have $\bigsqcup_{i \in \{1,\ldots,n\}} A_i \subseteq \bigsqcup_{i \in \mathbb{N}} A_i$ so that $B \setminus \bigcup_{i \in \mathbb{N}} A_i \subseteq B \setminus \bigcup_{i \in \{1,\ldots,n\}} A_i$ hence $\mu^*(B \setminus \bigcup_{i \in \mathbb{N}} A_i) \leqslant \mu^*\left(B \setminus \bigcup_{i \in \{1,\ldots,n\}} A_i\right)$. So

$$\begin{aligned}
\sum_{i=1}^{n} \mu^*\left(B \bigcap A_i\right) + \mu^*\left(B \setminus \bigcup_{i \in \mathbb{N}} A_i\right) \quad & \leqslant \quad \sum_{i=1}^{n} \mu^*\left(B \bigcap A_i\right) + \mu^*\left(B \setminus \bigcup_{i \in \{1,\ldots,n\}} A_i\right) \\
& \underset{\text{[eq: 20.177]}}{=} \mu^*(B)
\end{aligned} \tag{20.180}$$

Next

$$\sum_{i=1}^{\infty} \mu^*\left(B \bigcap A_i\right) + \mu^*\left(B \setminus \bigsqcup_{i \in \mathbb{N}} A_i\right) =$$

$$\lim_{n \to \infty} \left(\sum_{i=1}^{n} \mu^*\left(B \bigcap A_i\right)\right) + \mu^*\left(B \setminus \bigsqcup_{i \in \mathbb{N}} A_i\right) \underset{\text{[theorem: 20.78]}}{=}$$

$$\lim_{n \to \infty} \left(\sum_{i=1}^{n} \mu^*\left(B \bigcap A_i\right) + \mu^*\left(B \setminus \bigsqcup_{i \in \mathbb{N}} A_i\right)\right) \underset{\text{[theorem: 20.78]} \wedge \text{[eq: 20.180]}}{\leqslant}$$

$$\lim_{n \to \infty} \mu^*(B) \underset{\text{[theorem: 20.71]}}{=}$$

$$\mu^*(B)$$

proving that $\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{M}[\mu^*]$ and $B\in\mathcal{P}(X)$

$$\sum_{i=1}^{\infty}\mu^*(B\bigcap A_i)+\mu^*\left(B\setminus\bigsqcup_{i\in\mathbb{N}}A_i\right)\leqslant\mu^*(B) \tag{20.181}$$

Hence we have for $\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{M}[\mu^*]$ and $B\in\mathcal{P}(X)$ that

$$\begin{array}{rcl}\mu^*\left(B\bigcap\left(\bigsqcup_{i\in\mathbb{N}}A_i\right)\right)+\mu^*\left(B\setminus\bigsqcup_{i\in\mathbb{N}}A_i\right) & = & \\ \mu^*\left(\bigcup_{i\in\mathbb{N}}(B\bigcap A_i)\right)+\mu^*\left(B\setminus\bigsqcup_{i\in\mathbb{N}}A_i\right) & \underset{\mu^* \text{ is a outer measure}}{\leqslant} & \\ \sum_{i=1}^{\infty}\mu^*(B\bigcap A_i)+\mu^*\left(B\setminus\bigsqcup_{i\in\mathbb{N}}A_i\right) & \underset{[\text{eq: } 20.181]}{\leqslant} & \mu^*(B)\end{array}$$

So by [theorem: 20.198] it follows that $\bigsqcup_{i\in\mathbb{N}}A_i$ is $\mu^\star$-measureable. Hence we have proved that

$$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{M}[\mu^*] \text{ pairwise disjoint we have } \bigsqcup_{i\in\mathbb{N}}A_i\in\mathcal{M}[\mu^*] \tag{20.182}$$

As $\mathcal{M}[\mu^*]$ is a algebra [see eq: 20.176] we can use the alternative definition for a σ-algbra [see theorem: 20.135] and the above to prove that

$$\mathcal{M}[\mu^*] \text{ is a } \sigma\text{-algebra}$$

2. Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{M}[\mu^*]$ a pairwise disjoint family then for $B=\bigsqcup_{j\in\mathbb{N}}A_i\in\mathcal{P}(X)$ we have

$$\begin{array}{rcl}\sum_{i=1}^{\infty}(\mu^*)_{|\mathcal{M}[\mu^*]}(A_i) & = & \sum_{i=1}^{\infty}\mu^*(A_i)\\ & = & \sum_{i=1}^{\infty}\mu^*\left(\left(\bigcup_{j\in\mathbb{N}}A_j\right)\bigcap A_i\right)+0\\ & = & \sum_{i=1}^{\infty}\mu^*\left(\left(\bigcup_{j\in\mathbb{N}}A_j\right)\bigcap A_i\right)+\mu^*\left(\left(\bigcup_{j\in\mathbb{N}}A_j\right)\setminus\left(\bigcup_{j\in\mathbb{N}}A_j\right)\right)\\ & = & \sum_{i=1}^{\infty}\mu^*(B\bigcap A_i)+\mu^*\left(B\setminus\left(\bigcup_{j\in\mathbb{N}}A_j\right)\right)\\ & \underset{[\text{eq: } 20.181]}{\leqslant} & \mu^*(B)\\ & = & \mu^*\left(\bigsqcup_{i\in\mathbb{N}}A_i\right)\\ & = & (\mu^*)_{|\mathcal{M}[\mu^*]}\left(\bigsqcup_{i\in\mathbb{N}}A_i\right)\end{array}$$

As further by the sub-additivity of μ^* we have

$$(\mu^*)_{|\mathcal{M}[\mu^*]}\left(\bigsqcup_{i\in\mathbb{N}}A_i\right)=\mu^*\left(\bigcup_{i\in\mathbb{N}}A_i\right)\leqslant\sum_{i=1}^{\infty}\mu^*(A_i)=\sum_{i=1}^{\infty}(\mu^*)_{|\mathcal{M}[\mu^*]}(A_i)$$

it follows that

$$(\mu^*)_{|\mathcal{M}[\mu^*]}\left(\bigsqcup_{i\in\mathbb{N}}A_i\right)=\sum_{i=1}^{\infty}(\mu^*)_{|\mathcal{M}[\mu^*]}(A_i)$$

which together with $(\mu^*)_{|\mathcal{M}[\mu^*]}(\varnothing)=\mu^\star(\varnothing)=0$ proves

$$(\mu^*)_{|\mathcal{M}[\mu^*]} \text{ is a measure on } \mathcal{M}[\mu^*]$$

□

20.4.3 Uniqueness of measure extensions

Let X be a set, $\mathcal{G} \subseteq \mathcal{P}(X)$ and $\sigma[\mathcal{G}]$ is the σ-algebra generated by $\mathcal{G}$ [see theorem: 20.141] then if we have a set function on $\mathcal{G}$ that we extend to a measure on $\sigma[\mathcal{G}]$ we can ask ourselves if the extension is unique. The answer is true if $\mathcal{G}$ and the set function satisfies certain conditions. To prove this we introduce the concept of a Dynkin system.

20.4.3.1 Dynkin systems

Definition 20.201. (Dynkin System) *Let X be a set then $\mathfrak{D} \subseteq \mathcal{P}(X)$ is a **Dynkin system** on X if*

1. *$X \in \mathfrak{D}$*
2. *If $D \in \mathfrak{D}$ then $X \setminus D \in \mathfrak{D}$*
3. *If $\{D_i\}_{i \in \mathbb{N}} \subseteq \mathfrak{D}$ is a **pairwise disjoint family** then $\bigsqcup_{i \in \mathbb{N}} D_i \in \mathfrak{D}$*

***Note* 20.202.** *Using [definition: 20.132] every σ-algebra is automatical a Dynkin system however the opposite is in general not true.*

Some of the properties of a Dynkin system are similar to the properties of a σ-algebra.

Theorem 20.203. *Let X be a set and $\mathfrak{D}$ is a Dynkin system on X then we have:*

1. *$\varnothing \in \mathfrak{D}$*
2. *Let $n \in \mathbb{N}$ and $\{D_i\}_{i \in \{1,\ldots,n\}} \subseteq \mathfrak{D}$ pairwise disjoint then $\bigsqcup_{i \in \{1,\ldots,n\}} D_i \in \mathfrak{D}$*

Proof.

1. As $X \in \mathfrak{D}$ we have that $\varnothing = X \setminus X \in \mathfrak{D}$
2. Define $\{E_i\}_{i \in \mathbb{N}} \subseteq \mathfrak{D}$ by $E_i = \left\{ \begin{array}{l} \varnothing \text{ if } i \in \{n+1,\ldots,\infty\} \\ D_i \text{ if } i \in \{1,\ldots,n\} \end{array} \right.$. Let $i, j \in \mathbb{N}$ with $i \neq j$ then we have either

   $\boldsymbol{i \leqslant n \wedge j \leqslant n}$**.** Then $E_i \bigcap E_j = D_i \bigcap D_j = \varnothing$

   $\boldsymbol{n < i \wedge j \leqslant n}$**.** Then $E_i \bigcap E_j = \varnothing \bigcap D_j = \varnothing$

   $\boldsymbol{i \leqslant n \wedge n < j}$**.** Then $E_i \bigcap E_j = D_i \bigcap D_j = \varnothing$

   $\boldsymbol{n < i \wedge n < j}$**.** Then $E_i \bigcap E_j = \varnothing \bigcap \varnothing = \varnothing$

   which proves that $\{E_i\}_{i \in \mathbb{N}}$ is pairwise disjoint. So

   $$\bigsqcup_{i \in \mathbb{N}} E_i \in \mathfrak{D}$$

   If $x \in \bigcup_{i \in \{1,\ldots,n\}} D_i$ then $\exists i \in \{1,\ldots,n\}$ such that $x \in D_i = E_i$ so that $x \in \bigcup_{i \in \mathbb{N}} E_i$. If $x \in \bigcup_{i \in \mathbb{N}} E_i$ then there exist a $i \in \mathbb{N}$ such that $x \in E_i$, as $\forall i \in \{n+1,\ldots,\infty\}$ $E_i = \varnothing$ we must have that $i \in \{1,\ldots,n\}$ so that $E_i = D_i$ proving that $x \in \bigcup_{i \in \{1,\ldots,n\}} D_i$. Hence we have

   $$\bigsqcup_{i \in \{1,\ldots,n\}} D_i = \bigsqcup_{i \in \mathbb{N}} E_i \in \mathfrak{D}$$

□

Example 20.204. Let X be a set then $\mathcal{P}(X)$ is a Dynkin system on X

Proof. As $\mathcal{P}(X)$ is a σ-algebra on X [see example: 20.137] and every σ-algebra is a Dynkin system it follows that $\mathcal{P}(X)$ is a Dynkin system on X. □

Given a set of subsets of X we can, just like with σ-algebras, find the smallest Dynkin system that contains the set of subsets.

Lemma 20.205. *Let X be a set and $\mathcal{X}$ a non empty set of Dynkin systems on X then $\bigcap_{\mathfrak{D} \in \mathcal{X}} \mathfrak{D}$ is a Dynkin system on X.*

Proof. As $\mathcal{X} \neq \varnothing$ there exist a $\mathfrak{D} \in \mathcal{X}$ with $\mathfrak{D} \subseteq \mathcal{P}(X)$ so that we have by [theorem: 1.63] that $\bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D} \subseteq \mathfrak{D} \subseteq \mathcal{P}(X)$ proving that

$$\bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D} \subseteq \mathcal{P}(X)$$

As $\forall \mathcal{D} \in \mathfrak{D}$ $\mathcal{D}$ is a Dynkin system on X we have $X \in \mathcal{A}$ hence

$$X \in \bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D}$$

Let $A \in \bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D}$ then for $\mathcal{D} \in \mathcal{X}$ we have $A \in \mathcal{D}$ so that $X \setminus A \in \mathcal{D}$ which as $\mathcal{D} \in \mathcal{X}$ was chosen arbitrary proves that $X \setminus A \in \bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D}$. Hence as $A \in \bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D}$ was chosen arbitrary we have that

$$\forall A \in \bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D} \text{ we have } X \setminus A \in \bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D}$$

If $\{D_i\}_{i \in \mathbb{N}} \subseteq \bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D}$ is pairwise disjoint then for $\mathcal{D} \in \mathcal{X}$ we have $\{D_i\}_{i \in \mathbb{N}} \subseteq \mathcal{D}$ so that $\bigsqcup_{i \in \mathbb{N}} D_i \in \mathcal{D}$ which as $\mathcal{D} \in \mathcal{X}$ was chosen arbitrary proves that $\bigcup_{i \in \mathbb{N}} D_i \in \bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D}$. Hence as $\{D_i\}_{i \in \mathbb{N}} \subseteq \bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D}$ was chosen arbitrary we have that

$$\text{For every pairwise disjoint family } \{D_i\}_{i \in \mathbb{N}} \subseteq \bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D} \text{ we have } \bigcup_{i \in \mathbb{N}} D_i \in \bigcap_{\mathcal{D} \in \mathcal{X}} \mathcal{D}$$ □

Theorem 20.206. *Let X be a set $\mathcal{D} \subseteq \mathcal{P}(X)$ then there exist a **unique** Dynkin system $\delta[\mathcal{A}]$ on X such that:*

1. $\mathcal{D} \subseteq \delta[\mathcal{D}]$
2. *If $\mathfrak{D}'$ is another Dynkin system with $\mathcal{D} \subseteq \mathfrak{D}'$ then $\delta[\mathcal{D}] \subseteq \mathfrak{D}'$*
3. $\delta[\mathcal{D}] \subseteq \sigma[\mathcal{D}]$

This unique smallest Dynkin system that contains $\mathcal{D}$ is called the Dynkin system generated by $\mathcal{D}$

Proof.

1. Define $\mathcal{X} = \{\mathfrak{D} \subseteq \mathcal{P}(X) | \mathfrak{D}$ is a Dynkin system on X and $\mathcal{D} \subseteq \mathfrak{D}\}$ then as $\mathcal{P}(X)$ is a Dynkin system [see example: 20.204] and $\mathcal{D} \subseteq \mathcal{P}(X)$ it follows that $\mathcal{X} \neq \varnothing$. Hence by [lemma: 20.205] it follows that, if we take $\delta[\mathcal{D}] = \bigcap_{\mathfrak{D} \in \mathcal{X}} \mathfrak{D}$, that

   $$\delta[\mathcal{D}] \text{ is a Dynkin system on } X$$

   If $D \in \mathcal{D}$ then, as $\forall \mathfrak{D} \in \mathcal{X}$ we have by defintion of $\mathcal{X}$ that $\mathcal{D} \subseteq \mathfrak{D}$, it follows that $D \in \mathfrak{D}$, hence $D \in \bigcap_{\mathfrak{D} \in \mathcal{X}} \mathfrak{D}$. So that

   $$\mathcal{D} \subseteq \bigcap_{\mathfrak{D} \in \mathcal{X}} \mathfrak{D} = \delta[\mathcal{D}]$$

2. Assume that $\mathfrak{D}'$ is another Dynkin system on X such that $\mathcal{D} \subseteq \mathfrak{D}'$ then $\mathfrak{D}' \in \mathcal{X}$ so that by [theorem: 1.63]

   $$\delta[\mathcal{D}] = \bigcap_{\mathfrak{D} \in \mathcal{X}} \mathfrak{D} \subseteq \mathfrak{D}'$$

3. As $\mathcal{D} \subseteq \sigma[\mathcal{D}]$, $\sigma[\mathcal{D}]$ is a σ-algebra hence a Dynkin system on X it follows from (2) that

   $$\delta[\mathcal{D}] \subseteq \sigma[\mathcal{D}]$$ □

We show now the sufficient and necessary condition for a Dynkin system to be a σ-algebra.

Theorem 20.207. *Let X be a set and $\mathfrak{D}$ a Dynkin system on X then we have*

$$\mathfrak{D} \textit{ is a } \sigma\textit{-algebra} \Leftrightarrow \forall A, B \in \mathfrak{D} \textit{ we have } A \bigcap B \in \mathfrak{D} \textit{ [in other words } \mathfrak{D} \textit{ is } \cap\textit{-stable]}$$

Proof.

⇒. Let $A, B \in \mathfrak{D}$ then, as $\mathfrak{D}$ is a σ-algebra, it follows from [theorems: 20.135, 20.131] that $A \bigcap B \in \mathcal{A}$.

⇐. As $\mathfrak{D}$ is a Dynkin system and $\cap$-stable we have already that

$$X\in\mathfrak{D},\ \forall D\in\mathfrak{D} \text{ we have } X\setminus D\in\mathfrak{D} \text{ and } \forall A,B\in\mathfrak{D} \text{ we have } A\bigcap B\in\mathfrak{D} \tag{20.183}$$

Let $\{D_i\}_{i\in\mathbb{N}}\subseteq\mathfrak{D}$. Define

$$\{E_i\}_{i\in\mathbb{N}} \text{ by } E_i=\begin{cases} D_1 \text{ if } i=1\\ D_i\bigcap\left(\bigcap_{j\in\{1,\ldots,i-1\}}(X\setminus D_j)\right) \text{ if } i\in\{2,\ldots\infty\}\end{cases} \underset{\text{[eq: 20.183]}\wedge\text{[theorem: 20.111]}}{\in} \mathfrak{D}$$

Then $\forall i\in\mathbb{N}$ we have

$$E_i=\begin{cases} D_1\subseteq D_i \text{ if } i=1\\ D_i\bigcap\left(\bigcap_{j\in\{1,\ldots,i-1\}}(X\setminus D_j)\right)\subseteq D_i \text{ if } i\in\{2,\ldots,\infty\}\end{cases}\subseteq D_i \tag{20.184}$$

so that

$$\bigcup_{i\in\mathbb{N}} E_i\subseteq\bigcup_{i\in\mathbb{N}} D_i \tag{20.185}$$

If $x\in\bigcup_{i\in\mathbb{N}}D_i$ then there exist a $k\in\mathbb{N}$ such that $x\in D_k$. Define

$$S=\{i\in\{1,\ldots,k\}|x\in D_i\}\subseteq\{1,\ldots,k\}$$

then $k\in S\Rightarrow S\neq\varnothing$ and S is finite so that $m=\min(S)$ exists. For m we have either:

$m=1$. Then $x\in D_1=E_1\subseteq\bigcup_{i\in\mathbb{N}}E_i$

$1<m$. Then $x\in D_m$ and $\forall i\in\{1,\ldots,m-1\}$ $x\notin D_i\Rightarrow x\in X\setminus D_i$ so that

$$x\in D_m\bigcap\left(\bigcap_{i\in\{1,\ldots,m-1\}}(X\setminus D_i)\right)=E_m\subseteq\bigcup_{i\in\mathbb{N}}E_i$$

Hence $\bigcup_{i\in\mathbb{N}}D_i\subseteq\bigcup_{i\in\mathbb{N}}E_i$ and we have by [eq: 20.185] that

$$\bigcup_{i\in\mathbb{N}} E_i=\bigcup_{i\in\mathbb{N}} D_i \tag{20.186}$$

Let $i,j\in\mathbb{N}$ with $i\neq j$ then we may always assume that $i<j$ [otherwise exchange i and j]. Assume that $x\in E_i\bigcap E_j$ then $x\in E_i\underset{\text{[eq: 20.184]}}{\subseteq}D_i$ and $x\in E_j=D_j\bigcap\left(\bigcap_{k\in\{1,\ldots,j-1\}}(X\setminus D_k)\right)\subseteq(X\setminus D_i)\Rightarrow x\notin D_i$ so that we reach the contradiction $x\in D_i$ and $x\notin D_i$. Hence we have

$$\forall i,j\in\mathbb{N} \text{ with } i\neq j \text{ we have } E_i\bigcap E_j=\varnothing$$

So by the definition of a Dynkin system it follows that

$$\bigcup_{i\in\mathbb{N}} D_i\underset{\text{[eq: 20.186]}}{=}\bigcup_{i\in\mathbb{N}} E_i\in\mathfrak{D}$$

Finally by the above, [eq: 20.183] and [definition: 20.132] it follows that $\mathfrak{D}$ is a σ-algebra on X. □

Theorem 20.208. *If $\mathcal{D}\subseteq\mathcal{P}(\mathcal{A})$ is such that $\forall A,B\in\mathfrak{D}$ $A\bigcap B\in\mathcal{D}$ [in other words $\mathfrak{D}$ is $\cap$-stable] then $\delta(\mathcal{D})=\sigma(\mathcal{D})$*

Proof. Using [theorem: 20.206] it follows that

$$\delta[\mathcal{D}]\subseteq\sigma[\mathcal{D}] \tag{20.187}$$

For the opposite inclusion. Defiine for every $D\in\delta[\mathcal{D}]$

$$\mathfrak{D}_D=\left\{E\subseteq X|E\bigcap D\in\delta[\mathcal{D}]\right\} \tag{20.188}$$

As $X\subseteq X$ and $X\bigcap D=D\in\delta[\mathcal{D}]$ it follows that

$$X\in\mathfrak{D}_D \tag{20.189}$$

Let $E\in\mathfrak{D}_D$ then $X\setminus E\subseteq X$ and

$$\begin{aligned}
X\setminus((E\bigcap D)\bigcup(X\setminus D)) &\underset{[\text{theorem: } 1.31]}{=} (X\setminus(E\bigcap D))\bigcap(X\setminus(X\setminus D))\\
&\underset{[\text{theorem: } 1.33]\wedge D\subseteq X}{=} (X\setminus(E\bigcap D))\bigcap D\\
&\underset{[\text{theorem: } 1.31]}{=} ((X\setminus E)\bigcup(X\setminus D))\bigcap D\\
&\underset{[\text{theorem: } 1.30]}{=} ((X\setminus E)\bigcap D)\bigcup((X\setminus D)\bigcap D)\\
&= ((X\setminus E)\bigcap D)\bigcup\varnothing\\
&= (X\setminus E)\bigcap D
\end{aligned}$$

Further $(E\bigcap D)\bigcap(X\setminus D)=\varnothing$ so that

$$(X\setminus E)\bigcap D=X\setminus((E\bigcap D)\bigsqcup(X\setminus D)) \tag{20.190}$$

As $E\in\mathfrak{D}_D$ we have by [eq: 20.188] that $E\bigcap D\in\delta[\mathcal{D}]$, further as $\delta[\mathcal{D}]$ is a Dynkin system and $D\in\delta[\mathcal{D}]$ it follows that $X\setminus D\in\delta[\mathcal{D}]$ so that $(E\bigcap D)\bigsqcup(X\setminus D)\in\delta[\mathcal{D}]$ and finally that

$$X\setminus((E\bigcap D)\bigsqcup(X\setminus D))\in\delta[\mathcal{D}]$$

Hence by [eq: 20.190] it follows that

$$\forall E\in\mathfrak{D}_D \text{ that } X\setminus E\in\mathfrak{D}_D \tag{20.191}$$

Let $\{D_i\}_{i\in\mathbb{N}}\subseteq\mathfrak{D}_D$ be pairwise disjoint then $\forall i,\ j\in\mathbb{N}$ with $i\neq j$ we have $(D_i\bigcap D)\bigcap(D_j\bigcap D)\subseteq D_i\bigcap D_j=\varnothing$, further $\forall i\in\mathbb{N}$ we have by the definition of $\mathfrak{D}_D$ that $D_i\bigcap D\in\delta[\mathcal{D}]$. Hence $\{D_i\bigcap D\}_{i\in\mathbb{N}}\subseteq\delta[\mathcal{D}]$ is pairwise disjoint, so that, as $\delta[\mathcal{D}]$ is a Dynkin system, we have

$$\left(\bigsqcup_{i\in\mathbb{N}}D_i\right)\bigcap D=\bigsqcup_{i\in\mathbb{N}}(D_i\bigcap D)\in\delta[\mathcal{D}]$$

proving that

$$\bigsqcup_{i\in\mathbb{N}}D_i\in\mathfrak{D}_D \tag{20.192}$$

From [eqs: 20.189. 20.191 and 20.192] it follows that $\mathfrak{D}_D$ is a Dynkin system on X, hence

$$\forall D\in\delta[\mathcal{D}] \text{ we have that } \mathfrak{D}_D \text{ is a Dynkin system} \tag{20.193}$$

Let $G\in\mathcal{D}\subseteq\delta[\mathcal{D}]$ then by the hypothesis we have $\forall D\in\mathcal{D}$ that $G\bigcap D\in\mathcal{D}\Rightarrow D\in\mathfrak{D}_G$, hence $\mathcal{D}\subseteq\mathfrak{D}_G$. As $\mathfrak{D}_G$ is a Dynkin system and $\delta[\mathcal{D}]$ is the smallest Dynkin system containing $\mathcal{D}$ it follows that $\delta[\mathcal{D}]\subseteq\mathfrak{D}_G$. Hence we have proved that

$$\forall G\in\mathcal{D} \text{ we have } \delta[\mathcal{D}]\subseteq\mathfrak{D}_G \tag{20.194}$$

Let $G\in\mathcal{D}$ and $D\in\delta[\mathcal{D}]\underset{[\text{eq: } 20.194]}{\subseteq}\mathfrak{D}_G$ then by the definition of $\mathfrak{D}_G$ we have $G\bigcap D\in\delta[\mathcal{D}]$ so that $G\in\mathfrak{D}_D$. In other words $\forall D\in\delta[\mathcal{D}]$ we have $\forall G\in\mathcal{D}$ that $G\in\mathfrak{D}_D$ proving that

$$\forall D\in\delta[\mathcal{D}] \text{ we have } \mathcal{D}\subseteq\mathfrak{D}_D$$

As $\delta[\mathcal{D}]$ is the smallest Dynkin system containg $\mathcal{D}$ and $\mathfrak{D}_D$ is a Dynkin system [see eq: 20.193] it follows from the above that

$$\forall D\in\delta[\mathcal{D}] \text{ that } \delta[\mathcal{D}]\subseteq\mathfrak{D}_D$$

Let $A,B\in\delta[\mathcal{D}]$ then by the above $\delta[\mathcal{D}]\subseteq\mathfrak{D}_B$ and as $A\in\delta(\mathcal{D})$ we have $A\in\mathfrak{D}_B\underset{[\text{eq: } 20.188]}{\Rightarrow}A\bigcap B\in\delta[\mathcal{D}]$. So we have that

$$\forall A,B\in\delta[\mathcal{D}] \text{ we have } A\bigcap B\in\delta[\mathcal{D}] \text{ where } \delta[\mathcal{D}] \text{ is a Dynkin system}$$

Hence by [theorem: 20.207] it follows finally that

$$\delta[\mathcal{D}] \text{ is a } \sigma\text{-algebra}$$

□

20.4.3.2 Uniqueness theorem

We are now ready to prove that every set function on $\mathcal{G}\subseteq\mathcal{P}(X)$ has a unique extension to a measure on $\sigma[\mathcal{G}]$ if certain conditions are met.

Theorem 20.209. *Let X be a set, $\mathcal{G}\subseteq\mathcal{P}(X)$ such that:*

1. *$\forall A,B\in\mathcal{G}$ we have $A\bigcap B\in\mathcal{G}$ [in other words $\mathcal{G}$ is $\cap$-stable]*
2. *$\exists\{G_i\}_{i\in\mathbb{N}}\subseteq\mathcal{G}$ such that $\forall i\in\mathbb{N}$ we have $G_i\subseteq G_{i+1}$, $\mu(G_i)<\infty$ and $\bigcup_{i\in\mathbb{N}}G_i=X$*

Then if $\mu\colon\sigma[\mathcal{G}]\rightarrow[0,\infty]$ and $\nu\colon\sigma[\mathcal{G}]\rightarrow[0,\infty]$ are two measures defined on $\sigma[\mathcal{G}]$ [the σ-algebra generated from $\mathcal{G}$] such that $\forall A\in\mathcal{G}$ we have $\mu(A)=\nu(A)$ then

$$\mu=\nu$$

Proof. Given $n\in\mathbb{N}$ define

$$\mathfrak{D}_n=\left\{A\in\sigma[\mathcal{G}]|\mu\left(G_n\bigcap A\right)=\nu\left(G_n\bigcap A\right)\right\}\subseteq\sigma[\mathcal{G}] \tag{20.195}$$

We prove then that $\mathfrak{D}_n$ is a Dynkin system for every $n\in\mathbb{N}$. Take $G\in\mathcal{G}$ then $G,G_n\in\mathcal{G}$ so that by (1) $G_n\bigcap G\in\mathcal{G}$, as μ, ν are the same on $\mathcal{G}$ we have $\mu(G_n\bigcap G)=\nu(G_n\bigcap G)$. Hence we have

$$\mathcal{G}\subseteq\mathfrak{D}_n \tag{20.196}$$

As $X\in\sigma[\mathcal{G}]$ and $\mu(G_n\bigcap X)=\mu(G_n)\underset{G_n\in\mathcal{G}}{=}\nu(G_n)=\nu(G_n\bigcap X)$ it follows that

$$X\in\mathfrak{D}_n \tag{20.197}$$

Let $A\in\mathfrak{D}_n$ then we have $G_n\setminus(G_n\bigcap A)\underset{\text{[theorem: 1.31]}}{=}(G_n\setminus G_n)\bigcup(G_n\setminus A)=\varnothing\bigcup(G_n\setminus A)=G_n\setminus A$ so that

$$G_n\setminus\left(G_n\bigcap A\right)=G_n\setminus A \tag{20.198}$$

Further $\mu(G_n\bigcap A)\underset{\text{[theorem: 20.187]}}{\leqslant}\mu(G_n)\underset{(2)}{<}\infty$ and $\nu(G_n\bigcap A)\underset{\text{[theorem: 20.187]}}{\leqslant}\nu(G_n)\underset{(n)}{<}\infty$ so that by [theorem: 20.187]

$$\mu\left(G_n\setminus\left(G_n\bigcap A\right)\right)=\mu(G_n)-\mu\left(G_n\bigcap A\right)\wedge\nu\left(G_n\setminus\left(G_n\bigcap A\right)\right)=\nu(G_n)-\nu\left(G_n\bigcap A\right) \tag{20.199}$$

hence

$$\begin{aligned}
\mu\left(G_n\bigcap(X\setminus A)\right) &\underset{\text{[theorem: 1.33(8)]}}{=} \mu(G_n\setminus A)\\
&\underset{\text{[eq: 20.198]}}{=} \mu\left(G_n\setminus\left(G_n\bigcap A\right)\right)\\
&\underset{\text{[eq: 20.199]}}{=} \mu(G_n)-\mu\left(G_n\bigcap A\right)\\
&\underset{A\in\mathfrak{D}_n}{=} \mu(G_n)-\nu\left(G_n\bigcap A\right)\\
&\underset{G_n\in\mathcal{G}}{=} \nu(G_n)-\nu\left(G_n\bigcap A\right)\\
&\underset{\text{[eq: 20.199]}}{=} \nu\left(G_n\setminus\left(G_n\bigcap A\right)\right)\\
&\underset{\text{[eq: 20.198]}}{=} \nu(G_n\setminus A)\\
&\underset{\text{[theorem: 1.33]}}{=} \nu\left(G_n\bigcap(X\setminus A)\right)
\end{aligned}$$

proving that $(X\setminus A)\in\mathfrak{D}_n$, further as $A\in\mathfrak{D}_n\subseteq\sigma[\mathcal{G}]$ we have $X\setminus A\in\sigma[\mathcal{G}]$. Hence

$$\forall A\in\mathfrak{D}_n\text{ we have }X\setminus A\in\mathfrak{D}_n \tag{20.200}$$

Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathfrak{D}_n$ be a pairwise disjoint family. Then $\forall i\in\mathbb{N}$ with $i\neq j$ we have $(G_n\bigcap A_i)\bigcap(G_n\bigcap A_j)\subseteq A_i\bigcap A_j=\varnothing$ so that $\{G_n\bigcap A_i\}_{i\in\mathbb{N}}$ is pairwise disjoint. Hence

$$\mu\left(\bigsqcup_{i\in\mathbb{N}}\left(G_n\bigcap A_i\right)\right)=\sum_{i\in\mathbb{N}}\mu\left(G_n\bigcap A_i\right)\underset{A_i\in\mathfrak{D}_n}{=}\sum_{i\in\mathbb{N}}\nu\left(G_n\bigcap A_i\right)=\nu\left(\bigsqcup_{i\in\mathbb{N}}\left(G_n\bigcap A_i\right)\right)$$

which, as $G_n \bigcap \left(\bigsqcup_{i\in\mathbb{N}} A_i\right) = \bigsqcup_{i\in\mathbb{N}} (G_n \bigcap A_i)$ and $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathfrak{D}_n \subseteq \sigma[\mathcal{G}]$, proves that

$$\forall \{A_i\}_{i\in\mathbb{N}} \subseteq \mathfrak{D}_n \text{ we have } \bigsqcup_{i\in\mathbb{N}} A \in \mathfrak{D}_n \tag{20.201}$$

By [eqs: 20.197, 20.200 and 20.201] we have

$$\mathfrak{D}_n \text{ is a Dynkin system} \tag{20.202}$$

As $\forall A, B \in \mathcal{G}$ we have by (1) that $A \bigcap B \in \mathcal{G}$ we have by [theorem: 20.208] that

$$\delta[\mathcal{G}] = \sigma[\mathcal{G}] \tag{20.203}$$

As by [eq: 20.196] $\mathcal{G} \subseteq \mathfrak{D}_n$ and $\mathfrak{D}_n$ is a Dynkin system it follows that

$$\sigma[\mathcal{G}] \underset{[\text{eq: } 20.203]}{=} \delta[\mathcal{G}] \subseteq \mathfrak{D}_n \underset{[\text{eq: } 20.195]}{\subseteq} \sigma[\mathcal{G}]$$

Hence we have $\forall n \in \mathbb{N}$ that $\sigma[\mathcal{G}] = \mathfrak{D}_n$ so that using the definition of $\mathcal{D}_n$ [see eq: 20.195]

$$\forall n \in \mathbb{N} \text{ we have } \forall A \in \sigma[\mathcal{G}] \text{ that } \mu\left(G_n \bigcap A\right) = \nu\left(G_n \bigcap A\right) \tag{20.204}$$

Let $A \in \sigma[\mathcal{G}]$ then $\forall i \in \mathbb{N}$ we have $G_i \bigcap A \subseteq G_{i+1} \bigcap A \in \sigma[\mathcal{G}]$ and

$$A = A \bigcap X = A \bigcap \left(\bigcup_{i\in\mathbb{N}} G_i\right) = \bigcup_{i\in\mathbb{N}} \left(G_i \bigcap A\right) \tag{20.205}$$

so that by [theorem: 20.193] we have that

$$\lim_{i\to\infty} \mu\left(G_i \bigcap A\right) \text{ exists and } \lim_{i\to\infty} \mu\left(G_i \bigcap A\right) = \mu\left(\bigcup_{i\in\mathbb{N}} \left(G_i \bigcap A\right)\right) \underset{[\text{eq: } 20.205]}{=} \mu(A) \tag{20.206}$$

and

$$\lim_{i\to\infty} \nu\left(G_i \bigcap A\right) \text{ exists and } \lim_{i\to\infty} \nu\left(G_i \bigcap A\right) = \mu\left(\bigcup_{i\in\mathbb{N}} \left(G_i \bigcap A\right)\right) \underset{[\text{eq: } 20.205]}{=} \nu(A) \tag{20.207}$$

Hence we have for $A \in \sigma[\mathcal{G}]$ that

$$\mu(A) \underset{[\text{eq: } 20.206]}{=} \lim_{i\to\infty} \mu\left(G_i \bigcap A\right) \underset{[\text{eq: } 20.204]}{=} \lim_{i\to\infty} \nu\left(G_i \bigcap A\right) \underset{[\text{eq: } 20.207]}{=} \nu(A)$$

proving that

$$\mu = \nu$$

□

20.4.4 Constructing measures and σ-algebras

20.4.4.1 Constructing outer measures

Let X be a set then given $\mathcal{G} \subseteq \mathcal{P}(X)$ we want to extend a function $\mu: \mathcal{G} \rightarrow [0, \infty]$ to a measure on a σ-algebra containing $\mathcal{G}$. To use the Carathéodory theorem for this task we have to construct a outer measure out of μ. This is the subject of this section. First we need some definitions that specifies the minimum requirements for this task.

Definition 20.210. (Proto-Ring) *Let X be a set then $\mathcal{S} \subseteq \mathcal{P}(X)$ is a **proto-ring** on X if*

$$\varnothing \subseteq \mathcal{S}$$

Definition 20.211. (Proto-Measure) *Let X be a set then and $\mathcal{S}$ a proto-ring on X then a function $\mu: \mathcal{S} \rightarrow [0, \infty]$ is a **proto-measure** on $\mathcal{S}$ if*

$$\mu(\varnothing) = 0$$

Next we shown how to construct a outer-measure from a proto-measure.

Theorem 20.212. *Let X be a set, $\mathcal{S} \subseteq \mathcal{P}(X)$ a proto-ring and $\mu: \mathcal{S} \rightarrow [0, \infty]$ a proto-measure on $\mathcal{S}$ then*

1. $$\mu^{\ast}: \mathcal{P}(X) \rightarrow [0, \infty] \text{ defined by } \mu^{\ast}(A) = \begin{cases} \inf(\{\sum_{i=1}^{\infty} \mu(A_i) | \{A_i\}_{i \in \mathbb{N}} \in \mathcal{S}[A]\}) \text{ if } \mathcal{S}[A] \neq \varnothing \\ \infty \text{ if } \mathcal{S}[A] = \varnothing \end{cases}$$

   where

   $$\mathcal{S}[A] = \left\{ \{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{S} | A \subseteq \bigcup_{i \in \mathbb{N}} A_i \right\}$$

   is a outer measure.

2. *If additional we have that*

   a. *$\forall A, B \in S$ that $A \bigcap B \in \mathcal{S}$*

   b. *$\forall A, B \in \mathcal{S}$ with $A \subseteq B$ we have $\mu(A) \leqslant \mu(B)$*

   c. *$\forall \{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{S}$ with $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{S}$ we have $\mu(\bigcup_{i \in \mathbb{N}} A_i) \leqslant \sum_{i=1}^{\infty} \mu(A_i)$*

   then

   $$\forall A \in \mathcal{S} \text{ we have } \mu^{\ast}(A) = \mu(A)$$

***Note* 20.213.** *The outer measure defined in (1) is not neccesary a extension of μ, for that we need the extra conditions 2.a, 2.b, 2.c.*

Proof.

1. Define $\{N_i\}_{i \in \mathbb{N}} \subseteq \mathcal{S}$ by $N_i = \varnothing \in \mathcal{S}$ then $\varnothing \subseteq \bigcup_{i \in \mathbb{N}} N_i$ so that $\{N_i\}_{i \in \mathbb{N}} \in \mathcal{S}[\varnothing]$, hence $\mathcal{S}[\varnothing] \neq \oslash$. So $0 = \sum_{I=1}^{\infty} \mu(\varnothing) = \sum_{i=1}^{\infty} \mu(N_i) \in \{\sum_{i=1}^{\infty} \mu(A_i) | \{A_i\}_{i \in \mathbb{N}} \in \mathcal{S}[A]\}$, hence

   $$0 \leqslant \inf\left(\left\{ \sum_{i=1}^{\infty} \mu(A_i) | \{A_i\}_{i \in \mathbb{N}} \in \mathcal{S}[A] \right\} \right) \leqslant 0$$

   proving that $\mu^{\ast}(\varnothing) = \inf(\{\sum_{i=1}^{\infty} \mu(A_i) | \{A_i\}_{i \in \mathbb{N}} \in \mathcal{S}[A]\}) = 0$. Hence we have

   $$\mu^{\ast}(\varnothing) = 0 \tag{20.208}$$

   Let $A, B \in \mathcal{P}(X)$ with $A \subseteq B$. Then for $S[B]$ we have either:

   $\mathcal{S}[B] = \varnothing$. Then $\mu^{\ast}[A] \leqslant \infty = \mu^{\ast}[B]$

   $\mathcal{S}[B] \neq \varnothing$. If $\{A_i\}_{i \in \mathbb{N}} \in \mathcal{S}[B]$ then $\{A_i\}_{i \in \mathbb{N}} \subseteq S$ and $B \subseteq \bigcup_{i \in \mathbb{N}} A_i$ hence, as $A \subseteq B$ we have $A \subseteq \bigcup_{i \in \mathbb{N}} A_i$ so that $\{A_i\}_{i \in \mathbb{N}} \in \mathcal{S}[A]$. Hence

   $$\varnothing \neq \mathcal{S}[B] \subseteq S[A]$$

   From the above it follows that

   $$\left\{ \sum_{i=1}^{\infty} \mu(A_i) | \{A_i\}_{i \in \mathbb{N}} \in \mathcal{S}[B] \right\} \subseteq \left\{ \sum_{i=1}^{\infty} \mu(A_i) | \{A_i\}_{i \in \mathbb{N}} \in \mathcal{S}[A] \right\},$$

   so that by [theorem: 3.77]

   $$\begin{aligned} \mu^{\ast}[A] &= \inf\left(\left\{ \sum_{i=1}^{\infty} \mu(A_i) | \{A_i\}_{i \in \mathbb{N}} \in \mathcal{S}[A] \right\} \right) \\ &\leqslant \inf\left(\left\{ \sum_{i=1}^{\infty} \mu(A_i) | \{A_i\}_{i \in \mathbb{N}} \in \mathcal{S}[B] \right\} \right) \\ &= \mu^{\ast}[B] \end{aligned}$$

   To summarize

   $$\forall A, B \in \mathcal{P}(X) \text{ with } A \subseteq B \text{ we have } \mu^{\ast}(A) \leqslant \mu^{\ast}(B) \tag{20.209}$$

   Let $\{T_i\}_{i \in \mathbb{N}} \subseteq \mathcal{P}(X)$ then we have two cases to consider:

   $\exists i \in \mathbb{N}$ such that $\mu^{\ast}(T_i) = \infty$. Then $\mu^{\ast}(\bigcup_{i \in \mathbb{N}} T_i) \leqslant \infty = \sum_{i=1}^{\infty} \mu^{\ast}(A_i)$

$\forall i\in\mathbb{N}$ we have $\mu^*(T_i)<\infty$. Then we must have $\forall i\in\mathbb{N}$ that $\mathcal{S}[T_i]\neq\infty$ so that

$$\mu^*(T_i)=\inf\left(\left\{\sum_{i=1}^{\infty}\mu(A_i)|\{A_i\}_{i\in\mathbb{N}}\in\mathcal{S}[T_i]\right\}\right)$$

Take $\varepsilon\in\mathbb{R}^+$. Given $n\in\mathbb{N}$ we have, as $\mu^*(T_n)<\infty$, that $\mu^*(T_n)<\mu^*(T_n)+\frac{\varepsilon}{2^n}$, so as $\mu^*(T_n)=\inf(\{\sum_{i=1}^{\infty}\mu(A_i)|\{A_i\}_{i\in\mathbb{N}}\in\mathcal{S}[T_n]\})$, there exist a $\{I_{n,i}\}_{i\in\mathbb{N}}\in\mathcal{S}[T_n]$ with $T_n\subseteq\bigcup_{i\in\mathbb{N}}I_{n,i}$ such that $\mu^*(T_n)\leqslant\sum_{i=1}^{\infty}\mu(I_{n,i})<\mu^*(T_n)+\frac{\varepsilon}{2^n}$. Hence we have

$$\mu^*(T_n)\leqslant\sum_{i=1}^{\infty}\mu(I_{n,i})<\mu^*(T_n)+\frac{\varepsilon}{2^n}\text{ and }T_n\subseteq\bigcup_{i\in\mathbb{N}}I_{n,i} \tag{20.210}$$

As $\mathbb{N}\times\mathbb{N}$ is denumerable [see theorem: 6.63] there exist a bijection $\beta:\mathbb{N}\rightarrow\mathbb{N}\times\mathbb{N}$. Let $x\in\bigcup_{i\in\mathbb{N}}T_i$ then $\exists k\in\mathbb{N}$ such that $x\in T_k\underset{\text{[eq: 20.210]}}{\subseteq}\bigcup_{i\in\mathbb{N}}I_{k,i}$, hence $\exists l\in\mathbb{N}$ such that $x\in I_{k,l}$. As $(k,l)\in\mathbb{N}\times\mathbb{N}$ there exist a $i\in\mathbb{N}$ such that $\beta(i)=(k,l)$ so that $x\in I_{\beta(i)_1,\beta(i)_2}\subseteq\bigcup_{i\in\mathbb{N}}I_{\beta(i)_1,\beta(i)_2}$. To summarize we have proved that

$$\bigcup_{i\in\mathbb{N}}T_i\subseteq\bigcup_{i\in\mathbb{N}}I_{\beta(i)_1,\beta(i)_2}\text{ so that }\{I_{\beta(i)_1,\beta(i)_2}\}_{i\in\mathbb{N}}\in\mathcal{S}\left[\bigcup_{i\in\mathbb{N}}T_i\right] \tag{20.211}$$

Hence $\mathcal{S}[\bigcup_{i\in\mathbb{N}}T_i]\neq\varnothing$ so that

$$\begin{aligned}
\mu^*\left(\bigcup_{i\in\mathbb{N}}T_i\right) &= \inf\left(\left\{\sum_{i=1}^{\infty}\mu(A_i)|\{A_i\}_{i\in\mathbb{N}}\in\mathcal{S}\left[\bigcup_{i\in\mathbb{N}}T_i\right]\right\}\right)\\
&\underset{\text{[eq: 20.211]}}{\leqslant} \sum_{i=1}^{\infty}\mu(I_{\beta(i)_1,\beta(i)_2})\\
&\underset{\text{[lemma: 20.100]}}{=} \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\mu(I_{i,j})\right)\\
&\underset{\text{[eq: 20.210]}}{\leqslant} \sum_{i=1}^{\infty}\left(\mu^*(T_i)+\frac{\varepsilon}{2^i}\right)\\
&\underset{\text{[theorem: 20.91]}}{=} \sum_{i=1}^{\infty}\mu^*(T_i)+\sum_{i=1}^{\infty}\frac{\varepsilon}{2^i}\\
&\underset{\text{[theorem: 20.91]}}{=} \sum_{i=1}^{\infty}\mu^*(T_i)+\varepsilon\cdot\sum_{i=1}^{\infty}\frac{1}{2^i}\\
&= \sum_{i=1}^{\infty}\mu^*(T_i)+\varepsilon\cdot\lim_{n\rightarrow\infty}\sum_{i=1}^{n}\frac{1}{2^i}\\
&\underset{\text{[theorem: 20.68]}\wedge\text{[examle: 14.353]}}{=} \sum_{i=1}^{\infty}\mu^*(T_i)+\varepsilon\cdot\frac{\frac{1}{2}}{1-\frac{1}{2}}\\
&= \sum_{i=1}^{\infty}\mu^*(T_i)+\varepsilon
\end{aligned}$$

Hence we have

$$\forall\varepsilon\in\mathbb{R}^+\text{ that }\mu^*\left(\bigcup_{i\in\mathbb{N}}T_i\right)\leqslant\sum_{i=1}^{\infty}\mu^*(T_i)+\varepsilon$$

which, as $\varepsilon\in\mathbb{R}^+$ is choosen arbitrary, proves by [theorem: 20.40] that

$$\mu^*\left(\bigcup_{i\in\mathbb{N}}T_i\right)<\sum_{i=1}^{\infty}\mu^*(T_i).$$

To summarize we have in all cases that

$$\forall\{T_i\}_{i\in\mathbb{N}}\subseteq\mathcal{P}(X) \text{ we have } \mu^*\left(\bigcup_{i\in\mathbb{N}} T_i\right)\leqslant\sum_{i=1}^{\infty}\mu^*(T_i) \tag{20.212}$$

Taking in account the definition of a outer measure and [eqs: 20.208, 20.209 and 20.212] it follows that

$$\mu^* \text{ is a outer measure on } X$$

2. If $A\in\mathcal{S}$ then for $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{S}$ defined by $A_i=\left\{\begin{smallmatrix} A \text{ if } i=1 \\ \varnothing \text{ if } i\in\{2,\ldots,\infty\}\end{smallmatrix}\right.\in\mathcal{S}$ we have $A=\bigcup_{i\in\mathbb{N}}A_i$ so that $A\in S[A]\Rightarrow\mathcal{S}[A]\neq\varnothing$ and

$$\sum_{i=1}^{\infty}\mu(A_i)\underset{[\text{theorem: } 20.92]}{=}\sum_{i=1}^{1}\mu(A_i)+\sum_{i=2}^{\infty}\mu(A_i)=\mu(A_1)+\sum_{i=2}^{\infty}0=\mu(A)$$

so that $\mu(A)\in\{\sum_{i=1}^{\infty}\mu(A_i)|\{A_i\}_{i\in\mathbb{N}}\in S[A]\}$. Hence

$$\mu^*(A)=\inf\left(\left\{\sum_{i=1}^{\infty}\mu(A_i)|\{A_i\}_{i\in\mathbb{N}}\in S[A]\right\}\right)\leqslant\mu(A)$$

proving that

$$\forall A\in\mathcal{S} \text{ we have } \mu^*(A)\leqslant\mu(A) \tag{20.213}$$

For the opposite inequality, let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{S}[A]$ then $A\subseteq\bigcup_{i\in\mathbb{N}}A_i$ so that $\bigcup_{i\in\mathbb{N}}(A\bigcap A_i)=A\in\mathcal{S}$. By (2.a) we have $\{A\bigcap A_i\}_{i\in\mathbb{N}}\subseteq S$ and further

$$\mu(A)=\mu\left(\bigcup_{i\in\mathbb{N}}(A\bigcap A_i)\right)\underset{(2.c)}{\leqslant}\sum_{i=1}^{\infty}\mu\left(A\bigcap A_i\right)\underset{(2.b)}{\leqslant}\sum_{i=1}^{\infty}\mu(A_i)$$

Hence $\mu(A)$ is a lower bound of $\{\sum_{i=1}^{\infty}\mu(A_i)|\{A_i\}_{i\in\mathbb{N}}\in S[A]\}$, therefore $\mu(A)\leqslant\mu^*(A)$. Combining this with [eq: 20.213] gives finally

$$\mu(A)=\mu^*(A)$$

□

As is mentioned in [note: 20.213] if we want that the above theorem extend a proto-measure to a outer measure we need extra conditions on the proto-ring and the proto-measure. This motivates the definition of a semi-ring and a pre-measure below.

Definition 20.214. (Semi-Ring) *Let X be a set then $S\subseteq\mathcal{P}(X)$ is a* ***semi-ring*** *on X if*

1. $\varnothing\in\mathcal{S}$
2. *$\forall A,B\in\mathcal{S}$ we have $A\bigcap B\in\mathcal{S}$ [in other words $\mathcal{S}$ is $\bigcap$-stable]*
3. *$\forall A,B\in\mathcal{S}$ there exist a finite pairwise disjoint family $\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}$, $n\in\mathbb{N}$ such that*

$$A\setminus B=\bigsqcup_{i\in\{1,\ldots,n\}}A_i$$

Note 20.215. *As by (2) $\mathcal{S}$ is $\cap$-stable it follows from [lemma: 20.111] that $\forall\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}$, $n\in\mathbb{N}$ we have that $\bigcap_{i\in\{1,\ldots,n\}}A_i\in\mathcal{S}$.*

Note 20.216. *As $\varnothing\in\mathcal{S}$ it follows from the definition of a proto-ring [see definition: 20.210] that every semi-ring is a proto-ring.*

Definition 20.217. (Pre-Measure) *Let X be a set, $\mathcal{S}$ a semi-ring on X then a set function $\mu:\mathcal{S}\rightarrow[0,\infty]$ is a pre-measure if*

1. $\mu(\varnothing)=0$
2. *For every pairwise disjoint family $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{S}$ such that $\bigsqcup_{i\in\mathbb{N}}A_i\in\mathcal{S}$ we have that*

$$\mu\left(\bigsqcup_{i\in\mathbb{N}}A_i\right)=\sum_{i=1}^{\infty}\mu(A_i)$$

In other words $\mu: \mathcal{S} \rightarrow [0, \infty]$ is countable addive on $\mathcal{A}$]

Note 20.218. *As $\mu(\varnothing) = 0$ it follows from the definition of a proto-measure [see definition: 20.211] that that a pre-measure on a semi-ring is a proto-measure on a proto-ring.*

The above definitions do not exactly cover the extra conditions $1.a$, $2.b$ and $2.c$ in [theorem: 20.212]. The next theorems will ensure that this will be satisfied by semi-rings and pre-measures.

Lemma 20.219. *Let X be a set, $\mathcal{S}$ a semi-ring on X, $n \in \mathbb{N}$, $\{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S}$ and $A \in \mathcal{S}$ then there exist a pairwise disjoint family $\{B_i\}_{i \in \{1, \ldots, m\}} \subseteq \mathcal{S}$, $m \in \mathbb{N}$ such that*

$$A \setminus \bigcup_{i \in \{1, \ldots, n\}} A_i = \bigsqcup_{i \in \{1, \ldots, m\}} B_i$$

Proof. We use induction to prove this. Let $A \in \mathcal{S}$ and define

$$S_A = \left\{ n \in \mathbb{N} | \text{If } \{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S}_i \text{ then there exist a pairwise disjoint } \{B_i\}_{i \in \{1, \ldots, m\}} \subseteq \mathcal{S}, m \in \mathbb{N} \right.$$
$$\left. \text{such that } A \setminus \bigcup_{i \in \{1, \ldots, n\}} A_i = \bigsqcup_{i \in \{1, \ldots, m\}} B_i \right)$$

then we have:

$1 \in S_A$. If $\{A_i\}_{i \in \{1\}} \subseteq \mathcal{S}$ then $A \setminus \bigcup_{i \in \{1\}} A_i = A \setminus A_1$. As $\mathcal{S}$ is a semi-ring it follows that there exist a pairwise disjoint family $\{B_i\}_{i \in \{1, \ldots, m\}} \subseteq \mathcal{S}$, $m \in \mathbb{N}$ such that $A \setminus \bigcup_{i \in \{1\}} A_i = A \setminus A_1 = \bigsqcup_{i \in \{1, \ldots, m\}} B_i$ which proves that $1 \in \mathcal{S}$.

$n \in \mathcal{S}_A \Rightarrow n + 1 \in \mathcal{S}_A$. Let $\{A_i\}_{i \in \{1, \ldots, n+1\}} \subseteq \mathcal{S}$ then as $n \in S$ there exist a $\{B_i\}_{i \in \{1, \ldots, m\}} \subseteq \mathcal{S}$ pairwise disjoint such that

$$A \setminus \bigcup_{i \in \{1, \ldots, n\}} A_i = \bigsqcup_{i \in \{1, \ldots, m\}} B_i \tag{20.214}$$

Further we have

$$\begin{aligned}
A \setminus \bigcup_{i \in \{1, \ldots, n+1\}} A_i &= A \setminus \left(\left(\bigcup_{i \in \{1, \ldots, n\}} A_i \right) \bigcup A_{n+1} \right) \\
&\underset{\text{[theorem: 1.31]}}{=} \left(A \setminus \left(\bigcup_{i \in \{1, \ldots, n\}} A_i \right) \right) \setminus A_{n+1} \\
&\underset{\text{[eq: 20.214]}}{=} \left(\bigsqcup_{i \in \{1, \ldots, m\}} B_i \right) \setminus A_{n+1} \\
&\underset{\text{[theorem: 2.133]}}{=} \bigsqcup_{i \in \{1, \ldots, m\}} (B_i \setminus A_{n+1})
\end{aligned} \tag{20.215}$$

If $i, j \in \{1, \ldots, m\}$ with $i \neq j$ then we have $(B_i \setminus A_{n+1}) \bigcap (B_j \setminus A_{n+1}) \subseteq B_i \bigcap B_j = \varnothing$ proving that

$$\{B_i \setminus A_{n+1}\}_{i \in \{1, \ldots, m\}} \text{ is pairwise disjoint} \tag{20.216}$$

Let $i \in \{1, \ldots, m\}$ then as $\mathcal{S}$ is a semi-ring and $B_i, A_{n+1} \in \mathcal{S}$ there exist a pairwise disjoint $\{C_{i,j}\}_{j \in \{1, \ldots, n_i\}} \subseteq \mathcal{S}$ such that

$$B_i \setminus A_{n+1} = \bigsqcup_{j \in \{1, \ldots, n_i\}} C_{i,j} \tag{20.217}$$

Hence using [lemma: 20.113]

$$\bigsqcup_{i \in \{1, \ldots, m\}} (B_i \setminus A_{n+1}) = \bigsqcup_{\sigma \in \bigcup_{i \in \{1, \ldots, m\}} \{i\} \times \{1, \ldots, n_i\}} C_{\sigma_1, \sigma_2} \tag{20.218}$$

and

$$\{C_{\sigma_1, \sigma_2}\}_{\sigma \in \bigcup_{i \in \{1, \ldots, m\}} \{i\} \times \{1, \ldots, n_i\}} \subseteq \mathcal{S} \text{ is pairwise disjoint} \tag{20.219}$$

As $\bigcup_{i\in\{1,\ldots,m\}}\{i\}\times\{1,\ldots,n_i\}$ is finite [see theorems: 6.41, 6.36] there exist a bijection such that $\beta:\{1,\ldots,N\}\rightarrow\bigcup_{i\in\{1,\ldots,m\}}\{i\}\times\{1,\ldots,n_i\}$. Let $i,j\in\{1,\ldots,N\}$ with $i\neq j$ then, as β is injective, we have $\beta(i)\neq\beta(j)$. For $\beta(i)_1$, $\beta(j)_1$ we have either:

$\beta(i)_1=\beta(j)_1$. Then we must have that $\beta(i)_2\neq\beta(j)_2$ so that

$$C_{\beta(i)_1,\beta(i)_2}\bigcap C_{\beta(j)_1,\beta(j)_2}=C_{\beta(i)_1,\beta(i)_2}\bigcap C_{\beta(i)_1,\beta(j)_2}=\varnothing$$

$\beta(i)_1\neq\beta(j)_1$. Then we have

$$\begin{aligned}C_{\beta(i)_1,\beta(i)_2}\bigcap C_{\beta(j)_1,\beta(j)_2} &\subseteq \left(\bigcup_{k\in\{1,\ldots,n_{\beta(i)_1}\}}C_{\beta(i)_1,k}\right)\bigcap\left(\bigcup_{k\in\{1,\ldots,n_{\beta(j)_1}\}}C_{\beta(j)_1,k}\right)\\ &= B_{\beta(i)_1}\bigcap B_{\beta(j)_1}\\ &= \varnothing\end{aligned}$$

which proves that

$$\{C_{\beta(i)_1,\beta(i)_2}\}_{i\in\{1,\ldots,N\}}\subseteq\mathcal{S}\text{ is pairwise disjoint}\tag{20.220}$$

Further

$$\begin{aligned}A\setminus\bigcup_{i\in\{1,\ldots,n+1\}}A_i &\underset{[\text{eq: }20.215]}{=} \bigsqcup_{i\in\{1,\ldots,m\}}(B_i\setminus A_{n+1})\\ &\underset{[\text{eq: }20.217]}{=} \bigsqcup_{\sigma\in\bigcup_{i\in\{1,\ldots,m\}}\{i\}\times\{1,\ldots,n_i\}}C_{\sigma_1,\sigma_2}\\ &\underset{[\text{theorem: }2.119]}{=} \bigsqcup_{i\in\{1,\ldots,N\}}C_{\beta(i)_1,\beta(i)_2}\end{aligned}$$

which proves that $n+1\in\mathcal{S}$. □

Theorem 20.220. *Let X be a set, $\mathcal{S}$ a semi-ring on X and $\mu:\mathcal{S}\rightarrow[0,\infty]$ a pre-measure then $\forall\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq S$ pairwise disjoint, $n\in\mathbb{N}$ with $\bigcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{S}$ we have*

$$\mu\left(\bigcup_{i\in\{1,\ldots,n\}}A_i\right)=\sum_{i=1}^{n}A_i$$

In other words a pre-measure is additive.

Proof. As for a pre-measure we have $\mu(\varnothing)=0$ and $\mu:\mathcal{S}\rightarrow[0,\infty]$ is countable additive it follows from [theorem: 20.180] that $\mu:\mathcal{S}\rightarrow[0,\infty]$ is additive. In other words $\forall\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq S$ pairwise disjoint, $n\in\mathbb{N}$ with $\bigcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{S}$ we have

$$\mu\left(\bigcup_{i\in\{1,\ldots,n\}}A_i\right)=\sum_{i=1}^{n}A_i$$ □

Theorem 20.221. *Let X be a set, $\mathcal{S}$ a semi-ring on X and $\mu:\mathcal{S}\rightarrow[0,\infty]$ a additive set function [see definition: 20.116] then we have:*

1. *$\forall A\in\mathcal{S}$ and $\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}$, $n\in\mathbb{N}$* ***pairwise disjoint*** *such that $\bigcup_{i\in\{1,\ldots,n\}}A_i\subseteq A$ we have*

$$\sum_{i=1}^{n}\mu(A_i)\leqslant\mu(A)$$

2. *$\forall A,B\in\mathcal{S}$ with $A\subseteq B$ we have $\mu(A)\leqslant\mu(B)$*

Proof.

1. As $A\in\mathcal{S}$ and $\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}$, $n\in\mathbb{N}$ then by [lemma: 20.219] there exist a pairwise disjoint $\{B_i\}_{i\in\{1,\ldots,m\}}\subseteq\mathcal{S}$, $n\in\mathbb{N}$ such that

$$A\setminus\bigcup_{i\in\{1,\ldots,m\}}A_i=\bigsqcup_{i\in!,\ldots,m}B_i\tag{20.221}$$

Define

$$\{D_i\}_{i\in\{1,\ldots,n+m\}}\subseteq\mathcal{S} \text{ by } D_i=\begin{cases} A_i \text{ if } i\in\{1,\ldots,n\} \\ B_{i-n} \text{ if } i\in\{n+1,\ldots,n+m\}\end{cases}\subseteq\mathcal{S}$$

Let $i,j\in\{1,\ldots,n+m\}$ with $i\neq j$ then we can always assume that $i<j$ [otherwise exchange i and j]. For i we have either:

$i\in\{1,\ldots,n\}$. Then for j we have either:

$j\in\{1,\ldots,n\}$. Then $D_i\bigcap D_j=A_i\bigcap A_j \underset{\{A_i\}_{i\in\{1,\ldots,n\}} \text{ is pairwise disjoint}}{=} \varnothing$

$j\in\{n+1,\ldots,n+m\}$. Then

$$D_i\bigcap D_j=A_i\bigcap B_{j-n}\subseteq A_i\bigcap\left(A\setminus\bigcup_{i\in\{1,\ldots,n\}}A_i\right)=\varnothing$$

$i\in\{n+1,\ldots,n+m\}$. Then as $i<j$ we have $j\in\{n+1,\ldots,n+m\}$ so that

$$D_i\bigcap D_j=B_{i-n}\bigcap B_{j-n}\underset{i-n\neq j-n}{=}\varnothing$$

hence

$$\{D_i\}_{i\in\{1,\ldots,n+m\}}\subseteq\mathcal{S} \text{ is pairwise disjoint} \tag{20.222}$$

Further as $\bigsqcup_{i\in\{1,\ldots,n\}}A_i\subseteq A$ we have

$$\begin{aligned} A &\underset{[\text{theorem: } 1.35]}{=} \left(A\setminus\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)\bigcup\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right) \\ &= \left(\bigsqcup_{i\in\{1,\ldots,m\}}B_i\right)\bigsqcup\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right) \\ &= \left(\bigsqcup_{i\in\{1,\ldots,m\}}D_{i+n}\right)\bigsqcup\left(\bigsqcup_{i\in\{1,\ldots,n\}}D_i\right) \\ &\underset{[\text{theorem: } 2.119]}{=} \left(\bigsqcup_{i\in\{n+1,\ldots,n+m\}}D_i\right)\bigsqcup\left(\bigsqcup_{i\in\{1,\ldots,n\}}D_i\right) \\ &\underset{[\text{theorem: } 2.128]}{=} \bigsqcup_{i\in\{1,\ldots,n+m\}}D_i \end{aligned} \tag{20.223}$$

So as $\bigsqcup_{i\in\{1,\ldots,n+m\}}D_i=A\in\mathcal{S}$ we have

$$\begin{aligned} \mu(A) &\underset{[\text{eq: } 20.223]}{=} \mu\left(\bigsqcup_{i\in\{1,\ldots,n+m\}}D_i\right) \\ &\underset{\mu \text{ is additive}}{=} \sum_{i=1}^{n+m}\mu(D_i) \\ &= \sum_{i=1}^{n}\mu(D_i)+\sum_{i=n+1}^{n+m}\mu(D_i) \\ &= \sum_{i=1}^{n}\mu(A_i)+\sum_{i=n+1}^{n+m}\mu(D_i) \end{aligned}$$

so that by [theorem: 20.32] we have

$$\sum_{i=1}^{n}\mu(A_i)\leqslant\mu(A)$$

2. Let $A,B\in\mathcal{S}$ with $A\subseteq B$. Take then $\{A_i\}_{i\in\{1\}}\subseteq\mathcal{S}$ by $A_i=A$ then we have $\bigsqcup_{i\in\{1\}}A_i=A\subseteq B$, hence by (1) it follows that $\mu(A)=\sum_{i=1}^{1}\mu(A_i)\leqslant\mu(B)$ □

Corollary 20.222. *Let X be a set, $\mathcal{S}$ a semi-ring on X and $\mu: \mathcal{S} \rightarrow [0, \infty]$ a pre-measure on $\mathcal{S}$ [see definition: 20.217] then we have:*

1. $\forall A \in \mathcal{S}$ *and* $\{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S}$, $n \in \mathbb{N}$ *pairwise disjoint such that* $\bigcup_{i \in \{1, \ldots, n\}} A_i \subseteq A$ *we have*
$$\sum_{i=1}^{n} \mu(A_i) \leqslant \mu(A)$$
2. $\forall A, B \in \mathcal{S}$ *with* $A \subseteq B$ *we have* $\mu(A) \leqslant \mu(B)$

Proof. As a pre-measure is additive [see theorem: 20.220] this follows from the previous theorem [theorem: 20.221]. $\square$

Theorem 20.223. *Let X be a set, $\mathcal{S}$ a semi-ring on X and $\mu: \mathcal{S} \rightarrow [0, \infty]$ a pre-measure then*

$$\forall \{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{S} \text{ with } \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{S} \text{ we have } \mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) \leqslant \sum_{i=0}^{\infty} \mu(A_i)$$

In other words μ is countable sub-addtive.

Proof. Let $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{S}$ be a sequence such that $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{S}$. Define

$$\{B_i\}_{i \in \mathbb{N}} \text{ by } B_i = \begin{cases} A_1 \text{ if } i = 1 \\ A_i \setminus \bigcup_{j \in \{1, \ldots, i-1\}} A_j \text{ if } i \in \{2, \ldots, \infty\} \end{cases}$$

then by [lemma: 20.108]

$$\{B_i\}_{i \in \mathbb{N}} \text{ is pairwise disjoint and } \bigcup_{i \in \mathbb{N}} A_i = \bigsqcup_{i \in \mathbb{N}} B_i \text{ and } \forall i \in \mathbb{N}\ B_i \subseteq A_i \tag{20.224}$$

Note that the sets B_i are not elements of $\mathcal{S}$, however we will show that each B_i is a finite pairwise disjoint union of sets in $\mathcal{S}$ so that we can write $\bigcup_{i \in \mathbb{N}} A_i$ as countable pairwise disjoint union of elements of $\mathcal{S}$, allowing use to apply countable additivity. Let $i \in \mathbb{N}$ then we have either:

$i = 1$. Define $n_1 = 1$ and $\{C_{1,j}\}_{j \in \{1, \ldots, n_1\}} \subseteq \mathcal{S}$ by $C_{1,1} = A_1 \in \mathcal{S}$ then $\{C_{1,j}\}_{j \in \{1, \ldots, n_1\}} \subseteq S$ is pairwise disjoint and $B_1 = A_1 = \bigsqcup_{i \in \{1, \ldots, n_1\}} C_{1,j}$

$i \in \{2, \ldots, \infty\}$. By [lemma: 20.219] there exist a pairwise disjoint family $\{C_{i,j}\}_{j \in \{1, \ldots, n_i\}} \subseteq \mathcal{S}$ such that $B_i = A_i \setminus \bigcup_{j \in \{1, \ldots, i-1\}} A_j = \bigsqcup_{j \in \{1, \ldots, n_i\}} C_{i,j}$.

Hence we have

$$\forall i \in \mathbb{N}\ \exists \{C_{i,j}\}_{j \in \{1, \ldots, n_i\}} \subseteq \mathcal{S} \text{ pairwise disjoint such that } B_i = \bigsqcup_{j \in \{1, \ldots, n_i\}} C_{i,j} \tag{20.225}$$

By [corollary: 6.75] it follows that $\bigcup_{i \in \mathbb{N}} \{i\} \times \{1,..., n_i\}$ is denumerable, hence there exist a bijection

$$\beta: \mathbb{N} \rightarrow \bigcup_{i \in \mathbb{N}} \{i\} \times \{1, \ldots, n_i\}$$

Using the above and [eq: 20.225] allows us to use [corollary: 20.114] giving

$$\{C_{\beta(i)_1 \beta(i)_2}\}_{i \in \mathbb{N}} \subseteq \mathcal{S} \text{ is pairwise disjoint and } \bigcup_{i \in \mathbb{N}} A_i \underset{[\text{eq: } 20.224]}{=} \bigsqcup_{i \in \mathbb{N}} B_i = \bigsqcup_{i \in \mathbb{N}} C_{\beta(i)_1, \beta(i)_2} \tag{20.226}$$

So we have as $\bigsqcup_{i \in \mathbb{N}} C_{\beta(i)_1, \beta(i)_2} = \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{S}$ that

$$\begin{aligned} \mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) &\underset{[\text{eq: } 20.226]}{=} \mu\left(\bigsqcup_{i \in \mathbb{N}} C_{\beta(i)_1, \beta(i)_2}\right) \\ &\underset{\mu \text{ is a pre-measure}}{=} \sum_{i=1}^{\infty} \mu(C_{\beta(i)_1, \beta(i)_2}) \\ &\underset{[\text{theorem: } 20.99]}{=} \sum_{i=1}^{\infty} \left(\sum_{j=1}^{n_i} \mu(C_{i,j})\right) \end{aligned} \tag{20.227}$$

As $\forall i\in\mathbb{N}\ \bigcup_{j\in\{1,\ldots,n_i\}} C_{i,j} \underset{[\text{eq: } 20.225]}{=} B_i\subseteq A_i$ it follows from [corollary: 20.222(1)] and [eq: 20.226] that $\sum_{j=1}^{n_i}\mu(C_{i,j})\leqslant\mu(A_i)$, which combined with [eq: 20.227] and [theorem: 20.91] gives

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right)\leqslant\sum_{i=1}^{\infty} A_j$$

□

We can now combine the above theorems to prove a version of [theorem: 20.212] for semi-rings and pre-measures.

Theorem 20.224. *Let X be a set, $\mathcal{S}\subseteq\mathcal{P}(X)$ a semi-ring and $\mu:\mathcal{S}\to[0,\infty]$ a pre-measure on $\mathcal{S}$ then*

$$\mu^*:\mathcal{P}(X)\to[0,\infty] \textit{ defined by } \mu^*(A)=\begin{cases}\inf(\{\sum_{i=1}^{\infty}\mu(A_i)|\{A_i\}_{i\in\mathbb{N}}\in\mathcal{S}[A]\}) \textit{ if } \mathcal{S}[A]\neq\varnothing\\ \infty \textit{ if } \mathcal{S}[A]=\varnothing\end{cases}$$

where

$$\mathcal{S}[A]=\left\{\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{S}|A\subseteq\bigcup_{i\in\mathbb{N}} A_i\right\}$$

is a outer measure and

$$\forall A\in\mathcal{S} \textit{ we have } \mu^*(A)=\mu(A)$$

so that

$$\mu^* \textit{ is a extension of } \mu$$

Proof. As μ is a pre-measure [see definition: 20.217] we have that $\mu(\varnothing)=0$ hence

$$\mu \text{ is a proto-measure on } \mathcal{S}$$

Using [theorems: 20.222, 20.223] it follows that

$$\forall A,B\in\mathcal{S} \text{ with } A\subseteq B \text{ we have } \mu(A)\leqslant\mu(B)$$
$$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq S \text{ such that } \bigcup_{i\in\mathbb{N}} A_i\in\mathcal{S} \text{ we have } \mu\left(\bigcup_{i\in\mathbb{N}} A_i\right)\leqslant\sum_{i=1}^{\infty}\mu(A_i)$$

and as $\mathcal{S}$ is a semi-ring we have

$$\forall A,B\in\mathcal{S} \text{ we have } A\bigcap B\in\mathcal{S}$$

So all conditions for [theorem: 20.212] are satisfies so that

$$\mu^* \text{ is a outer measure on } S \text{ and } \forall A\in\mathcal{S} \text{ we have } \mu^*(A)=\mu(A)$$

□

In the above proof we use the fact that a pre-measure is countable sub-additive [see theorem 20.223] but never the countable additivity that is part of the definition of a pre-measure. So why do we define a pre-measure to be countable additive and not just countable sub-additive? The answer lies in proving the existance of a **unique extension** of a pre-measure on a semi-ring $\mathcal{S}$ to a measure on $\sigma[\mathcal{S}]$. To extend the pre-measure μ on $\mathcal{S}$ we first use the previous theorem [theorem: 20.212] to create a outer-measure μ^* that extend μ [here the need for countable sub-additivity], next use the Carathéodory theorem [theorem: 20.200] to find a measure on $\mathcal{M}[\mu^*]$. However to make sure that the constructed measure is also a measure on $\sigma[\mathcal{S}]$ we still have to prove that $\sigma[\mathcal{S}]\subseteq\mathcal{M}[\mu^*]$ and this is where we need additivity of μ. The countable additivity of a pre-measure ensures additional the additivity and countable sub-additivity of the pre-measure. Another way to look at this is the following, if μ on S is extended to a countable additive measure on $\sigma[\mathcal{S}]$ then we must have, as $\mathcal{S}\subseteq\sigma[\mathcal{S}]$, that μ is also countable additive.

Theorem 20.225. *Let X be a set, $\mathcal{S}$ a semi-ring on X and $\mu:\mathcal{S}\to[0,\infty]$ a pre-measure on $\mathcal{S}$ and define*

$$\mu^*:\mathcal{P}(X)\to[0,\infty] \text{ defined by } \mu^*(A)=\begin{cases}\infty \textit{ if } \mathcal{S}[A]=\varnothing\\ \inf(\{\sum_{i=1}^{\infty}\mu(A_i)|\{A_i\}_{i\in\mathbb{N}}\in\mathcal{S}[A]\})\end{cases}$$

where

$$\mathcal{S}[A]=\left\{\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{S}|A\subseteq\bigcup_{i\in\mathbb{N}}A_i\right\}$$

and

$$\mathcal{M}[\mu^*]=\left\{A\in\mathcal{P}(X)|\forall B\in\mathcal{P}(X)\text{ we have }\mu^*(B)=\mu^\star\left(B\bigcap A\right)+\mu^*(B\setminus A)\right\}$$

then

$$\mu^*\text{ is a outer measure}$$

$$\mathcal{M}[\mu^*]\text{ is a }\sigma\text{-algebra on }X$$

and

$$(\mu^*)_{|\mathcal{S}}=\mu\text{ [in other words }\mu^*\text{ extends }\mu$$

and

$$(\mu^*)_{|\mathcal{M}[\mu^*]}:\mathcal{M}[\mu^*]\rightarrow[0,\infty]\text{ is a measure on }\mathcal{M}[\mu^*]$$

and

$$\mathcal{S}\subseteq\mathcal{M}[\mu^*]$$

so that by [theorem: 20.141]

$$\sigma[\mathcal{S}]\subseteq\mathcal{M}[\mu^*]$$

and by [theorem: 20.185]

$$(\mu^*)_{|\sigma[\mathcal{S}]}:\sigma[\mathcal{S}]\rightarrow[0,\infty]\text{ is a measure on }\sigma[\mathcal{S}]$$

Hence

$$\langle X,\mathcal{M}[\mu^*],(\mu^*)_{\mathcal{M}[\mu^*]}\rangle\text{ and }\langle X,\sigma[\mathcal{S}],(\mu^*)_{|\sigma[\mathcal{S}]}\rangle\text{ are measure spaces}$$

where

$$\forall A\in\mathcal{S}\text{ we have }\mu(A)=\mu^*(A)=(\mu^*)_{|\mathcal{M}[\mu^*]}(A)=(\mu^*)_{|\sigma[\mathcal{S}]}$$

If additional there exist a $\{G_i\}_{i\in\mathbb{N}}\subseteq\mathcal{S}$ *such that* $\forall i\in\mathbb{N}\ G_i\subseteq G_{i+1}\wedge\mu(G_i)<\infty$ *and* $X=\bigcup_{i\in\mathbb{N}}G_i$ *then if* $\nu:\sigma[\mathcal{S}]\rightarrow[0,\infty]$ *is a another measure on* $\sigma[\mathcal{S}]$ *such that* $\nu_{|\mathcal{S}}=(\mu^*)_{|\mathcal{S}}$ *then*

$$\nu=(\mu^*)_{|\sigma[\mathcal{S}]}$$

In other words the extension of μ *to a measure* $(\mu^*)_{|\sigma[\mathcal{S}]}$ *on* $\sigma[\mathcal{S}]$ *is unique.*

Note 20.226. *We need additivity on* μ *to prove that* $\mathcal{S}\subseteq\mathcal{M}[\mu^*]$ *which is essential for the last part of the theorem.*

Proof. Using [theorem: 20.224] we have

$$\mu^*:\mathcal{P}(X)\rightarrow[0,\infty]\text{ defined by }\mu^*(A)=\begin{cases}\inf\left(\left\{\sum_{i=1}^\infty\mu(A_i)|\{A_i\}_{i\in\mathbb{N}}\in\mathcal{S}[A]\right\}\right)\text{ if }\mathcal{S}[A]\neq\varnothing\\\infty\text{ if }\mathcal{S}[A]=\varnothing\end{cases}\tag{20.228}$$

where

$$\mathcal{S}[A]=\left\{\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{S}|A\subseteq\bigcup_{i\in\mathbb{N}}A_i\right\}\tag{20.229}$$

is a outer measure and

$$\forall A\in\mathcal{S}\text{ we have }\mu^*(A)=\mu(A)\text{ hence }(\mu^*)_{|\mathcal{S}}=\mu\tag{20.230}$$

As μ^* is a outer measure we can use the Carathéodory [theorem: 20.200] so that

$$\mathcal{M}[\mu^*]=\left\{A\in\mathcal{P}(X)|\forall B\in\mathcal{P}(X)\text{ we have }\mu^*(B)=\mu^\star\left(B\bigcap A\right)+\mu^*(B\setminus A)\right\}$$

is a σ-algebra on X and

$$(\mu^*)_{|\mathcal{M}[\mu^*]}:\mathcal{M}[\mu^*]\rightarrow[0,\infty]\text{ is a measure on the }\sigma\text{-algebra }\mathcal{M}[\mu^*]\tag{20.231}$$

so that

$$\langle X,\mathcal{M}[\mu^*],(\mu^*)_{|\mathcal{M}[\mu^*]}\rangle\text{ a measure space}\tag{20.232}$$

Next we have to prove that $\mathcal{S} \subseteq \mathcal{M}[\mu^*]$. Let $R, T \in \mathcal{S}$ then as $\mathcal{S}$ is a semi-ring there exist a $m \in \mathbb{N}$, $\{S_i\}_{i \in \{1, \ldots, m\}} \subseteq \mathcal{S}$ pairwise disjoint such that

$$T \setminus R = \bigsqcup_{i \in \{1, \ldots, m\}} S_i \text{ and } R \bigcap T \in \mathcal{S} \tag{20.233}$$

Define now

$$\{D_i\}_{i \in \{1, \ldots, m+1\}} \subseteq \mathcal{S} \text{ by } D_i = \left\{ \begin{array}{l} S_i \text{ if } i \in \{1, \ldots, m\} \\ R \bigcap T \text{ if } i = m+1 \end{array} \right. \in \mathcal{S}$$

Let $i, j \in \{1, \ldots, m+1\}$ with $i \neq j$ then we may always assume that $i < j$ [otherwise exchange i and j], so that we have the following possibilities for j:

$\boldsymbol{j = m+1.}$ Then, as $i < j$, $i \in \{1, \ldots, m\}$ so that

$$\begin{array}{rcl} D_i \bigcap D_j & = & S_i \bigcap \left(R \bigcap T\right) \\ & \subseteq & \left(\bigsqcup_{i \in \{1, \ldots, m\}} S_i \right) \bigcap \left(R \bigcap T\right) \\ & \underset{[\text{eq: } 20.233]}{=} & (T \setminus R) \bigcap \left(R \bigcap T\right) \\ & \subseteq & (T \setminus R) \bigcap R \\ & = & \varnothing \end{array}$$

henve we have $D_i \bigcap D_j = \varnothing$.

$\boldsymbol{j \in \{1, \ldots, m\}.}$ Then, as $i < j$, $i \in \{1, \ldots, m\}$ so that $D_i \bigcap D_j = S_i \bigcap S = \varnothing$

which proves that

$$\{D_i\}_{i \in \{1, \ldots, m+1\}} \subseteq \mathcal{S} \text{ is pairwise disjoint}$$

Further

$$\begin{array}{rcl} T & = & (T \setminus R) \bigcup \left(T \bigcap R\right) \\ & = & \left(\bigcup_{i \in \{1, \ldots, m\}} S_i \right) \bigcup \left(T \bigcap R\right) \\ & = & \left(\bigcup_{i \in \{1, \ldots, m\}} D_i \right) \bigcup D_{m+1} \\ & = & \bigsqcup_{i \in \{1, \ldots, m+1\}} D_i \end{array}$$

Hence

$$\bigsqcup_{i \in \{1, \ldots, m+1\}} D_i = T \in \mathcal{S} \tag{20.234}$$

As by [theorem: 20.220] μ is additive on S we have

$$\begin{array}{rcl} \mu(T) & \underset{[\text{eq: } 20.234]}{=} & \mu\left(\bigsqcup_{i \in \{1, \ldots, m+1\}} D_i \right) \\ & \underset{\mu \text{ is additive}}{=} & \sum_{i=1}^{m+1} \mu(D_i) \\ & = & \mu(D_{m+1}) + \sum_{i=1}^{m} \mu(D_i) \\ & = & \mu\left(R \bigcap T\right) + \sum_{i=1}^{m} \mu(S_i) \end{array}$$

hence

$$\mu(T) = \mu\left(R \bigcap T\right) + \sum_{i=1}^{m} \mu(S_i) \tag{20.235}$$

Further as a outer measure is sub-additive [see definition: 20.195 and theorem: 20.122] we have

$$\begin{aligned}
\mu^*\left(R\bigcap T\right)+\mu^*(T\setminus R) &\underset{[\text{eq: } 20.233]}{=} \mu^*\left(R\bigcap T\right)+\mu^*\left(\bigsqcup_{i\in\{1,\ldots,m\}} S_i\right)\\
&\underset{\text{sub-additivity of } \mu^*}{\leqslant} \mu^*\left(R\bigcap T\right)+\sum_{i=1}^{m}\mu^*(S_i)\\
&\underset{[\text{eq: } 20.230]}{=} \mu\left(R\bigcap T\right)+\sum_{i=1}^{m}\mu(S_i)\\
&\underset{[\text{eq: } 20.235]}{=} \mu(T)
\end{aligned}$$

To summarize we have proved the following:

$$\forall R,S\in\mathcal{S} \text{ we have that } \mu^*\left(R\bigcap T\right)+\mu^*(T\setminus R)\leqslant\mu(T) \tag{20.236}$$

Let $A\in\mathcal{S}$ and take $B\in\mathcal{P}(X)$. For B we have either:

$\boldsymbol{\mathcal{S}[B]=\varnothing}$**.** Then $\mu^\star(B\setminus A)+\mu^*(B\bigcap A)\leqslant\infty=\mu^*(B)$

$\boldsymbol{S[B]\neq 0}$**.** Let $\{A_i\}_{i\in\mathbb{N}}\in\mathcal{S}[B]$ then we have

$$\begin{aligned}
\sum_{i=1}^{\infty}\mu^*\left(A_i\bigcap A\right)+\sum_{i=1}^{\infty}\mu^*(A_i\setminus A) &\underset{[\text{theorem: } 20.91]}{=}\\
\sum_{i=1}^{\infty}\left(\mu^*\left(A_i\bigcap A\right)+\mu^*(A_i\setminus A)\right) &\underset{[\text{theorem: } 20.91] \text{ and } [\text{eq: } 20.236]}{\leqslant} \sum_{i=1}^{\infty}\mu(A_i)
\end{aligned} \tag{20.237}$$

As $\{A_i\}_{i\in\mathbb{N}}\in\mathcal{S}[B]$ we have $B\subseteq\bigcup_{i\in\mathbb{N}}A_i$ so that

$$B\setminus A\subseteq\left(\bigcup_{i\in\mathbb{N}}A_i\right)\setminus A=\bigcup_{i\in\mathbb{N}}(A_i\setminus A) \text{ and } B\bigcap A\subseteq\left(\bigcup_{i\in\mathbb{N}}A_i\right)\bigcap A=\bigcup_{i\in\mathbb{N}}\left(A_i\bigcap A\right)$$

As μ^* is a outer measure hence monotone by definition [see definition: 20.195(2)] it follows from the above that

$$\mu^*(B\setminus A)\leqslant\mu^*\left(\bigcup_{i\in\mathbb{N}}(A_i\setminus A)\right) \text{ and } \mu^*\left(B\bigcap A\right)\leqslant\mu^*\left(\bigcup_{i\in\mathbb{N}}\left(A_i\bigcap A\right)\right) \tag{20.238}$$

Further using the sub-additivity of a outer measure [see definition: 20.195(3)] we have

$$\mu^*\left(\bigcup_{i\in\mathbb{N}}(A_i\setminus A)\right)\leqslant\sum_{i=1}^{\infty}\mu^*(A_i\setminus A) \text{ and } \mu^*\left(\bigcup_{i\in\mathbb{N}}\left(A_i\bigcap A\right)\right)\leqslant\sum_{i=1}^{\infty}\mu^*\left(A_i\bigcap A\right)$$

which combined with [eq: 20.238] gives

$$\mu^*(B\setminus A)\leqslant\sum_{i=1}^{\infty}\mu^*(A_i\setminus A) \text{ and } \mu^*\left(B\bigcap A\right)\leqslant\sum_{i=1}^{\infty}\mu^*\left(A_i\bigcap A\right)$$

so that

$$\begin{aligned}
\mu^*(B\setminus A)+\mu^*\left(B\bigcap A\right) &\leqslant \sum_{i=1}^{\infty}\mu^*(A_i\setminus A)+\sum_{i=1}^{\infty}\mu^*\left(A_i\bigcap A\right)\\
&\underset{[\text{eq: } 20.237]}{\leqslant} \sum_{i=1}^{\infty}\mu(A_i)
\end{aligned}$$

So $\mu^*(B\setminus A)+\mu^*(B\bigcap A)$ is a lower bound for $\{\sum_{i=1}^{\infty}\mu(A_i)|\{A_i\}_{i\in\mathbb{N}}\in\mathcal{S}[B]\}$ so that

$$\mu^*(B\setminus A)+\mu^*\left(B\bigcap A\right)\leqslant\inf\left(\left\{\sum_{i=1}^{\infty}\mu(A_i)|\{A_i\}_{i\in\mathbb{N}}\in\mathcal{S}[B]\right\}\right)\underset{[\text{eq: } 20.228]}{=}\mu^*(B)$$

So in all cases we have that $\mu^*(B\setminus A)+\mu^*(B\bigcap A)\leqslant\mu^*(B)$ which by [theorem: 20.198] means that $A\in\mathcal{M}[\mu^*]$. As $A\in\mathcal{S}$ was chosen arbitrary we conclude that

$$\mathcal{S}\subseteq\mathcal{M}[\mu^*] \tag{20.239}$$

As $\mathcal{M}[\mu^*]$ is a σ-algebra it follows from [theorem: 20.141] that

$$\mathcal{S}\subseteq\sigma[\mathcal{S}]\subseteq\mathcal{M}[\mu^*] \tag{20.240}$$

From the above, [eq: 20.231] and [theorem: 20.185] it follows that

$$(\mu^*)_{|\sigma[\mathcal{S}]} \underset{\text{[theorem: 2.86 (3)]}}{=} ((\mu^*)_{|\mathcal{M}[\mu^*]})_{|\sigma(\mathcal{S})} \text{ is a measure on } \sigma[\mathcal{S}]$$

making

$$\langle X,\sigma[\mathcal{S}],(\mu^*)_{|\sigma[\mathcal{S}]}\rangle \text{ a measure space}$$

Finally assume additional that there exist a $\{G_i\}_{i\in\mathbb{N}}\subseteq\mathcal{S}$ such that $\forall i\in\mathbb{N}\ G_i\subseteq G_{i+1}\wedge\mu(G_i)<\infty$. Let $\nu:\sigma[\mathcal{S}]\rightarrow[0,\infty]$ be a measure such that $\forall A\in\mathcal{S}$ we have $\mu(A)=\nu(A)$ then as $(\mu^*)_{|\sigma[\mathcal{S}]}(A)=\mu^*(A)\underset{\text{[eq: 20.230]}}{=}\mu(A)$ it follows that $\forall A\in\mathcal{S}\ (\mu^*)_{|\sigma[\mathcal{S}]}(A)=\nu(A)$. Hence by [theorem: 20.209] it follows that

$$(\mu^*)_{|\sigma[\mathcal{S}]}=\nu$$

proving uniqueness. □

20.4.4.2 Contents and pre-measures

So given a pre-measure μ on a semi-ring $\mathcal{S}$ we can extend μ to a measure on $\sigma[\mathcal{S}]$ and from there to a measure on $\mathcal{M}[\mathcal{S}]$. In the next section we show a way of proving that a additive set-function on a ring [see definition: 20.123] can be extended [if certain conditions are satisfied] to a pre-measure. First we show that every ring is actually a semi-ring.

Theorem 20.227. *Let X be a set, $\mathcal{R}\subseteq\mathcal{P}(X)$ a ring [see definition: 20.123] then $\mathcal{R}$ is a semi-ring [see definition: 20.214].*

Proof. As $\mathcal{R}$ is a ring we have by [theorem: 20.126] that

$$\varnothing\in\mathcal{R} \text{ and } \forall A,B\in\mathcal{R} \text{ we have } A\bigcap B\in\mathcal{R}$$

Further if $A,B\in\mathcal{R}$ then by the definition of a ring [see definition: 20.123] $A\setminus B\in\mathcal{R}$. So if we define $\{A_i\}_{i\in\{1\}}\subseteq\mathcal{R}$ by $A_1=A\setminus B$ we have that $A\setminus B=\bigsqcup_{i\in\{1\}}A_i$. Hence we have proved that

$$\forall A,B\in\mathcal{R}\ \ \exists\{A_i\}_{i\in\{1,\ldots,m\}}\subseteq\mathcal{R} \text{ pairwise disjointsuch that } A\setminus B=\bigsqcup_{i\in\{1,\ldots,m\}}A_i$$ □

Next we define what a **content** is.

Definition 20.228. *Let X be a set, $\mathcal{S}\subseteq\mathcal{P}(X)$ a semi-ring then $\mu:\mathcal{S}\rightarrow[0,\infty]$ is a* ***content*** *on $\mathcal{S}$ if*

1. $\mu(\varnothing)=0$
2. *$\forall\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}$, $n\in\mathbb{N}$ pairwise disjoint such that $\bigsqcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{S}$ we have $\mu(\bigsqcup_{i\in\{1,\ldots,n\}}A_i)=\sum_{i=1}^n\mu(A_i)$.*

A content is ***finite*** *if $\forall A\in\mathcal{S}\ \mu(A)<\infty$.*

***Note* 20.229.** *If $\mathcal{S}$ is a ring then (2) can be replaced by*

$$\forall\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S},\ n\in\mathbb{N} \text{ we have } \mu\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)=\sum_{i=1}^n\mu(A_i)$$

because [theorem: 20.126] ensures that $\bigsqcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{S}$.

Lemma 20.230. *Let X be a set $\mathcal{S}\subseteq\mathcal{P}(X)$ a semi-ring then a pre-measure $\mu:\mathcal{S}\rightarrow[0,\infty]$ is a content*

Proof. From the definition of a pre-measure [see definition: 20.217] it follows that

$$\mu(\varnothing)=0$$

Further $\forall\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}, n\in\mathbb{N}$ pairwise disjoint such that $\bigsqcup_{i\in\{1,\ldots,n\}}A_i\in\mathcal{S}$ we have by [theorem: 20.220] that

$$\mu\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)=\sum_{i=1}^{n}\mu(A_i)$$

□

We have a simpler equivalent definition of a content on a ring.

Theorem 20.231. *Let X be a set, $\mathcal{R}$ a ring on X and $\mu:\mathcal{R}\rightarrow[0,\infty]$ a function such that $\mu(\varnothing)=0$ [in other words μ is a proto-measure] then*

$$\mu \text{ is a content} \Leftrightarrow \forall A,B\in\mathcal{R} \text{ with } A\bigcap B=\varnothing \text{ we have } \mu\left(A\bigsqcup B\right)=\mu(A)+\mu(B)$$

Proof.

⇒. Let $A,B\in\mathcal{R}$ with $A\bigcap B=\varnothing$ and define $\{A_i\}_{i\in\{1,2\}}\subseteq\mathcal{R}$ by $A_1=A$ and $A_2=B$, then $\{A_i\}_{i\in\{1,2\}}$ is pairwise disjoint and $\mu(A\bigsqcup B)=\mu\left(\bigcup_{i\in\{1,2\}}A_i\right)=\sum_{i=1}^{2}\mu(A_i)=\mu(A)+\mu(B)$.

⇐. We prove this by induction so let

$$\mathcal{S}=\{n\in\mathbb{N}|\text{if } \{A_i\}_{i\in\{1,\ldots,n\}} \text{ is a pairwise disjoint family of sets in } \mathcal{R}\}$$

then we have

$1\in\mathcal{S}$. If $\{A_i\}_{i\in\{1,\ldots,1\}}$ is a pairwise disjoint family of sets in $\mathcal{R}$ then

$$\mu\left(\bigsqcup_{i\in\{1,\ldots,1\}}A_i\right)=\mu(A_1)=\sum_{i=1}^{1}\mu(A_i)$$

proving that $1\in\mathcal{S}$.

$n\in\mathcal{S}\Rightarrow n+1\in\mathcal{S}$. If $\{A_i\}_{i\in\{1,\ldots,n+1\}}$ is a pairwise disjoint family of sets in $\mathcal{R}$ then we have $\bigsqcup_{i\in\{1,\ldots,n+1\}}A_i=\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)\bigsqcup A_{n+1}$ so that

$$\begin{aligned}\mu\left(\bigsqcup_{i\in\{1,\ldots,n+1\}}A_i\right) &= \mu\left(\bigsqcup_{i\in\{1,\ldots,n+1\}}A_i\right)\\ &= \mu\left(\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)\bigsqcup A_{n+1}\right)\\ &= \mu\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)+\mu(A_{n+1})\\ &\underset{n\in\mathcal{S}}{=} \left(\sum_{i=1}^{n}\mu(A_i)\right)+\mu(A_{n+1})\\ &= \sum_{i=1}^{n+1}\mu(A_i)\end{aligned}$$

proving that $n+1\in\mathcal{S}$. □

A content has the following properties:

Theorem 20.232. *Let X be a set, $\mathcal{R}$ a ring on X and $\mu:\mathcal{R}\rightarrow[0,\infty]$ a content on $\mathcal{R}$ then we have:*

1. *$\forall A,B\in\mathcal{R}$ we have $\mu(A\bigcup B)+\mu(A\bigcap B)=\mu(A)+\mu(B)$*

2. *$\forall A, B \in \mathcal{R}$ with $A \subseteq B$ we have $\mu(A) \leqslant \mu(B)$ [μ is monotone].*
3. *$\forall A, B \in \mathcal{R}$ with $A \subseteq B$ and $\mu(A) < \infty$ we have $\mu(B \setminus A) = \mu(B) - \mu(A)$.*
4. *$\forall \{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{R}$, $n \in \mathbb{N}$ we have $\mu\left(\bigcup_{i \in \{1, \ldots, n\}} A_i\right) \leqslant \sum_{i=1}^{n} \mu(A_i)$ [sub-additivity]*
5. *$\forall \{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{R}$ pairwise disjoint such that $\bigsqcup_{i \in \mathbb{N}} A_i \in \mathcal{R}$ we have $\sum_{i=1}^{\infty} \mu(A_i) \leqslant \mu(\bigsqcup_{i \in \mathbb{N}} A_i)$.*

Proof. First as $\mathcal{R}$ is a ring we have that $\forall A, B \in \mathcal{R}$ $B \setminus A \in \mathcal{R}$, further

1. As $A \bigcup (B \setminus A) \underset{[\text{theorem: } 1.35]}{=} A \bigcup B$ with $A \bigcap (B \setminus A) \underset{[\text{theorem: } 1.35]}{=} \varnothing$ and μ is a content we have

$$\mu\left(A \bigcup B\right) = \mu(A) + \mu(B \setminus A) \tag{20.241}$$

Further $B \underset{[\text{theorem: } 1.35]}{=} (A \bigcap B) \bigcup (B \setminus A)$ with $(A \bigcap B) \bigcap (B \setminus A) \underset{[\text{theorem: } 1.35]}{=} \varnothing$ so that

$$\mu(B) = \mu\left(A \bigcap B\right) + \mu(B \setminus A) \tag{20.242}$$

For $\mu(B \setminus A)$ we have either:

$\boldsymbol{\mu(B \setminus A) < \infty}$**.** As μ is non negative we can sum [eq: 20.241] and [eq: 20.242] giving

$$\mu\left(A \bigcup B\right) + \mu\left(A \bigcap B\right) + \mu(B \setminus A) = \mu(B) + \mu(A) + \mu(B \setminus A)$$

and we can use [theorem: 20.28] to get

$$\mu\left(A \bigcup B\right) + \mu\left(A \bigcap B\right) = \mu(B) + \mu(A)$$

$\boldsymbol{\mu(B \setminus A) = \infty}$**.** Then

$$\mu\left(A \bigcup B\right) \underset{[\text{eq: } 20.241]}{=} \mu(A) + \mu(B \setminus A) = \mu(A) + \infty = \infty$$

and

$$\mu(B) \underset{[\text{eq: } 20.242]}{=} \mu\left(A \bigcap B\right) + \mu(B \setminus A) = \mu\left(A \bigcap B\right) + \infty = \infty$$

so that

$$\mu\left(A \bigcup B\right) + \mu\left(A \bigcap B\right) = \infty + \mu\left(A \bigcap B\right) = \infty = \mu(A) + \infty = \mu(A) + \mu(B)$$

So in all cases we have

$$\mu\left(A \bigcup B\right) + \mu\left(A \bigcap B\right) = \mu(B) + \mu(A)$$

2. If $A, B \in \mathcal{R}$ with $A \subseteq B$ then $A \bigcup B = B$ so that

$$\mu\,(B) = \mu\left(A \bigcup B\right) \underset{[\text{eq: } 20.241]}{=} \mu(A) + \mu(B \setminus A) \tag{20.243}$$

then by [theorem: 20.32] it follows that

$$\mu(A) \leqslant \mu(B)$$

3. As $A \subseteq B$ we have by [eq: 20.243] that $\mu(B) = \mu(A) + \mu(B \setminus A)$ so as $\mu(A) < \infty$ we can safely add $-\mu(A)$ to both sides giving

$$\mu(B) - \mu(A) = \mu(A) + \mu(B \setminus A) - \mu(A) = \mu(B \setminus A)$$

4. Let $\{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{R}$ and define

$$\{B_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{R} \text{ by } B_i = \begin{cases} A_1 \text{ if } i = 1 \\ A_i \setminus \bigcup_{j \in \{1, \ldots, i-1\}} A_j \underset{\mathcal{R} \text{ is a ring and [theorem: 20.126]}}{\in} \mathcal{R} \end{cases}$$

so that

$$\{B_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{R} \text{ and } \forall i \in \{1, \ldots, n\}\ B_i \subseteq A_i \underset{(2)}{\Rightarrow} \mu(B_i) \leqslant \mu(A_i) \tag{20.244}$$

Using [lemma: 20.108] we have that

$$\{B_i\}_{i\in\{1,\ldots,n\}} \text{ is pairwise disjoint and } \bigcup_{i\in\{1,\ldots,n\}} A_i = \bigsqcup_{i\in\{1,\ldots,n\}} B_i \tag{20.245}$$

Hence

$$\begin{aligned}\mu\left(\bigcup_{i\in\{1,\ldots,n\}} A_i\right) &\underset{[\text{eq: } 20.245]}{=} \mu\left(\bigsqcup_{i\in\{1,\ldots,n\}} B_i\right)\\ &\underset{[\text{definition: } 20.228]}{=} \sum_{i=1}^{n} \mu(B_i)\\ &\underset{[\text{eq: } 20.244]}{\leqslant} \sum_{i=1}^{n} \mu(A_i)\end{aligned}$$

5. Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R}$ be a pairwise disjoint sequence of sets in $\mathcal{R}$ such that $\bigcup_{i\in\mathbb{N}} A_i\in\mathcal{R}$. Let $n\in\mathbb{N}$ then we have $\bigsqcup_{i\in\{1,\ldots,n\}} A_i\subseteq\bigsqcup_{i\in\mathbb{N}} A_i$ so that

$$\sum_{i=1}^{n} \mu(A_i) \underset{[\text{definition: } 20.228]}{=} \mu\left(\bigsqcup_{i\in\{1,\ldots,n\}} A_i\right) \underset{(2)}{\leqslant} \mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \tag{20.246}$$

Hence $\mu(\bigcup_{i\in\mathbb{N}} A_i)$ is a upper bound of $\{\sum_{i=1}^{n}\mu(A_i)|n\in\mathbb{N}\}$ so that

$$\sup\left(\left\{\sum_{i=1}^{n} \mu(A_i)|n\in\mathbb{N}\right\}\right)\leqslant\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right)$$

Hence

$$\sum_{i=1}^{\infty} \mu(A_i) \underset{[\text{definition: } 20.87]}{=} \sup\left(\left\{\sum_{i=1}^{n} \mu(A_i)|n\in\mathbb{N}\right\}\right)\leqslant\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right)$$

□

Similar to [theorems: 20.193, 20.194] we have the following neccesary and sufficient conditions for a content to be a pre-measure.

Theorem 20.233. *Let X be a set, $\mathcal{R}\subseteq X$ a ring on X and $\mu:\mathcal{R}\rightarrow[0,\infty]$ a content then for the following statements:*

1. *μ is a pre-measure*
2. *$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R}$ such that $\forall i\in\mathbb{N}$ $A_i\subseteq A_{i+1}$ and $\bigcup_{i\in\mathbb{N}} A_i\in\mathcal{R}$ we have $\mu(\bigcup_{i\in\mathbb{N}} A_i)=\lim_{i\rightarrow\infty}\mu(A_i)$.*
3. *$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R}$ such that $\forall i\in\mathbb{N}$ $A_{i+1}\subseteq A_i$ and $\mu(A_i)<\infty$ and $\bigcap_{i\in\mathbb{N}} A_i\in\mathcal{R}$ we have $\mu(\bigcap_{i\in\mathbb{N}} A_i)=\lim_{i\rightarrow\infty}\mu(A_i)$.*
4. *$\forall\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R}$ such that $\forall i\in\mathbb{N}$ $A_{i+1}\subseteq A_i$ and $\mu(A_i)<\infty$ and $\bigcap_{i\in\mathbb{N}} A_i=\varnothing$ we have $\lim_{i\rightarrow\infty}\mu(A_i)=0$.*

we have $(1)\Leftrightarrow(2)\Rightarrow(3)\Leftrightarrow(4)$, *Further if μ is finite then* $(1)\Leftrightarrow(2)\Leftrightarrow(3)\Leftrightarrow(4)$

Proof. First as $\mathcal{R}$ is a ring we have by [theorem: 20.126] that

$$\forall\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{R} \bigcup_{i\in\{1,\ldots,n\}} A_i\in\mathcal{R} \text{ and } \bigcap_{i\in\{1,\ldots,n\}} A_i\in\mathcal{R} \tag{20.247}$$

(1) ⇒ (2). Let$\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R}$ be such that $\forall i\in\mathbb{N}$ $A_i\subseteq A_{i+1}$ and $\bigcup_{i\in\mathbb{N}} A_i\in\mathcal{R}$. Define

$$\{B_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R} \text{ by } B_i=\left\{\begin{array}{l}A_1 \text{ if } i=1\\ A_i\backslash A_{i-1} \text{ if } i\in\{2,\ldots,\infty\}\end{array}\right. \underset{\mathcal{R}\text{ is a ring}}{\in} \mathcal{R}$$

Then by the above definition we have

$$\forall i\in\mathbb{N}\; B_i\subseteq A_i \tag{20.248}$$

Let $i,j\in\mathbb{N}$ with $i\neq j$. We may always assume that $i<j$ [otherwise exchange i and j], then for i we have either

$i=1$. Then $1<j$ so that $B_1=A_1$ and $B_j=A_j\setminus A_{j-1}$. As $1<j\Rightarrow 1\leqslant j-1$ we have $A_1\subseteq A_{j-1}$ so that by [theorem: 1.33] $B_j=A_j\setminus A_{j-1}\subseteq A_j\setminus A_1=A_j\setminus B_1$. Hence

$$B_i\bigcap B_j=B_1\bigcap(A_j\setminus B_1)=\varnothing$$

$i\in\{2,\ldots,\infty\}$. Then $i,j\in\{2,...,\infty\}$ so that $B_i=A_i\setminus A_{i-1}\wedge B_j=A_j\setminus A_{j-1}$. As $i<j\Rightarrow i\leqslant j-1$ we have $B_i\subseteq A_i\subseteq A_{j-1}$ so that by [theorem: 1.33] $B_j=A_j\setminus A_{j-1}\subseteq A_j\setminus B_i$. Hence

$$B_j\bigcap B_i\subseteq(A_j\setminus B_i)\bigcap B_i=\varnothing$$

Hence it follows that

$$\{B_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R}\text{ is pairwise disjoint} \tag{20.249}$$

Let $n\in\mathbb{N}$. As $\forall i\in\{1,\ldots,n\}$ we have $B_i\underset{[\text{eq: } 20.248]}{\subseteq}A_i\subseteq A_n$ so that

$$\bigsqcup_{i\in\{1,\ldots,n\}}B_i\subseteq A_n \tag{20.250}$$

If $x\in A_n$ then $n\in\{i\in\{1,\ldots,n\}|x\in A_i\}$ a finite set so that $j=\min(\{i\in\{1,\ldots,n\}|x\in A_i\})$ exists. For j we have either:

$j=1$. Then $x\in A_1=B_1$ so that $x\in\bigsqcup_{i\in\{1,\ldots,n\}}B_i$

$1<j$. Then $x\in A_j$ and $x\notin A_{j-1}$ so that $x\in A_j\setminus A_{j-1}=B_j\subseteq\bigsqcup_{i\in\{1,\ldots,n\}}B_i$

hence $A_n\subseteq\bigsqcup_{i\in\{1,\ldots,n\}}B$ which combined with [eq: 20.250] we have

$$\forall n\in\mathbb{N}\text{ we have }A_n=\bigsqcup_{i\in\{1,\ldots,n\}}B_i \tag{20.251}$$

Let $x\in\bigcup_{n\in\mathbb{N}}A_n$ then $\exists n\in\mathbb{N}$ so that $x\in A_n$ and by [eq: 20.251] $\exists m\in\{1,\ldots,n\}\subseteq\mathbb{N}$ such that $x\in B_m$, hence $\bigcup_{n\in\mathbb{N}}A_n\subseteq\bigcup_{i\in\mathbb{N}}B_i$. As also $B_n\underset{[\text{eq: } 20.248]}{\subseteq}A_n\subseteq\bigcup_{n\in\mathbb{N}}A_n$ it follows that

$$\bigsqcup_{i\in\mathbb{N}}B_i=\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{S} \tag{20.252}$$

As μ is a pre-measure hence countable additive we have that

$$\begin{aligned}
\mu\left(\bigcup_{i\in\mathbb{N}}A_i\right) &= \mu\left(\bigsqcup_{i\in\mathbb{N}}B_i\right)\\
&= \sum_{i=1}^{\infty}\mu(B_i)\\
&\underset{[\text{definition: } 20.87]}{=} \lim_{i\rightarrow\infty}\sum_{j=1}^{i}\mu(B_i)\\
&\underset{[\text{theorem: } 20.220]}{=} \lim_{i\rightarrow\infty}\mu\left(\bigcup_{j=1}^{i}B_j\right)\\
&\underset{[\text{eq: } 20.251]}{=} \lim_{i\rightarrow\infty}\mu(A_i)
\end{aligned}$$

Hence we have

$$\mu\left(\bigcup_{i\in\mathbb{N}}A_i\right)=\lim_{i\rightarrow\infty}\mu(A_i)$$

(2) ⇒ (1). As for a content $\mu(\varnothing)=0$ we only have to prove countable additivity. Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R}$ be a pairwise disjoint sequence of sets in $\mathcal{R}$ such that $\bigcup_{i\in\mathbb{N}} A_i$. Define

$$\{B_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R} \text{ by } B_i=\bigsqcup_{j\in\{1,\ldots,i\}} A_j$$

Then we have

$$\forall i\in\mathbb{N}\ B_i=\bigsqcup_{j\in\{1,\ldots,i\}} A_i\subseteq\left(\bigsqcup_{j\in\{1,\ldots,i\}} A_i\right)\bigcup A_{i+1}=\bigsqcup_{j\in\{1,\ldots,i+1\}} A_i=B_{i+1} \tag{20.253}$$

If $x\in\bigsqcup_{i\in\mathbb{N}} A_i$ then $\exists i\in\mathbb{N}$ such that $x\in A_i\subseteq\bigsqcup_{j\in\{1,\ldots,i\}} A_i=B_i\subseteq\bigcup_{i\in\mathbb{N}} B_i$, hence $\bigcup_{i\in\mathbb{N}} A_i\subseteq\bigcup_{i\in\mathbb{N}} B_i$, as also $B_i=\bigcup_{j\in\{1,\ldots,i\}} A_j\subseteq\bigcup_{j\in\mathbb{N}} B_j$ it follows that $\bigcup_{i\in\mathbb{N}} B_i\subseteq\bigcup_{j\in\mathbb{N}} B_j$. Hence

$$\bigcup_{i\in\mathbb{N}} B_i=\bigsqcup_{i\in\mathbb{N}} A_i\in\mathcal{R} \tag{20.254}$$

Using (1) together with [eqs: 20.253, 20.254] it follows that

$$\mu\left(\bigcup_{i\in\mathbb{N}} B_i\right)=\lim_{i\rightarrow\infty}\mu(B_i)=\lim_{i\rightarrow\infty}\mu\left(\bigsqcup_{j\in\{1,\ldots,i\}} A_j\right) \tag{20.255}$$

A μ is a content hence additive we have $\forall i\in\mathbb{N}$ that $\mu\left(\bigsqcup_{j\in\{1,\ldots,i\}} A_j\right)=\sum_{j=1}^{i}\mu(A_j)$ so that by the above

$$\mu\left(\bigcup_{i\in\mathbb{N}} B_i\right)=\lim_{i\rightarrow\infty}\sum_{j=1}^{i}\mu(A_j)\underset{[\text{theorem: } 20.87]}{=}\sum_{i=1}^{\infty}\mu(A_i)$$

Finally using [eq: 20.254] it follows that

$$\mu\left(\bigsqcup_{i\in\mathbb{N}} A_i\right)=\sum_{i=1}^{\infty}\mu(A_i)$$

proving countable additivity.

(2) ⇒ (3). Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R}$ such that $\forall i\in\mathbb{N}\ \mu(A_i)<\infty$, $A_{i+1}\subseteq A_i$ and $\bigcap_{i\in\mathbb{N}} A_i\in\mathcal{R}$. Define

$$\{B_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R} \text{ by } B_i=A_1\setminus A_i\underset{\mathcal{R}\text{ is a ring}}{\in}\mathcal{R}$$

Then we have

$$\forall i\in\mathbb{N}\ B_i=A_1\setminus A_i\underset{[\text{theorem: } 1.33]}{\subseteq}A_1\setminus A_{i+1}=B_i \tag{20.256}$$

and

$$A_1\setminus\left(\bigcap_{i\in\mathbb{N}} A_i\right)\underset{[\text{theorem: } 2.132]}{=}\bigcup_{i\in\mathbb{N}}(A_1\setminus A_i)=\bigcup_{i\in\mathbb{N}} B_i \tag{20.257}$$

As $\mathcal{R}$ is a ring and $A_1,\bigcap_{i\in\mathbb{N}} A_i\in\mathcal{R}$ we have that $\bigcup_{i\in\mathbb{N}} B_i=A_1\setminus\left(\bigcap_{i\in\mathbb{N}} A_i\right)\in\mathcal{R}$. Hence we can apply (2)

$$\mu\left(A_1\setminus\bigcap_{i\in\mathbb{N}} A_i\right)=\mu\left(\bigcup_{i\in\mathbb{N}} B_i\right)\underset{(2)}{=}\lim_{i\rightarrow\infty}\mu(B_i)=\lim_{i\rightarrow\infty}\mu(A_1\setminus A_i) \tag{20.258}$$

Let $i\in\mathbb{N}$ then $\bigcap_{i\in\mathbb{N}} A_i\subseteq A_i$ so that by [theorem: 20.232 (2)] $\mu\left(\bigcap_{i\in\mathbb{N}} A_i\right)\leqslant\mu(A_i)<\infty$, further we have $A_1\setminus A_i\subseteq A_1\Rightarrow\mu(A_1\setminus A_i)\leqslant\mu(A_1)<\infty$. Hence using [theorem: 20.232(3)] we have

$$\forall i\in\mathbb{N} \text{ that } \mu(B_i)=\mu(A_1\setminus A_i)=\mu(A_1)-\mu(A_i) \tag{20.259}$$

and

$$\mu\left(\bigcup_{i\in\mathbb{N}} B_i\right)\underset{[\text{eq: } 20.257]}{=}\mu\left(A_1\setminus\bigcap_{i\in\mathbb{N}} A_i\right)=\mu(A_1)-\mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) \tag{20.260}$$

So

$$\mu(A_1) - \mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) \underset{[\text{eqs: } 20.258,\ 20.260]}{=} \lim_{i\to\infty} \mu(A_1 \backslash A_i)$$
$$\underset{[\text{eq; } 20.259]}{=} \lim_{i\to\infty} (\mu(A_1) - \mu(A_i))$$
$$\underset{[\text{theorem: } 20.78]}{=} \mu(A_1) - \lim_{i\to\infty} \mu(A_i)$$

As $\mu(A_1) < \infty \Rightarrow \mu(A_1) \in \mathbb{R}$ we can safely subtract $\mu(A)$ from both sides from the above equation to get $-\mu\left(\bigcap_{i\in\mathbb{N}} A_i = -\lim_{i\to\infty} \mu(A_i)\right.$ from which it follows that

$$\mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) = \lim_{i\to\infty} \mu(A_i)$$

(3) ⇒ (4). Let $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{R}$ be such that $\forall i \in \mathbb{N}\ A_{i+1} \subseteq A_i$ and $\mu(A_i) < \infty$ and $\bigcap_{i\in\mathbb{N}} A_i = \varnothing \in \mathcal{S}$ then by (3) we have

$$0 = \mu(\varnothing) = \mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) = \lim_{i\to\infty} \mu(A_i)$$

(4) ⇒ (3). Let $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{R}$ be such that $\forall i \in \mathbb{N}\ A_{i+1} \subseteq A_i$ and $\mu(A_i) < \infty$ and $\bigcap_{i\in\mathbb{N}} A_i \in \mathcal{R}$. Take $A = \bigcap_{i\in\mathbb{N}} A_i \in \mathcal{R}$ and define

$$\{B_i\}_{i\in\mathbb{N}} \subseteq \mathcal{R} \text{ by } B_i = A_i \backslash A \underset{\mathcal{R} \text{ is a ring}}{\in} \mathcal{R}$$

then we have

$$\forall i \in \mathbb{N} \text{ that } B_{i+1} = A_{i+1} \backslash A \subseteq A_i \backslash A = B_i$$

Further we have

$$\varnothing = A \backslash A = \left(\bigcap_{i\in\mathbb{N}} A_i\right) \backslash A \underset{[\text{theorem: } 2.133]}{=} \bigcap_{i\in\mathbb{N}} (A_i \backslash A) = \bigcap_{i\in\mathbb{N}} B_i$$

Hence using (4) we have

$$0 = \lim_{i\to\infty} \mu(B_i) = \lim_{i\to\infty} (A_i \backslash A) \tag{20.261}$$

As $A = \bigcap_{i\in\mathbb{N}} A_i \subseteq A_i$ it follows from [theorem: 20.232(2)] that $\mu(A) \leqslant \mu(A_i) < \infty$, hence we have by [theorem: 20.232(3)] that

$$\forall i \in \mathbb{N} \text{ we have } \mu(A_i \backslash A) = \mu(A_i) - \mu(A) \tag{20.262}$$

Hence we have

$$0 \underset{[\text{eq: } 20.261]}{=} \lim_{i\to\infty} \mu(A_i \backslash A)$$
$$\underset{[\text{eq: } 20.262]}{=} \lim_{i\to\infty} (\mu(A_i) - \mu(A))$$
$$\underset{[\text{theorem: } 20.78]}{=} \lim_{i\to\infty} \mu(A_i) - \mu(A)$$

As $\mu(A) < \infty \Rightarrow \mu(A) \in \mathbb{R}$ we have

$$\mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) = \mu(A) = \lim_{i\to\infty} \mu(A_i)$$

Now for the last part we have to prove that if μ is finite then (4) ⇒ (2) [because then from (1) ⇔ (2) ⇒ (3) ⇔ (4) it follows that (1) ⇔ (2) ⇔ (3) ⇔ 4].

(4) ⇒ (2). Let $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{R}$ is such that $\forall i \in \mathbb{N}$ we have $A_i \subseteq A_{i+1}$ and $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{R}$. Take $A = \bigcup_{i\in\mathbb{N}} A_i$ and define

$$\{B_i\}_{i\in\mathbb{N}} \subseteq \mathcal{R} \text{ by } B_i = A \backslash A_i \underset{\mathcal{R} \text{ is a ring}}{\in} \mathcal{R}$$

Then we have

$$\forall i\in\mathbb{N} \text{ that } B_{i+1}=A\setminus A_{i+1} \underset{[\text{theorem: } 1.33]}{\subseteq} A\setminus A_i=B_i \text{ and } \mu(B_i) \underset{\mu \text{ is finite}}{<} \infty$$

Further we have

$$\varnothing=A\setminus A=A\setminus\left(\bigcup_{i\in\mathbb{N}} A_i\right) \underset{[\text{theorem: } 2.133]}{=} \bigcap_{i\in\mathbb{N}} (A\setminus A_i)=\bigcap_{i\in\mathbb{N}} B_i$$

So using (4) we have that

$$\lim_{i\rightarrow\infty}\mu(A\setminus A_i)=\lim_{i\rightarrow\infty}\mu(B_i)=0 \tag{20.263}$$

Let $i\in\mathbb{N}$ then, as $A_i\subseteq\bigcup_{i\in\mathbb{N}} A_i=A$ and $\mu(A_i) \underset{\mu \text{ is finite}}{<} \infty$, it follows from [theorem: 20.232(3)] that $\mu(A\setminus A_i)=\mu(A)-\mu(A_i)$. Hence we have

$$\begin{aligned} 0 &\underset{[\text{eq: } 20.263]}{=} \lim_{i\rightarrow\infty}\mu(A\setminus A_i) \\ &= \lim_{i\rightarrow\infty}(\mu(A)-\mu(A_i)) \\ &\underset{[\text{theorem: } 20.78]}{=} \mu(A)-\lim_{i\rightarrow\infty}\mu(A_i) \end{aligned}$$

As μ is finite it follows from the above that

$$\mu\left(\bigsqcup_{i\in\mathbb{N}} A_i\right)=\mu(A)=\lim_{i\rightarrow\infty}\mu(A_i)$$

□

The above equivalences allows us to determine if a content on a ring is a pre-measure. However this is not always simple. If we assume a ring on a topological space then proving becomes easier if we make use of the properties of the topological space. For this we introduce approximation by compact classes. First recap the definition of a compact class [see definition: 14.242].

Definition 20.234. *Let X be a set then $\mathcal{C}\subseteq\mathcal{P}(X)$ is a **compact class** if $\forall\{K_i\}_{i\in\mathbb{N}}\subseteq\mathcal{C}$ with $\bigcap_{i\in\mathbb{N}} K_i=\varnothing$ there exist a $N\in\mathbb{N}$ such that $\bigcap_{i\in\{1,\ldots,N\}} K_i=\varnothing$.*

We have then the following theorem that can be used to determine if a content is a pre-measure.

Theorem 20.235. *Let X be a set, $\mathcal{R}$ a ring on X and $\mu: X\rightarrow[0,\infty]$ a **finite** content. If there exist a compact class $\mathcal{C}\subseteq\mathcal{P}(X)$ such that $\forall A\in\mathcal{R}$ and $\forall\varepsilon\in\mathbb{R}^+$ we have that $\exists K_\varepsilon\in\mathcal{C}$ and $\exists A_\varepsilon\in\mathcal{R}$ such that*

$$A_\varepsilon\subseteq K_\varepsilon\subseteq A \text{ and } \mu(A\setminus A_\varepsilon)<\varepsilon$$

then μ is a pre-measure. In other words if we can 'approximate' $\mathcal{R}$ by a complex class $\mathcal{C}$ then μ is a pre-measure.

Proof. Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R}$ be such that

$$\forall i\in\mathbb{N} \text{ we have } A_{i+1}\subseteq A_i \text{ and } \bigcap_{i\in\mathbb{N}} A_i=\varnothing \tag{20.264}$$

As μ is finite we have also

$$\forall i\in\mathbb{N} \text{ we have } \mu(A_i)<\infty \tag{20.265}$$

So if we prove that $\lim_{i\rightarrow\infty}\mu(A_i)=0$ then it follows from [theorem: 20.233] that μ is a pre-measure. Let $\varepsilon\in\mathbb{R}^+$ and $n\in\mathbb{N}$ then by the assumption there exist a $K_{n,\varepsilon}\in\mathcal{C}$ and a $R_{n,\varepsilon}\in\mathcal{R}$ such that $R_{n,\varepsilon}\subseteq K_{n,\varepsilon}\subseteq A_n$ and $\mu(A\setminus R_{n,\varepsilon})<\frac{\varepsilon}{2^n}$. Hence there exist a $\{K_{n,\varepsilon}\}_{n\in\mathbb{N}}\subseteq\mathcal{C}$ and $\{R_{n,e}\}_{n\in\mathbb{N}}\subseteq\mathcal{R}$ such that

$$\forall n\in\mathbb{N} \text{ we have } R_{n,\varepsilon}\subseteq K_{n,\varepsilon}\subseteq A_n \text{ and } \mu(A\setminus R_{n,\varepsilon})<\frac{\varepsilon}{2^n} \tag{20.266}$$

Then we have

$$\bigcap_{n\in\mathbb{N}} K_{n,\varepsilon}\subseteq\bigcap_{n\in\mathbb{N}} A_n \underset{[\text{eq: } 20.264]}{=} \varnothing$$

so that by the definition of a compact class there exist a $N_\varepsilon \in \mathbb{N}$ such that $\bigcap_{n\in\{1,\ldots,N_\varepsilon\}} K_{n,\varepsilon} = \varnothing$. As $\bigcap_{n\in\{1,\ldots,N_\varepsilon\}} R_{n,\varepsilon} \underset{\text{[eq: 20.266]}}{\subseteq} \bigcap_{n\in\{1,\ldots,N_\varepsilon\}} K_{n,\varepsilon}$ it follows that

$$\bigcap_{n\in\{1,\ldots,N_\varepsilon\}} R_{n,\varepsilon} = \bigcap_{n\in\{1,\ldots,N_\varepsilon\}} K_{n,\varepsilon} = \varnothing \tag{20.267}$$

Let $n \in \{1,\ldots,N_\varepsilon\}$ then $A_{N_\varepsilon} \underset{\text{[eq: 20.264]}}{\subseteq} A_n$ so that $A_{N_\varepsilon} \subseteq \bigcap_{n\in\{1,\ldots,N_\varepsilon\}} A_n \subseteq A_{N_\varepsilon}$ proving that

$$A_{N_\varepsilon} = \bigcap_{n\in\{1,\ldots,N_\varepsilon\}} A_n \tag{20.268}$$

Let $x \in \bigcap_{n\in\{1,\ldots,N_\varepsilon\}} A_n$. Assume that $x \notin \bigcup_{n\in\{1,\ldots,N_\varepsilon\}} (A_n \setminus R_{n,\varepsilon})$ then $\forall n \in \{1,\ldots,N_\varepsilon\}$ we have $x \notin A_n \setminus R_{n,\varepsilon}$, as $x \in \bigcap_{n\in\{1,\ldots,N_\varepsilon\}} A_n \subseteq A_n$, it follows that $x \in R_{n,\varepsilon}$. So $x \in \bigcap_{n\in\{1,\ldots,N_\varepsilon\}} R_{n,\varepsilon}$ contradicting [eq: 20.267]. Hence the assumption is wrong and we must have that $x \in \bigcup_{n\in\{1,\ldots,N_\varepsilon\}} (A_n \setminus R_{n,\varepsilon})$ proving

$$A_{N_\varepsilon} \underset{\text{[eq: 20.268]}}{=} \bigcap_{n\in\{1,\ldots,N_\varepsilon\}} A_n \subseteq \bigcup_{n\in\{1,\ldots,N_\varepsilon\}} (A_n \setminus R_{n,\varepsilon}) \underset{\mathcal{R}\text{ is a ring}}{\in} \mathcal{R} \tag{20.269}$$

Hence using the above we have

$$\begin{aligned}
\mu(A_{N_\varepsilon}) &\underset{\text{[theorem: 20.232(2)]}\wedge\text{[eq:20.269]}}{\leqslant} \mu\left(\bigcup_{n\in\{1,\ldots,N_\varepsilon\}} (A_n \setminus R_{n,\varepsilon})\right) \\
&\underset{\text{[definition: 20.228]}\wedge\text{[eq:20.269]}}{=} \sum_{i=1}^{N_\varepsilon} \mu(A_n \setminus R_{n,\varepsilon}) \\
&\underset{\text{[eq: 20.266]}}{\leqslant} \sum_{i=1}^{N_\varepsilon} \frac{\varepsilon}{2^n} \\
&= \varepsilon \cdot \sum_{i=1}^{N} \left(\frac{1}{2}\right)^n \\
&\underset{\text{[lemma: 14.352]}}{=} \varepsilon \cdot \left(\frac{1}{2}\right) \cdot \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} \\
&= \varepsilon \cdot \left(1 - \frac{1}{2^n}\right) \\
&< \varepsilon
\end{aligned}$$

So we have proved that

$\forall \varepsilon \in \mathbb{R}^+$ there exist a $N_\varepsilon \in \mathbb{N}$ such that $\forall n \in \{N_\varepsilon,\ldots,\infty\}$ we have $|\mu(A_n) - 0| = \mu(A_n) \leqslant \mu(A_{N_\varepsilon}) < \varepsilon$

which proves that $\lim_{n\to\infty}{}_{\mathbb{R}}\, \mu(A_n) = 0$. Using the fact that μ is finite and [theorem: 20.74] it follows that

$$\lim_{n\to\infty} \mu(A_i) = 0$$

So using [theorem: 20.233] it follows that μ is a pre-measure. □

The above theorem is especially usefull if we have a ring on a topological Hausdorff space because every set of compact sets is a compact class [see theorem: 14.243]. This is expressed in the following theorem.

Theorem 20.236. *Let $\langle X, \mathcal{T}\rangle$ be a Hausdorff topological space,, $\mathcal{R}$ a ring on X and $\mu: X \to [0,\infty]$ a **finite** content. If $\forall A \in \mathcal{R}$ and $\forall \varepsilon \in \mathbb{R}^+$ we have that $\exists K_\varepsilon$ compact and $\exists A_\varepsilon \in \mathcal{R}$ such that*

$$A_\varepsilon \subseteq K_\varepsilon \subseteq A \text{ and } \mu(A \setminus A_\varepsilon) < \varepsilon$$

then μ is a pre-measure.

Proof. If $\mathcal{C}=\{A\in\mathcal{P}(X)|A \text{ is compact}\}$ is the set of compact subsets of X then by [theorem: 14.243] $\mathcal{C}$ is a compact class. The proof follows then from [theorem: 20.235]. □

Proving that $\mathcal{R}\subseteq\mathcal{P}(X)$ is a ring is not always possible, especially proving that $\forall A,B\in\mathcal{R}$ we have that $A\setminus B\in\mathcal{R}$. For example the difference of two half-open intervals in $\mathcal{I}_{[[}$ is not always a half-open interval, hence $\mathcal{R}_{[[}$ is not a ring, however it will be proved later that $\mathcal{I}_{[[}$ is a semi-ring on $\mathbb{R}$. Luckely it turns out that a semi-ring with a content on it can easely be extended to a ring with a content on it. First we show how we can extend a semi-ring to a smallest ring containing the semi-ring.

Definition 20.237. *Let X be a set, $\mathcal{S}\subseteq\mathcal{P}(X)$ a semi-ring on X then $\mathcal{R}[\mathcal{S}]$ is defined by*

$$\mathcal{R}[\mathcal{S}]=\left\{\bigsqcup_{i\in\{1,\ldots,n\}} A_i|n\in\mathbb{N}\wedge\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq S \text{ is pairwise disjoint}\right\}$$

***Note* 20.238.** *As for $A,B\in\mathcal{S}$ we have by definition of a semi-ring that $\exists\{C_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}$, $n\in\mathbb{N}$ pairwise disjoint such that $A\setminus B=\bigsqcup_{i\in\{1,\ldots,n\}}C_i$ it follows that*

$$\forall A,B\in\mathcal{S} \text{ we have } A\setminus B\in\mathcal{S}$$

To prove that $\mathcal{R}[\mathcal{S}]$ we have to prove that $\varnothing\in\mathcal{R}[\mathcal{S}]$ and $\forall A,\, B\in\mathcal{R}[\mathcal{S}]$ that $A\bigcup B\in\mathcal{R}[\mathcal{S}]\wedge A\setminus B\in\mathcal{R}[\mathcal{S}]$. We start with proving that $\forall A,B\in\mathcal{R}[\mathcal{S}]$ $A\bigcup B\in\mathcal{R}[\mathcal{S}]$ using the following two lemma's.

Lemma 20.239. *Let X be a set, $\mathcal{S}\subseteq\mathcal{P}(X)$ a semi-ring on X, $A\in\mathcal{S}$ and $B\in\mathcal{R}[\mathcal{S}]$ then*

$$A\bigcup B\in\mathcal{R}[\mathcal{S}]$$

Proof. As $B\in\mathcal{R}[\mathcal{S}]$ there exist a pairwise disjoint $\{B_i\}_{i\in\{1,\ldots,,n\}}\subseteq\mathcal{S}$, $n\in\mathbb{N}$ such that

$$B=\bigsqcup_{i\in\{1,\ldots,n\}}B_i$$

Define

$$\{C_i\}_{i\in\{1,\ldots,n+1\}} \text{ by } C_i=\begin{cases} B_i\setminus A \text{ if } i\in\{1,\ldots,n\}\\ A \text{ if } i=n+1\end{cases}$$

Then if $x\in A\bigcup B$ we have either:

$x\in A$. Then $x\in C_{n+1}\subseteq\bigcup_{i\in\{1,\ldots,n+1\}}C_i$

$x\notin A$. Then we must have that $x\in B$ so that there exist a $i\in\{1,\ldots,n\}$ such that

$$x\in B_i\underset{x\notin A}{\Rightarrow}x\in B_i\setminus A\in C_i\subseteq\bigcup_{i\in\{1,\ldots,n+1\}}C_i$$

proving that

$$A\bigcup B\subseteq\bigcup_{i\in\{1,\ldots,n+1\}}C_i \tag{20.270}$$

Let $x\in\bigcup_{i\in\{1,\ldots,n+1\}}C_i$ then there exist a $i\in\{1,\ldots,n+1\}$ such that $x\in C_i$. For i we have either

$i\in\{1,\ldots,n\}$. Then $x\in C_i=B_i\setminus A\subseteq B_i\subseteq\bigcup_{i\in\{1,\ldots,n\}}B_i=B\subseteq A\bigcup B$

$i=n+1$. Then $x\in C_{n+1}=A\bigcup B$

hence $\bigcup_{i\in\{1,\ldots,n+1\}}C_i\subseteq A\bigcup B$. Combining this with [eq: 20.270] results in

$$A\bigcup B=\bigcup_{i\in\{1,\ldots,n+1\}}C_i \tag{20.271}$$

Let $i, j \in \{1, \ldots, n+1\}$ with $i \neq j$ then for i we have either:

$i \in \{1, \ldots, n\}$. Then for j we have either:

$j \in \{1, \ldots, n\}$. Then $C_i \bigcap C_j = (B_i \setminus A) \bigcap (B_j \setminus A) \subseteq B_i \bigcap B_j = \varnothing$

$j = n+1$. Then $C_i \bigcap C_j = (B_i \setminus A) \bigcap A = \varnothing$

$i = n+1$. Then as $i \neq j$ we have that $j \in \{1, \ldots, n\}$ so that $C_i \bigcap C_j = A \bigcap (B_j \setminus A) = \varnothing$

hence

$$\{C_i\}_{i \in \{1, \ldots, n+1\}} \text{ is pairwise disjoint so that } A \bigcup B = \bigsqcup_{i \in \{1, \ldots, n+1\}} C_i \tag{20.272}$$

Let $i \in \{1, \ldots, n+1\}$ then we have either:

$i \in \{1, \ldots, n\}$. As $A, B_i \in \mathcal{S}$ and S is a semi-ring there exist a pairwise disjoint $\{D_{i,j}\}_{j \in \{1, \ldots, n_i\}} \subseteq \mathcal{S}$ such that $C_i = B_i \setminus A = \bigsqcup_{i \in \{1, \ldots, n_i\}} D_{i,j}$.

$i = n+1$. Take $n_1 = 1$ and $\{D_{1,j}\}_{j \in \{1\}} \subseteq \mathcal{S}$ by $D_{1,1} = A$ so that $C_{n+1} = \bigsqcup_{i \in \{1, \ldots, n_i\}} D_{1,j}$

Summarized we have that

$$\forall i \in \{1, \ldots, n+1\} \ \exists \{D_{i,j}\}_{j \in \{1, \ldots, n\}} \subseteq \mathcal{S} \text{ with } C_i = \bigsqcup_{i \in \{1, \ldots, n_i\}} D_{i,j} \tag{20.273}$$

As $\bigcup_{i \in \{1, \ldots, n+1\}} \{i\} \times \{1, \ldots, n_i\}$ is finite by [theorems: 6.41, 6.36] there exists a bijection

$$\beta: \{1, \ldots, N\} \rightarrow \bigcup_{i \in \{1, \ldots, n+1\}} \{i\} \times \{1, \ldots, n_i\} \tag{20.274}$$

Using [eqs: 20.272, 20.273 and 20.274] allows us to apply [theorem: 20.114] that

$$\{D_{\beta(i)_1 \beta(i)2}\}_{i \in \{1, \ldots, N\}} \subseteq \mathcal{S} \text{ is pairwise disjoint and } \bigsqcup_{i \in \{1, \ldots, n+1\}} C_i = \bigsqcup_{i \in \{1, \ldots, N\}} D_{\beta(i)_1, \beta(i)_2}$$

Hence using [eq: 20.272] it follows that $A \bigcup B = \bigsqcup_{i \in \{1, \ldots, N\}} D_{\beta(i)_1, \beta(i)_2}$ which proves that

$$A \bigcup B \in \mathcal{S}[\mathcal{R}]$$

□

We use the above lemma in a proof by induction to prove that that $\mathcal{R}[\mathcal{S}]$ is $\cup$-stable.

Lemma 20.240. *Let X be a set, $\mathcal{S} \subseteq \mathcal{P}(X)$ a semi-ring on X and $A, B \in \mathcal{R}[\mathcal{S}]$ then $A \bigcup B \in \mathcal{R}[\mathcal{S}]$*

Proof. Let $B \in \mathcal{R}[\mathcal{S}]$ and define

$$\mathcal{S}_B = \left\{ n \in \mathbb{N} | \text{If } \{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S} \text{ pairwise disjoint then } \left(\bigsqcup_{i \in \{1, \ldots, n\}} A_i \right) \bigcup B \in \mathcal{R}[\mathcal{S}] \right\}$$

then we have:

$1 \in \mathcal{S}_B$. If $\{A_i\}_{i \in \{1\}} \subseteq \mathcal{S}$ is pairwise disjoint then, as $A_1 \in \mathcal{S}$, we have by [lemma: 20.239] that $\left(\bigsqcup_{i \in \{1\}} A_i\right) \bigcup B = A_1 \bigcup B \in \mathcal{R}[\mathcal{S}]$ which proves that $1 \in \mathcal{S}_B$.

$n \in \mathcal{S}_B \Rightarrow n+1 \in \mathcal{S}_B$. Let $\{A_i\}_{i \in \{1, \ldots, n+1\}} \subseteq \mathcal{S}$ be a pairwise disjoint family of sets in $\mathcal{S}$. As $\{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S}$ is pairwise disjoint and $n \in \mathcal{R}[\mathcal{S}]$ we have that

$$\left(\bigsqcup_{i \in \{1, \ldots, n\}} A_i \right) \bigcup B \in \mathcal{R}[\mathcal{S}]$$

Hence, as $A_{n+1} \in \mathcal{S}$, we have by the previous lemma [lemma: 20.239] that

$$A_{n+1} \bigcup \left(\left(\bigsqcup_{i \in \{1, \ldots, n\}} A_i \right) \bigcup B \right) \in \mathcal{R}[\mathcal{S}]$$

which as $A_{n+1}\bigcup\left(\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)\bigcup B\right)=\left(\bigsqcup_{i\in\{1,\ldots,n+1\}}A_i\right)\bigcup B$ proves that

$$\left(\bigsqcup_{i\in\{1,\ldots,n+1\}}A_i\right)\bigcup B$$

so that $n+1\in\mathcal{S}$.

Mathematical induction proves then that $\mathcal{S}_B=\mathbb{N}$. Finally if $A\in\mathcal{R}[\mathcal{S}]$ then there exist a $\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}$ such that $A=\bigsqcup_{i\in\{1,\ldots,n\}}A_i$, hence, as $n\in\mathbb{N}=\mathcal{S}_B$, it follows that $A\bigcup B=\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)\bigcup B\in\mathcal{R}[\mathcal{S}]$. □

Next we have to prove that $\forall A,B\in\mathcal{R}[\mathcal{S}]$.

Lemma 20.241. *Let X be a set, $\mathcal{S}\subseteq\mathcal{P}(X)$ a semi-ring on X, $A\in\mathcal{R}[\mathcal{S}]$ and $B\in\mathcal{S}$ then $A\setminus B\in\mathcal{R}[\mathcal{S}]$.*

Proof. We use mathematical induction in the proof. Define

$$S_B=\left\{n\in\mathbb{N}|\text{If } \{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}\text{ pairwise disjoint then }\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)\setminus B\in\mathcal{R}[\mathcal{S}]\right\}$$

then we have:

$1\in S_B$. Let $\{A_i\}_{i\in\{1\}}\subseteq\mathcal{S}$ pairwise disjoint then $\left(\bigsqcup_{i\in\{1\}}A_i\right)\setminus B=A_1\setminus B\in\mathcal{R}[S]$ [using the note in the definition of $\mathcal{R}[S]$ see [definition: 20.237]], hence $1\in S_B$.

$n\in S_B\Rightarrow n+1\in S_B$. Let $\{A_i\}_{i\in\{1,\ldots,n+1\}}\subseteq\mathcal{S}$ pairwise disjoint. Then we have

$$\begin{aligned}\left(\bigsqcup_{i\in\{1,\ldots,n+1\}}A_i\right)\setminus B &= \left(\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)\bigsqcup A_{n+1}\right)\setminus B\\ &= \left(\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)\setminus B\right)\bigsqcup(A_{n+1}\setminus B) \quad (20.275)\end{aligned}$$

As $n\in S_B$ we have that $\left(\bigcup_{i\in\{1,\ldots,n\}}A_i\right)\setminus B\in\mathcal{R}[\mathcal{S}]$ so that there exists a pairwise disjoint $\{C_i\}_{i\in\{1,\ldots,k\}}\subseteq\mathcal{S}$, $k\in\mathbb{N}$ such that

$$\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)\setminus B=\bigsqcup_{j\in\{1,\ldots,k\}}C_j \quad (20.276)$$

Further as $A_{n+1},B\in S$ and S is a semi-ring there exist a pairwise disjoint $\{D_i\}_{i\in\{1,\ldots,l\}}\subseteq\mathcal{S}$, $l\in\mathbb{N}$ such that

$$A_{n+1}\setminus B=\bigsqcup_{i\in\{1,\ldots,l\}}D_i \quad (20.277)$$

Define

$$\{E_i\}_{i\in\{1,\ldots,k+l\}}\subseteq\mathcal{S}\text{ by }E_i=\begin{cases}C_i\in\mathcal{S}\text{ if }i\in\{1,\ldots,k\}\\ D_{i-k}\in\mathcal{S}\text{ if }i\in\{k+1,\ldots,k+l\}\end{cases} \quad (20.278)$$

Let $i\in\{1,\ldots,k+l\}$ then we have either

$i\in\{1,\ldots,k\}$. Then

$$E_i=C_i\underset{[\text{eq: }20.276]}{\subseteq}\left(\bigsqcup_{i\in\{1,\ldots,n\}}A_i\right)\setminus B\underset{[\text{eq: }20.275]}{\subseteq}\left(\bigsqcup_{i\in\{1,\ldots,n+1\}}A_i\right)\setminus B$$

$i\in\{k+1,\ldots,k+l\}$. Then

$$E_i=D_{i-k}\underset{[\text{eq: }20.277]}{\subseteq}A_{n+1}\setminus B\underset{[\text{eq: }20.275]}{\subseteq}\left(\bigsqcup_{i\in\{1,\ldots,n+1\}}A_i\right)\setminus B$$

which proves that

$$\bigcup_{i\in\{1,\ldots,k+l\}} E_i \subseteq \left(\bigsqcup_{i\in\{1,\ldots,n+1\}} A_i\right)\setminus B \tag{20.279}$$

If $x\in\left(\bigsqcup_{i\in\{1,\ldots,n+1\}} A_i\right)\setminus B=\left(\left(\bigsqcup_{i\in\{1,\ldots,n\}} A_i\right)\setminus B\right)\bigsqcup(A_{n+1}\setminus B)$ then we have either:

$x\in\left(\bigsqcup_{i\in\{1,\ldots,n\}} A_i\right)\setminus B$. Then by [eq: 20.277] there exist a $i\in\{1,\ldots,k\}$ such that $x\in C_i \underset{\text{[eq: 20.278]}}{=} E_i\subseteq\bigcup_{i\in\{1,\ldots,k+l\}} E_i$

$x\in A_{n+1}\setminus B$. Then by [eq: 20.277] there exist a $i\in\{1,\ldots,l\}$ such that

$$x\in D_i=D_{(i+k)-k}=E_{i+k}\subseteq\bigcup_{i\in\{1,\ldots,k+l\}} E_i$$

which proves that $\left(\bigsqcup_{i\in\{1,\ldots,n+1\}} A_i\right)\setminus B\subseteq\bigcup_{i\in\{1,\ldots,k+l\}} E_i$. Combininig this with [eq: 20.279] proves that

$$\left(\bigsqcup_{i\in\{1,\ldots,n+1\}} A_i\right)\setminus B=\bigcup_{i\in\{1,\ldots,k+l\}} E_i \tag{20.280}$$

Let $i,j\in\{1,\ldots,k+l\}$ with $i\neq j$ then without losing generality we may assume that $i<j$ [otherwise exchange i and j]. For i we have either:

$i\in\{1,\ldots,k\}$. Then for j we have either:

$j\in\{1,\ldots,k\}$. Then $E_i\bigcap E_j=C_i\bigcap C_j=\varnothing$

$j\in\{k+1,\ldots,k+l\}$. Then

$$\begin{aligned}
E_i\bigcap E_j &= C_i\bigcap D_{j-k}\\
&\underset{\text{[eq: 20.276]}}{\subseteq} \left(\bigsqcup_{i\in\{1,\ldots,n\}} A_i\right)\bigcap D_{j-k}\\
&\underset{\text{[eq: 20.277]}}{\subseteq} \left(\bigsqcup_{i\in\{1,\ldots,n\}} A_i\right)\bigcap A_{n+1}\\
&= \bigsqcup_{i\in\{1,\ldots,n\}} \left(A_i\bigcap A_{n+1}\right)\\
&\underset{i\neq n+1}{=} \varnothing
\end{aligned}$$

$i\in\{k+1,\ldots,k+l\}$. As $i<j$ we have $j\in\{k+1,\ldots,k+l\}$ so that

$$E_i\bigcap E_j=D_{i-k}\bigcap D_{j-k}=\varnothing$$

which proves that

$$\{E_i\}_{i\in\{1,\ldots,k+l\}}\subseteq\mathcal{S} \text{ is pairwise disjoint and } \left(\bigsqcup_{i\in\{1,\ldots,n+1\}} A_i\right)\setminus B \underset{\text{[eq: 20.280]}}{=} \bigsqcup_{i\in\{1,\ldots,k+l\}} E_i$$

So using the definition of $\mathcal{R}[\mathcal{S}]$ it follows that $\left(\bigsqcup_{i\in\{1,\ldots,n+1\}} A_i\right)\setminus B\in\mathcal{R}[\mathcal{S}]$ which proves that $n+1\in\mathcal{S}_B$

By mathematical induction it follows that

$$S_B=\mathbb{N}$$

Finally let $A\in\mathcal{R}[\mathcal{S}]$ then there exist a pairwise disjoint $\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}$ such that $A=\bigsqcup_{i\in\{1,\ldots,n\}} A_i$. As $n\in\mathbb{N}=S_B$ we have that $A\setminus B=\left(\bigsqcup_{i\in\{1,\ldots,n\}} A_i\right)\setminus B\in\mathcal{R}[\mathcal{S}]$. □

Next we use mathematical induction to prove that $\forall A,B\in\mathcal{R}[S]$ we have $A\setminus B\in\mathcal{R}[\mathcal{S}]$.

Lemma 20.242. *Let X be a set, $\mathcal{S}\subseteq\mathcal{P}(X)$ a semi-ring on X then $\forall A,B\in\mathcal{R}[\mathcal{S}]$ we have that $A\setminus B\in\mathcal{R}[\mathcal{S}]$.*

Proof. Let $A\in\mathcal{R}[\mathcal{S}]$ and define

$$S_A=\left\{n\in\mathbb{N}|\text{If } \{B_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S} \text{ is pairwise disjoint then } A\setminus\bigsqcup_{i\in\{1,\ldots,n\}} B_i\in\mathcal{R}[\mathcal{S}]\right\}$$

then we have:

$1\in S_A$. Let $\{B_i\}_{i\in\{1\}}\subseteq\mathcal{S}$ be pairwise disjoint then by the previous lemma [lemma: 20.241] we have that $A\setminus\bigsqcup_{i\in\{1\}}B_i=A\setminus B_1\in\mathcal{R}[\mathcal{S}]$ which proves that $1\in\mathcal{S}_A$.

$n\in S_A\Rightarrow n+1\in S_A$. Let $\{B_i\}_{i\in\{1,\ldots,n+1\}}\subseteq\mathcal{S}$ be pairwise disjoint. Then

$$\begin{aligned} A\setminus\bigsqcup_{i\in\{1,\ldots,n+1\}} B_i &= A\setminus\left(\left(\bigsqcup_{i\in\{1,\ldots,n\}} B_i\right)\bigsqcup B_{n+1}\right)\\ &\underset{\text{[theorem: 1.31]}}{=} \left(A\setminus\bigsqcup_{i\in\{1,\ldots,n\}} B_i\right)\setminus B_{n+1}\end{aligned}$$

As $n\in S_A$ we have that $A\setminus\bigsqcup_{i\in\{1,\ldots,n\}}B_i\in\mathcal{R}[\mathcal{S}]$ so that by [lemma: 20.241] $\left(A\setminus\bigsqcup_{i\in\{1,\ldots,n\}}B_i\right)\setminus B_{n+1}\in\mathcal{R}[\mathcal{S}]$. Hence $A\setminus\bigsqcup_{i\in\{1,\ldots,n+1\}}B_i\in\mathcal{R}[\mathcal{S}]$ which proves that $n+1\in\mathcal{S}_A$.

Mathematical induction proves that

$$\mathcal{S}_A=\mathbb{N}$$

Finally if $B\in\mathcal{R}[\mathcal{S}]$ then by definition of $\mathcal{R}[\mathcal{S}]$ there exist a pairwise disjoint $\{B_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}$, $n\in\mathbb{N}$ such that $B=\bigsqcup_{i\in\{1,\ldots,n\}}B_i$. As $n\in\mathbb{N}=\mathcal{S}_A$ $A\setminus B=A\setminus\bigsqcup_{i\in\{1,\ldots,n\}}B_i\in\mathcal{R}[\mathcal{S}]$. □

We are now ready to prove that $\mathcal{R}[\mathcal{S}]$ is a ring

Theorem 20.243. *Let X be a set, $\mathcal{S}\subseteq\mathcal{P}(X)$ a semi-ring on X then $\mathcal{R}[\mathcal{S}]$ is a ring on X such that:*

1. $\mathcal{S}\subseteq\mathcal{R}[S]$
2. *If $\mathcal{T}$ is another ring on X with $\mathcal{S}\subseteq\mathcal{T}$ then $\mathcal{R}[\mathcal{S}]$*

In other words the set of all disjoint finite unions of sets in $\mathcal{S}$ is the smallest ring covering $\mathcal{S}$

Proof. Let Let $A\in\mathcal{S}$ and define $\{A_i\}_{i\in\{1\}}\subseteq\mathcal{S}$ by $A_1=A$ then $\{A_i\}_{i\in\{1\}}$ is pairwise disjoint and $A=\bigsqcup_{i\in\{1\}}A_i$, hence $A\in\mathcal{R}[\mathcal{S}]$ proving that

$$\mathcal{S}\subseteq\mathcal{R}[\mathcal{S}] \tag{20.281}$$

As $\mathcal{S}$ is a semi-ring we have $\varnothing\in\mathcal{S}$ so by the above we have

$$\varnothing\in\mathcal{R}[\mathcal{S}] \tag{20.282}$$

Let $A,B\in\mathcal{R}[\mathcal{S}]$ then by [lemma: 20.240] and [lemma:20.242] we have that

$$A\bigcup B\in\mathcal{R}[\mathcal{S}] \text{ and } A\setminus B\in\mathcal{R}[\mathcal{S}] \tag{20.283}$$

So by [eqs: 20.281, 20.282 and 20.283] it follows that

$$\mathcal{R}[\mathcal{S}] \text{ is a ring on } X$$

It remains to prove that $\mathcal{R}[\mathcal{S}]$ is the smallest ring containing $\mathcal{S}$. So let $\mathcal{T}$ be another ring such that $\mathcal{S}\subseteq\mathcal{T}$. If $A\in\mathcal{R}[\mathcal{S}]$ then then there exist a pairwise disjoint finite family $\{A_i\}_{i\in\{1,\ldots,k\}}\subseteq\mathcal{S}\subseteq\mathcal{T}$ such that $A=\bigsqcup_{i\in\{1,\ldots,k\}}A_i$. As $\mathcal{T}$ is a ring it follows from [theorem: 20.126] that $A=\bigsqcup_{i\in\{1,\ldots,k\}}A_i\in\mathcal{T}$ which proves that

$$\mathcal{R}[\mathcal{S}]\subseteq\mathcal{T}$$ □

Next we want to extend a content on a semi-ring $\mathcal{S}$ to a content on the ring $\mathcal{R}[\mathcal{S}]$.

Theorem 20.244. *Let X be a set, $\mathcal{S}$ a semi-ring on X and $\mu: \mathcal{S} \rightarrow [0, \infty]$ a content on $\mathcal{S}$ then*

1. *There exist a content $\nu: \mathcal{R}[\mathcal{S}] \rightarrow [0, \infty]$ such that $\nu_{|\mathcal{S}} = \mu$ [in other words ν is a extension of μ]*
2. *If $\omega: \mathcal{R}[\mathcal{S}] \rightarrow [0, \infty]$ is another content such that $\omega_{|\mathcal{S}} = \mu$ then $\omega = \nu$ [in other words the extension is unique].*
3. *If μ is finite then ν is finite.*

Proof.

1. First we proof existence. Let $A \in \mathcal{R}[\mathcal{S}]$ then there exist a pairwise disjoint finite family $\{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S}$, $n \in \mathbb{N}$ such that $A = \sqcup_{i \in \{1, \ldots, n\}} A_i$. As ν will be a content on $\mathcal{R}[\mathcal{S}]$ we must have that $\nu(A) = \sum_{i=1}^{n} \nu(A_i) \underset{A_i \in \Sigma}{=} \sum_{i=1}^{n} \mu(A_i)$. So it make sense to define

$$\nu: \mathcal{R}[\mathcal{S}] \rightarrow [0, \infty] \text{ by } \nu(A) = \sum_{i=1}^{n} \mu(A_i)$$

$$\text{where } \{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S}, n \in \mathbb{N} \text{ is pairwise disjoint and } A = \bigsqcup_{i \in \{1, \ldots, n\}} A_i$$

Of course this only works if $\nu(A)$ is independent of the chosen family $\{A_i\}_{i \in \{1, \ldots, n\}}$. So assume that for $A \in \mathcal{R}[\mathcal{S}]$ we have two pairwise disjoint families $\{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S}$, $n \in \mathbb{N}$ and $\{B_i\}_{i \in \{1, \ldots, m\}} \subseteq \mathcal{S}$, $m \in \mathbb{N}$ such that

$$A = \bigsqcup_{i \in \{1, \ldots, n\}} A_i = \bigsqcup_{i \in \{1, \ldots, m\}} B_i$$

Then we have

$$\begin{aligned}
\sum_{i=1}^{n} \mu(A_i) & \underset{A_i \bigcap A = A_i}{=} & \sum_{i=1}^{n} \mu\left(A_i \bigcap A\right) \\
& = & \sum_{i=1}^{n} \mu\left(A_i \bigcap \left(\bigsqcup_{j \in \{1, \ldots, m\}} B_j\right)\right) \\
& \underset{[\text{theorem: } 2.130]}{=} & \sum_{i=1}^{n} \mu\left(\bigsqcup_{j \in \{1, \ldots, m\}} \mu\left(A_i \bigcap B_j\right)\right) \\
& \underset{A_i \bigcap B_j \in \mathcal{S}}{=} & \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \mu\left(A_i \bigcap B_j\right)\right) \\
& \underset{[\text{theorem: } 20.96]}{=} & \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \mu\left(A_i \bigcap B_j\right)\right) \\
& \underset{A_i \bigcap B_j \in \mathcal{S}}{=} & \sum_{j=1}^{m} \mu\left(\bigsqcup_{i \in \{1, \ldots, n\}} \left(A_i \bigcap B_j\right)\right) \\
& \underset{[\text{theorem: } 2.130]}{=} & \sum_{j=1}^{m} \mu\left(\left(\bigsqcup_{i \in \{1, \ldots, n\}} A_i\right) \bigcap B_j\right) \\
& = & \sum_{j=1}^{m} \mu\left(A \bigcap B_j\right) \\
& = & \sum_{j=1}^{m} \mu(B_j)
\end{aligned}$$

proving that $\nu: \mathcal{R}[\mathcal{S}] \rightarrow [0, \infty]$ is well defined. Next we prove that ν is a extension of $\mathcal{S}$. Let $A \in \mathcal{S}$ then if we define $\{A_i\}_{i \in \{1\}} \subseteq \mathcal{S}$ by $A_1 = A$ then $\{A_i\}_{i \in \{1\}}$ is clearly disjoint and $\bigsqcup_{i \in \{1\}} A_i = A_1 = A$ so that $\nu(A) = \sum_{i=1}^{1} \mu(A_i) = \mu(A_1) = \mu(A)$ proving that

$$\nu_{|\mathcal{S}} = \mu \tag{20.284}$$

Next we use [theorem: 20.231] to prove that ν is a content on $\mathcal{R}[\mathcal{S}]$. As $\varnothing\in\mathcal{S}$ we have by [eq: 20.284] that

$$\nu(\varnothing)\underset{\varnothing\in\mathcal{S}}{=}\mu(\varnothing)=0 \tag{20.285}$$

Let $A,B\in\mathcal{R}[\mathcal{S}]$ such that $A\bigcap B=\varnothing$ then there exist pairwise disjoint families $\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}$, $n\in\mathbb{N}$ and $\{B_i\}_{i\in\{1,\ldots,m\}}\subseteq\mathcal{S}$, $m\in\mathbb{N}$ such that $A=\bigsqcup_{i\in\{1,\ldots,n\}}A_i$, $B=\bigsqcup_{i\in\{1,\ldots,m\}}$. Define

$$\{C_i\}_{i\in\{1,\ldots,n+m\}}\subseteq\mathcal{S}\text{ by }C_i=\begin{cases}A_i\text{ if }i\in\{1,\ldots,n\}\\ B_{i-n}\text{ if }i\in\{n+1,\ldots,n+m\}\end{cases}$$

Let $i,j\in\{1,\ldots,n+m\}$ with $i\neq j$ then we may assume that $i<j$ [otherwise exchange i and j], for i we have either:

$\boldsymbol{i\in\{1,\ldots,n\}}$**.** Then for j we have either:

$\boldsymbol{j\in\{1,\ldots,n\}}$**.** Then $C_i\bigcap C_j=A_i\bigcap A_j\underset{i\neq j}{=}\varnothing$.

$\boldsymbol{j\in\{n+1,\ldots,n+m\}}$**.** Then $C_i\bigcap C_j=A_{i-n}\bigcap B_{j-n}\subseteq A\bigcap B=\varnothing$.

$\boldsymbol{i\in\{n+1,\ldots,n+m\}}$**.** Then $j\in\{n+1,\ldots,n+m\}$ $C_i\bigcap C_j=A_{i-n}\bigcap B_{j-n}\underset{i-n\neq j-n}{=}\varnothing$

proving that

$$\{C_i\}_{i\in\{1,\ldots,n+m\}}\subseteq\mathcal{S}\text{ is pairwise disjoint} \tag{20.286}$$

Let $x\in A\bigcup B$ then either $x\in A=\bigcup_{i\in\{1,\ldots,n\}}A_i\Rightarrow\exists i\in\{1,\ldots,n\}$ with $x\in A_i=C_i$ or $x\in B=\bigcup_{i\in\{1,\ldots,m\}}B_i\Rightarrow\exists i\in\{1,\ldots,m\}$ with $x\in B_i=C_{i+n}$, hence $x\in\bigcup_{i\in\{1,\ldots,n+m\}}C_i$, proving that

$$A\bigsqcup B\subseteq\bigcup_{i\in\{1,\ldots,n+m\}}C_i. \tag{20.287}$$

For $i\in\{1,...,n+m\}$ we have either $i\in\{1,...,n\}$ then $C_i=A_i\subseteq A\bigcup B$ or $i\in\{n+1,\ldots,n+m\}$ then $C_i=B_{i-n}\subseteq A\bigcup B$, hence in all cases we have $C_i\subseteq A\bigcup B$. This proves that $\bigcup_{i\in\{1,\ldots,n+m\}}C_i\subseteq A\bigcup B$ which combines with [eq: 20.287] gives

$$A\bigsqcup B=\bigsqcup_{i\in\{1,\ldots,n+m\}}C_i \tag{20.288}$$

Hence we have

$$\begin{aligned}
\nu\left(A\bigcup B\right) &= \nu\left(\bigsqcup_{i\in\{1,\ldots,n+m\}}C_i\right)\\
&\underset{\text{def}}{=} \sum_{i=1}^{n+m}\mu(C_i)\\
&\underset{\text{[theorem: 11.23]}}{=} \sum_{i=1}^{n}\mu(C_i)+\sum_{i=n+1}^{n+m}\mu(C_i)\\
&= \sum_{i=1}^{n}\mu(A_i)+\sum_{i=n+1}^{n+m}\mu(B_{i-n})\\
&\underset{\text{[theorem: 11.12]}}{=} \sum_{i=1}^{n}\mu(A_i)+\sum_{i=1}^{m}\mu(B_i)\\
&= \nu(A)+\nu(B)
\end{aligned}$$

proving that

$$\forall A,B\in\mathcal{R}[\mathcal{S}]\text{ with }A\bigcap B=\varnothing\text{ we have }\nu\left(A\bigsqcup B\right)=\nu(A)+\nu(B) \tag{20.289}$$

Using [eqs: 20.285, 20.289] we can apply]theorem: 20.231] to prove that

$$\nu: \mathcal{R}[\mathcal{S}] \rightarrow [0, \infty] \text{ is a content on } \mathcal{R}[\mathcal{S}]$$

2. Let $\omega: \mathcal{R}[\mathcal{S}] \rightarrow [0, \infty]$ be another content on $\mathcal{R}[S]$ such that $\omega_{|\mathcal{S}} = \mu$. Let $A \in \mathcal{R}[\mathcal{S}]$ then there exist a pairwise disjoint family $\{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S}$, $n \in \mathbb{N}$ such that $A = \bigsqcup_{i \in \{1, \ldots, n\}} A_i$. Hence

$$\begin{array}{rcl} \omega(A) & = & \omega\left(\bigsqcup_{i \in \{1, \ldots, n\}} A_i\right) \\ & \underset{\omega \text{ is a content}}{=} & \sum_{i=1}^{n} \omega(A_i) \\ & \underset{\omega_{|\mathcal{S}} = \mu}{=} & \sum_{i=1}^{n} \mu(A_i) \\ & \underset{\text{def of } \nu}{=} & \nu(A) \end{array}$$

so that

$$\omega = \nu$$

3. If $A \in \mathcal{R}[\mathcal{S}]$ then there exist a finite pairwise disjoint family $\{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S}$, $n \in \mathbb{N}$ such that $A = \bigsqcup_{i \in \{1, \ldots, n\}} A_i$. Then $\nu(A) = \sum_{i=1}^{n} \mu(A) < \infty$ [because μ is assumed to be finite]. □

We can now rephrase [theorem: 20.235] to work with semi-rings instead of rings.

Theorem 20.245. *Let X be a set, $\mathcal{S}$ a semi-ring on X and $\mu: X \rightarrow [0, \infty]$ a **finite** content. If there exist a compact class $\mathcal{C} \subseteq \mathcal{P}(X)$ such that*

1. *$\forall \{K_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{C}$, $n \in \mathbb{N}$ we have that $\bigcup_{i \in \{1, \ldots, n\}} K_i \in \mathcal{C}$*
2. *$\forall A \in \mathcal{S}$ and $\forall \varepsilon \in \mathbb{R}^+$ we have that $\exists K_\varepsilon \in \mathcal{C}$ and $\exists A_\varepsilon \in \mathcal{S}$ such that*

$$A_\varepsilon \subseteq K_\varepsilon \subseteq A \text{ and } \mu(A) - \mu(A_\varepsilon) < \varepsilon$$

then

$$\mu \text{ is a pre-measure}$$

Proof. Using [theorem: 20.244] there exist a **finite** content $\overline{\mu}: \mathcal{R}[S] \rightarrow [0, \infty]$ on the ring $\mathcal{R}[\mathcal{S}]$ such that

$$(\overline{\mu})_{|\mathcal{S}_{[[}} = \mu \tag{20.290}$$

Let $A \in \mathcal{R}[\mathcal{S}]$ then by definition there exist a pairwise disjoint $\{A_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S}$, $n \in \mathbb{N}$ such that $A = \bigsqcup_{i \in \{1, \ldots, n\}} A_i$. Let $\varepsilon \in \mathbb{R}^+$ then by the hypothesis there exists $\{A_{i,\varepsilon}\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S}$ and $\{K_{i,\varepsilon}\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{C}$ such that

$$\forall i \in \{1, \ldots, n\} \text{ we have } A_{i,\varepsilon} \subseteq K_{i,\varepsilon} \subseteq A_i \text{ and } \mu(A_i) - \mu(A_{i,\varepsilon}) < \frac{\varepsilon}{2^n} \tag{20.291}$$

Define

$$K_\varepsilon = \bigcup_{i \in \{1, \ldots, n\}} K_{i,\varepsilon} \in \mathcal{C} \text{ [hypothesis]} \tag{20.292}$$

and

$$A_\varepsilon = \bigcup_{i \in \{1, \ldots, n\}} A_{i,\varepsilon} \in \mathcal{R}[\mathcal{S}] \text{ [because } \mathcal{S} \subseteq \mathcal{R}[\mathcal{S}] \text{ and } \mathcal{R}[\mathcal{S}] \text{ is a ring hence } \cup\text{-stable]} \tag{20.293}$$

then we have

$$A_\varepsilon = \bigcup_{i \in \{1, \ldots, n\}} A_{i,\varepsilon} \underset{\text{[eq: 20.291]}}{\subseteq} \bigcup_{i \in \{1, \ldots, n\}} K_{i,\varepsilon} = K_\varepsilon$$

and

$$K_\varepsilon = \bigcup_{i\in\{1,\ldots,n\}} K_{i,\varepsilon} \underset{\text{[eq: 20.291]}}{\subseteq} \bigcup_{i\in\{1,\ldots,n\}} A_i = A$$

so that we have

$$A_\varepsilon \in \mathcal{R}[\mathcal{S}],\ K_\varepsilon \in \mathcal{C} \text{ and } A_\varepsilon \subseteq K_\varepsilon \subseteq A \tag{20.294}$$

Next

$$\begin{aligned}
A \setminus A_\varepsilon &= \left(\bigcup_{i\in\{1,\ldots,n\}} A_i\right) \setminus \left(\bigcup_{j\in\{1,\ldots,n\}} A_{j,\varepsilon}\right) \\
&\underset{\text{[theorem: 2.133]}}{=} \bigcup_{i\in\{1,\ldots,n\}} \left(A_i \setminus \left(\bigcup_{j\in\{1,\ldots,n\}} A_{j,\varepsilon}\right)\right) \\
&\underset{\text{[theorem: 2.132]}}{=} \bigcup_{i\in\{1,\ldots,n\}} \left(\bigcap_{j\in\{1,\ldots,n\}} (A_i \setminus A_{j,\varepsilon})\right) \\
&\subseteq \bigcup_{i\in\{1,\ldots,n\}} (A_i \setminus A_{i,\varepsilon})
\end{aligned}$$

Further if $i, j \in \{1, \ldots, n\}$ with $i \neq j$ then $(A_i \setminus A_{i,\varepsilon}) \bigcap (A_j \setminus A_{j,\varepsilon}) \subseteq A_i \bigcap A_j = \varnothing$. To summarize we have that

$$A \setminus A_\varepsilon \subseteq \bigsqcup_{i\in\{1,\ldots,n\}} (A_i \setminus A_{i,\varepsilon}) \text{ and } \{A_i \setminus A_{i,\varepsilon}\}_{i\in\{1,\ldots,n\}} \text{ is pairwise disjoint} \tag{20.295}$$

As $\overline{\mu}$ is a **finite** content on the ring $\mathcal{R}[\mathcal{S}]$ and $\forall i \in \{1, \ldots, n\}$ $A_i \setminus A_{i,\varepsilon} \in \mathcal{R}[\mathcal{S}]$ [because $\mathcal{R}[\mathcal{S}]$ is a ring and $A, A_\varepsilon, A_i, A_{i,\varepsilon} \in \mathcal{R}[\mathcal{S}]$] we have

$$\begin{aligned}
\overline{\mu}(A \setminus A_\varepsilon) &\underset{\text{[theorem:20.232(2)]}}{\leqslant} \overline{\mu}\left(\bigsqcup_{i\in\{1,\ldots,n\}} (A_i \setminus A_{i,\varepsilon})\right) \\
&= \sum_{i=1}^n \overline{\mu}(A_i \setminus A_{i,\varepsilon}) \\
&\underset{\text{[theorem: 20.232(3)]}}{=} \sum_{i=1}^n (\overline{\mu}(A_i) - \overline{\mu}(A_{i,\varepsilon})) \\
&\underset{\overline{\mu} \text{ extends } \mu}{=} \sum_{i=1}^n (\mu(A_i) - \mu(A_{i,\varepsilon})) \\
&\underset{\text{[eq: 20.291]}}{\leqslant} \sum_{i=1}^m \frac{\varepsilon}{2^i} \\
&= \varepsilon \cdot \sum_{i=1}^n \left(\frac{1}{2}\right)^i \\
&\underset{\text{[lemma: 14.352]}}{=} \varepsilon \cdot \left(\frac{1}{2}\right) \cdot \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} \\
&= \varepsilon \cdot \left(1 - \left(\frac{1}{2}\right)^n\right) \\
&< \varepsilon
\end{aligned}$$

Hence

$$\overline{\mu}(A \setminus A_\varepsilon) < \varepsilon \tag{20.296}$$

Using [eq: 20.294] and [eq: 20.296] allows us to apply [theorem: 20.235] proving that

$$\overline{\mu} \text{ is a pre-measure on } \mathcal{R}[\mathcal{S}]$$

As μ is a content

$$\mu(\varnothing) = 0 \tag{20.297}$$

Let $\{A_i\}_{i\in\mathbb{N}}\subseteq\mathcal{S}$ be pairwise disjoint such that $\bigsqcup_{i\in\mathbb{N}}A_i\in\mathcal{S}$ then we have

$$\begin{aligned}\mu\left(\bigsqcup_{i\in\mathbb{N}}A_i\right) &\underset{[\text{eq: }20.290]}{=} \overline{\mu}\left(\bigsqcup_{i\in\mathbb{N}}A_i\right)\\ &\underset{\overline{\mu}\text{ is a pre-measure}}{=} \sum_{i=1}^{\infty}\overline{\mu}(A_i)\\ &\underset{[\text{eq: }20.290]}{=} \sum_{i=1}^{\infty}\mu(A_i)\end{aligned}$$

which together with [eq: 20.299] proves that μ is a pre-measure. □

If we deal with topological spaces then we have a even simpler test to see if the extension of a content on a semi-ring $\mathcal{S}$ to the ring $\mathcal{R}[\mathcal{S}]$.

Theorem 20.246. *Let $\langle X,\mathcal{T}\rangle$ be a topological space, $\mathcal{S}$ a semi-ring on X and $\mu: X\rightarrow[0,\infty]$ a **finite** content. If $\forall A\in\mathcal{S}$ and $\forall\varepsilon\in\mathbb{R}^+$ there exists a compact K_ε and a $A_\varepsilon\in\mathcal{S}$ such that*

$$A_\varepsilon\subseteq K_\varepsilon\subseteq A \text{ and } \mu(A)-\mu(A_\varepsilon)<\varepsilon$$

then

$$\mu:\mathcal{R}[\mathcal{S}]\rightarrow[0,\infty] \text{ is a pre-measure}$$

Proof. First using [theorem: 14.243] it follows that

$$\mathcal{C}=\{A\in\mathcal{P}(X)|A\text{ is compact}\}\text{ is a compact class} \tag{20.298}$$

Further:

1. $\forall\{K_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{C}$, $n\in\mathbb{N}$ we have by [theorem: 14.228] that $\bigcup_{i\in\{1,\ldots,n\}}K_i$ is compact hence $\bigcup_{i\in\{1,\ldots,n\}}K_i\in\mathcal{C}$.

2. $\forall A\in\mathcal{S}$ and $\forall\varepsilon\in\mathbb{R}^+$ we have by the hypothesis that $\exists K_\varepsilon\in\mathcal{C}$ and $\exists A_\varepsilon\in\mathcal{S}$ such that

$$A_\varepsilon\subseteq K_\varepsilon\subseteq A\text{ and }\mu(A)-\mu(A_\varepsilon)<\varepsilon$$

So using the previous theorem [theorem: 20.245] we have that μ is a pre-measure. □

20.4.4.3 Using the Carathéodory theorem on $\mathbb{R}$

We use now the theory developped in the previous sections to construct a σ-algebra and a measure on $\mathbb{R}$. The procedure that we will follow is the following:

1. Prove that $\mathcal{I}_{[[}=\{[a,b[|a,b\in\mathbb{R}\wedge[a,b]\neq\varnothing\}\bigcup\{\varnothing\}$ [see theorem: 20.150] is semi-ring.

2. Prove that $\text{len}:\mathcal{I}_{[[}\bigcup\{\varnothing\}\rightarrow[0,\infty]$ defines by $\text{len}(I)=\begin{cases} b-a \text{ if } I=[a,b[\in\mathcal{I}_{[[}\\ 0\text{ if } I=\varnothing\end{cases}$ [the function that calculates the lenght of a half-open interval] is a finite content on the semi-ring $\mathcal{I}_{[[}$.

3. Use [theorem: 20.245] to extend the function len to a pre-measure on the ring $\mathcal{R}[\mathcal{S}]$

4. Use [theorem: 20.225] a derivate of the Carathéodore Theorem to extend the pre-measure to a measure [the Lebesgue measure] on a σ-algebra [the Lebesgue algebra on $\mathbb{R}$] that contains $\sigma[\mathcal{I}_{[[}]=\mathcal{B}[\mathbb{R}]$ and whose restriction on $\mathcal{B}[\mathbb{R}]$ is also a measure.

Definition 20.247. *The set $\mathcal{S}_{[[}\subseteq\mathcal{P}(\mathbb{R})$ is defined by*

$$\mathcal{S}_{[[}=\{[a,b[|a,b\in\mathbb{R}\}$$

Theorem 20.248. *$S_{[[}=\mathcal{I}_{[[}\bigcup\{\varnothing\}$ and $\{\varnothing\}\bigcap\mathcal{I}_{[[}=\varnothing$ in other words*

$$\mathcal{S}_{[[}=\mathcal{I}_{[[}\bigsqcup\{\varnothing\}$$

where $\mathcal{I}_{[[} \underset{\text{[definition: 20.146]}}{=} \{[a,b[|a,b\in\mathbb{R}\wedge[a,b[\neq\varnothing\}$

Proof. If $I\in\mathcal{S}_{[[}$ then there exists $a,b\in\mathbb{R}$ such that $I=[a,b[$. For a,b we have either

$a<b$. Then $[a,b[\neq\varnothing$ so that $I\in\mathcal{I}_{[[}\subseteq\mathcal{I}_{[[}\bigcup\{\varnothing\}$

$b\leqslant a$. Then $[a,b[=\varnothing$ so that $I\in\{\varnothing\}=\mathcal{I}_{[[}\bigcup\{\varnothing\}$

hence we have that

$$\mathcal{S}_{[[}\subseteq\mathcal{I}_{[[}\bigcup\{\varnothing\} \tag{20.299}$$

If $I\in\mathcal{I}_{[[}\bigcup\{\varnothing\}$ then either we have:

$I\in\mathcal{I}_{[[}$. Then $\exists a,b\in\mathbb{R}$ such that $I=[a,b[$ so that $I\in\mathcal{S}_{[[}$.

$I\in\{\varnothing\}$. Then $I=\varnothing=[0,0[$ so that $I\in\mathcal{S}_{[[}$

which proves that $\mathcal{I}_{[[}\bigcup\{\varnothing\}\subseteq\mathcal{S}$. Combining this with [eq: 20.299] gives

$$S_{[[}=\mathcal{I}_{[[}\bigcup\{\varnothing\}$$

Further if $I\in\mathcal{I}_{[[}\bigcap\{\varnothing\}$ then $I=\varnothing$ and there exists a $a,b\in\mathbb{R}$ such that $I=[a,b[\neq\varnothing$ which is a contradiction. Hence

$$\mathcal{I}_{[[}\bigcap\{\varnothing\}=\varnothing$$

□

Theorem 20.249. $\mathcal{S}_{[[}=\mathcal{I}_{[[}\bigsqcup\{\varnothing\}$ where $\mathcal{I}_{[[}=\{[a,b[|a,b\in\mathbb{R}$ *such that* $[a,b[\neq\varnothing\}$ *is a semi-ring.*

Proof.

1. $\varnothing\in\mathcal{I}_{[[}\bigcup\{\varnothing\}=\mathcal{S}_{[[}$
2. If $[a,b[,\ [c,d[\in\mathcal{S}_{[[}$ then we have either:

   $[a,b[=\varnothing\wedge[c,d[=\varnothing$. Then $[a,b[\bigcap[c,d[=\varnothing\bigcap\varnothing=\varnothing\in\mathcal{S}_{[[}$.

   $[a,b[\in\mathcal{I}_{[[}\wedge[c,d[=\varnothing$. Then $[a,b[\bigcap[c,d[=[a,b[\bigcap\varnothing=\varnothing\in\mathcal{S}_{[[}$.

   $[a,b[=\varnothing\wedge[c,d[\in\mathcal{I}_{[[}$. Then $[a,b[\bigcap[c,d[=\varnothing\bigcap[c,d[=\varnothing\in\mathcal{S}_{[[}$.

   $[a,b[\in\mathcal{I}_{[[}\wedge[c,d[\in\mathcal{I}_{[[}$. Then for $[a,b[\bigcap[c,d[$ we have either

   $[a,b[\bigcap[c,d[=\varnothing$. Then $[a,b[\bigcap[c,d[=\varnothing\in\mathcal{S}_{[[}$

   $[a,b[\bigcap[c,d[\neq\varnothing$. Then for a we have either:

   $a\leqslant c$. For b we have either:

   $b\leqslant d$. Let $y\in[c,b[$ then $a\leqslant c\leqslant y<b\leqslant d$ so that $y\in[a,b[\wedge y\in[c,d[$ hence $y\in[a,b[\bigcap[c,d[$, proving that

$$[c,b[\subseteq[a,b[\bigcap[c,d[$$

   Let $y\in[a,b[\bigcap[c,d[$ then $y\in[a,b[\wedge y\in[c,d[$ so that

$$a\leqslant y<b\wedge c\leqslant y<d\Rightarrow c\leqslant y<b$$

   proving that $y\in[c,b[$ so that $[a,b[\bigcap[c,d[\subseteq[c,b[$. Hence

$$\varnothing\neq[a,b[\bigcap[c,d[=[c,b[\in\mathcal{I}_{[[}\subseteq\mathcal{S}_{[[}$$

   $d<b$. Let $y\in[c,d[$ then $a\leqslant c\leqslant y<d<b$ so that $y\in[a,b[$ hence $[c,d[\subset[a,b[$ so that

$$\varnothing\neq[c,d[=[a,b[\bigcap[c,d[$$

$\boldsymbol{c<a}$**.** For b we have either:

$\boldsymbol{b\leqslant d}$**.** Let $y\in[a,b[$ then $c<a\leqslant y<b\leqslant d$ so that $y\in[c,d[$ hence $[a,b[\subseteq[c,d[$ so that

$$\varnothing\neq[a,b[\subseteq[a,b[\bigcap\,[c,d[$$

$\boldsymbol{d<b}$**.** Let $y\in[a,d[$ then $c<a\leqslant y<d<b$ so that $y\in[a,b[\wedge[c,d[$ hence $y\in[a,b[\bigcap[c,d[$, proving that

$$\varnothing\neq[a,d[\subseteq[a,b[\bigcap\,[c,d[$$

Let $y\in[a,b[\bigcap[c,d[$ then $y\in[a,b[\wedge y\in[c,d[$ so that

$$a\leqslant y<b\wedge c\leqslant y<d\Rightarrow a\leqslant y<d$$

proving that $y\in[a,d[$ or $[a,b[\bigcap[c,d[\subseteq[a,d[$. Hence

$$\varnothing\neq[a,b[\bigcap\,[c,d[=[a,d[\in\mathcal{I}_{[[}\subseteq\mathcal{S}_{[[}$$

so in all cases we have that

$$[a,b[\bigcap\,[c,d[\in\mathcal{S}_{[[}$$

3. Let $[a,b[,[c,d[\in\mathcal{S}_{[[}$ then we have either:

$\boldsymbol{[a,b[=\varnothing\wedge[c,d[=\varnothing}$**.** Then $[a,b[\backslash[c,d[=\varnothing\backslash\varnothing=\varnothing=\bigsqcup_{i\in\{1\}}A_i$ where $\{A_i\}_{i\in\{1\}}\subseteq\mathcal{S}_{[[}$ is defined by $A_i=\varnothing$.

$\boldsymbol{[a,b[\in\mathcal{I}_{[[}\wedge[c,d[=\varnothing}$**.** Then $[a,b[\backslash[c,d[=[a,b[\backslash\varnothing=[a,b[=\bigsqcup_{i\in\{1\}}A_i$ where $\{A_i\}_{i\in\{1\}}\subseteq\mathcal{S}_{[[}$ is defined by $A_i=[a,b[$.

$\boldsymbol{[a,b[=\varnothing\wedge[c,d[\in\mathcal{I}_{[[}}$**.** Then $[a,b[\backslash[c,d[=\varnothing\backslash[c,d[=\varnothing=\bigsqcup_{i\in\{1\}}A_i$ where $\{A_i\}_{i\in\{1\}}\subseteq\mathcal{S}_{[[}$ is defined by $A_i=\varnothing$.

$\boldsymbol{[a,b[\in\mathcal{I}_{[[}\wedge[c,d[\in\mathcal{I}_{[[}}$**.** Then for $[a,b[\bigcap[c,d[$ we have either:

$\boldsymbol{[a,b[\bigcap[c,d[=\varnothing}$**.** Then $[a,b[\backslash[c,d[=[a,b[=\bigsqcup_{i\in\{1\}}A_i$ where $\{A_i\}_{i\in\{1\}}\subseteq\mathcal{S}_{[[}$ is defined by $A_i=[a,b[$.

$\boldsymbol{[a,b[\bigcap[c,d[\neq\varnothing}$**.** As $[a,b[\neq\varnothing\neq[c,d[$ we have

$$a<b\wedge c<d \tag{20.300}$$

further there exist a x such that

$$a\leqslant x<b\wedge c\leqslant x<d\Rightarrow a<d\wedge c<b \tag{20.301}$$

$\boldsymbol{a\leqslant c}$**.** Then for b we have either:

$\boldsymbol{b\leqslant d}$**.** Let $y\in[a,c[$ then $a\leqslant y<c\underset{\text{[eq: 20.301]}}{<}b$ so that $y\in[a,b[$. Assume that $y\in[c,d[$ then $c\leqslant y$ contradicting $y<c$ hence $y\notin[c,d[$ so that $y\in[a,b[\backslash[c,d[$. Hence

$$[a,c[\subseteq[a,b[\backslash[c,d[\tag{20.302}$$

On the other hand if $y\in[a,b[\backslash[c,d[$ we have $a\leqslant y<b\leqslant d$ and $\neg(c\leqslant y<d)\Rightarrow y<c\vee d\leqslant y$. Assume that $d\leqslant y$ then as also $y<b\leqslant d\Rightarrow y<d$ we would have a contradiction so that we must have $y<c$. Hence we have $a\leqslant y<c$ which proves that $[a,b[\backslash[c,d[\subseteq[a,c[$ so that, using [eq: 20.302] we have

$$[a,b[\backslash[c,d[=[a,c[\in\mathcal{S}_{[[}=\mathcal{I}_{[[}\bigcup\,\{\varnothing\}$$

So if we define $\{A_i\}_{i\in\{1\}}\subseteq\mathcal{S}_{[[}$ by $A_1=[a,c[$ we have

$$[a,b[\setminus[c,d[=\bigsqcup_{i\in\{1\}}A_i$$

$d<b$. Let $y\in[a,c[\bigcap[d,b[$ then $a\leqslant y<c\wedge d\leqslant y<b$, as by [eq: 20.300] $c<d$ we reach the contradictib $y<y$ so that

$$[a,c[\bigcap[d,b[=\varnothing \tag{20.303}$$

Let $y\in[a,c[\bigcup[d,b[$ then we have either:

$y\in[a,c[$. Then $a\leqslant y<c\underset{\text{[eq: 20.301]}}{<}b$ so that $y\in[a,b]$. Assume that $y\in[c,d[$ then $c\leqslant y<d$ contradicting $y<c$ hence we have $y\notin[c,d[$ so

$$y\in[a,b[\setminus[c,d]$$

$y\in[d,b[$. Then $a\underset{\text{[eq: 20.301]}}{<}d\leqslant y<b$ so that $y\in[a,b[$. Assume that $y\in[c,d[$ then $c\leqslant y<d$ contradicting $d\leqslant y$ hence we must have $y\notin[c,d[$ or

$$y\in[a,b[\setminus[c,d[$$

as in all cases we have $y\in[a,b[\setminus[c,d[$ it follows that

$$[a,c[\bigcup[d,b[\subseteq[a,b[\setminus[c,d[\tag{20.304}$$

On the other hand if $y\in[a,b[\setminus[c,d[$ then

$$\begin{aligned}a\leqslant y<b\wedge\neg(c\leqslant y\wedge y<d) &=\\ a\leqslant y<b\wedge(y<c\vee d\leqslant y) &=\\ (a\leqslant y<b\wedge y<c)\vee(a\leqslant y<b\wedge d\leqslant y)\end{aligned}$$

so that $y\in[a,c[\vee y\in[d,b[\Rightarrow y\in[a,c[\bigcup[d,b[$ which proves that $[a,b[\setminus[c,d[\subseteq[a,c[\bigcup[d,b[$. Combining this with [eq: 20.304]

$$[a,b[\setminus[c,d[=[a,c[\bigcup[d,b[$$

Hence if we define $\{A_i\}_{i\in\{1,2\}}\subseteq\mathcal{I}_{[[}\bigcup\{\varnothing\}$ by $A_1=[a,c[$, $A_2=[d,b[$ then $\{A_i\}_{i\in\{1,2\}}$ is pairwise disjoint and

$$[a,b[\setminus[c,d[=\bigsqcup_{i\in\{1,2\}}A_i$$

$c<a$. Then for b we have either:

$b\leqslant d$. If $y\in[a,b[$ then $c<a\leqslant y<b\leqslant d$ so that $y\in[c,d[$, hence $[a,b[\subseteq[c,d[$ so that $[a,b[\setminus[c,d[=\varnothing$. Hence if we define $\{A_i\}_{i\in\{1\}}\subseteq\mathcal{S}_{[[}$ by $A_1=\varnothing$ it follows that

$$[a,b[\setminus[c,d[=\bigsqcup_{i\in\{1\}}A_i$$

$d<b$. Let $y\in[d,b[$ then $a\underset{\text{[eq: 20.301]}}{<}d\leqslant y<b$ so that $y\in[a,b[$. Assume that $y\in[c,d[$ then $c\leqslant y<d$ contradicting $d\leqslant y$. Hence $y\in[a,b[\setminus[c,d[$ proving that

$$[d,b[\subseteq[a,b[\setminus[c,d] \tag{20.305}$$

If $y \in [a, b[\setminus [c, d]$ then

$$\begin{array}{rl} a \leqslant y < b \wedge \neg(c \leqslant y < d) & = \\ a \leqslant y < b \wedge (y < c \vee d \leqslant y) & = \\ (a \leqslant y < b \wedge y < c) \vee (a \leqslant y < b \wedge d \leqslant y) & \underset{c<a}{=} \\ (c < a \leqslant y < b \wedge y < c) \vee (c < a \leqslant y < b \wedge d \leqslant y) & \underset{c \nless c}{=} \\ c < a \leqslant y < b \wedge d \leqslant y & \end{array}$$

so that $y \in [d, b[$. Hence $[a, b[\setminus [c, d[\subseteq [d, b[$ which combined with [eq: 20.305] gives

$$[a, b[\setminus [c, d[= [d, b[$$

Define $\{A_i\}_{i \in \{1\}} \subseteq \mathcal{S}_{[[}$ by $A_1 = [d, b[$ then

$$[a, b[\setminus [c, d[= \bigsqcup_{i \in \{1\}} A_i$$

So in all cases we found a pairwise disjoint family $\{A_i\}_{i \in \{1, \ldots, m\}} \subseteq \mathcal{S}_{[[}$ such that

$$[a, b[\setminus [c, d[= \bigsqcup_{i \in \{1, \ldots, m\}} A_i$$

□

Theorem 20.250. *For $\mathcal{S}_{[[} = \mathcal{I}_{[[} \bigsqcup \{\varnothing\}$ where $\mathcal{I}_{[[} = \{[a, b[| a, b \in \mathbb{R}$ such that $[a, b[\neq \varnothing\}$ then*

$$\sigma[\mathcal{S}_{[[}] = \sigma[\mathcal{I}_{[[}]$$

so that using [theorem: 20.150]

$$\begin{array}{rcl} \sigma[\{C \subseteq \mathbb{R} | C \text{ is closed in } \mathcal{T}_{||}\}] & \underset{\text{[theorem: 20.143]}}{=} & \mathcal{B}[\mathbb{R}] \\ & = & \sigma[\mathcal{T}_{||}] \\ & = & \sigma[\mathcal{I}_{][}] \\ & = & \sigma[\mathcal{I}_{]]}] \\ & = & \sigma[\mathcal{I}_{[]}] \\ & = & \sigma[\mathcal{I}_{[[}] \\ & = & \sigma[\mathcal{S}_{[[}] \\ & = & \sigma[\{[a, \infty[| a \in \mathbb{R}\}] \\ & = & \sigma[\{]a, \infty[| a \in \mathbb{R}\}] \\ & = & \sigma[\{]-\infty, a] | a \in \mathbb{R}\}] \\ & = & \sigma[\{]-\infty, a[| a \in \mathbb{R}\}] \end{array}$$

Proof. As $\mathcal{I}_{[[} \subseteq \mathcal{S}_{[[} \subseteq \sigma[\mathcal{S}_{[[}]$ it follows from [theorem: 20.141] and the fact that $\sigma[\mathcal{S}_{[[}]$ is a σ-algebra that

$$\sigma[\mathcal{I}_{[[}] \subseteq \sigma[\mathcal{S}_{[[}] \tag{20.306}$$

Let $A \in \mathcal{S}_{[[}$ then either $A \in \mathcal{I}_{[[} \subseteq \sigma[\mathcal{I}_{[[}]$ or $A = \varnothing \in \sigma[\mathcal{I}_{[[}]$ which proves that $\mathcal{S}_{[[} \subseteq \sigma[\mathcal{I}_{[[}]$. Hence using [theorem: 20.141] and the fact that $\sigma[\mathcal{I}_{[[}]$ is a σ-algebra it follows that $\sigma[\mathcal{S}_{[[}] \subseteq \sigma[\mathcal{I}_{[[}]$. Combining this with [eq: 20.306] proves that

$$\sigma[\mathcal{S}_{[[}] = \sigma[\mathcal{I}_{[[}]$$

□

Definition 20.251. *We define*

$$\text{begin}: \mathcal{I}_{[[} \rightarrow \mathbb{R} \textit{ by } \text{begin}(I) = a \textit{ where } I = [a, b[$$

$$\text{end}: \mathcal{I}_{[[} \rightarrow \mathbb{R} \textit{ by } \text{end}(I) = b \textit{ where } I = [a, b[$$

***Note* 20.252.** *If* $\varnothing \neq I = [a, b[= [c, d[$ *then by [theorem: 3.144(7)] we have* $a = c \wedge b = d$ *so these functions are well defined.*

***Note* 20.253.** *As* $I \neq \varnothing$ *we have always that* $\text{begin}(I) < \text{end}(I)$

We define

$$\text{begin}: \mathcal{I}_{[[} \rightarrow \mathbb{R} \text{ by } \text{begin}(I) = a \text{ where } I = [a, b[$$

$$\text{end}: \mathcal{I}_{[[} \rightarrow \mathbb{R} \text{ by } \text{end}(I) = b \text{ where } I = [a, b[$$

Note 20.254. If $\varnothing \neq I = [a, b[= [c, d[$ then by [theorem: 3.144(7)] we have $a = c \wedge b = d$ these functions are well defined.

Note 20.255. As $I \neq \varnothing$ we have that $\text{begin}(I) < \text{end}(I)$

Using the above two functions we can now define the length function.

Definition 20.256. *We define*

$$\text{len}: \mathcal{S}_{[[} \rightarrow [0, \infty] \textit{ by } \text{len}(I) = \begin{cases} 0 \textit{ if } I = \varnothing \\ \text{end}(I) - \text{begin}(I) \textit{ if } I \in \mathcal{I}_{[[} \end{cases}$$

In other words the function len *calculates the length of a interval.*

To simplify the proof that len is a content we need to order the half open intervals based on their boundary points. We define now a total order on $\mathcal{I}_{[[}$ the set of half-open intervals.

Theorem 20.257. *The relation* $\preccurlyeq \subseteq \mathcal{I}_{[[} \times \mathcal{I}_{[[}$ *defined by*

$$[a, b[\preccurlyeq [c, d[\textit{ if } a < c \vee (a = c \wedge b \leqslant d)$$

is a total order on $\mathcal{I}_{[[}$ *making* $\langle \mathcal{I}_{[[}, \preccurlyeq \rangle$ *a totally ordered set.*

Proof. We have

reflexivity. Let $[a, b[\in \mathcal{I}_{[[}$ then as $a = a$ and $b = b \Rightarrow b \leqslant b$ it follows that $[a, b[\preccurlyeq [a, b[$ proving reflectivity.

anti-symmetry. Let $[a, b[, [c, d[\in \mathcal{I}_{[[}$ be such that $[a, b[\preccurlyeq [c, d[$ and $[c, d[\preccurlyeq [a, b[$. As $[a, b[\preccurlyeq [c, d[$ we have either:

$a < c$. Then as $[c, d[\preccurlyeq [a, b[$ we have either $c = a$ or $c < a$ which both give a contradiction, so this cases never occurs.

$a = c$. Then from $[a, b[\preccurlyeq [c, d[$ it follows that $b \leqslant d$ and from $[c, d[\preccurlyeq [a, b[$ it follows that $d \leqslant b$. Hence we have $a = c \wedge b = d$ so that $[a, b[= [c, d[$

transitivity. Assume that $[a, b[\preccurlyeq [c, d[$ and $[c, d[\preccurlyeq [e, f[$. As $[a, b[\preccurlyeq [c, d[$ we have either:

$a < c$. Then as $[c, d[\preccurlyeq [e, f[$ we have either:

$c < e$. Then $a < e$ so that $[a, b[\preccurlyeq [e, f[$

$c = e$. Then $a < e$ so that $[a, b[\preccurlyeq [e, f[$

$a = c$. Then as $[a, b[\preccurlyeq [c, d[$ we must have that

$$b \leqslant d$$

As $[c, d[\preccurlyeq [e, f[$ we have either:

$c < e$. Then $a < e$ so that $[a, b[\preccurlyeq [e, f[$.

$c = e$. Then as $[c, d[\preccurlyeq [e, f[$ we have that $d \leqslant f$ which combined with $a = c = e$ proves that $[a, b[\preccurlyeq [e, f[$.

So in all cases we have $[a, b[\preccurlyeq [e, f[$ proving transitivity.

totally. If $[a, b[, [c, d[$ then, as $a, b, c, d \in \mathbb{R}$ and $\langle \mathbb{R}, \leqslant \rangle$ is totally ordered, we have for a, c either:

$a < c$. Then $[a, b[\preccurlyeq [c, d[$

$a = c$. Then as $\langle \mathbb{R}, \leqslant \rangle$ is totally ordered we have for b, d either:

$b < d$. Then $[a, b[\preccurlyeq [c, d]$

$d \leqslant b$. Then $[c, d[\preccurlyeq [a, b[$

$c < a$. Then $[c, d[\preccurlyeq [a, b[$ □

So we can sort half open intervals which will be used in the next lemma.

Theorem 20.258. *Let $n \in \mathbb{N}$, $\{[a_i, b_i[\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{I}_{[[}$ pairwise disjoint such that*

$$\bigcup_{i \in \{1, \ldots, n\}} [a_i, b_i[\in \mathcal{I}_{[[}$$

then there exist a bijection $\sigma: \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ such that

$$\forall i \in \{1, \ldots, n-1\} \text{ we have } b_{\sigma(i)} = a_{\sigma(i+1)}$$

and

$$\bigcup_{i \in \{1, \ldots, n\}} [a_i, b_i[= [a_{\sigma(1)}, b_{\sigma(n)}]$$

Proof. For $n \in \mathbb{N}$ we have either:

$n = 1$. Take $\sigma = \mathrm{Id}_{\{1\}}: \{1\} \rightarrow \{1\}$ then σ is a bijection, further as $\{1, \ldots, n-1\} = \varnothing$ we have vacuously that $\forall i \in \{1, \ldots, n-1\}$ $b_{\sigma(i)} = a_{\sigma(i+1)}$, finally $\bigcup_{i \in \{1\}} [a_i, b_i[= [a_1, b_1[= [a_{\sigma(1)}, b_{\sigma(1)}[$.

$n \in \{2, \ldots, \infty\}$. As $\langle \mathcal{I}_{[[}, \preccurlyeq \rangle$ is totally ordered there exist by [corollary: 6.55] a bijection $\sigma: \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ such that

$$\forall i, j \in \{1, \ldots, n\} \text{ with } i \leqslant j \text{ we have } [a_{\sigma(i)}, b_{\sigma(i)}[\preccurlyeq [a_{\sigma(j)}, \beta_{\sigma(j)}[\tag{20.307}$$

Let $i, j \in \{1, \ldots, n\}$ with $i \leqslant j$ then as $[a_{\sigma(i)}, b_{\sigma(i)}[\preccurlyeq [a_{\sigma(j)}, \beta_{\sigma(j)}[$ we have either $a_{\sigma(i)} < a_{\sigma(j)} \Rightarrow a_{\sigma(i)} \leqslant a_{\sigma(j)}$ or $a_{\sigma(i)} = a_{\sigma(i+1)} \Rightarrow a_{\sigma(i)} \leqslant a_{\sigma(j)}$ so that

$$\forall i, j \in \{1, \ldots, n\} \text{ with } i \leqslant j \text{ we have } a_{\sigma(i)} \leqslant a_{\sigma(j)} \tag{20.308}$$

Futher by [theorem: 20.107] we have that

$$\{[a_{\sigma(i)}, b_{\sigma(i)}[\}_{i \in \{1, \ldots, n\}} \text{ is pairwise disjoint} \tag{20.309}$$

Assume that there exist a $k \in \{1, \ldots, n-1\}$ such that $a_{\sigma(k+1)} < b_{\sigma(k)}$. Then by [eq: 20.308] $a_{\sigma(k)} \leqslant a_{\sigma(k+1)} < b_{\sigma(k)}$ from which it follows that $a_{\sigma(k+1)} \in [a_{\sigma(k)}, b_{\sigma(k)}[\bigcap [a_{\sigma(k+1)}, b_{\sigma(k+1)}[$ contradicting $[a_{\sigma(k)}, b_{\sigma(k)}[\bigcap [a_{\sigma(k+1)}, b_{\sigma(k+1)}[\underset{\text{[eq: 20.309]}}{=} \varnothing$. Hence the assumtion is wrong and we conclude that

$$\forall i \in \{1, \ldots, n-1\} \text{ we have } b_{\sigma(i)} \leqslant a_{\sigma(i+1)} \tag{20.310}$$

As $\bigcup_{i\in\{1,\ldots,n\}}[a_i,b_i[\in\mathcal{I}_{[[}$ there exist a $c,d\in\mathbb{R}$ with $c<d$ such that

$$\bigcup_{i\in\{1,\ldots,n\}}[a_{\sigma(i)},b_{\sigma(i)}[\underset{[\text{theorem: } 2.119]}{=}\bigcup_{i\in\{1,\ldots,n\}}[a_i,b_i[=[c,d[\tag{20.311}$$

If $i=\{1,\ldots,n-1\}$ then by [eq: 20.310] we have $b_{\sigma(i)}\leqslant a_{\sigma(i+1)}$ and as $i\leqslant n-1<n\Rightarrow i+1\leqslant n$ it follows from [eq: 20.308] that $a_{\sigma(i+1)}\leqslant a_{\sigma(n)}<b_{\sigma(n)}$ so that $b_{\sigma(i)}<b_{\sigma(n)}$, hence

$$\forall i\in\{1,\ldots,n\} \text{ we have } b_{\sigma(i)}\leqslant b_{\sigma(n)} \tag{20.312}$$

Let $i\in\{1,\ldots,n\}$ then for $x\in[a_{\sigma(i)},b_{\sigma(i)}[$ we have $a_{\sigma(1)}\underset{[\text{eq: } 20.308]}{\leqslant}a_{\sigma(i)}<x<b_{\sigma(i)}\underset{[\text{eq: } 20.312]}{\leqslant}b_{\sigma(n)}$ so that $x\in[a_{\sigma(1)},b_{\sigma(n)}[$ hence $[a_{\sigma(i)},b_{\sigma(i)}[\subseteq[a_{\sigma(n)},b_{\sigma(n)}[$. So

$$[c,d[\underset{[\text{eq: } 20.311]}{=}\bigcup_{i\in\{1,\ldots,n\}}[a_{\sigma(i)},b_{\sigma(i)}[\subseteq[a_{\sigma(1)}b_{\sigma(n)}[\tag{20.313}$$

As $[c,d[\subseteq[a_{\sigma(1)}b_{\sigma(n)}[$ it follows from [theorem: 3.144(9)] that

$$a_{\sigma(1)}\leqslant c\wedge d\leqslant b_{\sigma(n)} \tag{20.314}$$

As $a_{\sigma(1)}\in[a_{\sigma(1)},b_{\sigma(1)}[\subseteq\bigcup_{i\in\{1,\ldots,n\}}[a_{\sigma(i)},b_{\sigma(i)}[\underset{[\text{eq: } 20.311]}{=}[c,d]$ it follows that $c\leqslant a_{\sigma(1)}$ which combined with [eq: 20.314] proves that

$$a_{\sigma(1)}=c \tag{20.315}$$

As $a_{\sigma(n)}\in[a_{\sigma(n)},b_{\sigma(n)}[\subseteq\bigcup_{i\in\{1,\ldots,n\}}[a_{\sigma(i)},b_{\sigma(i)}[\underset{[\text{eq:} 20.311]}{=}[c,d[$ we have $c\leqslant a_{\sigma(n)}<d$. Assume that $d<b_{\sigma(n)}$ then $a_{\sigma(n)}<d<b_{\sigma(n)}$, proving that $d\in[a_{\sigma(n)},b_{\sigma(n)}[\subseteq[c,d[$ leading to the contradiction $d<d$. Hence we must have $b_{\sigma(n)}\leqslant d$ which combined with [eq: 20.314] proves

$$b_{\sigma(n)}=d \tag{20.316}$$

Summarizing [eqs: 20.311, 20.315 and 20.316] we have

$$\bigsqcup_{i\in\{1,\ldots,n\}}[a_i,b_i[=\bigsqcup_{i\in\{1,\ldots,n\}}[a_{\sigma(i)},b_{\sigma(i)}[=[a_{\sigma(1)},b_{\sigma(n)}[\tag{20.317}$$

Assume now that there exist a $k\in\{1,\ldots,n-1\}$ such that $b_{\sigma(k)}\neq a_{\sigma(k+1)}$. If $a_{\sigma(k+1)}<b_{\sigma(k)}$ then by [eq: 20.310] $b_{\sigma(k)}\leqslant a_{\sigma(k+1)}$ and we reach the contradiction $a_{\sigma(k+1)}<a_{\sigma(k+1)}$ so we must have that $b_{\sigma(k)}<a_{\sigma(k+1)}$. Hence there exist a $x\in\mathbb{R}$ such that

$$b_{\sigma(k)}<x<a_{\sigma(k+1)} \text{ so that } x\notin[a_{\sigma(k)},b_{\sigma(k)}[\text{ and } x\notin[a_{\sigma(k+1)},b_{\sigma(k+1)}[\tag{20.318}$$

If $i\in\{1,\ldots,k-1\}$ then $b_{\sigma(i)}\underset{[\text{eq: } 20.310]}{\leqslant}a_{\sigma(i+1)}\underset{[\text{eq: } 20.308]}{\leqslant}a_{\sigma(k)}<b_{\sigma(k)}<x$ so that $x\notin[a_{\sigma(i)},b_{\sigma(i)}[$. Further if $i\in\{k+1,\ldots,n\}$ then $x<a_{\sigma(k+1)}\leqslant a_{\sigma(i)}$ so that $x\notin[a_{\sigma(i)},b_{\sigma(i)}[$. From this we conclude that $\forall i\in\{1,\ldots,n\}$ $x\notin[a_{\sigma(i)},b_{\sigma(i)}[$ or

$$x\notin\bigcup_{i\in\{1,\ldots,n\}}[a_{\sigma(i)},b_{\sigma(i)}[=[a_{\sigma(1)},b_{\sigma(n)}[\tag{20.319}$$

As $a_{\sigma(1)}\underset{[\text{eq: } 20.308]}{\leqslant}a_{\sigma(k)}<b_{\sigma(k)}<x$ and $x<a_{\sigma(k+1)}\underset{[\text{eq: } 20.308]}{\leqslant}a_{\sigma(n)}<b_{\sigma(n)}$ so that $x\in[a_{\sigma(1)},b_{\sigma(n)}[$ which contradicts [eq: 20.319]. Hence the assumption is false and we must have

$$\forall i\in\{1,\ldots,n-1\} \text{ that } b_{\sigma(i)}=a_{\sigma(i)}$$ □

We are now ready to prove that len is a content on the semi-ring $\mathcal{S}_{[[}$.

Lemma 20.259. *The length function* $\text{len}:\mathcal{S}_{[[}\to[0,\infty]$ *defined by* $\text{len}(I)=\begin{cases}0 \text{ if } I=\varnothing\\ \text{end}(I)-\text{begin}(I) \text{ if } I\in\mathcal{I}_{[[}\end{cases}$ *[see definition: 20.256] is a finite content on the semi-ring* $\mathcal{S}_{[[}$.

Proof. First by definition we have

$$\mathrm{len}(\varnothing)=0 \tag{20.320}$$

Second $\forall I\in\mathcal{S}_{[[}$ we have $\mathrm{len}(I)=\begin{cases} 0<\infty \text{ if } I=\varnothing \\ \mathrm{end}(I)-\mathrm{begin}(I)<\infty \text{ if } I\in\mathcal{I}_{[[} \end{cases}$ which proves that

$$\text{len is a finite} \tag{20.321}$$

Next we prove that len is additive on $\mathcal{I}_{[[}$. Let $\{I_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{I}_{[[}$, $n\in\mathbb{N}$ pairwise disjoint such that $\bigsqcup_{i\in\{1,\ldots,n\}}I_i\in\mathcal{I}_{[[}$ then we have for $n\in\mathbb{N}$ either:

$n=1$. Then $\mathrm{len}\left(\bigsqcup_{i\in\{1,\ldots,n\}}I_i\right)=\mathrm{len}\left(\bigsqcup_{i\in\{1\}}I_i\right)=\mathrm{len}(I_1)=\sum_{i=1}^{1}\mathrm{len}(I_i)=\sum_{i=1}^{n}\mathrm{len}(I_i)$

$1<n$. Then $\forall i\in\{1,\ldots,n\}$ there exists a_i, $b_i\in\mathbb{R}$ with $a_i<b_i$ such that $I_i=[a_i,b_i[$. Using [theorem: 20.258] there exist a bijection $\sigma:\{1,\ldots,n\}\to\{1,\ldots,n\}$ such that

$$\bigsqcup_{i\in\{1,\ldots,n\}}I_i=\bigsqcup_{i\in\{1,\ldots,n\}}[a_i,b_i[=\bigsqcup_{i\in\{1,\ldots,n\}}[a_{\sigma(i)},b_{\sigma(i)}[=[a_{\sigma(1)},b_{\sigma(n)}[\tag{20.322}$$

such that

$$\forall i\in\{1,\ldots,n-1\}\ b_{\sigma(i)}=a_{\sigma(i+1)} \tag{20.323}$$

Then

$$\begin{aligned}
\sum_{i=1}^{n}\mathrm{len}(I_i) &\underset{[\text{theorem: } 11.31]}{=} \sum_{i=1}^{n}\mathrm{len}(I_{\sigma(i)})\\
&= \sum_{i=1}^{n}\mathrm{len}([a_{\sigma(i)},b_{\sigma(i)}[)\\
&= \sum_{i=1}^{n}(b_{\sigma(i)}-a_{\sigma(i)})\\
&\underset{1<n}{=} (b_{\sigma(n)}-a_{\sigma(n)})+\sum_{i=1}^{n-1}(b_{\sigma(i)}-a_{\sigma(i)})\\
&\underset{[\text{eq: } 20.323]}{=} (b_{\sigma(n)}-a_{\sigma(n)})+\sum_{i=1}^{n-1}(a_{\sigma(i+1)}-a_{\sigma(i)})\\
&\underset{[\text{theorem: } 11.10]}{=} b_{\sigma(n)}-a_{\sigma(n)}+a_{\sigma(n)}-a_{\sigma(1)}\\
&= b_{\sigma(n)}-a_{\sigma(1)}\\
&= \mathrm{len}([a_{\sigma(1)},b_{\sigma(n)}[)\\
&\underset{[\text{eq: } 20.322]}{=} \mathrm{len}\left(\bigsqcup_{i\in\{1,\ldots,n\}}I_i\right)
\end{aligned}$$

To summarize

$$\forall\{I_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{I}_{[[} \text{ pairwise disjoint with } \bigsqcup_{i\in\{1,\ldots,n\}}I_i\in\mathcal{I}_{[[}\quad \mathrm{len}\left(\bigsqcup_{i\in\{1,\ldots,n\}}I_i\right)=\sum_{i=1}^{n}\mathrm{len}(I_i) \tag{20.324}$$

Next we extend the above to $\mathcal{S}_{[[}$. Let $\{I_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}_{[[}$, $n\in\mathbb{N}$ such that $\bigsqcup_{i\in\{1,\ldots,n\}}I_i\in\mathcal{S}_{[[}$. Define $N=\{i\in\{1,\ldots,n\}|I_i\neq\varnothing\}$ then for N we have either:

$N=\varnothing$. Then

$$\mathrm{len}\left(\bigsqcup_{i\in\{1,\ldots,n\}}I_i\right)=\mathrm{len}(\varnothing)=0=\sum_{i=1}^{n}0=\sum_{i=1}^{n}\mathrm{len}(I_i)$$

$N \neq \varnothing$. Then as $\varnothing \neq N$ is finite there exist a $m \in \{1, \ldots, n\}$ and a bijection $\beta: \{1, \ldots, m\} \rightarrow N$. Further we have

$$\begin{aligned}
\operatorname{len}\left(\bigsqcup_{i \in \{1, \ldots, n\}} I_i\right) &\underset{[\text{theorem: } 2.134]}{=} \operatorname{len}\left(\bigsqcup_{i \in N} I_i\right) \\
&\underset{[\text{theorem: } 2.119]}{=} \operatorname{len}\left(\bigsqcup_{i \in \{1, \ldots, m\}} I_{\sigma(i)}\right) \\
&\underset{[\text{eq: } 20.324]}{=} \sum_{i=1}^{m} \operatorname{len}(I_{\sigma(i)}) \\
&= \sum_{i \in \{1, \ldots, m\}} \operatorname{len}(I_{\sigma(i)}) \\
&\underset{[\text{theorem: } 11.36]}{=} \sum_{i \in N} \operatorname{len}(I_i) \\
&= \sum_{i \in N} \operatorname{len}(I_i) + \sum_{i \in \{1, \ldots, n\} \setminus N} 0 \\
&= \sum_{i \in N} \operatorname{len}(I_i) + \sum_{i \in \{1, \ldots, n\} \setminus N} \operatorname{len}(I_i) \\
&= \sum_{i \in \{1, \ldots, n\}} \operatorname{len}(I_i) \\
&= \sum_{i=1}^{n} \operatorname{len}(I_i)
\end{aligned}$$

So in all cases we have that

Let $\{I_i\}_{i \in \{1, \ldots, n\}} \subseteq \mathcal{S}_{[[}$, $n \in \mathbb{N}$ pairwise disjoint with $\bigsqcup_{i \in \{1, \ldots, n\}} I_i \in \mathcal{S}_{[[}$ then

$$\operatorname{len}\left(\bigsqcup_{i \in \{1, \ldots, n\}} I_i\right) = \sum_{i=1}^{n} \operatorname{len}(I_i) \tag{20.325}$$

So using [eqs: 20.320, 20.321 and 20.325] it follows that len is a finite content on the semi-ring $\mathcal{S}_{[[}$. □

Next we prove that len is a pre-measure on $\mathcal{S}_{[[}$.

Lemma 20.260. *The function* $\operatorname{len}: \mathcal{S}_{[[} \rightarrow [0, \infty]$ *[see definition: 20.256] is a pre-measure on the semi-ring* $\mathcal{S}_{[[}$

Proof. Consider the topological space $\langle \mathbb{R}, \mathcal{T}_{||} \rangle$. Let $A \in S_{[[}$ and $\varepsilon \in \mathbb{R}^+$ then for A we have either:

$A = \varnothing$. Take then $A_\varepsilon = \varnothing \in \mathcal{S}_{[[}$ and $K_\varepsilon = \varnothing$. Then K_ε is compact by [example: 14.226], $A_\varepsilon \subseteq K_\varepsilon \subseteq A$ and $\operatorname{len}(A) - \operatorname{len}(A_\varepsilon) = 0 - 0 = 0 < \varepsilon$.

$A \neq \varnothing$. Then there exists $a, b \in \mathbb{R}$ with $a < b$ such that $A = [a, b[$. Define then $A_\varepsilon = \left[a, b - \frac{\varepsilon}{2}\right[$ and $K_\varepsilon = \left[a, b - \frac{\varepsilon}{2}\right]$. Then by the Heine Borel theorem]theorem: 14.235] K_ε is compact. Further $A_\varepsilon \subseteq K_\varepsilon \subseteq A$ and $\operatorname{len}(A) - \operatorname{len}(A_\varepsilon) = \operatorname{len}([a, b[) - \operatorname{len}\left(\left[a, b - \frac{\varepsilon}{2}\right[\right) = b - a - \left(b - \frac{\varepsilon}{2} - a\right) = \frac{\varepsilon}{2} < \varepsilon$

Hence using [theorem: 20.246] it follows that

$$\overline{\operatorname{len}} \text{ is a pre-measure on } \mathcal{R}[\mathcal{S}]$$ □

Next we use a derivate of the Carathéodry theorem [theorem: 20.225] to extend len to the Lebesgue measure on Lebesgue measurable sets in $\mathbb{R}$.

Theorem 20.261. *If we define*

$$\lambda^*: \mathcal{P}[\mathbb{R}] \rightarrow [0, \infty] \text{ by } \lambda^*(A) = \begin{cases} \infty \text{ if } \mathcal{S}[A] = \varnothing \\ \inf\left(\left\{\sum_{i=1}^{\infty} \mu(A_i) | \{A_i\}_{i \in \mathbb{N}} \in \mathcal{S}[A]\right\}\right) \end{cases}$$

where

$$\mathcal{S}[A] = \left\{\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{S}_{[[} | A \subseteq \bigcup_{i \in \mathbb{N}} A_i\right\}$$

then

$$\lambda^* \text{ is a outer measure}$$

$$\mathcal{L}[\mathbb{R}] = \left\{A \in \mathcal{P}(\mathbb{R}) | \forall B \in \mathcal{P}(\mathbb{R}) \text{ we have } \lambda^*(B) = \lambda^*\left(B \bigcap A\right) + \lambda^*(B \setminus A)\right\}$$

is a σ-algebra on $\mathbb{R}$ such that

$$\sigma[\mathcal{S}_{[[}] \subseteq \mathcal{L}[\mathbb{R}]$$

and

$$\overline{\lambda} = (\lambda^*)_{|\mathcal{L}(\mathbb{R})}: \mathcal{L}[\mathbb{R}] \rightarrow [0, \infty] \text{ is a measure on } \mathcal{L}[\mathbb{R}]$$

and

$$\lambda = (\lambda^*)_{|\sigma[\mathcal{S}_{[[}]}: \sigma[\mathcal{S}_{[[}] \rightarrow [0, \infty] \text{ is a measure on } \sigma[\mathcal{S}_{[[}]$$

and

$$(\lambda^*)_{|\mathcal{S}_{[[}} = \text{len}$$

Further if $\mu: \sigma[\mathcal{S}_{[[}] \rightarrow [0, \infty]$ is another measure such that $\mu_{|S_{[[}} = \text{len}$ then

$$\mu = \lambda$$

We have then

$$\langle \mathbb{R}, \mathcal{L}[\mathbb{R}], \overline{\lambda} \rangle \text{ is a measure space called the Lebesgue measure space}$$

where

$$\mathcal{L}[\mathbb{R}] \text{ is called the set of Lebesgue measurable sets}$$

and

$$\overline{\lambda} \text{ the Lebesgue measure}$$

***Note* 20.262.** *As by [theorem: 20.250] $\mathcal{B}[\mathbb{R}] = \sigma[\mathcal{S}_{[[}]$ we have also that*

$$\mathcal{B}[\mathbb{R}] \subseteq \mathcal{L}[\mathbb{R}]$$

and

$$\lambda: \mathcal{B}[\mathbb{R}] \rightarrow [0, \infty] \text{ is a measure on } \mathcal{B}[\mathbb{R}]$$

So

$$\langle \mathbb{R}, \mathcal{B}[\mathbb{R}], \lambda \rangle \text{ is a measure space called the Borel measures space}$$

where

$$\mathcal{B}[\mathbb{R}] \text{ is called the set of Borel sets on } \mathbb{R}$$

and

$$\lambda \text{ is called the Borel measure}$$

Proof. Using [lemma: 20.260] we have that

$$\text{len is a pre-measure on the semi-ring } \mathcal{S}_{[[}$$

Define $\{[-i, i[\}_{i \in \mathbb{N}} \subseteq \mathcal{S}_{[[}$ then we have

$$\forall i \in \mathbb{N} \text{ that } [-i, i[\subseteq [-(i+1), i+1[\text{ and } \text{len}([-i, i[) = 2 \cdot i < \infty$$

If $x \in \mathbb{R}$ then by [theorem: 10.30] there exist $n, m \in \mathbb{N}$ such that $-n < x < m$. Let $i = \max(n, m)$ then $-i \leqslant -n < x < m < i$ so that $x \in [-i, i[$ from which it follows that $\mathbb{R} \subseteq \bigcup_{i \in \mathbb{N}} [-i, i[$. As $\forall i \in \mathbb{N}$ $[-i, i[\subseteq \mathbb{R}$ we have $\bigcup_{i \in \mathbb{N}} [-i, i[\subseteq \mathbb{R}$. Hence we have

$$\mathbb{R} = \bigcup_{i \in \mathbb{N}} [-i, i[$$

The proof follows then from applying [theorem: 20.225]. □

20.4.4.4 Product of semi-rings

If we want to use the Carathéodory theorem to create a measure on $\mathcal{B}[\mathbb{R}^n]$ we have to start with a semi-ring, as $\mathcal{B}[\mathbb{R}^n] = \sigma[\mathcal{R}^n]$ it seems sensible to prove that $\mathcal{R}^n \bigcup \{\varnothing\}$ is a semi-ring on $\mathbb{R}^n$. First note that $R \in \mathcal{R}^n \bigcup \{\varnothing\}$ can be written as $R = \prod_{i \in \{1, \ldots, n\}} I_i$ where $I_i \in \mathcal{T}_{[[}$ which suggest the following definition that extends this note to the more general cases of a product of a finite number of semi-rings.

Definition 20.263. *Let* $n \in \{2, \ldots, \infty\}$, $\{\langle X_i, \mathcal{S}_i \rangle\}_{i \in \{1, \ldots, n\}}$ *a finite family such that* $\forall i \in \{1, \ldots, n\}$ $\mathcal{S}_i \subseteq \mathcal{P}(X_i)$ *then* $\bigodot_{i \in \{1, \ldots, n\}} \mathcal{S}_i$ *is defined by*

$$\bigodot_{i \in \{1, \ldots, n\}} \mathcal{S}_i = \left\{ \prod_{i \in \{1, \ldots, n\}} A_i | \forall i \in \{1, \ldots, n\} \ A_i \in \mathcal{S}_i \right\}$$

***Note* 20.264.** *In the particular cases where* $n = 2$ *we note* $\bigodot_{i \in \{1,2\}} S_i$ *as* $\mathcal{S}_1 \odot \mathcal{S}_2$. *In other words*

$$S_1 \odot \mathcal{S}_2 = \{A \cdot B | A \in \mathcal{S}_1 \wedge B \in \mathcal{S}_2\}$$

Next we will prove using mathematical induction that $\bigodot_{i \in \{1, \ldots, n\}} \mathcal{S}_i$ is a semi-ring on $\prod_{i \in \{1, \ldots, \}} X_i$. The next lemma will be used in the base case and the induction step.

Lemma 20.265. *Let* X_1, X_2 *be two sets and* $\mathcal{S}_1 \subseteq \mathcal{P}(X_1)$, $\mathcal{S}_2 \subseteq \mathcal{P}(X_2)$ *be semi-rings on* X_1 *and* X_2 *respectively then* $\mathcal{S}_1 \odot \mathcal{S}_2$ *is a semi-ring on* $X_1 \cdot X_2$.

Proof. As $\mathcal{S}_1$, $\mathcal{S}_2$ are semi-rings we have that $\varnothing \in S_1 \wedge \varnothing \in S_2$ so that

$$\varnothing = \varnothing \cdot \varnothing \in \mathcal{S}_1 \odot \mathcal{S}_2 \tag{20.326}$$

Let $R, S \in \mathcal{S}_1 \odot \mathcal{S}_2$ then there exists $A, C \in \mathcal{S}_1$ and $B, D \in \mathcal{S}_2$ such that

$$R = A \cdot B \text{ and } S = C \cdot D$$

Then we have

$$\begin{array}{rcl} R \bigcap S & = & (A \cdot B) \bigcap (C \cdot D) \\ & \underset{[\text{theorem: } 6.84]}{=} & \left(A \bigcap C\right) \cdot \left(B \bigcap D\right) \end{array}$$

given that $\mathcal{S}_1, \mathcal{S}_2$ are semi-rings it follows that $A \bigcap C \in \mathcal{S}_1 \wedge B \bigcap D \in \mathcal{S}_2$ so that

$$R \bigcap S \in \mathcal{S}_1 \odot \mathcal{S}_2 \tag{20.327}$$

Using [theorem: 20.104] we have that

$$\begin{array}{rcl} R \setminus S & = & (A \cdot B) \setminus (C \cdot D) \\ & = & (A \setminus C) \cdot (B \setminus D) \bigcup (A \setminus C) \cdot \left(B \bigcap D\right) \bigcup \left(A \bigcap C\right) \cdot (B \setminus D) \end{array}$$

where

1. $((A \setminus C) \cdot (B \setminus D)) \bigcap ((A \setminus C) \cdot (B \bigcap D)) = \varnothing$
2. $((A \setminus C) \cdot (B \setminus D)) \bigcap ((A \bigcap C) \cdot (B \setminus D)) = \varnothing$
3. $((A \setminus C) \cdot (B \bigcap D)) \bigcap ((A \bigcap C) \cdot (B \setminus D)) = \varnothing$

So if we define $\{R_i\}_{i \in \{1,2,3\}}$ by

$$\begin{array}{rcl} R_1 & = & (A \setminus C) \cdot (B \setminus D) \\ R_2 & = & (A \setminus C) \cdot \left(B \bigcap D\right) \\ R_3 & = & \left(A \bigcap C\right) \cdot (B \setminus D) \end{array}$$

then we have

$$\{R_i\}_{i \in \{1,2,3\}} \text{ is a pairwise disjoint and } R \setminus S = \bigsqcup_{i \in \{1,2.3\}} R_i \tag{20.328}$$

Now for $i \in \{1, 2, 3\}$ we have either:

$\boldsymbol{i = 1}$**.** As $A, C \in \mathcal{S}_1$ a semi ring there exist a pairwise disjoint $\{A_i\}_{i \in \{1, \ldots, m_1\}} \subseteq \mathcal{S}_1$ such that

$$A \setminus C = \bigsqcup_{i \in \{1, \ldots, m_1\}} A_i \tag{20.329}$$

likewise as $B, D \in \mathcal{S}_1$ there exist a pairwise disjoint $\{B_i\}_{i \in \{1, \ldots, m_2\}} \subseteq \mathcal{S}_2$ such that

$$B \setminus D = \bigsqcup_{i \in \{1, \ldots, m_2\}} B_i \tag{20.330}$$

so that we have

$$\begin{array}{rcl} R_1 & = & (A\setminus C)\cdot(B\setminus D) \\ & = & \left(\bigsqcup_{i\in\{1,\ldots,m_1\}} A_i\right)\cdot\left(\bigsqcup_{j\in\{1,\ldots,m_2\}} B_j\right) \\ & \underset{\text{[theorem: 6.85]}}{=} & \bigcup_{i\in\{1,\ldots,m_1\}\times\{1,\ldots,m_2\}} (A_i\cdot B_j) \end{array}$$

By [theorem: 6.41] $\{1,\ldots,m_1\}\times\{1,\ldots,m_2\}$ is finite so there exist a bijection

$$\sigma:\{1,\ldots,N_1\}\rightarrow\{1,\ldots,m_1\}\times\{1,\ldots,m_2\}$$

then we have

$$R_1 = \bigcup_{(i,j)\in\{1,\ldots,m_1\}\times\{1,\ldots,m_2\}} (A_i\cdot B_j) \underset{\text{[theorem: 2.119]}}{=} \bigcup_{i\in\{1,\ldots,N_1\}} (A_{\sigma(i)_1}\cdot B_{\sigma(i)_2})$$

Define

$$\{R_{1,i}\}_{i\in\{1,\ldots,N_1\}} \text{ by } R_{1,i} = A_{\sigma(i)_1}\cdot B_{\sigma(i)_2}$$

then we have $\forall i\in\{1,\ldots,N_1\}$ that, as $A_{\sigma(i)_1}\in\mathcal{S}_1$, $B_{\sigma(i)_2}\in\mathcal{S}_2$ so that $R_{1,i} = A_{\sigma(i)_1}\cdot B_{\sigma(i)_2}\in S_1\odot\mathcal{S}_2$. Hence we have

$$\{R_{1,i}\}_{i\in\{1,\ldots,N_1\}}\subseteq\mathcal{S}_1\odot\mathcal{S}_2 \text{ and } R_1 = \bigcup_{i\in\{1,\ldots,N_1\}} R_{1,i}$$

Further if $i,j\in\{1,\ldots,N_1\}$ with $i\neq j$ we have, as σ is a bijection, that $\sigma(i)\neq\sigma(j)$, for $\sigma(i)_1$, $\sigma(j)_1$ we have either

$\boldsymbol{\sigma(i)_1=\sigma(j)_1}$. Then $\sigma(i)_2\neq\sigma(j)_2$ so that

$$\begin{array}{rcl} R_{1,i}\bigcap R_{1,j} & = & (A_{\sigma(i)_1}\cdot B_{\sigma(i)_2})\bigcap(A_{\sigma(j)_1}\cdot B_{\sigma(j)_2}) \\ & = & \left(A_{\sigma(i)_1}\bigcap A_{\sigma(j)_1}\right)\cdot\left(B_{\sigma(i)_2}\bigcap B_{\sigma(j)_2}\right) \\ & = & \left(A_{\sigma(i)_1}\bigcap A_{\sigma(j)_1}\right)\cdot\varnothing \\ & = & \varnothing \end{array}$$

$\boldsymbol{\sigma(i)_1\neq\sigma(j)_1}$. Then

$$\begin{array}{rcl} R_{1,i}\bigcap R_{1,j} & = & (A_{\sigma(i)_1}\cdot B_{\sigma(i)_2})\bigcap(A_{\sigma(j)_1}\cdot B_{\sigma(j)_2}) \\ & = & \left(A_{\sigma(i)_1}\bigcap A_{\sigma(j)_1}\right)\cdot\left(B_{\sigma(i)_2}\bigcap B_{\sigma(j)_2}\right) \\ & = & \varnothing\cdot\left(B_{\sigma(i)_2}\bigcap B_{\sigma(j)_2}\right) \\ & = & \varnothing \end{array}$$

which proves that

$$\{R_{1,i}\}_{i\in\{1,\ldots,N_1\}}\subseteq\mathcal{S}_1\odot\mathcal{S}_2 \text{ is pairwise disjoint and } R_1 = \bigsqcup_{i\in\{1,\ldots,N_1\}} R_{1,i}$$

$\boldsymbol{i=2}$. Then

$$\begin{array}{rcl} R_2 & = & (A\setminus C)\cdot\left(B\bigcap D\right) \\ & \underset{\text{[eq: 20.329]}}{=} & \left(\bigsqcup_{i\in\{1,\ldots,m_1\}} A_i\right)\cdot\left(B\bigcap D\right) \\ & \underset{\text{[theorem: 6.85]}}{=} & \bigcup_{i\in\{1,\ldots,m_1\}} \left(A_i\cdot\left(B\bigcap D\right)\right) \\ & = & \bigcup_{i\in\{1,\ldots,N_2\}} R_{2,i} \end{array}$$

where

$$N_2 = m_1 \text{ and } \{R_{2,i}\}_{i\in\{1,\ldots,N_2\}} \text{ is defined by } R_{2,i} = A_i\cdot\left(B\bigcap D\right)$$

Let $i \in \{1, \ldots, N_2\}$ then, as $A_i \in \mathcal{S}_1$, $B, D \in \mathcal{S}_2 \Rightarrow B \bigcap D \in \mathcal{S}_2$, $R_{2,i} = A_i \cdot (B \bigcap D) \in \mathcal{S}_1 \odot \mathcal{S}_2$ so that

$$\{R_{2,i}\}_{i \in \{1, \ldots, N_2\}} \subseteq S_1 \odot S_2$$

Further if $i, j \in \{1, \ldots, N_2\}$ with $i \neq j$ then we have

$$R_{2,i} \bigcap R_{2,j} = (A_i \cdot (B \bigcap D)) \bigcap (A_j \cdot (B \bigcap D)) = (A_i \bigcap A_j) \cdot (B \bigcap D) = \varnothing \cdot (B \bigcap D) = \varnothing$$

So we have proved that

$$\{R_{2,i}\}_{i \in \{1, \ldots, N_2\}} \subseteq \mathcal{S}_1 \odot \mathcal{S}_2 \text{ is pairwise disjoint and } R_2 = \bigsqcup_{i \in \{1, \ldots, N_2\}} R_{2,i}$$

$\boldsymbol{i = 3.}$ Then

$$\begin{array}{rcl} R_3 & = & (A \bigcap C) \cdot (B \setminus D) \\ & \underset{[\text{eq: } 20.330]}{=} & (A \bigcap C) \cdot \left(\bigsqcup_{i \in \{1, \ldots, m_2\}} B_i \right) \\ & \underset{[\text{theorem: } 6.85]}{=} & \bigcup_{i \in \{1, \ldots, m_2\}} ((A \bigcap C) \cdot B_i) \\ & = & \bigcup_{i \in \{1, \ldots, N_3\}} D_{3,i} \end{array}$$

where

$$N_3 = m_2 \text{ and } \{R_{3,i}\}_{i \in \{1, \ldots, N_3\}} \text{ is defined by } R_{3,i} = (A \bigcap C) \cdot B_i$$

Let $i \in \{1, \ldots, N_3\}$ then, as $B_i \in \mathcal{S}_2$, $A, C \in \mathcal{S}_1 \Rightarrow A \bigcap C \in \mathcal{S}_1$, $R_{3,i} = ((A \bigcap C) \cdot B_i) \in \mathcal{S}_1 \odot \mathcal{S}_2$ so that

$$\{R_{3,i}\}_{i \in \{1, \ldots, N_2\}} \subseteq S_1 \odot S_2$$

Further if $i, j \in \{1, \ldots, N_2\}$ with $i \neq j$ then we have

$$R_{3,i} \bigcap R_{3,j} = ((A \bigcap C) \cdot B_i) \bigcap ((A \bigcap C) \cdot B_j) = (A \bigcap C) \cdot (B_i \bigcap B_j) = (A \bigcap C) \cdot \varnothing = \varnothing$$

So we have proved that

$$\{R_{3,i}\}_{i \in \{1, \ldots, N_3\}} \subseteq \mathcal{S}_1 \odot \mathcal{S}_2 \text{ is pairwise disjoint and } R_3 = \bigsqcup_{i \in \{1, \ldots, N_3\}} R_{3,i}$$

To summarize the above, we have a pairwise disjoint $\{R_i\}_{i \in \{1,2,3\}}$ such that

$$R \setminus S \underset{[\text{eq: } 20.328[}{=} \bigsqcup_{i \in \{1,2,3\}} R_i$$

and $\forall i \in \{1, 2, 3\}$ there exist a pairwised disjoint $\{R_{i,j}\}_{j \in \{1, \ldots, N_i\}} \subseteq \mathcal{S}_1 \odot S_2$ such

$$R_i = \bigsqcup_{j \in \{1, \ldots, N_i\}} R_{i,j}$$

Further as $\bigcup_{i \in \{1,2,3\}} \{i\} \cdot \{1, \ldots, N_i\}$ is finite by [theorems: 6.41, 6.36] there exist a bijection

$$\sigma: \{1, \ldots, N\} \to \bigcup_{i \in \{1,2,3\}} \{i\} \cdot \{1, \ldots, N_i\}$$

Hence we can apply]corollary: 20.114] to get that

$$\{R_{\sigma(i)_1, \sigma(i)_2}\}_{i \in \{1, \ldots, N\}} \subseteq \mathcal{S}_1 \odot \mathcal{S}_2 \text{ is pairwise disjoint and } R \setminus S = \bigsqcup_{i \in \{1, \ldots, N\}} R_{\sigma(i)_1, \sigma(i)_2} \tag{20.331}$$

Finally from [eqs: 20.326, 20.327 and the above eq: 20.331] it follows that

$$\mathcal{S}_1 \odot \mathcal{S}_2 \text{ is a semi-ring on } X_1 \cdot X_2$$

□

If we want to use the above lemma in the proof by mathematical induction that $\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i$ is a semi-ring we have a small problem: $\prod_{i\in\{1,\ldots,n+1\}} X_i$ is not equal to $\left(\prod_{i\in\{1,\ldots,n\}} X_i\right)\cdot X_{n+1}$. Luckely we can prove that $\left(\prod_{i\in\{1,\ldots,n\}} X_i\right)\cdot X_{n+1}$ is bijective to $\prod_{i\in\{1,\ldots,n+1\}} X_i$. So it makes sense to see how a bijection can be used to generate a semi-ring on the image of the bijection based on a semi-ring on the domain of the bijection.

Theorem 20.266. *Let X, Y be sets, $\beta: X\rightarrow Y$ a bijection and $\mathcal{S}\subseteq X$ a semi-ring on X then $\{\beta(A)|A\in\mathcal{S}\}$ is a semi-ring on Y.*

Proof. As $\beta(\varnothing)=\varnothing$ it follows that

$$\varnothing\in\{\beta(A)|A\in\mathcal{S}\} \tag{20.332}$$

Let $R, S\in\{\beta(A)|A\in\mathcal{S}\}$ there exist $A, B\in\mathcal{S}$ such that $R=\beta(A)$, $S=\beta(B)$. Using [theorem: 2.95(4)] and the fact that β is a bijection hence injective we have that

$$R\bigcap S=\beta(A)\bigcap\beta(B)\underset{\text{[theorem: 2.95(4)]}}{=}\beta\left(A\bigcap B\right)$$

As $\mathcal{S}$ is a semi-ring $A\bigcap B\in\mathcal{S}$ so that $\beta(A\bigcap B)\in\{\beta(A)|A\in\mathcal{S}\}$ which combined with the above proves that

$$\forall R, S\in\{\beta(A)|A\in\mathcal{S}\}\text{ we have }R\bigcap S\in\{\beta(A)|A\in\mathcal{S}\} \tag{20.333}$$

Using [theorem: 2.94(5)] and the fact that β is injective it follows that

$$R\setminus S=\beta(A)\setminus\beta(B)\underset{\text{[theorem: 2.94(5)]}}{=}\beta(A\setminus B) \tag{20.334}$$

As $A, B\in\mathcal{S}$ and $\mathcal{S}$ is a semi-ring there exist a pairwise disjoint $\{A_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}$, $n\in\mathbb{N}$ such that

$$A\setminus B=\bigsqcup_{i\in\{1,\ldots,n\}} A_i \tag{20.335}$$

Let $i, j\in\{1,\ldots,n\}$ with $i\neq j$ then $\beta(A_i)\bigcap\beta(A_j)\underset{\text{[theorem: 2.95(4)]}}{=}\beta(A_i\bigcap A_j)=\beta(\varnothing)=\varnothing$ so that

$$\beta(A_i)_{i\in\{1,\ldots,n\}}\subseteq\{\beta(A)|A\in\mathcal{S}\}\text{ is pairwise disjoint} \tag{20.336}$$

Further

$$\bigsqcup_{i\in\{1,\ldots,n\}}\beta(A_i)\underset{\text{[theorem: 2.136]}}{=}\beta\left(\bigsqcup_{i\in\{1,\ldots,n\}} A_i\right)\underset{\text{[eq: 20.335]}}{=}\beta(A\setminus B)\underset{\text{[eq: 20.334]}}{=}R\setminus S \tag{20.337}$$

Define $\{R_i\}_{i\in\{1,\ldots,n\}}\subseteq\{\beta(A)|A\in\mathcal{S}\}$ by $R_i=\beta(A_i)$ then [eqs: 20.336, 20.337] can be summarized as

$$\forall R, S\in\{\beta(A)|A\in\mathcal{S}\}\ \exists\{R_i\}_{i\in\{1,\ldots,n\}}\subseteq\{\beta(A)|A\in\mathcal{S}\}\text{ such that }R\setminus S=\bigsqcup_{i\in\{1,\ldots,n\}} R_i \tag{20.338}$$

By [eqs: 20.332, 20.333 and 20.338] it follows that

$$\{\beta(A)|A\in\mathcal{S}\}\text{ is a semi-ring on }Y$$

□

Next we construct a bijection between $\left(\prod_{i\in\{1,\ldots,n\}} X_i\right)\cdot X_{n+1}$ and $\prod_{i\in\{1,\ldots,n+1\}} X_i$.

Lemma 20.267. *Let $n\in\mathbb{N}$, $\{\langle X_i,\mathcal{S}_i\rangle\}_{i\in\{1,\ldots,n+1\}}$ a finite family such that $\forall i\in\{1,\ldots,n\}$ $\mathcal{S}_i\subseteq\mathcal{P}(X_i)$ then*

1. $$\beta:\left(\prod_{i\in\{1,\ldots,n\}} X_i\right)\cdot X_{n+1}\rightarrow\prod_{i\in\{1,\ldots,n+1\}} X_i\text{ defined by }\beta(x,y)_i=\begin{cases} x_i \text{ if } i\in\{1,\ldots,n\}\\ y \text{ if } i=n+1\end{cases}$$

   is a bijection

2. $\bigodot_{i\in\{1,\ldots,n+1\}}\mathcal{S}_i=\left\{\beta(B)|B\in\left(\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i\right)\odot S_{n+1}\right\}$

Note 20.268. *Be carefull in the notation used in (2) $\beta(B)$ is the image of the set B in the domain $\left(\prod_{i\in\{1,\ldots,n\}} X_i\right)\cdot X_{n+1}$ of the bijection β. So $\bigodot_{i\in\{1,\ldots,n+1\}}\mathcal{S}_i$ is not equal to $\beta\left(\left(\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i\right)\odot S_{n+1}\right)$ which is by the way not well defined.*

Proof.

1. We have:

   injectivity. Assume that $\beta(x,y)=\beta(u,v)$ then $\forall i\in\{1,\ldots,n\}$ we have

   $$x_i=\beta(x,y)_i=\beta(u,v)_i=u_i$$

   so that $x=u$ and $y=\beta(x,y)_{n+1}=\beta(u,v)_{n+1}=v$. So $(x,y)=(u,v)$ which proves injectivity.

   surjectivity. Let $u\in\prod_{i\in[1,\ldots,n+1]} X_i$ and define $x\in\prod_{i\in\{1,\ldots,n\}} X_i$ by $x_i=u_i$ and $y\in X_{n+1}$ by $y=u_{n+1}$ then $\forall i\in\{1,\ldots,n+1\}$ we have

   $$\beta(x,y)_i=\begin{cases} x_i=u_i \text{ if } i\in\{1,\ldots,n\} \\ u_{n+1} \text{ if } i=n+1\end{cases}=u_i$$

   proving that $\beta(x,y)=u$

2. Let $\{A_i\}_{i\in\{1,\ldots,n+1\}}$ be a family of sets such that $\forall i\in\{1,\ldots,n+1\}$ $A_i\in S_i\subseteq\mathcal{P}(X_i)$. Let $u\in\beta\left(\left(\prod_{i\in\{1,\ldots,n\}} A_i\right)\cdot A_{n+1}\right)$ then $u=\beta(x,y)$ where $x\in\prod_{i\in\{1,\ldots,n\}} A_i\Rightarrow\forall i\in\{1,\ldots,n\}$ $x_i\in A_i$ and $y\in A_{n+1}$. So

   $$\forall i\in\{1,\ldots,n+1\}\ \beta(x,y)_i=\begin{cases} x_i\in A_i \text{ if } i\in\{1,\ldots,n\} \\ y\in A_i \text{ if } i=n+1\end{cases}\in A$$

   so that $u\in\prod_{i\in\{1,\ldots,n+1\}} A_i$, Hence

   $$\beta\left(\left(\prod_{i\in\{1,\ldots,n\}} A_i\right)\cdot A_{n+1}\right)\subseteq\prod_{i\in\{1,\ldots,n+1\}} A_i \tag{20.339}$$

   If $u\in\prod_{i\in\{1,\ldots,n+1\}} A_i$ then $\forall i\in\{1,\ldots,n+1\}$ we have $u_i\in A_i$. Define $x\in\prod_{i\in\{1,\ldots,n\}} X_i$ by $x_i=u_i\in A_i$ and $y=u_{n+1}\in A_{n_1}$ then $(x,y)\in\left(\prod_{i\in\{1,\ldots,n\}} A_i\right)\cdot A_{n+1}$. further

   $$\beta(x,y)=\begin{cases} x_i=u_i \text{ if } i\in\{1,\ldots,n\} \\ y=u_{n+1} \text{ if } i=n+1\end{cases}=u_i$$

   proving that $u=\beta(x,y)\in\beta\left(\left(\prod_{i\in\{1,\ldots,n\}} A_i\right)\cdot A_{n+1}\right)$. Hence

   $$\prod_{i\in\{1,\ldots,n+1\}} A_i\subseteq\beta\left(\left(\prod_{i\in\{1,\ldots,n\}} A_i\right)\cdot A_{n+1}\right)$$

   which combined with [eq: 20.339] proves that

   $$\prod_{i\in\{1,\ldots,n+1\}} A_i=\beta\left(\left(\prod_{i\in\{1,\ldots,n\}} A_i\right)\cdot A_{n+1}\right) \tag{20.340}$$

   Let now $A\in\bigodot_{i\in\{1,\ldots,n+1\}} S_i$ then $A=\prod_{i\in\{1,\ldots,n+1\}} A$ where $\forall i\in\{1,\ldots,n+1\}$ $A_i\in\mathcal{S}_i$. Then $\prod_{i\in\{1,\ldots,n\}} A_i\in\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i$ and $A_{n+1}\in S_{n+1}$ so that

   $$\left(\prod_{i\in\{1,\ldots,n\}} A_i\right)\cdot A_{n+1}\in\left(\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i\right)\odot\mathcal{S}_{n+1}$$

   Hence as

   $$A=\prod_{i\in\{1,\ldots,n+1\}} A_i \underset{\text{[eq: 20.340]}}{=} \beta\left(\left(\left(\prod_{i\in\{1,\ldots,n\}} A_i\right)\cdot A_{n+1}\right)\right)$$

it follows that $A\in\left\{\beta(B)|B\in\left(\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i\right)\odot S_{n+1}\right\}$ which proves that

$$\bigodot_{i\in\{1,\ldots,n+1\}} S_i\subseteq\left\{\beta(B)|B\in\left(\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i\right)\odot S_{n+1}\right\} \tag{20.341}$$

On the other hand if $B\in\left\{\beta(B)|B\in\left(\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i\right)\odot S_{n+1}\right\}$ then there exists a $A\in\left(\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i\right)\odot S_{n+1}$ such that $B=\beta(A)$. As $A\in\left(\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i\right)\odot S$ there exist a $\{A_i\}_{i\in\{1,\ldots,n\}}$ with $\forall i\in\{1,\ldots,n\}$ $A_i\in\mathcal{S}_i$ and a $A_{n+1}\in\mathcal{S}_{n+1}$ such that

$$B=\beta(A)=\beta\left(\left(\prod_{i\in\{1,\ldots,n\}} A_i\right)\cdot A_{n+1}\right)$$

Using [eq: 20.340] on the above proves that that

$$B=\prod_{i\in\{1,\ldots,n+1\}} A_i\in\bigodot_{i\in\{1,\ldots,n+1\}}\mathcal{S}_i$$

hence $\left\{\beta(B)|B\in\left(\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i\right)\odot S_{n+1}\right\}\subseteq\bigodot_{i\in\{1,\ldots,n+1\}}\mathcal{S}_i$ which combined with [eq: 20.341] proves that

$$\bigodot_{i\in\{1,\ldots,n+1\}} S_i=\left\{\beta(B)|B\in\left(\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i\right)\odot S_{n+1}\right\}$$

□

We are now ready to use mathematical induction to prove that $\bigodot_{i\in\{1,\ldots,n\}} S_i$ is a semi-ring on $\prod_{i\in\{1,\ldots,n\}} X_i$.

Theorem 20.269. *Let $n\in\mathbb{N}$, $\{\langle X_i,\mathcal{S}_i\rangle\}_{i\in\{1,\ldots,n\}}$ a finite family such that $\forall i\in\{1,\ldots,n\}$ $\mathcal{S}_i$ is a semi-ring on X_i then for $\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i$ defined by*

$$\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i=\left\{\prod_{i\in\{1,\ldots,n\}} A_i|\forall i\in\{1,\ldots,n\}\ A_i\in\mathcal{S}_i\right\}$$

we have that

$$\bigodot_{i\in\{1,\ldots,n\}} S_i \text{ is a semi-ring on } \prod_{i\in\{1,\ldots,n\}} X_i$$

Proof. We prove this by mathematical induction, so define

$$\mathcal{S}=\left\{n\in\mathbb{N}|\text{If } \{\langle X_i,\mathcal{S}_i\rangle\}_{i\in\{1,\ldots,n\}} \text{ is a family such that } \forall i\in\{1,\ldots,n\}\ \mathcal{S}_i \text{ is a semi-ring on } X_i \text{ then } \bigodot_{i\in\{1,\ldots,n\}} S_i \text{ is a semi-ring on } \prod_{i\in\{1,\ldots,n\}} X_i\right\}$$

then we have:

$1\in\mathcal{S}$. Let $\{\langle X_i,\mathcal{S}_i\rangle\}_{i\in\{1\}}$ be such that $\mathcal{S}_1$ is a semi-ring on X_1. Define

$$\beta: X_1\rightarrow\prod_{i\in\{1\}} X_i \text{ where } \beta(x) \text{ is defined by } \beta(x)_1=x \text{ [in other words } \beta(x)=(x)$$

then we have:

injectivity. If $\beta(x)=\beta(y)$ then $x=\beta(x)_1=\beta(y)_1=y$.

surjectivity. If $y\in\prod_{i\in\{1\}} X_i$ then for $x=y_1$ we have $\forall i\in\{1\}$ that $\beta(x)_1=x=y_1$ so that $\beta(x)=y$.

so that

$$\beta \text{ is a bijection}$$

Let $A \in \bigodot_{i \in \{1\}} S_i = \left\{\prod_{i \in \{1\}} A_i | \forall i \in \{1\}\ A_i \in S_i\right\}$ then $A = \prod_{i \in \{1\}} A_i$ where $A_1 \in \mathcal{S}_1$. If $x \in A = \prod_{i \in \{1\}} A_i$ then $x_1 \in A_i$ so that $x = \beta(x_1) \in \beta(A_1)$ and if $x \in \beta(A_1)$ then there exist a $a \in A_1$ such that $x = \beta(a) = (a)$ hence $x_1 = \beta(a)_1 = a \in A_i$ so that $x \in \prod_{i \in \{1\}} A_i = A$, proving that $A = \beta(A_1) \in \{\beta(B) | B \in \mathcal{S}_1\}$. Hence we have

$$\bigodot_{i \in \{1\}} S_i \subseteq \{\beta(B) | B \in \mathcal{S}_1\} \tag{20.342}$$

For the opposite inclusion, let $A \in \{\beta(B) | B \in \mathcal{S}_1\}$ then there exist a $A_1 \in S_1$ such that $A = \beta(A_1)$. If $x \in A = \beta(A_1)$ then there exist a $a \in A_1$ such that $x = \beta(a)$ so that $x_1 = \beta(a)_1 = a \in A_1$ hence $x \in \prod_{i \in \{1\}} A_i$ and if $x \in \prod_{i \in \{1\}} A_i$ then $x_1 \in A_1$ so that $x = \beta(x_1) \in \beta(A_i)$, proving that $A = \prod_{i \in \{1\}} A_i \in \bigodot_{i \in \{1\}} S_i$. Hence we have that $\{\beta(B) | B \in \mathcal{S}_1\} \subseteq \bigodot_{i \in \{1\}} S_i$ which combined with [eq: 20.342] proves that

$$\bigodot_{i \in \{1\}} S_i = \{\beta(B) | B \in \mathcal{S}_1\}$$

Finally as $\mathcal{S}_1$ is a semi-ring it follows from [lemma: 20.266] that $\bigodot_{i \in \{1\}} \mathcal{S}_i$ is a semi-ring which proves that $1 \in S$.

$2 \in \mathcal{S}$. If $\{\langle X_i, \mathcal{S}_i \rangle\}_{i \in \{1,2\}}$ is such that $\mathcal{S}_1$ is a semi-ring on X_1 and $\mathcal{S}_2$ is a semi-ring on X_2 then by [theorem: 20.263] $\bigodot_{i \in \{1,2\}} \mathcal{S}_i = \mathcal{S}_1 \odot \mathcal{S}_2$ is a semi-ring on $X_1 \cdot X_2 = \prod_{i \in \{1,2\}} X_i$. Hence $2 \in \mathcal{S}$.

$n \in \mathcal{S} \Rightarrow n+1 \in \mathcal{S}$. Let $\{\langle X_i, \mathcal{S}_i \rangle\}_{i \in \{1,\ldots,n+1\}}$ be a finite family such that $\forall i \in \{1,\ldots,n+1\}$ we have that $\mathcal{S}_i$ is a semi-ring on X_i. As $n \in \mathcal{S}$ it follows that $\bigodot_{i \in \{1,\ldots,n\}} \mathcal{S}_i$ is a semi-ring on $\prod_{i \in \{1,\ldots,n\}} X_i$. As also $\mathcal{S}_{n+1}$ is a semi-ring on X_{n+1} it follows from [theorem: 20.263] that

$$\left(\bigodot_{i \in \{1,\ldots,n\}} \mathcal{S}_i\right) \odot S_{n+1} \text{ is a semi-ring on } \left(\prod_{i \in \{1,\ldots,n\}} X_i\right) \cdot X_{n+1}$$

Using the previous lemma [lemma: 20.267] there exist a bijection

$$\beta: \left(\prod_{i \in \{1,\ldots,n\}} X_i\right) \cdot X_{n+1} \to \prod_{i \in \{1,\ldots,n+1\}} X_i$$

such that

$$\bigodot_{i \in \{1,\ldots,n+1\}} \mathcal{S}_i = \left\{\beta(B) | B \in \left(\bigodot_{i \in \{1,\ldots,n\}} \mathcal{S}_i\right) \odot S_{n+1}\right\}$$

By [theorem: 20.266] it follows then that $\left\{\beta(B) | B \in \left(\bigodot_{i \in \{1,\ldots,n\}} \mathcal{S}_i\right) \odot S_{n+1}\right\}$ is a semi-ring on $\prod_{i \in \{1,\ldots,n+1\}} X_i$. Hence $\bigodot_{i \in \{1,\ldots,n+1\}} \mathcal{S}_i$ is a semi-ring on $\prod_{i \in \{1,\ldots,n+1\}} X_i$ proving that $n+1 \in \mathcal{S}$

□

Corollary 20.270. *Let $n \in \mathbb{N}$, X a set and $\mathcal{S}$ a semi-ring on X then*

$$\mathcal{S}^n = \left\{\prod_{i \in \{1,\ldots,n\}} A_i | \forall i \in \{1,\ldots,n\}\ A_i \in \mathcal{S}\right\} = \left\{\prod_{i \in \{1,\ldots,m\}} A_i | \{A_i\}_{i \in \{1,\ldots,n\}} \subseteq \mathcal{S}\right\}$$

is a semi-ring on X^n

Proof. Define $\{\langle X_i, \mathcal{S}_i \rangle\}_{i \in \{1,\ldots,n\}}$ by $X_i = X$ and $\mathcal{S}_i = \mathcal{S}$. Then by the previous theorem [theorem: 20.269] we have that $\bigodot_{i \in \{1,\ldots,n\}} \mathcal{S}_i$ is a semi-ring on $\prod_{i \in \{1,\ldots,n\}} X_i$. Now by definition we have that

$$X^n \underset{\text{definition}}{=} \prod_{i \in \{1,\ldots,n\}} X_i$$

and

$$\begin{aligned}\bigodot_{i\in\{1,\ldots,n\}}\mathcal{S}_i &= \left\{\prod_{i\in\{1,\ldots,n\}} A_i|\forall i\in\{1,\ldots,n\}\ A_i\in\mathcal{S}_i\right\}\\ &\underset{\mathcal{S}_i=\mathcal{S}}{=} \left\{\prod_{i\in\{1,\ldots,n\}} A_i|\forall i\in\{1,\ldots,n\}\ A_i\in\mathcal{S}\right\}\\ &= \mathcal{S}^n\end{aligned}$$

so that $\mathcal{S}^n$ is a semi-ring on X^n. □

20.4.4.5 Semi-ring on $\mathbb{R}^n$

We are now ready to prove that $\mathcal{R}^n\bigcup\{\varnothing\}$ is a semi-ring.

Theorem 20.271. *Let $n\in\mathbb{N}$ then we have*

$$\mathcal{R}^n=\left\{\prod_{i\in\{1,\ldots,n\}} I_i|\forall i\in\{1,\ldots,n\}\ I_i\in\mathcal{I}_{[[}\right\}\underset{\text{def}}{=}\left\{\prod_{i\in\{1,\ldots,n\}} I_i|\{I_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{I}_{[[}\right\}$$

$$\mathcal{R}^n\bigcap\{\varnothing\}=\varnothing$$

$$\mathcal{R}^n\bigcup\{\varnothing\}=\left\{\prod_{i\in\{1,\ldots,n\}} I_i|\forall i\in\{1,\ldots,n\}\ I_i\in\mathcal{S}_{[[}\right\}\underset{\text{def}}{=}\left\{\prod_{i\in\{1,\ldots,n\}} I_i|\{I_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{S}_{[[}\right\}$$

Proof. If $R\in\mathcal{R}^n$ then by [definition: 20.164] there exists $a,b\in\mathbb{R}^n$ with $a<b\Rightarrow a_i<b_i$ such that

$$R=[a,b[\underset{[\text{definition: } 20.161]}{=}\prod_{i\in\{1,\ldots,n\}}[a_i,b_i[$$

As $\forall i\in\{1,\ldots,n\}\ a_i<b_i$ hence $[a_i,b_i[\in\mathcal{I}_{[[}$ it follows that $R\in\left\{\prod_{i\in\{1,\ldots,n\}} I_i|\forall i\in\{1,\ldots,n\}\ I_i\in\mathcal{I}_{[[}\right\}$ proving that

$$\mathcal{R}^n\subseteq\left\{\prod_{i\in\{1,\ldots,n\}} I_i|\forall i\in\{1,\ldots,n\}\ I_i\in\mathcal{I}_{[[}\right\} \tag{20.343}$$

Further if $R\in\left\{\prod_{i\in\{1,\ldots,n\}} I_i|\forall i\in\{1,\ldots,n\}\ I_i\in\mathcal{I}_{[[}\right\}$ then there exists a $\{I_i\}_{i\in\{1,\ldots,n\}}$ with $\forall i\in\{1,\ldots,n\}\ I_i\in\mathcal{I}_{[[}$ such that $R=\prod_{i\in\{1,\ldots,n\}} I_i$.. Hence $\forall i\in\{1,\ldots,n\}$ there exists $a_i,b_i\in\mathbb{R}$ with $a_i<b_i$ such that $I_i=[a_i,b_i[$. Define $a=(a_1,\ldots,a_n)\in\mathbb{R}^n$, $(b_1,\ldots,b_n)\in\mathbb{R}^n$ then $a<b$ and $[a,b[\underset{[\text{definition: } 20.161]}{=}\prod_{i\in\{1,\ldots,n\}}[a_i,b_i[=\prod_{i\in\{1,\ldots,n\}} I_i=R$ from which it follows that $R\in\mathcal{R}^n$. So $\left\{\prod_{i\in\{1,\ldots,n\}} I_i|\forall i\in\{1,\ldots,n\}\ I_i\in\mathcal{I}_{[[}\right\}\subseteq\mathcal{R}^n$, combining this with [eq: 20.343] gives

$$\mathcal{R}^n=\left\{\prod_{i\in\{1,\ldots,n\}} I_i|\forall i\in\{1,\ldots,n\}\ I_i\in\mathcal{I}_{[[}\right\} \tag{20.344}$$

Let $R\in\mathcal{R}^n$ then there exists $\{I_i\}_{i\in\{1,\ldots,n\}}$ with $I_i\in\mathcal{I}_{[[}$ [so that $I_i\neq\varnothing$] such that $R=\prod_{i\in\{1,\ldots,n\}} I_i$. Applying [theorem: 3.109] it follows that $R\neq\varnothing$ so that

$$\mathcal{R}^n\bigcap\{\varnothing\}=\varnothing \tag{20.345}$$

If $R\in\mathcal{R}^n\bigcup\{\varnothing\}$ then we have either:

- $\boldsymbol{R\in\mathcal{R}^n}$**.** Then by [eq: 20.344] $R=\prod_{i\in\{1,\ldots,n\}} I_i$ where $\forall i\in\{1,\ldots,n\}\ I_i\in\mathcal{I}_{[[}\subseteq\mathcal{S}_{[[}$ so that $R\in\left\{\prod_{i\in\{1,\ldots,n\}} I_i|\forall i\in\{1,\ldots,n\}\ I_i\in\mathcal{S}_{[[}\right\}$.
- $\boldsymbol{R=\varnothing}$**.** Define $\{I_i\}_{i\in\{1,\ldots,n\}}$ by $I_i=\varnothing\in\mathcal{S}_{[[}$ then $\varnothing\underset{[\text{theorem: } 2.138]}{=}\prod_{i\in\{1,\ldots,n\}} I_i$ so that $R\in\left\{\prod_{i\in\{1,\ldots,n\}} I_i|\forall i\in\{1,\ldots,n\}\ I_i\in\mathcal{S}_{[[}\right\}$.

hence we have that

$$\mathcal{R}^n \bigcup \{\varnothing\} \subseteq \left\{ \prod_{i \in \{1,\ldots,n\}} I_i | \forall i \in \{1,\ldots,n\}\ I_i \in \mathcal{S}_{[[} \right\} \tag{20.346}$$

Let $R \in \left\{ \prod_{i \in \{1,\ldots,n\}} I_i | \forall i \in \{1,\ldots,n\}\ I_i \in \mathcal{S}_{[[} \right\}$ then $R = \prod_{i \in \{1,\ldots,n\}} I_i$ where $\forall i \in \{1,\ldots,n\}$ $I_i \in \mathcal{S}_{[[}$. Then we have either:

$\forall i \in \{1,\ldots,n\}$ $I_i \neq \varnothing$. Then by [theorem: 20.248] it follows that $\forall i \in I$ $I_i \in \mathcal{I}_{[[}$ so that $I_i = [a_i, b_i[\neq \varnothing \Rightarrow a_i < b_i$, hence for $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}$ we have $a < b$ so that

$$R = \prod_{i \in \{1,\ldots,n\}} [a, b[\in \mathcal{R}^n \subseteq \mathcal{R}^n \bigcup \{\varnothing\}$$

$\exists i \in \{1,\ldots,n\}$ such that $I_i = \varnothing$. Then

$$R = \prod_{i \in \{1,\ldots,n\}} A_i \underset{\text{[theorem: 2.138]}}{=} \varnothing \in \mathcal{R}^n \bigcap \{\varnothing\}$$

proving that $\left\{ \prod_{i \in \{1,\ldots,n\}} I_i | \forall i \in \{1,\ldots,n\}\ I_i \in \mathcal{S}_{[[} \right\} \subseteq \mathcal{R}^n \bigcap \{\varnothing\}$. Combining this with [eq: 20.346] proves finally that

$$\mathcal{R}^n \bigcup \{\varnothing\} = \left\{ \prod_{i \in \{1,\ldots,n\}} I_i | \forall i \in \{1,\ldots,n\}\ I_i \in \mathcal{S}_{[[} \right\}$$ □

A direct consequence of the above theorem is the following

Theorem 20.272. *Let $n \in \mathbb{N}$ then $\mathcal{R}^n \bigsqcup \{\varnothing\}$ is a semi-ring on $\mathbb{R}^n$*

Proof. First we have

$$\mathcal{R}^n \bigsqcup \{\varnothing\} \underset{\text{[theorem: 20.271]}}{=} \mathcal{R}^n \bigcup \{\varnothing\} = \left\{ \prod_{i \in \{1,\ldots,n\}} I_i | \forall i \in \{1,\ldots,n\}\ I_i \in \mathcal{S}_{[[} \right\} = (\mathcal{S}_{[[})^n$$

so that by [theorem: 20.270]

$$\mathcal{R}^n \bigsqcup \{\varnothing\} \text{ is a semi-ring}$$ □

Theorem 20.273. *Let $n \in \mathbb{N}$ then we have that*

$$\sigma[\{C \subseteq \mathbb{R}^n | C \text{ is closed in } \mathcal{T}_{\|\|^n}\}] = \beta[\mathbb{R}^n] = \sigma[\mathcal{R}^n] = \sigma\left[\mathcal{R}^n \bigcup \{\varnothing\}\right]$$

Proof. By [theorem: 20.177] we only have to prove that $\sigma[\mathcal{R}^n] = \sigma[\mathcal{R}^n \bigsqcup \{\varnothing\}]$. As $\mathcal{R}^n \subseteq \mathcal{R}^n \bigcup \{\varnothing\} \subseteq \sigma[\mathcal{R}^n \bigcup \{\varnothing\}]$ and $\sigma[\mathcal{R}^n \bigcup \{\varnothing\}]$ is a σ-algebra it follows from [theorem: 20.141] that

$$\sigma[\mathcal{R}^n] \subseteq \sigma\left[\mathcal{R}^n \bigsqcup \{\varnothing\}\right] \tag{20.347}$$

Let $A \in \mathcal{R}^n \bigsqcup \{\varnothing\}$ then either $A \in \mathcal{R}^n \subseteq \sigma[\mathcal{R}_n]$ or $A = \varnothing \in \sigma[\mathcal{R}_n]$ so that $\mathcal{R}^n \bigsqcup \{\varnothing\} \subseteq \sigma[\mathcal{R}^n]$, as $\sigma[\mathcal{R}^n]$ is a σ-algebra it follows from [theorem: 20.141] that $\sigma[\mathcal{R}^n \bigsqcup \{\varnothing\}] \subseteq \sigma[\mathcal{R}^n]$. Combining this with [eq: 20.347] proves

$$\sigma[\mathcal{R}^n] = \sigma\left[\mathcal{R}^n \bigsqcup \{\varnothing\}\right]$$ □

20.4.4.6 Content on the semi-ring $\mathcal{R}^n \bigcup \{\varnothing\}$

Definition 20.274. (Content of a rectangle) *Let $n \in \mathbb{N}$ then we define*

$$\mathrm{vol}^n: \mathcal{R}^n \bigsqcup \{\varnothing\} \to [0, \infty] \text{ by } \mathrm{vol}^n(R) = \begin{cases} 0 \text{ if } R = \varnothing \\ v^n(R) \text{ if } R \neq \varnothing \end{cases}$$

where $v^n(R): \mathcal{R}^n \to [0, \infty]$ is defined by $v^n(R) = \prod_{i=1}^n (\mathrm{begin}(R)_i - \mathrm{end}(R)_i) \in \mathbb{R}^+$ [see definition: 20.168].

***Note* 20.275.** *Using the definition it follows that* $\mathrm{vol}^n(R)$ *is finite and*

$$\forall R \in \mathcal{R}^n \bigcup \{\varnothing\} \text{ we have } \mathrm{vol}^n(R) = 0 \Leftrightarrow R \in \mathcal{R}^n$$

Next it remains to prove that vol^n is actual a content on $\mathcal{R}^n \bigcup \{\varnothing\}$, the most difficult part is proving additivity of vol^n. To illustrate how to prove this consider the following 2-dimesional case [see figure:] where

$$R = \bigsqcup_{i \in \{1, \ldots, 4\}} R_i$$

then we have to prove that

$$\mathrm{vol}^2(R) = \sum_{i=1}^{4} \mathrm{vol}^2(R_i)$$

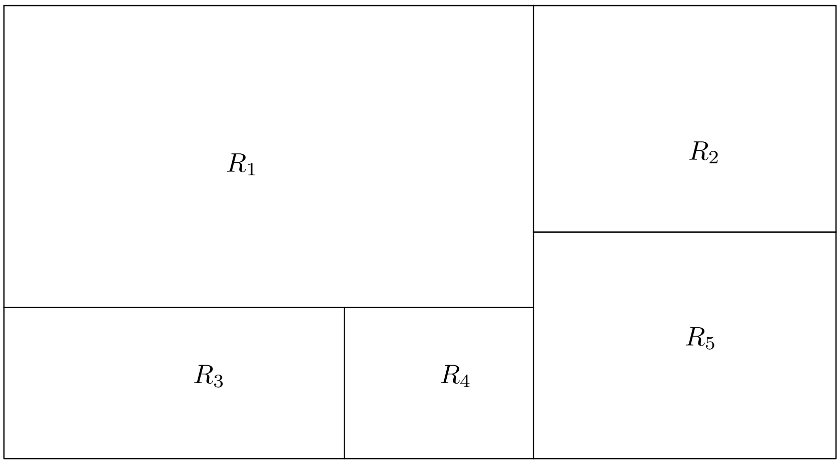

Figure 20.1.

To do this we divide R_1, R_2, R_3, R_4 and R_5 in sub-cubes by partitioning their sides in sub intervals.

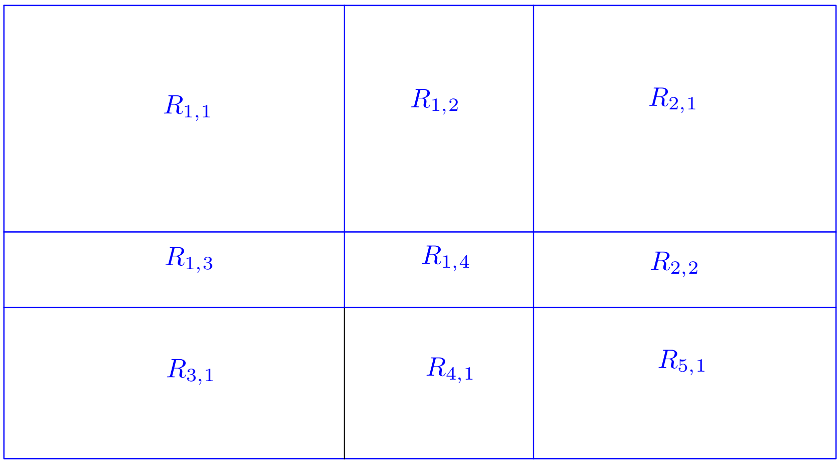

Figure 20.2.

so that

$$\begin{aligned} R_1 &= R_{1,1} + R_{1,2} + R_{1,3} + R_{1,4} \\ R_2 &= R_{2,1} + R_{2,2} \\ R_3 &= R_{3,1} \\ R_4 &= R_{4,1} \\ R_5 &= R_{5,1} \end{aligned}$$

then we prove that

$$\begin{aligned}\operatorname{vol}^2(R) &= \operatorname{vol}^2(R_{1,1})+\operatorname{vol}^2(R_{1,2})+\operatorname{vol}^2(R_{1,3})+\operatorname{vol}^2(R_{1,4})+\operatorname{vol}^2(R_{2,1})+\operatorname{vol}^2(R_{2,2})+\operatorname{vol}^2(R_{3,1})+\\ &\quad \operatorname{vol}^2(R_{3,1})+\operatorname{vol}^2(R_{4,1})+\operatorname{vol}^2(R_{5,1})\end{aligned}$$

and apply associativity to get

$$\begin{aligned}\operatorname{vol}^2(R) &= (\operatorname{vol}^2(R_{1,1})+\operatorname{vol}^2(R_{1,2})+\operatorname{vol}^2(R_{1,3})+\operatorname{vol}^2(R_{1,4}))+(\operatorname{vol}^2(R_{2,1})+\operatorname{vol}^2(R_{2,2}))+\\ &\quad (\operatorname{vol}^2(R_{3,1}))+(\operatorname{vol}^2(R_{3,1}))+(\operatorname{vol}^2(R_{4,1}))+(\operatorname{vol}^2(R_{5,1}))\\ &= \operatorname{vol}^2(R_1)+\operatorname{vol}^2(R_2)+\operatorname{vol}^2(R_3)+\operatorname{vol}^2(R_4)+\operatorname{vol}^2(R_5)\end{aligned}$$

which proves additivity. First we show that every increasing finite family of real numbers can be used to construct a pairwise disjoint union of half open intervals that is itself a half open interval. This will be used later to turn a union of half open intervals that is itself a half open interval in a disjoint union of half open intervals.

Lemma 20.276. *Let $n,m\in\mathbb{N}$ with $n\leqslant m$, $\{a_i\}_{i\in\{n,\ldots,m\}}\subseteq\mathbb{R}$ such that $\forall i\in\{n,\ldots,m-1\}$ we have $a_i\leqslant a_{i+1}$ then*

$$\{[a_i,a_{i+1}[\}_{i\in\{n,\ldots,m-1\}} \text{ is pairwise disjoint}$$

and

$$\bigsqcup_{i\in\{n,\ldots,m-1\}}[a_i,a_{i+1}[=[a_n,a_m[$$

Proof. Let $i,j\in\{n,\ldots,m-1\}$ with $i\neq j$ then we may assume that $i<j$ [otherwise exchange i and j]. Assume that $x\in[a_i,a_{i+1}[\bigcap[a_j,a_{j+1}[$ then $a_i\leqslant x<a_{i+1}\underset{i<j\Rightarrow i+1\leqslant j}{\leqslant}a_j\leqslant x<a_{j+1}$ leading to the contradiction $x<x$, hence the assumption is wrong proving that $[a_i,a_{i+1}[\bigcap[a_j,a_{j+1}[=\varnothing$. So

$$\{[a_i,a_{i+1}[\}_{i\in\{n,\ldots,m-1\}} \text{ is pairwise disjoint}$$

Let $x\in[a_n,a_m[$ then $a_n\leqslant x<a_m$ so that $n\in\{i\in\{n,\ldots,m\}|a_i\leqslant x\}$ a finite set. Hence $M=\max(\{i\in\{n,\ldots,m\}|a_i\leqslant x\})$ exist. As $x<a_m\Rightarrow a_m\nleqslant x$ we have $M\neq m$ so that $M\in\{n,\ldots,m-1\}\Rightarrow M+1\in\{n,\ldots,m\}$. As $M=\max(\{i\in\{n,\ldots,m\}|a_i\leqslant x\})$ it follows that $a_M\leqslant x<a_{M+1}$ so that $x\in[a_M,a_{M+1}[\subseteq\bigsqcup_{i\in\{n,\ldots,m-1\}}[a_i,a_{i+1}[$. Hence we have proved that

$$[a_n,a_m[\subseteq\bigsqcup_{i\in\{n,\ldots,m-1\}}[a_i,a_{i+1}[\tag{20.348}$$

Let $x\in\bigsqcup_{i\in\{n,\ldots,m-1\}}[a_i,a_{i+1}[$ then there exist a $i\in\{n,\ldots,m-1\}$ such that $a_i\leqslant x<a_{i+1}$. As $n\leqslant i$ and $i\leqslant m-1<m$ we have by the hypothesis that $a_n\leqslant a_i\wedge a_{i+1}\leqslant a_m$ so that $a_n\leqslant x<a_m$ or $x\in[a_n,a_m[$. Hence $\bigsqcup_{i\in\{n,\ldots,m-1\}}[a_i,a_{i+1}[\subseteq[a_n,a_m[$ which combined with [eq: 20.348] gives

$$\bigsqcup_{i\in\{n,\ldots,m-1\}}[a_i,a_{i+1}[=[a_n,a_m[$$ □

Next we show that a half open interval that is a union of half open subintevals is the disjoint union of sub intervals that are based on the boundary points of the half open subintervalas. For example:

$$\begin{aligned}[-4,2[&= [-3,2[\bigcup[-1,1[\bigcup[-4,0[\bigcup[0,2[\\ &= [-4,-3[\bigsqcup[-3,-1[\bigsqcup[-1,0[\bigsqcup[0,1[\bigsqcup[1,2[\end{aligned}$$

to do thist we sort the boundary points $\{-3,2,-1,1,-4,0,2\}$ giving the sorted set

$$\{-4,-3,-1,0,1,2\}$$

and use the the previous lemma to create the final list of intervals

$$[-4,-3[,[-3,-1[,[-1,0[,[0,1[,[1,2[$$

Lemma 20.277. *Let $n\in\mathbb{N}$ and $\{[a_i,b_i[\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{I}_{[[}$ such that $\bigcup_{i\in\{1,\ldots,n\}}[a_i,b_i[=[a,b[$ then if we take*

$$A=\{\alpha_i|i\in\{1,\ldots,n\}\}\bigcup\{b_i|i\in\{1,\ldots,n\}\}$$

there exists a $m\in\mathbb{N}\setminus\{1\}$ and a bijection $\beta:\{1,\ldots,m\}\rightarrow A$ such that:

1. $\beta(1)=a\wedge\beta(m)=b$
2. $\forall i,j\in\{1,\ldots,m\}$ *with* $i<j$ *we have* $\beta(i)<\beta(j)$
3. $\forall x,y\in A$ *with* $x<y$ *we have* $\beta^{-1}(x)<\beta^{-1}(y)$
4. $\forall i\in\{1,\ldots,n\}$ *we have* $\beta^{-1}(a_i)<\beta^{-1}(b_i)$
5. $\forall i\in\{1,\ldots,n\}$ $\{[\beta(j),\beta(j+1)[\}_{j\in\{\beta^{-1}(a_i),\ldots,\beta^{-1}(b_i)-1\}}$ *is pairwise disjoint and* $[a_i,b_i[=\bigsqcup_{j\in\{\beta^{-1}(a_i),\ldots,\beta^{-1}(b_i)-1\}}[\beta(j),\beta(j+1)[$
6. $\{[\beta(i),\beta(i+1)[\}_{i\in\{1,\ldots,m-1\}}$ *is pairwise disjoint and* $\bigsqcup_{i\in\{1,\ldots,m-1\}}[\beta(i),\beta(i+1)[=[a,b[$

Proof. Define $A=\{a_i|i\in\{1,\ldots,n\}\}\bigcup\{b_i|i\in\{1,\ldots,n\}\}$. As $[a_1,b_1[\neq\varnothing$ we have that $[a,b[=\bigcup_{i\in\{1,\ldots,n\}}[a_i,b_i[\neq\varnothing$ so that

$$a<b \tag{20.349}$$

Assume that $a\notin A$. As $a\in[a,b[=\bigcup_{i\in\{1,\ldots,n\}}[a_i,b_i[$ there exist a $i\in\{1,\ldots,n\}$ such that $a\in[a_i,b_i[\Rightarrow a_i\leqslant a<b_i\underset{a\notin A\Rightarrow a\neq a_i}{\Rightarrow}a_i<a<b_i$. Using the density of the real numbers [theorem: 10.32] there exist a $t\in\mathbb{R}$ such that $a_i<t<a<b_i\Rightarrow t\notin[a,b[\wedge t\in[a_i,b_i[\subseteq[a,b[$ a contradiction. So the assumption is wrong and we have

$$a\in A \tag{20.350}$$

Assume that $b\notin A$. Given $i\in\{1,\ldots,n\}$ we have $[a_i,b_i[\subseteq\bigcup_{i\in\{1,\ldots,n\}}[a_i,b_i[\subseteq[a,b[$ so that by [theorem: 3.144] $b_i\leqslant b\underset{b\notin A\Rightarrow b\neq b_i}{\Rightarrow}b_i<b$. As also $a<b$ it follows that $\max(\{a\}\bigcup\{b_i|i\in\{1,\ldots,n\}\})<b$. Hence using the density of the real numbers [theorem: 10.32] there exists a $t\in\mathbb{R}$ with $\max(\{a\}\bigcup\{b_i|i\in\{1,\ldots,n\}\})<t<b$. Hence $\forall i\in\{1,\ldots,n\}$ $t\notin[a_i,b_i[\wedge a<t<b$. So $t\notin\bigcup_{i\in\{1,\ldots,n\}}[a_i,b_i[=[a,b[\wedge t\in[a,b[$ again a contradiction. Hence the assumption is wrong and we have

$$b\in A \tag{20.351}$$

As $\forall i\in\{1,\ldots,n\}$ we have $[a_i,b_i[\subseteq\bigcup_{i\in\{1,\ldots,n\}}[a_i,b_i[=[a,b[$ it follows from theorem: 3.144] that $a\leqslant a_i<b_i\leqslant b$ proving that

$$a=\min(A)\wedge b=\max(A)$$

Using [theorems: 6.34, 6.45] it follows that A is finite and as $a,b\in A$ $\operatorname{card}(A)\in\mathbb{N}$, hence using [theorem: 10.96] it follows that there exist a bijection

$$\beta:\{1,\ldots,m\}\rightarrow A\text{ where }m=\operatorname{card}(A)\text{ such that }\forall i\in\{1,\ldots,m-1\}\ \beta(i)<\beta(i+1) \tag{20.352}$$

As $\forall i\in\{1,\ldots,m\}$ we have $\beta(1)\leqslant\beta(i)$ and $\beta(i)\leqslant\beta(m)$ it follows that $\beta(1)=\min(A)=a\wedge\beta(m)=\max(A)=b$, hence

$$a=\beta(1)\wedge b=\beta(m) \tag{20.353}$$

Let $x,y\in A$ such that $x<y$. If $\beta^{-1}(y)\leqslant\beta^{-1}(x)$ it follows that $y=\beta(\beta^{-1}(y))\leqslant\beta(\beta^{-1}(x))=x$ contradicting $x\leqslant y$. Hence we have

$$\forall x,y\in A\text{ with }x<y\text{ we have }\beta^{-1}(x)<\beta^{-1}(y) \tag{20.354}$$

Let $i\in\{1,\ldots,n\}$ then $a_i,b_i\in A$ and $a_i<b_i$ [as $[a_i,b_i[\in\mathcal{I}_{[[}$] so that by the above $\beta^{-1}(a_i)<\beta^{-1}(b_i)$. In other words

$$\forall i\in\{1,\ldots,n\}\text{ we have }\beta^{-1}(a_i)<\beta^{-1}(b_i) \tag{20.355}$$

Let $i\in\{1,\ldots,n\}$ then for $j\in\{\beta^{-1}(a_i),\ldots,\beta^{-1}(b_i)-1\}$ we have $\beta(j)<\beta(j)$ so that by [lemma: 20.276] we have

$$\{[\beta(j),\beta(j+1)[\}_{j\in\{\beta^{-1}(a_i),\ldots,\beta^{-1}(b_i)-1\}}\text{ is pairwise disjoint} \tag{20.356}$$

and

$$\bigsqcup_{j\in\{\beta^{-1}(a_i),\ldots,\beta^{-1}(b_i)-1\}}[\beta(j),\beta(j+1)[=[\beta(\beta^{-1}(a_i)),\beta(\beta^{-1}(b_i))[=[a_i,b_i[\tag{20.357}$$

Using [lemma: 20.276] on [eq: 20.352] gives

$$\{[\beta(i),\beta(i-1)[\}_{i\in\{1,\ldots,m-1\}} \text{ is pairwise disjoint} \tag{20.358}$$

and

$$\bigsqcup_{i\in\{1,\ldots,m-1\}}[\beta(i),\beta(i+1)[=[\beta(1),\beta(n)[\underset{\text{[eq: 20.353]}}{=}[a,b[\tag{20.359}$$

So (1) is proved by [eq: 20.353], (2) is proved by [eq: 20.352], (3) is proved by [eq: 20.354], (4) is proved by [eq: 20.355], (5) by [eqs: 20.356, 20.357] and finally (6) by [eqs: 20.358, 20.359]. □

Next we show how the volume of a half open rectangle partioned by its sides can be calculated. Giving the follow example of a partioned rectangle R

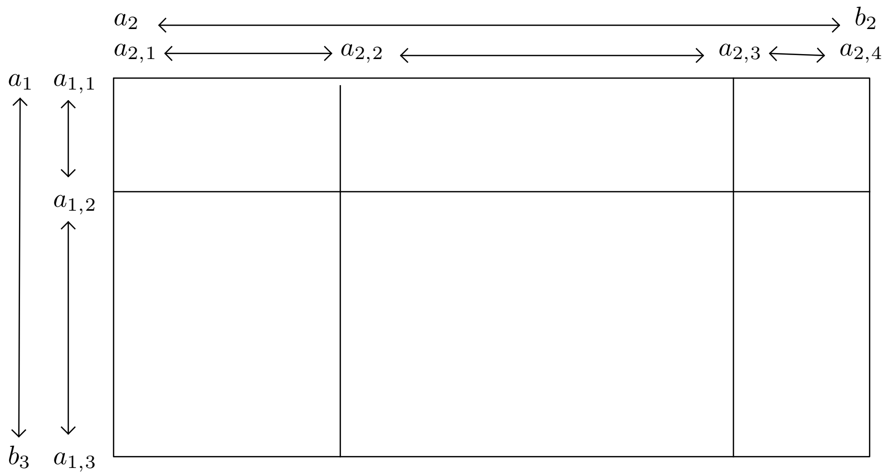

Figure 20.3.

we have then

$$\begin{aligned}
\prod_{i\in\{1,\ldots,n\}}[a_i,b_i[&= [a_1,b_1[\cdot[a_2,b_2[\\
&= [a_{1,1},a_{1,2}[\cdot[a_{2,1},a_{2,2}[\bigsqcup[a_{1,1},a_{1,2}[\cdot[a_{2,2},a_{2,3}[\bigsqcup[a_{1,1},a_{1,2}[\cdot[a_{2,3},a_{2,4}[\bigsqcup\\
&\quad [a_{1,2},a_{1,3}[\cdot[a_{2,1},a_{2,2}[\bigsqcup[a_{1,2},a_{1,3}[\cdot[a_{2,2},a_{2,3}[\bigsqcup[a_{1,2},a_{1,3}[\cdot[a_{2,3},a_{2,4}[\\
&= \bigsqcup_{\rho\in\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}}[a_{1,\rho_1},a_{1,\rho_1+1}[\cdot[a_{2,\rho_2},a_{2,\rho_2+1}[\\
&= \bigsqcup_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}}\left(\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[\right)
\end{aligned}$$

and

$$\begin{aligned}
\prod_{i\in\{1,\ldots,n\}}(b_i-a_i) &= (b_1-a_1)\cdot(b_2-a_2)\\
&= (a_{1,2}-a_{1,1})\cdot(a_{2,2}-a_{2,1})+(a_{1,2}-a_{1,1})\cdot(a_{2,3}-a_{2,2})+(a_{1,2}-a_{1,1})\cdot(a_{2,4}-\\
&\quad a_{2,3})+(a_{1,3}-a_{1,2})\cdot(a_{2,2}-a_{2,1})+(a_{1,3}-a_{1,2})\cdot(a_{2,3}-a_{2,2})+(a_{1,3}-a_{1,2})\cdot\\
&\quad (a_{2,4}-a_{2,3})\\
&= \bigsqcup_{\rho\in\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}}(a_{1,\rho_1+1}-a_{1,\rho_1})\cdot(a_{2,\rho_2+1}-a_{2,\rho_2})\\
&= \bigsqcup_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}}\left(\prod_{i\in\{1,\ldots,n\}}(a_{i,\rho_i+1}-a_{i,\rho_i})\right)
\end{aligned}$$

Lemma 20.278. *Let $n\in\mathbb{N}$, $\{[a_i,b_i[\}_{i\in\{1,\ldots,n\}}\subseteq\mathcal{I}_{[[}$ such that $\forall i\in\{1,\ldots,n\}$ there exists $n_i,m_i\in\mathbb{N}$ with $n_i\leqslant m_i$ and a $\{a_{i,j}\}_{j\in\{n_i,\ldots,m_i\}}\subseteq\mathbb{R}$ such that $\forall j\in\{n_i,\ldots,m_i-1\}$ $a_{i,j}\leqslant a_{i,j+1}$ and $a_{i,n_i}=a_i\wedge a_{i,m_i}=b_i$ then*

$$\left\{\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[\right\}_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}}\text{ is pairwise disjoint}$$

$$\prod_{i\in\{1,\ldots,n\}}[a_i,b_i[=\bigsqcup_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}}\left(\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[\right)$$

$$\prod_{i=1}^{n}(b_i-a_i)=\sum_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}}\left(\prod_{i\in\{1,\ldots,n\}}(a_{i,\rho_i+1}-a_{i,\rho_i})\right)$$

Proof. Assume that $\exists i\in\{1,\ldots,n\}$ such that $n_i=m_i$ then $a_i=a_{i,n_i}=a_{o,m_i}=b_i$ contradicting $a_i<b_i$ [because $[a_i,b_i[\neq\varnothing$], hence a assumption is false and we have

$$\forall i\in\{1,\ldots,n\}\text{ we have }n_i<m_i\text{ or }n_i\leqslant m_i-1\tag{20.360}$$

Let $i\in\{1,\ldots,n\}$ then, as $\forall j\in\{n_i,\ldots,m_i-1\}$ $a_{i,j}\leqslant a_{i,j+1}$ it follows form [lemma: 20.276] that

$$\{[a_{i,j},a_{i,j+1}[\}_{j\in\{n_i\ldots,m_i-1\}}\text{ is pairwise disjoint}\tag{20.361}$$

and

$$[a_i,b_i[=[a_{i,n_i},a_{i,m_i}[=\bigsqcup_{j\in\{n_i,\ldots,m_i-1\}}[a_{i,j},a_{i,j+1}[\tag{20.362}$$

Let $x\in\prod_{i\in\{1,\ldots,n\}}[a_i,b_i[$ then $\forall i\in\{1,\ldots,n\}$ we have $x_i\in[a_i,b_i[$, hence by the above there exist a $\rho_i\in\{n_i,\ldots,m_i-1\}$ such that $x_i\in[a_{i,\rho_i},a_{i,\rho_i+1}[$. So for $\rho=(\rho_1,\ldots,\rho_n)\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}$ it follows that $\forall i\in\{1,\ldots,n\}$ $x_i\in[a_{i,\rho_i},a_{i,\rho_i+1}[$ proving that $x\in\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[$. Hence we have

$$\prod_{i\in\{1,\ldots,n\}}[a_i,b_i[\subseteq\bigcup_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}}\left(\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[\right)\tag{20.363}$$

If $x\in\bigcup_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}}\left(\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[\right)$ then there exist a $\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}$ such that $x\in\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[$. Hence $\forall i\in\{1,\ldots,n\}$ we have $x_i\in[a_{i,\rho_i},a_{i,\rho_i+1}[\underset{\text{[eq: 20.362]}}{\subseteq}[a_i,b_i[$ proving that $x\in\prod_{i\in\{1,\ldots,n\}}[a_i,b_i[$. So it follows that $\bigcup_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}}\left(\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[\right)\subseteq\prod_{i\in\{1,\ldots,n\}}[a_i,b_i[$ which combined with [eq: 20.363] proves that

$$\prod_{i\in\{1,\ldots,n\}}[a_i,b_i[=\bigcup_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}}\left(\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[\right)\tag{20.364}$$

Let $\rho,\sigma\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}$ with $\rho=\sigma$ then there exist a $k\in\{1,\ldots,n\}$ such that $\rho(l)\neq\sigma(k)$. Assume that that there exist a $x\in\left(\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[\right)\bigcap\left(\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[\right)$ then $x_k\in[a_{k,\rho_k},a_{k,\rho_k+1}[\bigcap[a_{k,\sigma_k},a_{k,\sigma_k+1}[\underset{\rho_k\neq\sigma_k}{=}\varnothing$ which is a contradiction, hence the assumption is wrong and we must have that $\left(\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[\right)\bigcap\left(\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[\right)=\varnothing$. So we have

$$\left\{\prod_{i\in\{1,\ldots,n\}}[a_{i,\rho_i},a_{i,\rho_i+1}[\right\}_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i-1\}}\text{ is pairwise disjoint}\tag{20.365}$$

Further we have

$$\begin{aligned}
\prod_{i=1}^{n}(b_i-a_i) &\underset{[\text{definition: } 20.256]}{=} \prod_{i=1}^{n}\text{len}([a_i,b_i[)\\
&\underset{[\text{eq: } 20.361,\ 20.362]}{=} \prod_{i=1}^{n}\text{len}\left(\bigsqcup_{j\in\{n_i,\ldots,m_i\}}[a_{i,j},a_{i,j+1}[\right)\\
&\underset{[\text{lemma: } 20.259]}{=} \prod_{i=1}^{n}\left(\sum_{j\in\{n_i,\ldots,m_i\}}\text{len}([a_{i,j},a_{i,j+1}])\right)\\
&\underset{[\text{theorem: } 11.49]}{=} \sum_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i\}}\left(\prod_{i\in\{1,\ldots,n\}}\text{len}([a_{i,\rho_i},a_{i,\rho_i+1}[)\right)\\
&= \sum_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{n_i,\ldots,m_i\}}\left(\prod_{i\in\{1,\ldots,n\}}(a_{i,\rho_i+1}-a_{i,\rho_i})\right)
\end{aligned}$$

which proves the last part of the theorem. □

The following lemma shows that if the union of half open rectangles is a half open rectangle then each side of the of the open rectangle is a union of the corresponding sides of the subrectangles. For example in the following 2-dimesional we have that

$$\begin{aligned}
[a_1,b_1[&= [a_{1,1},b_{1,1}[\bigcup[a_{2,1},b_{2,1}[\bigcup[a_{3,1},b_{3,1}[\bigcup[a_{4,1},b_{5,1}[\bigcup[a_{5,1},b_{5,1}[\\
&= \bigcup_{i\in\{1,\ldots,5\}}[a_{i,1},b_{i,1}[\\
[a_2,b_2[&= [a_{1,2},b_{1,2}[\bigcup[a_{2,2},b_{2,2}[\bigcup[a_{3,2},b_{3,2}[\bigcup[a_{4,2}],b_{4,2}[\bigcup[a_{5,2},b_{5,2}[\\
&= \bigcup_{i\in\{1,\ldots,b\}}[a_{i,2},b_{i,2}[
\end{aligned}$$

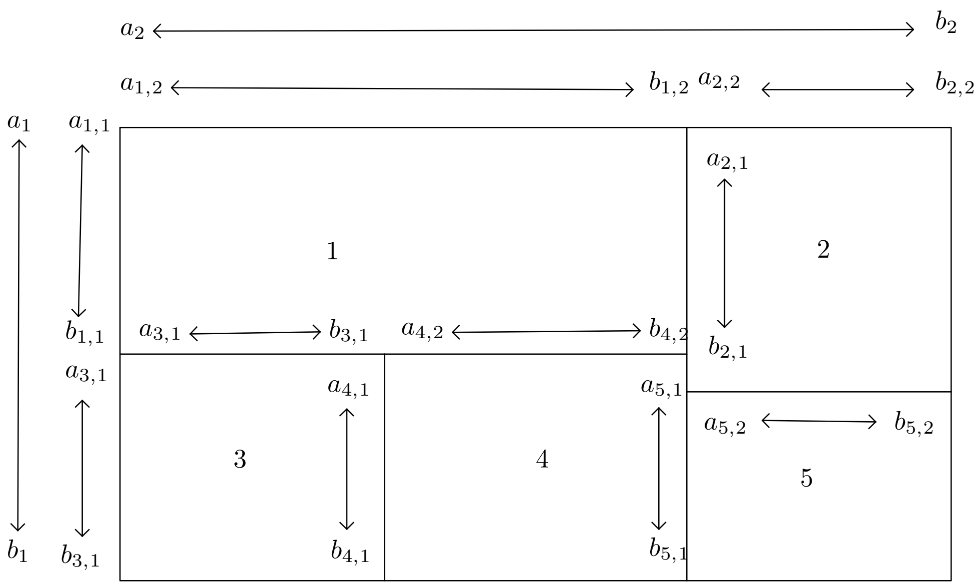

Figure 20.4.

Lemma 20.279. *Let $n,m\in\mathbb{N}$ and let $\left\{\prod_{j\in\{1,\ldots,n\}}[a_{i,j},b_{i,j}[\right\}_{i\in\{1,\ldots,m\}}\subseteq\mathcal{R}^n$ such that*

$$\bigcup_{i\in\{1,\ldots,m\}}\left(\prod_{j\in\{1,\ldots n\}}[a_{i,j},b_{i,j}[\right)=\prod_{j\in\{1,\ldots,n\}}[a_j,b_j[\in\mathcal{R}^n$$

then $\forall j\in\{1,\ldots,n\}$ *we have*

$$[a_j,b_j[=\bigcup_{i\in\{1,\ldots,m\}}[a_{i,j},b_{i,j}[$$

Proof. Take $j\in\{1,\ldots,n\}$. If $t\in\bigcup_{i\in\{1,\ldots,m\}}[a_{i,j},b_{i,j}[$ then there exist a $i\in\{1,\ldots,m\}$ such that $t\in[a_{i,j},b_{i,j}[$. As $\prod_{j\in\{1,\ldots n\}}[a_{i,j},b_{i,j}[\in\mathcal{R}^n$ $\prod_{j\in\{1,\ldots n\}}[a_{i,j},b_{i,j}[\neq\varnothing$ it follows from [theorems: 3.109, 3.108] that the projection function π_j: $\prod_{j\in\{1,\ldots n\}}[a_{i,j},b_{i,j}[\rightarrow[a_{i,j},b_{i,j}[$ is a surjection hence there exists a $x\in\prod_{j\in\{1,\ldots n\}}[a_{i,j},b_{i,j}[$ such that $x_j=\pi_j(x)=t$. As

$$\prod_{j\in\{1,\ldots n\}}[a_{i,j},b_{i,j}[\subseteq\bigsqcup_{i\in\{1,\ldots,m\}}\left(\prod_{j\in\{1,\ldots n\}}[a_{i,j},b_{i,j}[\right)=\prod_{j\in\{1,\ldots,n\}}[a_j,b_j[$$

we have that $x\in\prod_{j\in\{1,\ldots,n\}}[a_j,b_j[$ so that $t=\pi_j(x)\in[a_j,b_j[$. Hence we have

$$\bigcup_{i\in\{1,\ldots,m\}}[a_{i,j},b_{i,j}[\subseteq[a_j,b_j[\tag{20.366}$$

Let $t\in[a_j,b_j[$. As $\prod_{j\in\{1,\ldots,n\}}[a_j,b_j[\in\mathcal{R}^n$ $\prod_{j\in\{1,\ldots,n\}}[a_j,b_j[\neq\varnothing$ so that by [theorems: 3.109, 3.108] the projection function π_j: $\prod_{j\in\{1,\ldots,n\}}[a_j,b_j[\rightarrow[a_j,b_j[$ is a surjection, hence there exist a $x\in\prod_{j\in\{1,\ldots,n\}}[a_j,b_j[=\bigsqcup_{i\in\{1,\ldots,m\}}\left(\prod_{j\in\{1,\ldots n\}}[a_{i,j},b_{i,j}[\right)$ such that $t=\pi_j(x)$. Then there exists a $i\in\{1,\ldots,m\}$ such that $x\in\prod_{j\in\{1,\ldots n\}}[a_{m,j},b_{m,j}[$. So it follows that $t=\pi_j(x)\in[a_{m,j},b_{m,j}[\subseteq\bigcup_{i\in\{1,\ldots,m\}}[a_{i,j},b_{i,j}[$ which proves that $[a_j,b_j[\subseteq\bigcup_{i\in\{1,\ldots,m\}}[a_{i,j},b_{i,j}[$. Combining this with [eq: 20.366] proves that

$$[a_j,b_j[=\bigcup_{i\in\{1,\ldots,m\}}[a_{i,j},b_{i,j}[$$

□

We show now that the volume of the pairwise distinct union of half open non empty rectangles is equal to the sum of the volumes of thise reactangles if this distinct union is also a half open rectangle.

Lemma 20.280. *Let* $n,\ m\in\mathbb{N}$, $\{\prod_{j\in\{1,\ldots,n\}}[a_{i,j},b_{i,j}[\}_{i\in\{1,\ldots,m\}}\subseteq\mathcal{R}^n$ *pairwise disjoint such that* $\bigsqcup_{i\in\{1,\ldots,m\}}\left(\prod_{j\in\{1,\ldots n\}}[a_{i,j},b_{i,j}[\right)=\prod_{i\in\{1,\ldots,n\}}[a_i,b_i[\in\mathcal{R}^n$ *then we have*

$$\prod_{i\in\{1,\ldots,n\}}(b_i-a_i)=\sum_{i=1}^m\left(\prod_{j=1}^n(b_{i,j}-a_{i,j})\right)$$

Proof. First using the previous lemma [lemma: 20.279] we have that

$$\forall j\in\{1,\ldots,n\}\ [a_j,b_j[=\bigcup_{i\in\{1,\ldots,m\}}[a_{i,j},b_{i,j}[\tag{20.367}$$

Let $j\in\{1,\ldots,n\}$. Using [lemma: 20.277] there exists a $n_j\in\mathbb{N}$ and a bijection

$$\beta_j\colon\{1,\ldots,n_j\}\rightarrow\{a_{i,j}|i\in\{1,\ldots,n_j\}\}\bigcup\{b_{i,j}|i\in\{1,\ldots,n_j\}\} \tag{20.368}$$

such that

$$\beta_j(1)=a_j\wedge\beta_j(n_j)=b_j \tag{20.369}$$

$$\forall k,l\in\{1,\ldots,n_j\}\text{ with }k<l\text{ we have }\beta_j(k)<\beta_j(l) \tag{20.370}$$

$$\forall x,y\in\{a_{i,j}|i\in\{1,...,n_j\}\}\bigcup\{b_{i,j}|i\in\{1,...,n_j\}\}\text{ with }x<y\text{ we have }(\beta_j)^{-1}(x)<(\beta_j)^{-1}(y) \tag{20.371}$$

$$\forall i\in\{1,\ldots,m\}\text{ we have }(\beta_j)^{-1}(a_{i,j})<(\beta_j)^{-1}(b_{i,j}) \tag{20.372}$$

$$\forall i\in\{1,\ldots,m\}\quad[a_{i,j},b_{i,j}[=\bigsqcup_{l\in\{(\beta_j)^{-1}(a_{i,j}),\ldots,(\beta_j)^{-1}(b_{i,j})-1\}}[\beta_j(l),\beta_j(l+1)[\tag{20.373}$$

$$[a_j,b_j[=\bigsqcup_{i\in\{1,\ldots,n_j\}}[\beta_i(i),\beta_j(i+1)[\tag{20.374}$$

Because of [eqs: 20.369 and 20.370] we can apply the previous lemma [lemma: 20.278] giving

$$\prod_{j\in\{1,\ldots,n\}}[a_j,b_j[=\bigsqcup_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{1,\ldots,n_j-1\}}\left(\prod_{j\in\{1,\ldots,n\}}[\beta_j(\rho_j)-\beta_j(\rho_j+1)[\right) \tag{20.375}$$

$$\prod_{j=1}^{n}(b_j-a_j)=\sum_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{1,\ldots,n_j-1\}}\left(\prod_{j\in\{1,\ldots,n\}}(\beta_j(\rho_j+1)-\beta_j(\rho_j))\right) \tag{20.376}$$

Let $i\in\{1,\ldots,m-1\}$. As $a_{i,j}=\beta_j((\beta_j)^{-1}(a_{i,j}))$ and $b_{i,j}=\beta_j((\beta_j)^{-1}(b_{i,j}))$ and $\forall i\in\{(\beta_j)^{-1}(a_{i,j}),\ldots,(\beta_j)^{-1}(b_{i,j})-1\}$ we have by [eq: 20.370] that $\beta_j(i)<\beta_j(i+1)$ we can use [[lemma: 20.278] again to get

$$\prod_{j\in\{1,\ldots,n\}}[a_{i,j},b_{i,j}[\;=\;\bigsqcup_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{(\beta_j)^{-1}(a_{i,j}),\ldots,(\beta_j)^{-1}(b_{i,j})-1\}}\left(\prod_{j=1}^{n}[\beta_j(\rho_j),\beta_j(\rho_j)+1[\right) \tag{20.377}$$

$$\prod_{j\in\{1,\ldots,n\}}(b_{i,j}-a_{i,j})\;=\;\sum_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{(\beta_j)^{-1}(a_{i,j}),\ldots,(\beta_j)^{-1}(b_{i,j})-1\}}\left(\prod_{j=1}^{n}(\beta_j(\rho_j+1)-\beta_j(j))\right) \tag{20.378}$$

To relate [eq: 20.375] to [eq: 20.378] we prove now the following:

$$\bigsqcup_{i\in\{1,\ldots,m\}}\left(\prod_{j\in\{1,\ldots,n\}}\{(\beta_j)^{-1}(a_{i,j}),\ldots,(\beta_j)^{-1}(b_{i,j})-1\}\right)=\prod_{j\in\{1,\ldots,n\}}\{1,\ldots,n_j-1\} \tag{20.379}$$

Proof. Let $i\in\{1,\ldots,m\}$ and $j\in\{1,\ldots,n\}$. From [eq: 20.368] it follows that $(\beta_j)^{-1}(a_{i,j})$, $(\beta_j)^{-1}(b_{i,j})\in\{1,\ldots,m_j\}$, further by [eq: 20.372] we have $(\beta_j)^{-1}(a_{i,j})\leqslant(\beta_j)^{-1}(b_{i,j})-1$ so that $\{(\beta_j)^{-1}(a_{i,j}),\ldots,(\beta_j)^{-1}(b_{i,j})-1\}\subseteq\{1,\ldots,m_j-1\}$. Applying [theorem: 2.141] gives us then $\prod_{j\in\{1,\ldots,n\}}\{(\beta_j)^{-1}(a_{i,j}),\ldots,(\beta_j)^{-1}(b_{i,j})-1\}\subseteq\prod_{j\in\{1,\ldots,m\}}\{1,\ldots,n_j-1\}$ hence we have

$$\bigcup_{i\in\{1,\ldots,m\}}\left(\prod_{j\in\{1,\ldots,n\}}\{(\beta_j)^{-1}(a_{i,j}),\ldots,(\beta_j)^{-1}(b_{i,j})-1\}\right)\subseteq\prod_{j\in\{1,\ldots,n\}}\{1,\ldots,n_j-1\} \tag{20.380}$$

For the opposite inclusion let $\rho\in\prod_{j\in\{1,\ldots,n\}}\{1,\ldots,n_j-1\}$. For $j\in\{1,\ldots,n\}$ $\rho_j\in\{1,\ldots,n_j-1\}$ we have by [eq: 20.370] that $\beta_j(\rho_j)<\beta_j(\rho_j+1)$ so that $\beta_j(\rho_j)\in[\beta_j(\rho_j),\beta_j(\rho_j)[$. Hence

$$\begin{aligned}
(\beta_1(\rho_1),\ldots,\beta_n(\rho_n)) &\in \prod_{j\in\{1,\ldots,n\}}[\beta_j(\rho_j),\beta_j(\rho_j)[\\
&\subseteq \bigsqcup_{\rho\in\prod_{i\in\{1,\ldots,n\}}\{1,\ldots,n_j-1\}}\left(\prod_{j\in\{1,\ldots,n\}}[\beta_j(\rho_j),\beta_j(\rho_j+1)[\right)\\
&\underset{[\text{eq: }20.375]}{=} \prod_{j\in\{1,\ldots,n\}}[a_j,b_j[\\
&\underset{\text{hypothesis}}{=} \bigsqcup_{i\in\{1,\ldots,m\}}\left(\prod_{j\in\{1,\ldots n\}}[a_{i,j},b_{i,j}[\right)
\end{aligned}$$

Hence

$$\exists k \in \{1, \ldots, m\} \text{ such that } (\beta_1(\rho_1), \ldots, \beta_n(\rho_n)) \in \prod_{j \in \{1, \ldots n\}} [a_{k,j}, b_{k,j}[\tag{20.381}$$

Assume now that $\exists j \in \{1,\ldots,n\}$ such that $b_{k,j} < \beta_j(\rho_j + 1)$ then $\beta_j(\rho_j) \underset{\text{[eq: 20.381]}}{<} b_{k,j} < \beta_j(\rho_j + 1)$ so that by [eq: 20.371] $(\beta_j)^{-1}(\beta_j(\rho_j)) < (\beta_j)^{-1}(b_{k,j}) < (\beta_j)^{-1}(\rho_j + 1)$, hence $\rho_j < (\beta_j)^{-1}(b_{k,j}) < \rho_j + 1$ from which it follows that $0 < (\beta_j)^{-1}(b_{k,j}) - 1 < 1$ which, as $(\beta_j)^{-1}(b_{k,j})$ is a whole number [because $(\beta_j)^{-1}(b_{k,j}) \in \{1, \ldots, n_j\}$], is impossible. Hence the assumption is wrong and we must have that

$$\forall j \in \{1, \ldots, n\} \ \beta_j(\rho_j + 1) \leqslant b_{k,j} \tag{20.382}$$

So given $j \in \{1, \ldots, n\}$ we hav $a_{k,j} \underset{\text{[eq: 20.381]}}{\leqslant} \beta_j(\rho_j) \underset{\text{[eq: 20.370]}}{<} \beta_j(\rho_j + 1) \underset{\text{[eq: 20.382}}{\leqslant} b_{k,j}$. applying then [eq: 20.371] proves that $(\beta_j)^{-1}(a_{k,j}) \leqslant \rho_j < \rho_j + 1 \leqslant (\beta_j)^{-1}(b_{k,j})$, hence $(\beta_j)^{-1}(a_{k,j}) \leqslant \rho_j \leqslant (\beta_j)^{-1}(b_{k,j}) - 1$ proving that $\rho_j \in \{(\beta_j)^{-1}(a_{k,j}), (\beta_j)^{-1}(b_{k,j}) - 1\}$ from which it follows that $\rho \in \prod_{j \in \{1, \ldots, n\}} \{(\beta_j)^{-1}(a_{k,j}), (\beta_j)^{-1}(b_{k,j}) - 1\}$. This prove that

$$\prod_{j \in \{1, \ldots, n\}} \{1, \ldots, n_j - 1\} \subseteq \bigcup_{i \in \{1, \ldots, m\}} \left(\prod_{j \in \{1, \ldots, n\}} \{(\beta_j)^{-1}(a_{i,j}), \ldots, (\beta_j)^{-1}(b_{i,j}) - 1\} \right)$$

which combined with [eq: 20.380] results in

$$\prod_{j \in \{1, \ldots, n\}} \{1, \ldots, n_j - 1\} = \bigcup_{i \in \{1, \ldots, m\}} \left(\prod_{j \in \{1, \ldots, n\}} \{(\beta_j)^{-1}(a_{i,j}), \ldots, (\beta_j)^{-1}(b_{i,j}) - 1\} \right) \tag{20.383}$$

To finish the prove for [eq: 20.379] we must prove that it is a pairwise disjoint union. So let k, $l \in \{1, \ldots, m\}$ with $k \neq l$ and assume that

$$\exists r \in \prod_{j \in \{1, \ldots, n\}} \{(\beta_j)^{-1}(a_{k,j}), \ldots, (\beta_j)^{-1}(b_{k,j}) - 1\} \bigcap \prod_{j \in \{1, \ldots, n\}} \{(\beta_j)^{-1}(a_{l,j}), \ldots, (\beta_j)^{-1}(b_{l,j}) - 1\}$$

then $\forall j \in \{1, \ldots, n\}$

$$(\beta_j)^{-1}(a_{k,j}) \leqslant r \leqslant (\beta_j)^{-1}(b_{k,j}) - 1 \wedge (\beta_j)^{-1}(a_{l,j}) \leqslant r \leqslant (\beta_j)^{-1}(b_{l,j}) - 1$$

or

$$(\beta_j)^{-1}(a_{k,j}) \leqslant r < (\beta_j)^{-1}(b_{k,j}) \wedge (\beta_j)^{-1}(a_{l,j}) \leqslant r < (\beta_j)^{-1}(b_{l,j})$$

applying then [eq: 20.370] gives

$$a_{k,j} \leqslant \beta_j(r) < b_{k,j} \wedge a_{l,j} \leqslant \beta_j(r) < b_{l,j} \text{ or } \beta_j(r) \in [a_{k,j}, b_{k,j}[\wedge \beta_j(r) \in [a_{l,j}, b_{l,j}[$$

proving that $(\beta_1(r), \ldots, \beta_n(r)) \in \left(\prod_{i \in \{1, \ldots, n\}} [a_{k,i}, b_{k,i}[\right) \bigcap \left(\prod_{i \in \{1, \ldots, n\}} [a_{l,i}, b_{l,i}[\right)$ which as by the hypothesis $(\beta_1(r), \ldots, \beta_n(r)) \in \left(\prod_{i \in \{1, \ldots, n\}} [a_{k,i}, b_{k,i}[\right) \bigcap \left(\prod_{i \in \{1, \ldots, n\}} [a_{l,i}, b_{l,i}[\right) = \varnothing$ is a contradiction hence we must have that

$$\prod_{j \in \{1, \ldots, n\}} \{(\beta_j)^{-1}(a_{k,j}), \ldots, (\beta_j)^{-1}(b_{k,j}) - 1\} \bigcap \prod_{j \in \{1, \ldots, n\}} \{(\beta_j)^{-1}(a_{l,j}), \ldots, (\beta_j)^{-1}(b_{l,j}) - 1\}$$

the above together with [eq: 20.383] proves finally [eq: 20.379]. □

Finally we are able to finish the prove of the lemma.

$$\prod_{i=1}^{n} (b_i - a_i) \underset{[\text{eq: } 20.376]}{=}$$

$$\sum_{\rho \in \prod_{i \in \{1,\ldots,n\}} \{1,\ldots,n_j-1\}} \left(\prod_{j \in \{1,\ldots,n\}} (\beta_j(\rho_j+1) - \beta_j(\rho_j)) \right) \underset{\substack{[\text{eq: } 20.379] \\ [\text{theorem: } 11.44]}}{=}$$

$$\sum_{i \in \{1,\ldots,m\}} \left(\sum_{\rho \in \prod_{j \in \{1,\ldots,n\}} \{(\beta_j)^{-1}(a_{i,j}),\ldots,(\beta_j)^{-1}(b_{i,j})-1\}} (\beta_j(\rho_j+1) - \beta_j(\rho_j)) \right) \underset{[\text{eq: } 20.378]}{=}$$

$$\sum_{i \in \{1,\ldots,m\}} \left(\prod_{j \in \{1,\ldots,n\}} (b_{i,j} - a_{i,j}) \right)$$

□

Now we are finally ready to prove that $\mathrm{vol}^n : \mathcal{R}^n \bigcup \{\varnothing\} \to [0, \infty]$ is a finite content.

Theorem 20.281. *Let $n \in \mathbb{N}$ then $\mathrm{vol}^n : \mathcal{R}^n \bigcup \{\varnothing\} \to [0, \infty]$ [see definition: 20.274] is a finite content on the semi-ring $\mathcal{R}^n \bigcup \{\varnothing\}$ on $\mathbb{R}^n$ [see theorem: 20.272]*

Proof. First from the definition of vol^n [see definition: 20.274] it follows that

$$\mathrm{vol}^n \text{ is finite} \tag{20.384}$$

and

$$\mathrm{vol}^n(\varnothing) = \varnothing \tag{20.385}$$

Let $m \in \mathbb{N}$ and $\{R_i\}_{i \in \{1,\ldots,m\}} \subseteq \mathcal{R}^n \bigcup \{\varnothing\}$ be pairwise disjoint such that $\bigsqcup_{i \in \{1,\ldots,m\}} R_i \in \mathcal{R}^n \bigcup \{\varnothing\}$ then we have two possibilities for $\bigsqcup_{i \in \{1,\ldots,m\}} R_i$

$\boldsymbol{\bigsqcup_{i \in \{1,\ldots,m\}} R_i = \varnothing}$**.** Then we must have $\forall i \in \{1,\ldots,m\}$ that $R_i = \varnothing$ so that

$$\mathrm{vol}^n\left(\bigsqcup_{i \in \{1,\ldots,m\}} R_i \right) = \mathrm{vol}^n(\varnothing) = 0 = \sum_{i=1}^{n} 0 = \sum_{i=1}^{n} \mathrm{vol}^n(\varnothing) = \sum_{i=1}^{n} \mathrm{vol}^n(R_i)$$

$\boldsymbol{\bigsqcup_{i \in \{1,\ldots,m\}} R_i \neq \varnothing}$**.** Then for $R = \bigsqcup_{i \in \{1,\ldots,m\}} R_i$ there exist a $\{[a_i, b_i[\}_{i \in \{1,\ldots,n\}} \subseteq \mathcal{I}_{[[}$ such that

$$R = \prod_{i \in \{1,\ldots,n\}} [a_i, b_i[\in \mathcal{R}^n \tag{20.386}$$

Then there exist a $i \in \{1,\ldots,m\}$ such that $R_i \neq \varnothing$ so that $I = \{i \in \{1,\ldots,m\} | R_i \neq \varnothing\} \neq \varnothing$ hence there exist a $k \in \mathbb{N}$ and a bijection

$$\beta : \{1,\ldots,k\} \to I$$

Then $\forall i \in \{1,\ldots,k\}$ $R_{\beta(i)} \neq \varnothing$ and $\forall i, j \in \{1,\ldots,k\}$ with $i \neq j$ we have, as β is injective, that $\beta(i) \neq \beta(j)$ so that $R_{\beta(i)} \bigcap R_{\beta(j)} = \varnothing$. So that

$$\{R_{\beta(i)}\}_{i \in \{1,\ldots,k\}} \subseteq \mathcal{R}^n \text{ is pairwise disjoint} \tag{20.387}$$

Let $R = \bigsqcup_{i \in \{1,\ldots,m\}} R_i$ then we have

$$R = \bigsqcup_{i \in \{1,\ldots,m\}} R_i \underset{[\text{theorem: } 2.134]}{=} \bigsqcup_{i \in I} R_i \underset{[\text{theorem: } 2.119]}{=} \bigsqcup_{i \in \{1,\ldots k\}} R_{\beta(i)} \tag{20.388}$$

Further as $\{R_{\beta(i)}\}_{i \in \{1,\ldots,k\}} \subseteq \mathcal{R}^n$ we have $\forall i \in \{1,\ldots,k\}$ that

$$R_{\beta(i)} = \Pi_{j \in \{1,\ldots,n\}} [a_{i,j}, b_{i,j}[\subseteq \mathcal{R}n. \tag{20.389}$$

So that

$$\prod_{i\in\{1,\ldots,n\}} [a_i,b_i[\underset{[\text{eq: } 20.386]}{=} \bigsqcup_{i\in\{1,\ldots,k\}} \left(\prod_{j\in\{1,\ldots,n\}} [a_{i,j},b_{i,j}[\right) \tag{20.390}$$

Then we have

$$\begin{aligned}
\mathrm{vol}^n(R) &= \prod_{i=1}^n (\mathrm{begin}(R)_i = \mathrm{end}(R)_i)\\
&= \prod_{i=1}^n (b_i - a_i)\\
&\underset{\substack{[\text{eq: } 20.390]\\ [\text{lemma: } 20.280]}}{=} \sum_{i=1}^k \left(\prod_{j=1}^n (b_{i,j} - a_{i,j})\right)\\
&\underset{[\text{eq: } 20.389]}{=} \sum_{i=1}^k \left(\prod_{j=1}^n (\mathrm{begin}(R_{\beta(i)})_j - \mathrm{end}(R_{\beta(j)})_j)\right)\\
&= \sum_{i=1}^k \mathrm{vol}^n(R_{\beta(i)})\\
&= \sum_{i\in\{1.,\ldots,k\}} \mathrm{vol}^n(R_{\beta(i)})\\
&\underset{[\text{theorem: } 11.36]}{=} \sum_{i\in I} \mathrm{vol}^n(R_i)\\
&= \sum_{i\in I} \mathrm{vol}^n(R_i) + 0\\
&\underset{i\notin I = R_i = \varnothing}{=} \sum_{i\in I} \mathrm{vol}^n(R_i) + \sum_{i\in\{1,\ldots,m\}\setminus I} \mathrm{vol}^n(R_i)\\
&= \sum_{i\in\{1,\ldots,m\}} \mathrm{vol}^n(R_i)\\
&= \sum_{i=1}^m \mathrm{vol}^n(R_i)
\end{aligned}$$

Hence

$$\mathrm{vol}^n\left(\bigsqcup_{i\in\{1,\ldots,m\}} R_i\right) = \sum_{i=1}^n \mathrm{vol}^n(R_i)$$

which finished the proof of the theorem. □

Next we use [theorem: 20.246] to prove that vol^n is actually a pre-measure on the semi-ring $\mathcal{R}^n \bigcup \{\varnothing\}$. To do this we need the following lemma that will allows us to 'aproximate' a half open rectangle by a compact set.

Lemma 20.282. *Let $n\in\mathbb{N}$, $\varepsilon\in\mathbb{R}^+$ and $\{x_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathbb{R}^+$ then $\exists\{\varepsilon_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathbb{R}^+$ with $0<\varepsilon_i<x_i$ such that $\prod_{i=1}^n x_i < (\prod_{i=1}^n (x_i-\varepsilon_i))+\varepsilon$*

Proof. We use induction on n. Let $\varepsilon\in\mathbb{R}^n$ and define

$$S_\varepsilon = \left\{ n\in\mathbb{N} | \text{If } \{x_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathbb{R}^+ \text{ then } \exists\{\varepsilon_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathbb{R}^+ \text{ with } \forall i\in\{1,\ldots,n\}\ 0<\varepsilon_i<x_i \text{ such that } \prod_{i=1}^n x_i < \left(\prod_{i=1}^n (x_i-\varepsilon_i)\right)+\varepsilon \right\}$$

then we have:

$1\in S_\varepsilon$. If $\{x_i\}_{i\in\{1,\ldots,1\}}\subseteq\mathbb{R}^+$ then $x_1\in\mathbb{R}^+$, define $\varepsilon_1=\min\left(\frac{x_1}{2},\frac{\varepsilon}{2}\right)\in\mathbb{R}^+$ then $0<\varepsilon_1\leqslant\frac{x_1}{2}<x_1$ and $\prod_{i=1}^{1} x_i-\prod_{i=1}^{1}(x_i-\varepsilon_i)=x_1-(x_1-\varepsilon_1)=\varepsilon_1\leqslant\frac{\varepsilon}{2}<\varepsilon$ so that $\prod_{i=1}^{1} x_i<\left(\prod_{i=1}^{1}(x_i-\varepsilon_i)\right)+\varepsilon$ proving that $1\in\mathcal{S}$.

$n\in S_\varepsilon\Rightarrow n+1\in S_\varepsilon$. Assume that $\{x_i\}_{i\in\{1,\ldots,n+1\}}\subseteq\mathbb{R}^+$ then as $n\in\mathcal{S}$ there exists $\{\varepsilon_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathbb{R}^+$ with $0<\varepsilon_i<x_i$ and $\prod_{i=1}^{n} x_i<\left(\prod_{i=1}^{n}(x_i-\varepsilon_i)\right)+\frac{\varepsilon}{2\cdot x_{n+1}}$ so that as $0<x_{n+1}$ we have

$$x_{n+1}\cdot\prod_{i=1}^{n} x_i-x_{n+1}\cdot\prod_{i=1}^{n}(x_i-\varepsilon_i)<\frac{\varepsilon}{2} \tag{20.391}$$

Take now $\varepsilon_{n+1}=\min\left(\frac{\varepsilon}{4\cdot\prod_{i=1}^{n} x_i},\frac{x_{n+1}}{2}\right)$ then $0<\varepsilon_{n+1}\leqslant\frac{x_{n+1}}{2}<x_{n+1}$ and $\varepsilon_{n+1}\cdot\left(\prod_{i=1}^{n} x_i\right)\leqslant\frac{\varepsilon}{4}<\frac{\varepsilon}{2}$. Now

$$\begin{aligned}
\prod_{i=1}^{n+1} x_i-\prod_{i=1}^{n+1}(x_i-\varepsilon_i) &= x_{n+1}\cdot\prod_{i=1}^{n} x_i-(x_{n+1}-\varepsilon_{n+1})\cdot\prod_{i=1}^{n}(x_i-\varepsilon_i)\\
&= x_{n+1}\cdot\left(\prod_{i=1}^{n} x_i-\prod_{i=1}^{n}(x_i-\varepsilon_i)\right)+\varepsilon_{n+1}\cdot\prod_{i=1}^{n}(x_i-\varepsilon_i)\\
&\leqslant x_{n+1}\cdot\left(\prod_{i=1}^{n} x_i-\prod_{i=1}^{n}(x_i-\varepsilon_i)\right)+\frac{\varepsilon}{4}\\
&\underset{[\text{eq: } 20.391]}{<} \frac{\varepsilon}{2}+\frac{\varepsilon}{4}\\
&< \varepsilon
\end{aligned}$$

proving that $n+1\in S_\varepsilon$. □

Now we can prove that the volume function is a pre-measure.

Theorem 20.283. *Let $n\in\mathbb{N}$ then the function $\mathrm{vol}^n:\mathcal{R}^n\bigcup\{\varnothing\}\to[0,\infty]$ is a finite pre-measure on the semi-ring $\mathcal{R}^n\bigcup\{\varnothing\}$.*

Proof. Consider the topological space $\langle\mathbb{R}^n,\|\|_n\rangle$ based on the canonical topology $\|\|_n$ [see theorem: 14.95]. Let $A\in\mathcal{R}^n\bigcup\{\varnothing\}$ then for A we have either:

$A=\varnothing$. Take then $A_\varepsilon=\varnothing$ and $K_\varepsilon=\varnothing$. Then K_ε is compact by [example: 14.226], $A_\varepsilon\subseteq K_\varepsilon\subseteq A$ and $\mathrm{vol}^n(A)-\mathrm{vol}^n(A_\varepsilon)=0-0=0<\varepsilon$.

$A\neq\varnothing$. Then there exiss $a,b\in\mathbb{R}^n$ with $a<b$ such that $A=[a,b[=\prod_{i\in\{1,\ldots,n\}}[a_i,b_i[$. As $\forall i\in\{1,\ldots,n\}$ $a_i<b_i\Rightarrow b_i-a_i\in\mathbb{R}^+$ we can use [lemma: 20.282] there exists a $\{\varepsilon_i\}_{i\in\{1,\ldots,n\}}\subseteq\mathbb{R}^+$ such that

$$\forall i\in\{1,\ldots,n\}\ 0<\varepsilon_i<b_i-a_i \text{ and } \prod_{i=1}^{n}(b_i-a_i)<\left(\prod_{i=1}^{n}((b_i-a_i)-\varepsilon_i)\right)+\varepsilon \tag{20.392}$$

From the above it follows that $\forall i\in\{1,\ldots,n\}$ $a_i<b_i-\varepsilon_i$ so that

$$A_\varepsilon=\prod_{i\in\{1,\ldots,n\}}[a_i,b_i-\varepsilon_i[\in\mathcal{R}^n\subseteq\mathcal{R}^n\bigcup\{\varnothing\}$$

Further by [theorem: 14.275]

$$K_\varepsilon=\prod_{i\in\{1,\ldots,n\}}[a,b_i-\varepsilon_i] \text{ is compact}$$

As $\forall i\in\{1,\ldots,n\}$ $[a_i,b_i-\varepsilon_i[\subseteq[a_i,b_i-\varepsilon_i]\subseteq[a_i,b_i]$ it follows from [theorem: 2.141] that

$$A_\varepsilon\subseteq K_\varepsilon\subseteq A$$

Further

$$\mathrm{vol}^n(A) - \mathrm{vol}^n(A_\varepsilon) = \prod_{i=1}^n (b_i - a_i) - \prod_{i=1}^n ((b_i - a_i) - \varepsilon_i) \underset{[\text{eq: } 20.392]}{<} \varepsilon$$

So in all cases we found a $A_\varepsilon \in \mathcal{R}^n \bigcup \{\varnothing\}$ and a compact set K_ε such that $A_\varepsilon \subseteq K_\varepsilon \subseteq A$ and $\mathrm{vol}^n(A) - \mathrm{vol}^n(A_\varepsilon) < \varepsilon$. Finally using [theorem: 20.246] it follows that

$$\mathrm{vol}^n : \mathcal{R}^n \bigcup \{\varnothing\} \to [0, \infty] \text{ is a pre-measure on } \mathcal{R}^n \bigcup \{\varnothing\}$$

which is finite by [theorem: 20.281]. □

Next we use a derivate of the Carathéodry theorem [theorem: 20.225] to extend vol^n to the Lebesgue measure on Lebesgue measurable sets in $\mathbb{R}^n$.

Theorem 20.284. *If we define*

$$(\lambda^n)^* : \mathcal{P}[\mathbb{R}^n] \to [0, \infty] \text{ by } (\lambda^n)^*(A) = \begin{cases} \infty \text{ if } \mathcal{S}[A] = \varnothing \\ \inf\left(\{\sum_{i=1}^\infty \mu(A_i) | \{A_i\}_{i \in \{1, \ldots, n\}} \in \mathcal{S}[A]\right) \end{cases}$$

where

$$\mathcal{S}[A] = \left\{ \{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{R}^n \bigsqcup \{\varnothing\} | A \subseteq \bigcup_{i \in \mathbb{N}} A_i \right\}$$

then

$$(\lambda^n)^* \text{ is a outer measure}$$

$$\mathcal{L}[\mathbb{R}^n] = \left\{ A \in \mathcal{P}[\mathbb{R}^n] | \forall B \in \mathcal{P}(\mathbb{R}^n) \text{ we have } (\lambda^n)^*(B) = (\lambda^n)^*\left(B \bigcap A\right) + (\lambda^n)^*(B \setminus A) \right\}$$

is a σ-algebra on $\mathbb{R}^n$ such that

$$\sigma\left[\mathcal{R}^n \bigcup \{\varnothing\}\right] \subseteq \mathcal{L}[\mathbb{R}^n]$$

and

$$\overline{\lambda^n} = ((\lambda^n)^\star)_{|\mathcal{L}(\mathbb{R}^n)} : \mathcal{L}[\mathbb{R}^n] \to [0, \infty] \text{ is a measure on } \mathcal{L}[\mathbb{R}^n]$$

and

$$\lambda^n = ((\lambda^n)^\star)_{|\sigma[\mathcal{R}^n \bigsqcup \{\varnothing\}]} : \sigma\left[\mathcal{R}^n \bigsqcup \{\varnothing\}\right] \to [0, \infty] \text{ is a measure on } \sigma\left[\mathcal{R}^n \bigsqcup \{\varnothing\}\right]$$

and

$$((\lambda^n)^\star)_{|\mathcal{R}^n \bigsqcup \{\varnothing\}} = \mathrm{vol}^n$$

Further if $\mu : \sigma[\mathcal{R}^n \bigsqcup \{\varnothing\}] \to [0, \infty]$ is another measure such that $\mu_{|\mathcal{R}^n \bigsqcup \{\varnothing\}} = \mathrm{vol}^n$ then

$$\mu = \lambda^n$$

We have then

$$\langle \mathbb{R}^n, \mathcal{L}[\mathbb{R}^n], \overline{\lambda^n} \rangle \text{ is a measure space called the Lebesgue mesaure space on } \mathbb{R}^n$$

where

$$\mathcal{L}[\mathbb{R}^n] \text{ is called the set of Lebesgue measurable sets in } \mathbb{R}^n$$

and

$$\overline{\lambda^n} \text{ is called the Lebesgue mearuse on } \mathbb{R}^n$$

***Note* 20.285.** *As by [theorem: 20.273] $\mathcal{B}[\mathbb{R}^n] = \sigma[\mathcal{R}^n \bigsqcup \{\varnothing\}]$ we have also*

$$\mathcal{B}[\mathbb{R}^n] \subseteq \mathcal{L}[\mathbb{R}^n]$$

and

$$\lambda^n : \mathcal{B}[\mathbb{R}^n] \to [0, \infty] \text{ is a measure on } \mathcal{B}[\mathbb{R}^n]$$

so that

$$\langle \mathbb{R}^n, \mathcal{B}[\mathbb{R}^n], \lambda^n \rangle \text{ is a measure space called the Borel measure space on } \mathbb{R}^n$$

and

$$\lambda^n \text{ is called the Borel measure on } \mathbb{R}^n$$

Proof. Using [theorem: 20.283] we have that

$$\mathrm{vol}^n: \mathcal{R}^n \bigsqcup \{\varnothing\} \to [0, \infty] \text{ is a pre-measure on } \mathbb{R}^n \tag{20.393}$$

Define now $\{G_i\}_{i\in\mathbb{N}} \subseteq \mathcal{R}^n$ by $G_i = \prod_{j\in\{1,\ldots,n\}} [-i, i[$. Then as $\forall i \in \mathbb{N}$ $[-i, i[\subseteq [-(i+1), i+1[$ it follows from [theorem: 2.141] that $G_i \subseteq G_{i+1}$, hence we have

$$\forall i \in \mathbb{N} \text{ we have } G_i \subseteq G_{i+1} \text{ and as } \mathrm{vol}^n \text{ is finite that } \mathrm{vol}^n(G_i) < \infty \tag{20.394}$$

Let $x \in \mathbb{R}^n$ then $\forall i \in \{1, \ldots, n\}$ we have by [theorem: 10.30] that there exists $n_i, m_i \in \mathbb{N}$ such that $-n_i < x_i < m_i$. Take $k = \max(\{n_i | i \in \mathbb{N}\} \bigcup \{m_i | i \in \mathbb{N}\}) \in \mathbb{N}$ then $\forall i \in \mathbb{N}$ we have $-k < n_i < x_i < m_i < k$ so that $x \in G_k \subseteq \bigcup_{i\in\mathbb{N}} G_i \subseteq \mathbb{R}^n$. Hence we have

$$\mathbb{R}^n = \bigcup_{i\in\mathbb{N}} G_i \tag{20.395}$$

The proof follows then from [eqs: 20.393, 20.394, 20.395] and [theorem: 20.225]. □

20.4.5 Completeness of measures

Definition 20.286. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space then $B \in \mathcal{P}(X)$ is* ***μ-negligible*** *[sometimes also called μ-null] if there exist a $A \in \mathcal{S}$ with $\mu(A) = 0$ such that $B \subseteq A$. The set of all μ-negligible sets is noted as $\mathcal{N}_\mu$ hence*

$$\mathcal{N}_\mu = \{A \in \mathcal{P}(X) | \exists B \in \mathcal{A} \text{ with } \mu(B) = \varnothing \text{ such that } A \subseteq B\}$$

We have the following properties for μ-negligible sets.

Theorem 20.287. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measurable space then we have*

1. $\varnothing \in \mathcal{N}_\mu$
2. *If $A \in \mathcal{N}_\mu$ then for every $B \subseteq A$ we have $B \in \mathcal{N}_\mu$*
3. *If $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{N}_\mu$ then we have $\bigcup_{i\in\mathbb{N}} A_i \subseteq \mathcal{N}_\mu$ and $\bigcap_{i\in\mathbb{N}} A_i \subseteq \mathcal{N}_\mu$*

Proof.

1. This is trivial as $\varnothing \subseteq \varnothing \in \mathcal{A}$ and $\mu(\varnothing) = 0$.
2. Let $A \in \mathcal{N}_\mu$ and $B \subseteq A$. As $A \in \mathcal{N}_\mu$ there exists a $C \in \mathcal{A}$ with $\mu(C) = 0$ such that $A \subseteq C$ which as $B \subseteq A$ proves that $B \subseteq C$ so that $B \in \mathcal{N}_\mu$.
3. Assume that $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{N}_\mu$ then there exists a $\{B_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$ such that $\forall i \in \mathbb{N}$ $\mu(B_i) = 0$ and $A_i \subseteq B_i$. Hence $\bigcup_{i\in\mathbb{N}} A_i \subseteq \bigcup_{i\in\mathbb{N}} B_i$ and using [theorem: 20.192] we have that

$$\mu\left(\bigcup_{i\in\mathbb{N}} B_i\right) \leqslant \sum_{i=1}^{\infty} \mu(B_i) = \sum_{i=1}^{\infty} 0 = 0$$

which proves that

$$\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{N}_\mu$$

Finally as $\bigcap_{i\in\mathbb{N}} A_i \subseteq A_1 \in \mathcal{N}_\mu$ it follows from (2) that $\bigcap_{i\in\mathbb{N}} A_i \in \mathcal{N}_\mu$. □

Definition 20.288. (Complete Measures) *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space then μ is* ***complete*** *if $\forall A \in \mathcal{A}$ with $\mu(A) = 0$ we have $\forall B \in \mathcal{P}(X)$ with $B \subseteq A$ that $B \in \mathcal{A}$. Every measure space with a complete measure is called a* ***complete measure space****.*

Theorem 20.289. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space then*

$$\mu \text{ is complete} \Leftrightarrow \mathcal{N}_\mu \subseteq \mathcal{A}$$

Proof.

⇒. If $A \in \mathcal{N}_\mu$ then there exists a $B \in \mathcal{A}$ with $\mu(B) = 0$ such that $A \subseteq B$. As μ is complete, $B \in \mathcal{A}$, $\mu(B) = 0$ and $A \subseteq B$ it follows that $A \in \mathcal{A}$. Hence $\mathcal{N}_A \subseteq \mathcal{A}$

⇐. Let $A \in \mathcal{A}$ with $\mu(A) = 0$ then for $B \in \mathcal{P}(X)$ with $B \subseteq A$ we have by definition of $\mathcal{N}_\mu$ that $B \in \mathcal{N}_\mu$ hence, as $\mathcal{N}_\mu \subseteq \mathcal{A}$, it follows that $B \in \mathcal{A}$ which proves that μ is complete. □

We prove now that every measure space constructed via the Carathéodory is a complete measure space.

Theorem 20.290. *Let X be a set and $\mu^*: \mathcal{P}(X) \to [0, \infty]$ is a **outer measure** then the measure space*

$$\langle X, \mathcal{M}[\mu^*], (\mu^*)_{|\mathcal{M}[\mu^*]}\rangle \text{ [see theorem: 20.200]}$$

where

$$\mathcal{M}[\mu^*] \underset{\text{[definition: 20.197]}}{=} \left\{A \in \mathcal{P}(X) | \forall B \in P(X) \text{ we have } \mu^*(B) = \mu^*\left(B \bigcap A\right) + \mu^*(B \setminus A)\right\}$$

is a complete measure space.

Proof. If $A \in \mathcal{N}_{(\mu^*)_{|\mathcal{M}[\mu^*]}}$ then there exists a $B \in \mathcal{M}[\mu^*]$ with $\mu^*(B) = (\mu^*)_{|\mathcal{M}[\mu^*]}(B) = 0$ such that $A \subseteq B$. Hence by the definition of a outer measure [see definition: 20.195] we have that $\mu^*(A) \leqslant \mu^*(B) = 0$ which proves that $\mu^*(A) = 0$. Applying then [theorem: 20.199] proves that $A \in \mathcal{M}[\mu^*]$. Hence $\mathcal{N}_{(\mu^*)_{|\mathcal{M}[\mu^*]}} \subseteq \mathcal{M}[\mu^*]$ which by the previous theorem [theorem: 20.289] proves that $\langle X, \mathcal{M}[\mu^*], (\mu^*)_{|\mathcal{M}[\mu^*]}\rangle$ is a complete measure space. □

Corollary 20.291. *We have that*

1. *$\left\langle \mathbb{R}, \mathcal{L}[\mathbb{R}], \overline{\lambda}\right\rangle$ is a complete measure space.*
2. *Let $n \in \mathbb{N}$ then $\langle \mathbb{R}, \mathcal{L}[\mathbb{R}^n], \overline{\lambda^n}\rangle$ is a complete measure space*

Proof.

1. This follows from the previous theorem [theorem: 20.261] and the definition of $\langle \mathbb{R}, \mathcal{L}[\mathbb{R}], \overline{\lambda}\rangle$ [see theorem: 20.261].
2. This follows from the previous theorem [theorem: 20.261] and the definition of $\langle \mathbb{R}^n, \mathcal{L}[\mathbb{R}^n], \overline{\lambda^n}\rangle$ [see theorem: 20.284]. □

Sometimes we have to deal with arbitrary subsets of sets with measure zero, it is then desirable to that there exists a extension of measure space to a bigger measure space in which these sets are elements of the bigger σ-algebra. To do this we introduce the concept of the **completion** of a measure space.

Definition 20.292. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space then the **completion** of $\mathcal{A}$ noted as $\mathcal{A}_\mu$ is defined by*

$$\mathcal{A}_\mu = \{A \in \mathcal{P}(X) | \exists E, F \in \mathcal{A} \text{ such that } E \subseteq A \subseteq F \text{ and } \mu(F \setminus E) = 0\}$$

elements of $\mathcal{A}_\mu$ are called μ-measurable sets.

***Note* 20.293.** *As by [theorem: 20.135] a σ-algebra is a algebra if follows from [theorem: 20.130] that $\forall E, F \in \mathcal{A}$ $\mu \setminus F \in \mathcal{A}$ so the above is well defined.*

Next we define a set function of $\overline{\mu}: \mathcal{A}_\mu \to [0, \infty]$, first we need a little lemma needed for the existence of such a function.

Lemma 20.294. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space $A \in \mathcal{P}(X)$ and $E, F \in \mathcal{A}$ such that $E \subseteq A \subseteq F$ and $\mu(F \setminus E) = 0$ then*

$$\{\mu(B) | B \in \mathcal{A} \text{ with } B \subseteq A\} \neq \varnothing \text{ [so that } \sup(\{\mu(B) | B \in \mathcal{A} \text{ with } B \subseteq A\}) \text{ exists]}$$

and

$$\mu(F) = \mu(E) = \sup(\{\mu(B) | B \in \mathcal{A} \text{ with } B \subseteq A\})$$

Proof. As $E \in \mathcal{A} \in \mathcal{A}$ and $E \subseteq A$ it follows that $\mu(E) \in \{\mu(B) | B \in \mathcal{A} \text{ with } B \subseteq A\}$ hence

$$\{\mu(B) | B \in \mathcal{A} \text{ with } B \subseteq A\} \neq \varnothing \tag{20.396}$$

and

$$\mu(E) \leqslant \sup(\{\mu(B) | B \in \mathcal{A} \text{ with } B \subseteq A\}) \tag{20.397}$$

As $E \subseteq A \subseteq F$ we have $E \subseteq F$ so that by [theorem: 1.35] $F = (F \setminus E) \bigsqcup E$ hence $\mu(F) = \mu(F \setminus E) + \mu(E) = 0 + \mu(E) = \mu(E)$ proving that

$$\mu(E) = \mu(F) \tag{20.398}$$

If $B \in \mathcal{A}$ with $B \subseteq A$ then $B \subseteq F$ so that by [theorem: 20.187] $\mu(B) \leqslant \mu(F) = \mu(E)$ hence

$$\sup(\{\mu(B) | B \in \mathcal{A} \text{ with } B \subseteq A\}) \leqslant \mu(E)$$

which combining with [eq: 20.397] proves that

$$\sup(\{\mu(B) | B \in \mathcal{A} \text{ with } B \subseteq A\})$$

□

Theorem 20.295. *Let $\langle X, \mathcal{A}, \mu\rangle$ be a measure space then if we define*

$$\overline{\mu}: \mathcal{A}_\mu \rightarrow [0, \infty] \text{ by } \overline{\mu}(A) = \sup(\{\mu(B) | B \in \mathcal{A} \text{ with } B \subseteq A\})$$

then we have

$$\forall A \in \mathcal{A}_\mu \text{ we have if } E, F \in \mathcal{A} \text{ satsifies } E \subseteq A \subseteq F \text{ and } \mu(F \setminus E) = 0 \text{ then } \mu(E) = \mu(F) = \overline{\mu}(A)$$

and

$$\langle X, \mathcal{A}_\mu, \overline{\mu}\rangle \text{ is a measure space}$$

and

$$\mathcal{A} \subseteq \mathcal{A}_\mu$$

and

$$\mu = (\overline{\mu})_{|\mathcal{A}}$$

In other words the measure space $\langle X, \mathcal{A}_\mu, \overline{\mu}\rangle$ is a extension of the measure space $\langle X, \mathcal{A}, \mu\rangle$. $\langle X, \mathcal{A}_\mu, \overline{\mu}\rangle$ is called the ***completion of the measure space*** *$\langle X, \mathcal{A}, \mu\rangle$*

Note 20.296. *$\overline{\mu}$ is welll defined because if $A \in \mathcal{A}_\mu$ then there exists $E, F \in \mathcal{A}$ with $E \subseteq A \subseteq F$ and $\mu(F \setminus E) = 0$ so that by [lemma: 20.294] $\{\mu(B) | B \in \mathcal{A} \text{ with } B \subseteq A\} \neq \varnothing$.*

Proof. Let $A \in \mathcal{A}_\mu$ then if $E, F \in \mathcal{A}$ is such that $E \subseteq A \subseteq F$ and $\mu(F \setminus E) = 0$ it follows from [lemma: 20.294] that

$$\mu(E) = \mu(F) = \sup(\{\mu(B) | B \in \mathcal{A} \text{ with } B \subseteq A\}) = \overline{\mu}(A) \tag{20.399}$$

Next we prove that $\mathcal{A}_\mu$ contains $\mathcal{A}$ and is a σ-algebra. Let $A \in \mathcal{A}$ and take $E = \varnothing \in \mathcal{A}$,, $F = A \in \mathcal{A}$ then $E = \varnothing \subseteq A \subseteq A = F$ and $\mu(F \setminus E) = \mu(\varnothing) = 0$ so that $A \in \mathcal{A}_\mu$. Hence we have

$$\mathcal{A} \subseteq \mathcal{A}_\mu \tag{20.400}$$

As $\mathcal{A}$ is a σ-algebra $\varnothing \in \mathcal{A}$ hence using the above it follows that

$$\varnothing \in \mathcal{A}_\mu \tag{20.401}$$

Let $A \in \mathcal{A}_\mu$ then there exists $E, F \in \mathcal{A}$ such that $E \subseteq A \subseteq F$ and $\mu(F \setminus E) = 0$. As $\mathcal{A}$ is a σ-algebra we have that

$$X \setminus E, X \setminus F \in \mathcal{A} \tag{20.402}$$

further applying [theorem: 1.33] gives

$$X \setminus F \subseteq X \setminus A \subseteq X \setminus E \tag{20.403}$$

Next

$$\begin{aligned}
x \in (X \setminus E) \setminus (X \setminus F) &\Leftrightarrow x \in X \setminus E \wedge x \notin X \setminus F \\
&\Leftrightarrow (x \in X \wedge x \notin E) \wedge \neg(x \in X \wedge x \notin F) \\
&\Leftrightarrow (x \in X \wedge x \notin E) \wedge (x \notin X \vee x \in F) \\
&\Leftrightarrow (x \in X \wedge x \notin E \wedge x \notin X) \vee (x \in X \wedge x \notin E \wedge x \in F) \\
&\Leftrightarrow x \in X \wedge x \notin E \wedge x \in F \\
&\underset{F \subseteq X}{\Leftrightarrow} x \notin E \wedge x \in F \\
&\Leftrightarrow F \setminus E
\end{aligned}$$

so that $(X \setminus E) \setminus (X \setminus F) = F \setminus E$ hence

$$\mu((X \setminus E) \setminus (X \setminus F)) = \mu(F \setminus E) = 0 \tag{20.404}$$

From [eqs: 20.402, 20.403 and 20.404] it follows that $X \setminus A \in \mathcal{A}_\mu$ so that

$$\forall A \in \mathcal{A}_\mu \text{ we have } X \setminus A \in \mathcal{A}_\mu \tag{20.405}$$

Let $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}_\mu$ then there exists $\{E_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$, $\{F_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ such that $\forall i \in \mathbb{N}$ $E_i \subseteq A_i \subseteq F_i$ and $\mu(F_i \setminus E_i) = 0$. Then we have

$$\bigcup_{i \in \mathbb{N}} E_i \subseteq \bigcup_{i \in \mathbb{N}} A_i \subseteq \bigcup_{i \in \mathbb{N}} F_i \text{ and, as } \mathcal{A} \text{ is a } \sigma\text{-algebra, } \bigcup_{i \in \mathbb{N}} E_i, \bigcup_{i \in \mathbb{N}} F_i \in \mathcal{A} \tag{20.406}$$

If $x \in (\bigcup_{i \in \mathbb{N}} F_i) \setminus (\bigcup_{i \in \mathbb{N}} E_i)$ then $x \in \bigcup_{i \in \mathbb{N}} F_i$ and $x \notin \bigcup_{i \in \mathbb{N}} E_i$, hence there exist a $i \in \mathbb{N}$ such that $x \in F_i$ and $\forall j \in \mathbb{N}$ $x \notin E_j$ in particular $x \notin E_i$, so $x \in F_i \setminus E_i$. So $(\bigcup_{i \in \mathbb{N}} F_i) \setminus (\bigcup_{i \in \mathbb{N}} E_i) \subseteq \bigcup_{i \in \mathbb{N}} (F_i \setminus E_i)$ where as $\mathcal{A}$ is a σ-algebra $F_i \setminus E_i \in \mathcal{A}$, hence

$$\mu\left(\left(\bigcup_{i \in \mathbb{N}} F_i\right) \setminus \left(\bigcup_{i \in \mathbb{N}} E_i\right)\right) \leqslant \mu\left(\bigcup_{i \in \mathbb{N}} (F_i \setminus E_i)\right) \underset{\text{[theorem: 20.192]}}{\leqslant} \sum_{i=1}^{\infty} \mu(F_i \setminus E_i) = 0 \tag{20.407}$$

From [eq: 20.406] and [eq: 20.407] it follows that $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}_\mu$ hence

$$\forall \{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}_\mu \text{ we have } \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}_\mu \tag{20.408}$$

From [eqs: 20.401, 20.405 and 20.408] it follows that

$$\mathcal{A}_\mu \text{ is a } \sigma\text{-algebra} \tag{20.409}$$

Now as $\varnothing \in \mathcal{A}$, $\varnothing \subseteq \varnothing \subseteq \varnothing$ and $\mu(\varnothing \setminus \varnothing) = 0$ it follows from [eq: 20.399] that

$$\overline{\mu}(\varnothing) = \mu(\varnothing) = 0 \tag{20.410}$$

If additional $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}_\mu$ is pairwise disjoint then $\forall i \in \mathbb{N}$ $E_i \subseteq A_i \subseteq F_i$ and $\mu(F_i \setminus E_i) = 0$. Let $i, j \in \mathbb{N}$ with $i \neq j$ then $E_i \bigcap E_j \subseteq A_i \bigcap A_j = \varnothing$ which proves that $\{E_i\}_{i \in \mathbb{N}}$ is pairwise disjoint. So

$$\begin{aligned}
\overline{\mu}\left(\bigsqcup_{i \in \mathbb{N}} A_i\right) &\underset{\text{[eq: 20.406, 20.407, 20.399]}}{=} \mu\left(\bigsqcup_{i \in \mathbb{N}} E_i\right) \\
&\underset{\mu \text{ is a measure}}{=} \sum_{i=1}^{\infty} \mu(E_i) \\
&\underset{\text{[eq: 20.399]}}{=} \sum_{i=1}^{\infty} \overline{\mu}(A_i)
\end{aligned}$$

The above together with [eq: 20.410] proves that

$$\overline{\mu} \text{ is a measure so that } \langle X, \mathcal{A}_\mu, \overline{\mu}\rangle \text{ is a measure space} \tag{20.411}$$

Let $A\in\mathcal{A}$ then as $A\subseteq A\subseteq A$ and $\mu(A\setminus A)=\mu(\varnothing)=0$ we have by [eq: 20.399] that $\overline{\mu}(A)=\mu(A)$, hence

$$(\overline{\mu})_{|A}=\mu \tag{20.412}$$

Let $A\in\mathcal{N}_{\overline{\mu}}$ then there exist a $B\in\mathcal{A}_\mu$ with $\overline{\mu}(B)=0$ such that $A\subseteq B$. As $B\in\mathcal{A}_\mu$ there exists $E,F\in\mathcal{A}$ with $E\subseteq B\subseteq F$ and $\mu(F\setminus E)=0$, hence by [eq: 20.399] $\mu(F)=\overline{\mu}(B)=0$. Next as $\varnothing\subseteq A\subseteq B\subseteq F$ and $\mu(F\setminus\varnothing)=\mu(F)=0$ it follows that $A\in\mathcal{A}_\mu$. This proves that $\mathcal{N}_{\overline{\mu}}\subseteq\mathcal{A}_\mu$ so that by [theorem: 20.289]

$$\langle X, \mathcal{A}_\mu, \overline{\mu}\rangle \text{ is complete}$$ □

20.5 Measurable Functions

20.5.1 Measurable function definition

We define now the functions that can be integrated.

Definition 20.297. (Measurable Function) *Let $\langle X,\mathcal{A}\rangle$, $\langle Y,\mathcal{B}\rangle$ be two measurable spaces and $A\in\mathcal{A}$ then a function*

$$f: A\rightarrow Y$$

is ***$\mathcal{A},\mathcal{B}$-measurable*** *if $\forall B\in\mathcal{B}$ we have that $f^{-1}(B)\in\mathcal{A}$. The set of $\mathcal{A},\mathcal{B}$-measurable function graphs is noted as $\mathfrak{M}[A,\mathcal{A},Y,\mathcal{B}]$ hence*

$$\mathfrak{M}[A,\mathcal{A},Y,\mathcal{B}]=\{f\in Y^A|f: A\rightarrow Y \text{ is } \mathcal{A},\mathcal{B}\text{-measurable}\}$$

***Note* 20.298.** *A set can have many σ-algebras defined on it, so the same function can at the same time be measurable and not measurable, depending on the σ-algebras used.*

Example 20.299. Let $\langle X,\mathcal{A}\rangle$, $\langle Y,\mathcal{B}\rangle$ be measurable spaces, $A\in\mathcal{A}$, $y\in Y$ and $C_y: A\rightarrow Y$ defined by $C_y(x)=y$ [the constant function then C_y is $\mathcal{A},\mathcal{B}$-measurable.

Proof. Let $B\in\mathcal{B}$ then we have two cases to check

$y\in B$. Then for $x\in A$ we have $C_y(x)=y\in B$ so that $C_y^{-1}(B)=A\in\mathcal{A}$

$y\notin B$. Assume that $x\in C_y^{-1}(B)$ then $C_y(x)=y\in B$ contradicting $y\notin B$, hence $C_y^{-1}(B)=\varnothing\in\mathcal{A}$

so in all cases $C_y^{-1}(B)\in\mathcal{A}$ which proves that C_y is $\mathcal{A},\mathcal{B}$-measurable. □

We have the following more specialized cases of measurable functions

Definition 20.300. *Let $\langle X,\mathcal{T}_X\rangle$, $\langle Y,\mathcal{T}_Y\rangle$ be two topological spaces, $A\in\mathcal{B}[X,\mathcal{T}_X]$ then a $\mathcal{B}[X,\mathcal{T}_X]$, $\mathcal{B}[Y,\mathcal{T}_Y]$-measurable $f: X\rightarrow Y$ function is called a* ***Borel measurable*** *function.*

Example 20.301. Let $A\in\mathcal{B}[\mathbb{R}]$ then a $\mathcal{B}[\mathbb{R}],\mathcal{B}[\overline{\mathbb{R}}]$-measurable function $f: A\rightarrow\overline{\mathbb{R}}$ is a Borel measurable function.

Example 20.302. Let $A\in\mathcal{B}[\mathbb{R}^n]$ then a $\mathcal{B}[\mathbb{R}^n],\mathcal{B}[\overline{\mathbb{R}}]$-measurable funciton $f: A\rightarrow\overline{\mathbb{R}}$ is a Borel measurable function.

For integration we are only interested in functions to either $\mathbb{R}$ or $\overline{\mathbb{R}}$ so we use some simplified notation for this.

Definition 20.303. *Let $\langle X, \mathcal{A}\rangle$ be a measurable space, $A \in \mathcal{A}$ then we define:*

1. *A $\mathcal{A}, \mathcal{B}[\overline{\mathbb{R}}]$-function $f: A \rightarrow \overline{\mathbb{R}}$ function is called a **$\mathcal{A}$-measurable** function. The set of $\mathcal{A}$-measurable function graphs is noted as $\overline{\mathfrak{M}[A, \mathcal{A}]}$ hence*

$$\overline{\mathfrak{M}[A, \mathcal{A}]} = \{f \in \overline{\mathbb{R}}^A | f: A \rightarrow \overline{\mathbb{R}} \text{ is } \mathcal{A}, \mathcal{B}[\overline{\mathbb{R}}]\text{-measurable}\}$$

2. *A $\mathcal{A}, \mathcal{B}[\mathbb{R}]$-function $f: A \rightarrow \mathbb{R}$ function is called a **finite $\mathcal{A}$-measurable** function hence*

$$\mathfrak{M}[A, \mathcal{A}] = \{f \in \mathbb{R}^A | f: A \rightarrow \mathbb{R} \text{ is } \mathcal{A}, \mathcal{B}[\mathbb{R}]\text{-measurable}\}$$

Note that in the previous we have to duplicate our statements between real functions and extended real functions. The following proposition allows us to significantly reduce the number of proofs and statements for measurability of real functions or extended real functions.

Theorem 20.304. *Let $\langle X, \mathcal{A}\rangle$ be a measurable space, $A \in \mathcal{A}$ then we have:*

1. $\mathfrak{M}[A, \mathcal{A}] \subseteq \overline{\mathfrak{M}[A, \mathcal{A}]}$
2. *If $f \in \overline{\mathfrak{M}[A, \mathcal{A}]}$ and $f(A) \subseteq \mathbb{R}$ then $f \in \mathfrak{M}[A, \mathcal{A}]$*

Proof.

1. Let $f \in \mathfrak{M}[A, \mathcal{A}]$ then $f: A \rightarrow \mathbb{R}$ is $\mathcal{A}, \mathcal{B}[\mathbb{R}]$-measurable. Let $B \in \mathcal{B}[\overline{\mathbb{R}}]$ then by [theorem: 20.156] $B \bigcap \mathbb{R} \in \mathcal{B}[\mathbb{R}]$ so that $f^{-1}(B \bigcap \mathbb{R}) \in \mathcal{A}$. Further

$$f^{-1}\left(B \bigcap \mathbb{R}\right) \underset{\text{[theorem: 2.95]}}{=} f^{-1}(B) \bigcap f^{-1}(\mathbb{R}) = f^{-1}(B) \bigcap A = f^{-1}(B)$$

   proving that $f^{-1}(B) \in \mathcal{A}$, hence $f: A \rightarrow \overline{\mathbb{R}}$ is $\mathcal{A}, \mathcal{B}[\overline{\mathbb{R}}]$-measurable so that $f \in \overline{\mathfrak{M}[A, \mathcal{A}]}$. Hence

$$\mathfrak{M}[A, \mathcal{A}] \subseteq \overline{\mathfrak{M}[A, \mathcal{A}]}$$

2. If $f \in \overline{\mathfrak{M}[A, \mathcal{A}]}$ then $f: A \rightarrow \overline{\mathbb{R}}$ is $\mathcal{A}, \mathcal{B}[\overline{\mathbb{R}}]$-measurable. Let $B \in \mathcal{B}[\mathbb{R}]$ then as $B \bigcap \mathbb{R} = \mathbb{R}$ we have by [theorem: 20.156] that $B \in \mathcal{B}[\overline{\mathbb{R}}]$ so that $f^{-1}(B) \in \mathcal{A}$. As $f(A) \subseteq \mathbb{R}$ the function $f: A \rightarrow \mathbb{R}$ is well defined and is $\mathcal{A}, \mathcal{B}[\mathbb{R}]$-measurable, hence $f \in \mathfrak{M}[A, \mathcal{A}]$. □

For the cases $X = \mathbb{R}$, $X = \mathbb{R}^n$ and we have also some extra notations.

Definition 20.305. *Let $A \in \mathcal{B}[\mathbb{R}]$ then a $\mathcal{B}[\mathbb{R}], \mathcal{B}[\overline{\mathbb{R}}]$-measurable function $f: A \rightarrow \overline{\mathbb{R}}$ is by definition a Borel measurable function and the set of $\mathcal{B}[\mathbb{R}], \mathcal{B}[\overline{\mathbb{R}}]$-measurable function graphs is noted as $\mathfrak{M}[A]$ or $\mathfrak{M}$ if $A = \mathbb{R}$ hence*

$$\mathfrak{M}[A] = \{f \in \overline{\mathbb{R}}^A | f: A \rightarrow \overline{\mathbb{R}} \text{ is } \mathcal{B}[\mathbb{R}], \mathcal{B}[\overline{\mathbb{R}}]\text{-measurable}\}$$

$$\mathfrak{M} = \{f \in \overline{\mathbb{R}}^{\mathbb{R}} | f: \mathbb{R} \rightarrow \overline{\mathbb{R}} \text{ is } \mathcal{B}[\mathbb{R}], \mathcal{B}[\overline{\mathbb{R}}]\text{-measurable}\}$$

Definition 20.306. *Let $n \in \mathbb{N}$, $A \in \mathcal{B}[\mathbb{R}^n]$ then a $\mathcal{B}[\mathbb{R}^n], \mathcal{B}[\overline{\mathbb{R}}]$-measurable function $f: A \rightarrow \overline{\mathbb{R}}$ is by definition a Borel measurable function and the set of $\mathcal{B}[\mathbb{R}^n], \mathcal{B}[\overline{\mathbb{R}}]$-measurable function graphs is noted as $\mathfrak{M}^n[A]$ or $\mathfrak{M}^n$ if $A = \mathbb{R}^n$ hence*

$$\mathfrak{M}^n[A] = \{f \in \overline{\mathbb{R}}^A | f: A \rightarrow \overline{\mathbb{R}} \text{ is } \mathcal{B}[\mathbb{R}^n], \mathcal{B}[\overline{\mathbb{R}}]\text{-measurable}\}$$

$$\mathfrak{M}^n = \{f \in \overline{\mathbb{R}}^{R^n} | f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}} \text{ is } \mathcal{B}[\mathbb{R}^n], \mathcal{B}[\overline{\mathbb{R}}]\text{-measurable}\}$$

Next we define Lebesgue measurable functions.

Definition 20.307. *Let $A \in \mathcal{L}[\mathbb{R}]$ then a $\mathcal{L}[\mathbb{R}], \mathcal{B}[\overline{\mathbb{R}}]$-measurable function $f: A \rightarrow \overline{\mathbb{R}}$ is called a Lebesgue measurable function and the set of $\mathcal{L}[\mathbb{R}], \mathcal{B}[\overline{\mathbb{R}}]$-measurable function is noted as $\mathfrak{L}[A]$ or $\mathfrak{L}$ if $A = \mathbb{R}$ hence*

$$\mathfrak{L}[A] = \{f \in \overline{\mathbb{R}}^A | f: A \rightarrow \overline{\mathbb{R}} \text{ is } \mathcal{L}[\mathbb{R}], \mathcal{B}[\overline{R}]\text{-measurable}\}$$

$$\mathfrak{L} = \{f \in \overline{\mathbb{R}}^R | f: \mathbb{R} \rightarrow \overline{\mathbb{R}} \text{ is } \mathcal{L}[\mathbb{R}], \mathcal{B}[\overline{R}]\text{-measurable}\}$$

Definition 20.308. *Let $n \in \mathbb{N}$, $A \in \mathcal{L}[\mathbb{R}^n]$ then a $\mathcal{L}[\mathbb{R}^n], \mathcal{B}[\overline{\mathbb{R}}]$-measurable function $f: A \rightarrow \overline{\mathbb{R}}$ is called a Lebesgue measurable function and the set of $\mathcal{L}[\mathbb{R}^n], \mathcal{B}[\overline{\mathbb{R}}]$-measurable function is noted as $\mathfrak{L}^n[A]$ or $\mathfrak{L}^n$ if $A = \mathbb{R}^n$ hence*

$$\mathfrak{L}^n[A] = \{f \in \overline{\mathbb{R}}^A | f: A \rightarrow \overline{\mathbb{R}} \text{ is } \mathcal{L}[\mathbb{R}], \mathcal{B}[\overline{R}]\text{-measurable}\}$$

$$\mathfrak{L}^n = \{f \in \overline{\mathbb{R}}^R | f: \mathbb{R} \rightarrow \overline{\mathbb{R}} \text{ is } \mathcal{L}[\mathbb{R}], \mathcal{B}[\overline{R}]\text{-measurable}\}$$

Note that Borel measurable implies Lebesgue measurability.

Theorem 20.309. *We have*

1. *If $A \in \mathcal{B}[\mathbb{R}]$ then a Borel measurable function $f: A \rightarrow \overline{\mathbb{R}}$ is Lebesgue measurable, in other words*

$$\mathfrak{M}[A] \subseteq \mathfrak{L}[A]$$

2. *Let $n \in \mathbb{N}$ and $A \in \mathcal{B}[\mathbb{R}]$ then a Borel measurable function $f: A \rightarrow \overline{\mathbb{R}}$ is Lebesgue measurable, in other words*

$$\mathfrak{M}^n[A] \subseteq \mathfrak{L}^n[A]$$

Proof.

1. Using [theorem: 20.261] $\mathcal{B}[\mathbb{R}] \subseteq \mathcal{L}[\mathbb{R}]$ hence $A \in \mathcal{L}[\mathbb{R}]$. Further if $B \in \mathcal{B}[\overline{\mathbb{R}}]$ then as $f \in \mathfrak{M}[A]$ we have that $f^{-1}(B) \in \mathcal{B}[\mathbb{R}]$ hence $f^{-1}(B) \in \mathcal{L}[\mathbb{R}]$ so that $f: A \rightarrow \overline{\mathbb{R}}$ is $\mathcal{L}[\mathbb{R}], \mathcal{B}[\mathbb{R}]$-measurable which proves that $f \in \mathfrak{L}[\mathbb{R}]$. Hence

$$\mathfrak{M}[A] \subseteq \mathfrak{L}[A]$$

2. Using [theorem: 20.284] $\mathcal{B}[\mathbb{R}^n] \subseteq \mathcal{L}[\mathbb{R}^n]$ hence $A \in \mathcal{L}[\mathbb{R}^n]$. Further if $B \in \mathcal{B}[\overline{\mathbb{R}^n}]$ then as $f \in \mathfrak{M}^n[A]$ we have that $f^{-1}(B) \in \mathcal{B}[\mathbb{R}^n]$ hence $f^{-1}(B) \in \mathcal{L}[\mathbb{R}^n]$ so that $f: A \rightarrow \overline{\mathbb{R}}$ is $\mathcal{L}[\mathbb{R}^n], \mathcal{B}[\mathbb{R}]$-measurable which proves that $f \in \mathfrak{L}^n[\mathbb{R}]$. Hence

$$\mathfrak{M}^n[A] \subseteq \mathfrak{L}^n[A]$$ □

In case of generated σ-algebras we have a easier way of checking that a function is measurable as is shown in the next theorem.

Theorem 20.310. *Let $\langle X, \mathcal{A}\rangle$ be a measurable space, $A \in \mathcal{A}$, Y a set and $\mathcal{B} \subseteq \mathcal{P}(Y)$ so that $\langle Y, \sigma[\mathcal{B}]\rangle$ is a measurable space and $f: A \rightarrow Y$ a function then*

$$f \text{ is } \mathcal{A}, \sigma[\mathcal{B}]\text{-measurable} \Leftrightarrow \forall B \in \mathcal{B} \text{ we have } f^{-1}(B) \in \mathcal{A}$$

Proof.

⇒. If f is $\mathcal{A}, \sigma[\mathcal{B}]$-measurable then if $B \in \mathcal{B}$ we have, as $\mathcal{B} \subseteq \sigma[\mathcal{B}]$, that $B \in \sigma[\mathcal{B}]$, hence $f^{-1}(B) \in \mathcal{A}$.

⇐. Define $\mathcal{H} = \{B \in \mathcal{P}(Y) | f^{-1}(B) \in \mathcal{A}\}$ and prove that $\mathcal{H}$ is a σ-algebra containing $\mathcal{B}$. If $B \in \mathcal{B}$ then by the hypothesis $f^{-1}(B) \in \mathcal{A}$ so that $\mathcal{B} \in \mathcal{H}$, hence

$$\mathcal{B} \subseteq \mathcal{H} \tag{20.413}$$

As $f^{-1}(Y) = A \in \mathcal{A}$ we have also

$$Y \in \mathcal{H} \tag{20.414}$$

Let $B \in \mathcal{H}$ then $f^{-1}(B) \in \mathcal{A}$ so that using [theorems: 20.131, 20.135] we have also that $A \setminus f^{-1}(B) \in \mathcal{A}$. As $f^{-1}(Y \setminus B) \underset{\text{[theorem: 2.94(4]}}{=} A \setminus f^{-1}(B)$ it follows that $f^{-1}(Y \setminus B) \in \mathcal{A}$ from which it follows that $Y \setminus B \in \mathcal{H}$. Hence

$$\forall B \in \mathcal{H} \text{ we have that } Y \setminus B \in \mathcal{H} \tag{20.415}$$

Let $\{B_i\}_{i \in \mathbb{N}} \subseteq \mathcal{H}$ then $\forall i \in \mathbb{N}$ $f^{-1}(B_i) \in \mathcal{A}$. As $f^{-1}(\bigcup_{i \in \mathbb{N}} B_i) \underset{\text{[theorem: 2.136]}}{=} \bigcup_{i \in \mathbb{N}} f^{-1}(B_i)$ and $\mathcal{A}$ is a σ-algebra it follows that $f^{-1}(\bigcup_{i \in \mathbb{N}} B_i) \in \mathcal{A}$ so that $\bigcup_{i \in \mathbb{N}} B_i \in \mathcal{H}$. Hence

$$\forall \{B_i\}_{i \in \mathbb{N}} \subseteq \mathcal{H} \text{ we have that } \bigcup_{i \in \mathbb{N}} B_i \in \mathcal{H} \tag{20.416}$$

From [eqs: 20.413, 20.414, 20.415 and 20.416] it follows that $\mathcal{H}$ is a σ-algebra containing $\mathcal{B}$ hence, using [theorem: 20.141], it follows that $\sigma[\mathcal{B}] \subseteq \mathcal{H}$. So $\forall B \in \sigma[\mathcal{B}]$ we have $B \in \mathcal{H}$ meaning that $f^{-1}(B) \in \mathcal{A}$ which proves that f is $\mathcal{A}, \sigma[\mathcal{B}]$-measurable. □

Most books give a alternative definition for [finite] $\mathcal{A}$-measurable functions based on the following definitions.

Definition 20.311. *Let X be a set then we define*

1. *If $f: X \rightarrow \overline{\mathbb{R}}$,$g: X \rightarrow \overline{\mathbb{R}}$ then given $x \in \overline{\mathbb{R}}$ we define*

$$\begin{aligned}
\{f < x\} &\underset{\text{def}}{=} f^{-1}([-\infty, x[) = \{t \in X | f(t) < x\} \\
\{f \leqslant x\} &\underset{\text{def}}{=} f^{-1}([-\infty, x]) = \{t \in X | f(t) \leqslant x\} \\
\{f > x\} &\underset{\text{def}}{=} f^{-1}(]x, \infty]) = \{t \in X | x < f(t)\} \\
\{f \geqslant x\} &\underset{\text{def}}{=} f^{-1}([x, \infty]) = \{t \in X | x \leqslant f(t)\} \\
\{f = x\} &\underset{\text{def}}{=} f^{-1}(\{x\}) = \{t \in X | x = f(t)\} \\
\{f \leqslant g\} &\underset{\text{def}}{=} \{x \in X | f(x) \leqslant g(x)\} \\
\{f < g\} &\underset{\text{def}}{=} \{x \in X | f(x) < g(x)\} \\
\{f = g\} &\underset{\text{def}}{=} \{x \in X | f(x) = g(x)\}
\end{aligned}$$

2. *If $f: X \rightarrow \mathbb{R}$, $g: X \rightarrow \mathbb{R}$ then given $x \in \mathbb{R}$ we define*

$$\begin{aligned}
\{f < x\} &\underset{\text{def}}{=} f^{-1}(]-\infty, x[) = \{t \in X | f(t) < x\} \\
\{f \leqslant x\} &\underset{\text{def}}{=} f^{-1}(]-\infty, x]) = \{t \in X | f(t) \leqslant x\} \\
\{f > x\} &\underset{\text{def}}{=} f^{-1}(]x, \infty[) = \{t \in X | x < f(t)\} \\
\{f \geqslant x\} &\underset{\text{def}}{=} f^{-1}([x, \infty[) = \{t \in X | x \leqslant f(t)\} \\
\{f = x\} &\underset{\text{def}}{=} f^{-1}(\{x\}) = \{t \in X | x = f(t)\} \\
\{f \leqslant g\} &\underset{\text{def}}{=} \{x \in X | f(x) \leqslant g(x)\} \\
\{f < g\} &\underset{\text{def}}{=} \{x \in X | f(x) < g(x)\} \\
\{f = g\} &\underset{\text{def}}{=} \{x \in X | f(x) = g(x)\}
\end{aligned}$$

Lemma 20.312. *Let X be a set, $f: X \rightarrow \overline{\mathbb{R}}$ and $x \in \mathbb{R}$ then*

1. $\{f \geqslant x\} = \bigcap_{n \in \mathbb{N}} \left\{f > x - \frac{1}{n}\right\}$
2. $X \setminus \{f \geqslant x\} = \{f < x\}$
3. $\{f \leqslant x\} = \bigcap_{n \in \mathbb{N}} \left\{f < x + \frac{1}{n}\right\}$
4. $X \setminus \{f \leqslant x\} = \{f > x\}$

Proof.

1. Let $n \in \mathbb{N}$ then if $t \in \{f \geqslant x\}$ we have that $x \leqslant f(t) \underset{x \in \mathbb{R}}{\Rightarrow} x - \frac{1}{n} < f(t) \Rightarrow t \in \left\{x - \frac{1}{n} < f\right\}$ proving

$$\{f \geqslant x\} \subseteq \bigcap_{n \in \mathbb{N}} \left\{f > x - \frac{1}{n}\right\} \tag{20.417}$$

If $t \in \bigcap_{n \in \mathbb{N}} \left\{f > x - \frac{1}{n}\right\}$ then $\forall n \in \mathbb{N}$ we have $x - \frac{1}{n} < f(t)$. Assume that $f(t) < x$ then by [theorem: 10.30] there exists a $n \in \mathbb{N}$ such that $f(t) + \frac{1}{n} < x \underset{x \in \mathbb{R}}{\Rightarrow} f(t) < x - \frac{1}{n}$ a contradiction, so we must have $x \leqslant f(t)$ or $t \in \{f \geqslant x\}$. So $\bigcap_{n \in \mathbb{N}} \left\{f > x - \frac{1}{n}\right\} \subseteq \{f \geqslant x\}$. Combining this with [eq: 20.417] proves

$$\{f \geqslant x\} = \bigcap_{n \in \mathbb{N}} \left\{f > x - \frac{1}{n}\right\}$$

2.

$$\begin{array}{rcl} X\setminus\{f\geqslant x\} & = & X\setminus f^{-1}([x,\infty]) \\ & = & f^{-1}(\overline{\mathbb{R}})\setminus f^{-1}(]x,\infty]) \\ & \underset{[\text{theorem: } 2.94]}{=} & f^{-1}(\overline{\mathbb{R}}\setminus[x,\infty]) \\ & = & f^{-1}([-\infty,x[) \\ & = & \{f<x\} \end{array}$$

3. Let $n\in\mathbb{N}$ then if $t\in\{f\leqslant x\}$ we have that $f(t)\leqslant x \underset{x\in\mathbb{R}}{\Rightarrow} f(t)<x+\frac{1}{n}$ or $t\in\left\{f<x+\frac{1}{n}\right\}$. So

$$\{f\leqslant x\}\subseteq\bigcap_{n\in\mathbb{N}}\left\{f<x+\frac{1}{n}\right\} \tag{20.418}$$

Let $t\in\bigcap_{n\in\mathbb{N}}\left\{f<x+\frac{1}{n}\right\}$ then $\forall n\in\mathbb{N}$ we have $f(t)<x+\frac{1}{n}$. Assume that $f(t)>x$ then by [theorem: 10.30] there exists a $n\in\mathbb{N}$ such that $x+\frac{1}{n}<f(t)$ a contradiction hence $f(t)\leqslant x$ or $t\in\{f\leqslant x\}$. Combining this with [eq: 20.418] proves that

$$\{f\leqslant x\}=\bigcap_{n\in\mathbb{N}}\left\{f<x+\frac{1}{n}\right\}$$

4.

$$\begin{array}{rcl} X\setminus\{f\leqslant x\} & = & X\setminus f^{-1}([-\infty,x]) \\ & = & f^{-1}(\overline{\mathbb{R}})\setminus f^{-1}([-\infty,x]) \\ & \underset{[\text{theorem: } 2.94]}{=} & f^{-1}(\overline{\mathbb{R}}\setminus[-\infty,x]) \\ & = & f^{-1}(]x,\infty]) \\ & = & \{x<f\} \end{array}$$

□

We have a similar lemma for finite functions.

Lemma 20.313. *Let X be a set, $f\colon X\rightarrow\mathbb{R}$ and $x\in\mathbb{R}$ then*

1. $\{f\geqslant x\}=\bigcap_{n\in\mathbb{N}}\left\{f>x-\frac{1}{n}\right\}$
2. $X\setminus\{f\geqslant x\}=\{f<x\}$
3. $\{f\leqslant x\}=\bigcap_{n\in\mathbb{N}}\left\{f<x+\frac{1}{n}\right\}$
4. $X\setminus\{f\leqslant x\}=\{f>x\}$

Proof.

1. Let $n\in\mathbb{N}$ then if $t\in\{f\geqslant x\}$ we have that $x\leqslant f(t) \underset{x\in\mathbb{R}}{\Rightarrow} x-\frac{1}{n}<f(t)\Rightarrow t\in\left\{x-\frac{1}{n}<f\right\}$ proving

$$\{f\geqslant x\}\subseteq\bigcap_{n\in\mathbb{N}}\left\{f>x-\frac{1}{n}\right\} \tag{20.419}$$

If $t\in\bigcap_{n\in\mathbb{N}}\left\{f>x-\frac{1}{n}\right\}$ then $\forall n\in\mathbb{N}$ we have $x-\frac{1}{n}<f(t)$. Assume that $f(t)<x$ then by [theorem: 10.30] there exists a $n\in\mathbb{N}$ such that $f(t)+\frac{1}{n}<x \underset{x\in\mathbb{R}}{\Rightarrow} f(t)<x-\frac{1}{n}$ a contradiction, so we must have $x\leqslant f(t)$ or $t\in\{f\geqslant x\}$. So $\bigcap_{n\in\mathbb{N}}\left\{f>x-\frac{1}{n}\right\}\subseteq\{f\geqslant x\}$. Combining this with [eq: 20.419] proves

$$\{f\geqslant x\}=\bigcap_{n\in\mathbb{N}}\left\{f>x-\frac{1}{n}\right\}$$

2.

$$\begin{array}{rcl} X\setminus\{f\geqslant x\} & = & X\setminus f^{-1}([x,\infty[) \\ & = & f^{-1}(\mathbb{R})\setminus f^{-1}(]x,\infty[) \\ & \underset{\text{[theorem: 2.94]}}{=} & f^{-1}(\mathbb{R}\setminus[x,\infty[) \\ & = & f^{-1}(]-\infty,x[) \\ & = & \{f<x\} \end{array}$$

3. Let $n\in\mathbb{N}$ then if $t\in\{f\leqslant x\}$ we have that $f(t)\leqslant x \underset{x\in\mathbb{R}}{\Rightarrow} f(t)<x+\frac{1}{n}$ or $t\in\left\{f<x+\frac{1}{n}\right\}$. So

$$\{f\leqslant x\}\subseteq\bigcap_{n\in\mathbb{N}}\left\{f<x+\frac{1}{n}\right\} \tag{20.420}$$

Let $t\in\bigcap_{n\in\mathbb{N}}\left\{f<x+\frac{1}{n}\right\}$ then $\forall n\in\mathbb{N}$ we have $f(t)<x+\frac{1}{n}$. Assume that $f(t)>x$ then by [theorem: 10.30] there exists a $n\in\mathbb{N}$ such that $x+\frac{1}{n}<f(t)$ a contradiction hence $f(t)\leqslant x$ or $t\in\{f\leqslant x\}$. Combining this with [eq: 20.420] proves that

$$\{f\leqslant x\}=\bigcap_{n\in\mathbb{N}}\left\{f<x+\frac{1}{n}\right\}$$

4.

$$\begin{array}{rcl} X\setminus\{f\leqslant x\} & = & X\setminus f^{-1}(]-\infty,x]) \\ & = & f^{-1}(\mathbb{R})\setminus f^{-1}(]-\infty,x]) \\ & \underset{\text{[theorem: 2.94]}}{=} & f^{-1}(\mathbb{R}\setminus]-\infty,x]) \\ & = & f^{-1}(]x,\infty[) \\ & = & \{x<f\} \end{array}$$

□

We are now ready to state the following equivalences for $\mathcal{A}$-measurability [which is used in many books about measure theory].

Theorem 20.314. *Let $\langle X,\mathcal{A}\rangle$ be a measurable set, $A\in\mathcal{A}$ and $f:A\rightarrow\overline{\mathbb{R}}$ then we have the following equivalences:*

1. *f is $\mathcal{A}$-measurable*
2. *$\forall x\in\mathbb{R}$ we have $\{f>x\}\in\mathcal{A}$*
3. *$\forall x\in\mathbb{R}$ we have $\{f\geqslant x\}\in\mathcal{A}$*
4. *$\forall x\in\mathbb{R}$ we have $\{f<x\}\in\mathcal{A}$*
5. *$\forall x\in\mathbb{R}$ we have $\{f\leqslant x\}\in\mathcal{A}$*

Proof.

1 ⇒ 2. Take $x\in\mathbb{R}$ then $]x,\infty]\in\sigma[\{]a,\infty]|a\in\mathbb{R}\}] \underset{\text{[theorem: 20.156]}}{=} \mathcal{B}[\overline{\mathbb{R}}]$ and we have, as f is $\mathcal{A}$-measurable, that $\{f>x\}=f^{-1}(]x,\infty])\in\mathcal{A}$

2 ⇒ 1. Let $B\in\{]a,\infty]|a\in\mathbb{R}\}$ then $B=]x,\infty]$ where $x\in\mathbb{R}$. hence

$$f^{-1}(B)=f^{-1}(]x,\infty])=\{f>x\}\in\mathcal{A}.$$

So using [theorem: 20.310] together with $\sigma[\{]a,\infty]|a\in\mathbb{R}\}] \underset{\text{[theorem: 20.156]}}{=} \mathcal{B}[\overline{\mathbb{R}}]$ it follows that f is $\mathcal{A},\mathcal{B}[\overline{\mathbb{R}}]$-measurable or in other words $\mathcal{A}$-measurable.

2 ⇒ 3. Let $x\in\mathbb{R}$ then using (2) we have $\forall n\in\mathbb{N}$ that $\left\{f>x-\frac{1}{n}\right\}\in\mathcal{A}$, hence by [theorem: 20.135] it follows that $\bigcap_{n\in\mathbb{N}}\left\{f>x-\frac{1}{n}\right\}\in\mathcal{A}$. Using [lemma: 20.312] $\{f\geqslant x\}=\bigcap_{n\in\mathbb{N}}\left\{f>x-\frac{1}{n}\right\}$ proving that $\{f\geqslant x\}\in\mathcal{A}$.

3 ⇒ 4. Let $x\in\mathbb{R}$ then $\{f<x\}\underset{[\text{lemma: } 20.312]}{=} A\setminus\{f\geqslant x\}\in\mathcal{A}$ [as $A,\{f\geqslant x\}\in\mathcal{A}$, together with [theorems: 20.130, 20.135]].

4 ⇒ 5. Let $x\in\mathbb{R}$ then by (4) we have that $\forall n\in\mathbb{N}$ that $\left\{f<x+\frac{1}{n}\right\}\in\mathcal{A}$, hence by [theorem: 20.135] it follows that $\bigcap_{n\in\mathbb{N}}\left\{f<x+\frac{1}{n}\right\}\in\mathcal{A}$. Using [lemma: 20.312] $\{f\leqslant x\}=\bigcap_{n\in\mathbb{N}_0}\left\{f<x+\frac{1}{n}\right\}$ proving that $\{f\leqslant x\}\in\mathcal{A}$

5 ⇒ 2. Let $x\in\mathbb{R}$ then $\{f>x\}\underset{[\text{lemma: } 20.312]}{=} A\setminus\{f\leqslant x\}\in\mathcal{A}$ [as $A\in\mathcal{A}$ together with [theorems: 20.130, 20.135]]. □

We have a similar theorem for finite functions.

Theorem 20.315. *Let $\langle X,\mathcal{A}\rangle$ be a measurable space, $A\in\mathcal{A}$ and $f\colon A\to\mathbb{R}$ then we have the following equivalences:*

1. *f is finite $\mathcal{A}$-measurable*
2. *$\forall x\in\mathbb{R}$ we have $\{f>x\}\in\mathcal{A}$*
3. *$\forall x\in\mathbb{R}$ we have $\{f\geqslant x\}\in\mathcal{A}$*
4. *$\forall x\in\mathbb{R}$ we have $\{f<x\}\in\mathcal{A}$*
5. *$\forall x\in\mathbb{R}$ we have $\{f\leqslant x\}\in\mathcal{A}$*

Proof.

1 ⇒ 2. Take $x\in\mathbb{R}$ then $]x,\infty[\,\in\sigma[\{]a,\infty[|a\in\mathbb{R}\}]\underset{[\text{theorem: } 20.150]}{=}\mathcal{B}[\mathbb{R}]$ hence we have, as f is finite $\mathcal{A}$-measurable, that $\{f>x\}=f^{-1}(]x,\infty[)\in\mathcal{A}$

2 ⇒ 1. Let $B\in\{]a,\infty|a\in\mathbb{R}[\}$ then there exist a $x\in\mathbb{R}$ such that $B=]x,\infty[$. So

$$f^{-1}(B)=f^{-1}(]x,\infty[)=\{f>x\}\in\mathcal{A}$$

So using [theorem: 20.310] together with $\mathcal{B}[\mathbb{R}]\underset{[\text{theorem: } 20.150]}{=}\sigma[\{]a,\infty|a\in\mathbb{R}[\}]$ it follows that $f=\mathcal{A},\mathcal{B}[\mathbb{R}]$-measurable or finite $\mathcal{A}$-measurable.

2 ⇒ 3. Let $x\in\mathbb{R}$ then using (2) we have $\forall n\in\mathbb{N}$ that $\left\{f>x-\frac{1}{n}\right\}\in\mathcal{A}$, hence by [theorem: 20.135] it follows that $\bigcap_{n\in\mathbb{N}}\left\{f>x-\frac{1}{n}\right\}\in\mathcal{A}$. Using [lemma: 20.313] $\{f\geqslant x\}=\bigcap_{n\in\mathbb{N}_0}\left\{f>x-\frac{1}{n}\right\}$ proving that $\{f\geqslant x\}\in\mathcal{A}$.

3 ⇒ 4. Let $x\in\mathbb{R}$ then $\{f<x\}\underset{[\text{lemma: } 20.313]}{=} A\setminus\{f\geqslant x\}\in\mathcal{A}$ [as $A,\{f\geqslant x\}\in\mathcal{A}$ together with [theorems: 20.130, 20.135]]

4 ⇒ 5. Let $x\in\mathbb{R}$ then by (4) we have that $\forall n\in\mathbb{N}$ that $\left\{f<x+\frac{1}{n}\right\}\in\mathcal{A}$ hence by [theorem: 20.135] it follows that $\bigcap_{n\in\mathbb{N}}\left\{f<x+\frac{1}{n}\right\}\in\mathcal{A}$. Using [lemma: 20.313] $\{f\leqslant x\}=\bigcap_{n\in\mathbb{N}}\left\{f<x+\frac{1}{n}\right\}$ proving that $\{f\leqslant x\}\in\mathcal{A}$

5 ⇒ 2. Let $x\in\mathbb{R}$ then $\{f>x\}\underset{[\text{lemma: } 20.313]}{=} A\setminus\{f\leqslant x\}\in\mathcal{A}$ [as $\{f\leqslant x\},A\in\mathcal{A}$ together with [theorems: 20.130, 20.135]] □

20.5.2 Measurable function properties

Theorem 20.316. *Let $\langle X,\mathcal{A}\rangle$, $\langle Y,\mathcal{B}\rangle$ be measurable spaces, $A,B\in\mathcal{A}$ with $B\subseteq A$ and $f\colon A\to Y$ a $\mathcal{A},\mathcal{B}$-measurable function then $f_{|B}\colon B\to Y$ is $\mathcal{A},\mathcal{B}$-measurable.*

Proof. Let $C\in\mathcal{B}$ then $(f_{|B})^{-1}(C)\underset{[\text{theorem: } 2.86]}{=}B\bigcap f^{-1}(C)\in\mathcal{A}$ [as $B,f^{-1}(C)\in\mathcal{A}$]. □

Theorem 20.317. *Let $\langle X,\mathcal{T}_X\rangle$, $\langle Y,\mathcal{T}_Y\rangle$ be topological spaces and $f\colon X\to Y$ a continuous function then f is $\mathcal{B}[X,\mathcal{T}_X],\mathcal{B}[Y,\mathcal{T}_Y]$-measurable. In other words continuous functions are Borel measurable.*

Proof. If $U \in \mathcal{T}_Y$ then as f is continuous we have $f^{-1}(U) \in \mathcal{T}_X \subseteq \mathcal{B}[X, \mathcal{T}_X]$ so that by [theorem: 20.310] we have that f is $\mathcal{B}[X, \mathcal{T}_X], \mathcal{B}[Y, \mathcal{T}_Y]$-measurable. □

Theorem 20.318. *Let $\langle X, \mathcal{A}\rangle$, $\langle Y, \mathcal{B}\rangle$,$\langle Z, \mathcal{C}\rangle$ be measurable spaces, $A \in \mathcal{A}$, $f: A \to Y$ a $\mathcal{A}, \mathcal{B}$-measurable function, $B \in \mathcal{B}$ such that $f(A) \subseteq \mathcal{B}$ and $g: B \to Z$ a $\mathcal{B}, \mathcal{C}$-measurable function then*

$$g \circ f: A \to Z$$

is a $\mathcal{A}, \mathcal{C}$ measurable function.

Proof. Take $B \in \mathcal{C}$ then, as g is $\mathcal{B}, \mathcal{C}$-measurable, $g^{-1}(B) \in \mathcal{B}$. As f is $\mathcal{A}, \mathcal{B}$-measurable it follows that $f^{-1}(g^{-1}(B)) \in \mathcal{A}$. So, as $(g \circ f)^{-1}(B) \underset{\text{[theorem: 2.23]}}{=} f^{-1}(g^{-1}(B))$ it follows that $(g \circ f)^{-1}(B) \in \mathcal{A}$. Hence $g \circ f$ is $\mathcal{A}, \mathcal{C}$-measurable. □

Note that although the composition of measurable functions is measurable this is not the case for Lebesgue measurable functions. For example if $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable functions then f is $\mathcal{L}[\mathbb{R}], \mathcal{B}[\mathbb{R}]$-measurable and g is $\mathcal{L}[\mathbb{R}], \mathcal{B}[\mathbb{R}]$–measurable not $\mathcal{B}[\mathbb{R}]$, $\mathcal{B}[\mathbb{R}]$-measurable, so we can not apply the previous theorem. If however g is $\mathcal{B}[\mathbb{R}], \mathcal{B}[\mathbb{R}]$-measurable then $g \circ f$ is by the previous theorem $\mathcal{L}[\mathbb{R}], \mathcal{B}[\mathbb{R}]$-measurable hence Lebesgue measurable.

Theorem 20.319. *Let $\langle X, \mathcal{A}\rangle$ be a measurable space and let $f: X \to \overline{\mathbb{R}}$ and $g: X \to \overline{\mathbb{R}}$ be $\mathcal{A}$-measurable functions then we have:*

1. $\{f < g\} \in \mathcal{A}$
2. $\{f \leqslant g\} \in \mathcal{A}$
3. $\{f = g\} \in \mathcal{A}$
4. $\{f \neq g\} \in \mathcal{A}$
5. *If $x \in \overline{\mathbb{R}}$ then $\{f = x\} \in \mathcal{A}$*

Proof.

1. Let $t \in \{f < g\}$ then $f(t) < g(t)$. Using the density of the rational numbers in the extended real numbers [see theorem: 20.12] there exist a $q \in \mathbb{Q}$ such that $f(t) < q < g(t)$, so that $t \in \{f < q\}$ and $t \in \{g > q\}$, proving $t \in \{f < q\} \bigcap \{g > q\}$ Hence we have

   $$\{f < g\} \subseteq \bigcup_{q \in \mathbb{Q}} (\{f < q\} \bigcap \{g > q\}) \tag{20.421}$$

   If $t \in \bigcup_{q \in \mathbb{Q}} (\{f < q\} \bigcap \{g > q\})$ then there exist a $q \in Q$ such that $t \in \{f < q\} \bigcap \{g > q\}$, hence $f(t) < q \wedge g(t) < q \Rightarrow f(t) < g(t)$ proving that $t \in \{f < g\}$. Hence $\bigcup_{q \in \mathbb{Q}} (\{f < q\} \bigcap \{g > q\}) \subseteq \{f < g\}$ which combined with [eq: 20.421] proves that

   $$\{f < g\} = \bigcup_{q \in \mathbb{Q}} (\{f < q\} \bigcap \{g > q\})$$

   Using [theorem: 20.314] we have that $\forall q \in Q$ $\{f < q\}, \{g > q\} \in \mathcal{A} \Rightarrow \{f < q\} \bigcap \{g > q\} \in \mathcal{A}$. As $\mathbb{Q}$ is denumerable [see theorem: 10.5] it follows from [theorem: 20.138] that $\bigcup_{q \in \mathbb{Q}} (\{f < q\} \bigcap \{g > q\}) \in \mathcal{A}$ so that

   $$\{f < g\} \in \mathcal{A}$$

2. We have

   $$\begin{aligned} t \in (X \setminus \{g < f\}) &\Leftrightarrow t \in X \wedge \neg(g(t) < f(t)) \\ &\Leftrightarrow t \in X \wedge f(t) \leqslant g(t) \\ &\Leftrightarrow t \in \{f \leqslant g\} \end{aligned}$$

   proving that $X \setminus \{g < f\} = \{f \leqslant g\}$. Using (1) we have $\{g < f\} \in \mathcal{A}$ hence $X \setminus \{g < f\} \in \mathcal{A}$ which proves that

   $$\{f \leqslant g\} \in \mathcal{A}$$

3. We have

$$\begin{aligned} t\in(\{f\leqslant g\}\setminus\{f<g\}) &\Leftrightarrow t\in\{f\leqslant g\}\wedge t\notin\{f<g\}\\ &\Leftrightarrow t\in X\wedge f(t)\leqslant g(t)\wedge\neg(f(t)<g(t))\\ &\Leftrightarrow t\in X\wedge f(t)\leqslant g(t)\wedge g(t)\leqslant f(t)\\ &\Leftrightarrow t\in X\wedge f(t)=g(t)\\ &\Leftrightarrow t\in\{f=g\}\end{aligned}$$

proving that $\{f=g\}=\{f\leqslant g\}\setminus\{f>g\}\in\mathcal{A}$ [using (1), (2) and [theorems: 20.131, 20.135]

4. We have

$$\begin{aligned} t\in X\setminus\{f=g\} &\Leftrightarrow t\in X\wedge\neg(f(t)=g(t))\\ &\Leftrightarrow t\in X\wedge f(t)\neq g(t)\\ &\Leftrightarrow t\in\{f\neq g\}\end{aligned}$$

proving that $\{f\neq g\}=X\setminus\{f=g\}\in\mathcal{A}$ [using (3)]

5. Let $x\in\overline{\mathbb{R}}$ then as $\{x\}$ is closed [see theorem: 20.49], we have by 20.143] that $\{x\}\in\mathcal{B}[\overline{\mathbb{R}}]$ which as f is $\mathcal{A}$-measurable proves that $\{f=x\}=f^{-1}(\{x\})\in\mathcal{A}$. □

Theorem 20.320. *Let $\langle X,\mathcal{A}\rangle$ be a measurable space, $A\in\mathcal{A}$, $f\colon A\rightarrow\overline{\mathbb{R}}$ a $\mathcal{A}$-measurable function and $g\colon A\rightarrow\overline{\mathbb{R}}$ a $\mathcal{A}$-measurable function then*

1. *$f\vee g\colon A\rightarrow[0,\infty]$ defined by $(f\vee g)(x)=\max(f(x),g(x))$ is a $\mathcal{A}$-measurable function.*
2. *$f\wedge g\colon A\rightarrow[0,\infty]$defined by $(f\wedge g)(x)=\min(f(x),g(x))$ is a $\mathcal{A}$-measurable function.*

Proof.

1. Let $x\in\mathbb{R}$ then

$$\begin{aligned} t\in\{(f\vee g)<x\} &\Leftrightarrow (f\vee g)(t)<x\\ &\Leftrightarrow \max(f(t),g(t))<x\\ &\Leftrightarrow f(t)<x\wedge g(t)<x\\ &\Leftrightarrow t\in\{f<x\}\wedge t\in\{g<x\}\\ &\Leftrightarrow t\in\{f<x\}\bigcap\{g<x\}\end{aligned}$$

which proves that

$$\{(f\vee g)<x\}=\{f<x\}\bigcap\{g<x\}$$

Using [theorem: 20.314] and the fact that f,g are $\mathcal{A}$-measurable it follows that $\{f<x\}$, $\{g<x\}\in\mathcal{A}$ from which it follows, using the above, that $\{(f\vee g)<x\}\in\mathcal{A}$. Applying [theorem: 20.314] again proves that $f\vee g$ is $\mathcal{A}$-measurable.

2. Let $x\in\mathbb{R}$ then

$$\begin{aligned} t\in\{(f\wedge g)>x\} &\Leftrightarrow (f\wedge g)(t)>x\\ &\Leftrightarrow \min(f(t),g(t))>x\\ &\Leftrightarrow f(t)>x\wedge g(t)>x\\ &\Leftrightarrow t\in\{f>x\}\vee t\in\{g>x\}\\ &\Leftrightarrow t\in\{f>x\}\bigcap\{g>x\}\end{aligned}$$

which proves that

$$\{(f\vee g)>x\}=\{f>x\}\bigcap\{g>x\}$$

Using [theorem: 20.314] and the fact that f,g are $\mathcal{A}$-measurable it follows that $\{f>x\}$, $\{g>x\}\in\mathcal{A}$ from which it follows, using the above, that $\{(f\vee g)>x\}\in\mathcal{A}$. Applying [theorem: 20.314] again proves that $f>g$ is $\mathcal{A}$-measurable. □

Using mathematical induction we can extend the above to a finite family of $\mathcal{A}$-measurable functions.

Corollary 20.321. *Let $\langle X,\mathcal{A}\rangle$ be a measurable space, $A\in\mathcal{A}$, $n\in\mathbb{N}$ and $\{f_i\}_{i\in\{1,\ldots,n\}}\subseteq\overline{\mathfrak{M}[A,\mathcal{A}]}$ then*

1. *$\bigvee_{i=1}^n f_i: A\rightarrow\overline{\mathbb{R}}$ defined by $(\bigvee_{i=1}^n f_i)(x)=\max(\{f_i(x)|i\in\{1,\ldots,n\}\})$ is $\mathcal{A}$-measurble. In other words $\bigvee_{i=1}^n f_i\in\overline{\mathfrak{M}[A,\mathcal{A}]}$*
2. *$\bigwedge_{i=1}^n f_i: A\rightarrow\overline{\mathbb{R}}$ defined by $(\bigwedge_{i=1}^n f_i)(x)=\min(\{f_i(x)|i\in\{1,\ldots,n\}\})$ is $\mathcal{A}$-measurable. In other words $\bigwedge_{i=1}^n f_i\in\overline{\mathfrak{M}[A,\mathcal{A}]}$.*

Proof. We use recursion to prove this.

1. Let

$$S=\left\{n\in\mathbb{N}|\text{If } \{f_i\}_{i\in\{1,\ldots,n\}}\subseteq\overline{\mathfrak{M}[A,\mathcal{A}]}\text{ then }\bigvee_{i=1}^n f_i\in\overline{\mathfrak{M}[A,\mathcal{A}]}\right\}$$

then we have:

$1\in S$. If $\{f_i\}_{i\in\{1\}}\subseteq\overline{\mathfrak{M}[A,\mathcal{A}]}$ then $(\bigvee_{i=1}^1 f)(x)=\max(\{f_1(x)\})=f_1(x)$ so that $\bigvee_{i=1}^1 f_i=f_1\in\overline{\mathfrak{M}[A,\mathcal{A}]}$, hence $1\in S$

$n\in S\Rightarrow n+1\in S$. Let $\{f_i\}_{i\in\{1,\ldots,n+1\}}\subseteq\overline{\mathfrak{M}[A,\mathcal{A}]}$ then $\forall x\in A$ we have

$$\begin{aligned}\left(\bigvee_{i=1}^{n+1} f_i\right)(x) &= \max(\{f_i(x)|i\in\{1,\ldots,n+1\}\})\\ &\underset{[\text{corollary: } 3.67]}{=} \max(\max(\{f_i(x)|i\in\{1,\ldots,n+1\}\}),f_{n+1}(x))\\ &= \max\left(\left(\bigvee_{i=1}^n f_i\right)(x),f_{n+1}(x)\right)\\ &= \left(\left(\bigvee_{i=1}^n f_i\right)\vee f_{n+1}\right)(x)\end{aligned}$$

so that

$$\bigvee_{i=1}^{n+1} f_i=\left(\bigvee_{i=1}^n f_i\right)\vee f_{n+1}$$

As $n\in S$ we have that $\bigvee_{i=1}^n f_i\in\overline{\mathfrak{M}[A,\mathcal{A}]}$, hence, using [theorem: 20.320] it follows that $(\bigvee_{i=1}^{n+1} f_i)\in\overline{\mathfrak{M}[A,\mathcal{A}]}$, proving $n+1\in\mathcal{S}$.

2. Let

$$S=\left\{n\in\mathbb{N}|\text{If } \{f_i\}_{i\in\{1,\ldots,n\}}\subseteq\overline{\mathfrak{M}[A,\mathcal{A}]}\text{ then }\bigwedge_{i=1}^n f_i\in\overline{\mathfrak{M}[A,\mathcal{A}]}\right\}$$

then we have:

$1\in S$. If $\{f_i\}_{i\in\{1\}}\subseteq\overline{\mathfrak{M}[A,\mathcal{A}]}$ then $(\bigwedge_{i=1}^1 f)(x)=\min(\{f_1(x)\})=f_1(x)$ so that $\bigwedge_{i=1}^1 f_i=f_1\in\overline{\mathfrak{M}[A,\mathcal{A}]}$, hence $1\in S$

$n\in S\Rightarrow n+1\in S$. Let $\{f_i\}_{i\in\{1,\ldots,n+1\}}\subseteq\overline{\mathfrak{M}[A,\mathcal{A}]}$ then $\forall x\in A$ we have

$$\begin{aligned}\left(\bigwedge_{i=1}^{n+1} f_i\right)(x) &= \min(\{f_i(x)|i\in\{1,\ldots,n+1\}\})\\ &\underset{[\text{corollary: } 3.67]}{=} \min(\min(\{f_i(x)|i\in\{1,\ldots,n+1\}\}),f_{n+1}(x))\\ &= \min\left(\left(\bigwedge_{i=1}^n f_i\right)(x),f_{n+1}(x)\right)\\ &= \left(\left(\bigwedge_{i=1}^n f_i\right)\wedge f_{n+1}\right)(x)\end{aligned}$$

so that

$$\bigwedge_{i=1}^{n+1} f_i = \left(\bigwedge_{i=1}^{n} f_i\right) \vee f_{n+1}$$

As $n \in S$ we have that $\bigwedge_{i=1}^{n} f_i \in \overline{\mathfrak{M}[A, \mathcal{A}]}$, hence, using [theorem: 20.320] it follows that $(\bigwedge_{i=1}^{n+1} f_i) \in \overline{\mathfrak{M}[A, \mathcal{A}]}$, proving $n + 1 \in \mathcal{S}$. □

Theorem 20.322. *Let $\langle X, \mathcal{A}\rangle$ be a measurable space, $A \in \mathcal{A}$, $f: A \to \overline{\mathbb{R}}$ and $g: A \to \overline{\mathbb{R}}$ $\mathcal{A}$-measurable functions such that $f(A) \subseteq [0, \infty] \wedge g(A) \subseteq [0, \infty]$ then the function*

$$f + g: A \to \overline{\mathbb{R}} \text{ where } (f + g)(x) = f(x) + g(x)$$

is $\mathcal{A}$-measurable.

Proof. Let $x \in \mathbb{R}$. If $t \in \{f + g < x\}$ then $f(t) + g(t) < x$. As $0 \leqslant f(t) + g(t) < x \in \mathbb{R}$ $f(t), g(t) \in \mathbb{R}$ so that $f(t) < x - g(t)$. Using the density of $\mathbb{Q}$ in $\mathbb{R}$ [see theorem: 10.32] there exists a $q \in \mathbb{Q}$ such that $f(t) < q < x - g(t)$ hence

$$f(t) < q \wedge g(t) < x - q \Rightarrow t \in \{f < q\} \bigcap \{g < x - q\} \subseteq \bigcup_{q \in \mathbb{Q}} \left(\{f < q\} \bigcap \{g < x - q\}\right)$$

which proves that

$$\{f + g < x\} \subseteq \bigcup_{q \in \mathbb{Q}} \left(\{f < q\} \bigcap \{g < x - q\}\right) \tag{20.422}$$

If $t \in \bigcup_{q \in \mathbb{Q}} (\{f < q\} \bigcap \{g < x - q\})$ then there exists a $q \in \mathbb{Q}$ such that $t \in \{f < q\} \bigcap \{g < x - q\}$ hence $f(t) < q \wedge g(t) < x - q \Rightarrow f(t) + g(t) < x - q + q = x$ or $t \in \{f + g < x\}$. So $\bigcup_{q \in \mathbb{Q}} (\{f < q\} \bigcap \{g < x - q\}) \subseteq \{f + g < x\}$ which combined with [eq: 20.422] results in

$$\{f + g < x\} = \bigcup_{q \in \mathbb{Q}} \left(\{f < q\} \bigcap \{g < x - q\}\right)$$

Let $q \in \mathbb{Q}$ then as f, g is $\mathcal{A}$-measurable $\{f < q\}, \{g < x - q\} \in \mathcal{A} \Rightarrow \{f < q\} \bigcap \{g < x - q\} \in \mathcal{A}$, using [theorem: 20.138] and the fact that $\mathbb{Q}$ is countable [see theorem: 10.5] it follows that $\bigcup_{q \in \mathbb{Q}} (\{f < q\} \bigcap \{g < x - q\}) \in \mathcal{A}$ hence we have

$$\{f + g < x\} \in \mathcal{A}$$

As $x \in \mathbb{R}$ was choosen arbitrary it follows from the above that $f + g$ is $\mathcal{A}$-measurable. □